DE GRUYTER

Journal of Numerical Mathematics 'Just Accepted' Papers

Journal of Numerical Mathematics 'Just Accepted' Papers are papers published online, in advance of appearing in the print journal. They have been peer-reviewed, accepted and are online published in manuscript form, but have not been copy edited, typeset, or proofread. Copy editing may lead to small differences between the Just Accepted version and the final version. There may also be differences in the quality of the graphics. When papers do appear in print, they will be removed from this feature and grouped with other papers in an issue.

Journal of Numerical Mathematics 'Just Accepted' Papers are citable; the online publication date is indicated on the Table of Contents page, and the article's Digital Object Identifier (DOI), a unique identifier for intellectual property in the digital environment (e.g., 10.1515/jnma-2016-xxxx), is shown at the top margin of the title page. Once an article is published as **Journal of Numerical Mathematics 'Just Accepted' Paper** (and before it is published in its final form), it should be cited in other articles by indicating author list, title and DOI.

After a paper is published in **Journal of Numerical Mathematics 'Just Accepted' Paper** form, it proceeds through the normal production process, which includes copy editing, typesetting and proofreading. The edited paper is then published in its final form in a regular print and online issue of **Journal of Numerical Mathematics**. At this time, the **Journal of Numerical Mathematics 'Just Accepted' Paper** version is replaced on the journal web site by the final version of the paper with the same DOI as the **Journal of Numerical Mathematics 'Just Accepted' Paper** version.

Disclaimer

Journal of Numerical Mathematics 'Just Accepted' Papers have undergone the complete peer-review process. However, none of the additional editorial preparation, which includes copy editing, typesetting and proofreading, has been performed. Therefore, there may be errors in articles published as **Journal of Numerical Mathematics 'Just Accepted' Papers** that will be corrected in the final print and online version of the Journal. Any use of these articles is subject to the explicit understanding that the papers have not yet gone through the full quality control process prior to advanced publication.

Journal of Numerical Mathematics
DOI 10.1515/jnma-2016-1103

Fictitious domain method with boundary value correction using penalty-free Nitsche method

Thomas Boiveau, Erik Burman, Susanne Claus and
Mats G. Larson

Abstract. In this paper, we consider a fictitious domain approach based on a Nitsche type method without penalty. To allow for high order approximation using piecewise affine approximation of the geometry we use a boundary value correction technique based on Taylor expansion from the approximate to the physical boundary. To ensure stability of the method a ghost penalty stabilization is considered in the boundary zone. We prove optimal error estimates in the H^1-norm and estimates suboptimal by $\mathcal{O}(h^{\frac{1}{2}})$ in the L^2-norm. The suboptimality is due to the lack of adjoint consistency of our formulation. Numerical results are provided to corroborate the theoretical study.

Keywords. Nitsche's method, fictitious domain method, boundary value correction.

2010 Mathematics Subject Classification. 65N12, 65N30, 65N85.

1 Introduction

Mesh generation is an important challenge in computational mechanics, in fact for complex geometries this can be highly nontrivial. In some cases for time dependent problems, such as a solid body embedded in a flow, the geometry of the problem changes each time step imposing conintuous remeshing, at least locally. The main idea of the fictitious domain method [1, 7, 8, 11–14, 18] is to relax the constraint that imposes the mesh to fit with the computational domain. In fact the principle is to embed the computational domain in a mesh that is easy to generate, without matching the elements with the boundary. In the early developments of fictitious domain [11], the method was faced with the choice of either integrating the equations over the whole computational mesh including the nonphysical part

This work received funding from EPSRC (award number EP/J002313/1) which is gratefully acknowledged. The Author, S. Claus, gratefully acknowledges the financial support provided by the Welsh Government and Higher Education Funding Council for Wales through the Sêr Cymru National Research Network in Advanced Engineering and Materials. The author M. G. Larson gratefully acknowledges the financial support from the Swedish Foundation for Strategic Research Grant No. AM13-0029, the Swedish Research Council Grant No. 2011-4992, and Swedish strategic research programme eSSENCE.

or only integrate inside the physical domain. In the first case, the method is robust but inaccurate, the second approach is accurate but can generate bad conditioning of the system matrix depending on how the boundary crosses the mesh. As a fix to solve the conditioning problem a boundary penalty term was introduced in [3] the effect of this term is that it extends the stability in the physical domain to the whole mesh domain, provided the distance from the mesh boundary to the physical boundary is $\mathcal{O}(h)$.

Nitsche's method was first introduced for the weak imposition of the boundary conditions in [19] and designed to be consistent and preserve the symmetry of the original problem. The stability of the method relies on a penalty term that needs to be sufficiently large. In the context of fictitious domain methods Nitsche's method can suffer from instability for certain mesh boundary configurations. A solution to this problem using the ghost penalty approach was suggested in [8, 18].

In [10] a non-symmetric version was proposed where the penalty parameter only needs to be strictly greater than zero for the stability to be ensured. The possibility of considering the penalty parameter equal to zero for the non-symmetric case was suggested in [15], however, coercivity cannot be proved for this non-symmetric penalty-free method and stability was not established. In [4] the stability of the nonsymmetric Nitsche's method without penalty for elliptic problems was proved, using an inf-sup argument, drawing on earlier work on discontinuous Galerkin methods [16]. Recently the work on penalty free methods has been extended to compressible and incompressible elasticity in [2]. The penalty-free method can be seen as a Lagrange multiplier method where the Lagrange multipliers has been replaced by the the boundary fluxes of the discrete elliptic operator. In multiphysics problems and particularly for fluid-structure interaction the loose coupling of this method appears to have some advantages that has been observed numerically in [6].

We consider a cut finite element method (CutFEM) [5] in the fictitious domain fashion, the implementation of this method often requires an approximation of the physical domain due to the boundary that can arbitrarily cut through the elements of the mesh. In this paper we propose a method to control the error introduced by this approximation of the physical domain. We follow the method that has been developed in [9] where a piecewise affine approximate geometry was used for the integration of the equation with a correction based on Taylor expansion from the approximate to the physical boundary to improve order when higher polynomial orders are used. In this work we use the penalty-free Nitsche's method to impose the boundary conditions. This eliminates one penalty parameter at the price of loss of symmetry of the algebraic system and $\mathcal{O}(h^{\frac{1}{2}})$ suboptimality in the L^2-norm. We believe that the method nevertheless may be of interest, in particular for problems

that are not symmetric, such as the advection–diffusion equation. We present and analyse the method in the two-dimensional case, but the results hold also in the three dimensional case.

We end this section by introducing our model problem. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ and exterior unit normal n. The Poisson problem is given by:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$

where $f \in L^2(\Omega)$ the given body force and $g \in H^{\frac{3}{2}}(\Omega)$ the boundary condition. The following regularity estimate holds

$$\|u\|_{H^{s+2}(\Omega)} \lesssim \|f\|_{H^s(\Omega)} \quad , \qquad s \geqslant -1. \tag{1}$$

In this paper C will be used as a generic positive constant that may change at each occurrence, we use the notation $a \lesssim b$ for $a \leqslant Cb$. We will also use $a \sim b$ to denote $a \lesssim b$ and $b \lesssim a$.

2 Preliminaries

Let $\{\mathcal{T}_h\}_h$ be a family of quasi-uniform and shape regular triangulations. In a generic sense a node of the triangulation is designated by x_i, K denotes a triangle of $\mathcal{T}_h$ and F denotes a face of a triangle K. $h_K = \operatorname{diam}(K)$ is the diameter of K and $h = \max_{K\in\mathcal{T}_h} h_K$ the mesh parameter for a given triangulation $\mathcal{T}_h$. $\mathbb{P}_p(K)$ defines the space of polynomials of degree less than or equal to p on the element K, $\Omega_{\mathcal{T}}$ is the domain covered by the mesh $\mathcal{T}_h$, let us introduce the following finite element space

$$V_h^p = \left\{ v \in H^1(\Omega_{\mathcal{T}}) : v|_K \in \mathbb{P}_p(K) \ \ \forall K \in \mathcal{T}_h \right\}.$$

For simplicity we will write the L^2-norm on a domain Θ, $\|\cdot\|_{L^2(\Theta)}$ as $\|\cdot\|_\Theta$. The domain Ω is embedded in a mesh $\mathcal{T}_h$. Figure 1 gives an example of a simple configuration.

We define ρ as the signed distance function, negative on the inside and positive on the outside of Ω. The tubular neighbourhood of Γ is defined as $U_\delta(\Gamma) = \{x \in \mathbb{R}^2 : |\rho(x)| < \delta\}$. We consider a constant $\delta_0 > 0$ such that the closest point mapping $p(x) : U_{\delta_0}(\Gamma) \to \Gamma$ is well defined and we have the identity $p(x) = x - \rho(x)n(p(x))$. We suppose that δ_0 is chosen small enough such that $p(x)$ is a bijection. Let the polygonal domain Ω_h with boundary Γ_h, be a domain approximating Ω. For simplicity we will assume that the discrete domain is defined by the zero levelset of the nodal interpolant of ρ on V_h^1. Then on a triangle

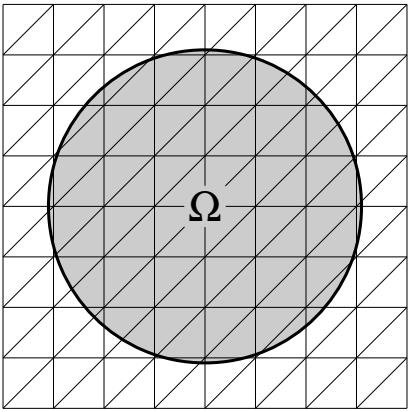


Figure 1. Domain Ω embedded in a background mesh $\mathcal{T}_h$.

K cut by the boundary, Γ_h restricted to K is a straight line. The discrete normal n_h denotes the exterior unit normal to Γ_h. Observe that n_h is constant on $\Gamma_h|_K$. We define $p_h(x,\varsigma) = x + \varsigma n_h(x)$, the function ϱ_h is defined by $\varrho_h : \Gamma_h \longrightarrow \mathbb{R}$ such that $p_h(x, \varrho_h(x)) \in \Gamma$ for all $x \in \Gamma_h$, for simplicity let $p_h(x) = p_h(x, \varrho_h(x))$, Figure 2 shows $p_h(x)$ for a value of $x \in \Gamma_h$. Observe that ϱ_h is well defined for h

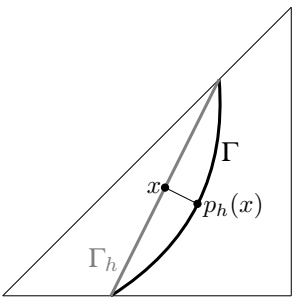


Figure 2. Representation of Γ and Γ_h on one element.

small enough (see [9]). We assume that $p_h(x,\varsigma) \in U_{\delta_0}(\Omega)$ for all $x \in \Gamma_h$ and all ς between 0 and $\varrho_h(x)$. By our definition of Ω_h we have

$$\delta_h = \|\varrho_h\|_{L^\infty(\Gamma_h)} = \mathcal{O}(h^2). \tag{2}$$

Let ω be a subset of $\Omega_{\mathcal{T}}$, we define

$$\mathcal{K}_h(\omega) = \left\{K \in \mathcal{T}_h \mid \overline{K} \cap \overline{\omega} \neq \varnothing\right\}, \qquad \mathcal{N}_h(\omega) = \cup_{K\in\mathcal{K}_h(\omega)} K,$$

$\mathcal{K}_h(\omega)$ defines a submesh of $\mathcal{T}_h$ and $\mathcal{N}_h(\omega)$ the union of the elements of this submesh. Let the norm

$$\|v\|_{\mathcal{N}_h(\omega)} = \left(\sum_{K\in\mathcal{K}_h(\omega)} \|v\|_K^2\right)^{\frac{1}{2}}.$$

We will use the notations

$$\begin{aligned}
\mathcal{K}_h &= \mathcal{K}_h(\Omega \cup \Omega_h),\\
\mathcal{K}_\Gamma &= \mathcal{K}_h(\Gamma) \cup \mathcal{K}_h(\Gamma_h),\\
\mathcal{N}_h &= \mathcal{N}_h(\Omega \cup \Omega_h).
\end{aligned}$$

We now recall the following trace inequalities for $v \in H^1(\mathcal{N}_h)$

$$\|v\|_{\partial K} \lesssim (h_K^{-\frac{1}{2}}\|v\|_K + h_K^{\frac{1}{2}}\|\nabla v\|_K) \quad \forall K \in \mathcal{K}_h, \tag{3}$$

$$\|v\|_{K\cap\Gamma} \lesssim (h_K^{-\frac{1}{2}}\|v\|_K + h_K^{\frac{1}{2}}\|\nabla v\|_K) \quad \forall K \in \mathcal{K}_h. \tag{4}$$

Inequality (4) has been shown in [13], we note that this is also true if we consider Γ_h instead of Γ. For $v_h \in V_h^p$ the following inverse estimate holds

$$\|\nabla v_h\|_K \lesssim h_K^{-1}\|v_h\|_K \quad \forall K \in \mathcal{K}_h. \tag{5}$$

The following inequality has been shown in [9] for all $v \in H^1(\Omega_h)$

$$\|v\|_{\Omega_h\setminus\Omega}^2 \lesssim \delta_h^2\|\nabla v \cdot n\|_{\Omega_h\setminus\Omega}^2 + \delta_h\|v\|_{\Gamma_h}^2. \tag{6}$$

The ghost penalty [3] is introduced to ensure the well conditioning of the system matrix, it also provides the control of the gradient in case of small cut elements

$$J_h(u_h, v_h) = \gamma_g \sum_{F\in\mathcal{F}_G}\sum_{l=1}^{p} h^{2l-1}\langle[\![D_{n_F}^l u_h]\!]_F, [\![D_{n_F}^l v_h]\!]_F\rangle_F,$$

with $\mathcal{F}_G = \{F \in \mathcal{K}_\Gamma \mid F \cap (\Omega \cup \Omega_h) \neq \varnothing\}$, γ_g the ghost penalty parameter and n_F the unit normal to the face F with fixed but arbitrary orientation. $D_{n_F}^l$ is the partial derivative of order l in the direction n_F and $[\![w]\!]_F = w_F^+ - w_F^-$, with $w_F^\pm = \lim_{s\to 0^+} w(x \mp s n_F)$ is the jump across a face F. The following estimate has been shown in [18] for all $v_h \in V_h^p$

$$\|\nabla v_h\|_{\mathcal{N}_h}^2 \lesssim \|\nabla v_h\|_{\Omega_h}^2 + J_h(v_h, v_h) \lesssim \|\nabla v_h\|_{\mathcal{N}_h}^2. \tag{7}$$

We now construct an interpolation operator π_h. Let $\mathbb{E}$ be an H^s-extension on $\mathcal{N}_h$, $\mathbb{E} : H^s(\Omega) \to H^s(\mathcal{N}_h)$ such that for all $w \in (\mathbb{E}w)|_\Omega = w$ and

$$\|\mathbb{E}w\|_{H^s(\mathcal{N}_h)} \lesssim \|w\|_{H^s(\Omega)} \qquad \forall w \in H^s(\Omega), s \geqslant 0. \tag{8}$$

For simplicity we will write w instead of $\mathbb{E}w$. Let $\pi_h^* : H^s(\mathcal{N}_h) \to V_h^p$ be the Lagrange interpolant, we construct the interpolation operator π_h such that

$$\pi_h u = \pi_h^* \mathbb{E} u. \tag{9}$$

We have the interpolation estimate for $0 \leqslant r \leqslant s \leqslant p+1$,

$$\|u - \pi_h^* u\|_{H^r(K)} \lesssim h^{s-r} |u|_{H^s(K)} \qquad \forall K \in \mathcal{K}_h. \tag{10}$$

Using the estimate (8) together with (10) we have

$$\Big(\sum_{K \in \mathcal{N}_h} \|u - \pi_h u\|^2_{H^r(K)} \Big)^{\frac{1}{2}} \lesssim h^{s-r} |u|_{H^s(\Omega)}. \tag{11}$$

Let us introduce the norms

$$\begin{aligned} |||w|||_h^2 &= \|\nabla w\|^2_{\Omega_h} + h^{-1}\|w\|^2_{\Gamma_h} + J_h(w,w), \\ \|w\|_*^2 &= |||w|||_h^2 + h\|\nabla w \cdot n_h\|^2_{\Gamma_h} + h^{-1}\|T_{1,k}(w)\|^2_{\Gamma_h}, \end{aligned}$$

with the Taylor expansion defined for $0 \leqslant m \leqslant k$ such that

$$T_{m,k}(u)(x) = \sum_{i=m}^{k} \frac{D^i_{n_h} u(x)}{i!} \varrho_h^i(x), \tag{12}$$

$D^i_{n_h}$ is the derivative of order i in the direction n_h. The interpolation properties introduced above give us

$$|||u - \pi_h u|||_h \leqslant \|u - \pi_h u\|_* \lesssim h^p |u|_{H^{p+1}(\Omega)}, \tag{13}$$

the key point in showing this property is obtained using the trace inequality (3)

$$h^{-1}\|T_{1,k}(u - \mathcal{I}_h u)\|^2_{\Gamma_h} \lesssim h^{-2}\|T_{1,k}(u - \mathcal{I}_h u)\|^2_{\mathcal{N}_h} + \|\nabla T_{1,k}(u - \mathcal{I}_h u)\|^2_{\mathcal{N}_h},$$

and for all $K \in \mathcal{N}_h$

$$\begin{aligned} h^{-1}\|T_{1,k}(u - \mathcal{I}_h u)\|_K &\lesssim \sum_{i=1}^{k} \delta_h^i h^{-1}\|u - \mathcal{I}_h u\|_{H^i(K)}, \\ \|\nabla T_{1,k}(u - \mathcal{I}_h u)\|_K &\lesssim \sum_{i=1}^{k} \delta_h^i \|u - \mathcal{I}_h u\|_{H^i(K)}, \end{aligned}$$

using the estimate (11) with $\delta_h < h$ the estimate is recovered. The other terms are straightforward to handle using the interpolation properties.

3 Finite element formulation

Here we use the boundary value correction approach from [9], we write the extensions of f and u respectively as $f = \mathbb{E}f$ and $u = \mathbb{E}u$.

$$\begin{aligned}(f, v)_{\Omega_h} &= (f + \Delta u, v)_{\Omega_h} - (\Delta u, v)_{\Omega_h} \\ &= (f + \Delta u, v)_{\Omega_h \backslash \Omega} + (\nabla u, \nabla v)_{\Omega_h} - \langle \nabla u \cdot n_h, v \rangle_{\Gamma_h}.\end{aligned}$$

We know that $f + \Delta u = 0$ on Ω. On $\Omega_h \backslash \Omega$ we have $f + \Delta u = \mathbb{E}f + \Delta \mathbb{E}u \neq 0$. Let us enforce weakly the boundary condition $u = g$ on Γ by adding a consistent boundary term

$$\begin{aligned}(f, v)_{\Omega_h} = (f + \Delta u, v)_{\Omega_h \backslash \Omega} + (\nabla u, \nabla v)_{\Omega_h} - \langle \nabla u \cdot n_h, v \rangle_{\Gamma_h} \\ + \langle \nabla v \cdot n_h, u \circ p_h - g \circ p_h \rangle_{\Gamma_h}.\end{aligned}$$

Remark that this is equivalent to the penalty-free Nitsche's method [2,4]. However, we cannot access $u \circ p_h$, so we use a Taylor approximation in the direction n_h (12)

$$u \circ p_h(x) \approx T_{0,k}(u)(x). \tag{14}$$

We note that in (12) we could replace n_h by $n \circ p$ and ϱ_h by ϱ if these quantities were available (as mentioned in [9]). Adding and subtracting the Taylor expansion in the Nitsche antisymmetric term and rearranging we obtain

$$\begin{aligned}(\nabla u, \nabla v)_{\Omega_h} - \langle \nabla u \cdot n_h, v \rangle_{\Gamma_h} + \langle \nabla v \cdot n_h, T_{0,k}(u) \rangle_{\Gamma_h} + (f + \Delta u, v)_{\Omega_h \backslash \Omega} \\ + \langle \nabla v \cdot n_h, u \circ p_h - T_{0,k}(u) \rangle_{\Gamma_h} = (f, v)_{\Omega_h} + \langle \nabla v \cdot n_h, g \circ p_h \rangle_{\Gamma_h}.\end{aligned} \tag{15}$$

The discrete formulation is obtained by dropping the terms $(f + \Delta u, v)_{\Omega_h \backslash \Omega}$ and $\langle \nabla v \cdot n_h, u \circ p_h - T_{0,k}(u) \rangle_{\Gamma_h}$. Find $u_h \in V_h^p$

$$A_h(u_h, v_h) + J_h(u_h, v_h) = L_h(v_h) \qquad \forall v_h \in V_h^p, \tag{16}$$

with the linear forms

$$\begin{aligned}A_h(u_h, v_h) &= (\nabla u_h, \nabla v_h)_{\Omega_h} - \langle \nabla u_h \cdot n_h, v_h \rangle_{\Gamma_h} + \langle \nabla v_h \cdot n_h, T_{0,k}(u_h) \rangle_{\Gamma_h}, \\ L_h(v_h) &= (f, v_h)_{\Omega_h} + \langle \nabla v_h \cdot n_h, g \circ p_h \rangle_{\Gamma_h}.\end{aligned}$$

Using the definition of the Taylor expansion, the bilinear form A_h can be written as

$$\begin{aligned}A_h(u_h, v_h) = (\nabla u_h, \nabla v_h)_{\Omega_h} - \langle \nabla u_h \cdot n_h, v_h \rangle_{\Gamma_h} + \langle \nabla v_h \cdot n_h, u_h \rangle_{\Gamma_h} \\ + \langle \nabla v_h \cdot n_h, T_{1,k}(u_h) \rangle_{\Gamma_h}.\end{aligned}$$

The terms that has been dropped in the discrete formulation are defined as

$$B_h(u, v_h) = (f + \Delta u, v_h)_{\Omega_h \setminus \Omega} + \langle \nabla v_h \cdot n_h, u \circ p_h - T_{0,k}(u) \rangle_{\Gamma_h} \quad \forall v_h \in V_h^p. \tag{17}$$

In Section 4 we show an inf-sup condition for the discrete formulation (16). In Section 5 the high order terms of $B_h(u, v_h)$ are treated. Section 6 presents the error estimates.

4 Inf-sup condition on Ω_h

We assume that Ω_h is defined by the zero level set of the nodal interpolant $\mathcal{I}_h \rho$ of the distance function ρ. We also assume that h is small enough so that a band of elements in $\mathcal{K}_h(\Gamma_h)$ is in the tubular $U_{\delta_0}(\Gamma)$ and that in every $K \in \mathcal{K}_h(\Gamma_h)$ there holds

$$\|\rho - \mathcal{I}_h \rho\|_{L^\infty(K)} + h \|\nabla(\rho - \mathcal{I}_h \rho)\|_{L^\infty(K)} \leqslant c_\rho h$$

where the constant c_ρ only depends on the regularity of the interface and, since ρ is a distance function, for $x \in \mathcal{K}_h(\Gamma_h)$ we have

$$1 \leqslant |\nabla \rho(x)| \leqslant C_1, \tag{18}$$

with $C_1 > 1$ a constant of order 1. We now introduce boundary patches that will be useful for the upcoming inf-sup analysis. Let us consider the set $\mathcal{K}_h(\Gamma_h)$ and split it into N_p smaller disjoint sets of elements $\mathcal{K}_j$ with $j = 1, \ldots, N_p$, then we define

$$P_j = \mathcal{K}_j \cup \{K \in \mathcal{T}_h : K \cap \Omega_h \neq \varnothing, \exists K' \in \mathcal{K}_j \text{ such that } K \cap K' \neq \varnothing\}.$$

This means that P_j consists of the elements on $\mathcal{K}_j$ and its neighbours that intersect Ω_h (we assume here that the mesh is truncated beyond $\mathcal{K}_h(\Gamma_h)$ so that there are no exterior neighbours, otherwise it is straightforward to handle them separately). Observe that the patches P_j overlap. For each patch P_j we define the faces F_j^1 and F_j^2 where $\partial P_j \cap \Gamma_h \neq \varnothing$. We define the interior elements of the patch by

$$\mathcal{K}_j^\circ := \{K \in \mathcal{K}_h(\Gamma_h) \cap P_j : K \cap (F_j^1 \cup F_j^2) = \varnothing\}.$$

Let I_{P_j} be the set of vertices $\{x_i\}$ in the patch P_j and the cardinality of I_{P_j} is N_{P_j}. We define the set of mesh vertices I_j that are in the interior of the patch P_j or on the outer boundary,

$$I_j = \left\{x_i \in K : K \in \mathcal{K}_j^\circ\right\},$$

Figure 3 shows an example of a patch. Let $\Gamma_j = \Gamma_h \cap \mathcal{K}_j$ denote the part of the boundary included in the patch P_j, for all j, the patch P_j has the following properties

$$\text{meas}_1(\Gamma_j) \sim h \quad \text{and} \quad \text{meas}_2(P_j) \sim h^2. \tag{19}$$

In (19) we can control the constant in both relations by choosing the patches to contain more elements (but uniformly under refinement).

The function v_Γ is defined such that $v_\Gamma = \sum_{j=1}^{N_p} v_j$, for each patch P_j, the function v_j has the form $v_j = \zeta\tilde{\varphi}_j$ with $\zeta \in \mathbb{R}$ to be defined later. Let $\tilde{\varphi}_j \in V_h^1$ be defined for each node $x_i \in \mathcal{T}_h$ such that

$$\tilde{\varphi}_j(x_i) = \begin{cases} 0 & \text{for} \quad x_i \in I_{P_j} \backslash I_j \\ -\rho(x_i) & \text{for} \quad x_i \in I_j, \end{cases}$$

with $i = 1, \ldots, N_{P_j}$. By the Poincaré inequality on a patch P_j the following

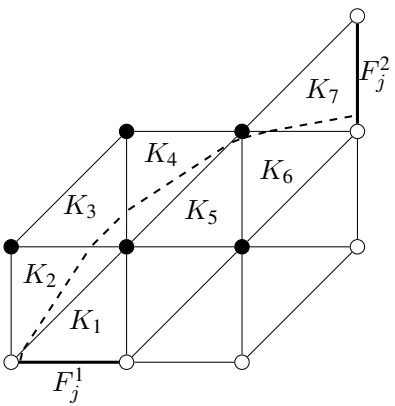


Figure 3. Example of a patch P_j, in this case $K_3 \cup K_4 \cup K_5 = \mathcal{K}_j = \mathcal{K}_j^\circ$, $\tilde{\varphi}_j$ is equal to zero on the nonfilled nodes, I_j are the filled nodes.

inequality holds

$$\|v_j\|_{P_j} \lesssim h\|\nabla v_j\|_{P_j}. \tag{20}$$

Lemma 4.1. *For every patch P_j with $1 \leqslant j \leqslant N_p$; $\forall r_j \in \mathbb{R}$ there exists $v_j \in V_h^p$ as defined above with ζ such that*

$$\text{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla v_j \cdot n_h \, \mathrm{d}s = r_j, \tag{21}$$

and the following property holds

$$\|\nabla v_j\|_{P_j} \lesssim \|h^{\frac{1}{2}} r_j\|_{\Gamma_j}. \tag{22}$$

Proof. The functions v_j and $\tilde{\varphi}_j$ are defined as previously described. Let

$$\Xi_j = \text{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla\tilde{\varphi}_j \cdot n_h \, \mathrm{d}s.$$

We will first prove that for shape regular meshes, Ξ_j is strictly negative and bounded away from zero uniformly in h, provided the hidden constant of the upper bound in (19) is chosen large enough. Let $K_1, \ldots, K_m$ be the triangles crossed by Γ_h within a patch P_j, the numbering of the crossed elements is done in order following a path of Γ_j as in figure 3. The assumption (19) says that the number of these triangles should be uniformly bounded by some m. We will show that the upper bound m only depends on the shape regularity of the mesh. The mesh size on the other hand must be small enough to resolve the boundary. The triangles K_1, K_2, K_{m-1} and K_m will have nodes on the boundary of ∂P_i where $\tilde{\varphi} = 0$ (on F_j^1 or F_j^2). Let us merge K_1 and K_2 (resp. K_{m-1} and K_m) into one quadrilateral element $K_1^\square$ (resp. $K_m^\square$). Observe that for the triangles K_i, $i = 3, ..., m-2$ there holds $K_i \in \mathcal{K}_j^\circ$ and with $\mathcal{I}_h$ denoting the standard nodal interpolant on piecewise linear functions,

$$(n_h \cdot \nabla\tilde{\varphi}_j|_{K_i}) = |\nabla\mathcal{I}_h\rho|_{K_i}|.$$

On $K_1^\square$ and $K_m^\square$ the following upper bound holds trivially

$$(n_h \cdot \nabla\tilde{\varphi}_j|_{K_i^\square}) < |\nabla\tilde{\varphi}_j|_{K_i^\square}|, \quad i = 1, m.$$

Consider now

$$\int_{\Gamma_h \cap P_j} \nabla\tilde{\varphi}_j \cdot n_h \, \mathrm{d}s = \sum_{i=1}^{m} \int_{\Gamma_j \cap K_i} \nabla\tilde{\varphi}_j \cdot n_h \, \mathrm{d}s = T_1^\square + T_m^\square + \sum_{i=3}^{m-2} T_i.$$

For $i \in \{3, \ldots, m-2\}$ using (18) we have

$$\begin{aligned} T_i = |\nabla\pi_h\rho|_{K_i}|\text{meas}(\Gamma_j \cap K_i) &\geqslant \text{meas}(\Gamma_j \cap K_i)(1 - |\nabla(1-\mathcal{I}_h)\rho|_{K_i}|) \\ &\geqslant (1 - c_\rho h)\text{meas}(\Gamma_j \cap K_i). \end{aligned}$$

For $i \in \{1, m\}$ we have

$$\begin{aligned} T_i^\square \leqslant h^{\frac{1}{2}}\|\nabla\tilde{\varphi}_j\|_{\Gamma_j \cap K_i^\square} \lesssim \|\nabla\tilde{\varphi}_j\|_{K_i^\square} \leqslant C_i h^{-1}\|\tilde{\varphi}_j\|_{K_i^\square} &\leqslant C_i h^{-1/2}\|\mathcal{I}_h\rho\|_{\partial K_i^\square \cap \partial\mathcal{K}_j^\circ} \\ \leqslant C_i h^{-1/2}(\|\rho\|_{\partial K_i^\square \cap \partial\mathcal{K}_j^\circ} + \|\rho - \mathcal{I}_h\rho\|_{\partial K_i^\square \cap \partial\mathcal{K}_j^\circ}) &\leqslant c_\partial h. \end{aligned}$$

The right hand side of these two inequalities depend only on the shape regularity of the mesh. Hence, using that $\mathrm{meas}(\Gamma_j \cap \mathcal{K}_j^\circ) \geqslant \mathrm{meas}(\Gamma_j) - 2h$

$$\mathrm{meas}(\Gamma_j)^{-1} \sum_{i=1}^{m} \int_{\Gamma_j \cap K_i} \nabla \tilde{\varphi}_j \cdot n_h \, \mathrm{d}s \\ \geqslant (1 - c_\rho h)(1 - 2h\mathrm{meas}(\Gamma_j)^{-1}) - 2c_\partial h \mathrm{meas}(\Gamma_j)^{-1}.$$

Assume now that h is sufficiently small so that $(1 - c_\rho h) > \frac{1}{2}$ then

$$\mathrm{meas}(\Gamma_j)^{-1} \sum_{i=1}^{m} \int_{\Gamma_j \cap K_i} \nabla \tilde{\varphi}_j \cdot n_h \, \mathrm{d}s \geqslant \frac{1}{2} - (2c_\partial + 1) h \mathrm{meas}(\Gamma_j)^{-1}.$$

If the lower constant of the left relation of (19) is larger than $4(2c_\partial + 1)$ we may conclude that

$$\Xi_j = \mathrm{meas}(\Gamma_j)^{-1} \sum_{i=1}^{m} \int_{\Gamma_j \cap K_i} \nabla \tilde{\varphi}_j \cdot n_h \, \mathrm{d}s \geqslant \frac{1}{4},$$

which shows the uniform lower bound. Note that the constant c_∂ only depends on the curvature of the boundary and the mesh geometry. Thanks to this lower bound we may define the normalised function φ_j by

$$\varphi_j = \Xi_j^{-1} \tilde{\varphi}_j.$$

By definition there holds

$$\mathrm{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla \varphi_j \cdot n_h \, \mathrm{d}s = 1, \tag{23}$$

and

$$\|\nabla \varphi_j\|_{P_j} = \Xi_j^{-1} \|\nabla \tilde{\varphi}_j\|_{P_j}. \tag{24}$$

The right hand side can be bounded as follows

$$\|\nabla \tilde{\varphi}_j\|_{P_j} \leqslant \|\nabla \tilde{\varphi}_j\|_{\mathcal{K}_j^\circ} + \|\nabla \tilde{\varphi}_j\|_{P_j \setminus \mathcal{K}_j^\circ} = T_1 + T_2,$$

$$T_1 \leqslant \|\nabla(\pi_h \rho - \rho)\|_{\mathcal{K}_j^\circ} + \|\nabla \rho\|_{\mathcal{K}_j^\circ} \leqslant C h^{\frac{1}{2}} \mathrm{meas}(\Gamma_j)^{\frac{1}{2}},$$

$$T_2 \leqslant C_i h^{-1} \|\tilde{\varphi}_j\|_{P_i \setminus \mathcal{K}_j^\circ} \leqslant C h^{-1/2} \|\mathcal{I}_h \rho\|_{\partial \mathcal{K}_j^\circ} \\ \leqslant C h^{-1/2} (\|\rho\|_{\partial \mathcal{K}_j^\circ} + \|\mathcal{I}_h \rho - \rho\|_{\partial \mathcal{K}_j^\circ}) \leqslant C h^{\frac{1}{2}} \mathrm{meas}(\Gamma_j)^{\frac{1}{2}}.$$

We conclude that

$$\|\nabla\varphi_j\|_{P_j} \leqslant Ch^{\frac{1}{2}}\text{meas}(\Gamma_j). \tag{25}$$

Let $v_j = r_j\varphi_j$, then condition (21) is verified considering (23). The upper bound (22) is obtained using (25), (19) and

$$\|\nabla v_j\|_{P_j} = |r_j|\|\nabla\varphi_j\|_{P_j} \lesssim \text{meas}(\Gamma_j)^{\frac{1}{2}}|r_j|h^{\frac{1}{2}} = \|h^{\frac{1}{2}}r_j\|_{\Gamma_j}.$$

□

Lemma 4.2. *For $u_h, v_h \in V_h^p$ with $v_h = u_h + \alpha v_\Gamma$, there exists a positive constant β_0 such that the following inequality holds*

$$\beta_0|||u_h|||_h^2 \leqslant A_h(u_h, v_h) + J_h(u_h, v_h).$$

Proof. Using (7) it is straightforward to obtain

$$\begin{aligned}(A_h + J_h)(u_h, u_h) &= (\nabla u_h, \nabla u_h)_{\Omega_h} + \langle\nabla u_h \cdot n_h, T_{1,k}(u_h)\rangle_{\Gamma_h} + J_h(u_h, u_h)\\ &\gtrsim \|\nabla u_h\|_{\mathcal{N}_h}^2 + \langle\nabla u_h \cdot n_h, T_{1,k}(u_h)\rangle_{\Gamma_h}.\end{aligned}$$

We bound the second term using the trace inequality (4), inverse inequality (5) and (2)

$$\begin{aligned}\langle\nabla u_h \cdot n_h, T_{1,k}(u_h)\rangle_{\Gamma_h} &\leqslant \|\nabla u_h \cdot n_h\|_{\Gamma_h}\|T_{1,k}(u_h)\|_{\Gamma_h}\\ &\lesssim h^{-1}\|\nabla u_h\|_{\mathcal{N}_h(\Gamma_h)}\|T_{1,k}(u_h)\|_{\mathcal{N}_h(\Gamma_h)}\\ &\lesssim \gamma(h)\|\nabla u_h\|_{\mathcal{N}_h(\Gamma_h)}^2,\end{aligned}$$

where $\gamma(h) = \sum_{i=1}^k \left(\frac{\delta_h}{h}\right)^i$, note that $\mathcal{O}(\gamma(h)) = h$. Considering v_Γ as defined in Section 2, for the j^{th} patch P_j we get

$$\begin{aligned}\alpha(A_h + J_h)(u_h, v_j) = \alpha(\nabla u_h, \nabla v_j)_{P_j\cap\Omega_h} - \alpha\langle\nabla u_h \cdot n_h, v_j\rangle_{\Gamma_j} + \alpha J_h(u_h, v_j)\\ + \alpha\langle\nabla v_j \cdot n_h, u_h\rangle_{\Gamma_j} + \alpha\langle\nabla v_j \cdot n_h, T_{1,k}(u_h)\rangle_{\Gamma_j},\end{aligned}$$

applying Cauchy-Schwarz inequality together with (7) we can write

$$\begin{aligned}&\alpha(\nabla u_h, \nabla v_j)_{P_j\cap\Omega_h} + \alpha J_h(u_h, v_j)\\ &\quad\leqslant \alpha\|\nabla u_h\|_{P_j\cap\Omega_h}\|\nabla v_j\|_{P_j\cap\Omega_h} + \alpha J_h(u_h, u_h)^{\frac{1}{2}}J_h(v_j, v_j)^{\frac{1}{2}},\\ &\quad\leqslant \epsilon\|\nabla u_h\|_{P_j}^2 + \frac{C\alpha^2}{4\epsilon}\|\nabla v_j\|_{P_j}^2.\end{aligned}$$

Using the trace inequality (4), the inverse inequality (5) and the inequality (20) we can write the following

$$\begin{aligned}\alpha\langle\nabla u_h\cdot n_h, v_j\rangle_{\Gamma_j} &\leqslant \alpha\|\nabla u_h\cdot n_h\|_{\Gamma_j}\|v_j\|_{\Gamma_j} \lesssim \alpha h^{-1}\|\nabla u_h\|_{P_j}\|v_j\|_{P_j}\\ &\lesssim \alpha\|\nabla u_h\|_{P_j}\|\nabla v_j\|_{P_j} \leqslant \epsilon\|\nabla u_h\|^2_{P_j} + \frac{C\alpha^2}{4\epsilon}\|\nabla v_j\|^2_{P_j}.\end{aligned}$$

Let us consider the average $\overline{u}^j = \text{meas}(\Gamma_j)^{-1}\int_{\Gamma_j} u_h \,\mathrm{d}s$. Using Lemma 4.1 and choosing $r_j = h^{-1}\overline{u}^j$ we get the inequality

$$\|\nabla v_j\|_{P_j} \lesssim \|h^{-\frac{1}{2}}\overline{u}^j\|_{\Gamma_j}. \tag{26}$$

Our choice of r_j allows us to write

$$\alpha\langle\nabla v_j\cdot n_h, u_h\rangle_{\Gamma_j} = \alpha\|h^{-\frac{1}{2}}\overline{u}^j\|^2_{\Gamma_j} + \alpha\langle\nabla v_j\cdot n_h, u_h-\overline{u}^j\rangle_{\Gamma_j}.$$

It is straightforward to observe

$$\|u_h-\overline{u}^j\|_{\Gamma_j} \leqslant Ch\|\nabla u_h\|_{\Gamma_j}. \tag{27}$$

Combining this result with the trace and inverse inequalities, we can show

$$\alpha\langle\nabla v_j\cdot n_h, u_h-\overline{u}^j\rangle_{\Gamma_j} \leqslant \epsilon\|\nabla u_h\|^2_{P_j} + \frac{C\alpha^2}{4\epsilon}\|\nabla v_j\|^2_{P_j}.$$

Using (2) once again

$$\alpha\langle\nabla v_j\cdot n_h, T_{1,k}(u_h)\rangle_{\Gamma_j} \leqslant \epsilon\gamma(h)^2\|\nabla u_h\|^2_{P_j} + \frac{C\alpha^2}{4\epsilon}\|\nabla v_j\|^2_{P_j}.$$

Since each term has been bounded, we can now get back to

$$\begin{aligned}(A_h+J_h)(u_h,v_h) \geqslant (C-\gamma(h))\|\nabla u_h\|^2_{\mathcal{N}_h} + \alpha\sum_{j=1}^{N_p}\|h^{-\frac{1}{2}}\overline{u}^j\|^2_{\Gamma_j}\\ - 4\epsilon\sum_{j=1}^{N_p}\|\nabla u_h\|^2_{P_j} - \frac{C\alpha^2}{\epsilon}\sum_{j=1}^{N_p}\|\nabla v_j\|^2_{P_j}.\end{aligned}$$

Using (26) and rearranging, we obtain

$$(A_h+J_h)(u_h,v_h) \geqslant (C-\gamma(h)-4\epsilon)\|\nabla u_h\|^2_{\mathcal{N}_h} + \alpha\Big(1-\frac{C\alpha}{\epsilon}\Big)\sum_{j=1}^{N_p}\|h^{-\frac{1}{2}}\overline{u}^j\|^2_{\Gamma_j}.$$

Using (27), the trace inequality and the inverse inequality we can show

$$\|h^{-\frac{1}{2}}\overline{u}^j\|^2_{\Gamma_j} \geqslant \|h^{-\frac{1}{2}}u_h\|^2_{\Gamma_j} - C'\|\nabla u_h\|^2_{P_j}.$$

Using this result together with (7) we obtain

$$(A_h + J_h)(u_h, v_h) \geqslant (C - \gamma(h) - 4\epsilon - C'\alpha)\left(\|\nabla u_h\|^2_{\Omega_h} + J_h(u_h, u_h)\right) + \alpha\Big(1 - \frac{C\alpha}{\epsilon}\Big)\sum_{j=1}^{N_p}\|h^{-\frac{1}{2}}u_h\|^2_{\Gamma_j}.$$

It is easy to choose ϵ and α such that the two terms of this expression are positive, for example, by choosing $\epsilon = \frac{1}{16}$ we obtain $\alpha = \min(\frac{C-\gamma(h)-\frac{1}{4}}{C'}, \frac{1}{16C})$. □

Theorem 4.3. *There exists a positive constant β such that for all function $u_h \in V_h^p$ the following inequality holds*

$$\beta|||u_h|||_h \leqslant \sup_{v_h \in V_h^p} \frac{A_h(u_h, v_h) + J_h(u_h, v_h)}{|||v_h|||_h}.$$

Proof. Considering Lemma 4.2 the only thing to show is $|||v_h|||_h \lesssim |||u_h|||_h$,

$$|||v_h|||_h \leqslant |||u_h|||_h + |||v_\Gamma|||_h \quad \text{with} \quad |||v_\Gamma|||_h \leqslant \sum_{j=1}^{N_p}|||v_j|||_h,$$

$$|||v_j|||^2_h = \|\nabla v_j\|^2_{P_j\cap\Omega_h} + \|h^{-\frac{1}{2}}v_j\|^2_{\Gamma_j} + J_h(v_j, v_j).$$

Using the trace inequality together with (20) and (26) we observe

$$\|h^{-\frac{1}{2}}v_j\|^2_{\Gamma_j} \lesssim \|\nabla v_j\|^2_{P_j} \lesssim \|h^{-\frac{1}{2}}\overline{u}^j\|_{\Gamma_j} \lesssim \|h^{-\frac{1}{2}}u_h\|_{\Gamma_j}.$$

We conclude using (7). □

5 Boundary value correction

The goal of this section is bound the two high order terms that has been dropped in the finite element formulation (16).

Theorem 5.1. *Let B_h be the bilinear form as defined in* (17) *the following holds* $\forall v_h \in V_h^p$

$$B_h(u, v_h) \lesssim (h^{-\frac{1}{2}}\delta_h^{k+1}\sup_{0\leqslant t\leqslant\delta_0}\|D^{k+1}u\|_{\Gamma_t} + \delta_h^{l+1}\sup_{0\leqslant t\leqslant\delta_0}\|D_n^l(f + \Delta u)\|_{\Gamma_t})|||v_h|||_h.$$

Proof. Using the Cauchy-Schwarz inequality

$$|B_h(u,v_h)| \leqslant \|u\circ p_h(x) - T_{0,k}(u)(x)\|_{\Gamma_h}\|\nabla v_h\cdot n_h\|_{\Gamma_h} + \|f+\Delta u\|_{\Omega_h\backslash\Omega}\|v_h\|_{\Omega_h\backslash\Omega}.$$

By definition of the Taylor approximation we have

$$|u\circ p_h(x) - T_{0,k}(u)(x)| = \Big|\int_0^{\varrho_h(x)} D_{n_h}^{k+1}u(x(s))(\varrho_h(x)-s)^k\,\mathrm{d}s\Big|.$$

Let $I_x = [0,\varrho_h(x)]$ and $I_{\delta_h} = [0,\max_{x\in\Gamma_h}\varrho_h(x)]$, using the Cauchy Schwarz inequality

$$\begin{aligned}
\|u\circ p_h(x) - T_{0,k}(u)(x)\|_{\Gamma_h}^2 &\leqslant \int_{\Gamma_h}\|D_{n_h}^{k+1}u\|_{I_x}^2\|(\varrho_h(x)-s)^k\|_{I_x}^2\,\mathrm{d}s\\
&\leqslant \int_{\Gamma_h}\|D_{n_h}^{k+1}u\|_{I_{\delta_h}}^2|\varrho_h(x)|^{2k+1}\,\mathrm{d}s\\
&\leqslant \delta_h^{2k+1}\|D^{k+1}u\|_{U_{\delta_h}(\Gamma_h)}^2\\
&\leqslant \delta_h^{2k+2}\sup_{0\leqslant t\leqslant\delta_0}\|D^{k+1}u\|_{\Gamma_t}^2,
\end{aligned}$$

where $\Gamma_t = \{x\in\Omega\ :\ |\rho(x)| = t\}$ is the levelset with distance t to the boundary Γ, following the approach from [9]. Suppose that

$$f+\Delta u \in H^{l+1}(U_{\delta_0}(\Omega)).$$

This property holds if $f\in H^{l+1}(\Omega)$ by applying (1) and (8). Using $\Omega_h\backslash\Omega\subset U_\delta(\Gamma)$ and $\delta\sim\delta_h$ it follows that

$$\|f+\Delta u\|_{\Omega_h\backslash\Omega}\lesssim\delta_h^l\|D_n^l(f+\Delta u)\|_{\Omega_h\backslash\Omega}\lesssim\delta_h^{l+\frac12}\sup_{0\leqslant t\leqslant\delta_0}\|D_n^l(f+\Delta u)\|_{\Gamma_t}.$$

Then we can bound the bilinear form B_h, using $\|v\|_{\Omega_h\backslash\Omega}\lesssim\delta_h^{1/2}|||v|||_h$ (deduced from (6)) the trace and inverse inequalities $\forall v_h\in V_h^p$

$$\begin{aligned}
&|B_h(u,v_h)|\\
&\lesssim \delta_h^{k+1}\sup_{0\leqslant t\leqslant\delta_0}\|D^{k+1}u\|_{\Gamma_t}\|\nabla v_h\cdot n_h\|_{\Gamma_h} + \delta_h^{l+\frac12}\sup_{0\leqslant t\leqslant\delta_0}\|D_n^l(f+\Delta u)\|_{\Gamma_t}\|v_h\|_{\Omega_h\backslash\Omega}\\
&\lesssim (h^{-\frac12}\delta_h^{k+1}\sup_{0\leqslant t\leqslant\delta_0}\|D^{k+1}u\|_{\Gamma_t} + \delta_h^{l+1}\sup_{0\leqslant t\leqslant\delta_0}\|D_n^l(f+\Delta u)\|_{\Gamma_t})|||v_h|||_h.
\end{aligned}\tag{28}$$

□

6 A priori error estimate

The formulation (16) satisfies the following consistency relation (Galerkin orthogonality).

Lemma 6.1. *Let $u_h \in V_h^p$ be the solution of (16) and $u \in H^2(\mathcal{N}_h)$ be the solution of (1). Then*

$$A_h(u - u_h, v_h) - J_h(u_h, v_h) + B_h(u, v_h) = 0 \quad , \quad \forall v_h \in V_h^p.$$

Proof. Subtracting (15) and (16) this is straightforward. □

Lemma 6.2. *Let $w \in H^2(\mathcal{N}_h) + V_h^p$ and $v_h \in V_h^p$. There exists a positive constant M such that*

$$A_h(w, v_h) \leqslant M \|w\|_* |\!|\!| v_h |\!|\!|_h.$$

Proof. Using the Cauchy-Schwarz inequality, trace inequality and inverse inequality we have

$$\begin{aligned}(\nabla w, \nabla v_h)_{\Omega_h} - \langle \nabla w \cdot n_h, v_h \rangle_{\Gamma_h} + \langle \nabla v_h \cdot n_h, T_{0,k}(w) \rangle_{\Gamma_h} \\ \lesssim \|\nabla w\|_{\Omega_h} \|\nabla v_h\|_{\Omega_h} + \|\nabla w \cdot n_h\|_{\Gamma_h} \|v_h\|_{\Gamma_h} \\ + \|\nabla v_h\|_{\Omega_h} h^{-\frac{1}{2}} (\|w\|_{\Gamma_h} + \|T_{1,k}(w)\|_{\Gamma_h}).\end{aligned}$$

□

Using the inf-sup condition from Section 4 and the estimate obtained in Section 5 the error in the triple norm can now be estimated.

Proposition 6.3. *Let $u \in H^{p+1}(\mathcal{N}_h)$ be the solution of (1) and $u_h \in V_h^p$ the solution of (16). Then*

$$\begin{aligned}|\!|\!| u - u_h |\!|\!|_h \lesssim h^p \|u\|_{H^{p+1}(\Omega)} + h^{-\frac{1}{2}} \delta_h^{k+1} \sup_{0 \leqslant t \leqslant \delta_0} \|D^{k+1} u\|_{\Gamma_t} \\ + \delta_h^{l+1} \sup_{0 \leqslant t \leqslant \delta_0} \|D_n^l (f + \Delta u)\|_{\Gamma_t}.\end{aligned}$$

p	h^p	k	$h^{-\frac{1}{2}}\delta_h^{k+1}$	$l+1$	δ_h^{l+1}
1	h^1	0	$h^{1.5}$	1	h^2
2	h^2	1	$h^{3.5}$	2	h^4
3	h^3	2	$h^{5.5}$	3	h^6
4	h^4	3	$h^{7.5}$	4	h^8

Table 1. Order of the terms in the estimation of the H^1-error depending on p, k and l, assuming $\delta_h = \mathcal{O}(h^2)$.

Proof. Using the Galerkin orthogonality of Lemma 6.1 we obtain

$$\begin{aligned} &A_h(u_h - \pi_h u, v_h) + J_h(u_h - \pi_h u, v_h) \\ &\quad = A_h(u - \pi_h u, v_h) - J_h(\pi_h u, v_h) + (f + \Delta u, v_h)_{\Omega_h \setminus \Omega} \\ &\qquad + \langle \nabla v_h \cdot n_h, u \circ p_h - T_{0,k}(u) \rangle_{\Gamma_h}. \end{aligned}$$

Using this result, Theorem 4.3, the Lemma 6.2 and $J_h(u, v_h) = 0$ given by the regularity of u, we can write

$$\begin{aligned} \beta |\!|\!| u_h - \pi_h u |\!|\!|_h &\leqslant \frac{A_h(u - \pi_h u, v_h) - J_h(\pi_h u, v_h) + B_h(u, v_h)}{|\!|\!| v_h |\!|\!|_h} \\ &\leqslant M \|u - \pi_h u\|_* + \frac{J_h(\pi_h u - u, \pi_h u - u)^{\frac{1}{2}} J_h(v_h, v_h)^{\frac{1}{2}} + B_h(u, v_h)}{|\!|\!| v_h |\!|\!|_h} \\ &\leqslant (M+1) \|u - \pi_h u\|_* + \frac{B_h(u, v_h)}{|\!|\!| v_h |\!|\!|_h} \end{aligned}$$

Applying the triangle inequality we can write

$$\begin{aligned} |\!|\!| u - u_h |\!|\!|_h &\leqslant |\!|\!| u - \pi_h u |\!|\!|_h + |\!|\!| u_h - \pi_h u |\!|\!|_h \\ &\leqslant |\!|\!| u - \pi_h u |\!|\!|_h + \frac{1}{\beta} \Big((M+1) \|u - \pi_h u\|_* + \frac{B_h(u, v_h)}{|\!|\!| v_h |\!|\!|_h} \Big), \end{aligned}$$

Using the estimate (13) we conclude applying Theorem 5.1. □

The next proof requires these two inequalities and the assumption $\delta_h \lesssim h^2$

$$\|T_{1,k}(u - u_h)\|_{\Gamma_h} \lesssim h^{p+1} \|u\|_{H^{p+1}(\Omega)} + (h^{-\frac{3}{2}} \delta_h) h |\!|\!| u - u_h |\!|\!|_h \tag{29}$$

$$h^{\frac{1}{2}} \|\nabla(u - u_h) \cdot n_h\|_{\Gamma_h} \lesssim h^p \|u\|_{H^{p+1}(\Omega)} + |\!|\!| u - u_h |\!|\!|_h. \tag{30}$$

Inequality (29) is proved in [9] and (30) can be shown using the trace and inverse inequalities in the following way

$$\begin{aligned} h^{\frac{1}{2}}\|\nabla(u-u_h)\cdot n_h\|_{\Gamma_h} &\lesssim \|\nabla(u-u_h)\|_{\mathcal{N}(\Gamma_h)} + h\|D^2(u-u_h)\|_{\mathcal{N}(\Gamma_h)} \\ &\lesssim \|\nabla(u-\pi_h u)\|_{\mathcal{N}(\Gamma_h)} + h\|D^2(u-\pi_h u)\|_{\mathcal{N}(\Gamma_h)} \\ &\quad + \|\nabla(u_h-\pi_h u)\|_{\mathcal{N}(\Gamma_h)} + h\|D^2(u_h-\pi_h u)\|_{\mathcal{N}(\Gamma_h)} \\ &\lesssim h^p\|u\|_{H^{p+1}(\Omega)} + \|\nabla(u_h-\pi_h u)\|_{\mathcal{N}(\Gamma_h)} \\ &\lesssim h^p\|u\|_{H^{p+1}(\Omega)} + \|\nabla(u-u_h)\|_{\mathcal{N}(\Gamma_h)}. \end{aligned}$$

Theorem 6.4. *Let $u \in H^{p+1}(\mathcal{N}_h)$ be the solution of (1) and $u_h \in V_h^p$ the solution of (16). We assume $\delta_h \lesssim h^2$. Then*

$$\begin{aligned} \|u-u_h\|_{\Omega_h} \lesssim h^{p+\frac{1}{2}}\|u\|_{H^{p+1}(\Omega)} + \delta_h^{k+1}\sup_{0\leqslant t\leqslant\delta_0}\|D^{k+1}u\|_{\Gamma_t} \\ + h^{\frac{1}{2}}\delta_h^{l+1}\sup_{0\leqslant t\leqslant\delta_0}\|D_n^l(f+\Delta u)\|_{\Gamma_t}. \end{aligned}$$

p	$h^{p+\frac{1}{2}}$	k	δ_h^{k+1}	$l+1$	$h^{\frac{1}{2}}\delta_h^{l+1}$
1	$h^{1.5}$	*0*	h^2	*1*	$h^{2.5}$
2	$h^{2.5}$	*1*	h^4	*2*	$h^{4.5}$
3	$h^{3.5}$	*2*	h^6	*3*	$h^{6.5}$
4	$h^{4.5}$	*3*	h^8	*4*	$h^{8.5}$

Table 2. Order of the terms in the estimation of the L^2-error depending on p, k and l, assuming $\delta_h = \mathcal{O}(h^2)$.

Proof. We define the function ψ such that

$$\psi = \begin{cases} u-u_h & \text{in} \quad \Omega_h \\ 0 & \text{in} \quad \Omega\backslash\Omega_h. \end{cases}$$

Let z satisfy the adjoint problem

$$\begin{aligned} -\Delta z &= \psi \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \Gamma, \end{aligned}$$

z is extended to $U_{\delta_0}(\Omega)$ using the extension operator. In this framework, the following estimates hold

$$\|z\|_{H^2(\Omega)} \lesssim \|\psi\|_{\Omega\cap\Omega_h}, \tag{31}$$

$$\|z\|_{\Omega_h\setminus\Omega} \lesssim \delta_h\|\nabla z\cdot n\|_{U_{\delta_h}(\Gamma)}, \tag{32}$$

$$\|z\|_{\Gamma_h} \lesssim \delta_h^{\frac{1}{2}}\|\nabla z\cdot n\|_{U_{\delta_h}(\Gamma)}. \tag{33}$$

Inequalities (32) and (33) has been shown in [9]. Using integration by parts, the L^2-error on Ω_h can be written as

$$\begin{aligned}\|u-u_h\|_{\Omega_h} &= (u-u_h,\psi+\Delta z)_{\Omega_h} - (u-u_h,\Delta z)^2_{\Omega_h}\\ &= (u-u_h,\psi+\Delta z)_{\Omega_h\setminus\Omega} + (\nabla(u-u_h),\nabla z)_{\Omega_h} - \langle u-u_h,\nabla z\cdot n_h\rangle_{\Gamma_h}\\ &= (u-u_h,\psi+\Delta z)_{\Omega_h\setminus\Omega} + A_h(u-u_h,z) + \langle\nabla(u-u_h)\cdot n_h,z\rangle_{\Gamma_h}\\ &\quad - 2\langle\nabla z\cdot n_h,u-u_h\rangle_{\Gamma_h} - \langle\nabla z\cdot n_h,T_{1,k}(u-u_h)\rangle_{\Gamma_h}.\end{aligned}$$

Using (6), the property of the extension operator (8) and (31), we can write

$$\begin{aligned}(u-u_h,&\psi+\Delta z)_{\Omega_h\setminus\Omega} \leqslant \|u-u_h\|_{\Omega_h\setminus\Omega}\|\psi+\Delta z\|_{\Omega_h\setminus\Omega}\\ &\leqslant (\delta_h^2\|\nabla(u-u_h)\cdot n\|^2_{\Omega_h\setminus\Omega} + \delta_h\|u-u_h\|^2_{\Gamma_h})^{\frac{1}{2}}(\|\psi\|_{\Omega_h\setminus\Omega} + \|\Delta z\|_{\Omega_h\setminus\Omega})\\ &\lesssim (\delta_h^2 + h\delta_h)^{\frac{1}{2}}|||u-u_h|||_h(\|u-u_h\|_{\Omega_h\setminus\Omega} + \|z\|_{H^2(\Omega)})\\ &\lesssim (\delta_h^2 + h\delta_h)^{\frac{1}{2}}|||u-u_h|||_h\|u-u_h\|_{\Omega_h}.\end{aligned}$$

Using the interpolant defined by (9) we obtain

$$A_h(u-u_h,z) \leqslant |A_h(u-u_h,z-\pi_h z) + A_h(u-u_h,\pi_h z)|,$$

by Cauchy-Schwarz inequality, (29), (30) and (31) we have

$$\begin{aligned}&A_h(u-u_h,z-\pi_h z)\\ &\lesssim (|||u-u_h|||_h + h^{\frac{1}{2}}\|\nabla(u-u_h)\cdot n_h\|_{\Gamma_h} + h^{-\frac{1}{2}}\|T_{1,k}(u-u_h)\|_{\Gamma_h})|||z-\pi_h z|||_*\\ &\lesssim ((1+h^{-1}\delta_h)h|||u-u_h|||_h + h^{p+1}\|u\|_{H^{p+1}(\Omega)})\|u-u_h\|_{\Omega_h}.\end{aligned}$$

The Galerkin orthogonality of Lemma 6.1 allows us to write

$$\begin{aligned}A_h(u-u_h,\pi_h z) &\lesssim |B_h(u,\pi_h z) + J_h(u_h,\pi_h z)|\\ &\lesssim |B_h(u,\pi_h z) + J_h(u_h,u_h)^{\frac{1}{2}}J_h(\pi_h z,\pi_h z)^{\frac{1}{2}}|.\end{aligned}$$

From [8] and the properties of z we have

$$J_h(\pi_h z, \pi_h z)^{\frac{1}{2}} = J_h(\pi_h z - z, \pi_h z - z)^{\frac{1}{2}} \lesssim h\|z\|_{H^2(\Omega)} \lesssim h\|u - u_h\|_{\Omega_h},$$

we also have the upper bound

$$J_h(u_h, u_h)^{\frac{1}{2}} \lesssim |||u_h - \pi_h u|||_h + J_h(\pi_h u, \pi_h u)^{\frac{1}{2}} \lesssim |||u_h - \pi_h u|||_h + h^p\|u\|_{H^{p+1}(\Omega)}.$$

Then using the proof of Proposition 6.3 we have

$$|J_h(u_h, \pi_h z)| \lesssim \Big(h^{p+1}\|u\|_{H^{p+1}(\Omega)} + h^{\frac{1}{2}}\delta_h^{k+1} \sup_{0\leqslant t\leqslant\delta_0} \|D^{k+1}u\|_{\Gamma_t} + h\delta_h^{l+1} \sup_{0\leqslant t\leqslant\delta_0} \|D_n^l(f + \Delta u)\|_{\Gamma_t}\Big)\|u - u_h\|_{\Omega_h}.$$

Using equation (28) the term $B_h(u, \pi_h z)$ can also be bounded with

$$|B_h(u, \pi_h z)| \lesssim \delta_h^{k+1} \sup_{0\leqslant t\leqslant\delta_0} \|D^{k+1}u\|_{\Gamma_t}\|\nabla\pi_h z \cdot n_h\|_{\Gamma_h} + \delta_h^{l+\frac{1}{2}} \sup_{0\leqslant t\leqslant\delta_0} \|D_n^l(f + \Delta u)\|_{\Gamma_t}\|\pi_h z\|_{\Omega_h\setminus\Omega}.$$

Using the global trace inequality $\|\nabla z \cdot n_h\|_{\Gamma_h} \lesssim \|z\|_{H^2(\Omega_h)}$, (8) and (31) we can write

$$\begin{aligned}\|\nabla\pi_h z \cdot n_h\|_{\Gamma_h} &\leqslant \|\nabla(\pi_h z - z) \cdot n_h\|_{\Gamma_h} + \|\nabla z \cdot n_h\|_{\Gamma_h}\\ &\lesssim h^{-\frac{1}{2}}|||\pi_h z - z|||_* + \|z\|_{H^2(\Omega_h)}\\ &\lesssim h^{\frac{1}{2}}\|z\|_{H^2(\Omega)} + \|z\|_{H^2(\Omega_h)}\\ &\lesssim \|u - u_h\|_{\Omega_h}.\end{aligned}$$

Using inequalities (8), (31) and (32) we obtain

$$\begin{aligned}\|\pi_h z\|_{\Omega_h\setminus\Omega} &\leqslant \|\pi_h z - z\|_{\Omega_h\setminus\Omega} + \|z\|_{\Omega_h\setminus\Omega}\\ &\leqslant h^2\|z\|_{H^2(\Omega)} + \delta_h\|\nabla z\|_{U_{\delta_h}(\Gamma)}\\ &\leqslant h^2\|u - u_h\|_{\Omega_h}.\end{aligned}$$

Then we obtain the upper bound

$$|B_h(u, \pi_h z)| \lesssim \Big(\delta_h^{k+1} \sup_{0\leqslant t\leqslant\delta_0} \|D^{k+1}u\|_{\Gamma_t} + h^2\delta_h^{l+\frac{1}{2}} \sup_{0\leqslant t\leqslant\delta_0} \|D_n^l(f + \Delta u)\|_{\Gamma_t}\Big)\|u - u_h\|_{\Omega_h},$$

and

$$A_h(u-u_h,z) \lesssim \Big(h^{p+1}\|u\|_{H^{p+1}(\Omega)} + \delta_h^{k+1}\sup_{0\leqslant t\leqslant\delta_0}\|D^{k+1}u\|_{\Gamma_t} + h^2\delta_h^{l+\frac{1}{2}}\sup_{0\leqslant t\leqslant\delta_0}\|D_n^l(f+\Delta u)\|_{\Gamma_t}\Big)\|u-u_h\|_{\Omega_h}.$$

Using (33) and (31) and the we have

$$\|z\|_{\Gamma_h} \lesssim \delta_h^{\frac{1}{2}}\|\nabla z\cdot n\|_{U_{\delta_h}(\Gamma)} \lesssim \delta_h \sup_{0\leqslant t\leqslant\delta_0}\|\nabla z\cdot n\|_{\Gamma_t} \lesssim \delta_h\|z\|_{H^2(\Omega)} \lesssim \delta_h\|u-u_h\|_{\Omega_h}.$$

Using this result with (30) and (29) we have

$$\begin{aligned}
&|\langle\nabla(u-u_h)\cdot n_h, z\rangle_{\Gamma_h} - \langle\nabla z\cdot n_h, T_{1,k}(u-u_h)\rangle_{\Gamma_h}|\\
&\quad\leqslant \|\nabla(u-u_h)\cdot n_h\|_{\Gamma_h}\|z\|_{\Gamma_h} + \|\nabla z\cdot n_h\|_{\Gamma_h}\|T_{1,k}(u-u_h)\|_{\Gamma_h}\\
&\quad\lesssim (h^p\|u\|_{H^{p+1}(\Omega)} + |||u-u_h|||_h)h^{-\frac{1}{2}}\|z\|_{\Gamma_h} + \|z\|_{H^2(\Omega_h)}\|T_{1,k}(u-u_h)\|_{\Gamma_h}\\
&\quad\lesssim (h^{-\frac{3}{2}}\delta_h(h^{p+1}\|u\|_{H^{p+1}(\Omega)} + h|||u-u_h|||_h) + \|T_{1,k}(u-u_h)\|_{\Gamma_h})\|u-u_h\|_{\Omega_h}\\
&\quad\lesssim ((h^{-\frac{1}{2}}\delta_h)|||u-u_h|||_h + h^{p+1}\|u\|_{H^{p+1}(\Omega)})\|u-u_h\|_{\Omega_h}.
\end{aligned}$$

Also

$$\begin{aligned}
|\langle\nabla z\cdot n_h, u-u_h\rangle_{\Gamma_h}| &\leqslant \|\nabla z\cdot n_h\|_{\Gamma_h}\|u-u_h\|_{\Gamma_h}\\
&\leqslant \|z\|_{H^2(\Omega_h)}h^{\frac{1}{2}}|||u-u_h|||_h\\
&\leqslant \|u-u_h\|_{\Omega_h}h^{\frac{1}{2}}|||u-u_h|||_h.
\end{aligned}$$

Using $\delta_h \lesssim h^2$ we obtain the bound

$$\begin{aligned}
&\langle\nabla(u-u_h)\cdot n_h, z\rangle_{\Gamma_h} - \langle\nabla z\cdot n_h, T_{1,k}(u-u_h)\rangle_{\Gamma_h} - 2\langle\nabla z\cdot n_h, u-u_h\rangle_{\Gamma_h}\\
&\lesssim \Big(h^{p+1}\|u\|_{H^{p+1}(\Omega)} + h^{\frac{1}{2}}|||u-u_h|||_h\Big)\|u-u_h\|_{\Omega_h}\\
&\lesssim \Big(h^{p+\frac{1}{2}}\|u\|_{H^{p+1}(\Omega)} + \delta_h^{k+1}\sup_{0\leqslant t\leqslant\delta_0}\|D^{k+1}u\|_{\Gamma_t}\\
&\qquad + h^{\frac{1}{2}}\delta_h^{\frac{1}{2}}\delta_h^{l+\frac{1}{2}}\sup_{0\leqslant t\leqslant\delta_0}\|D_n^l(f+\Delta u)\|_{\Gamma_t}\Big)\|u-u_h\|_{\Omega_h}.
\end{aligned}$$

The theorem follows. □

7 Numerical Results and Discussion

We will consider 3 examples of increasing complexity to corroborate the theoretical findings in the previous sections. The exact boundary of the domain Ω is described using analytical expressions of level set functions whose zero level set describes the boundary. We first consider a circular domain and a domain with convex and concave boundaries with zero Dirichlet boundary conditions and then a flower shape domain with non-zero Dirichlet boundary conditions. We will demonstrate the effect of the boundary value correction terms for polynomial order 2 and 3. In all examples, we set the ghost penalty parameter $\gamma_p = 0.1$.

7.1 Reference Solution in Circle with Zero Dirichlet Boundary Conditions

In our first example, we consider a circular domain described by the zero level set of

$$\phi = R - 1$$

where $R = \sqrt{x^2 + y^2}$. We investigate the convergence of the numerical solution to the following analytical solution

$$u(x, y) = \cos(\pi \frac{R^2}{2}),$$

which we prescribed using

$$f(x, y) = \pi^2 R^2 \cos\left(\pi\left(\frac{R^2}{2}\right)\right) + 2\pi \sin\left(\pi\left(\frac{R^2}{2}\right)\right).$$

The solution and the linear approximation of the domain are depicted in Figure 4. Figure 5 shows the convergence rates of the numerical solution in the H^1 and L^2-norm. The order of convergence is optimal when a Taylor expansion of first order is used ($k = 1$). Adding terms beyond the first order term in the Taylor expansion does not yield any improvment in the rate of convergence ($k > 1$).

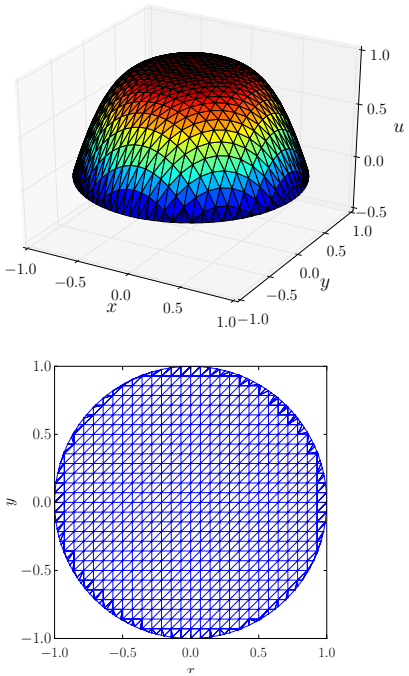


Figure 4. Reference solution u and the cut finite element mesh in the circle geometry.

7.2 Reference Solution in Torus with Zero Dirichlet Boundary Conditions

Next, we consider a domain with convex and concave boundaries given by the zero level set of the function

$$\phi = (R - 0.75)\,(R - 0.25)\,. \tag{34}$$

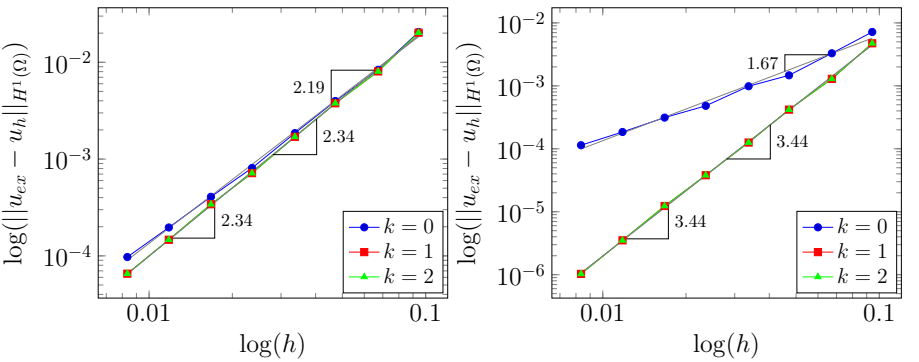
We set

$$f = 20\left(4 - \frac{1}{R}\right) \tag{35}$$

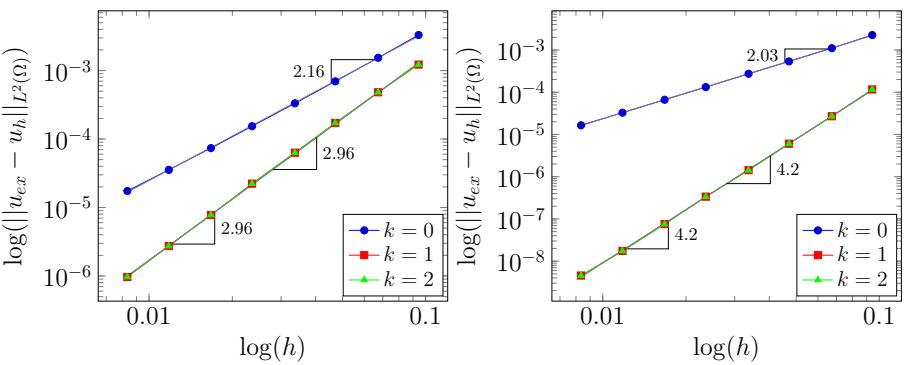
and obtain the analytical solution

$$u = 20\,(0.75 - R)\,(R - 0.25)\,. \tag{36}$$

as shown in Figure 6. The convergence rates shown in Figure 7 are optimal for $p = 2$, $p = 3$ when a first order Taylor expansion is used in the boundary value correction terms.



(a) $p = 2$ (b) $p = 3$



(c) $p = 2$ (d) $p = 3$

Figure 5. Convergence rates for the first reference solution in the circle in the H_1 norm for $p = 2$, (b), $p = 3$ for Taylor expansions of order $k = 0, 1, 2$ and in the L_2 norm for (c) $p = 2$,(d) $p = 3$.

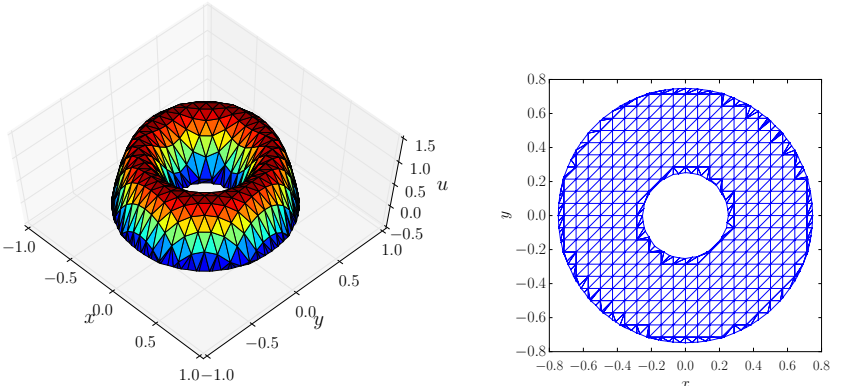
7.3 Reference Solution in Flower Shape with Non-Zero Dirichlet Boundary Conditions

In our final example, we consider a flower like shaped domain [17] defined by

$$\phi = (R^2 - r_\theta)(R^2 - (1.0/6.0)^2) \tag{37}$$

with $r_\theta = r_0 + 0.1\sin(\omega\theta)$, $r_0 = 1/2$, $\omega = 8$ and $\theta = \arctan(x/y)$. We investigate the convergence rates of our numerical solution with respect to

$$u = \cos\left(\pi\frac{x}{2}\right)\cos\left(\pi\frac{y}{2}\right) \tag{38}$$



(a) (b)

Figure 6. Reference solution u in a torus-like shaped geometry and the cut finite element mesh of the domain.

$$f = \frac{\pi^2}{2} \cos\left(\pi \frac{x}{2}\right) \cos\left(\pi \frac{y}{2}\right) \tag{39}$$

The reference solution and the cut mesh are shown in Figure 8. Figure 9 shows the convergence rate for $p = 2$ and $p = 3$.

7.4 3D Solution in an Ellipsoid

We compute the reference solution

$$u = \cos(\pi \frac{\hat{r}^2}{2}) \tag{40}$$

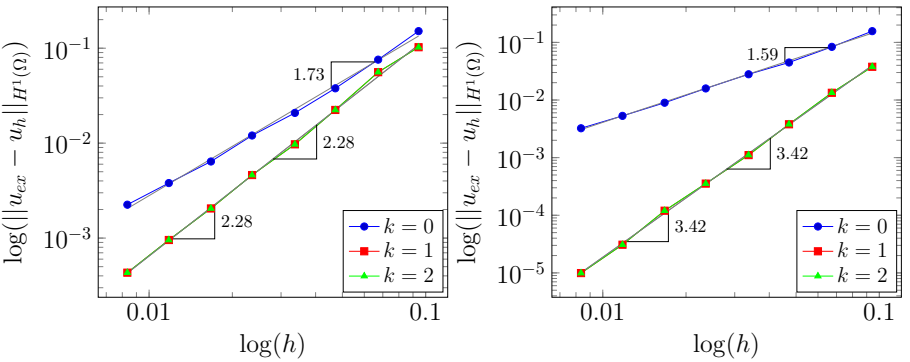
with

$$\hat{r} = \sqrt{\frac{x^2}{(3.0/4.0)^2} + \frac{y^2}{(1.0/2.0)^2} + \frac{z^2}{(1.0/2.0)^2}} \tag{41}$$

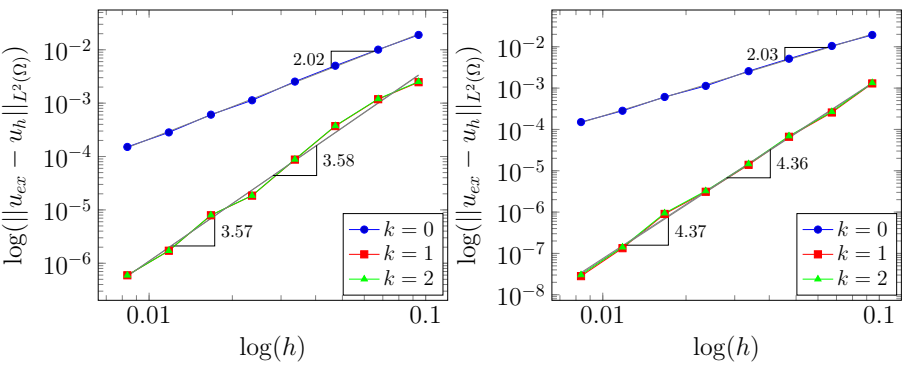
in a 3D ellipsoid given by the function

$$\phi = \hat{r} - 1. \tag{42}$$

Figure 10 shows the solution in the ellipsoid and Figure 11 shows the convergence for the solution for $p = 2$ and $k = 1$ demonstrating the optimal convergence rate of the numerical solution as predicted by the estimates of the previous section.



(a) $p = 2$ (b) $p = 3$



(c) $p = 2$ (d) $p = 3$

Figure 7. Convergence rates for the reference solution in the domain including convex and concave boundaries in the H_1 norm for (a) $p = 2$, (b), $p = 3$ for Taylor expansions of order $k = 0, 1, 2$ and in the L_2 norm for (c) $p = 2$,(d) $p = 3$.

Bibliography

[1] P. Angot, H. Lomenède and I. Ramière, *A general fictitious domain method with non-conforming structured meshes*, Finite volumes for complex applications IV, ISTE, London, 2005, pp. 261–272.

[2] T. Boiveau and E. Burman, A penalty-free Nitsche method for the weak imposition of boundary conditions in compressible and incompressible elasticity, *IMA J. Numer. Anal.* **36** (2016), 770–795.

[3] E. Burman, Ghost penalty, *C. R. Math. Acad. Sci. Paris* **348** (2010), 1217–1220.

[4] E. Burman, A penalty free nonsymmetric Nitsche-type method for the weak imposition of boundary conditions, *SIAM J. Numer. Anal.* **50** (2012), 1959–1981.

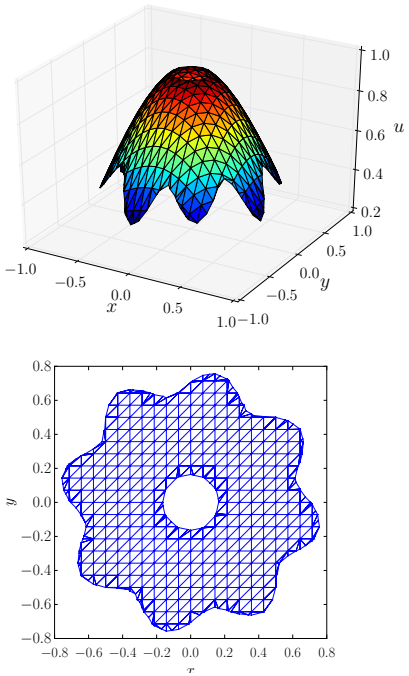


Figure 8. Reference solution u in flower geometry and the cut finite element mesh of the domain.

[5] E. Burman, S. Claus, P. Hansbo, M. G. Larson and A. Massing, CutFEM: Discretizing geometry and partial differential equations, *International Journal for Numerical Methods in Engineering* **104** (2015), 472–501.

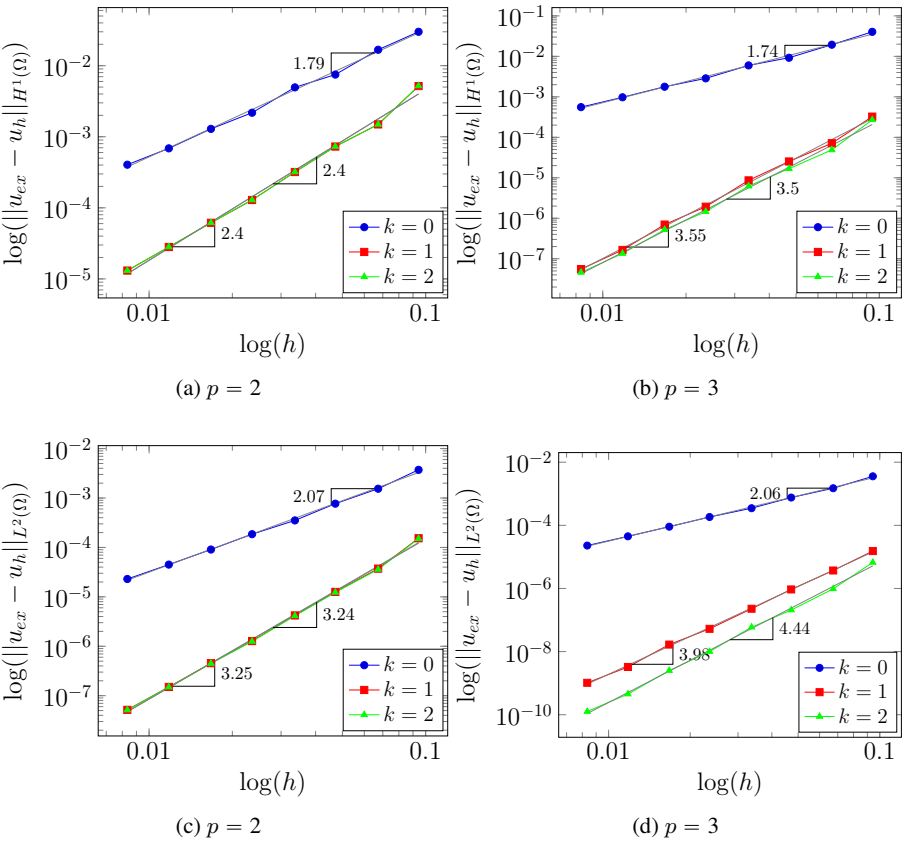
[6] E. Burman and M. A. Fernández, Explicit strategies for incompressible fluid-structure interaction problems: Nitsche type mortaring versus Robin-Robin coupling, *Internat. J. Numer. Methods Engrg.* **97** (2014), 739–758.

[7] E. Burman and P. Hansbo, Fictitious domain finite element methods using cut elements: I. A stabilized Lagrange multiplier method, *Comput. Methods Appl. Mech. Engrg.* **199** (2010), 2680–2686.

[8] E. Burman and P. Hansbo, Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method, *Appl. Numer. Math.* **62** (2012), 328–341.

[9] E. Burman, P. Hansbo and M. G. Larson, A Cut Finite Element Method with Boundary Value Correction, (2015).

[10] J. Freund and R. Stenberg, On weakly imposed boundary conditions for second order problems, in: *Proceedings of the Ninth International Conference on Finite Elements in Fluids*, pp. 327–336, Università di Padova, 1995.



(a) $p = 2$ (b) $p = 3$

(c) $p = 2$ (d) $p = 3$

Figure 9. Convergence rates for the reference solution in the flower shaped domain in the H_1 norm for (a) $p = 2$, (b), $p = 3$ for Taylor expansions of order $k = 0, 1, 2$ and in the L_2 norm for (c) $p = 2$,(d) $p = 3$.

[11] V. Girault and R. Glowinski, Error analysis of a fictitious domain method applied to a Dirichlet problem, *Japan Journal of Industrial and Applied Mathematics* **12** (1995), 487–514 (English).

[12] V. Girault, R. Glowinski and T. W. Pan, *A fictitious-domain method with distributed multiplier for the Stokes problem*, Applied nonlinear analysis, Kluwer/Plenum, New York, 1999, pp. 159–174.

[13] A. Hansbo and P. Hansbo, An unfitted finite element method, based on Nitsche's method, for elliptic interface problems, *Comput. Methods Appl. Mech. Engrg.* **191** (2002), 5537–5552.

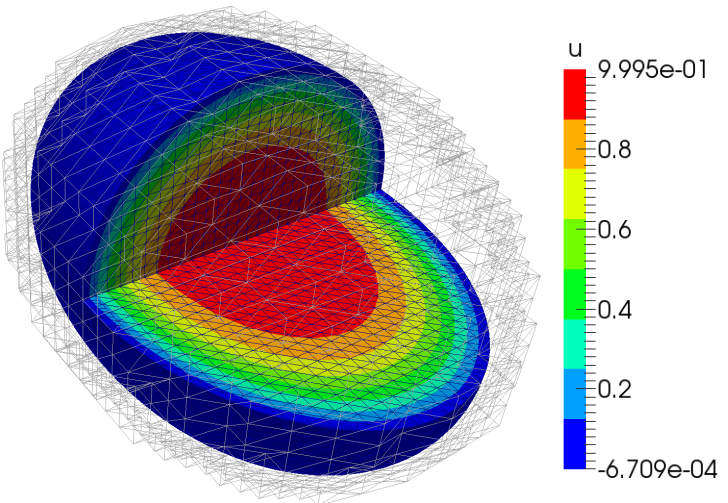


Figure 10. Reference solution u in a 3D ellipsoid.

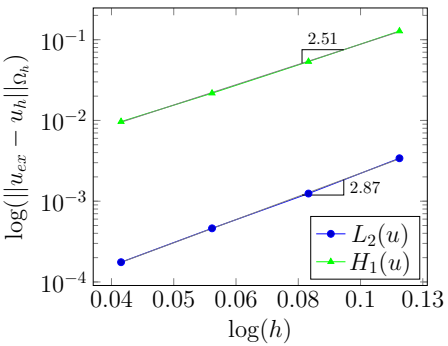


Figure 11. Error for $p = 2$ in ellipsoid for $k = 1$.

[14] J. Haslinger and Y. Renard, A New Fictitious Domain Approach Inspired by the Extended Finite Element Method, *SIAM Journal on Numerical Analysis* **47** (2009), 1474–1499.

[15] T. J. R. Hughes, G. Engel, L. Mazzei and M. G. Larson, *A comparison of discontinuous and continuous Galerkin methods based on error estimates, conservation, robustness and efficiency*, Discontinuous Galerkin methods (Newport, RI, 1999), Lect. Notes Comput. Sci. Eng. 11, Springer, Berlin, 2000, pp. 135–146.

[16] M. G. Larson and A. J. Niklasson, Analysis of a nonsymmetric discontinuous Galerkin method for elliptic problems: stability and energy error estimates, *SIAM J. Numer. Anal.* **42** (2004), 252–264 (electronic).

[17] C. Lehrenfeld, High order unfitted finite element methods on level set domains using isoparametric mappings, *Computer Methods in Applied Mechanics and Engineering* **300** (2016), 716–733.

[18] A. Massing, M. G. Larson, A. Logg and M. E. Rognes, A Stabilized Nitsche Fictitious Domain Method for the Stokes Problem, *Journal of Scientific Computing* **61** (2014), 604–628 (English).

[19] J. Nitsche, Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* **36** (1971), 9–15 (German).

Received **???**.

Author information

Thomas Boiveau, Department of Mathematics, University College London, Gower Street, London, UK-WC1E 6BT, United Kingdom.
E-mail: thomas.boiveau.12@ucl.ac.uk

Erik Burman, Department of Mathematics, University College London, Gower Street, London, UK-WC1E 6BT, United Kingdom.

Susanne Claus, Cardiff School of Engineering, Cardiff University, Queen's Buildings, The Parade, Cardiff CF24 3AA, Wales, United Kingdom.

Mats G. Larson, Department of Mathematics and Mathematical Statistics, Umeå University, SE-90187 Umeå, Sweden.