ON THE ASYMPTOTIC GROWTH OF BLOCH–KATO–SHAFAREVICH–TATE GROUPS OF MODULAR FORMS OVER CYCLOTOMIC EXTENSIONS

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Abstract. We study the asymptotic behaviour of the Bloch–Kato–Shafarevich–Tate group of a modular form f over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ under the assumption that f is non-ordinary at p. In particular, we give upper bounds of these groups in terms of Iwasawa invariants of Selmer groups defined using p-adic Hodge Theory. These bounds have the same form as the formulae of Kobayashi, Kurihara and Sprung for supersingular elliptic curves.

1. Introduction

Let p be an odd prime and f a normalised new cuspidal modular eigenform of weight $k \geq 2$, and p an odd prime which does not divide the level of f. For notational simplicity, we assume in this introduction that all the Fourier coefficients of f lie in $\mathbb{Z}$. We let V_f be the *cohomological* p-adic Galois representation attached to f (so the determinant of V_f is χ^{1-k} times a finite-order character). Then V_f has Hodge–Tate weights $\{0, 1-k\}$, where our convention[1] is that the Hodge–Tate weight of the cyclotomic character is 1. Let T_f be the canonical $G_{\mathbb{Q}}$-stable $\mathbb{Z}_p$-lattice in V_f defined by Kato [Kat04, 8.3].

Let K_∞ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and write K_n for the unique sub-extension of degree p^n. Our aim is to study the asymptotic behaviour of the Bloch–Kato–Shafarevich–Tate groups $Ш(K_n, T_f(j))$ (with $j \in [1, k-1]$), whose definition we shall recall below.

When $k = 2$, the form f corresponds to an isogeny class of elliptic curves, and we may choose a curve $\mathcal{E}$ in this isogeny class such that $T_f(1) = T_p(\mathcal{E})$, where the latter is the p-adic Tate module of $\mathcal{E}$. In this case it can be shown that the group $Ш(K_n, T_f(1))$ is the quotient of the classical p-primary Shafarevich–Tate group $Ш_p(K_n, \mathcal{E})$ by its maximal divisible subgroup; hence if the latter group is finite (which is a well-known conjecture), the two groups are equal.

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[1]This is usual in p-adic Hodge theory, but the opposite convention appears to be common in papers on modularity lifting.

The ordinary case. The behaviour of the Selmer and Shafarevich–Tate groups over the cyclotomic extension depends sharply on whether $\mathcal{E}$ has ordinary or supersingular reduction at p. If $\mathcal{E}$ is ordinary, then the p-Selmer group

$$\mathrm{Sel}_p(K_\infty, \mathcal{E}) = \varinjlim_n \mathrm{Sel}_p(K_n, \mathcal{E})$$

of A over K_∞ is cotorsion over the Iwasawa algebra $\mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/\mathbb{Q})]\!]$, by a theorem of Kato [Kat04, Theorem 17.4]. By Mazur's control theorem [Maz72], this implies that **if** the groups $Ш_p(K_n, \mathcal{E})$ are finite for all n, then we must have

$$\mathrm{len}_{\mathbb{Z}_p} Ш_p(K_n, \mathcal{E}) = \mu p^n + \lambda n + O(1),$$

for some Iwasawa invariants μ and λ associated to $\mathrm{Sel}_p(\mathcal{E}/K_\infty)$.

The supersingular case. The case of supersingular elliptic curves with $a_p(\mathcal{E}) = 0$ has been studied by Kurihara [Kur02] and Kobayashi [Kob03]. Suppose that $Ш_p(K_n, \mathcal{E})$ is finite for all n and write $s_n(\mathcal{E}) = \mathrm{len}_{\mathbb{Z}_p} Ш_p(K_n, \mathcal{E})$. They showed that for n sufficiently large,

$$s_n(\mathcal{E}) - s_{n-1}(\mathcal{E}) = q_n + \lambda_\pm + \mu_\pm(p^n - p^{n-1}) - r_\infty(\mathcal{E}),$$

where q_n is an explicit sum of powers of p, $r_\infty(\mathcal{E})$ is the rank of $\mathcal{E}$ over K_∞, $\lambda_\pm$ and $\mu_\pm$ are the Iwasawa invariants of some cotorsion signed Selmer groups, and the sign $\pm$ depends on the parity of n.

For supersingular elliptic curves with $a_p(\mathcal{E}) \neq 0$ (which can only occur when $p = 2$ or 3), Sprung [Spr13] proved a similar formula:

$$s_n(\mathcal{E}) - s_{n-1}(\mathcal{E}) = q_n^\star + \lambda_\star + \mu_\star(p^n - p^{n-1}) - r_\infty(\mathcal{E}),$$

for $n \gg 0$, where $q_n^\star$ is again an explicit sum of powers of p, $\star \in \{\#, \flat\}$, $\lambda_\star$ and $\mu_\star$ are Iwasawa invariants of some cotorsion Selmer groups defined in [Spr12] and the choice of $\star$ depends on the "modesty algorithm". An analytic version of this formula has been generalised to arbitrary weight 2 modular forms in [Spr15].

Higher weights. The main result of the present article is that a similar formula for modular forms of higher weight would give us an upper bound on the growth of the Bloch–Kato–Shafarevich–Tate groups. Suppose that $\mathrm{ord}_p(a_p(f)) > \frac{k-1}{2p}$ and $3 \leq k \leq p$, where $a_p(f)$ is the p-th Fourier coefficient of the modular form f. We shall see below that the Selmer coranks

$$r_n(f) = \mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}(K_n, T_f(j))$$

stabilise for $n \gg 0$, and we define $r_\infty(f)$ to be the limiting value (see Proposition 5.4). We define

$$s_n(f) = \mathrm{len}_{\mathbb{Z}_p} Ш_p(K_n, T_f(j))$$

(which is finite by definition). We prove the inequality (see Theorem 5.5 for the precise statement)

$$s_n(f) - s_{n-1}(f) \leq q_n^\star + \lambda_\star + \mu_\star(p^n - p^{n-1}) + \kappa - r_\infty(f),$$

for $n \gg 0$, where $q_n^\star$ is once again a sum of powers of p that depends on k and the parity of n, $\lambda_\star$ and $\mu_\star$ are the Iwasawa invariants of the Selmer groups defined in [LLZ10] for some choice of basis of the Wach module of T_f, κ is some integer that depends on the image of some Coleman maps that we shall review in §3 of this article and the choice of $\star$ is given by an explicit algorithm (similar to the "modesty algorithm" of Sprung).

The fact that we have an inequality is a result of the growth of the logarithmic matrix contributed from the twists of $T_f(i)$ for $i \neq j$. In the appendix to this paper, we relate the defect of this inequality to the Tamagamwa numbers of $T_f(j)$ using the method developed by Perrin-Riou in [PR03].

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2. Background from p-adic Hodge theory

We recall the necessary notation and definitions from p-adic analysis and p-adic Hodge theory. For more details see [LLZ11, §1.3]. We fix (for the duration of this article) a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, which will be the coefficient field for all the representations we shall consider.

2.1. Iwasawa algebras and distribution algebras.

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. This group is isomorphic to a direct product $\Delta \times \Gamma_1$, where Δ is a finite group of order $p-1$ and $\Gamma_1 = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$. We choose a topological generator γ of Γ_1, which determines an isomorphism $\Gamma_1 \cong \mathbb{Z}_p$. We also fix a finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ which will be our field of coefficients (i.e. we will consider representations of Galois groups on E-vector spaces).

We write $\Lambda = \mathcal{O}[\![\Gamma]\!]$, the Iwasawa algebra of Γ. The subalgebra $\mathcal{O}[\![\Gamma_1]\!]$ can be identified with the formal power series ring $\mathcal{O}[\![X]\!]$, via the isomorphism sending γ_1 to $1+X$; this extends to an isomorphism

$$\Lambda = \mathcal{O}[\Delta][\![X]\!]. \tag{2.1}$$

For a character η of Δ and a Λ-module M, M^η denotes its η-isotypic component, which is regarded as an $\mathcal{O}[\![X]\!]$-module. For $n \geq 1$, we write Γ_n for the subgroup $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_{p^n}))$ and $\Lambda_n = \mathcal{O}[\Gamma/\Gamma_n]$. Note that

$$\Lambda_n = \mathcal{O}[\Delta][\![X]\!]/(\omega_{n-1}(X)),$$

where $\omega_{n-1}(X)$ denotes the polynomial $(1+X)^{p^{n-1}} - 1$.

We may consider Λ as a subring of the ring $\mathcal{H}$ of locally analytic E-valued distributions on Γ. The isomorphism (2.1) extends to an identification between $\mathcal{H}$ and the subring of power series $F \in E[\Delta][\![X]\!]$ which converge on the open unit disc $|X| < 1$.

2.2. Power series rings.

Let $\mathbb{A}^+_{\mathbb{Q}_p} = \mathcal{O}[\![\pi]\!]$, where π is a formal variable. We equip this ring with a $\mathcal{O}$-linear *Frobenius endomorphism* φ, defined by $\pi \mapsto (1+\pi)^p - 1$, and with an $\mathcal{O}$-linear action of Γ defined by $\pi \mapsto (1+\pi)^{\chi(\sigma)} - 1$ for $\sigma \in \Gamma$, where χ denotes the p-adic cyclotomic character.

The Frobenius φ has a left inverse ψ, satisfying

$$(\varphi \circ \psi)(f)(\pi) = \frac{1}{p} \sum_{\zeta : \zeta^p = 1} f\left(\zeta(1+\pi) - 1\right).$$

The map ψ is not a morphism of rings, but it is $\mathcal{O}$-linear, and commutes with the action of Γ.

We write $\mathbb{B}^+_{\mathbb{Q}_p} = \mathbb{A}^+_{\mathbb{Q}_p}[1/p] \subset E[\![\pi]\!]$, and

$$\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} = \{F(\pi) \in E[\![\pi]\!] : F \text{ converges on the open unit disc}\},$$

so there are natural inclusions

$$\mathbb{A}^+_{\mathbb{Q}_p} \hookrightarrow \mathbb{B}^+_{\mathbb{Q}_p} \hookrightarrow \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}.$$

The actions of φ, ψ, and Γ extend to these larger rings (via the same formulae as before). We shall write $q = \varphi(\pi)/\pi \in \mathbb{A}^+_{\mathbb{Q}_p}$, and $t = \log(1+\pi) \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$.

2.3. **The Mellin transform.** The action of Γ on $1+\pi \in (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}$ extends to an isomorphism of Λ-modules

$$\begin{aligned}\mathfrak{M} : \Lambda &\xrightarrow{\cong} (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}\\ 1 &\longmapsto 1+\pi,\end{aligned}$$

called the *Mellin transform*. This can be further extended to an isomorphism of $\mathcal{H}$-modules

$$\mathcal{H} \xrightarrow{\cong} (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$$

which we denote by the same symbol.

Theorem 2.1. *For every $n \ge 1$, the Mellin transform induces an isomorphism of Λ-modules*

$$\Lambda_n \cong (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}/\varphi^n(\pi)(\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}.$$

Proof. If $\mu \in \omega_{n-1}(X)\Lambda$, then $\mathfrak{M}(\mu) \in \varphi^n(\pi)(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$, by [LLZ10, Theorem 5.4]. However, $\varphi^n(\pi)$ is a monic polynomial in π, so if an element of $\mathbb{A}^+_{\mathbb{Q}_p}$ is divisible by $\varphi^n(\pi)$ in $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, it is divisible by the same element in $\mathbb{A}^+_{\mathbb{Q}_p}$. Hence the Mellin transform induces a map $\Lambda_n \to (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}/\varphi^n(\pi)(\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}$; and this map is surjective, because the Mellin transform itself is surjective. Since both sides are free $\mathcal{O}$-modules of the same rank, namely $(p-1)p^n$, it follows that the map must in fact be an isomorphism. □

We write ∂ for the differential operator $(1+\pi)\frac{\mathrm{d}}{\mathrm{d}\pi}$ on $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, and Tw for the ring automorphism of $\mathcal{H}$ defined by $\sigma \mapsto \chi(\sigma)\sigma$ for $\sigma \in \Gamma$. Then one has the compatibility relation

$$\mathfrak{M} \circ \mathrm{Tw} = \partial \circ \mathfrak{M}.$$

Let $u = \chi(\gamma)$ be the image of our topological generator γ under the cyclotomic character, so that Tw maps X to $u(1+X)-1$. If $m \ge 0$ is an integer, we define $\omega_{n,m}(X) = \omega_n(u^{-m}(1+X)-1)$ and $\tilde{\omega}_{n,m} = \prod_{i=0}^m \omega_{n,i}$. By exactly the same argument as Theorem 2.1, this gives the following isomorphism of Λ-modules

$$\Lambda_{n,m} \coloneqq \Lambda/\tilde{\omega}_{n-1,m}\Lambda \cong (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}/\varphi^n(\pi^{m+1})(\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}. \tag{2.2}$$

We will need below the following technical result, regarding the interaction between Mellin transforms and the Iwasawa invariants of power series. We recall the *Weierstrass preparation theorem*, which states that any $F \in \mathcal{O}[\![X]\!]$ can be factorized uniquely as

$$F(X) = \varpi^{\mu(F)} \cdot (X^{\lambda(F)} + \varpi G(X)) \cdot u(X),$$

where ϖ is a uniformizer of $\mathcal{O}$, $\lambda(F)$ and $\mu(F)$ are non-negative integers, $G \in \mathcal{O}[X]$ is a polynomial of degree $< \lambda(F)$, and $u \in \mathcal{O}[\![X]\!]^\times$. The quantities $\lambda(F)$ and $\mu(F)$ are called the *Iwasawa invariants* of F.

It is clear that, for $x \in \mathcal{O}_{\mathbf{C}_p}$ with $\mathrm{ord}_p(x) > 0$, we have the lower bound

$$\mathrm{ord}_p F(x) \ge \min\left(\tfrac{\mu+1}{e}, \tfrac{\mu}{e} + \lambda \,\mathrm{ord}_p(x)\right), \tag{2.3}$$

where $e = 1/\operatorname{ord}_p(\varpi)$ is the absolute ramification degree of F. Moreover, if $\operatorname{ord}_p(x)$ is sufficiently small (depending on F), this lower bound is an equality (it suffices to take $\operatorname{ord}_p(x) < 1/(e\lambda)$).

Proposition 2.2. *Let $f \in \mathbb{A}^+_{\mathbb{Q}_p}$, and let g be the unique element of $\Lambda(\Gamma_1)$ such that $\mathfrak{M}(g) = (1+\pi)\varphi(f)$. Then the λ- and μ-invariants of f (as an element of $\mathcal{O}[\![\pi]\!]$) coincide with those of g (as an element of $\mathcal{O}[\![X]\!]$).*

Proof. This is a consequence of Proposition 7.2 of [LZ12], which shows that for any $f \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and $g \in \mathcal{H}$ such that $\mathfrak{M}(g) = (1+\pi)\varphi(f)$, and any real s with $0 < s < 1$, we have $v_s(f) = v_s(g)$, where

$$v_s(f) \coloneqq \inf\{\operatorname{ord}_p f(x) : \operatorname{ord}_p(x) \geq s\}.$$

When $f \in \mathcal{O}[\![X]\!]$ and s is sufficiently small, $v_s(f)$ is determined by the Iwasawa invariants of f: from the inequality (2.3) and the discussion following, we have $v_s(f) = \frac{1}{e}\mu(f) + \lambda(f)s$ for any $s < \frac{1}{e\lambda(f)}$. So the cited proposition implies the equalities $\lambda(f) = \lambda(g)$ and $\mu(f) = \mu(g)$. □

2.4. **Crystalline representations and Wach modules.** Fontaine has defined a certain topological $\mathbb{Q}_p$-algebra $\mathbb{B}_{\mathrm{cris}}$, equipped with an action of $G_{\mathbb{Q}_p}$, a filtration $\mathrm{Fil}^\bullet$, and a Frobenius endomorphism φ.

For any p-adic representation V of $G_{\mathbb{Q}_p}$, we define the *crystalline Dieudonné module* of V by

$$\mathbb{D}_{\mathrm{cris}}(V) = \left(V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}}\right)^{G_{\mathbb{Q}_p}}.$$

The space $\mathbb{D}_{\mathrm{cris}}(V)$ inherits a filtration and a Frobenius endomorphism from those of $\mathbb{B}_{\mathrm{cris}}$. It is known that $\dim_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V) \leq \dim_{\mathbb{Q}_p} V$, and we say V is *crystalline* if equality holds. If in fact V is an E-linear representation, then $\mathbb{D}_{\mathrm{cris}}(V)$ is naturally an E-vector space (and its filtration and Frobenius are E-linear).

Definition 2.3. *Let $a \leq b$ be integers. A* Wach module *over $\mathbb{B}^+_{\mathbb{Q}_p}$ with weights in $[a,b]$ is a finite free $\mathbb{B}^+_{\mathbb{Q}_p}$-module N, equipped with an action of Γ and a Frobenius*

$$\varphi : N[1/\pi] \to N[1/\varphi(\pi)]$$

compatible with those of $\mathbb{B}^+_{\mathbb{Q}_p}$, satisfying the following conditions:

- *Γ acts trivially on $N/\pi N$,*
- *$\varphi(\pi^b N) \subseteq \pi^b N$,*
- *if $\varphi^*(\pi^b N)$ is the $\mathbb{B}^+_{\mathbb{Q}_p}$-submodule of $\pi^b N$ generated by $\varphi(\pi^b N)$, then the quotient $\pi^b N/\varphi^*(\pi^b N)$ is killed by q^{b-a}.*

Cf. [Ber04, Definition III.4.1]. In *op.cit.* it is shown how to attach to every crystalline E-linear representation V of $G_{\mathbb{Q}_p}$ a Wach module $\mathbb{N}(V)$ over $\mathbb{B}^+_{\mathbb{Q}_p}$, in such a way that there is a canonical isomorphism

$$\mathbb{N}(V) \otimes_{\mathbb{B}^+_{\mathbb{Q}_p}} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t] \cong \mathbb{D}_{\mathrm{cris}}(V) \otimes_E \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t].$$

Moreover, the definition of Wach modules also makes sense integrally, i.e. over $\mathbb{A}^+_{\mathbb{Q}_p}$; and we may associate to each $\mathcal{O}$-lattice T in V that is stable under $G_{\mathbb{Q}_p}$ an integral Wach module $\mathbb{N}(T) \subset \mathbb{N}(V)$ (Lemme II.1.3 of *op.cit.*).

Definition 2.4. *We say V satisfies the* Fontaine–Laffaille condition *if it is crystalline and has Hodge–Tate weights in $[a, a+(p-1)]$ for some $a \in \mathbb{Z}$.*

If V satisfies the Fontaine–Laffaille condition, and V is irreducible of dimension ≥ 2, then one has a particularly convenient parametrisation of $G_{\mathbb{Q}_p}$-stable lattices in V. We say a $\mathcal{O}$-lattice $M \subset \mathbb{D}_{\mathrm{cris}}(V)$ is a *strongly divisible lattice* if the equality

$$\varphi\left(M \cap \operatorname{Fil}^i \mathbb{D}_{\mathrm{cris}}(V)\right) \subset p^i M$$

holds for all $i \in \mathbb{Z}$. Then there is a bijection $T \mapsto \mathbb{D}_{\mathrm{cris}}(T)$ between $G_{\mathbb{Q}_p}$-stable lattices in V, and strongly divisible lattices in $\mathbb{D}_{\mathrm{cris}}(V)$, given by defining $\mathbb{D}_{\mathrm{cris}}(T)$ to be the image of $\mathbb{N}(T)$ in $\mathbb{N}(V)/\pi\mathbb{N}(V) \cong \mathbb{D}_{\mathrm{cris}}(V)$; cf. [Ber04, Propositions V.2.1 & V.2.3].

We shall need below the following technical result.

Theorem 2.5. *Let T be a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}$-lattice in a crystalline E-linear representation V. Then $(\varphi^*\mathbb{N}(T))^{\psi=0}$ is a free Λ-module of rank $d = \dim_E V$. Moreover, if $\{n_1, \dots, n_d\}$ is an $\mathbb{A}^+_{\mathbb{Q}_p}$-basis of $\mathbb{N}(T)$ which satisfies the condition*

$$(\gamma - 1)n_i \in \pi^2\mathbb{N}(T)$$

for all i, then $\{(1+\pi)\varphi(n_i) : i = 1, \dots, d\}$ is a Λ-module basis of $(\varphi^*\mathbb{N}(T))^{\psi=0}$.

Proof. This is shown in the course of the proof of Theorem 3.5 of [LLZ10]. The condition on the basis modulo π^2 is the conclusion of Lemma 3.9 in *op. cit.* □

2.5. **Iwasawa cohomology and the Fontaine isomorphism.** If V is an E-linear p-adic representation of $G_{\mathbb{Q}_p}$, and $T \subset V$ is a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice, then we define *Iwasawa cohomology* groups by

$$H^i_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) = \varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T)$$

(where the inverse limit is with respect to the corestriction maps). These groups are finitely-generated Λ-modules, zero unless $i \in \{1, 2\}$. If $H^0(\mathbb{Q}_p(\mu_{p^\infty}), T/pT) = 0$, which is the case in our applications below, then H^2_{Iw} is zero, and H^1_{Iw} is a free Λ-module of rank equal to the $\mathcal{O}$-rank of T.

The following theorem is the starting-point for our study of Iwasawa cohomology:

Theorem 2.6 (Fontaine–Berger)**.** *If V is crystalline with all Hodge–Tate weights ≥ 0, and V has no non-zero quotient on which $G_{\mathbb{Q}_p}$ acts trivially, then there is a canonical Λ-module isomorphism*

$$h^1_T : \mathbb{N}(T)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T).$$

See [CC99, §II.1], where it is shown that (for any T) there is an isomorphism $\mathbb{D}(T)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)$ where $\mathbb{D}(T)$ is the (φ, Γ)-module of T; and [Ber03, §A], where it is shown that $\mathbb{N}(T)^{\psi=1} = \mathbb{D}(T)^{\psi=1}$ under the above hypotheses.

3. Wach modules and Coleman maps

3.1. **Review on the definition of Coleman maps.** Let $f = \sum a_n q^n$ be a normalised new cuspidal modular eigenform of weight $k \geq 3$ (note that the case $k = 2$ can be dealt with using the method of Sprung in [Spr13]), nebentypus ε and level N with $(p, N) = 1$. We take E to be the completion of the smallest number field containing all the coefficients of f at some fixed prime above p. We assume that f is non-ordinary at p, and that $k \leq p$. We write T_f for the $\mathcal{O}$-linear representation of

$G_{\mathbb{Q}}$ associated to f as defined by Kato [Kat04, 8.3]. It is crystalline, with Hodge–Tate weights 0 and $1-k$. We fix an integer $j \in [1, k-1]$ and write $T = T_f(j)$ and $\mathcal{T} = T_f(k-1)$. Note that $T = \mathcal{T}(j-k+1)$.

The representation $T/\varpi T$ (where ϖ is a uniformiser of $\mathcal{O}$) is irreducible as a representation of $G_{\mathbb{Q}_p}$, so in particular we have

$$H^0(\mathbb{Q}_p(\mu_{p^\infty}), T/\varpi T) = 0.$$

Both T_f and $\mathcal{T}$ are $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattices in crystalline representations of $G_{\mathbb{Q}_p}$, so we may consider their Wach modules and Dieudonné modules. By [Ber04, Proposition III.2.1], there are inclusions of $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-modules

$$\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{A}^+_{\mathbb{Q}_p}} \mathbb{N}(\mathcal{T}) \subset \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathcal{O}} \mathbb{D}_{\mathrm{cris}}(\mathcal{T}),$$
$$\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathcal{O}} \mathbb{D}_{\mathrm{cris}}(T_f) \subset \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{A}^+_{\mathbb{Q}_p}} \mathbb{N}(T_f),$$

where the elementary divisors of the inclusions are given by 1 and $(t/\pi)^{k-1}$ in both cases.

Lemma 3.1. *There exists an $\mathcal{O}$-basis $\mathfrak{v}_1, \mathfrak{v}_2$ of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ such that $\mathfrak{v}_1 \in \mathrm{Fil}^0\, \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ and $\mathfrak{v}_2 = \varphi(\mathfrak{v}_1)$, where φ is the Frobenius action on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$.*

Proof. The Fontaine–Laffaille condition of [FL82] implies that for all integers i

(a) $\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ is a direct summand of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$;
(b) $\varphi(\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathcal{T})) \subset p^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$;
(c) $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \sum_i p^{-i} \varphi(\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathcal{T}))$.

The Hodge–Tate weights of $\mathcal{T}$ are 0 and $k-1$, so $\mathrm{Fil}^0\, \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ is of rank 1, say $\mathrm{Fil}^0\, \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \mathfrak{v}_1$ and (b) says that $\mathfrak{v}_2 := \varphi(\mathfrak{v}_1) \in \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$. Furthermore, (a) tells us that there exists some $\mathfrak{v}' \in \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ such that

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \mathfrak{v}_1 \oplus \mathcal{O} \cdot \mathfrak{v}'.$$

By (c), we have

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \varphi(\mathfrak{v}_1) + p^{k-1} \varphi(\mathbb{D}_{\mathrm{cris}}(\mathcal{T})).$$

Combing the last two equations gives

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \varphi(\mathfrak{v}_1) \oplus \mathcal{O} \cdot p^{k-1} \varphi(\mathfrak{v}'). \tag{3.1}$$

Let D be the $\mathcal{O}$-lattice generated by $\mathfrak{v}_1$ and $\mathfrak{v}_2$. Note that (3.1) implies that

$$\mathfrak{v}' \in D + \mathcal{O} \cdot p^{k-1} \varphi(\mathfrak{v}'). \tag{3.2}$$

As $\mathfrak{v}_2 = \varphi(\mathfrak{v}_1)$ and

$$\varphi^2 - \frac{a_p}{p^{k-1}} \varphi + \frac{\varepsilon(p)}{p^{k-1}} = 0$$

on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$, we have $p^{k-1}\varphi(\mathfrak{v}_2) = a_p \mathfrak{v}_2 - \varepsilon(p)\mathfrak{v}_1$. In particular, this implies that $p^{k-1}\varphi(D) \subset D$. Hence, we may iterate the inclusion (3.2) to deduce that

$$\mathfrak{v}' \in D + \mathcal{O} \cdot (p^{k-1}\varphi)^n(\mathfrak{v}')$$

for all $n \geq 0$. However, as f is non-ordinary at p, $p^{k-1}\varphi$ is an $\mathcal{O}$-operator on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ with strictly positive slope. This implies that $(p^{k-1}\varphi)^n \to 0$ as $n \to \infty$, which forces that $\mathfrak{v}' \in D$. Hence, $D = \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ as required. ∎

We fix an $\mathcal{O}$-basis $\mathfrak{v}_1, \mathfrak{v}_2$ of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$, as given by Lemma 3.1. Since $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathbb{N}(\mathcal{T})/\pi\mathbb{N}(\mathcal{T})$, this basis can be lifted to a basis $\mathfrak{n}_1, \mathfrak{n}_2$ of $\mathbb{N}(\mathcal{T})$ as an $\mathbb{A}_{\mathbb{Q}_p}^+$-module. There is a change of basis matrix $M \in M_{2\times 2}(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})$ such that

$$(3.3) \qquad \begin{pmatrix}\mathfrak{n}_1 & \mathfrak{n}_2\end{pmatrix} = \begin{pmatrix}\mathfrak{v}_1 & \mathfrak{v}_2\end{pmatrix} M$$

and $M \equiv I_2 \mod \pi$, where I_2 is the 2×2 identity matrix. We write $v_i = \mathfrak{v}_i \cdot t^{k-j-1}e_{-k+j+1}$, $n_i = \mathfrak{n}_i \cdot \pi^{k-j-1}e_{-k+j+1}$, $v_{f,i} = \mathfrak{v}_i \cdot t^{k-1}e_{1-k}$ and $n_{f,i} = \mathfrak{n}_i \cdot \pi^{k-1}e_{1-k}$ for the corresponding bases of $\mathbb{D}_{\mathrm{cris}}(T)$, $\mathbb{N}(T)$, $\mathbb{D}_{\mathrm{cris}}(T_f)$ and $\mathbb{N}(T_f)$ respectively. Here e_r denotes a basis of the Tate motive $\mathcal{O}(\chi^r)$ for $r \in \mathbb{Z}$. By [Ber04, proof of Proposition V.2.3] and [Lei15, Proposition 4.2], we may choose our bases so that

$$(3.4) \qquad M \equiv I_2 \mod \pi^{k-1}$$

and that the matrices of φ with respect to $v_{1,f}, v_{2,f}$ and $n_{1,f}, n_{2,f}$ are given by

$$\begin{pmatrix} 0 & -\varepsilon(p) \\ p^{k-1} & a_p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\varepsilon(p) \\ (\delta q)^{k-1} & a_p \end{pmatrix}$$

respectively, where $\delta = p/(q - \pi^{p-1}) \in (\mathbb{A}^+_{\mathbb{Q}_p})^\times$. Then, the matrices of φ with respect to $\mathfrak{v}_1, \mathfrak{v}_2$ and $\mathfrak{n}_1, \mathfrak{n}_2$ are given by

$$A = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{p^{k-1}} \\ 1 & \frac{a_p}{p^{k-1}} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{q^{k-1}} \\ \delta^{k-1} & \frac{a_p}{q^{k-1}} \end{pmatrix}.$$

Definition 3.2. *We define the* logarithmic matrix $M_{\log}$ *(with respect to the chosen bases) to be* $\mathfrak{M}^{-1}\left((1+\pi)A\varphi(M)\right)$.

Theorem 3.3. *Let $\mathfrak{n}_1, \mathfrak{n}_2$ be the basis of $\mathbb{N}(\mathcal{T})$ chosen above. Then, $(1+\pi)\varphi(\mathfrak{n}_1), (1+\pi)\varphi(\mathfrak{n}_2)$ form a Λ-basis of $(\varphi^*\mathbb{N}(\mathcal{T}))^{\psi=0}$.*

Proof. Let $\gamma \in \Gamma$ be a topological generator. Then, (3.3) tells us that

$$\begin{pmatrix}\gamma\cdot\mathfrak{n}_1 & \gamma\cdot\mathfrak{n}_2\end{pmatrix} = \begin{pmatrix}\mathfrak{v}_1 & \mathfrak{v}_2\end{pmatrix}\gamma(M).$$

This gives the equation

$$\begin{pmatrix}\gamma\cdot\mathfrak{n}_1 & \gamma\cdot\mathfrak{n}_2\end{pmatrix} = \begin{pmatrix}\mathfrak{n}_1 & \mathfrak{n}_2\end{pmatrix} M^{-1}\cdot\gamma(M).$$

Hence, for both $i = 1, 2$, we have

$$(1-\gamma)\mathfrak{n}_i \in \pi^{k-1}\mathbb{N}(\mathcal{T})$$

thanks to (3.4). As we assume that $k \geq 3$, we have in particular

$$(1-\gamma)\mathfrak{n}_i \in \pi^2\mathbb{N}(\mathcal{T}),$$

which is the condition required in Theorem 2.5[2]. Therefore, our result follows. □

Recall from [LLZ10, Remark 3.4] that for all $z \in \mathbb{N}(\mathcal{T})^{\psi=1}$, we have $(1-\varphi)z \in (\varphi^*\mathbb{N}(\mathcal{T}))^{\psi=0}$. The latter is free of rank 2 over Λ, with basis $(1+\pi)\varphi(\mathfrak{n}_1), (1+\pi)\varphi(\mathfrak{n}_2)$ as given by Theorem 3.3. This allows us to define the Coleman maps (again, with respect to our chosen bases) as follows.

[2]This is the only place where we use the assumption that $k \geq 3$.

Definition 3.4. *For $i \in \{1,2\}$, we define the Λ-homomorphisms $\mathrm{Col}_i : \mathbb{N}(\mathcal{T})^{\psi=1} \to \Lambda$ given by the relation*

$$(1-\varphi)z = \sum_{i=1}^{2} \mathrm{Col}_i(z) \cdot (1+\pi)\varphi(\mathfrak{n}_i) = \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot M_{\log} \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix}.$$

Let $h^1_{\mathcal{T}} : \mathbb{N}(\mathcal{T})^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$ be the Λ-isomorphism given by Theorem 2.6. By an abuse of notation, we shall write $\mathrm{Col}_1, \mathrm{Col}_2$ for the compositions $\mathrm{Col}_1 \circ (h^1_{\mathcal{T}})^{-1}$ and $\mathrm{Col}_2 \circ (h^1_{\mathcal{T}})^{-1}$ as well.

3.2. **A finite projection of the Coleman maps.**

Definition 3.5. *For each $n \geq 1$, we define $H_n = \varphi^{n-1}(P^{-1}) \cdots \varphi(P^{-1})$ and $\mathscr{H}_n = \mathfrak{M}^{-1}\left((1+\pi)H_n\right)$.*

Remark 3.6. *Note that $H_n \in \mathbb{A}^+_{\mathbb{Q}_p}$, and $\mathscr{H}_n \in \Lambda$; and $H_1 = \mathscr{H}_1 = 1$.*

Lemma 3.7. *We have the congruence*

$$M_{\log} \equiv A^n \cdot \mathscr{H}_n \mod \tilde{\omega}_{n-1,k-2}(X)\mathcal{H}.$$

Proof. From (3.3), we have the relation

$$MP = A\varphi(M),$$

which we may rewrite as $M = A\varphi(M)P^{-1}$. On iteration, we have

$$M = A^{n-1}\varphi^{n-1}(M)\varphi^{n-2}(P^{-1}) \cdots \varphi(P^{-1})P^{-1}.$$

By (3.4), we have $\varphi^{n-1}(M) = 1 \bmod \varphi^{n-1}(\pi^{k-1})$, so this implies that

$$M \equiv A^{n-1}\varphi^{n-2}(P^{-1}) \cdots \varphi(P^{-1})P^{-1} \mod \varphi^{n-1}(\pi^{k-1}).$$

This implies that

$$\varphi(M) \equiv A^{n-1} \cdot H_n \mod \varphi^n(\pi^{k-1}).$$

Hence the result by (2.2). □

Lemma 3.8. *For all $n \geq 1$ and $z \in \mathbb{N}(\mathcal{T})^{\psi=1}$, $(1 \otimes \varphi^{-n}) \circ (1-\varphi)z$ is congruent to an element in $\Lambda_{n,k-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ modulo $\tilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$.*

Proof. By Lemma 3.7 and the equation in Definition 3.4, we have the congruence

$$(1-\varphi)z \equiv \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot A^n \cdot \mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix} \mod \tilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}).$$

If we apply $(1 \otimes \varphi^{-n})$ to both sides, we obtain

$$\left(1 \otimes \varphi^{-n}\right) \circ (1-\varphi)z \equiv \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot \mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix} \mod \tilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}).$$

As $\mathscr{H}_n$, $\mathrm{Col}_1(z)$ and $\mathrm{Col}_2(z)$ are all defined over Λ, we see that $(1 \otimes \varphi^{-n}) \circ (1-\varphi)z$ is indeed congruent to an element in $\Lambda_{n,p-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$. □

This allows us to give the following definition.

Definition 3.9. *For $n \geq 1$, define*

$$\begin{aligned} \underline{\mathrm{Col}}_n : H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}) &\to \Lambda_{n,k-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) \\ z &\mapsto \left(1 \otimes \varphi^{-n}\right) \circ (1-\varphi) \circ (h^1_{\mathcal{T}})^{-1}(z) \mod \tilde{\omega}_{n-1,k-2}(X). \end{aligned}$$

We recall that $h^1_{\mathcal{T}}$ is an isomorphism by Theorem 2.6. Therefore, Lemma 3.8 tells us that the map $\underline{\mathrm{Col}}_n$ is well-defined.

For an integer m, we define the twisting map

$$\mathrm{Tw}_m := \mathrm{Tw}^{-m} \otimes t^{-m} e_m : \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) \to \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}(m)).$$

Consider the twisting map $\mathrm{Tw}^{k-j-1} : \sigma \mapsto \chi^{k-j-1}(\sigma)\sigma$ on Λ. Since $k-j-1 \leq k-1$, $\mathrm{Tw}^{k-j-1}(\tilde{\omega}_{n-1,k-2}(X))$ is divisible by $\omega_{n-1}(X)$. Hence, Tw^{k-j-1} induces a natural map $\Lambda_{n,k-2} \to \Lambda_n$. Therefore, we may define

$$\begin{aligned}\underline{\mathrm{Col}}_{T,n} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) \to & \Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T) \\ z \mapsto & \mathrm{Tw}_{-k+j+1} \circ \underline{\mathrm{Col}}_n(z \cdot e_{k-j-1}) \mod \omega_{n-1}(X),\end{aligned}$$

on identifying $H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) \cdot e_{k-j-1}$ with $H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$.

Lemma 3.10. *The map $\underline{\mathrm{Col}}_{T,n}$ defines a Λ_n-homomorphism from $H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ to $\Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T)$.*

Proof. We note that $\underline{\mathrm{Col}}_{T,n}$ is a Λ-homomorphism since both $\underline{\mathrm{Col}}_n$ and $x \mapsto \mathrm{Tw}^{k-j-1} \circ (x \cdot e_{k-j-1})$ are Λ-linear. The fact that $\underline{\mathrm{Col}}_{T,n}$ factors through $H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ follows from the equation $H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)_{\Gamma_n} = H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ (because of the vanishing of $H^2_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)$). □

We have the explicit formula

$$\underline{\mathrm{Col}}_{T,n}(z) \equiv \begin{pmatrix} v_1 & v_2 \end{pmatrix} \cdot \mathrm{Tw}^{k-1-j} \left(\mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z \cdot e_{k-1-j}) \\ \mathrm{Col}_2(z \cdot e_{k-1-j}) \end{pmatrix} \right) \mod \omega_{n-1}(X)\Lambda \otimes \mathbb{D}_{\mathrm{cris}}(T), \tag{3.5}$$

by Lemma 3.7 and the expansion of $1 - \varphi$ as given in Definition 3.4.

We now modify the definition of $\underline{\mathrm{Col}}_{T,n}$ to define a map that lands in Λ_n. For any $u \in \mathbb{Z}_p^\times$, we define $\underline{\mathrm{Col}}_{T,n,u} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) \to \Lambda_n$ to be the composition of $\underline{\mathrm{Col}}_{T,n}$ and the linear functional on $\Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T) \to \Lambda_n$ given by $a \cdot v_1 + b \cdot v_2 \mapsto a + ub$. More explicitly, (3.5) tells us that $\underline{\mathrm{Col}}_{T,n,u}$ is given by

$$\underline{\mathrm{Col}}_{T,n,u}(z) \equiv \begin{pmatrix} 1 & u \end{pmatrix} \cdot \mathrm{Tw}^{k-1-j} \left(\mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z \cdot e_{k-1-j}) \\ \mathrm{Col}_2(z \cdot e_{k-1-j}) \end{pmatrix} \right) \mod \omega_{n-1}(X)\Lambda. \tag{3.6}$$

Note that Lemma 3.10 tells us that $\underline{\mathrm{Col}}_{T,n,u}$ is Λ_n-linear.

3.3. **Analysis of Bloch–Kato subgroups via Coleman maps.** If F is a finite extension of $\mathbb{Q}_p$, we write $H^1_f(F, T) \subset H^1(F, T)$ for the usual Bloch–Kato subgroup from [BK90] and $H^1_{/f}(F, T)$ denotes the quotient $H^1(F, T)/H^1_f(F, T)$. The goal of this section is to study $H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T)$ via the map $\underline{\mathrm{Col}}_{T,n,u}$.

Let $\mathcal{T}^*$ be the $\mathcal{O}$-linear dual of $\mathcal{T}$. For each $n \geq 1$, we define the pairing

$$\begin{aligned}\langle \sim, \sim \rangle_n : H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}) \times H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}^*(1)) \to & \Lambda_n \\ (x, y) \mapsto & \sum_{\sigma \in \Gamma/\Gamma_n} [x, y^\sigma]_n \sigma,\end{aligned}$$

where $[\sim, \sim]_n$ is the standard cup-product pairing

$$H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}) \times H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}^*(1)) \to \mathcal{O}.$$

On taking inverse limits, this induces a pairing

$$\langle \sim, \sim \rangle : H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}) \times H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1)) \to \Lambda.$$

It is semi-linear over Λ with respect to the involution on Λ (which we denote by $\tilde{\iota}$) in the following sense:

$$\langle \sigma x, y \rangle = \sigma \langle x, y \rangle, \quad \langle x, \sigma y \rangle = \sigma^{\tilde{\iota}} \langle x, y \rangle$$

We may extend the pairing $\langle \sim, \sim \rangle$ by semi-linearity to

$$\left(\mathcal{H} \otimes_{\mathcal{O}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})\right) \times \left(\mathcal{H} \otimes_{\mathcal{O}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1))\right) \to \mathcal{H},$$

which is again denoted by $\langle \sim, \sim \rangle$ by an abuse of notation.

Recall that in [PR94], Perrin-Riou defined the big exponential map

$$\Omega_{\mathcal{T}^*(1),1} : (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1)) \to \mathcal{H} \otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1)).$$

By [LLZ11, proof of Proposition 4.8], for all $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$,

$$(\mathfrak{M}^{-1} \otimes 1)(1 - \varphi)z = \sum_{i=1}^{2} \langle z, \Omega_{\mathcal{T}^*(1),1}((1+\pi) \otimes \mathfrak{v}'_i) \rangle \mathfrak{v}_i$$

where $\mathfrak{v}'_1, \mathfrak{v}'_2$ is the dual basis of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1))$ to $\mathfrak{v}_1, \mathfrak{v}_2$ with respect to the natural pairing

$$[\sim, \sim] : \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) \times \mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1)) \to \mathcal{O}.$$

Therefore,

$$\begin{aligned} \underline{\mathrm{Col}}_n(z) &= \sum_{i=1}^{2} \langle z, \Omega_{\mathcal{T}^*(1),1}((1+\pi) \otimes \mathfrak{v}'_i) \rangle \varphi^{-n}(\mathfrak{v}_i) \mod \tilde{\omega}_{n-1,k-2} \\ &= \sum_{i=1}^{2} \langle z, \Omega_{\mathcal{T}^*(1),1}((1+\pi) \otimes (p\varphi)^n(\mathfrak{v}'_i)) \rangle \mathfrak{v}_i \mod \tilde{\omega}_{n-1,k-2} \end{aligned}$$

as the dual of φ^{-1} with respect to $[\sim, \sim]$ is $p\varphi$. This description allows us to make the following choice of u to describe the kernel of $\underline{\mathrm{Col}}_{T,n,u}$.

Proposition 3.11. *There exists* $u \in \mathbb{Z}_p^\times$ *such that* $\ker(\underline{\mathrm{Col}}_{T,n,u}) = H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$.

Proof. Write $v' = (\mathfrak{v}'_1 + u\mathfrak{v}'_2) \cdot t^{-k+j+1} e_{k-j-1} \in \mathbb{D}_{\mathrm{cris}}(T^*(1))$ and let $z \in H^1(\mathbb{Q}_p(\mu_{p^n}), T)$. If θ is a Dirichlet character of conductor $p^m > 1$, we have the interpolation formula of Perrin-Riou [PR94, §3.2.3] (see also [Lei11, §3.2])

$$\frac{\theta\left(\underline{\mathrm{Col}}_{T,n,u}(z)\right)}{(-1)^{k-j-1}(k-j-1)!} = \sum_{\sigma \in \Gamma/\Gamma_m} \frac{\theta^{-1}(\sigma)}{\tau(\theta)} [\exp^*_{T,m}(z^\sigma), p^n \varphi^{n-m}(v')], \tag{3.7}$$

where $\exp^*_{T,m} : H^1(\mathbb{Q}_p(\mu_{p^m}), T) \to \mathbb{Q}_p(\mu_{p^m}) \otimes \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$ is the Bloch–Kato dual exponential map and $\tau(\theta)$ is the Gauss sum of θ. There is a similar formula when θ is the trivial character on replacing φ^{-m} by $\left(1 - \frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1}$. We note that here $\exp^*_{T,m}(z)$ is the shorthand for $\exp^*_{T,m} \circ \mathrm{cor}_{n/m}(z)$, where $\mathrm{cor}_{n/m}$ denotes the the corestriction map $H^1(\mathbb{Q}_p(\mu_{p^n}), T) \to H^1(\mathbb{Q}(\mu_{p^m}), T)$. Recall that $\exp^*_{T,n}$ factors through $H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T)$. Therefore, we see that $H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$ is contained in $\ker(\underline{\mathrm{Col}}_{T,n,u})$.

We choose u so that $\varphi^{n-m}(v')$, $1 \le m \le n$ and $\varphi^n \left(1 - \frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1}(v')$ are not contained inside $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V)$. We note that such u exists since all maps are surjective on $\mathbb{D}_{\mathrm{cris}}(V)$ and $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V)$ is of dimension one. Let v'' be any $\mathcal{O}$-basis of $\mathbb{D}_{\mathrm{cris}}(T)/\,\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(T)$. In particular, for each $m \geq 1$, there exists a non-zero constant $c_m \in \mathcal{O}$ such that $\varphi^{n-m}(v') \equiv c_m v''$ and $\varphi^n\left(1 - \frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1}(v') \equiv c_0 v''$ modulo $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(T)$.

Suppose that $\underline{\mathrm{Col}}_{T,n,u}(z) = 0$. From (3.7), we deduce that

$$\sum_{\sigma \in \Gamma/\Gamma_n} \theta^{-1}(\sigma)[\exp^*_{T,n}(z^\sigma), v''] = 0$$

for all characters θ on Γ/Γ_n. By the independence of the characters, this implies that $[\exp^*_{T,n}(z^\sigma), v''] = 0$ for all σ. In particular, z is contained in the kernel of $\exp^*_{T,n}$, which is $H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$. ☐

Corollary 3.12. *For any $u \in \mathbb{Z}_p^\times$ that satisfies the condtion of Proposition 3.11, $\underline{\mathrm{Col}}_{T,n,u}$ induces an injection of Λ_n-modules*

$$H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T) \hookrightarrow \Lambda_n,$$

whose cokernel is finite.

Proof. The injectivity is given by Proposition 3.11. By [BK90, Theorem 4.1], $H^1_f(\mathbb{Q}_p(\mu_{p^n}), V)$ is isomorphic to $\mathbb{D}_{\mathrm{cris}}(V)/\,\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\mu_{p^n})$. Hence, by duality $H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), V)$ is isomorphic to $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\mu_{p^n})$. Therefore, the finiteness of the cokernel follows from the fact that the two sides have the same $\mathbb{Z}_p$-rank. ☐

We remark that our map $\underline{\mathrm{Col}}_{T,n,u}$ does depend on the choice of u. But it does not affect our calculations later, see the proof of Proposition 4.11 below.

4. Results on p-adic valuations

4.1. **Review of Kobayashi rank.**

Given an $\mathcal{O}$-module N, we shall write $\mathrm{len}(N)$ for the $\mathcal{O}$-length of N. We fix a family of primitive p^n-th root of unity ζ_{p^n} and write $\epsilon_n = \zeta_{p^n} - 1$.

Definition 4.1. *Let $N = (N_n)$ be an inverse system of finitely generated $\mathcal{O}$-modules with transition maps $\pi_n : N_n \to N_{n-1}$. If π_n has finite kernel and cokernel, the Kobayashi rank ∇N_n is defined as*

$$\nabla N_n \coloneqq \mathrm{len}(\ker \pi_n) - \mathrm{len}(\mathrm{coker}\,\pi_n) + \mathrm{rank}_{\mathcal{O}} N_{n-1}.$$

If L is an $\mathcal{O}[\![X]\!]$-module, we define $\nabla_n L$ to be $\nabla\left(L/\omega_n(X)L\right)$, with the connecting map given by the natural projection $L/\omega_n(X)L \to L/\omega_{n-1}(X)L$, if its kernel and cokernel are finite.

Lemma 4.2. *Let $F \in \mathcal{O}[\![X]\!]$ be a non-zero element. Let N be the inverse limit defined by $N_n = \mathcal{O}[\![X]\!]/(F, \omega_n)$, where the the connecting maps are the natural projections.*

(a) *Suppose that $F(\epsilon_n) \neq 0$, then ∇N_n is defined and is equal to $\mathrm{ord}_{\epsilon_n} F(\epsilon_n)$.*
(b) *When n is sufficiently large, then ∇N_n is defined. Furthermore,*

$$\nabla N_n = e \times \mathrm{ord}_{\epsilon_n} F(\epsilon_n) = e\lambda(F) + (p^n - p^{n-1})\mu(F),$$

where e is the ramification index of $E/\mathbb{Q}_p$ and $\lambda(F)$, $\mu(F)$ are the Iwasawa invariants as defined in §2.3 above.

(c) *If L is a finitely generated torsion $\mathcal{O}[\![X]\!]$-module, then $\nabla_n L$ is defined for $n \gg 0$ and its value is given by*

$$\lambda(L) + (p^n - p^{n-1})\mu(L),$$

where $\lambda(L)$ and $\mu(L)$ are the λ- and μ-invariants of a generator of the characteristic ideal of L.

Proof. This follows from the same proof as [Kob03, Lemma 10.5]. □

We write p^r for the size of the residue field of E. The following lemma allows us to relate the growth in the size of a tower of finite $\mathcal{O}$-modules and Kobayashi ranks.

Lemma 4.3. *Suppose that $N = (N_n)$ is an inverse limit of finite $\mathcal{O}$-modules such that $|N_n| = p^{s_n}$ for some integer $s_n \in r\mathbb{Z}$ for all $n \geq 1$. Then, $r\nabla N_n = s_n - s_{n-1}$.*

Proof. Since N_{n-1} is finite, we have

$$\begin{aligned}\nabla N_n &= \mathrm{len}(\ker \pi_n) - \mathrm{len}(\mathrm{coker}\, \pi_n)\\ &= (\mathrm{len}(N_n) - \mathrm{len}(\mathrm{Im}\, \pi_n)) - (\mathrm{len}(N_{n-1}) - \mathrm{len}(\mathrm{Im}\, \pi_n))\\ &= \mathrm{len}(N_n) - \mathrm{len}(N_{n-1}).\end{aligned}$$

In general, if L is a finite $\mathcal{O}$-module, then $|L| = p^{r\mathrm{len}(L)}$. Hence the result. □

Finally, we prove a lemma on p-adic valuations that will be needed later.

Lemma 4.4. *Let $F \in \mathcal{O}[\![X]\!]$ be non-zero. Then for all sufficiently large integers n we have*

$$\mathrm{ord}_p\, F(\epsilon_n) = \mathrm{ord}_p\, \mathfrak{M}(F)(\epsilon_{n+1}).$$

Moreover, for $n \gg 0$ we also have

$$\mathrm{ord}_p\, F(\epsilon_n) = \mathrm{ord}_p\, \mathrm{Tw}(F)(\epsilon_n).$$

Proof. We may write $\mathfrak{M}(F) = (1+\pi)\varphi(G)$ for some $G \in \mathbb{A}^+_{\mathbb{Q}_p}$. By Proposition 2.2, F and G have the same Iwasawa invariants, so $\mathrm{ord}_p\, F(\epsilon_n) = \mathrm{ord}_p\, G(\epsilon_n)$ for $n \gg 0$. This implies the first part of the lemma since $(1+\pi)\varphi(G)(\epsilon_{n+1}) = \zeta_{p^{n+1}} G(\epsilon_n)$. The second part of the lemma follows from the fact that Tw preserves μ- and λ-invariants. □

4.2. **Calculations on evaluation matrices.** From now on, we shall write $v = \mathrm{ord}_p(a_p)$, where a_p is the p-th Fourier coefficient of f. Following [Spr13, §4.1], given any 2×2 matrix $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defined over $\overline{\mathbb{Q}_p}$, we write $\mathrm{ord}_p(\phi) = \begin{pmatrix} \mathrm{ord}_p(a) & \mathrm{ord}_p(b) \\ \mathrm{ord}_p(c) & \mathrm{ord}_p(d) \end{pmatrix}$.

Lemma 4.5. *Let $1 \leq i \leq n-2$, then*

$$\mathrm{ord}_p \left(\varphi^i(P^{-1})(\epsilon_n)\right) = \begin{pmatrix} v & 0 \\ \frac{k-1}{p^{n-i-1}} & \infty \end{pmatrix}.$$

Proof. Recall that

$$P = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{q^{k-1}} \\ \delta^{k-1} & \frac{a_p}{q^{k-1}} \end{pmatrix},$$

so its inverse is given by

$$P^{-1} = \begin{pmatrix} \frac{a_p}{\delta^{k-1}\varepsilon(p)} & \frac{1}{\delta^{k-1}} \\ -\frac{q^{k-1}}{\varepsilon(p)} & 0 \end{pmatrix}.$$

Therefore, our result follows from the fact that $\delta \in \mathbb{Z}_p^\times$, $\varepsilon(p) \in \mathcal{O}^\times$ and $\varphi^i(q)$ is equal to the p^{i+1}-cyclotomic polynomial, so $\varphi^i(q)(\epsilon_n) = \frac{\zeta_{p^{n-i-1}}-1}{\zeta_{p^{n-i}}-1}$ whose p-adic valuation is $1/p^{n-i-1}$. □

Proposition 4.6. *Assume that $2v > \frac{k-1}{p}$. For all $n \geq 1$,*

$$\mathrm{ord}_p\left(H_n(\epsilon_n)\right) = \begin{cases} \begin{pmatrix} v + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}} & \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix} & \text{if } n \text{ is odd.} \\ \begin{pmatrix} \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}} & v + \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

Proof. By Lemma 4.5, we have

$$\mathrm{ord}_p\left(H_n(\epsilon_n)\right) = \begin{pmatrix} v & 0 \\ \infty & \infty \end{pmatrix} \begin{pmatrix} v & 0 \\ \frac{k-1}{p} & \infty \end{pmatrix} \cdots \begin{pmatrix} v & 0 \\ \frac{k-1}{p^{n-1}} & \infty \end{pmatrix}.$$

In particular,

$$\mathrm{ord}_p\left(H_{n+1}(\epsilon_{n+1})\right) = \mathrm{ord}_p\left(H_n(\epsilon_n)\right) \begin{pmatrix} v & 0 \\ \frac{k-1}{p^n} & \infty \end{pmatrix}. \tag{4.1}$$

Therefore,

$$\mathrm{ord}_p(H_1(\epsilon_1)) = \begin{pmatrix} v & 0 \\ \infty & \infty \end{pmatrix} \quad \text{and} \quad \mathrm{ord}_p(H_2(\epsilon_2)) = \begin{pmatrix} \frac{k-1}{p} & v \\ \infty & \infty \end{pmatrix}$$

since $2v > \frac{k-1}{p}$ by our assumption.

Suppose that

$$\mathrm{ord}_p(H_{2\ell-1}(\epsilon_{2\ell-1})) = \begin{pmatrix} v + \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i}} & \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i-1}} \\ \infty & \infty \end{pmatrix},$$

$$\mathrm{ord}_p(H_{2\ell}(\epsilon_{2\ell})) = \begin{pmatrix} \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}} & v + \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix}$$

for some integer $\ell \geq 1$. By (4.1), we have first of all

$$\mathrm{ord}_p(H_{2\ell+1}(\epsilon_{2\ell+1})) = \begin{pmatrix} v + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} & \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}} \\ \infty & \infty \end{pmatrix}$$

because $\sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} < \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}}$. On applying (4.1) again, we have

$$\mathrm{ord}_p(H_{2\ell+2}(\epsilon_{2\ell+2})) = \begin{pmatrix} \sum_{i=1}^{\ell+1} \frac{k-1}{p^{2i-1}} & v + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix}$$

thanks to our assumption that $2v > \frac{k-1}{p}$, which implies that

$$2v + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} > \sum_{i=1}^{\ell+1} \frac{k-1}{p^{2i-1}}.$$

Therefore, our result follows from induction. □

For $i = 1, 2$, we fix two elements $F_1, F_2 \in \mathcal{O}[\![X]\!]$ with μ_i and λ_i being its μ- and λ-invariants.

Corollary 4.7. *Under the condition that* $2v > \frac{k-1}{p}$*, for* $n \gg 0$ *we have the formulae*

$$\operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1}\cdot F_1(\epsilon_n)\right) = \begin{cases} \lambda_1 + (p^n - p^{n-1})\left(\frac{\mu_1}{e} + v + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}}\right) & n \text{ odd},\\ \lambda_1 + (p^n - p^{n-1})\left(\frac{\mu_1}{e} + \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}}\right) & n \text{ even},\end{cases}$$

$$\operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2}\cdot F_2(\epsilon_n)\right) = \begin{cases} \lambda_2 + (p^n - p^{n-1})\left(\frac{\mu_2}{e} + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}}\right) & n \text{ odd},\\ \lambda_2 + (p^n - p^{n-1})\left(\frac{\mu_2}{e} + v + \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}}\right) & n \text{ even}.\end{cases}$$

Proof. By Lemma 4.4, $\operatorname{ord}_p \mathscr{H}_{n+1}(\epsilon_n) = \operatorname{ord}_p H_n(\epsilon_n)$. Hence, our result follows from combining Proposition 4.6 with Lemma 4.2(b). □

Corollary 4.8. *Suppose that* $2v > \frac{k-1}{p}$*. For* $n \gg 0$ *and* n *odd, we have*

$$\operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1}\cdot F_1(\epsilon_n)\right) < \operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2}\cdot F_2(\epsilon_n)\right) \quad \text{if } \frac{\mu_1}{e} + v + \frac{k-1}{p+1} \le \frac{\mu_2}{e}$$

$$\operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1}\cdot F_1(\epsilon_n)\right) > \operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2}\cdot F_2)(\epsilon_n)\right) \quad \text{if } \frac{\mu_1}{e} + v + \frac{k-1}{p+1} > \frac{\mu_2}{e}.$$

For $n \gg 0$ *and* n *even, we have*

$$\operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1}\cdot F_1(\epsilon_n)\right) < \operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2}\cdot F_2(\epsilon_n)\right) \quad \text{if } \frac{\mu_1}{e} < \frac{\mu_2}{e} + v + \frac{k-1}{p+1}$$

$$\operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1}\cdot F_1(\epsilon_n)\right) > \operatorname{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2}\cdot F_2(\epsilon_n)\right) \quad \text{if } \frac{\mu_1}{e} \ge \frac{\mu_2}{e} + v + \frac{k-1}{p+1}.$$

Proof. Note that

$$\sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}} - \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}} > 0 \quad \text{and} \quad \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}} - \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}} > 0$$

and that both sequences are strictly increasing and tend to $\frac{k-1}{p+1}$ as $n \to \infty$. Hence the result. □

4.3. **Some global Iwasawa modules.** For $n \ge 0$ let us write $K_n = \mathbb{Q}(\mu_{p^n})$.

Definition 4.9 (cf. [Kat04, §12.2])**.** *For* $m \ge 0$*, we define*

$$\mathbb{H}^m(T) \coloneqq \varprojlim_n H^m_{\text{ét}}\Big(\operatorname{Spec} \mathcal{O}_{K_n}[1/p], j_* T\Big),$$

where the inverse limit is respect to the corestriction maps, and j *is the inclusion map* $\operatorname{Spec} K_n \hookrightarrow \operatorname{Spec} \mathcal{O}_{K_n}[1/p]$.

By [Kat04, 12.4(3)], the modules $\mathbb{H}^m(T)$ are finitely-generated over Λ, and are zero unless $m \in \{1, 2\}$; and $\mathbb{H}^1(T)$ is free of rank 1 over Λ. We fix an element $\mathbf{z} \in \mathbb{H}^1(T)$ so that $\mathbb{H}^1(T) = \Lambda \cdot \mathbf{z}$. Tensoring with the basis vector e_{k-1-j} of $\mathcal{O}(k-1-j)$ gives a bijection

$$\mathbb{H}^1(T) \cong \mathbb{H}^1(\mathcal{T}),$$

and (in a slight abuse of notation) we shall write $\operatorname{Col}_i(\mathbf{z})$ for the image of $\mathbf{z}\cdot e_{k-1-j}$ under Col_i composed with the localization map $\mathbb{H}^1(\mathcal{T}) \to H^1_{\text{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$.

Definition 4.10. *For $i=1,2$ and η a Dirichlet character modulo p. Let μ_i^η be the μ-invariant of $\mathrm{Col}_i(\mathbf{z})^\eta$. For each $n \geq 1$, we define an integer $\tau(n,\eta) \in \{1,2\}$ by*

$$\begin{cases} 1 & \text{if } \frac{\mu_1^\eta}{e} + v + \frac{k-1}{p+1} \leq \frac{\mu_2^\eta}{e} \text{ and } n \text{ odd or } \frac{\mu_1^\eta}{e} < \frac{\mu_2^\eta}{e} + v + \frac{k-1}{p+1} \text{ and } n \text{ even,} \\ 2 & \text{otherwise.} \end{cases}$$

Furthermore, we write $q_n^* = \mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,\tau(n,\eta)}(\epsilon_n)\right)$.

Note in particular that q_n^* is a sum of some powers of p, together with possibly v, as given by Proposition 4.6. Furthermore, Corollary 4.8 tells us that

$$\mathrm{ord}_{\epsilon_n}\left(\sum_{i=1}^{2}(\mathscr{H}_{n+1})_{1,i}\cdot \mathrm{Col}_i(\mathbf{z})^\eta(\epsilon_n)\right) = q_n^* + \mathrm{ord}_{\epsilon_n}\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta(\epsilon_n). \tag{4.2}$$

4.4. **Analysis of some local Iwasawa modules.** For $n \geq 1$, we define

$$\mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^n})) = \mathrm{coker}\left(\mathbb{H}^1(T)_{\Gamma_n} \to H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}),T)\right),$$

which gives an inverse limit with the connecting maps given by the corestriction maps. We would like to study $\nabla\mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ for a fixed Dirichlet character η modulo p.

Proposition 4.11. *Suppose that $\underline{\mathrm{Col}}_1(\mathbf{z})^\eta$ and $\underline{\mathrm{Col}}_2(\mathbf{z})^\eta$ are non-zero. For $n \gg 0$, $\nabla\mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined, and its value is bounded above by*

$$\nabla_n\mathcal{X}_{\mathrm{loc}}^\eta \leq eq_n^* + \nabla_n(\mathcal{O}[\![X]\!]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta).$$

Proof. Recall from Corollary 3.12, we have the injection

$$\underline{\mathrm{Col}}_{T,n+1,u} : H^1_{/f}(\mathbb{Q}_p(\mu_{p^{n+1}}),T) \hookrightarrow \Lambda_{n+1}.$$

On taking Γ_n-coinvariants, the *same* map (*not* $\underline{\mathrm{Col}}_{T,n,u}$) induces an injection

$$\underline{\mathrm{Col}}_{T,n+1,u} : H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}),T) \hookrightarrow \Lambda_n,$$

which admits the same description as (3.6). We write coker_{n+1} and coker_n for the cokernels of these two maps respectively. Then, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{/f}(\mathbb{Q}_p(\mu_{p^{n+1}}),T) & \xrightarrow{\underline{\mathrm{Col}}_{T,n+1,u}} & \Lambda_{n+1} & \longrightarrow & \mathrm{coker}_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\pi} & & \\ 0 & \longrightarrow & H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}),T) & \xrightarrow{\underline{\mathrm{Col}}_{T,n+1,u}} & \Lambda_n & \longrightarrow & \mathrm{coker}_n & \longrightarrow & 0, \end{array}$$

where the vertical maps are all natural projections. This gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^{n+1}})) & \longrightarrow & \Lambda_{n+1}/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})) & \longrightarrow & \mathrm{coker}_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\pi} & & \\ 0 & \longrightarrow & \mathcal{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^n})) & \longrightarrow & \Lambda_n/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})) & \longrightarrow & \mathrm{coker}_n & \longrightarrow & 0. \end{array}$$

Recall from Corollary 3.12 that coker_{n+1} is finite (in particular, coker_n too). Hence, on taking η-isotypic components, $\nabla\,\mathrm{coker}_{n+1}^\eta$ (with respect to π) is defined. In fact, it is given by $\mathrm{len}(\ker\pi^\eta)$, which is ≥ 0.

Furthermore, recall that we assume $\mathrm{Col}_i(\mathbf{z})^\eta \neq 0$ for $i=1,2$. Proposition 4.6 tells us that the second row of $\mathscr{H}_{n+1}(\epsilon_n)$ is 0. So, the formulae (3.6) and (4.2) imply that $\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})(\epsilon_n) \neq 0$ when $n \gg 0$. Hence, $\nabla\left(\Lambda_{n+1}/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z}))\right)^\eta = \nabla_n\left(\mathcal{O}[\![X]\!]/\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})^\eta\right)$ is defined. Its value is given by

$$eq_n^* + \nabla_n(\mathcal{O}[\![X]\!]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta),$$

thanks to Lemma 4.2.

Therefore, the fact that the Kobayashi rank ∇ respects short exact sequences ([Kob03, Lemma 10.4]) tells us that $\nabla\mathcal{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^{n+1}}))^\eta$ is defined and its value is equal to

$$\nabla_n\left(\mathcal{O}[\![X]\!]/\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})^\eta\right) - \mathrm{len}(\ker\pi^\eta).$$

Hence the result. □

This can be considered as a weakened version of the modesty proposition [Spr13, Proposition 3.10]. In the $k=2$ case, equality holds because the projection π turns out to be an injection (see [Kob03, Lemma 10.7] and [Spr13, Lemma 4.12]).

5. Selmer groups and Shafarevich–Tate groups

5.1. **Signed Selmer groups.** Let $T^\vee$ be the Pontryagin dual of T. As in [LLZ10], the Coleman maps allow us to define the Selmer groups

$$\mathrm{Sel}_i(T^\vee/\mathbb{Q}(\mu_{p^\infty})) = \ker\left(\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^\infty})) \to \frac{H^1(\mathbb{Q}_p(\mu_{p^\infty}),T^\vee)}{\ker(\mathrm{Col}_i)^\perp}\right)$$

for $i=1,2$. Here $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$ is the Bloch–Kato Selmer group from [BK90]. We shall write $\mathcal{X}(\mathbb{Q}(\mu_{p^n})) = \mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^n}))^\vee$ for $n \geq 1$.

Let $\mathcal{X}_i$ be the Pontryagin dual of $\mathrm{Sel}_i(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$. We subsequently assume that for any Dirichlet character η that factors through $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, both $\mathcal{X}_1^\eta$ and $\mathcal{X}_2^\eta$ are $\mathcal{O}[\![X]\!]$-torsion. Note that this is the case if either $k \geq 3$ or $a_p = 0$ by [LLZ10, Theorem 6.5]. In particular, $\nabla_n\mathcal{X}_i^\eta$ are defined for $n \gg 0$ by Lemma 4.2(c).

We have the Poitou-Tate exact sequence (see for example [LLZ10, (61)])

$$\mathbb{H}^1(T) \to \mathrm{Im}\,\mathrm{Col}_i \to \mathcal{X}_i \to \mathcal{X}_0 \to 0, \tag{5.1}$$

where $\mathcal{X}_0$ is $\mathbb{H}^2(T)$ and can be realized as the Pontryagin dual of the zero Selmer group $\mathrm{Sel}_0(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$, which is defined to be

$$\ker\left(H^1(\mathbb{Q}(\mu_{p^\infty}),T^\vee) \to \prod_v H^1(\mathbb{Q}(\mu_{p^\infty})_v,T^\vee)\right),$$

where v runs through all places of $\mathbb{Q}(\mu_{p^\infty})$. Note that $\mathcal{X}_0$ is a torsion Λ-module by [Kat04, Theorem 12.4] and hence $\nabla_n\mathcal{X}_0^\eta$ is defined for $n \gg 0$ by Lemma 4.2(c). Note that (5.1) gives the short exact sequence

$$0 \to \frac{\mathrm{Im}\,\mathrm{Col}_i}{(\mathrm{Col}_i(\mathbf{z}))} \to \mathcal{X}_i \to \mathcal{X}_0 \to 0.$$

Hence, our assumption that $\mathcal{X}_i^\eta$ be torsion implies that $\mathrm{Col}_i(\mathbf{z})^\eta \neq 0$. In particular, Proposition 4.11 applies.

Recall from [LLZ11, §5] that $\mathrm{Im}\,\mathrm{Col}_i^\eta$ is pseudo-isomorphic to $\prod_m(X-\chi(\gamma)^m+1)\mathcal{O}[\![X]\!]$, where m runs through some subset of $\{0,1,\ldots,k-2\}$ depending on i and η. Let us write $\kappa_i(\eta)$ for the cardinality of this subset and write $\kappa(n,\eta) = \kappa_{\tau(n,\eta)}(\eta)$. We have the following generalization of [Spr13, Proposition 3.11].

Proposition 5.1. *For $i=1,2$, η any Dirichlet character modulo p and $n\gg 0$,*

$$\nabla_n\mathcal{X}_i^\eta=\nabla_n(\Lambda/\mathrm{Col}_i(\mathbf{z}))^\eta+\nabla_n\mathcal{X}_0^\eta-e\kappa_i(\eta).$$

Proof. The following sequence

$$0\to\left(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta\to\left(\frac{\Lambda}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta\to\frac{\mathcal{O}[\![X]\!]}{\prod_m(X-\chi(\gamma)^m+1)\mathcal{O}[\![X]\!]}\to G\to 0$$

is exact, where G is some finite subgroup. In particular, $\nabla_nG=0$ for $n\gg 0$. We may work out the Kobayashi rank of the second last term using Lemma 4.2(b). Recall from [Kob03, Lemma 10.4] that Kobayashi ranks respect exact sequences, therefore,

$$\nabla_n\left(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta+e\kappa_i(\eta)=\nabla_n\left(\frac{\Lambda}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta.$$

From (5.1), we have furthermore the following exact sequence

$$0\to\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})}\to\mathcal{X}_i\to\mathcal{X}_0\to 0,$$

which implies that

$$\nabla_n\left(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta+\nabla_n\mathcal{X}_0^\eta=\nabla_n\mathcal{X}_i^\eta.$$

Combing the two equations gives our result. □

Remark 5.2. *Let μ_0^η be the μ-invariant of $\mathcal{X}_0^\eta$. For $i=1,2$, let $\tilde{\mu}_i^\eta$ be the μ-invariant of $\mathcal{X}_i^\eta$. Then, Proposition 5.1 implies that $\tilde{\mu}_i^\eta=\mu_i^\eta-\mu_0^\eta$. In particular, $\mu_1^\eta-\mu_2^\eta=\tilde{\mu}_1^\eta-\tilde{\mu}_2^\eta$. Therefore, we may replace μ_1^η and μ_2^η by $\tilde{\mu}_1^\eta$ and $\tilde{\mu}_2^\eta$ respectively in Definition 4.10. In other words, we may define $\tau(n,\eta)$ using the μ-invariants of the dual Selmer groups $\mathcal{X}_i$, instead of* $\mathrm{Col}_i(\mathbf{z})$.

Corollary 5.3. *For $n\gg 0$, $\nabla\mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined. Furthermore, its value is bounded above by*

$$eq_n^*+\nabla_n\mathcal{X}_{\tau(n,\eta)}^\eta+e\kappa(n,\eta).$$

Proof. Let $\mathcal{Y}(\mathbb{Q}(\mu_{p^n}))=\mathrm{coker}(H^1(G_{n,S},T)\to H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}),T))$ and $\mathcal{X}_0(\mathbb{Q}(\mu_{p^n}))=\mathrm{Sel}_0(T^\vee/\mathbb{Q}(\mu_{p^n}))^\vee$. As a consequence of the Poitou–Tate exact sequence, we have the short exact sequence

$$0\to\mathcal{Y}(\mathbb{Q}(\mu_{p^n}))\to\mathcal{X}(\mathbb{Q}(\mu_{p^n}))\to\mathcal{X}_0(\mathbb{Q}(\mu_{p^n}))\to 0$$

(c.f. [Kob03, (10.35)]). But Proposition 10.6 in *op. cit.* says that

- $\nabla\mathcal{Y}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined for $n\gg 0$ and is equal to $\nabla\mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$;
- $\nabla\mathcal{X}_0(\mathbb{Q}(\mu_{p^{n+1}}))^\eta=\nabla_n\mathcal{X}_0^\eta$.

Therefore,

$$\nabla\mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta=\nabla\mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta+\nabla_n\mathcal{X}_0^\eta$$

and our result follows from Propositions 4.11 and 5.1. □

5.2. **Bloch–Kato–Shafarevich–Tate groups.** Let L be a number field. We recall that the Bloch–Kato–Shafarevich–Tate group of $T^\vee$ over L is defined to be

$$Ш(L,T^\vee)=\frac{\mathrm{Sel}(T^\vee/L)}{\mathrm{Sel}(T^\vee/L)_{\mathrm{div}}}, \tag{5.2}$$

where $(\star)_{\mathrm{div}}$ denotes the maximal divisible subgroup of $\star$. (See e.g. [BK90, Remark 5.15.2]). If f corresponds to an elliptic curve $\mathcal{E}$ and the p-primary part of the classical Shafarevich–Tate group $\mathcal{E}$ is finite, then the two definitions of (p-primary) Shafarevich–Tate groups agree.

Proposition 5.4. *There exists integers $n_0^\eta, r_\infty^\eta \geq 0$ such that*

$$\mathrm{corank}_{\mathcal{O}}\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\eta = r_\infty^\eta$$

for all $n\geq n_0^\eta$.

Proof. By Corollary 5.3, $\nabla\mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined for $n\gg 0$. In particular, the kernel and cokernel of the connecting map

$$\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\vee\to\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n}}))^\vee$$

are finite for $n\gg0$. In particular, $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))$ and $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n}}))$ must have the same $\mathbb{Z}_p$-corank. □

This implies that $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\eta_{\mathrm{div}}\cong(E/\mathcal{O})^{\oplus r_\infty^\eta}$ (as $\mathbb{Z}_p$-modules) for $n\gg0$. Combined this with (5.2), we obtain the following short exact sequence of $\mathbb{Z}_p$-modules

$$0\to(E/\mathcal{O})^{\oplus r_\infty^\eta}\to\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\eta\to Ш(\mathbb{Q}(\mu_{p^{n+1}}),T^\vee)^\eta\to0.$$

Therefore, on taking Pontryagin duals, we deduce that

$$\nabla\mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta=r_\infty^\eta+\nabla Ш(\mathbb{Q}(\mu_{p^{n+1}}),T^\vee)^\eta.$$

From Corollary 5.3, we deduce that

$$\nabla Ш(\mathbb{Q}(\mu_{p^{n+1}}),T^\vee)^\eta\leq eq_n^*+\nabla_n\mathcal{X}^\eta_{\tau(n,\eta)}+e\kappa(n,\eta)-r_\infty^\eta.$$

Therefore, we obtain the following theorem on applying Lemma 4.3.

Theorem 5.5. *Let $\#Ш(\mathbb{Q}(\mu_{p^n}),T^\vee)^\eta=p^{s_n^\eta}$. For $n\gg0$,*

$$s_{n+1}^\eta-s_n^\eta\leq r\left(eq_n^*+\nabla_n\mathcal{X}^\eta_{\tau(n,\eta)}+e\kappa(n,\eta)-r_\infty^\eta\right),$$

where r is the integer so that the residue field of E has cardinality p^r.

Using Lemma 4.2, we may rewrite this formula as

$$s_{n+1}^\eta-s_n^\eta\leq d\left(q_n^*+\lambda_{\tau(n,\eta)}+(p^n-p^{n-1})\frac{\mu_{\tau(n,\eta)}}{e}+\kappa(n,\eta)-\frac{r_\infty^\eta}{e}\right),$$

where $d=[E:\mathbb{Q}_p]$.

Appendix A. Growth of Tamagawa numbers over cyclotomic extensions

We let $T = T_f(j)$ and $\mathcal{T} = T_f(k-1)$ be the representations studied in the main part of the article. In particular, we assume all the previous hypotheses on T and $\mathcal{T}$ are satisfied throughout. Furthermore, we shall assume that the eigenvalues of φ on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ are not integral powers of p. For notational simplicity, we shall assume that the coefficient field E is $\mathbb{Q}_p$ throughout.

Recall the Perrin-Riou p-adic regulator

$$\mathcal{L}_{\mathcal{T}} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}) \to \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$$

defined by $\mathfrak{M}^{-1} \circ (1-\varphi) \circ (h^1_{\mathcal{T}})^{-1}$, which is the map used to define the Coleman maps in Definition 3.4. We have the following interpolation formula

Proposition A.1. *Let $n \geq 1$. For any $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$, $i \geq 0$ and a Dirichlet character δ of conductor p^n, we have*

$$\mathcal{L}_{\mathcal{T}}(z)(\chi^i\delta) = \begin{cases} i!(1-\varphi)(1-p^{-1}\varphi^{-1})^{-1}\left(\exp^*(z_{0,-i})\right)\cdot t^{-i}e_i & \text{if } n=0,\\ \frac{i!p^n}{\tau(\delta)}\varphi^n\left(\exp^*(\tilde{e}_\delta \cdot z_{n,-i})\right)\cdot t^{-i}e_i & \text{otherwise,}\end{cases}$$

where $\tau(\delta)$ is the Gauss sum of δ, $z_{n,-i}$ is the projection of z in $H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}(-i))$ and $\tilde{e}_\delta$ is the element $\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p)} \delta^{-1}(\sigma)\sigma$.

Proof. This is a slight reformulation of [LZ14, Theorem B.5] since we have the equation

$$\varphi(t^{-i}e_i) = p^{-i}\cdot t^{-i}e_i.$$

□

Corollary A.2. *Let $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathcal{T})$. Then, $\mathcal{L}_{\mathcal{T}}(z)(\chi^i\delta) = 0$ if and only if $\tilde{e}_\delta \cdot z_{n,-i} \in \tilde{e}_\delta \cdot H^1_f(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}(-i))$.*

Proof. This is because our assumption on the eigenvalues of φ implies that $(1-\varphi)(1-p^{-1}\varphi^{-1})^{-1}$ and φ^n are both invertible. □

We write $K = \mathbb{Q}(\mu_{p^n})$ and $\Delta_K = \mathrm{Gal}(K/\mathbb{Q})$. For each character δ on Δ_K, we write p^{n_δ} for its conductor. Let K_p be the completion of K at the unique place above p (which may be identified with $\mathbb{Q}_p(\mu_{p^n})$). We fix a basis v for $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(T)$ and its dual v' in $\mathbb{D}_{\mathrm{cris}}(T^*(1))/\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(T^*(1))$. We have the definition of the Tamagawa number as defined by Bloch–Kato [BK90]:

$$\mathrm{Tam}(T/K) = [H^1_f(K_p, T) : \mathcal{O}_{K_p}\cdot v]L_p(T,1),$$

where $L_p(T,1)$ is the Euler factor of the complex L-function $L_p(T,1)$ at p and we identify $\mathcal{O}_{K_v}v$ with its image under the Bloch–Kato exponential map. We may decompose the Tamagawa number into isotypic components, namely

$$\mathrm{Tam}(T/K) = \prod_\eta \mathrm{Tam}(T/K)^\eta,$$

where the product runs through all the Dirichlet characters modulo p and $\mathrm{Tam}(T/K)^\eta$ is given by

$$[H^1_f(K_p, T)^\eta : \left(\mathcal{O}_{K_p}\cdot v\right)^\eta]L_p(T(\eta),1),$$

which we may identify with $\mathrm{Tam}(T(\eta)/K^\Delta)$.

Lemma A.3. *Let d_K be the discriminant of K. Then, we have the formula*

$$\mathrm{Tam}(T/K) = |d_K|_p^{-1}[\mathcal{O}_{K_p}\cdot v : H^1_{/f}(K_p,T)]L_p(T,1),$$

where we identify $H^1_{/f}(K_p,T)$ with its image under the Bloch–Kato dual exponential map.

Proof. This follows from the commutative diagram

$$\begin{array}{ccc}
\left(K_p\otimes \mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(T)\right)\times\left(K_p\otimes \frac{\mathbb{D}_{\mathrm{cris}}(T^*(1)}{\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(T^*(1))}\right) & \longrightarrow & K_p \\
\uparrow \exp^* \qquad\qquad\qquad \downarrow \exp & & \downarrow \mathrm{Tr}_{K_p/\mathbb{Q}_p} \\
\left(\mathbb{Q}_p\otimes H^1_{/f}(K_p,T)\right)\times\left(\mathbb{Q}_p\otimes H^1_f(K_p,T^*(1))\right) & \longrightarrow & \mathbb{Q}_p.
\end{array}$$

□

Take $\mathbf{z}$ to be a Λ-generator of $\mathbb{H}^1(T)$ as in the main part of the article. This gives a Λ-basis $\mathbf{z}\cdot e_{k-j-1}$ of $\mathbb{H}^1(\mathcal{T})$. We shall write $\mathcal{L}_T(\mathbf{z})$ for $\mathrm{Tw}_{-k+j+1}\circ\mathcal{L}_{\mathcal{T}}(\mathbf{z}\cdot e_{k-j-1})$ and

$$\tilde{v}_K = \bigotimes_{\delta\in\hat{\Delta}_K}\left(\varphi^{n_\delta}(1-\delta(p)\varphi)(1-p^{-1}\bar{\delta}(p)\varphi^{-1})^{-1}v\right).$$

Theorem A.4. *Suppose that $\mathcal{L}_T(\mathbf{z})(\delta)\neq 0$ for all $\delta\in\hat{\Delta}_K$. Then,*

$$\bigotimes_{\delta\in\hat{\Delta}_K}\mathcal{L}_T(\mathbf{z})(\delta)\sim_p \frac{\mathrm{Tam}(T/K)}{L_p(T,1)}\prod_\delta\left[e_\delta H^1_{/f}(K_p,T) : e_\delta\mathbf{z}_K\right]\tilde{v}_K.$$

Here, we write $a\sim_p b$ if a and b have the same p-adic valuation.

Proof. Let $\mathbf{z}_K$ be the projection of $\mathbf{z}$ in $H^1(K_p,T)$. For each character of Δ_K, we write $e_\delta=\sum_{\sigma\in\Delta_K}\delta^{-1}(\sigma)\sigma$ and let K_δ for the subfield of K defined by the kernel of δ. Our assumption means that $e_\delta\cdot\mathbf{z}_K\notin e_\delta\cdot H^1_f(\mathbb{Q}_p(\mu_{p^n}),T)$ for all δ by Corollary A.2. Note that $\sum e_\delta=[K:\mathbb{Q}]$. On applying Proposition A.1, we deduce that

$$\begin{aligned}
\bigotimes_{\delta\in\hat{\Delta}_K}\mathcal{L}_T(\mathbf{z})(\delta) &\sim_p \prod_\delta\left[\frac{e_\delta}{[K:\mathbb{Q}]}\mathcal{O}[\Delta_K]v : e_\delta\mathcal{O}[\Delta_K]\frac{p^{n_\delta}}{\tau(\delta)}\exp^*(\mathbf{z}_K)\right]\tilde{v}_K\\
&\sim_p \prod_\delta p^{n_\delta}\left[e_\delta\mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K:\mathbb{Q}]} : e_\delta\mathcal{O}[\Delta_K]\right]\\
&\qquad\times[e_\delta\mathcal{O}[\Delta_K]v : e_\delta\mathcal{O}[\Delta_K]\exp^*(\mathbf{z}_K)]\,\tilde{v}_K.
\end{aligned}$$

Note that the factor $(k-j-1)!$ does not appear because of the Fontaine–Laffaille condition. Now, [Gil79, Proposition 1] tells us that

$$\left[e_\delta\mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K:\mathbb{Q}]} : e_\delta\mathcal{O}[\Delta_K]\right]=[K:K_\delta]\left[e_\delta\mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K_\delta:\mathbb{Q}]} : e_\delta\mathcal{O}[\Delta_K]\right]=1.$$

Therefore, we deduce from the conductor-discriminant formula that

$$\bigotimes_{\delta\in\hat{\Delta}_K}\mathcal{L}_T(\mathbf{z})(\delta)\sim_p |d_K|_p^{-1}\prod_\delta[e_\delta\mathcal{O}[\Delta_K]v : e_\delta\mathcal{O}[\Delta_K]\exp^*(\mathbf{z}_K)]\,\tilde{v}_K.$$

Combining this with Lemma A.3 gives us the result. □

Remark A.5. *There is in fact a similar formula without assuming the non-vanishing of $\mathbb{I}_{\mathrm{arith}}(T)(\delta)$. It would involve Perrin-Riou's p-adic height. See* [PR03, p.180].

Corollary A.6. *Let η be a Dirichlet character modulo p. Under the conditions of Theorem A.4, we have*

$$\nabla_n \mathcal{X}^\eta_{\mathrm{loc}} + b^\eta_{n+1} - b^\eta_n = q^*_n + \nabla_n(\mathbb{Z}_p[\![X]\!]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta) + p^{n-1}(p-1)n(k-j-1)$$

for $n \gg 0$, where $\tau(n,\eta)$ is as defined in Definition 4.10 and b^η_i denotes the p-adic valuation of $\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^i})^\eta)$ for $i = n, n+1$.

Proof. Let Δ_{n+1} be the set of Dirichlet characters of conductor p^{n+1} whose Δ-component is η. Its cardinality is given by $p^{n-1}(p-1)$. By Theorem A.4, we have

$$\bigotimes_{\delta\in\Delta_{n+1}} \mathcal{L}_T(\mathbf{z})(\delta) \sim_p \frac{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^{n+1}}))^\eta}{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^\eta} \prod_{\delta\in\Delta_{n+1}} \left[e_\delta H^1_{/f}(K_p,T) : e_\delta \mathbf{z}_K\right] \varphi^{n+1}(v)^{\otimes|\Delta_{n+1}|}.$$

This gives

$$\bigotimes_{\delta\in\Delta_{n+1}} \varphi^{-n-1}\circ\mathcal{L}_T(\mathbf{z})(\delta) \sim_p \frac{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^{n+1}}))^\eta}{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^\eta} \prod_{\delta\in\Delta_{n+1}} \left[e_\delta H^1_{/f}(K_p,T) : e_\delta \mathbf{z}_K\right] v^{\otimes|\Delta_{n+1}|}. \tag{A.1}$$

Note that $\varphi^{-n-1}\circ\mathrm{Tw}_{-k+j+1} = p^{(n+1)(k-j-1)}\mathrm{Tw}_{-k+j+1}\circ\varphi^{-n-1}$. The terms appearing on the left-hand side are therefore simply $p^{(n+1)(k-j-1)}\underline{\mathrm{Col}}_{T,n+1}(\mathbf{z})(\delta)$. Therefore, the p-adic valuation of the left-hand side of (A.1) is given by

$$p^{n-1}(p-1)(n+1)(k-j-1) + q^*_n + \mathrm{ord}_{\epsilon_n}\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta(\epsilon_n)$$

thanks to (4.2). Hence the result. □

The proof of our Proposition 4.11 implies that the defect of our inequality in Theorem 5.5 is in fact given by the length of $\ker\pi^\eta$, where π is some projection map. We see here that we may in fact relate this defect to the Tamagawa numbers, namely,

$$\mathrm{len}_{\mathbb{Z}_p}\ker\pi^\eta = b^\eta_{n+1} - b^\eta_n - p^{n-1}(p-1)n(k-j-1).$$

Let t^η_n be the integer $s^\eta_n + b^\eta_n$, which is the p-adic valuation of $\#\text{Ш}(\mathbb{Q}(\mu_{p^n}), T^\vee)^\eta \times \mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^\eta$. The Bloch–Kato conjecture predicts that this quantity should be related to the leading coefficient of the complex L function of T at 1. Theorem 5.5 tells us that we have the equality

$$t^\eta_{n+1} - t^\eta_n = q^*_n + \nabla_n\mathcal{X}^\eta_{\tau(n,\eta)} + \kappa(n,\eta) - r^\eta_\infty + p^{n-1}(p-1)n(k-j-1).$$

for $n \gg 0$.

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