

USING GOOGLE SKETCHUP TO DEVELOP STUDENTS' EXPERIENCES OF DIMENSION IN GEOMETRY

Nicole Panorkou (Corresponding author)

Montclair State University, Department of Mathematical Sciences
1 Normal Avenue, Montclair, NJ07043, United States

panorkoun@mail.montclair.edu

Dave Pratt

UCL Institute of Education, Department of Curriculum, Pedagogy and
Assessment, 20 Bedford Way, London, WC1H 0AL, United Kingdom

d.pratt@ioe.ac.uk

***Abstract** Dimension is a powerful mathematical construct that is rarely taught or researched explicitly. The study reported here explored how the software Google SketchUp can facilitate students' experiences of dimension. Clinical interviews based around carefully designed tasks were conducted with 10-year old students. This paper reports evidence from the data on how the activity setting including SketchUp's dimensional tools identified in the software prompted the construction of ideas about dimension. More specifically, children expressed intuitive ideas about an object/space's freedom to move within a space/object of higher dimension and its capacity to house other objects/spaces of lower dimension.*

Keywords: dimension, geometry, situated abstractions, Google SketchUp

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UCL Institute of Education, Department of Curriculum, Pedagogy and
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In this study, we aimed to create new ways through which young children might experience the mathematical notion of dimension. We show how children as young as ten years of age were able to work creatively and imaginatively in a three-dimensional (rendered in 2D) computer-based drawing environment. We analyse the properties of the tools and tasks within that environment that seemed to trigger novel expressions of dimension. First, we need to consider how dimension is encountered informally in everyday settings and formally in scientific contexts.

1 Dimension as a quality of object and space

Our experiences of dimension are in some sense differentiated between a lived-in unformalised world and an artificial mathematized world, where space is expressed, for example, in terms of geometry and dimension. For instance, in mathematics, dimension is a powerful mathematical construct integrated within coordinate geometry, topology, vectors, projective geometry, statistics, graphs, and calculus (Banchoff 1990). At the same time, children already have an idea of geometry and dimension before entering primary school. Estimating how much water will fit in a glass, cutting enough paper to cover the presents, and drawing their house and family are just some of the many situations in which children experience dimension, though unaware of the mathematical connection.

In everyday life and in scientific endeavour, dimension is sometimes referred to as a *quality of an object*. For example, in everyday life, dimension might be used to describe the size of a box as, say, 15cm by 10cm by 5cm or a 3D television. In mathematics, dimension might refer to a line as one dimensional and a filled-in square as two dimensional.

On the other hand, dimension might be a *quality of space*. For example, even in everyday life there is talk of 3D space. In mathematics, dimension might be expressed as a quality of space by using coordinates to locate a given point in space (Cartesian approach). Roughly speaking, the dimension of a space is the minimum number of coordinates needed to specify every point within it. Similarly in vector geometry, the dimension of a vector space V is the cardinality (i.e. the number of vectors) of a basis of V . In other words, dimension can be expressed as the least number of linearly independent vectors needed to span space.

In fact the qualities of object and space easily become blurred. A line can be considered as a 1-dimensional space if we think about a point moving on it, because there is only one direction in which the point can move (regarding back and forth as positive and negative movements of one direction). The same line, however, can exist in a plane and might then be expressed as a relationship between two variables (x,y). In the first case, the line is a space and in the second it is an object within a higher dimensional space.

From another perspective, dimension portrays qualities of *freedom* and *capacity*. There is a sense in which space/objects with greater dimension have the capacity to incorporate space/objects of lower dimension; for instance a plane can ‘house’ a line or a point. At the same time, a freedom quality is expressed by recognizing that the greater the dimension of a space/object, the more are the degrees of freedom for movement within that space/object.

2 Children’s experience of ‘dimension’

Although much research has been conducted on the teaching and learning of geometry, most of this research refers either to general facts of geometric thinking (e.g. Fujita and Jones 2002; Gravemeijer 1998; Gutierrez 1996a) or specifically to 2-dimensional geometry (e.g. Clements et al. 1999) or 3-dimensional geometry (e.g. Pittalis and Christou 2010). In other words, little research has been conducted explicitly on dimension. Nevertheless, the relationship, or ambiguity, between the material and theoretical interpretation of geometry and dimension is a recurrent theme in the literature; for instance, geometric objects such as lines and planes are treated in the everyday world as if they can be observed in reality but they have a theoretical instantiation in mathematics.

Although geometry deals in abstractions as much as any other branch of mathematics – points, lines, and planes are just as much things of the imagination as are polynomials – it is natural to relate *visually* to geometric objects (Goldenberg et al. 1998, p. 21)

This ambiguity leads to misinterpretations of representations. These misinterpretations occur mainly because it was not evident for the students that representations present a loss of information and “that the drawing cannot, by itself, substitute for the object” (Parzysz 1988, p.85). Parzysz (1988) identified

two main problems of representations: coding (producing) representations and decoding (reading) representations. In terms of coding representations, when Parzysz (1988) asked sixth grade students to represent a square-based regular pyramid in a 2D drawing, he noted that:

For these pupils, there is actually no conscious and deliberate choice of a perspective, as well as a position of the object which “presents in order to see”, but a use- more or less important and more or less empirical- of perspective effects, with a concern for preserving some properties of the object (p. 90).

Difficulties exist also while decoding representations, especially when “interpreting two-dimensional diagrams of three dimensional objects” (French 2004, p. 21). As Parzysz (1988) noted students confuse the drawn 3D figure with a 2D figure having the same representation. French (2004) gave an example of a cube that can be equally seen as a two dimensional picture of a hexagon (Figure 1).

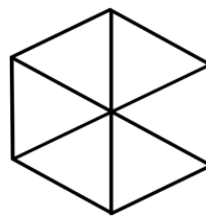


Figure 1: A cube or a hexagon? Considering the above as a 2D drawing, its perimeter is a perfect regular hexagon

Parzysz (1988) argued that this happens because “what ones *knows* of a 3D object comes into conflict with what one *sees* of it” (p. 84). In their longitudinal study of children’s reasoning about 2D figures and 3D solids, Lehrer et al. (1998) concluded that children tend to relate to known figures (e.g., “looks like a pushed-in rectangle”) and view figures dynamically (morphing via non rigid transformations by ‘pushing’ or ‘pulling’ at vertices). In contrast, the 2D representations create a static and restricted view of shapes that conflicts with the dynamic perceptions that students already have about the subject.

Problems of decoding were recorded also when students tried to make sense of space in a representation. Parzysz (1988) asked sixth grade students to state which points are located on the plane below (Figure 2) and found that “if the representation of the point is interior to the representation of the plane, then the point itself is seen as belonging to the plane” (p. 83).



Figure 2: Which points are on plane P ? (Reproduction from original: Parzysz, 1988, p. 83)

The above findings show that representations restrict students' dynamic perspective of shape and space and as a result impede the development of students' ability of dimensional deconstruction, the analytical breaking down of a solid object into interrelated parts based on relationships (Duval 2005; Laborde and Laborde 2011) as in studies showing the limitation in young students' judgements of volume by ignoring some vital dimensions. Piaget (Piaget 1968; Piaget et al. 1960) recognised that primary school children appeared to use only the height of the container when making volume judgments. For example, these children thought that the volume had been reduced when the liquid was poured into a wider glass. Raghbir's (1999) research showed that these difficulties continue to exist even in some adults. Raghbir (1999) observed that when trying to estimate the volume, people's judgements are likely to be based on one or two dimensions only. His study revealed that the volume perception is mostly influenced by the 'height' on its own or in a combination with 'width'. Consequently, taller shapes are perceived larger than shorter ones without reference to other dimensions.

Similarly to Piaget and Raghbir's studies, while exploring the notion of definition within mathematics, Morgan (2005) argued that students and teachers identify dimension with regards to 'thickness.' Her study also showed that they construct "a multi-faceted notion of dimension" (p. 104), including (among others) a description of 2D as 'flat' and 3D as 'solid', and considering 3D as having something extra compared to 2D. Additionally, Lehrer et al. (1998) found that students describe 2D and 3D shapes in the same manner. They pointed out that children tend to view 2D and 3D figures as 'slanty' or 'pointy', count sides or vertices, or "describe figures in terms of other properties such as parallelism remained stable" (Lehrer et al. 1998).

The above studies show evidence that the way 2D and 3D shapes/spaces are represented and experienced forms students' construction of the notion of dimension. Panorkou (2011) reported supporting evidence when she studied 10-year old students' experiences of dimension through three approaches: (i) using the software Elica (<http://www.elica.net/site/museum/dalest/dalest.html>); (ii) using a sorting task of physical objects (a variety of 2D and 3D shapes that students had to sort and discuss their similarities and differences); (iii) through

discussion based on the film Flatland (Travis 2007). This study showed that students talked of dimension in five specific ways: (i) as a type of measurement (e.g. the more vertices or edges an object has, the greater its dimension); (ii) as a domain incorporating shapes (e.g. dimension is a ‘place’ with different shapes); (iii) as an object having materialistic attributes (e.g. the more dimension, the thicker the object); (iv) as a property for distinguishing between 2D and 3D shapes (e.g. a square could be in 2D and a cube could be a square seen in 3D); (v) as creating a hierarchy of relationships among shapes (point C line, line C square, square C cube). The current study took place shortly after that reported above and so the five specific ways of talking about dimension provided a basis for knowing what meanings students might have about dimension.

It is instructive to note that some elements in how dimension is used in scientific endeavour were absent, or not clearly articulated, from students’ expressions of dimension in the Panorkou study. In particular, Panorkou (2011) argued that children sometimes talked of dimension as if it were a property of objects, while at other times they referred to it as a property of space, without indicating any appreciation of the flexibility between the two. What is more, as Panorkou (2011) noted, the design of the three settings (Elica, physical objects, Flatland) did not offer opportunities for the students to actively experience dimension as a quality of freedom and capacity. And yet, Panorkou (2011) observed that the frequency of generating expressions of dimension belonging to a specific type was dependent on the task on which the individuals were working, therefore acknowledging the significance of the setting in shaping the creation of those expressions.

The significance of the setting to the formation of experience about shape and space was also noted by Ferrara and Mammana (2014) who argued that students’ experiences with 2D and 3D shapes through physical manipulation prevent them from discovering the crucial relationships between geometric 2D and 3D figures. They pointed out that “the visual and cognitive potential of interlacing related but different figures is offered by the use of DGEs, allowing for moving back and forth between plane and space” (p. 57). In exploring quadrilaterals and tetrahedra with high school students utilizing Cabri 2D and 3D, Ferrara and Mammana (2014) noted that the environment of the software:

encouraged the students to take on multiple perspectives, as if they were taking on various physical positions from which to see a figure, as bodily projecting themselves both beyond and around it. So, they engaged the students in dynamic visual experiences with the diagrams, effecting new kinds of vision that pushed them towards a search for similarities and differences, invariants and changes, between quadrilaterals and tetrahedra. This engagement spoke directly to students’ enhanced visual skills, so that they not only came to see the quadrilateral *as* a tetrahedron, but also to see them, when thought of as represented in a diagram, as “the same thing” (p. 64)

The Ferrara & Mammana's (2014) study illustrated the power of digital environments, such as Cabri 3D, for developing dynamic experiences of shape and space. Indeed, we recognise in what Ferrara and Mammana (2014) call 'interlacing related but different figures' and 'moving back and forth between plane and space' the flexibility to view geometric entities as objects or spaces and the qualities of freedom or capacity. We conjectured that enhanced flexibility might lead to greater facility in encoding and decoding representations and that new specially designed settings might offer experiences to support such flexibility. More precisely our research questions asked: (i) What new meanings might young students articulate for dimension beyond the five specific ways listed above; (ii) How do the features of the setting support these novel meanings?

3 A framework for researching dimension

The idea of researching "experience" derived from an acknowledgment that, although it is not possible to gain direct access to the child's mind, it is possible to observe their experiences. We might interpret experience as a realistic articulation of thinking. While experience is evident, thinking from the researcher's perspective is an inference. From the perspective that even very young children might have experiences of dimension that might be seen by expert others as the situated root of sophisticated mathematical concepts, we set out to identify a setting which might offer experiences of dimension that stimulated the articulation of new meanings for dimension. Whilst we recognised that the articulation of such meanings would not provide evidence that the children 'understood' dimension, our objective was to understand how such meanings could be shaped by appropriate tools in the setting.

We intended that, by providing new tools, students' understanding of dimension could be perturbed, an action referred to as 'thinking-in-change' by Noss and Hoyles (1996). Noss and Hoyles regard the setting for these perturbing tools as a 'window' through which the researcher is able to examine the thinking process. We have seen how thinking about dimension is sensitive to the context and so the design of the window in our study was critical. We have interpreted Ferrara and Mammana (2014) as proposing that the affordances of digital technology can offer the flexibility we seek to move between plane and space so that the child might code or decode representations as either an object's freedom to move within a space or the capacity of a space to house an object. We now wish to be more precise at a theoretical level about how the use of digital tools might support enhanced flexibility.

The integration of computers into students' learning has been described as a complex process involving *instrumental genesis* (Verillon and Rabardel 1995). Instrumental genesis is a dialectical process comprising of how an individual

assimilates an artefact (instrumentation) and how the specific artefact is transformed for specific uses (instrumentalisation) (Artigue 2002). The result of instrumental genesis is the “construction of personal schemes or, more generally, the appropriation of social pre-existing schemes” (Artigue 2002, p. 250). However, this process of instrumental genesis does not necessarily produce individual understandings and ways of working that align to standard mathematics and can be described by the general notion of ‘scheme’. Following Hoyles, Noss and Kent’s (2004) view that there is a need for describing and validating the computationally mediated mathematical abstraction that can be divergent from the standard mathematical norms, we were interested in this study in how students articulate connections between the setting in which they explore and develop meaning. In other words, our focus was on the *process* of abstracting knowledge within the specific context, an idea described by Hoyles, Noss & Kent (2004) as *webbing*:

[...] the *process* of abstraction of mathematical properties and invariants that is key, and this is *necessarily* both situated and shaped by the tools being used, the users’ relationship to the tools — including whether the users judge them to be expressive of their developing mathematical meanings — and ultimately whether these meanings are valued and judged by the community (e.g. the classroom) to be mathematical (p. 314).

Building on the idea of webbing, Hoyles and Noss (1992) introduced the notion of situated abstraction in order “to describe how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed” (Noss and Hoyles 1996, p. 122). They refer to *situated abstractions* as generalisations that students form in order to act in specific mathematical contexts (Hoyles & Noss 1992), consequently embedded in the particular context in which they took place. The main idea underlying the notion of situated abstractions is that:

[...] conceptions of mathematics might be situated — in terms of language and connectivity and with the context of their genesis, means of expression and use (that is, with the artefacts, goals and discourse that form part of the activity) — and yet are abstract in that they extend beyond immediate concerns to more general conceptions of knowledge, that is, they can be “mapped onto” parts of formal mathematics (Hoyles et al. 2004, p. 312).

Adopting the theoretical framework of webbing and situated abstractions, we refer to the enhanced flexibility that we seek as the expansion or refinement of the child’s *contextual neighbourhood* of situated abstractions for dimension, where the contextual neighbourhood “captures the domain over which the idea has been encountered and found to be powerful by the child in explaining the on-screen behaviour” (Pratt and Noss 2010, p. 94). (Unfortunately, there is scope for confusion in this use of terminology. The reader will note later that our tasks involved the children in building a neighbourhood consisting of streets and

buildings so we will refer to ‘contextual neighbourhood’ as ‘c.n.’ for the remainder of the paper to avoid confusion over the use of the term ‘neighbourhood’.)

We also recognised the significance of feedback within the window in response to student activity. Feedback is believed to be one of the most significant aspects of the qualities of a window as it increases “the likelihood that children would identify the lack of explanatory power of their fragmented knowledge pieces in certain situations” (Pratt and Noss 2010, p. 90). As Healy and Hoyles (2002) argued, “guided by the visual feedback resulting from their programming activity, users can edit or debug their symbolic constructions until they are satisfied with their final product” (p. 236) and this is where learning and expansion of c.n.s occurs.

Another critical element of the design of the window lies in its capacity to support expressiveness by the students (diSessa et al. 1995; Hoyles and Noss 2008). A window can be expressive in the sense that it can present abstract mathematical ideas in a concrete form. In this study, the window needed to somehow embed the notion of dimension to support concretising of dimension by the student (Wilensky 1991). We intended that students would express their ideas about dimension and use feedback to develop the c.n. across which dimension made sense. At the same time, such expressivity would make the student’s ideas more ‘observable’ to the researcher.

With in mind the two key criteria of feedback and expressivity, we chose the software Google SketchUp (<http://sketchup.google.com/>) as a suitable environment for such young students to experience dimension as capacity of an object or freedom within a space. We can now be more precise about our research questions, which were:

- a) What new scientific meanings for dimension might young students articulate beyond the five specific ways listed above when using the technological tool, Google SketchUp?
- b) How do the constraints and affordances of Google SketchUp shape thinking-in-change about dimension?

4 The opportunities and constraints of Google SketchUp

Our choice for using SketchUp was influenced both from the geometry that the design of SketchUp embeds but also from the ways that the environment supports students’ perceptions of the key features of this geometry. Our decisions were based on Jones, Mackrell and Stevenson’s (2010) analysis of principles for designing digital technologies, which are based on “how the opening software screen both orients the user to 3D space, and provides a framework for the creation of 3D figures and structures” (p. 53). These principles include a) the way(s) objects ‘look’ on screen pointing to the idea of perspective and b) the

way(s) objects can be constructed, manipulated and dragged using a mouse (Jones et al. 2010).

In SketchUp, the basic window is coloured green and light blue in a way to represent the ground floor and sky. There are three axes by default: the red, the blue and the green axes. The axes are perpendicular and their extensions are marked with dotted lines (negative direction). There is also a person standing ‘on the ground’, the latter being defined by the red and the green axes (Figure 3).

After exploring the software further we decided to have an open task for the students in order to give them the opportunity to explore different features of SketchUp. The aim of the task was for students to use SketchUp in order to build a neighbourhood, modeling in that way a real world phenomenon (Noss and Hoyles 2006; Simpson et al. 2005) and providing opportunities for the students to resolve illusions articulated as errors in coding and decoding when modeling a 3D world on a 2D screen.

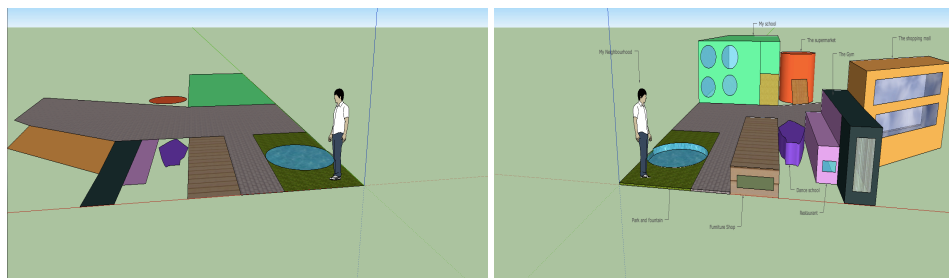


Figure 3: Example of 2D and 3D neighbourhoods in SketchUp

In designing the learning task, we considered the affordance of the environment to motivate the students to create their own structures. Our notion of structure was based on Jones, Mackrell & Stevenson’s (2010) observations in Cabri 3D:

Such structures included models of ‘realworld’ objects and/or objects that moved, with the creation of such structures necessitating the use of a range of mathematics. [...] For instance, in order for an object to have a particular visual property when viewed from all angles (such as a segment being perpendicular to the base plane) the object needs to be constructed using the mathematical tool which creates the desired relationship (p. 56).

Similar to the notion of mathematical tool mentioned by Jones, Mackrell & Stevenson (2010), SketchUp incorporates what we came to call *dimensional tools*, tools whose use was likely to raise opportunities for expanding or refining the c.n. of situated abstractions for dimension. These tools had an *expressive power* (Abelson and diSessa 1986) in that the students used them to build their ideas about a neighbourhood and they provided constructive feedback to the user in the way that they were rendered on the screen according to their orientation, and connectivity to other objects on the screen.

The dimensional tools of SketchUp addressed formal and informal ideas of freedom and capacity. Informally, playing with the tools opens up ideas about space containing equal or lower dimensional objects and generated by combinations of those objects. For example, in SketchUp the user can create line segments parallel to the three axes using the Line Tool. The colour of these segments mirrors the colour of the axis to which they are parallel (blue, red, green). For creating a 3D shape (e.g. a cube) the user has to utilise all three different colours of line segments (blue, red, green) in order to give length, width and height to the cube. Informally, the user might recognise that a cube can be created by an appropriate combination of 1-dimensional line segments. The role of the ‘dimensional’ tools incorporated into SketchUp was to make those mathematical ideas apparent; in other words, it became the communication medium between the students and the researcher’s intentions by making “visible its operations and how they are integrated with the embedded context” (Orhun 1995, p. 310).

In setting out the designer’s perspective on how SketchUp might contribute to the expansion or refinement of the child’s c.n. for dimension, we detail four of its most significant tools (for our purposes). For each tool, we discuss expressivity and feedback and then identify the situated abstractions for dimension that might be articulated by students while working with those tools. It is perhaps worth mentioning that the techniques described, though difficult to express in writing, became quite intuitive for the students through use.

4.1 Line tool

Expressivity: students used the Line tool to render lines, thus creating 2D shapes that would perhaps represent buildings or streets in their neighbourhood. Feedback: while drawing, the colour of the line changes to mirror that of any parallel axis (red, blue or green). If a line is drawn that is not parallel to any of the default axes then it has a black colour.

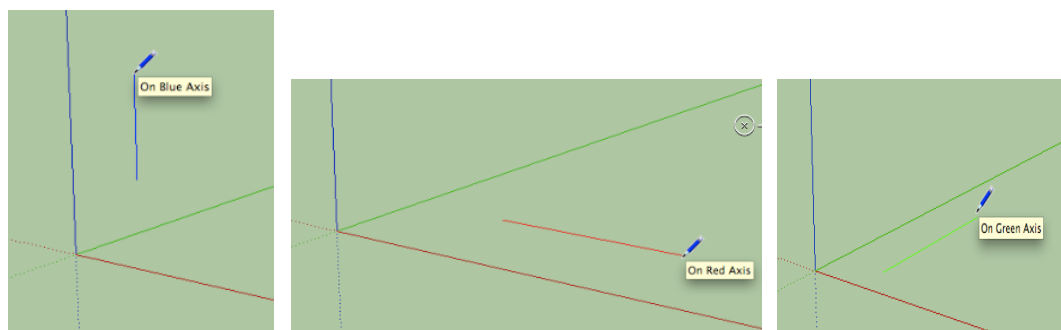


Figure 4: Line tool

The Line tool facilitates the user in creating line segments in different positions or directions and with various orientations in space. It also gives the opportunity to the user to create one-dimensional objects in three different spaces (planes),

therefore embedding the object-space duality of dimension with the idea of freedom to move and capacity to house. Our conjecture was that, by creating objects/spaces with the line tool, the child's attention would be drawn to the relationship with the axes, perhaps leading to an expansion or refinement of the child's c.n. of situated abstractions for the directional, positional and orientational aspects of dimension.

4.2 Shaded Surfaces tool

This tool is an extension of the Line tool. Expressivity: students could create a surface by drawing a closed loop of edges using the Line tool, perhaps to imagine walls of a building. Feedback: if this loop of edges is co-planar, the surface created is shaded. If not shaded, one of the edges belongs to another plane.

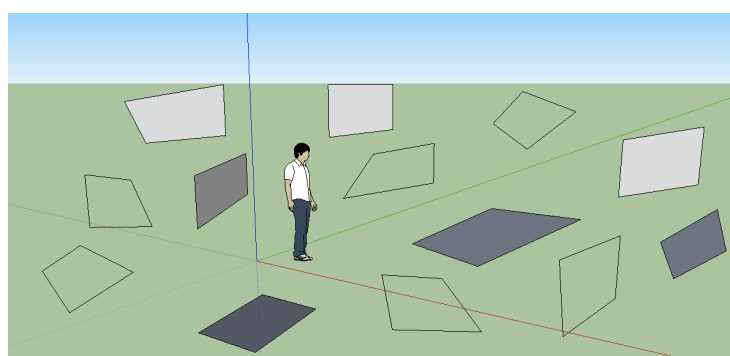


Figure 5: Shaded Surfaces tool

Furthermore, as soon as a closed co-planar shape is created, its surface is shaded light grey, dark grey or white (see Figure 6), according to its position in terms of the plane on which the surface is situated.

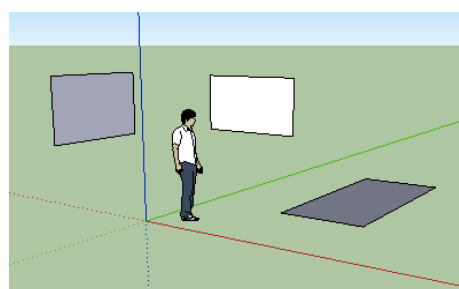


Figure 6: Dark grey, light grey and white shapes

This tool incorporates not only the object-space duality as discussed above for the Line tool but also the notion of generating and positioning surfaces. For instance, to create a 2D shape, a student needed to use line segments in two different directions whereas a 3D shape required line segments in three directions, possibly one parallel to each of the three different axes. If the line is not coloured, it is not parallel to an axis and thus impedes the creation of surfaces. If a shape was created by edges which were not co-planar then it did not have a shade, and also:

- (a) it could not be coloured afterwards using another tool (not detailed here);
- (b) it could not be pushed/pulled into 3D (see Push/Pull tool below).

We conjectured that the limitations of this tool offered debugging opportunities (Edwards and Benedickt 1995; Papert 1980) for the children to identify and fix problems. We anticipated that such debugging might lead to expansion or refinement of the c.n. of situated abstractions for the cardinality of dimension.

4.3 Orbit tool

Expressivity: students used the Orbit tool to ‘move’ in 3D space. Orbiting is like holding an object and turning it around. When using this tool, the user clicks and holds the mouse button and moves the mouse around. Where the user’s cursor is on the screen, it will be the centre of rotation. **Feedback:** the Orbit tool enables experience of co-planarity or lack of it. Students used the Orbit tool to move their object around and see whether the loop of edges they created was co-planar.

In Sketch up, while moving with the Orbit tool, first the user defines the height (up-down) of rotation and then drags towards an arbitrary direction. If the height remains fixed then a plane of rotation is created. However, usually the height changes because of the use of the unsteady mouse. In addition, since the orbit tool changes the position of the eye relative to the object, the user sees the object from different perspectives. As well as being a key tool for noticing coding or decoding errors, we conjectured that use of the Orbit tool might lead to the expansion or refinement of the c.n. of situated abstraction for the plane of rotation and perspective.

4.4 Push/Pull tool

This tool extrudes a 2D shape into three dimensions in a direction that is perpendicular to the surface. **Expressivity:** students used this tool for creating 3D buildings in their neighbourhood by pushing and pulling 2D surfaces. 2D surfaces can be created by using the Line tool as described above or by using specially designed tools for creating rectangles and circles respectively. Surfaces are extruded along a direction parallel to the red, green or blue axes (normal to the plane being extruded). **Feedback:** when for example extruding a circle, the colour of the cursor showed the direction that the extruding would go using the push/pull tool. For instance, if the cursor were blue that means that the extruding would go in the direction of the blue axis (Figure 7) as this is normal to the plane being extruded.

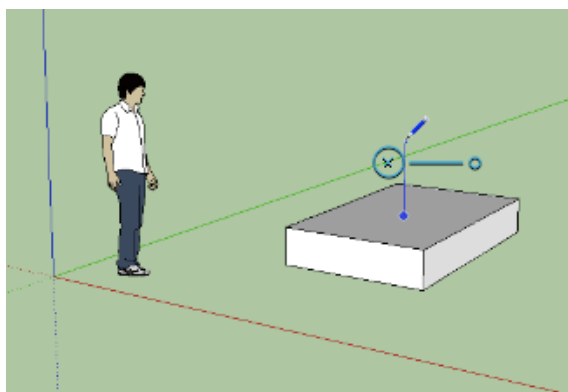


Figure 7: Example showing Push/Pull tool

The Push/Pull tool emphasises movement between the plane and space, proposed as an important affordance of digital technology by Ferrara and Mannama (2014). We therefore conjectured that this tool might support the expansion or refinement of the c.n. of situated abstractions for how dimension distinguishes and connects different objects and spaces.

5 Method

Experiences of dimension were generated from twelve 10-year old students who worked in six pairs (Pairs A-F). The students chosen were upper-middle ability students in mathematics in their teachers' views. Working in pairs was chosen in order for the students to work with a class partner with whom they felt comfortable, according to their teacher, and also because their natural discussion and argument would externalise their thoughts, helping the researcher to interpret the activity. The specific students were taught the UK's curriculum as described by the National Numeracy Strategy (DfEE 1999). The Measures, Shape and Space strand of the National Numeracy Strategy consists of four main targets: a) measurement of length, mass, capacity, perimeter, area and time, b) direction, angle, c) position, including coordinates and d) properties of 2D and 3D shapes, including symmetry. We considered that by age 10, the students were likely to be able to articulate their own thinking about dimension but that, since the students at this age had not yet experienced formal geometric teaching, the meanings that might emerge would be highly sensitive to how their informal experiences responded to the features in Google SketchUp. We also took the five types of meaning for dimension listed above, which arose in the previously reported study, as a basis for what we might expect to be articulated in the current study. In terms of observing thinking-in-change, we were particularly interested to observe when and how new meanings for dimension emerged during the interactions with Google SketchUp.

The students had not used Google SketchUp before. The semi-structured clinical interview (Opper 1977; Piaget 1976; Hunting 1997), based on the completion of

tasks using Google SketchUp, was the main method in this study. Each interview lasted around 120-180 minutes. (It is perhaps testimony to the engaging nature of the tasks that these young students were able to concentrate for such an extended period.)

As described previously, the central task of the interview was for students to build their imaginary neighbourhood (*Task A*). Students were shown the 3D version of the neighbourhood as in Figure 3 and were asked: “Here is an example of a neighbourhood a pair of students designed. They used special software for designing. The task is to draw your own version of a neighbourhood like the example. For building the neighbourhood, students first designed it like this (the researcher shows the 2D example of the neighbourhood in Figure 3) and then used this special tool (Push/Pull tool) to make it like that (like the 3D example)”.

The aim of this main task (Task A) was to offer students the opportunity to work in a simulated 3D environment and experience the capability this facility had to offer. More specifically, it included experiences about:

- a) the restrictions of movement in a 2D compared to a 3D environment (working with 2 or 3 axes; working on a plane; working in space);
- b) the restrictions of vision in a 2D compared to a 3D environment;
- c) how the objects were created in a 2D and a 3D environment (by aggregating other shapes; by dragging lower dimensional shapes);
- d) which objects can exist in a 2D and/or in a 3D environment.

The main task offered them the chance to explore both the 2D and the 3D design and make comparisons pointing to their potentials and constraints. It involved working both in different spaces (2D floor – 3D space) and with different objects (lines, 2D shapes, 3D shapes) and experiencing the flexibility to move back and forth between spaces and examine ‘interlacing related but different figures’ as described by Ferrara and Mammana (2014). The quality of dimension as freedom could be identified through the experiences a) and b) above. Similarly, the duality of object/space could be identified through the c) and d) type of experiences, by generating different shapes and then identifying the lower-dimensional shapes that were used for the generation.

Our goal in these interviews was to allow the students to be in control of their explorations, decision-makings and consequently construction of knowledge. The interview was conducted by the first-named author acting as a participant observer. Most of the interventions involved probing the students to further explain their actions in words while avoiding changing the direction of students’ thinking. However, some interventions were less neutral and more experimental by intending to make some change in the direction of activity, especially when students were clearly stuck on a path with no apparent learning potential.

For instance, an additional task, the Incomplete Frames task (*Task B*), was created to enrich the interview plan by being more specific, leading students to the

exploration of specific mathematical notions especially in relation to the decoding of specific representations and perspective (French 2004; Parzysz, 1998). As described above, for a surface to be shaded it has to be created using the Shaded Surfaces tool with a closed loop of co-planar edges. Students were presented some incomplete frames as shown in Figure 8a and they were asked to complete them as in Figure 8b. Then they were asked to use the Orbit tool to explore why some of the surfaces remained uncoloured.

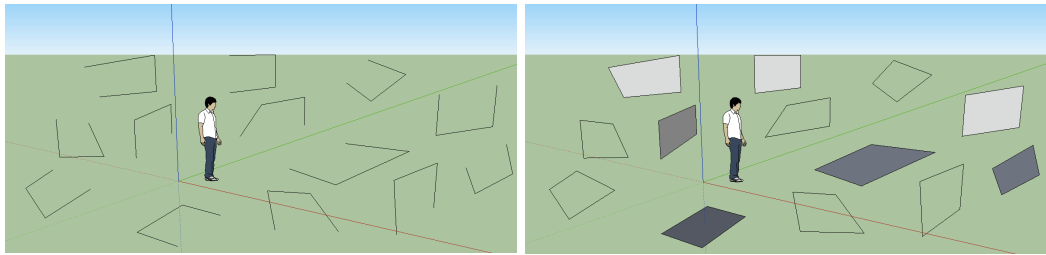


Figure 8a

Figure 8b

The Incomplete Frames task was brought into play according to the needs of the students, which could be either during or after Task A. In other words, this task was complementary because it offered space for further exploration of specific tools or even notions.

Additionally, during Task A, we used some prompting questions with the students as an extra reinforcement of the notions of direction, position and orientation, asking the students to: (i) discuss the construction of shapes in SketchUp ('How many ways are there to create a rectangle/ a cube?'); (ii) reflect on the differences between 2D and 3D objects/spaces and make predictions of other dimensional environments ('What would a 1D/0D world look like?' 'What shapes might exist in this world and why?' 'What would be the difference between this world and their 2D/3D world?'); (iii) reflect on how their thinking may have changed throughout the process ('What do we mean by dimension?' 'What is an example of a 2D/3D shape?' 'What is an example of a 2D/3D world?'). We also used the 2D and 3D pictures of a neighbourhood (Figure 3) and asked the students to comment on the difference between the two worlds/pictures, and on the different shapes that exist in each world/picture.

The aim of analysis was to describe the nature of students' reasoning of dimension and to uncover the constraints and affordances of Google SketchUp for thinking-in-change about dimension. Consequently, the unit of analysis was the variation of situated abstractions about dimension, as articulated by the students. In order for an excerpt to be considered as a situated abstraction of dimension, it had to relate to the definition(s) of dimension as extracted from the literature. For example, the situated abstraction "*We have flat shapes and 3D shapes*" was relevant to Morgan's (2005) study and was included. In contrast, some excerpts did not qualify because they referred to mathematics or geometry in general or were even considered to be irrelevant to our study. For example, students talked

of the 'D' following 1, 2 and 3 as showing dimension, and that, without it, the naming of the shapes would have no meaning i.e. 2/3 shapes instead of 2D shapes/3D shapes. Even though the way we name the symbol of dimension is an important notion, and it is mentioned in some excerpts, it was considered irrelevant to include all the excerpts referring to it.

After collecting all the relevant situated abstractions of dimension, we used thematic analysis (Braun and Clarke 2006) to examine themes and patterns across the situated abstractions based on their similarities and differences. For example, the students' situated abstractions "*Two/three different coloured lines were needed for creating a 2D/3D shape*" and "*3D shapes can be created by pushing/pulling 2D shapes in 3 directions,*" describe dimension in terms of freedom to create objects so they were placed in the same theme category. On the contrary, the situated abstraction "*A 3D object can go inside the 3D neighbourhood and its buildings but it can't go inside the 2D one*" shows an articulation of dimension in terms of the capacity to house objects, and therefore was placed in a different category. Our attention was focused on how the students' pre-existing experiences of dimension, articulated through the particular situations they had experienced, might be broadened through the situations presented to them in this particular study.

6 Describing dimensional experience

In total, the analysis revealed 190 situated abstractions articulated by the twelve students. The students' experiences of dimension were sorted into two categories of description, which roughly speaking address situated abstractions of the object/space's: (i) freedom to move, and (ii) capacity to house. The freedom to move idea incorporates experiences of direction, position and orientation whereas capacity integrates experiences of containment. By containment one can see the space as incorporating objects; in this sense the space contains the objects. At the same time, the space can be thought of as generated by the objects. The following paragraphs describe students' articulations of each of the categories by presenting some thinking-in-change episodes and giving specific examples of situated abstractions that Pairs A-F expressed during the clinical interviews. (Phrases in the protocol below that are inserted inside square brackets are intended as explanatory commentary and were not part of the students' articulations.)

6.1 Freedom to move

The majority of the situated abstractions (156 out of 190 situated abstractions) expressed situated abstractions of dimension in terms of direction, position and orientation. Direction, position and orientation can be considered as essential components characterising a particular object/space. In this section, we present the findings on these components and then present discussion of these components.

At the beginning of the task, students positioned shapes in more than one plane and they were able to distinguish their orientation in space using the characterisations of ‘flat’, and ‘coming out’ or ‘standing up’. For example, when Pair D used the special tool for creating rectangles, they distinguished ‘flat’ shapes from ‘non-flat’ ones (Figure 9):

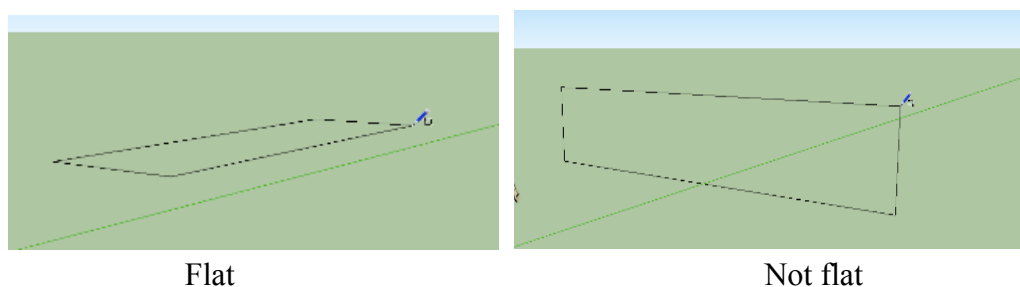


Figure 9: Flat and non-flat shapes

Shortly after, the same pair argued that the difference between the two rectangles in their neighbourhood and the new circle they created is that the former were flat while the latter was ‘wrong’ because “*it is coming towards you*” (Figure 10).

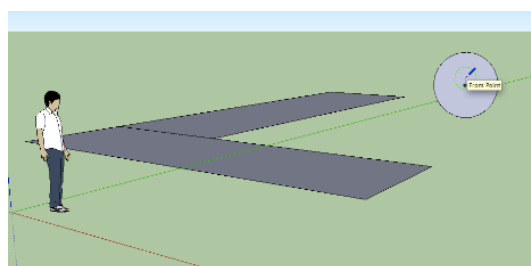


Figure 10: Characterising shapes in different planes

While exploring the Push/Pull tool for extruding their rectangles (see Push/Pull tool description previously), students began to identify which rectangles would go ‘up’ (extruded in the direction parallel to the blue axis) with the Push/Pull tool:

R: Which do you think will go up with the Push/Pull?

S1 and S2: This one, and that one and that one [showing the dark grey rectangles in Figure 11]

[They used the Push/Pull tool to show that indeed they went up.]

R: How do you know which ones they are?

S2: Because they are flat on the ground, they are flat.

S1: The other ones are not flat.

[Excerpt 1, Pair D]

Subsequently, students were shown a different dark grey rectangle, which was not on the ground but was nevertheless going up with the Push/Pull tool:

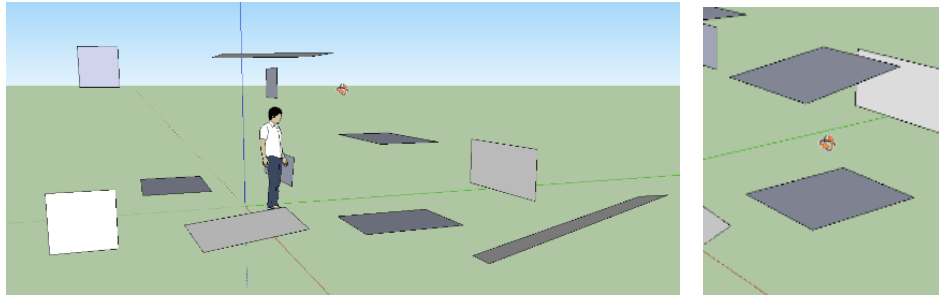


Figure 11: Students identify which rectangles will go ‘up’ with the push/pull tool

R: Is this on the ground? [the dark grey rectangle on the top]

Both: No.

R: But it is going up...

S2: It is because they are going up. In different angles they go in different ways.

R: What is similar between this and this? [the two rectangles above] *They both go up, but this is not on the floor* [the top one]

S1: Oh I got it. It has to be flat, it doesn't matter if it is on the ground or not, it just has to be flat.

R: What about this? [a light grey vertical rectangle shown above] *Is this not flat? It looks flat to me...* [I used the Orbit tool to show them that it is flat]

S1: It is flat but it is like straight like that [showing a vertical direction with his hands] *Instead of being this way* [showing vertical movement with a paper] *it has to be that way* [showing the “flat” position of a paper]
[Excerpt 2, Pair D]

Excerpt 2 illustrates that students acknowledged that shapes could face in ‘different ways’ and they also referred to ‘shapes in different angles’. Between them, Excerpts 1 and 2 demonstrated that the students were able to distinguish shapes within the Google SketchUp environment according to their orientation (see situated abstraction a) in Table 1). They used the situated language ‘shapes were put in different ways’ to describe ‘in different planes’. It is an abstraction in the sense that the students demonstrated many times their ability to apply the rule across the Google SketchUp environment but situated insofar as the language was very couched in the terminology that SketchUp uses. We had no evidence to suggest that the c.n for this situated abstraction went beyond this setting and the students would presumably need to have further experiences before such a development was likely but it is clear, we believe, that this new meaning has a scientific quality that separates it from any of the five meanings previously reported in the literature.

After exploring the tasks and the tools further, students were also able to relate the colour of the shape’s shade to the way it would go with the Push/Pull i.e. “*Maybe white* [surface colour] *is going like the red* [axis], *the light grey like the green* [axis] *and those like the blue* [axis]”. (Red, green and blue here refers to the axes

that the extruding direction of each surface type is parallel to.) Students also defined parallel shapes arguing that if they used the Push/Pull they would go in the same direction.

Furthermore, while using the Circle tool, the idea of extrusion was noted as a transformation from 2D to 3D. Students related the colour of the circle's cursor to the direction they would go with the Push/Pull tool (See description of Push/Pull tool regarding the colour of the cursor for extruding) and they acknowledged the three (or six if the negative directions were included) ways of movement in space. The students exercised control over the process of pushing/pulling 2D shapes to transform them into 3D shapes in order to create the buildings in the neighbourhood they desired. By their actions, they demonstrated a sense of the difference between 2D and 3D and how there was freedom to extrude the shape in three directions (see situated abstraction b) in Table 1). Of course, the heuristic they applied generally throughout the setting was nevertheless restricted to that setting.

While exploring the Line tool to create shapes (see Line tool description previously), students noticed the different colours the lines had arguing that the different colours of the lines follow different directions. Subsequently, their thinking began to change as they recognised that the colour of the lines depended on the colour of the axis of the same direction:

R: Why does it change colour [the line]?

S2: I think it has to do with these lines when it is going in the direction of those lines.

R: Which lines?

S2: These lines [showing the red axis].

R: The red?

S2: The red and the green [the axes].

[Excerpt 3: Pair D]

Pair D also pointed out that some lines were parallel and that even though they did not have the same length, they were of the “*same type of shape*”. Their thinking was further developed when the students used green and blue lines to create a rectangle and their shape was shaded (see the description of the Shaded Surfaces tool for explanation of when a surface drawn is shaded). We then asked them to explain how it was possible for a shape to be shaded, if it was drawn by using the Line tool. (Before they argued that only the special rectangle and circle tools was able to create shaded shapes.) They pointed out that this happened because they used coloured lines to draw the rectangle:

S2: It's because the green and the blue, if you do it by using the green and the blue then...

S1: The red, the green and the blue.

R: Try it.

S2: For example if you do that is black, and black [They drew a rectangle made out of black lines]. Now it's blue, red, red, and blue. It's coloured! [They drew a second rectangle with coloured lines] (Figure 12).

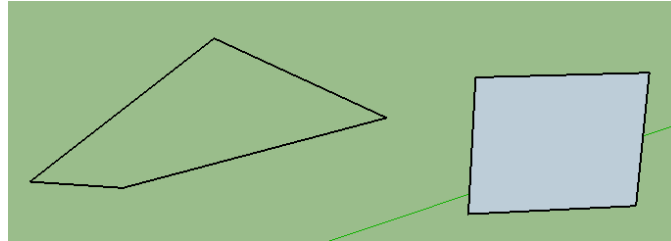


Figure 12: Designing with coloured lines vs. designing with black lines

R: Why does this happen?

S1: Because if it is coloured [the line segments] it would make a coloured shape.

[Excerpt 4: Pair D]

After using the Orbit tool to look around they realised that their first shape (the one with the black lines) was not actually a proper rectangle because “*it wasn't drawn properly*”:

S1: No, not wasn't drawn properly... because when I was watching while S2 was drawing, I saw that when she drew it, it was black and black and black [the lines] while when she did it here it was different colours, it was red and blue.

[Excerpt 5: Pair D]

While drawing their next shape, a rectangle, they used red, blue, red lines and a black one, and thus the shape was not shaded. However, they identified that this happened “*because one of the lines was not coloured*” [Pair D]. Up to this point, students were able to find a relationship between the colour of the lines and whether the shape created was shaded or not. However, in order for a shape to be shaded it had to belong to only one plane (see Shaded Surfaces tool before). Initially, they did not notice that. Therefore, we decided to draw a rectangle by using all the three different colours of lines. The rectangle looked perfect, created by coloured lines but it was not shaded! The students used the Orbit tool to turn around and saw that indeed the rectangle that looked perfect was not even a rectangle:

S1: It's not symmetry! [He then uses the Orbit tool] See! It's bending...

R: Why? They were coloured [the lines]!

S1: I think the reason why, it's because they have to be two same colours. Because that's what happened with that [a previous coloured rectangle], it was red there and blue there.

R: Try to do it.

S1: Blue, red, blue, red! Yes! I've told you! [It turned a shaded colour]

S2: So it has to be two of the same colour.

[Excerpt 6: Pair D]

As shown in Excerpt 6, students concluded that in order for a rectangle to be shaded it had to be made out of only two colours of lines, and noticed the symmetry of the colour of lines within this combination (Figure 13).

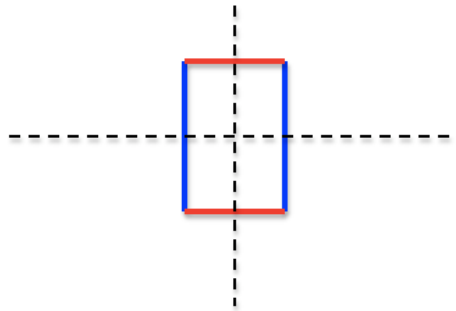


Figure 13: Symmetry in the colour of lines that create a surface

During this thinking-in-change point, we prompted the students to use the Line tool to create a cube and recognized that for the cube they needed to use lines of all the three colours:

R: What colour of lines did you use?

S2: Red and green.

S1: No, red, green and blue to make a cube.

R: So for making a cube you need 3 colours? Because if you remember for creating a rectangle you said you need 2 colours.

S2: No, I think we need 2 colours. I remember when we did the first square I only used green and red...

R: Yes but when you created the cube?

S2: We used all the different colours.

[Excerpt 7, Pair D]

The two/three *different colours* of lines showed the different directions that the line could go. At the same time, it expressed a generalisation of how lines could generate 2D and 3D shapes respectively. By acknowledging that they need two different coloured lines to create a 2D shape and three coloured lines to create a 3D shape and also by talking about the symmetry of those colours (see situated abstractions c) and d) in Table 1), students showed articulations of new meanings which were different than the ones reported previously in the Panorkou (2011) study. Although both situated and shaped by the SketchUp environment, these meanings were applied in a consistent way, which suggested they might be useful resources on which to build in the future.

Additionally, when Pair B was asked to explain how many different coloured lines were going to be used for creating a 1D space/object, they argued: *“Because if you realise for 3D we used all of them [all different colours], but for 2D only 2 of them, and 1D it would be good if we use one.”* Following a similar path, Pair D concluded that 1D would be just a “line on its own.” Likewise, Pairs E and F expressed this generalisation in terms of how many ‘lines’ shapes had in various dimensions (e.g. *“You know 3D has 3 lines, 2D has 2 lines, then maybe 1D has 1 line”* and *“it [1D] would be just one line, 2D two lines, 3D three lines, 4D would have 4 lines.”*).

A new meaning was also evident when students concluded there were two ways for creating a 2D shape and three ways of creating a 3D shape (see situated abstraction e) in Table 1):

R: How can we create a square?

S: Press the rectangle button or drawing with the lines.

R: And a cube?

S: By using the rectangle button and pull up or drawing the rectangle and pull up or you draw the whole thing, or use just rectangles.

[Excerpt 8, Pair F]

Students argued that 2D shapes can be created by the use of lines or rectangles. For 3D shapes, one way was with the Push/Pull tool arguing that *“when it is flat 2D and then you push it you make it, it will turn 3D”* (Pair D). A second way was by using a combination of rectangles (2D shapes) to create a cube. The third way was by arguing that three different coloured lines could combine and result in a ‘proper’ shape.

While drawing the buildings in their neighbourhood, students talked of dimension as positioning, arguing that in 2 dimensions there were 2 ways of positioning things while in 3 dimensions there were 3 ways. Pair E even argued that dimension was position and how things were placed. After the students finished their neighbourhood design, a discussion followed about the 2D and the 3D versions of their neighbourhood. Pair B recognised the difference in moving around in the 2D and 3D neighbourhoods respectively:

S: Because, that’s 2D, you go only left and right and up and down, while if you are in 3D you go everywhere. For example, let’s say a house if it was 2D, you wouldn’t be able to go into the house.

[Excerpt 9, Pair B]

Similarly, Pair D talked about the bug as being able to walk on anything ‘he’ wanted to in a 2D neighbourhood. The use of the Orbit tool to turn their 3D buildings around in order to be able to draw windows at the back was a sign of familiarity of moving in 3D space, showing the flexibility of generating and

positioning shapes in 3D (see situated abstractions f) and g) in Table 1), meanings that were not evident in previous studies (for instance, the five meanings that the original Panorkou study showed and the difficulties in perspective and coding/decoding of representations that Parzysz reported.)

6.1.1 Discussion on ‘freedom to move’

The excerpts above showed examples of thinking-in-change episodes illustrating the process of students’ development of dimensional abstractions during the activity with Google SketchUp. Table 1 shows an overview of students’ situated abstractions on the object/space’s freedom to move, pointing to the ideas of direction, position and orientation.

Table 1: Sample of students’ situated abstractions on ‘freedom to move’

- a) *Shapes can be distinguished according to their orientation (flat or 2D vs. standing up, coming out, kind of 3D but flat shapes; right or wrong shapes).*
- b) *We can create 3D shapes by pushing/pulling 2D shapes in 3 directions.*
- c) *We need two different coloured lines to create a 2D shape. For creating a 3D shape we need to use all three different colours of lines.*
- d) *The colour of the lines used should be symmetrical (as in Figure 13) in order for a ‘proper shape’ (surface) to be created.*
- e) *We can create a cube by using lines and rectangles; We can create a rectangle by using lines.*
- f) *In 2 dimensions there are two ways of positioning things while in 3 dimensions there are 3 ways.*
- g) *In 2D, we go only left and right and up and down, while if we are in 3D we go everywhere.*

The situated abstractions in Table 1 show that students became familiar with working in a 3D environment and, more specifically, generating and positioning 2D and 3D objects in 2D and 3D spaces. In contrast to previous literature on students’ difficulties of decoding representations, situated abstractions *a* and *f* show that students were able to describe how objects were placed in space by drawing on the ideas of position and orientation. Although Parzysz (1988) identified the difficulties of locating objects in different planes, the students in this study showed that they were able to locate objects in different domains. The use of SketchUp made this possible by presenting objects in different shades depending on the plane they are on. Additionally, difficulties of perspective of 2D/3D objects (see for example French 2004; Parzysz 1988 previously) were eliminated because students were able to use the Orbit tool as a key tool for looking at their shapes from different perspectives and noticing coding and decoding errors.

Situated abstraction *b* shows articulation of the extrusion of 2D shapes following the direction that is perpendicular to their surface. By using the Push/Pull tool students experienced movement between plane and space and were able to connect different objects and spaces. Students' thinking was challenged and changed to include the generation of objects/spaces using two or three coloured line segments for creating 2D and 3D objects/spaces respectively (Situated abstractions *c* and *d*). By using the Line tool to create line segments in space, students' experienced the directional, positional and orientational aspects of dimension and also identified the degrees of freedom for generating objects within each space.

These situated abstractions together with situated abstraction *e*, show students' ability of coding (producing) representations of 2D/3D objects/spaces, surely a precursor to a capability for dimensional deconstruction (Duval 2005; Laborde and Laborde 2011). Although these abstractions are situated in terms of language and connectivity to the SketchUp environment, the fact that the students were consistently able to create cubes and lines from lower dimension objects suggests that this resource has potential for abstraction to a wider c.n. that would be recognised as truly mathematical.

The constraints of SketchUp in terms of both creating objects and moving within each domain, led to articulations of dimension as degrees of freedom of an object/space. Two dimensions and three dimensions offer two and three degrees of freedom respectively. This relationship was expressed for both creating shapes (e.g. situated abstraction *e*) and for moving within the space (e.g. situated abstraction *g*).

6.2 Capacity to house

A noteworthy amount of students' situated abstractions (34 out of 190 situated abstractions) referred to dimension as capacity to house. Students' experiences of dimension as capacity seem to reflect the idea of capacity as containment. In this section, we present examples of students' articulations and then present discussion of these ideas.

By using the Orbit tool, students recognised that although a shape would look flat in 3D, it could be disconnected or twisted (see Shaded Surfaces tool previously). For example, Pair B characterised the shapes, which belonged to more than one plane as 'wrong', 'not completed', 'not done properly', and 'wonky' and, when they finally created a shaded surface, they argued that it was because it was 'good', 'right', it was done 'neatly', 'it is perfect' and 'it does not change with orbit'. Students constructed new meanings about representations of shapes and perspective by acknowledging that the additional direction of movement allows the possibility of illusory effects and also the additional freedom throws up the possibility of 2D shapes that are twisted (see situated abstractions *a*) and *b*) in Table 2). This increasingly helped them through the activity with SketchUp to

avoid coding and decoding errors and difficulties in perspective as noted in previous studies (e.g. French 2004; Parzysz 1988). Thus, a key idea about dimension seems to be that, in some sense, it depicts the level of capacity of the space and, more specifically, the containment of that space (edges could be twisted) and the generation of shapes (shapes could be disconnected).

After the students finished their neighbourhood design, a discussion followed about the 2D and the 3D versions of their neighbourhood (Figure 14).

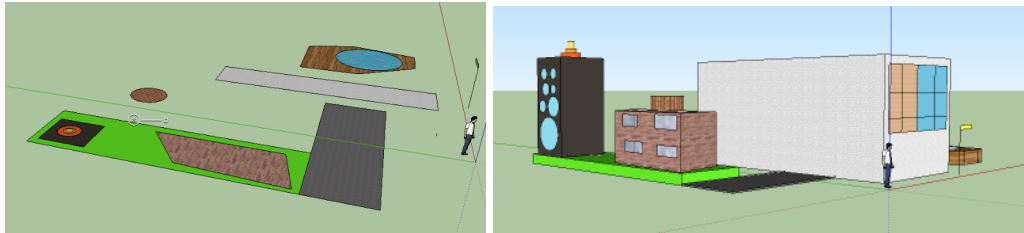


Figure 14: Example of 2D and 3D neighbourhoods constructed by one pair of students

Pair D, for instance, argued that 2D is “flat”, “you can see the flat top”, compared to 3D where “you can actually see a big piece compared to that [the 2D]” and “you can actually see everything inside”. They also pointed out that if we turned the two neighbourhoods into a top view, then they would look the same. They gave the example of a table to support their argument:

S1: Like this table is like that [showing the table] if you look here you see just that [the surface] but if you look underneath you see the whole thing.

R: So what is that table?

S1 & S2: 3D.

R: And what can be 2D in that table?

S1: That [showing the surface]

R: Oh the surface?

Students 1 & 2: Yeah the surface.

S2: The surface is always 2D.

S1: But everything else is 3D.

[Excerpt 10, Pair D]

Students thought of 3 dimensions as offering something more than 2 dimensions, like showing the whole picture of something. In their experiences, 2D was incorporated into 3D; in other words, 2D was ‘housed’ by 3D i.e. surface-whole table, flat top - everything inside (see situated abstractions c) - f) in Table 2). For example, in Excerpt 9 (Pair B) students argued that a 3D object can go inside the 3D neighbourhood and its buildings but it cannot go inside the 2D one. In the Panorkou (2011) study, students identified dimension as a domain incorporating shapes and as a description of those shapes. During activity with SketchUp, students constructed new more holistic meanings about the relationship between

2D and 3D spaces and actively experienced the flexibility of moving between plane and space.

Furthermore, students felt the need to define the planes with which they were working/creating. They gave names to their shapes depending on the colour of the lines they used to create them. For example, Pair A distinguished between the ‘floor level’ shapes which were created with the use of red and green lines from the ‘red-blue’ lined shapes, and this distinction helped them in creating the cube using only lines. Similarly, Pair F argued, “*the floor can be the green thing.*” Pair C even connected the SketchUp domain to their familiar coordinate plane (see description of UK’s national curriculum previously), arguing that the red line could be the Y-axis and the green the X-axis.

It is worth mentioning here that one of the students in Pair D also expressed the idea of the three axes as forming a cube:

S: You see like... this is the bottom [showing the space between the green and the red axes]. It is not a whole cube but a part of it. This is the middle [origin point] and then you go down down, down [across the green line] that’s the edge, that’s like down, that’s the corner [showing the angle created by the blue and the green] and that’s the other line [the red axis].
[Excerpt 11, Pair D]

The student above thought of the axes as forming a cube (object), which included many other objects and of course the whole design of their neighbourhood (space). The idea of the axes forming a cube was also noted by Pair B, pointing out that some shapes were out of the cube and some in the cube. Thus, these statements (see situated abstraction g) in Table 2) were a way of expressing dimension both as a quality of object and space, bringing out the idea of lower dimensional objects create higher dimensional objects/spaces (axes creating a cube) but at the same time showing that higher dimensional objects/spaces (a cube) can incorporate lower dimensional objects/spaces (1D/2D/3D objects). Students also talked about the reverse situations of whether it was possible to have higher dimensional spaces/objects incorporated into lower dimensional spaces/objects or having higher dimensional spaces/objects generating lower dimensional spaces/objects.

6.2.1 Discussion of ‘capacity to house’

The above thinking-in-change episodes are examples of how students’ thinking on dimension had developed through activity to include generalizations based on the object/space’s capacity to house equal and lower dimensional objects/spaces. Table 2 presents a sample of students’ expressions of dimension as capacity.

Table 2: Sample of students’ situated abstractions on ‘capacity to house’

- a) *Shapes that look flat in 3D can be twisted and disconnected.*
- b) *The 'proper' and 'neat' surfaces are shaded compared to the unshaded ones that are 'twisting' and 'not properly done'*
- c) *If we put both 2D and 3D into a top view, they look the same.*
- d) *A surface is 2D while the whole object (to which the surface belongs) is 3D.*
- e) *Lines can be drawn on various surfaces; 2D is included into the 3D*
- f) *A 3D object can go inside the 3D neighbourhood and its buildings but it can't go inside the 2D one.*
- g) *The three axes in SketchUp form a cube.*

Situated abstractions *a*, *b* and *g*, show that by coding representations of surfaces, students were able to generalize how lines can be composed to create a surface in space. These expressions show that the students see objects/spaces as created (generated) in various ways from lower dimensional objects, in particular by lines. These articulations show that object/space was thought of as generated by the objects through the way that they span it. Through their experience with SketchUp, students were able to recognize the different dimensions that are composed for defining a space (e.g. situated abstraction *g*), a finding that contrasts the literature that states that students' judgments of volume is focused on a single dimension (e.g. Piaget 1968; Piaget et al., 1960; Raghbir 1999).

Situated abstractions *c*, *d* and *e*, show that students identified space as incorporating objects, illustrating students' capability of dimensional deconstruction by making connections between the surfaces of the 3D shapes and the 3D shapes themselves. By moving back and forth between plane and space, students were able to recognize the 'interlacing related but different' shapes, a notion identified by Ferrara and Mammana (2014). The use of the Push/Pull tool together with the Orbit tool, helped students in making those connections by allowing them to view shapes dynamically, comparable to their intuitions noted in the literature (e.g. Lehrer et al., 1998). These abstractions, together with situated abstraction *f*, illustrate students' ability to overcome difficulties of coding and decoding of representations and describe objects/spaces in terms of their capacity to house other objects/spaces. Although the above abstractions were situated to the SketchUp context and relationships, we recognize their abstraction across the setting as they extend their meanings beyond naive conceptions and the difficulties noted in the literature.

7 Final discussion

The findings in the study reported here showed that further aspects of dimension can be experienced even by very young students in situated ways through a setting incorporating SketchUp alongside carefully designed tasks and probing questions.

The dimensional tool was an artificial construct (in the sense that the designer created it) that was used to trigger affective responses at the same time as shaping conceptual change. The constraint of working with the dimensional tools perturbed students' experiences and directed them towards looking for patterns and creating generalisations that were in the form of abstractions created *in situ*. As a result, this study confirmed the key significance of the design of the setting in the formation of experience of dimension.

These children working in this setting did not seem very often afflicted by the difficulties reported elsewhere, and, where problems were apparent, the children were able to work through them by trying out ideas in SketchUp using the dimensional tools. More specifically, whereas we might have expected problems in coding (Parzysz 1988) when the children were expressing their ideas about their neighbourhoods, the use of the Line and Shaded Surface tools enabled them to create two and three dimensional shapes and debug them when the feedback did not match their expectations. Similarly, the use of the Orbit tool enabled the children to observe their creations from many perspectives, thus ameliorating decoding problems that might have been expected when viewing a three dimensional shape on a two dimensional screen. A key affordance of the setting seemed to be in the way it supported a flexibility in thinking. For example, the Push/Pull tool helped the children's webbing (or what Ferrara and Mammana, 2014, called 'interlacing') of two and three dimensions, articulated as the situated abstractions reported in Tables 1 and 2.

We recognise that it is not possible to map out all the possible ways of experiencing a phenomenon. Thus, the outcomes consist of a *subset* of all the possible ways of experiencing the phenomenon. Other experiences of dimension are not excluded from this categorisation. The sample of children was not controlled for gender or cultural background, although each of these variables might have an effect in the way a child articulates expressions of dimension. Therefore, it is possible that there are further differences to be observed in how children experience dimension, particularly among children with different backgrounds or who are less competent at school (the children chosen for this study were of upper-middle ability according to their teachers). Furthermore, this study demonstrated that the children were largely oriented towards the way the phenomenon of dimension was presented in an environment like SketchUp and this showed that the setting in which these experiences take place influences their type. Thus, we would argue here that it is possible in other settings to have more types of dimensional experience but the evidence we have presented illustrates two specific categories.

Students' expressions articulated the duality of dimension as expressing both freedom to move and capacity to house. Their articulations of experience showed that they were able to form situated abstractions, which we recognise as 'roots' (Mason et al. 1985) of advanced mathematical ideas such as vector space, and we

speculate that these situated abstractions may act as the starting point for ‘routes’ (Mason et al. 1985) towards those ideas. Students’ situated abstractions of dimension as freedom to move (as listed in Table 1) could be interpreted as roots of the attributes of vectors such as direction, position and orientation. Similarly, students’ situated abstractions of capacity (as listed in Table 2) could be interpreted as roots of the idea of vector space.

Of course, we do not claim that students as young as these understand vectors or vector spaces. The students’ situated abstractions noted above are truly humble expressions of these sophisticated mathematical ideas. It is the authors, and not the students, who make the connections between the students’ humble articulations and these sophisticated mathematical constructs. Nevertheless, the potential significance of such early intuitive learning of situated abstractions of dimension is nicely captured in the following aspirational question:

Research into language acquisition indicates that [...] if a child is not introduced early to other languages, he or she will experience much more difficulty in learning a second tongue. Might the same be true with respect to mathematical perceptions? If we wait until students have developed a great deal of arithmetic sophistication (and a great many misconceptions) before we encourage them to think about solid objects and the interaction between different dimensions, we may be depriving them of the chance to appreciate the full power and scope of geometry (Banchoff 1990).

8 References

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