Doctoral Thesis

# Penalty-free Nitsche method for interface problems in computational mechanics

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*A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy.*

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# Declaration of Authorship

I, Thomas Boiveau confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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# *Abstract*

Faculty of Mathematical & Physical Sciences
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Doctor of Philosophy

**Penalty-free Nitsche method for interface problems in computational mechanics**

by Thomas Boiveau

Nitsche's method is a penalty-based method to enforce weakly the boundary conditions in the finite element method. In this thesis, we consider a penalty-free version of Nitsche's method, we prove its stability and convergence in various frameworks. The idea of the penalty-free method comes from the nonsymmetric version of the Nitsche's method where the penalty parameter has been set to zero; it can be seen as a Lagrange multiplier method, where the Lagrange multiplier has been replaced by the boundary fluxes of the discrete elliptic operator. The main observation is that although coercivity fails, inf-sup stability can be proven. The study focuses on compressible and incompressible elasticity. An unfitted framework is considered when the computational mesh does not fit with the physical domain (fictitious domain method). The penalty-free Nitsche's method is also used to enforce the coupling for interface problems when the mesh fits the interface (nonconforming domain decomposition) or not (unfitted domain decomposition). Fluid structure interaction is also investigated, a new fully discrete implicit scheme is introduced.

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# Chapter 1

# Introduction

## 1.1 Finite element method

### 1.1.1 Principle

A large number of physical phenomena are described by differential equations. Unfortunately these equations are not solvable analytically except in some very simple cases. Over the years, several methods have been developed to approximate solutions of differential equations. In this thesis, we focus on the finite element method, but there are other popular methods, such as the finite difference method, the finite volume method or the spectral method. The classical finite element method is based on a decomposition of the physical domain into non-overlapping elements. An approximation of the solution is constructed from the contributions of each of these elements using piecewise polynomial expansion functions. As an example, we take $\Omega$ a one dimensional domain decomposed in five elements called respectively $K_1, \ldots, K_5$, the lengths of the elements are not necessarily equal. The set of elements defines a mesh of $\Omega$. Figure 1.1 represents an example

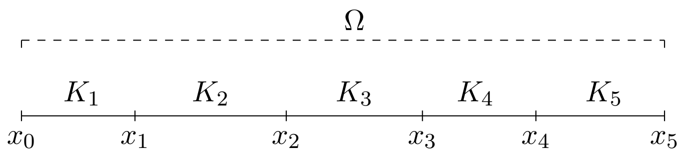



FIGURE 1.1: One dimensional mesh.

of domain decomposed into discrete elements. The functions $\phi_0, \ldots, \phi_5$ are called nodal basis functions, each function $\phi_j$ is defined such that

$$\phi_j(x_i) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

For a first order approximation of the solution, each function $\phi_j$ is continuous and piece-wise linear. Let $h_j = |K_j|$, the function $\phi_j$ is defined as

$$\phi_j(x) = \begin{cases} \frac{x-x_{j-1}}{h_j} & x \in K_j, \\ \frac{x_{j+1}-x}{h_{j+1}} & x \in K_{j+1}, \\ 0 & x \notin \{K_j, K_{j+1}\}. \end{cases}$$

Figure 1.2, represents the basis functions for the first order case on the one dimensional mesh of $\Omega$. The approximated solution of a given problem obtained by the finite element

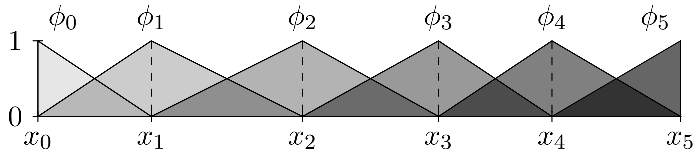



FIGURE 1.2: One dimensional domain, first order basis functions.

method is a linear combination of the nodal basis functions, it can be written as

$$u_h(x) = \sum_{i=0}^{5} u_i \phi_i(x). \tag{1.1}$$

The approximation of the solution $u_h$ is obtained using a finite element formulation.

### 1.1.2 Finite element formulation

Let us consider the Poisson problem with zero Dirichlet boundary condition as a model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with $\partial\Omega$ the boundary of $\Omega$ and $f \in L^2(\Omega)$. Multiplying by a test function $v$ and integrating over the domain, we obtain

$$(-\Delta u, v)_\Omega = (f, v)_\Omega,$$

where $(a, b)_\Omega = \int_\Omega ab \, \mathrm{d}x$. Using integration by parts the problem becomes

$$(\nabla u, \nabla v)_\Omega - \langle \nabla u \cdot n, v \rangle_{\partial\Omega} = (f, v)_\Omega,$$

where $n$ is the outward unit normal vector to the boundary $\partial\Omega$ and $\langle a, b \rangle_{\partial\Omega} = \int_{\partial\Omega} ab \, \mathrm{d}s$. Let us define the bilinear form $a(u, v) = (\nabla u, \nabla v)_\Omega$ and let

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\},$$

we choose $v \in H_0^1(\Omega)$, then we obtain the following weak formulation: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v)_\Omega \qquad \forall v \in H_0^1(\Omega).$$

The boundary term vanishes since $v|_{\partial\Omega} = 0$. This abstract problem is discretised using the Galerkin method. The general principle is to replace the functional spaces by finite dimensional spaces. Let $K$ denote a generic interval in the partitioning of $\Omega$ and $\mathbb{P}_k(K)$ a polynomial of global degree at most $k$ on $K$, then we define the space of piecewise continuous polynomials

$$V_{h,0}^k = \{v_h \in H_0^1(\Omega) : v_h|_K \in \mathbb{P}_k(K) \quad \forall K\} \quad k \geq 1.$$

The finite element formulation of the problem can be written as: find $u_h \in V_{h,0}^k$

$$a(u_h, v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_{h,0}^k. \tag{1.2}$$

### 1.1.3 The linear system

By substituting the decomposition of $u_h$ (1.1) in the formulation (1.2) and choosing $v_h = \phi_j$ for $j = 1, \ldots, 4$, the problem becomes

$$\sum_{i=1}^{4} u_i(\phi_i', \phi_j')_\Omega = (f, \phi_j)_\Omega \qquad j = 1, \ldots, 4.$$

Note that $\phi_j \in V_{h,0}^1$ for $j = 1, \ldots, 4$. Let $A_{ij} = (\phi_i', \phi_j')_\Omega$ and $b_j = (f, \phi_j)_\Omega$. We remark that $\boldsymbol{A}$ is a symmetric matrix, the system becomes

$$\sum_{i=1}^{4} A_{ij} u_i = b_j \qquad j = 1, \ldots, 4,$$

this is equivalent to the linear system

$$\boldsymbol{A}\boldsymbol{u}_h = \boldsymbol{b}, \tag{1.3}$$

where $\boldsymbol{u}_h = (u_1, \ldots, u_4)^T$ and $\boldsymbol{b} = (b_1, \ldots, b_4)^T$. The matrix $\boldsymbol{A}$ is the stiffness matrix,

$$\boldsymbol{A} = \begin{pmatrix} \frac{1}{h_1} + \frac{1}{h_2} & \frac{-1}{h_2} & 0 & 0 \\ \frac{-1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & \frac{-1}{h_3} & 0 \\ 0 & \frac{-1}{h_3} & \frac{1}{h_3} + \frac{1}{h_4} & \frac{-1}{h_4} \\ 0 & 0 & \frac{-1}{h_4} & \frac{1}{h_4} + \frac{1}{h_5} \end{pmatrix}.$$

Finding the approximation $\boldsymbol{u}_h$ is equivalent to solving the linear system (1.3).

## 1.2 Nitsche type methods

Let $\Omega$ be a two dimensional polygonal domain, we consider the following Poisson problem with Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

with $f \in L^2(\Omega)$ and $g \in H^{\frac{3}{2}}(\partial\Omega)$ or $g \in H^{\frac{1}{2}}(\partial\Omega)$.

### 1.2.1 Weak imposition of Dirichlet boundary conditions

The standard method to impose a homogeneous Dirichlet boundary condition in finite element is to eliminate the degrees of freedom associated with the boundary in the discrete linear system. The inhomogeneous case is slightly more technical but follows the same principle introducing a lifting operator at the boundary (see for example [46]). In other words the boundary condition is imposed in the finite element space as in Section 1.1.2. Another approach is to impose weakly the boundary condition, several techniques can be used, here we give a brief description of the main methods.

- The penalty method proposed by Babuška [6, 11] considers a penalised version of the problem (1.4)

$$\begin{aligned} -\Delta u_\epsilon &= f \quad \text{in } \Omega, \\ \epsilon^{-1}(u_\epsilon - g) + \nabla u_\epsilon \cdot n &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

  with $\epsilon > 0$. The corresponding formulation is: find $u_\epsilon \in H^1(\Omega)$ such that

$$(\nabla u_\epsilon, \nabla v)_\Omega + \epsilon^{-1}\langle u_\epsilon, v\rangle_{\partial\Omega} = (f, v)_\Omega + \epsilon^{-1}\langle g, v\rangle_{\partial\Omega} \quad \forall v \in H^1(\Omega).$$

  The weak imposition is done via the terms $\epsilon^{-1}\langle u_\epsilon, v\rangle_{\partial\Omega}$ and $\epsilon^{-1}\langle g, v\rangle_{\partial\Omega}$.

- The Lagrange multiplier method also introduced by Babuška [5, 23, 96, 97, 100] requires the use of Lagrange multipliers in the formulation: find $(u, \lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega + \langle \lambda, v\rangle_{\partial\Omega} + \langle \mu, u\rangle_{\partial\Omega} = (f, v)_\Omega + \langle \mu, g\rangle_{\partial\Omega} \quad \forall (v, \mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega). \tag{1.5}$$

  Note that the boundary flux $\nabla u \cdot n$ has been replaced by $-\lambda$. The terms $\langle \mu, u\rangle_{\partial\Omega}$ and $\langle \mu, g\rangle_{\partial\Omega}$ are enforcing weakly the boundary condition.

- The method introduced by Barbosa and Hughes [78, 8, 9, 100] that considers an alternative of (1.5) by introducing the following additional term at the discrete level

$$\gamma h\langle \lambda_h + \nabla u_h \cdot n, \mu_h + \nabla v_h \cdot n\rangle_{\partial\Omega},$$

with $\gamma$ the stabilisation parameter and $h$ the parameter associated to the space discretisation (maximal element diameter).

- Nitsche's method [92, 4] is a consistent penalty based method. Let $V_h$ be the $H^1$-conforming finite element space fitted to $\Omega$ where $h$ is the maximal element diameter. The corresponding finite element formulation is written as: find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h, \tag{1.6}$$

where the linear forms are defined by

$$\begin{aligned} A_h(u_h, v_h) &= (\nabla u_h, \nabla v_h)_\Omega - \langle \nabla u_h \cdot n, v_h \rangle_{\partial\Omega} - \langle \nabla v_h \cdot n, u_h \rangle_{\partial\Omega} + \gamma \langle h^{-1} u_h, v_h \rangle_{\partial\Omega}, \\ L_h(v_h) &= (f, v_h)_\Omega - \langle \nabla v_h \cdot n, g \rangle_{\partial\Omega} + \gamma \langle h^{-1} g, v_h \rangle_{\partial\Omega}, \end{aligned}$$

where $\gamma > 0$. In the bilinear form $A_h$ the two first terms are classically obtained by integration by parts, the third term preserves symmetry and the fourth term ensures the coercivity for $\gamma$ big enough. The corresponding terms are added to the linear form $L_h$ to enforce weakly the boundary condition.

### 1.2.2 The penalty-free Nitsche method

Pursuing the idea to relax the constraint on $\gamma$ Freund and Stenberg suggested a nonsymmetric version of Nitsche's method [56]. The formulation (1.6) is modified such that

$$\begin{aligned} A_h(u_h, v_h) &= (\nabla u_h, \nabla v_h)_\Omega - \langle \nabla u_h \cdot n, v_h \rangle_{\partial\Omega} + \langle \nabla v_h \cdot n, u_h \rangle_{\partial\Omega} + \gamma \langle h^{-1} u_h, v_h \rangle_{\partial\Omega}, \\ L_h(v_h) &= (f, v_h)_\Omega + \langle \nabla v_h \cdot n, g \rangle_{\partial\Omega} + \gamma \langle h^{-1} g, v_h \rangle_{\partial\Omega}. \end{aligned}$$

The only difference compared to the classical Nitsche's method is that the terms $\langle \nabla v_h \cdot n, u_h \rangle_{\partial\Omega}$ and $\langle \nabla v_h \cdot n, g \rangle_{\partial\Omega}$ are added instead of being subtracted. In this case the coercivity is straightforward to show as

$$A_h(u_h, u_h) = (\nabla u_h, \nabla u_h)_\Omega + \gamma \langle h^{-1} u_h, u_h \rangle_{\partial\Omega} \geq C(\|\nabla u_h\|_{0,\Omega}^2 + h^{-1} \|u_h\|_{0,\partial\Omega}^2),$$

where $C$ is a positive constant. The advantage of the nonsymmetric version is that no lower bound has to be respected for the penalty parameter to ensure coercivity, it only needs to be strictly larger than zero. The symmetric and nonsymmetric versions of Nitsche's method were further discussed by Hughes and co-workers in [77] where the possibility of using the nonsymmetric version with $\gamma = 0$ was mentioned. Penalty-free nonsymmetric methods have indeed been advocated for the discontinuous Galerkin method [93, 81, 63, 40]. In [26], Burman proved that the nonsymmetric Nitsche's method is stable without penalty for scalar elliptic problems. The linear forms associated to the penalty-free Nitsche's method are such that

$$A_h(u_h, v_h) = (\nabla u_h, \nabla v_h)_\Omega - \langle \nabla u_h \cdot n, v_h \rangle_{\partial\Omega} + \langle \nabla v_h \cdot n, u_h \rangle_{\partial\Omega},$$
$$L_h(v_h) = (f, v_h)_\Omega + \langle \nabla v_h \cdot n, g \rangle_{\partial\Omega}.$$

The main observation for this method is that although coercivity fails when the penalty parameter is set to zero, the formulation can be proven to be inf-sup stable. It leads to a method that is stable without any unknown parameter and without introducing additional degrees of freedom. In terms of solvers, some solvers cannot be employed (ex: Crout, Cholesky, Conjugate Gradients) due to the nonsymmetry of the method, it is also known that in the case of inf-sup stable formulations for saddle point systems Krylov-Schur solver fails to converge [88]. Optimal convergence of the error can be shown in the $H^1$-norm, however the lack of adjoint-consistency of the nonsymmetric formulation leads to a suboptimality of order $\mathcal{O}(h^{\frac{1}{2}})$ for the $L^2$-error. By looking at (1.5) we remark that the nonsymmetric version of the Nitsche's method without penalty can be seen as a Lagrange multiplier method where the Lagrange multiplier has been replaced by the boundary fluxes of the discrete elliptic operator.

### 1.2.3 A brief comparison

In this section, we compare the different versions of Nitsche's method for the Poisson problem (1.4). Let us consider the following manufactured solution that is used for the computations

$$u = \sin(\pi x)\sin(2\pi y).$$

We approximate this solution using the three versions of Nitsche's method presented in the previous sections and compare the $L^2$ and $H^1$-errors. First we compare the slopes of convergence, we choose arbitrarily a penalty parameter for both penalised formulations and we obtain Figure 1.3. The slopes of convergence of the $L^2$-error shows an optimal

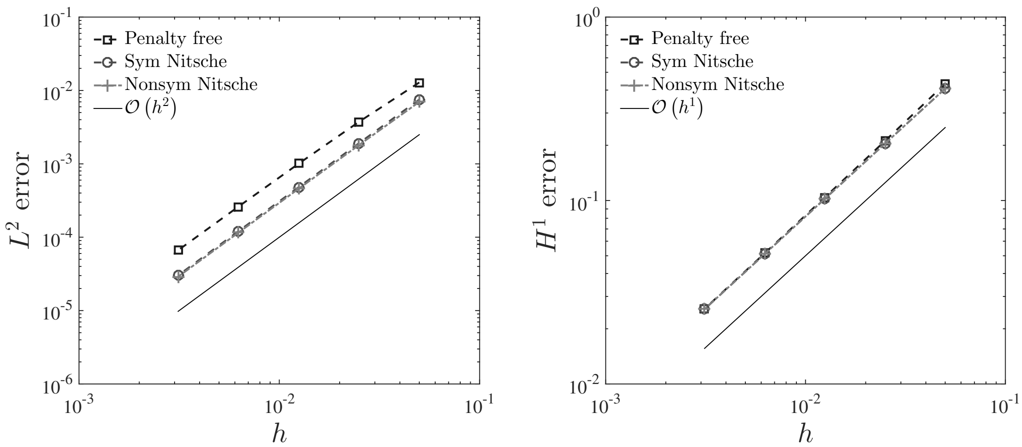



FIGURE 1.3: Comparison of Nitsche's methods, piecewise affine approximation, error versus the maximal element diameter, $\gamma = 10$.

order of convergence $\mathcal{O}(h^2)$ for the three methods. The formulations that consider a penalty parameter (symmetric and nonsymmetric) give very similar results, in fact the difference between the two slopes is negligible. The penalty-free method gives an error slightly bigger than the penalised methods. The $H^1$-error shows an optimal order of convergence for each method. The difference of the error between the three methods is negligible. In Figure 1.3 the penalty parameter is considered high enough to ensure stability for both penalised versions, it is therefore interesting to study the influence of the penalty parameter for a fixed mesh (Figure 1.4). The $L^2$ and $H^1$-errors of the penalty-

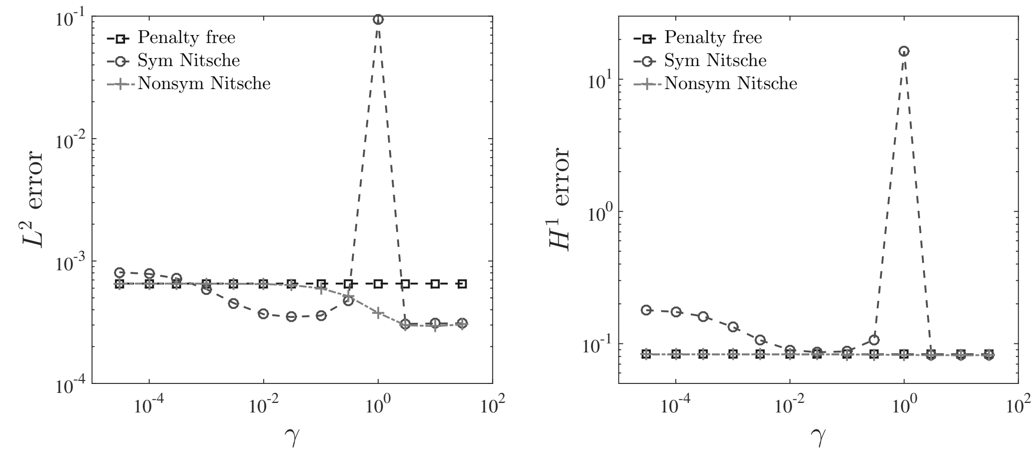



FIGURE 1.4: Comparison of Nitsche's methods, piecewise affine approximation, error versus the penalty term $\gamma$.

free scheme show a constant error because there is no penalty parameter involved by definition. Both $L^2$ and $H^1$-errors of the symmetric formulation reach a peak for $\gamma = 1$ that is due to an eigenvalue of the finite element matrix. For $\gamma \geq 10$ both penalised methods gives the same error (case of Figure 1.3). For $\gamma \in [10^{-3}, 10^{-1}]$ the $L^2$-error given by the symmetric scheme is smaller than for the other schemes. The penalised nonsymmetric scheme converges to the penalty-free scheme as $\gamma$ decreases. For the $H^1$-error the difference between the penalty-free scheme and the nonsymmetric penalised scheme is very small. For $\gamma \geq 10$ and $\gamma \in [10^{-2}, 10^{-1}]$ the $H^1$-error given by the symmetric scheme is very similar to the other methods.

### 1.2.4 First proof of stability

In this section we show the inf-sup condition for a formulation that involves the penalty-free Nitsche's method for a one dimensional case. We aim to introduce the general concepts that will be used to prove the stability of the penalty-free schemes that will be studied in the four following chapters. In what follows, we will consider the usual Sobolev spaces $H^s(\omega)$ with $(s \geq 0)$, with norm $\|\cdot\|_{s,\omega}$ and semi-norm $|\cdot|_{s,\omega}$, details are provided in Appendix A. Let $\Omega = [0,1]$ be a one dimensional domain, the Poisson problem with Dirichlet boundary conditions (1.4) can be rewritten as

$$\begin{aligned} -u'' &= f \quad \text{in } [0,1], \\ u(0) = u(1) &= g. \end{aligned}$$

The domain $\Omega$ is partitioned into $n > 3$ intervals $K_1, \dots, K_n$ of length $h = |K_i|$ for all $i \in 1, \dots, n$, we introduce the finite element space

$$V_{h,1\mathrm{D}}^k = \{v_h \in H^1(\Omega) : v_h|_{K_i} \in \mathbb{P}_k(K_i), \forall i \in [1,n]\}.$$

The penalty-free formulation is written as: find $u_h \in V_{h,1\mathrm{D}}^k$ such that

$$(u_h', v_h')_{[0,1]} - [u_h' v_h]_0^1 + [u_h v_h']_0^1 = (f, v_h)_{[0,1]} + [g v_h']_0^1 \qquad v_h \in V_{h,1\mathrm{D}}^k.$$

Let us define the bilinear form

$$A_h(u_h, v_h) = (u_h', v_h')_{[0,1]} - [u_h' v_h]_0^1 + [u_h v_h']_0^1.$$

We choose the function $v_h$ such that $v_h = u_h + \alpha v_\Gamma$ and we define $v_\Gamma$ to be zero everywhere except for the elements that have one vertex on the boundary. Let

$$\begin{aligned} v_\Gamma(0) &= u_h(0), \qquad & v_\Gamma'(0) &= -h^{-1} u_h(0), \\ v_\Gamma(1) &= u_h(1), \qquad & v_\Gamma'(1) &= h^{-1} u_h(1). \end{aligned} \tag{1.7}$$

By applying the definition of $v_h$ we obtain

$$A_h(u_h, v_h) = A_h(u_h, u_h) + \alpha A_h(u_h, v_\Gamma),$$

clearly we have

$$A_h(u_h, u_h) = \|u_h'\|_{0,\Omega}^2.$$

Also, we can write

$$A_h(u_h, v_\Gamma) = (u_h', v_\Gamma')_\Omega - u_h'(1) v_\Gamma(1) + u_h'(0) v_\Gamma(0) + u_h(1) v_\Gamma'(1) - u_h(0) v_\Gamma'(0).$$

Using the Cauchy Schwarz inequality and the definition on $v_\Gamma$ we obtain

$$\begin{aligned} (u_h', v_\Gamma')_\Omega \le \|u_h'\|_{0,\Omega} \|v_\Gamma'\|_{0,\Omega} &\le \|u_h'\|_{0,\Omega} (h^{-1} u_h(1)^2 + h^{-1} u_h(0)^2)^{\frac{1}{2}} \\ &\le \|u_h'\|_{0,\Omega} h^{-\frac{1}{2}} \|u_h\|_{0,\partial\Omega}. \end{aligned}$$

The trace inequality and the inverse inequality (these two inequalities will be introduced in the next chapter by Lemmas 2.0.1 and 2.0.2) tell us that

$$\|u_h'\|_{0,\partial\Omega} \lesssim h^{-\frac{1}{2}} \|u_h'\|_{0,\Omega}$$

using this result, the consistency term becomes

$$u_h'(1)v_\Gamma(1) - u_h'(0)v_\Gamma(0) = u_h'(1)u_h(1) - u_h'(0)u_h(0) \leq \|u_h'\|_{0,\partial\Omega}\|u_h\|_{0,\partial\Omega}$$
$$\lesssim \|u_h'\|_{0,\Omega}h^{-\frac{1}{2}}\|u_h\|_{0,\partial\Omega}.$$

The antisymmetric Nitsche term can be expressed as

$$u_h(1)v_\Gamma'(1) - u_h(0)v_\Gamma'(0) = h^{-1}u_h(1)^2 + h^{-1}u_h(0)^2 = h^{-1}\|u_h\|^2_{0,\partial\Omega}.$$

Then we obtain the following using the Young's inequality ($ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$ with $\epsilon > 0$)

$$A_h(u_h, v_h) \gtrsim (1-\epsilon)\|u_h'\|^2_{0,\Omega} + \alpha\Big(1 - \frac{\alpha}{\epsilon}\Big)h^{-1}\|u_h\|^2_{0,\partial\Omega} \gtrsim \|u_h\|^2_{1,h}, \tag{1.8}$$

note that for $\alpha$ and $\epsilon$ well chosen the lower bound of $A_h$ is in fact positive. The norm $\|\cdot\|_{1,h}$ is defined as

$$\|w\|^2_{1,h} = \|\nabla w\|^2_{0,\Omega} + h^{-1}\|w\|^2_{0,\partial\Omega}, \tag{1.9}$$

in the one dimensional framework we have $\nabla w = w'$. Also, using similar arguments as previously we obtain

$$\|v_h\|^2_{1,h} \lesssim \|u_h\|^2_{1,h} + \alpha(\|v_\Gamma'\|^2_{0,\Omega} + h^{-1}\|v_\Gamma\|^2_{0,\partial\Omega}) \lesssim \|u_h\|^2_{1,h} + h^{-1}\|u_h\|^2_{0,\partial\Omega} \lesssim \|u_h\|^2_{1,h},$$

combining this result with (1.8) we get the inf-sup condition

$$\beta\|u_h\|_{1,h} \leq \sup_{v_h \in V^k_{h,1\mathrm{D}}} \frac{A_h(u_h, v_h)}{\|v_h\|_{1,h}}.$$

For the proof of the two dimensional case, the value of $v_\Gamma$ at the boundary (1.7) is replaced by an average of $u_h$ over patches of boundary elements; this leads to additional terms that have to be controlled but the principle of the proof remains the same. In this thesis we consider the two dimensional case in order to reduce the amount of technicalities in the theoretical proofs.

## 1.3 Motivations

In the previous sections we have introduced the penalty-free Nitsche's method as a way to impose the boundary conditions weakly when the triangulation fits the physical domain, this case is considered in Chapter 2. Nitsche's method may also be used in several other configurations.

### 1.3.1 Fictitious domain method

Mesh generation is an important challenge in computational mechanics, in fact for complex geometries this can be highly nontrivial. In some cases for time dependent problems, such as a solid body embedded in a flow, the geometry of the problem may be modified for each time step which implies that the mesh must be modified at each time step. The main idea of the fictitious domain method [59, 60, 67, 3, 74, 36, 37, 29] is to relax the constraint that imposes the mesh to fit with the physical domain. In fact the principle of the fictitious domain approach is to embed the physical domain in a mesh that is easy to generate, the elements do not need to match with the boundary as shown in Figure 1.5. In the early developments of fictitious domain [59], the method was faced with the

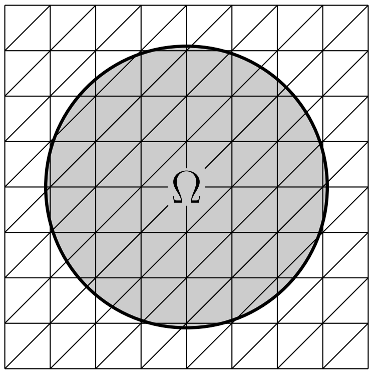



FIGURE 1.5: Fictitious domain, $\Omega$ is embedded in a background mesh.

choice of either integrating the equations over the whole computational mesh including the nonphysical part, or only integrate inside the physical domain. In the first case, the method is robust but inaccurate, the second approach is accurate but can generate bad conditioning of the system matrix depending on how the boundary crosses the mesh. The ghost penalty [25] has been introduced to avoid this problem, this simple trick improves robustness without loss of accuracy. The fictitious domain approach is considered in Chapter 3 for the Poisson problem [37], but also for compressible and incompressible elasticity [13, 38, 89].

### 1.3.2 Domain decomposition

In domain decomposition the physical domain is partitioned into multiple subdomains, in this thesis we are interested in the coupling at the interface between two subdomains, we will therefore split the physical domain into exactly two domains $\Omega_1$ and $\Omega_2$ with a common interface. A first approach is to consider $\Omega_1$ and $\Omega_2$ meshed independently this case is commonly called nonconforming domain decomposition (see Figure 1.6). To handle this configuration, iterative procedures can be considered using the standard Schwarz alternating method [87]. Another approach is to consider Lagrange multiplier [86, 17, 84] for the coupling at the interface, here we consider the Nitsche's method [16, 70]. The Nitsche's method has been applied to nonconforming domain decomposition with its symmetric and nonsymmetric version in [16] for the Poisson problem. The method has been extended using a weighted average of the fluxes at the interface for the advection-diffusion-reaction problem in [41]. Several difficulties can be handled by taking

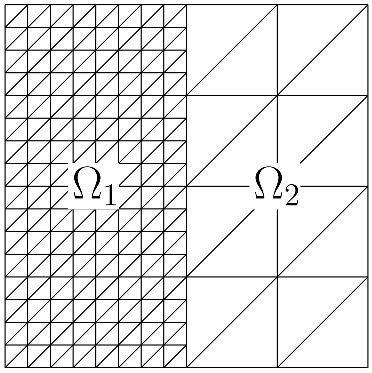



FIGURE 1.6: Nonconforming domain decomposition, $\Omega_1$ and $\Omega_2$ are meshed independently.

the right choice of weights [47, 10, 41]. In Chapter 4 we consider domain decomposition with discontinuous material parameters for the Poisson problem, the study is extended to compressible elasticity [57, 61] and incompressible elasticity [14].

### 1.3.3 Unfitted domain decomposition

The domain decomposition study is extended to the unfitted framework. The physical domain is decomposed in two subdomains as in the previous section, however the specificity here is that both subdomains are meshed with one triangulation, the interface between the two subdomains is not necessarily fitting with the elements of the mesh (see Figure 1.7). As for nonconforming domain decomposition, the Lagrange multiplier

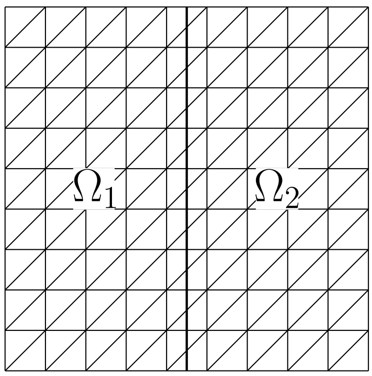



FIGURE 1.7: Unfitted domain decomposition, $\Omega_1$ and $\Omega_2$ and the computational mesh.

method may be used to handle the coupling at the interface [73], here we are interested in using the Nitsche's method [67, 69] as we want to investigate the penalty-free Nitsche's method. Using the tools introduced in Chapter 3, the domain decomposition approach of Chapter 4 is transposed to the unfitted domain decomposition framework of Chapter 5. Compressible elasticity is considered [13, 68] as well as incompressible elasticity [43, 94, 90, 65, 72].

### 1.3.4 Fluid structure interaction

Nitsche type methods can be used in a fluid-structure interaction framework [71, 91] to handle the coupling at the fluid-solid interface. In this thesis we consider the special case of a viscous incompressible fluid and an elastic structure when the fluid and solid densities are close (i.e. the fluid added-mass acting on the structure is strong) [50, 30]. In such configuration, explicit couplings [48, 83, 42] are known to produce numerical instabilities. Alternatively, semi-implicit [50, 51, 1] and implicit [104, 58, 53] approaches can be considered to handle these instabilities. In [30] a stabilised explicit coupling scheme for fluid-structure interaction based on Nitsche's method has been introduced, in order to get optimal accuracy, a defect correction approach is used [101]. An analysis of this explicit scheme with defect correction is done in [32]. The schemes considered in these contributions rely on the classical version of the Nitsche's method. In [33] the penalty-free Nitsche's method has been investigated, in this article numerical observations show that the defect correction is no longer needed to recover optimal accuracy in the penalty-free case. Figure 1.8 is extracted from [33], it shows the displacement energy norm error for both classical Nitsche and penalty-free Nitsche without defect correction; the penalty-free scheme shows optimal convergence which is not the case for the clas-

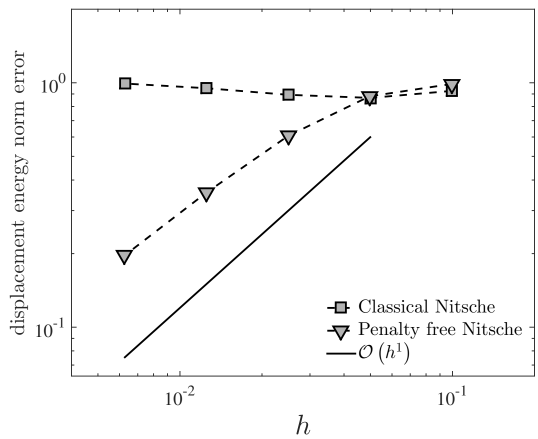



FIGURE 1.8: Extracted from [33], error versus the meshsize, classical Nitsche compared to penalty-free Nitsche.

sical scheme. In [34] the study has been extended to the unfitted framework, explicit and implicit strategies are considered including a convergence analysis for the implicit scheme for the classical Nitsche case. In Chapter 6 we propose a fully discrete implicit scheme based on the penalty-free Nitsche's method. Given the convergence properties observed numerically for the explicit scheme in the Figure 1.8, it is therefore interesting to study the penalty-free version of the Nitsche based schemes to understand its stability mechanisms in general.

# Chapter 2

# Weak imposition of boundary conditions

In this chapter, we use the penalty-free Nitsche's method to impose weakly the boundary conditions when the computational mesh is fitted to the physical domain. The study of the Poisson problem is used as an extended introduction, in fact this case has already been considered in [26]. The study is extended to compressible and incompressible elasticity following [20]. Let $\Omega$ be a convex bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Let $\{\Gamma_i\}_i$ be the sides to the polygonal domain $\Omega$ such that $\partial\Omega = \cup_i \Gamma_i$. The set $\{\mathcal{T}_h\}_h$ defines a family of quasi-uniform and shape regular triangulations fitted to $\Omega$. In a generic sense we define $K$ as the triangles in a triangulation $\mathcal{T}_h$ and $h_K = \operatorname{diam}(K)$ is the diameter of $K$. We define the shape regularity as the existence of a constant $c_\rho \in \mathbb{R}_+^*$ for the family of triangulations such that, with $\rho_K$ the radius of the largest circle inscribed in an element $K$, there holds

$$\frac{h_K}{\rho_K} \leq c_\rho \quad \forall K \in \mathcal{T}_h. \tag{2.1}$$

We define $h = \max_{K\in\mathcal{T}_h} h_K$ as the mesh parameter for a given triangulation $\mathcal{T}_h$. $\mathbb{P}_k(K)$ defines the space of all polynomials of degree less than or equal to $k$ on the element $K$. We define $V_h^k$ the finite element space of continuous piecewise polynomial functions

$$V_h^k = \{v_h \in H^1(\Omega) : v_h|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\} \quad k \geq 1.$$

The vector $n$ denotes the outward unit normal to the boundary $\partial\Omega$ and $\tau$ denotes the tangent unit vector to $\partial\Omega$. $C$ is used as a generic positive constant that may change at each occurrence, $C$ is independent of the meshsize and physical parameters, it only depends on the shape regularity of the mesh considered, given the assumptions made above we have $C = \mathcal{O}(1)$. We will use the notation $a \lesssim b$ for $a \leq Cb$. The following results will be useful in the analysis, proofs can be found in [21].

**Lemma 2.0.1.** *There exists $C_T \in \mathbb{R}_+$ such that for all $w \in H^1(K)$ and for all $K \in \mathcal{T}_h$, the trace inequality holds*

$$\|w\|_{0,\partial K} \leq C_T(h_K^{-\frac{1}{2}}\|w\|_{0,K} + h_K^{\frac{1}{2}}\|\nabla w\|_{0,K}).$$

**Lemma 2.0.2.** *There exists $C_I \in \mathbb{R}_+$ such that for all $w_h \in \mathbb{P}_k(K)$ and for all $K \in \mathcal{T}_h$, the inverse inequality holds*

$$\|\nabla w_h\|_{0,K} \leq C_I h_K^{-1} \|w_h\|_{0,K}.$$

## 2.1 Poisson problem

The Poisson problem with Dirichlet boundary conditions is given by

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

with $f \in L^2(\Omega)$ and $g \in H^{\frac{3}{2}}(\partial\Omega)$. The weak formulation of the problem can be expressed as: find $u \in H^1_g(\Omega)$ such that

$$a(u, v) = (f, v)_\Omega \qquad \forall v \in H^1_0(\Omega),$$

with

$$a(u, v) = (\nabla u, \nabla v)_\Omega,$$

and $H^1_g(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = g\}$. After suitable suitable modifications in order to handle the inhomogeneous Dirichlet boundary conditions, the well-posedness of this problem follows from the Lax-Milgram Lemma (see appendix C), we also have the elliptic regularity estimate [64]

$$\|u\|_{2,\Omega} \lesssim \|f\|_{0,\Omega} + \|g\|_{\frac{3}{2},\partial\Omega}. \tag{2.3}$$

The finite element formulation obtained using the penalty-free Nitsche's method reads: find $u_h \in V_h^k$ such that

$$A_h(u_h, v_h) = L_h(v_h) \qquad \forall v_h \in V_h^k, \tag{2.4}$$

where

$$\begin{aligned} A_h(u_h, v_h) &= a(u_h, v_h) - \langle \nabla u_h \cdot n, v_h \rangle_{\partial\Omega} + \langle \nabla v_h \cdot n, u_h \rangle_{\partial\Omega}, \\ L_h(v_h) &= (f, v_h)_\Omega + \langle \nabla v_h \cdot n, g \rangle_{\partial\Omega}. \end{aligned}$$

### 2.1.1 A useful boundary mortaring

Anticipating the inf-sup analysis we introduce patches of boundary elements as in [26, 20] for the construction of special functions in the finite element space $V_h^k$ that will serve for the proof of stability. We regroup the boundary elements in closed, disjoint patches $P_j$ with boundary $\partial P_j$, $j = 1, ..., N_p$. $N_p$ defines the total number of patches. The boundary elements are the elements with either a face or a vertex on the boundary. Every boundary element is a member of exactly one patch $P_j$. The number of elements necessary in each patch is always at least two and upper bounded by a constant depending only on the shape regularity parameter $c_\rho$. Let $F_j = \partial P_j \cap \partial\Omega$, we assume that every $\Gamma_i$ is partitioned by at least one $F_j$. We assume that each $F_j$ has at least one inner node but in some case they may need up to three inner nodes in the case $F_j = \Gamma_i$. Let $P = \cup_j P_j$. For each $F_j$ there exists two positive constants $c_1$, $c_2$ such that for all $j$

$$c_1 h \leq \text{meas}(F_j) \leq c_2 h. \tag{2.5}$$

Figure 2.1 gives a representation of a patch as defined above with four inner nodes. Let $\chi_j \in V_h^1$ be defined for each node $x_i \in \mathcal{T}_h$ such that for each patch $P_j$

$$\chi_j(x_i) = \begin{cases} 0 & \text{for} \quad x_i \in \Omega \backslash \mathring{F}_j \\ 0 & \text{for} \quad x_i \in K \text{ such that } K \text{ has all its vertices on } \partial\Omega \\ 1 & \text{for} \quad x_i \in \mathring{F}_j, \end{cases}$$

with $i = 1, \ldots, N_n$. Here $N_n$ is the number of nodes in the triangulation $\mathcal{T}_h$ and $\mathring{F}_j$ defines the interior of the face $F_j$.

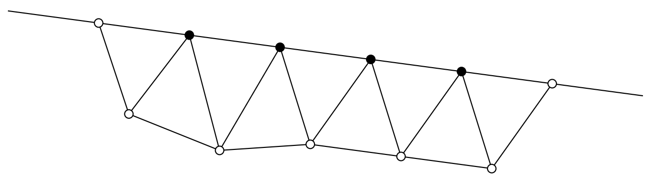


FIGURE 2.1: Example of a patch $P_j$, the function $\chi_j$ is equal to 0 on the nonfilled nodes, 1 on the filled nodes.

**Lemma 2.1.1.** *Assume that, for all $P_j$, $\partial P_j$ meets $\partial\Omega$ at an angle smaller than $\frac{\pi}{2}$. For any given vector $(r_j)_{j=1}^{N_p} \in \mathbb{R}^{N_p}$ there exists $\varphi_r \in V_h^1$ such that for all $1 \leq j \leq N_p$ there holds*

$$\text{meas}(F_j)^{-1} \int_{F_j} \nabla\varphi_r \cdot n \, \mathrm{d}s = r_j, \tag{2.6}$$

*and, if $r(x) : \partial\Omega \to \mathbb{R}$ denotes the function such that $r|_{F_i} = r_i$,*

$$\|\varphi_r\|_{1,h} \lesssim \Big( \sum_{j=1}^{N_p} h \|r\|_{0,F_j}^2 \Big)^{\frac{1}{2}} \tag{2.7}$$

*Proof.* Let

$$\Xi_j = \text{meas}(F_j)^{-1} \int_{F_j} \nabla\chi_j \cdot n \, \mathrm{d}s.$$

The normalised function $\varphi_j$ is defined such that

$$\varphi_j = \Xi_j^{-1} \chi_j.$$

The following lower bound that holds uniformly in $j$ and $h$ tells us that $\varphi_j$ is well defined

$$0 < C_\Xi \leq \Xi_j h.$$

The constant $C_\Xi$ depends only on the local geometry of the patches $P_j$. By definition there holds

$$\text{meas}(F_j)^{-1} \int_{F_j} \nabla \varphi_j \cdot n \, \mathrm{d}s = 1, \tag{2.8}$$

using the inverse inequality of Lemma 2.0.2 we obtain

$$\|\nabla \varphi_j\|_{0,\Omega} \lesssim h^{-1} \Xi_j^{-1} \|\chi_j\|_{0,P_j} \lesssim h^{-1} \Xi_j^{-1} \text{meas}(P_j)^{\frac{1}{2}} \lesssim C_\Xi^{-1} h. \tag{2.9}$$

Let

$$\varphi_r = \sum_{j=1}^{N_p} r_j \varphi_j,$$

then condition (2.6) is verified considering (2.8). The upper bound (2.7) is obtained using (2.9), (2.5) and

$$h^{-1} \|\varphi_r\|^2_{0,\partial\Omega} = \sum_{j=1}^{N_p} h^{-1} \|r_j \varphi_j\|^2_{0,F_j} \lesssim \sum_{j=1}^{N_p} h^{-1} r_j^2 \Xi_j^{-2} \|\chi_j\|^2_{0,F_j} \lesssim C_\Xi^{-2} \sum_{j=1}^{N_p} h \|r\|^2_{0,F_j}.$$

□

**Remark 2.1.1.** *In the previous Lemma the assumption that every patch meets the domain boundary at an angle smaller than $\pi/2$ is very restrictive, however for larger angles the Lemma can always be made to hold by making the patches wider. We also note that under the shape regularity defined above there is an upper limit on how big this angle can become. The conclusion of these two points is that the analysis is always valid for fine enough meshes under the shape regularity assumptions.*

The projection of $u$ on constant functions on the interval $I$ is defined as

$$\overline{u}^I = \text{meas}(I)^{-1} \int_I u \, \mathrm{d}s. \tag{2.10}$$

**Lemma 2.1.2.** *For any function $u_h \in V_h^k$ the following inequalities are true*

$$h \|\nabla u_h \cdot \tau\|_{0,F_j} \gtrsim \|u_h - \overline{u_h}^{F_j}\|_{0,F_j}. \tag{2.11}$$

$$h^{-\frac{1}{2}} \|\overline{u_h}^{F_j}\|_{0,F_j} \geq h^{-\frac{1}{2}} \|u_h\|_{0,F_j} - C' \|\nabla u_h\|_{0,P_j}. \tag{2.12}$$

*Proof.* The inequality (2.11) can be shown by defining $x_0 \in F_j$ such that $(u_h - \overline{u_h}^{F_j})(x_0) = 0$, then for any $x \in F_j$

$$(u_h - \overline{u_h}^{F_j})(x) = \int_{x_0}^{x} \nabla u_h \cdot \tau \, \mathrm{d}s,$$

using the Cauchy-Schwarz inequality it follows that

$$\|u_h - \overline{u_h}^{F_j}\|_{0,F_j} \lesssim \Big(\int_{F_j}\Big(\int_{F_j}|\nabla u_h\cdot\tau|\ \mathrm{d}s\Big)^2 \mathrm{d}s\Big)^{\frac{1}{2}} \lesssim h^{\frac{1}{2}}\|\nabla u_h\cdot\tau\|_{0,F_j}\Big(\int_{F_j}\mathrm{d}s\Big)^{\frac{1}{2}}.$$

The inequality (2.12) is shown in the following way, the triangle inequality gives

$$h^{-\frac{1}{2}}\|u_h\|_{0,F_j} \le h^{-\frac{1}{2}}\|u_h - \overline{u_h}^{F_j}\|_{0,F_j} + h^{-\frac{1}{2}}\|\overline{u_h}^{F_j}\|_{0,F_j},$$

considering the inequality (2.11) the trace inequality and the inverse inequality we can write

$$\|u_h - \overline{u_h}^{F_j}\|_{0,F_j} \lesssim h^{\frac{1}{2}}\|\nabla u_h\|_{0,P_j}.$$

□

### 2.1.2 Inf-sup stability

In this section we prove the inf-sup condition using the boundary mortaring defined in Section 2.1.1. We follow the same principles as for the one dimensional case of Section 1.2.4.

**Theorem 2.1.1.** *There exists $\beta > 0$ such that for all functions $u_h \in V_h^k$ the following inequality holds*

$$\beta\|u_h\|_{1,h} \le \sup_{v_h\in V_h^k}\frac{A_h(u_h,v_h)}{\|v_h\|_{1,h}},$$

*with $\beta = \mathcal{O}(1)$.*

*Proof.* Let $v_h = u_h + \alpha\sum_{j=1}^{N_p} v_j$, such that $v_j = \nu_j\chi_j$, with $\nu_j \in \mathbb{R}$, each $v_j$ has the property

$$\mathrm{meas}(F_j)^{-1}\int_{F_j}\nabla v_j\cdot n\ \mathrm{d}s = h^{-1}\overline{u_h}^{F_j}. \tag{2.13}$$

Applying Lemma 2.1.1 with $\varphi_r = v_j$ and $r_j = h^{-1}\overline{u_h}^{F_j}$ we get

$$\|\nabla v_j\|_{0,P_j} \lesssim h^{-\frac{1}{2}}\|\overline{u_h}^{F_j}\|_{0,F_j}. \tag{2.14}$$

Replacing $v_h$ in the bilinear form

$$A_h(u_h,v_h) = A_h(u_h,u_h) + \alpha\sum_{j=1}^{N_p}A_h(u_h,v_j).$$

Clearly we have

$$A_h(u_h,u_h) = \|\nabla u_h\|^2_{0,\Omega},$$

and

$$A_h(u_h,v_j) = (\nabla u_h,\nabla v_j)_{P_j} - \langle\nabla u_h\cdot n, v_j\rangle_{F_j} + \langle\nabla v_j\cdot n, u_h\rangle_{F_j}.$$

Using (2.13), we can write

$$\langle \nabla v_j \cdot n, u_h \rangle_{F_j} = h^{-1} \| \overline{u_h}^{F_j} \|^2_{0,F_j} + \langle \nabla v_j \cdot n, u_h - \overline{u_h}^{F_j} \rangle_{F_j}.$$

Using (2.11), the Cauchy-Schwarz inequality, the trace inequality of Lemma 2.0.1 and the inverse inequality of Lemma 2.0.2, we obtain

$$\langle \nabla v_j \cdot n, u_h - \overline{u_h}^{F_j} \rangle_{F_j} \lesssim \| \nabla u_h \|_{0,P_j} \| \nabla v_j \|_{0,P_j}.$$

Applying the Poincaré inequality on each patch $P_j$ we remark that the function $v_j$ has the following property

$$\| v_j \|_{0,P_j} \lesssim h \| \nabla v_j \|_{0,P_j}, \tag{2.15}$$

using this result, we obtain

$$|(\nabla u_h, \nabla v_j)_{P_j} - \langle \nabla u_h \cdot n, v_j \rangle_{F_j}| \lesssim \| \nabla u_h \|_{0,P_j} \| \nabla v_j \|_{0,P_j}.$$

It allows us to write for every $j = 1, \ldots, N_p$

$$A_h(u_h, v_j) \geq h^{-1} \| \overline{u_h}^{F_j} \|^2_{0,F_j} - C \| \nabla u_h \|_{0,P_j} \| \nabla v_j \|_{0,P_j},$$

Using inequality (2.14) it becomes

$$A_h(u_h, v_j) \geq h^{-1} \| \overline{u_h}^{F_j} \|^2_{0,F_j} - C \| \nabla u_h \|_{0,P_j} h^{-\frac{1}{2}} \| \overline{u_h}^{F_j} \|_{0,F_j}.$$

Summing over the patches and using the Young's inequality

$$\begin{aligned} A_h(u_h, v_h) &\geq \| \nabla u_h \|^2_{0,\Omega} + \alpha \sum_{j=1}^{N_p} h^{-1} \| \overline{u_h}^{F_j} \|^2_{0,F_j} - C\alpha \sum_{j=1}^{N_p} \| \nabla u_h \|_{0,P_j} h^{-\frac{1}{2}} \| \overline{u_h}^{F_j} \|_{0,F_j} \\ &\geq (1-\epsilon) \| \nabla u_h \|^2_{0,\Omega} + \alpha \Big( 1 - \frac{C\alpha}{4\epsilon} \Big) \sum_{j=1}^{N_p} h^{-1} \| \overline{u_h}^{F_j} \|^2_{0,F_j}. \end{aligned}$$

Choosing $\epsilon = \frac{1}{4}$ and using (2.12) we obtain

$$A_h(u_h, v_h) \geq \Big( \frac{3}{4} - C'\alpha \Big) \| \nabla u_h \|^2_{0,\Omega} + \frac{\alpha}{2} (1 - C\alpha) \sum_{j=1}^{N_p} h^{-1} \| u_h \|^2_{0,F_j},$$

taking $\alpha < \min(\frac{3}{4C'}, \frac{1}{C})$ we can write

$$A_h(u_h, v_h) \gtrsim \| u_h \|^2_{1,h}.$$

Using that $(a+b)^2 \leq 2a^2 + 2b^2$ we can show

$$\begin{aligned}
\|v_h\|_{1,h}^2 &= \|\nabla(u_h + \alpha \sum_{j=1}^{N_p} v_j)\|_{0,\Omega}^2 + h^{-1}\|u_h + \alpha \sum_{j=1}^{N_p} v_j\|_{0,\partial\Omega}^2 \\
&= \|\nabla u_h\|_{0,\Omega\setminus P}^2 + \sum_{j=1}^{N_p} \|\nabla(u_h + \alpha \sum_{j=1}^{N_p} v_j)\|_{0,P_j}^2 + h^{-1} \sum_{j=1}^{N_p} \|u_h + \alpha \sum_{j=1}^{N_p} v_j\|_{0,F_j}^2 \\
&\leq \|\nabla u_h\|_{0,\Omega\setminus P}^2 + 2\|\nabla u_h\|_{0,P}^2 + 2\alpha^2 \sum_{j=1}^{N_p} \|\nabla v_j\|_{0,P_j}^2 + 2h^{-1}(\|u_h\|_{0,\partial\Omega}^2 + \alpha^2 \sum_{j=1}^{N_p} \|v_j\|_{0,F_j}^2) \\
&\lesssim \|u_h\|_{1,h}^2 + \alpha^2 \sum_{j=1}^{N_p} (\|\nabla v_j\|_{0,P_j}^2 + h^{-1}\|v_j\|_{0,F_j}^2) \\
&\lesssim \|u_h\|_{1,h}^2 + \alpha^2 \sum_{j=1}^{N_p} \|v_j\|_{1,h}^2,
\end{aligned} \tag{2.16}$$

with

$$\|v_j\|_{1,h}^2 = \|\nabla v_j\|_{0,P_j}^2 + h^{-1}\|v_j\|_{0,F_j}^2.$$

Using (2.15) and (2.14) we have

$$h^{-1}\|v_j\|_{0,F_j}^2 \lesssim \|\nabla v_j\|_{0,P_j}^2 \lesssim h^{-1}\|\overline{u_h}^{F_j}\|_{0,F_j}^2 \lesssim h^{-1}\|u_h\|_{0,F_j}^2,$$

we obtain $\|v_h\|_{1,h} \lesssim \|u_h\|_{1,h}$. □

### 2.1.3 A priori error estimate

The following consistency relation characterizes the Galerkin orthogonality.

**Lemma 2.1.3.** *If $u \in H^2(\Omega)$ is the solution of (2.2) and $u_h \in V_h^k$ the solution of (2.4) then*

$$A_h(u - u_h, v_h) = 0, \qquad \forall v_h \in V_h^k.$$

*Proof.* $A_h(u, v_h) = L_h(v_h) = A_h(u_h, v_h),\ \forall v_h \in V_h^k$. □

We introduce an auxiliary norm in order to study the a priori error estimate

$$\|w\|_*^2 = \|w\|_{1,h}^2 + h\|\nabla w \cdot n\|_{0,\partial\Omega}^2,$$

where $\|\cdot\|_{1,h}$ is defined by (1.9).

**Lemma 2.1.4.** *For all $w \in H^2(\Omega) + V_h^k$ and $v_h \in V_h^k$, there exists a positive constant $M$ such that*

$$A_h(w, v_h) \leq M\|w\|_*\|v_h\|_{1,h}.$$

*Proof.* The proof is straightforward using the Cauchy-Schwarz inequality. □

**Theorem 2.1.2.** *Let $u \in H^{k+1}(\Omega)$ be the solution of* (2.2) *and $u_h \in V_h^k$ the solution of* (2.4)*, then there holds*

$$\|u - u_h\|_{1,h} \lesssim \inf_{w_h \in V_h^k} \|u - w_h\|_*.$$

*Proof.* Let $w_h \in V_h^k$, the triangle inequality gives us

$$\|u - u_h\|_{1,h} \leq \|u - w_h\|_{1,h} + \|w_h - u_h\|_{1,h}.$$

Using the Galerkin orthogonality of Lemma 2.1.3, the Theorem 2.1.1 and the Lemma 2.1.4 we can write

$$\beta \|u_h - w_h\|_{1,h} \leq \sup_{v_h \in V_h^k} \frac{A_h(u - w_h, v_h)}{\|v_h\|_{1,h}} \leq M \|u - w_h\|_*.$$

Note that $\|u - w_h\|_{1,h} \leq \|u - w_h\|_*$, taking the inf over all $w_h$ we obtain

$$\|u - u_h\|_{1,h} \leq \Big(1 + \frac{M}{\beta}\Big) \inf_{w_h \in V_h^k} \|u - w_h\|_*.$$

□

Let $\pi_h^k$ denote the nodal interpolant, we have the following approximation property for $u \in H^{k+1}(\Omega)$

$$\|u - \pi_h^k u\|_{0,K} + h_K \|\nabla(u - \pi_h^k u)\|_{0,K} + h_K^2 \|D^2(u - \pi_h^k u)\|_{0,K} \lesssim h_K^{k+1} |u|_{k+1,K}. \quad (2.17)$$

**Corollary 2.1.1.** *Let $u \in H^{k+1}(\Omega)$ be the solution of* (2.2) *and $u_h \in V_h^k$ the solution of* (2.4)*, then there holds*

$$\|u - u_h\|_{1,h} \lesssim h^k |u|_{k+1,\Omega}.$$

*Proof.* Using (2.17) and the trace inequality of Lemma 2.0.1 we have

$$h^{-\frac{1}{2}} \|u - \pi_h^k u\|_{0,\partial\Omega} \lesssim h^{-1} \|u - \pi_h^k u\|_{0,\Omega} + \|\nabla(u - \pi_h^k u)\|_{0,\Omega} \lesssim h^k |u|_{k+1,\Omega},$$

and

$$\begin{aligned} h^{\frac{1}{2}} \|\nabla(u - \pi_h^k u) \cdot n\|_{0,\partial\Omega} \lesssim & \|\nabla(u - \pi_h^k u)\|_{0,\Omega} \\ & + h \Big( \sum_{K \in \mathcal{T}_h} \|D^2(u - \pi_h^k u)\|_{0,K}^2 \Big)^{\frac{1}{2}} \lesssim h^k |u|_{k+1,\Omega}. \end{aligned}$$

Then we deduce

$$\|u - \pi_h^k u\|_* \lesssim h^k |u|_{k+1,\Omega}, \quad (2.18)$$

the claim follows by using Theorem 2.1.2 with $w_h = \pi_h^k u$. □

**Proposition 2.1.1.** *Let $u \in H^{k+1}(\Omega)$ be the solution of* (2.2) *and $u_h \in V_h^k$ the solution of* (2.4)*, then there holds*

$$\|u - u_h\|_{0,\Omega} \lesssim h^{k+\frac{1}{2}} |u|_{k+1,\Omega}.$$

*Proof.* Let $z$ satisfy the adjoint problem

$$\begin{aligned} -\Delta z &= u - u_h && \text{in } \Omega, \\ z &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We also have

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 = (u - u_h, -\Delta z)_\Omega &= (\nabla(u - u_h), \nabla z)_\Omega - \langle (u - u_h), \nabla z \cdot n \rangle_{\partial\Omega} \\ &= A_h(u - u_h, z) - 2\langle (u - u_h), \nabla z \cdot n \rangle_{\partial\Omega}. \end{aligned}$$

By Lemma 2.1.3, using $(z - \pi_h^1 z)|_{\partial\Omega} \equiv 0$ and the estimate (2.18) we can write

$$\begin{aligned} A_h(u - u_h, z) &= A_h(u - u_h, z - \pi_h^1 z) \\ &= (\nabla(u - u_h), \nabla(z - \pi_h^1 z))_\Omega + \langle (u - u_h), \nabla(z - \pi_h^1 z) \cdot n \rangle_{\partial\Omega} \\ &\leq \|u - u_h\|_{1,h} \|z - \pi_h^1 z\|_* \\ &\lesssim h \|u - u_h\|_{1,h} |z|_{2,\Omega}. \end{aligned}$$

The global trace inequalities $\|\nabla z \cdot n\|_{0,\partial\Omega} \lesssim \|z\|_{2,\Omega}$ leads to

$$|\langle (u - u_h), \nabla z \cdot n \rangle_{\partial\Omega}| \lesssim h^{\frac{1}{2}} \|u - u_h\|_{1,h} \|z\|_{2,\Omega}.$$

Then we obtain

$$\|u - u_h\|_{0,\Omega}^2 \lesssim (h + h^{\frac{1}{2}}) h^k |u|_{k+1,\Omega} \|z\|_{2,\Omega}.$$

We conclude by applying the regularity estimate (2.3) ($\|z\|_{2,\Omega} \lesssim \|u - u_h\|_{0,\Omega}$ for the adjoint problem). □

**Remark 2.1.2.** *The convergence of the $L^2$-error suffers of suboptimality of order $\mathcal{O}(h^{\frac{1}{2}})$ due to the lack of adjoint consistency of the nonsymmetric formulation.*

## 2.2 Compressible elasticity

In this section we extend the method to compressible elasticity, we consider the problem in two dimensions. The main difficulty is that the problem considers a deformation tensor, a variant of the Korn's inequality is shown to handle this problem. The compressible elasticity problem with Dirichlet boundary condition is given by

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{g} && \text{on } \partial\Omega. \end{aligned} \tag{2.19}$$

with $\boldsymbol{f} \in [L^2(\Omega)]^2$, $\boldsymbol{g} \in [H^{\frac{3}{2}}(\partial\Omega)]^2$ and the stress tensor

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda(\nabla \cdot \boldsymbol{u})\mathbb{I}_{2\times 2},$$

with $\mu$ and $\lambda$ are the Lamé coefficients and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ the deformation tensor. The weak formulation of this problem gives: find $\boldsymbol{u} \in [H^1_g(\Omega)]^2$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall \boldsymbol{v} \in [H^1_0(\Omega)]^2,$$

with

$$a(\boldsymbol{u}, \boldsymbol{v}) = (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_\Omega + (\lambda\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_\Omega.$$

The well-posedness of this problem follows from the Lax-Milgram Lemma. The following elliptic regularity estimate holds

$$\mu\|\boldsymbol{u}\|_{2,\Omega} + (\lambda + \mu)\|\nabla \cdot \boldsymbol{u}\|_{1,\Omega} \lesssim \|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{\frac{3}{2},\partial\Omega}. \tag{2.20}$$

this result is shown in [22], the proof involves the use of Korn's inequality (see Appendix A). We define the space $W^k_h = [V^k_h]^2$, applying the penalty-free Nitsche's method to the compressible elasticity problem (2.19), we obtain the following finite element formulation: find $\boldsymbol{u}_h \in W^k_h$

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = L_h(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in W^k_h, \tag{2.21}$$

where the linear forms $A_h$ and $L_h$ are defined as

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &= a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \boldsymbol{u}_h), \\ L_h(\boldsymbol{v}_h) &= (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega + b(\boldsymbol{v}_h, \boldsymbol{g}). \end{aligned}$$

The bilinear form $b$ is defined as

$$b(\boldsymbol{u}_h, \boldsymbol{v}_h) = \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h) \cdot \boldsymbol{n}, \boldsymbol{v}_h\rangle_{\partial\Omega} + \langle \lambda\nabla \cdot \boldsymbol{u}_h, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\Omega}.$$

### 2.2.1 A new Korn's inequality

In order to handle the deformation tensor in the proof of the inf-sup condition, we need a particular form of Korn's inequality. First let us split the boundary into $N_b \geq 3$ smaller smooth sections $\{\Upsilon_i\}_{1\leq i\leq N_b}$ of the boundary with meas$(\Upsilon_i) \gtrsim h$, these smaller sections can be for example the faces of the polygonal boundary or the faces $\{F_j\}_{1\leq j\leq N_p}$ of the patches. To prove this alternative form of the Korn's inequality we need to define the following seminorm

$$|\boldsymbol{u}|^2_\Gamma = \sum_{i=1}^{N_b} \int_{\Upsilon_i} (\overline{\boldsymbol{u}}^{\Upsilon_i})^2 \, \mathrm{d}s \qquad \forall \boldsymbol{u} \in [H^1(\Omega)]^2. \tag{2.22}$$

Note that $\boldsymbol{u}$ has two components, $\overline{\boldsymbol{u}}^{\Upsilon_i}$ is the average of $\boldsymbol{u}$ on $\Upsilon_i$.

**Proposition 2.2.1.** *For all* $\boldsymbol{u} \in [H^1(\Omega)]^2$ *the seminorm* (2.22) *is a norm on* RM *with*

$$\mathrm{RM} = \{\boldsymbol{u} : \boldsymbol{u} = \boldsymbol{c} + b(x_2, -x_1)^T, \boldsymbol{c} \in \mathbb{R}^2, b \in \mathbb{R}\}.$$

*Proof.* Since $|\boldsymbol{u}|_\Gamma$ is a seminorm, we only need to show that $|\boldsymbol{u}|_\Gamma = 0 \Rightarrow \boldsymbol{u} = 0 \; \forall \boldsymbol{u} \in \mathrm{RM}$. First note that

$$|\boldsymbol{u}|_\Gamma = 0 \Rightarrow \sum_{i=1}^{N_b} \int_{\Upsilon_i} (\overline{\boldsymbol{u}}^{\Upsilon_i})^2 \, \mathrm{d}s = 0 \Rightarrow \overline{\boldsymbol{u}}^{\Upsilon_i} = 0.$$

We also know that $\boldsymbol{u} \in \mathrm{RM}$ then

$$\boldsymbol{u} = \begin{pmatrix} c_1 + bx_2 \\ c_2 - bx_1 \end{pmatrix}, \qquad \overline{\boldsymbol{u}}^{\Upsilon_i} = \begin{pmatrix} c_1 + b\overline{x_2}^{\Upsilon_i} \\ c_2 - b\overline{x_1}^{\Upsilon_i} \end{pmatrix}.$$

Considering $\Upsilon_k$ and $\Upsilon_l$ with $k \neq l$ we obtain

$$\begin{aligned} c_1 + b\overline{x_2}^{\Upsilon_k} &= 0, \\ c_2 - b\overline{x_1}^{\Upsilon_k} &= 0, \\ c_1 + b\overline{x_2}^{\Upsilon_l} &= 0, \\ c_2 - b\overline{x_1}^{\Upsilon_l} &= 0. \end{aligned}$$

Observe that we only need $\overline{x_1}^{\Upsilon_k} \neq \overline{x_1}^{\Upsilon_l}$ or $\overline{x_2}^{\Upsilon_k} \neq \overline{x_2}^{\Upsilon_l}$ to obtain $c_1 = c_2 = b = 0$. There exist $k$ and $l$ such that one of these condition is always true since $(\overline{x_1}^{\Upsilon_k}, \overline{x_2}^{\Upsilon_k})$ and $(\overline{x_1}^{\Upsilon_l}, \overline{x_2}^{\Upsilon_l})$ are respectively the mid-points of the sides $\Upsilon_k$ and $\Upsilon_l$. This implies

$$|\boldsymbol{u}|_\Gamma = 0 \Rightarrow \boldsymbol{u} = 0.$$

□

**Theorem 2.2.1.** *There exists a positive constant* $C_K$ *such that* $\forall \boldsymbol{u} \in [H^1(\Omega)]^2$

$$C_K \|\boldsymbol{u}\|_{1,\Omega} \leq \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega} + |\boldsymbol{u}|_\Gamma.$$

*Proof.* This proof is inspired by the proof of the Korn's inequality in [21]. First we define

$$[\tilde{H}^1(\Omega)]^2 = \big\{\boldsymbol{u} \in [H^1(\Omega)]^2 : \int_\Omega \boldsymbol{u} \, \mathrm{d}x = 0, \int_\Omega \mathrm{rot}\; \boldsymbol{u} \, \mathrm{d}x = 0\big\}.$$

We know that, $[H^1(\Omega)]^2 = [\tilde{H}^1(\Omega)]^2 \times \mathrm{RM}$. Therefore, given any $\boldsymbol{u} \in [H^1(\Omega)]^2$, there exists a unique pair $(\boldsymbol{z}, \boldsymbol{w}) \in [\tilde{H}^1(\Omega)]^2 \times \mathrm{RM}$ such that

$$\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{w}.$$

By the Open Mapping Theorem (theorem 15, chapter 15 of [82]) there exists a positive constant $C_1$ such that

$$C_1(\|\boldsymbol{z}\|_{1,\Omega} + \|\boldsymbol{w}\|_{1,\Omega}) \leq \|\boldsymbol{u}\|_{1,\Omega}. \tag{2.23}$$

We establish the theorem by contradiction. If the inequality that we want to show does not hold for any positive constant $C_K$, then there exists a sequence $\{\boldsymbol{u}_n\} \subseteq [H^1(\Omega)]^2$ such that

$$\|\boldsymbol{u}_n\|_{1,\Omega} = 1, \tag{2.24}$$

and

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_n)\|_{0,\Omega} + |\boldsymbol{u}_n|_\Gamma < \frac{1}{n}. \tag{2.25}$$

For each $n$, let $\boldsymbol{u}_n = \boldsymbol{z}_n + \boldsymbol{w}_n$, where $\boldsymbol{z}_n \in [\tilde{H}^1(\Omega)]^2$ and $\boldsymbol{w}_n \in \mathrm{RM}$, then

$$\|\boldsymbol{\varepsilon}(\boldsymbol{z}_n)\|_{0,\Omega} = \|\boldsymbol{\varepsilon}(\boldsymbol{u}_n)\|_{0,\Omega} < \frac{1}{n}.$$

The second Korn's inequality then implies that $\boldsymbol{z}_n \longrightarrow 0$ in $[H^1(\Omega)]^2$. It follows from (2.23) and (2.24) that $\{\boldsymbol{w}_n\}$ is a bounded sequence in $[H^1(\Omega)]^2$. But since RM is finite dimensional, $\{\boldsymbol{w}_n\}$ has a convergent subsequence $\{\boldsymbol{w}_{n_j}\}$ in $[H^1(\Omega)]^2$. Then the subsequence $\{\boldsymbol{u}_{n_j} = \boldsymbol{z}_{n_j} + \boldsymbol{w}_{n_j}\}$ converges in $[H^1(\Omega)]^2$ to some $\boldsymbol{u} = \lim_{n_j \to \infty} \boldsymbol{w}_{n_j} \in \mathrm{RM}$ and we obtain

$$\|\boldsymbol{u}\|_{1,\Omega} = 1, \tag{2.26}$$

and

$$|\boldsymbol{u}|_\Gamma = 0.$$

The Proposition 2.2.1 tells us that $|\cdot|_\Gamma$ is a norm on RM and therefore

$$|\boldsymbol{u}|_\Gamma = 0 \Leftrightarrow \boldsymbol{u} = 0,$$

which contradicts the equation (2.26). □

### 2.2.2 Inf-sup stability

In this section we show the inf-sup condition for the formulation (2.21). We introduce the following two dimensional rotation transformation to reduce the technicalities.

**Definition 2.2.1.** *The rotation transformation in two dimensions can be written as*

$$\begin{aligned} \mathcal{R} : [L^2(\hat{\Omega})]^2 &\longrightarrow [L^2(\Omega)]^2 \\ \hat{\boldsymbol{z}} &\longmapsto \boldsymbol{z} = \mathcal{R}(\hat{\boldsymbol{z}}) = A\hat{\boldsymbol{z}}, \end{aligned}$$

*with $A$ a rotation matrix and $\hat{\boldsymbol{z}}$ the rotated quantity of $\boldsymbol{z}$.*

This two-dimensional rotation is used to transform the generic fixed frame $(x, y)$ into a rotated frame $(\xi, \eta)$ associated to each side $\Gamma_i$ of $\partial\Omega$. This rotated frame has its first component tangent to the side $\Gamma_i$ of the polygonal boundary and its second component normal to this same side $\Gamma_i$. A function $\boldsymbol{z} = (z_1, z_2)$ expressed in the two-dimensional rotated frame has the following properties

$$\hat{z}_1 = \boldsymbol{z} \cdot \boldsymbol{\tau}, \qquad \hat{z}_2 = \boldsymbol{z} \cdot \boldsymbol{n}.$$

The hat denotes a value expressed in the rotated frame $(\xi, \eta)$. Figure 2.2 represents schematically how is defined this frame for a side $\Gamma_i$. The function $\boldsymbol{v}_j$ is defined such

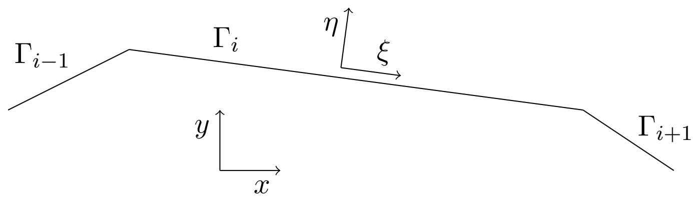



Figure 2.2: Representation of the rotated frame $(\xi, \eta)$, the first component of the frame is tangent to the side $\Gamma_i$ and the second component is normal to the side $\Gamma_i$.

that $\boldsymbol{v}_j = (\alpha_1 v_{j1}, \alpha_2 v_{j2})^T$. For simplicity of notation we will use $v_1$, $v_2$ respectively instead of $v_{j1}$, $v_{j2}$. We define $v_1 = \nu_{j1}\chi_j$ and $v_2 = \nu_{j2}\chi_j$ with $\nu_{j1}, \nu_{j2} \in \mathbb{R}$ and $\chi_j$ as defined in Section 2.1.1. In order to be able to use Lemma 2.1.1, the function $\boldsymbol{v}_j$ has the properties

$$\text{meas}(F_j)^{-1} \int_{F_j} \frac{\partial \hat{v}_1}{\partial \eta} \, \mathrm{d}s = h^{-1} \overline{\hat{u}_1}^{F_j}, \qquad \text{meas}(F_j)^{-1} \int_{F_j} \frac{\partial \hat{v}_2}{\partial \eta} \, \mathrm{d}s = h^{-1} \overline{\hat{u}_2}^{F_j}, \tag{2.27}$$

with $\hat{\boldsymbol{u}}_h = (\hat{u}_1, \hat{u}_2)^T$. Note that the projection defined by (2.10) is used. Using Lemma 2.1.1 it is straightforward to show

$$\|\hat{\nabla} \hat{v}_1\|_{0,P_j} \lesssim h^{-\frac{1}{2}} \|\overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{\tau}\|_{0,F_j}, \qquad \|\hat{\nabla} \hat{v}_2\|_{0,P_j} \lesssim h^{-\frac{1}{2}} \|\overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{n}\|_{0,F_j}. \tag{2.28}$$

We first give two technical Lemmas, proofs are provided in appendix D.

**Lemma 2.2.1.** *There exists $C > 0$ independent of $h$, $\mu$ and $\lambda$, but not of the mesh geometry, $\forall \boldsymbol{u}_h \in W_h^k$, on each patch $P_j$ for $\boldsymbol{v}_j \in W_h^1$ as defined above and $\forall \epsilon, \alpha_1, \alpha_2 \in \mathbb{R}_+^*$, such that*

$$\langle \lambda \nabla \cdot \boldsymbol{v}_j, \boldsymbol{u}_h \cdot \boldsymbol{n} \rangle_{F_j} \gtrsim \alpha_2 \Big(1 - \frac{C\alpha_2}{4\epsilon}\Big) \frac{\lambda}{h} \|\overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{n}\|_{0,F_j}^2 - \frac{C\alpha_1^2}{4\epsilon} \frac{\lambda}{h} \|\overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{\tau}\|_{0,F_j}^2 - 2\epsilon\lambda \|\nabla \boldsymbol{u}_h\|_{0,P_j}^2.$$

**Lemma 2.2.2.** *There exists $C > 0$ independent of $h$, $\mu$ and $\lambda$, but not of the mesh geometry, $\forall \boldsymbol{u}_h \in W_h^k$, on each patch $P_j$ for $\boldsymbol{v}_j \in W_h^1$ as defined above and $\forall \epsilon, \alpha_1, \alpha_2 \in \mathbb{R}_+^*$,* *such that*

$$\langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_j)\cdot\boldsymbol{n},\boldsymbol{u}_h\rangle_{F_j} \geq \alpha_2\Big(2-\frac{5C\alpha_2}{4\epsilon}\Big)\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j} + \alpha_1\Big(1-\frac{C\alpha_1}{4\epsilon}\Big)\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} - 3\epsilon\mu\|\nabla\boldsymbol{u}_h\|^2_{0,P_j}.$$

We introduce the following norms

$$|||\boldsymbol{w}|||^2 = \mu(\|\nabla\boldsymbol{w}\|^2_{0,\Omega}+h^{-1}\|\boldsymbol{w}\|^2_{0,\partial\Omega}) + \lambda(\|\nabla\cdot\boldsymbol{w}\|^2_{0,\Omega}+h^{-1}\|\boldsymbol{w}\cdot\boldsymbol{n}\|^2_{0,\partial\Omega}),$$
$$\|\boldsymbol{w}\|^2_* = |||\boldsymbol{w}|||^2 + \mu h\|\nabla\boldsymbol{w}\cdot\boldsymbol{n}\|^2_{0,\partial\Omega} + \lambda h\|\nabla\cdot\boldsymbol{w}\|^2_{0,\partial\Omega}.$$

Observe that these are norms by the Poincaré inequality.

**Lemma 2.2.3.** *For $\boldsymbol{u}_h,\boldsymbol{v}_h\in W_h^k$ with $\boldsymbol{v}_h=\boldsymbol{u}_h+\sum_{j=1}^{N_p}\boldsymbol{v}_j$, $\boldsymbol{v}_j$ defined by (2.27), there exists positive constants $\beta_0$ and $h_0$ such that the following inequality holds for $h<h_0$*

$$\beta_0|||\boldsymbol{u}_h|||^2 \leq A_h(\boldsymbol{u}_h,\boldsymbol{v}_h).$$

*Proof.* Decomposing the bilinear form, we can write the following

$$A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = A_h(\boldsymbol{u}_h,\boldsymbol{u}_h) + \sum_{j=1}^{N_p}A_h(\boldsymbol{u}_h,\boldsymbol{v}_j).$$

Clearly we have

$$A_h(\boldsymbol{u}_h,\boldsymbol{u}_h) = 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \lambda\|\nabla\cdot\boldsymbol{u}_h\|^2_{0,\Omega},$$

and

$$A_h(\boldsymbol{u}_h,\boldsymbol{v}_j) = (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h),\boldsymbol{\varepsilon}(\boldsymbol{v}_j))_{P_j} - \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\cdot\boldsymbol{n},\boldsymbol{v}_j\rangle_{F_j} + \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_j)\cdot\boldsymbol{n},\boldsymbol{u}_h\rangle_{F_j} + (\lambda\nabla\cdot\boldsymbol{u}_h,\nabla\cdot\boldsymbol{v}_j)_{P_j} - \langle\lambda\nabla\cdot\boldsymbol{u}_h,\boldsymbol{v}_j\cdot\boldsymbol{n}\rangle_{F_j} + \langle\lambda\nabla\cdot\boldsymbol{v}_j,\boldsymbol{u}_h\cdot\boldsymbol{n}\rangle_{F_j}.$$

Using the Cauchy-Schwarz inequality and (2.28), we can write the two terms defined over $P_j$ as

$$(2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h),\boldsymbol{\varepsilon}(\boldsymbol{v}_j))_{P_j} \leq \epsilon\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,P_j} + \frac{C\alpha_1^2}{\epsilon}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \frac{C\alpha_2^2}{\epsilon}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j},$$
$$(\lambda\nabla\cdot\boldsymbol{u}_h,\nabla\cdot\boldsymbol{v}_j)_{P_j} \leq \epsilon\lambda\|\nabla\boldsymbol{u}_h\|^2_{0,P_j} + \frac{C\alpha_1^2}{4\epsilon}\frac{\lambda}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \frac{C\alpha_2^2}{4\epsilon}\frac{\lambda}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j}.$$

Combining the inequality (2.15) with the trace and inverse inequalities, followed by (2.28) we obtain

$$\langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\cdot\boldsymbol{n},\boldsymbol{v}_j\rangle_{F_j} \leq \epsilon\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,P_j} + \frac{C\alpha_1^2}{\epsilon}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \frac{C\alpha_2^2}{\epsilon}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j},$$
$$\langle\lambda\nabla\cdot\boldsymbol{u}_h,\boldsymbol{v}_j\cdot\boldsymbol{n}\rangle_{F_j} \leq \epsilon\lambda\|\nabla\boldsymbol{u}_h\|^2_{0,P_j} + \frac{C\alpha_1^2}{4\epsilon}\frac{\lambda}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \frac{C\alpha_2^2}{4\epsilon}\frac{\lambda}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j}.$$

Considering Lemmas 2.2.1 and 2.2.2 we can write

$$\begin{aligned}
&A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)\\
&\quad \geq 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \lambda\|\nabla\cdot\boldsymbol{u}_h\|^2_{0,\Omega} - 2\epsilon\sum_{j=1}^{N_p}\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,P_j} - (3\epsilon\mu + 4\epsilon\lambda)\sum_{j=1}^{N_p}\|\nabla\boldsymbol{u}_h\|^2_{0,P_j}\\
&\qquad + \alpha_1\Big(1 - \alpha_1\frac{9C}{4\epsilon}\Big)\sum_{j=1}^{N_p}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \alpha_2\Big(2 - \alpha_2\frac{13C}{4\epsilon}\Big)\sum_{j=1}^{N_p}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j}\\
&\qquad + \alpha_1\Big(-\alpha_1\frac{3C}{4\epsilon}\Big)\sum_{j=1}^{N_p}\frac{\lambda}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \alpha_2\Big(1 - \alpha_2\frac{3C}{4\epsilon}\Big)\sum_{j=1}^{N_p}\frac{\lambda}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j}.
\end{aligned}$$

Using the Korn's inequality of Theorem 2.2.1 with $\Upsilon_j = F_j$ in the definition of the seminorm $|\cdot|_\Gamma$ we obtain the following bound

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \sum_{j=1}^{N_p}\|\overline{\boldsymbol{u}_h}^{F_j}\|^2_{0,F_j} \geq C_K\|\boldsymbol{u}_h\|^2_{1,\Omega} \qquad \forall\boldsymbol{u}_h \in W_h^k.$$

Using this result, we can write

$$\begin{aligned}
A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &\geq \lambda\|\nabla\cdot\boldsymbol{u}_h\|^2_{0,\Omega} + 2C_K\mu\|\nabla\boldsymbol{u}_h\|^2_{0,\Omega\setminus P} + (2\mu C_K - 5\epsilon\mu - 4\epsilon\lambda)\sum_{j=1}^{N_p}\|\nabla\boldsymbol{u}_h\|^2_{0,P_j}\\
&\quad + \Big(\Big(\alpha_1\Big(1 - \alpha_1\frac{9C}{4\epsilon}\Big) - 2h\Big)\mu - \alpha_1^2\frac{3C}{4\epsilon}\lambda\Big)\sum_{j=1}^{N_p}h^{-1}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j}\\
&\quad + \Big(\Big(\alpha_2\Big(2 - \alpha_2\frac{13C}{4\epsilon}\Big) - 2h\Big)\mu + \alpha_2\Big(1 - \alpha_2\frac{3C}{4\epsilon}\Big)\lambda\Big)\sum_{j=1}^{N_p}h^{-1}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j}.
\end{aligned}$$

Considering the inequality (2.12) we obtain

$$\begin{aligned}
A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &\geq \lambda\|\nabla\cdot\boldsymbol{u}_h\|^2_{0,\Omega} + 2C_K\mu\|\nabla\boldsymbol{u}_h\|^2_{0,\Omega\setminus P} + (C_a - C_b - C_c)\sum_{j=1}^{N_p}\|\nabla\boldsymbol{u}_h\|^2_{0,P_j}\\
&\quad + \frac{C_b}{2}\sum_{j=1}^{N_p}h^{-1}\|\boldsymbol{u}_h\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \frac{C_c}{2}\sum_{j=1}^{N_p}h^{-1}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|^2_{0,F_j},
\end{aligned}$$

with the constants

$$\begin{aligned}
C_a &= 2\mu C_K - 5\epsilon\mu - 4\epsilon\lambda,\\
C_b &= \Big(\alpha_1\Big(1 - \alpha_1\frac{9C}{4\epsilon}\Big) - 2h\Big)\mu - \alpha_1^2\frac{3C}{4\epsilon}\lambda,\\
C_c &= \Big(\alpha_2\Big(2 - \alpha_2\frac{13C}{4\epsilon}\Big) - 2h\Big)\mu + \alpha_2\Big(1 - \alpha_2\frac{3C}{4\epsilon}\Big)\lambda.
\end{aligned}$$

First we choose $\epsilon = \frac{\mu C_K}{5\mu+4\lambda}$ so that $C_a = \mu C_K$. Let $h < h_0$ such that $C_b$ and $C_c$ are positive respectively for

$$\frac{4\mu^2 C_K}{(9C\mu + 3C\lambda)(5\mu + 4\lambda)} > \alpha_1, \qquad \frac{4\mu C_K(2\mu + \lambda)}{(13C\mu + 3C\lambda)(5\mu + 4\lambda)} > \alpha_2.$$

$C_a - C_b - C_c$ will be positive for

$$\frac{C_K}{2} > \alpha_1, \qquad \frac{\mu C_K}{2(2\mu + \lambda)} > \alpha_2.$$

By looking at the order of the constants, we can see that $\beta_0 = \mathcal{O}\big(\frac{\mu}{\lambda+\mu}\big)$ and $h_0 = \mathcal{O}(\frac{\mu^2}{(\lambda+\mu)^2})$. If $\lambda$ is large compared to $\mu$, $h_0$ has to be very small. This reflects the locking phenomena that is well known for finite element method using low order $H^1$-conforming spaces. $\square$

**Theorem 2.2.2.** *There exists positive constants $\beta$ and $h_0$ such that for all $\boldsymbol{u}_h \in W_h^k$ and for $h < h_0$, the following inequality holds*

$$\beta |||\boldsymbol{u}_h||| \leq \sup_{\boldsymbol{v}_h \in W_h^k} \frac{A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||}.$$

*Proof.* Considering Lemma 2.2.3, the only remaining thing to show is

$$|||\boldsymbol{v}_h||| \leq C|||\boldsymbol{u}_h|||. \tag{2.29}$$

Using the definition of the test function as in the previous proof, similarly as (2.16) using the linearity of the divergence operator we have

$$|||\boldsymbol{v}_h|||^2 \lesssim |||\boldsymbol{u}_h|||^2 + \sum_{j=1}^{N_p} |||\boldsymbol{v}_j|||^2.$$

By definition

$$|||\boldsymbol{v}_j|||^2 = \mu(\|\nabla \boldsymbol{v}_j\|_{0,P_j}^2 + h^{-1}\|\boldsymbol{v}_j\|_{0,F_j}^2) + \lambda(\|\nabla \cdot \boldsymbol{v}_j\|_{0,P_j}^2 + h^{-1}\|\boldsymbol{v}_j \cdot \boldsymbol{n}\|_{0,F_j}^2).$$

We observe that $\|\overline{\boldsymbol{u}_h}^{F_j}\|_{0,F_j} \lesssim \|\boldsymbol{u}_h\|_{0,F_j}$, using this results and recalling (2.28) it gives the appropriate upper bounds considering the definition of $\boldsymbol{v}_j$

$$\begin{aligned} \sum_{j=1}^{N_p} \mu\|\nabla \boldsymbol{v}_j\|_{0,P_j}^2 &\lesssim |||\boldsymbol{u}_h|||^2, \\ \sum_{j=1}^{N_p} \lambda\|\nabla \cdot \boldsymbol{v}_j\|_{0,P_j}^2 \leq \sum_{j=1}^{N_p} \lambda\|\nabla \boldsymbol{v}_j\|_{0,P_j}^2 &\lesssim |||\boldsymbol{u}_h|||^2. \end{aligned} \tag{2.30}$$

Using the trace inequality for the boundary terms and the inequality (2.15) we can write

$$\begin{aligned}\sum_{j=1}^{N_p} \mu h^{-1} \|\boldsymbol{v}_j\|_{0,F_j}^2 &\lesssim \sum_{j=1}^{N_p} \mu \|\nabla \boldsymbol{v}_j\|_{0,P_j}^2 \lesssim |||\boldsymbol{u}_h|||^2, \\ \sum_{j=1}^{N_p} \lambda h^{-1} \|\boldsymbol{v}_j \cdot \boldsymbol{n}\|_{0,F_j}^2 &\lesssim \sum_{j=1}^{N_p} \lambda \|\nabla \boldsymbol{v}_j\|_{0,P_j}^2 \lesssim |||\boldsymbol{u}_h|||^2.\end{aligned} \tag{2.31}$$

Note that $\beta = \mathcal{O}(\frac{\mu}{\lambda+\mu})$. □

### 2.2.3 A priori error estimate

Using the stability proven in the previous section we may deduce the a priori error estimate in the $H^1$-norm. We first prove the consistency of the method in the form of a Galerkin orthogonality.

**Lemma 2.2.4.** *If $\boldsymbol{u} \in [H^2(\Omega)]^2$ is the solution of* (2.19) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (2.21) *the following property holds*

$$A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in W_h^k.$$

*Proof.* We observe that $A_h(\boldsymbol{u}, \boldsymbol{v}_h) = L_h(\boldsymbol{v}_h) = A_h(\boldsymbol{u}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in W_h^k$. □

**Lemma 2.2.5.** *Let $\boldsymbol{w} \in [H^2(\Omega)]^2 + W_h^k$ and $\boldsymbol{v}_h \in W_h^k$, there exists a positive constant $M$ such that*

$$A_h(\boldsymbol{w}, \boldsymbol{v}_h) \leq M \|\boldsymbol{w}\|_* |||\boldsymbol{v}_h|||.$$

*Proof.* Using the Cauchy-Schwarz inequality it is straightforward to write

$$\begin{aligned}(\lambda \nabla \cdot \boldsymbol{w}, \nabla \cdot \boldsymbol{v}_h)_\Omega + (2\mu \boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_\Omega &\lesssim \|\boldsymbol{w}\|_* |||\boldsymbol{v}_h|||, \\ \langle \lambda \nabla \cdot \boldsymbol{w}, \boldsymbol{v}_h \cdot \boldsymbol{n} \rangle_{\partial\Omega} + \langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{w}) \cdot \boldsymbol{n}, \boldsymbol{v}_h \rangle_{\partial\Omega} &\lesssim \|\boldsymbol{w}\|_* |||\boldsymbol{v}_h|||.\end{aligned}$$

The trace and inverse inequalities allow us to write

$$\begin{aligned}\langle \lambda \nabla \cdot \boldsymbol{v}_h, \boldsymbol{w} \cdot \boldsymbol{n} \rangle_{\partial\Omega} &\lesssim \lambda^{\frac{1}{2}} \|\nabla \boldsymbol{v}_h\|_{0,\Omega} \lambda^{\frac{1}{2}} h^{-\frac{1}{2}} \|\boldsymbol{w} \cdot \boldsymbol{n}\|_{0,\partial\Omega} \lesssim \|\boldsymbol{w}\|_* |||\boldsymbol{v}_h|||, \\ \langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial\Omega} &\lesssim \mu^{\frac{1}{2}} \|\nabla \boldsymbol{v}_h\|_{0,\Omega} \mu^{\frac{1}{2}} h^{-\frac{1}{2}} \|\boldsymbol{w}\|_{0,\partial\Omega} \lesssim \|\boldsymbol{w}\|_* |||\boldsymbol{v}_h|||.\end{aligned}$$

□

**Theorem 2.2.3.** *If $\boldsymbol{u} \in [H^{k+1}(\Omega)]^2$ is the solution of* (2.19) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (2.21) *with $h < h_0$, then there holds*

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq C_\beta \inf_{\boldsymbol{w}_h \in W_h^k} \|\boldsymbol{u} - \boldsymbol{w}_h\|_*,$$

*where $C_\beta$ is a positive constant that depends on the mesh geometry.*

*Proof.* Let $\boldsymbol{w}_h \in W_h^k$, the triangle inequality gives us

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq |||\boldsymbol{u} - \boldsymbol{w}_h||| + |||\boldsymbol{w}_h - \boldsymbol{u}_h|||.$$

Using Theorem 2.2.2, the Galerkin orthogonality of Lemma 2.2.4, and the Lemma 2.2.5 we deduce

$$\beta |||\boldsymbol{u}_h - \boldsymbol{w}_h||| \leq \sup_{\boldsymbol{v}_h \in W_h^k} \frac{A_h(\boldsymbol{u} - \boldsymbol{w}_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||} \leq M \|\boldsymbol{u} - \boldsymbol{w}_h\|_*.$$

Note that $|||\boldsymbol{u} - \boldsymbol{w}_h||| \leq \|\boldsymbol{u} - \boldsymbol{w}_h\|_*$, taking the inf over all $\boldsymbol{w}_h$ we obtain

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq \Big(1 + \frac{M}{\beta}\Big) \inf_{\boldsymbol{w}_h \in W_h^k} \|\boldsymbol{u} - \boldsymbol{w}_h\|_*,$$

and $C_\beta = \mathcal{O}(\beta^{-1})$. □

Let $i_{\mathsf{SZ}}^k$ denote the Scott-Zhang interpolant [99]. The approximation property of the interpolant may be written for each $K \in \mathcal{T}_h$ and $\boldsymbol{u} \in [H^{k+1}(\Omega)]^2$

$$\|\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u}\|_{0,K} + h_K \|\nabla(\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u})\|_{0,K} + h_K^2 \|D^2(\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u})\|_{0,K} \lesssim h_K^{k+1} |\boldsymbol{u}|_{k+1,S_K}, \quad (2.32)$$

with $S_K = \text{interior}(\cup\{\overline{K}_i | \overline{K}_i \cap \overline{K} \neq \emptyset, K_i \in \mathcal{T}_h\})$.

**Corollary 2.2.1.** *If $\boldsymbol{u} \in [H^{k+1}(\Omega)]^2$ is the solution of* (2.19) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (2.21) *with $h < h_0$, then there holds*

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq C_{\mu\lambda} h^k |\boldsymbol{u}|_{k+1,\Omega},$$

*where $C_{\mu\lambda}$ is a positive constant that depends on $\mu$, $\lambda$ and the mesh geometry.*

*Proof.* Using the trace inequality of Lemma 2.0.1 and (2.32) we have

$$h^{\frac{1}{2}} \|\nabla \cdot (\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u})\|_{0,\partial\Omega} \lesssim \|\nabla(\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u})\|_{0,\Omega} + h\Big(\sum_{K \in \mathcal{T}_h} \|D^2(\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u})\|_{0,K}^2\Big)^{\frac{1}{2}} \lesssim h^k |\boldsymbol{u}|_{k+1,\Omega},$$

using similar arguments as in the proof of Corollary 3.2.1 and noting that

$$\|\nabla \cdot (\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u})\|_{0,\Omega} \lesssim \|\nabla(\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u})\|_{0,\Omega}, \qquad \|(\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u}) \cdot \boldsymbol{n}\|_{0,\partial\Omega} \lesssim \|\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u}\|_{0,\partial\Omega},$$

we obtain the estimate

$$\|\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u}\|_* \lesssim (\mu^{\frac{1}{2}} + \lambda^{\frac{1}{2}}) h^k |\boldsymbol{u}|_{k+1,\Omega}. \quad (2.33)$$

The claim follows by using Theorem 2.2.3 with $\boldsymbol{w}_h = i_{\mathsf{SZ}}^k \boldsymbol{u}$. The constant in the estimate satisfies $C_{\mu\lambda} = \mathcal{O}(\beta^{-1}(\lambda^{\frac{1}{2}} + \mu^{\frac{1}{2}}))$. □

**Proposition 2.2.2.** *Let* $\boldsymbol{u} \in [H^{k+1}(\Omega)]^2$ *be the solution of* (2.19) *and* $\boldsymbol{u}_h$ *the solution of* (2.21) *with* $h < h_0$*, then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq C'_{\mu\lambda} h^{k+\frac{1}{2}} |\boldsymbol{u}|_{k+1,\Omega},$$

*where* $C'_{\mu\lambda}$ *is a positive constant that depends on* $\mu$*,* $\lambda$ *and the mesh geometry.*

*Proof.* Let $\boldsymbol{z}$ satisfy the adjoint problem

$$\begin{aligned} -2\mu\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{z}) - \lambda\nabla(\nabla\cdot\boldsymbol{z}) &= \boldsymbol{u} - \boldsymbol{u}_h \quad &&\text{in } \Omega, \\ \boldsymbol{z} &= 0 &&\text{on } \partial\Omega. \end{aligned}$$

Then we can write

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{0,\Omega} &= (\boldsymbol{u} - \boldsymbol{u}_h, -2\mu\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{z}) - \lambda\nabla(\nabla\cdot\boldsymbol{z}))_\Omega \\ &= (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z}))_\Omega + (\lambda\nabla\cdot(\boldsymbol{u} - \boldsymbol{u}_h), \nabla\cdot\boldsymbol{z})_\Omega \\ &\quad - \langle 2\mu(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z})\cdot\boldsymbol{n}\rangle_{\partial\Omega} - \langle\lambda(\boldsymbol{u} - \boldsymbol{u}_h)\cdot\boldsymbol{n}, \nabla\cdot\boldsymbol{z}\rangle_{\partial\Omega} \\ &= A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}) - 2\langle 2\mu(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z})\cdot\boldsymbol{n}\rangle_{\partial\Omega} - 2\langle\lambda(\boldsymbol{u} - \boldsymbol{u}_h)\cdot\boldsymbol{n}, \nabla\cdot\boldsymbol{z}\rangle_{\partial\Omega}. \end{aligned}$$

By Lemma 2.2.4, using $(\boldsymbol{z} - i^1_{\mathsf{SZ}}\boldsymbol{z})|_{\partial\Omega} \equiv 0$ and (2.33) we deduce that

$$\begin{aligned} A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}) &= A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z} - i^1_{\mathsf{SZ}}\boldsymbol{z}) \\ &= (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z} - i^1_{\mathsf{SZ}}\boldsymbol{z}))_\Omega + (\lambda\nabla\cdot(\boldsymbol{u} - \boldsymbol{u}_h), \nabla\cdot(\boldsymbol{z} - i^1_{\mathsf{SZ}}\boldsymbol{z}))_\Omega \\ &\quad + \langle 2\mu(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z} - i^1_{\mathsf{SZ}}\boldsymbol{z})\cdot\boldsymbol{n}\rangle_{\partial\Omega} + \langle\lambda(\boldsymbol{u} - \boldsymbol{u}_h)\cdot\boldsymbol{n}, \nabla\cdot(\boldsymbol{z} - i^1_{\mathsf{SZ}}\boldsymbol{z})\rangle_{\partial\Omega} \\ &\lesssim (\lambda^{\frac{1}{2}} + \mu^{\frac{1}{2}})h|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!||\boldsymbol{z}|_{2,\Omega}. \end{aligned}$$

The global trace inequalities $\|\boldsymbol{\varepsilon}(\boldsymbol{z})\cdot\boldsymbol{n}\|_{0,\partial\Omega} \lesssim \|\boldsymbol{z}\|_{2,\Omega}$ and $\|\nabla\cdot\boldsymbol{z}\|_{0,\partial\Omega} \lesssim \|\boldsymbol{z}\|_{2,\Omega}$, lead to

$$|\langle 2\mu(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z})\cdot\boldsymbol{n}\rangle_{\partial\Omega}| + |\langle\lambda(\boldsymbol{u} - \boldsymbol{u}_h)\cdot\boldsymbol{n}, \nabla\cdot\boldsymbol{z}\rangle_{\partial\Omega}| \lesssim (\lambda^{\frac{1}{2}} + \mu^{\frac{1}{2}})h^{\frac{1}{2}}|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|\|\boldsymbol{z}\|_{2,\Omega}.$$

Then we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{0,\Omega} \lesssim C_{\mu\lambda}(\lambda^{\frac{1}{2}} + \mu^{\frac{1}{2}})(h + h^{\frac{1}{2}})h^k|\boldsymbol{u}|_{k+1,\Omega}\|\boldsymbol{z}\|_{2,\Omega}.$$

The regularity estimate (2.20) applied to the adjoint problem tells us that

$$\mu\|\boldsymbol{z}\|_{2,\Omega} \lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega},$$

the claim follows, note that $C'_{\mu\lambda} = \mathcal{O}((1 + \frac{\lambda}{\mu})^2)$. □

## 2.3 Incompressible elasticity

In this section we study the case of incompressible elasticity, for this problem the inf-sup condition must be shown to hold simultaneously for the displacment and the pressure.

We choose to work with equal order interpolation, the pressure is stabilised using a Galerkin least squares stabilisation. The incompressible elasticity problem with Dirichlet boundary condition is given by

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial\Omega. \end{aligned} \tag{2.34}$$

with $\boldsymbol{f} \in [L^2(\Omega)]^2$, $\boldsymbol{g} \in [H^{\frac{3}{2}}(\partial\Omega)]^2$ and the stress tensor

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbb{I}_{2\times 2}.$$

To determine uniquely $p$ we assume that $\int_\Omega p \, \mathrm{d}x = 0$. We have the following weak formulation: find $(\boldsymbol{u}, p) \in [H^1_g(\Omega)]^2 \times L^2(\Omega)$ such that

$$a[(\boldsymbol{u}, p), (\boldsymbol{v}, q)] = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall (\boldsymbol{v}, q) \in [H^1_0(\Omega)]^2 \times L^2(\Omega),$$

with

$$a[(\boldsymbol{u}, p), (\boldsymbol{v}, q)] = (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_\Omega - (p, \nabla \cdot \boldsymbol{v})_\Omega + (\nabla \cdot \boldsymbol{u}, q)_\Omega.$$

The well-posedness of this problem follows from the Lax-Milgram Lemma, we also have the regularity estimate [2]

$$\mu\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \lesssim \|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{\frac{3}{2},\partial\Omega}. \tag{2.35}$$

Let us define the space $Q = \{q \in L^2(\Omega),\ \int_\Omega q \, \mathrm{d}x = 0\}$. In order to introduce the discrete formulation, we define the finite element space

$$Q^k_h = \{q_h \in Q : q_h|_K \in \mathbb{P}_k(K) \ \ \forall K \in \mathcal{T}_h\}, \quad k \geq 1.$$

Applying the penalty-free Nitsche's method to the incompressible elasticity problem (2.34), the following finite element formulation is obtained: find $\boldsymbol{u}_h \in W^k_h$ and $p_h \in Q^k_h$ such that

$$A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = L_h(\boldsymbol{v}_h, q_h) \qquad \forall (\boldsymbol{v}_h, q_h) \in W^k_h \times Q^k_h, \tag{2.36}$$

where the linear forms $A_h$ and $L_h$ are defined as

$$\begin{aligned} A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] &= a[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] - b(\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) + b(\boldsymbol{v}_h, \boldsymbol{u}_h, q_h) \\ &\quad + S_h(\boldsymbol{u}_h, p_h, q_h), \\ L_h(\boldsymbol{v}_h, q_h) &= (\boldsymbol{f}, \boldsymbol{v}_h + \gamma_p \mu^{-1} h^2 \nabla q_h)_\Omega + b(\boldsymbol{v}_h, \boldsymbol{g}, q_h). \end{aligned}$$

The bilinear form $b$ is defined as

$$b(\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) = \langle (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h \mathbb{I}_{2\times 2}) \cdot \boldsymbol{n}, \boldsymbol{v}_h \rangle_{\partial\Omega}.$$

$S_h$ denotes the stabilisation term, different strategies can be used to stabilise the pressure [18, 24, 79, 80, 35, 15] here we consider a Galerkin least squares stabilisation [89, 12, 55]

$$S_h(\boldsymbol{u}_h, p_h, q_h) = \frac{\gamma_p}{\mu} \sum_{K \in \mathcal{T}_h} \int_K h^2(-2\mu \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \nabla p_h) \nabla q_h \, \mathrm{d}x.$$

This term is necessary as we want to use equal order interpolation. Note that if compatible function spaces are chosen this stabilisation term is not needed.

### 2.3.1 Inf-sup stability

We proceed similarly as the compressible case, we introduce the following norms for any functions $(\boldsymbol{w}, \varrho) \in [H^1(\Omega)]^2 \times L^2(\Omega)$ as

$$\begin{aligned} |||(\boldsymbol{w}, \varrho)|||^2 &= \mu(\|\nabla \boldsymbol{w}\|_{0,\Omega}^2 + h^{-1}\|\boldsymbol{w}\|_{0,\partial\Omega}^2) + h^2\mu^{-1}\|\nabla \varrho\|_{0,\Omega}^2, \\ \|(\boldsymbol{w}, \varrho)\|_*^2 &= |||(\boldsymbol{w}, \varrho)|||^2 + \mu h \|\nabla \boldsymbol{w} \cdot \boldsymbol{n}\|_{0,\partial\Omega}^2 + \mu^{-1}\|\varrho\|_{0,\Omega}^2 + \mu^{-1} h \|\varrho\|_{0,\partial\Omega}^2 + \mu h^{-2}\|\boldsymbol{w}\|_{0,\Omega}^2 \\ &\quad + h^2 \mu \sum_{K \in \mathcal{T}_h} \|\nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\|_{0,K}^2. \end{aligned}$$

**Lemma 2.3.1.** *For $\boldsymbol{u}_h, \boldsymbol{v}_h \in W_h^k$ with $\boldsymbol{v}_h = \boldsymbol{u}_h + \sum_{j=1}^{N_p} \boldsymbol{v}_j$, $\boldsymbol{v}_j$ defined by equations (2.27), and $q_h = p_h$, there exists positive constants $\beta_0$ and $h_0$ such that the following inequality holds for $h < h_0$*

$$\beta_0 |||(\boldsymbol{u}_h, p_h)|||^2 \leq A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)].$$

*Proof.* Decomposing the bilinear form, we can write the following

$$A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] + \sum_{j=1}^{N_p} A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_j, 0)].$$

Using the Cauchy-Schwarz inequality and the inverse inequality we can write

$$\begin{aligned} &A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] \\ &\qquad \geq 2\mu \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2 - \sum_{K \in \mathcal{T}_h} 2h \|\nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,K} \gamma_p h \|\nabla p_h\|_{0,K} + \frac{\gamma_p}{\mu} h^2 \|\nabla p_h\|_{0,\Omega}^2 \\ &\qquad \geq 2(1 - \epsilon')\mu \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2 + \frac{\gamma_p}{\mu}\Big(1 - \frac{C\gamma_p}{4\epsilon'}\Big) h^2 \|\nabla p_h\|_{0,\Omega}^2. \end{aligned}$$

The second part can be written as

$$A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_j, 0)] = (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_j))_{P_j} + (\nabla p_h, \boldsymbol{v}_j)_{P_j} \\ - \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\cdot\boldsymbol{n}, \boldsymbol{v}_j\rangle_{F_j} + \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_j)\cdot\boldsymbol{n}, \boldsymbol{u}_h\rangle_{F_j}.$$

We want to obtain a lower bound for each term, most of the terms have been studied in the compressible case. The lower bound of the only remaining term can be found using (2.28) and the inequality (2.15), we get

$$(\nabla p_h, \boldsymbol{v}_j)_{P_j} \leq \frac{\epsilon}{\mu}h^2\|\nabla p_h\|^2_{0,P_j} + \frac{C\alpha_1^2\mu}{2\epsilon}h^{-1}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \frac{C\alpha_2^2\mu}{2\epsilon}h^{-1}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j}.$$

Then we get

$$\begin{aligned} &A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] \\ &\quad\geq 2(1-\epsilon')\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \frac{\gamma_p}{\mu}\Big(1 - \frac{C\gamma_p}{4\epsilon'}\Big)h^2\|\nabla p_h\|^2_{0,\Omega} \\ &\quad - 2\epsilon\sum_{j=1}^{N_p}\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,P_j} - \frac{\epsilon}{\mu}\sum_{j=1}^{N_p}h^2\|\nabla p_h\|^2_{0,P_j} - 3\epsilon\sum_{j=1}^{N_p}\mu\|\nabla\boldsymbol{u}_h\|^2_{0,P_j} \\ &\quad + \alpha_1\Big(1 - \alpha_1\frac{11C}{4\epsilon}\Big)\sum_{j=1}^{N_p}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \alpha_2\Big(2 - \alpha_2\frac{15C}{4\epsilon}\Big)\sum_{j=1}^{N_p}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|^2_{0,F_j}. \end{aligned}$$

Similarly as for the compressible case, using the Theorem 2.2.1 and the inequality (2.12) we obtain

$$\begin{aligned} A_h[(\boldsymbol{u}_h, p_h),&(\boldsymbol{v}_h, q_h)] \\ &\geq C_a\mu\|\nabla\boldsymbol{u}_h\|^2_{\Omega\setminus P} + C_b\frac{h^2}{\mu}\|\nabla p_h\|^2_{\Omega\setminus P} + (C_c - C_e - C_f)\sum_{j=1}^{N_p}\mu\|\nabla\boldsymbol{u}_h\|^2_{0,P_j} \\ &+ C_d\sum_{j=1}^{N_p}\frac{h^2}{\mu}\|\nabla p_h\|^2_{0,P_j} + \frac{C_e}{2}\sum_{j=1}^{N_p}\frac{\mu}{h}\|\boldsymbol{u}_h\cdot\boldsymbol{\tau}\|^2_{0,F_j} + \frac{C_f}{2}\sum_{j=1}^{N_p}\frac{\mu}{h}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|^2_{0,F_j}, \end{aligned}$$

with the constants

$$\begin{aligned} C_a &= 2C_K(1-\epsilon'), \\ C_b &= \gamma_p\Big(1 - \frac{C\gamma_p}{4\epsilon'}\Big), \\ C_c &= 2C_K(1-\epsilon') - 5\epsilon, \\ C_d &= \gamma_p\Big(1 - \frac{C\gamma_p}{4\epsilon'}\Big) - \epsilon, \\ C_e &= \alpha_1\Big(1 - \alpha_1\frac{11C}{4\epsilon}\Big) - 2h(1-\epsilon'), \\ C_f &= \alpha_2\Big(2 - \alpha_2\frac{15C}{4\epsilon}\Big) - 2h(1-\epsilon'). \end{aligned}$$

We choose $\epsilon = \frac{\gamma_p^2}{4}$ and $\epsilon' = \frac{1}{4}$. Taking $\gamma_p < \frac{1}{C+\frac{1}{4}}$, $C_e$ and $C_f$ are positive, for $h < h_0$ $C_e$ and $C_f$ will be positive respectively for

$$\frac{\gamma_p^2}{11C} > \alpha_1, \qquad \frac{2\gamma_p^2}{15C} > \alpha_2.$$

$C_c - C_e - C_f$ will be positive for

$$\sqrt{\frac{2C_K}{5}} > \gamma_p, \qquad \frac{C_K}{2} > \alpha_1, \qquad \frac{C_K}{4} > \alpha_2.$$

$h_0$ is the biggest value of $h$ that can be considered, we observe that $\beta_0 = \mathcal{O}(1)$, $h_0 = \mathcal{O}(1)$. □

**Remark 2.3.1.** *Contrary to the case of compressible elasticity the conditions on the constants are independent of the physical parameters, this reflects that the mixed method is locking free.*

**Theorem 2.3.1.** *There exists positive constants $\beta$ and $h_0$ such that for all functions $(\boldsymbol{u}_h, p_h) \in W_h^k \times Q_h^k$ and for $h < h_0$, the following inequality holds*

$$\beta |||(\boldsymbol{u}_h, p_h)||| \leq \sup_{(\boldsymbol{v}_h, q_h) \in W_h^k \times Q_h^k} \frac{A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)]}{|||(\boldsymbol{v}_h, q_h)|||}.$$

*Proof.* Considering Lemma 2.3.1, we need to show

$$|||(\boldsymbol{v}_h, q_h)||| \lesssim |||(\boldsymbol{u}_h, p_h)|||.$$

Using the definition of the test functions, using similar argument as for (2.16) we have

$$|||(\boldsymbol{v}_h, q_h)|||^2 \lesssim |||(\boldsymbol{u}_h, p_h)|||^2 + \sum_{j=1}^{N_p} |||(\boldsymbol{v}_j, 0)|||^2.$$

By definition we have

$$|||(\boldsymbol{v}_j, 0)|||^2 = \mu(\|\nabla \boldsymbol{v}_j\|_{0,\Omega}^2 + h^{-1}\|\boldsymbol{v}_j\|_{0,\partial\Omega}^2).$$

The claim follows from (2.30) and (2.31). Note that $\beta = \mathcal{O}(1)$. □

### 2.3.2 A priori error estimate

In order to show the a priori error estimate, we state the following Galerkin orthogonality.

**Lemma 2.3.2.** *If $(\boldsymbol{u}, p) \in [H^2(\Omega)]^2 \times H^1(\Omega)$ is the solution of (2.34) and $(\boldsymbol{u}_h, p_h) \in W_h^k \times Q_h^k$ the solution of (2.36) the following property holds*

$$A_h[(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}_h, q_h)] = 0, \qquad \forall (\boldsymbol{v}_h, q_h) \in W_h^k \times Q_h^k.$$

**Lemma 2.3.3.** *Let* $(\boldsymbol{w}, \varrho) \in ([H^2(\Omega)]^2 + W_h^k) \times (H^1(\Omega) + Q_h^k)$ *and* $(\boldsymbol{v}_h, q_h) \in W_h^k \times Q_h^k$ *there exists a positive constant* $M$ *such that*

$$A_h[(\boldsymbol{w}, \varrho), (\boldsymbol{v}_h, q_h)] \leq M \|(\boldsymbol{w}, \varrho)\|_* |||(\boldsymbol{v}_h, q_h)|||.$$

*Proof.* The proof of the Lemma 2.2.5 gives us the desired upper bound for the terms $(2\mu\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_\Omega$, $\langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{w}) \cdot \boldsymbol{n}, \boldsymbol{v}_h\rangle_{\partial\Omega}$ and $\langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_h) \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\Omega}$. The integration by parts gives

$$(\nabla\varrho, \boldsymbol{w})_\Omega = \langle \varrho \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\Omega} - (\varrho, \nabla \cdot \boldsymbol{w})_\Omega.$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\langle \varrho \cdot \boldsymbol{n}, \boldsymbol{v}_h\rangle_{\partial\Omega} - (\varrho, \nabla \cdot \boldsymbol{v}_h)_\Omega - (\nabla q_h, \boldsymbol{w})_\Omega \lesssim \|(\boldsymbol{w}, \varrho)\|_* |||(\boldsymbol{v}_h, q_h)|||,\\
&\frac{\gamma_p}{\mu} \sum_{K\in\mathcal{T}_h} (h^2(-2\mu\nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) + \nabla\varrho), \nabla q_h)_K \lesssim \|(\boldsymbol{w}, \varrho)\|_* |||(\boldsymbol{v}_h, q_h)|||.
\end{aligned}$$

Note that the second line corresponds to the stabilisation term. □

**Theorem 2.3.2.** *If* $(\boldsymbol{u}, p) \in [H^{k+1}(\Omega)]^2 \times H^k(\Omega)$ *is the solution of* (2.34) *and* $(\boldsymbol{u}_h, p_h) \in W_h^k \times Q_h^k$ *the solution of* (2.36) *with* $h < h_0$*, then there holds*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leq C_\beta \inf_{(\boldsymbol{w}_h,\varrho_h)\in W_h^k\times Q_h^k} \|(\boldsymbol{u} - \boldsymbol{w}_h, p - \varrho_h)\|_*.$$

*where* $C_\beta$ *is a positive constant that depends on the mesh geometry.*

*Proof.* Let $(\boldsymbol{w}_h, \varrho_h) \in W_h^k \times Q_h^k$ using the triangle inequality we get

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leq |||(\boldsymbol{u} - \boldsymbol{w}_h, p - \varrho_h)||| + |||(\boldsymbol{w}_h - \boldsymbol{u}_h, \varrho_h - p_h)|||.$$

Using the Theorem 2.3.1, the Galerkin orthogonality and the Lemma 2.3.3 we obtain

$$\begin{aligned}
\beta|||(\boldsymbol{u}_h - \boldsymbol{w}_h, p_h - \varrho_h)||| &\leq \sup_{(\boldsymbol{v}_h,q_h)\in W_h^k\times Q_h^k} \frac{A_h[(\boldsymbol{u} - \boldsymbol{w}_h, p - \varrho_h), (\boldsymbol{v}_h, q_h)]}{|||(\boldsymbol{v}_h, q_h)|||}\\
&\leq M\|(\boldsymbol{u} - \boldsymbol{w}_h, p - \varrho_h)\|_*.
\end{aligned}$$

Note that $|||\boldsymbol{u} - \boldsymbol{w}_h||| \leq \|\boldsymbol{u} - \boldsymbol{w}_h\|_*$, taking the inf over all $\boldsymbol{w}_h$ we obtain

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leq \left(1 + \frac{M}{\beta}\right) \inf_{(\boldsymbol{w}_h,\varrho_h)\in W_h^k\times Q_h^k} \|(\boldsymbol{u} - \boldsymbol{w}_h, p - \varrho_h)\|_*.$$

We note that $C_\beta = \mathcal{O}(1)$. □

Let $i_{\mathsf{SZ}}^k$ denote the Scott-Zhang interpolant [99], for each $K \in \mathcal{T}_h$ and $p \in H^k(\Omega)$ we have

$$\|p - i_{\mathsf{SZ}}^k p\|_K + h_K\|\nabla(p - i_{\mathsf{SZ}}^k p)\|_K \lesssim h_K^k |p|_{k,S_K}. \tag{2.37}$$

**Corollary 2.3.1.** *If* $(\boldsymbol{u}, p) \in [H^{k+1}(\Omega)]^2 \times H^k(\Omega)$ *is the solution of* (2.34) *and* $(\boldsymbol{u}_h, p_h) \in W_h^k \times Q_h^k$ *the solution of* (2.36) *with* $h < h_0$*, then there holds*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leq h^k (C_{u\mu} |\boldsymbol{u}|_{k+1,\Omega} + C_{p\mu} |p|_{k,\Omega}).$$

*where* $C_{u\mu}$ *and* $C_{p\mu}$ *are positive constants that depends on* $\mu$ *and the mesh geometry.*

*Proof.* Using (2.32) and (2.37) we have

$$\begin{aligned} h\|\nabla(p - i_{\mathsf{SZ}}^k p)\|_{0,\Omega} &\lesssim h^k |p|_{k,\Omega}, \\ \|p - i_{\mathsf{SZ}}^k p\|_{0,\Omega} &\lesssim h^k |p|_{k,\Omega}, \\ h^{-1}\|\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u}\|_{0,\Omega} &\lesssim h^k |\boldsymbol{u}|_{k+1,\Omega}, \\ h\Big(\sum_{K\in\mathcal{T}_h} \|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u})\|_{0,K}^2\Big)^{\frac{1}{2}} &\lesssim h^k |\boldsymbol{u}|_{k+1,\Omega}. \end{aligned}$$

Using the trace inequality of Lemma 2.0.1 and (2.37) we have

$$h^{\frac{1}{2}}\|p - i_{\mathsf{SZ}}^k p\|_{0,\partial\Omega} \lesssim \|\nabla(p - i_{\mathsf{SZ}}^k p)\|_{0,\Omega} + h\Big(\sum_{K\in\mathcal{T}_h} \|D^2(p - i_{\mathsf{SZ}}^k p)\|_{0,K}^2\Big)^{\frac{1}{2}} \lesssim h^k |p|_{k,\Omega}.$$

Using the proof of Corollary 2.2.1 we deduce that

$$\|(\boldsymbol{u} - i_{\mathsf{SZ}}^k \boldsymbol{u}, p - i_{\mathsf{SZ}}^k p)\|_* \lesssim h^k (\mu^{\frac{1}{2}} |\boldsymbol{u}|_{k+1,\Omega} + \mu^{-\frac{1}{2}} |p|_{k,\Omega}).$$

Then we use Theorem 2.3.2 with $\boldsymbol{w}_h = i_{\mathsf{SZ}}^k \boldsymbol{u}$ and $\varrho_h = i_{\mathsf{SZ}}^k p$ to conclude. The constants are such that $C_{u\mu} = \mu^{\frac{1}{2}}$ and $C_{p\mu} = \mathcal{O}(\mu^{-\frac{1}{2}})$. □

**Remark 2.3.2.** *The convergence of the* $L^2$*-error of the displacement with the order* $\mathcal{O}(h^{k+\frac{1}{2}})$ *may be proven similarly as in Proposition 2.2.2 using the regularity* (2.35)*.*

**Proposition 2.3.1.** *Let* $(\boldsymbol{u}, p) \in [H^{k+1}(\Omega)]^2 \times H^k(\Omega)$ *be the solution of* (2.34) *and* $(\boldsymbol{u}_h, p_h) \in W_h^k \times Q_h^k$ *the solution of* (2.36) *with* $h < h_0$*, then*

$$\|p - p_h\|_{0,\Omega} \leq h^k (C'_{u\mu} |u|_{k+1,\Omega} + C'_{p\mu} |p|_{k,\Omega}),$$

*where* $C'_{u\mu}$ *and* $C'_{p\mu}$ *are positive constants that depends on* $\mu$ *and the mesh geometry.*

*Proof.* By the surjectivity of the divergence operator $\nabla\cdot : [H_0^1(\Omega)]^2 \to L_0^2(\Omega)$ (see, [62]), there exists $\boldsymbol{v}_p \in [H_0^1(\Omega)]^2$ such that $\nabla\cdot\boldsymbol{v}_p = p - p_h$. Therefore we may write (using the

Lemma 2.3.2 and observing that $(\boldsymbol{v}_p - i_{\mathsf{SZ}}^k \boldsymbol{v}_p)|_{\partial\Omega} = 0)$

$$\begin{aligned}
\|p - p_h\|_{0,\Omega}^2 &= (p - p_h, \nabla \cdot \boldsymbol{v}_p)_\Omega + A_h[(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (i_{\mathsf{SZ}}^k \boldsymbol{v}_p, 0)] \\
&= (p - p_h, \nabla \cdot (\boldsymbol{v}_p - i_{\mathsf{SZ}}^k \boldsymbol{v}_p))_\Omega \\
&\quad + (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(i_{\mathsf{SZ}}^k \boldsymbol{v}_p))_\Omega + \langle 2\mu \boldsymbol{\varepsilon}(i_{\mathsf{SZ}}^k \boldsymbol{v}_p) \cdot \boldsymbol{n}, \boldsymbol{u} - \boldsymbol{u}_h \rangle_{\partial\Omega} \\
&= -(\nabla(p - p_h), \boldsymbol{v}_p - i_{\mathsf{SZ}}^k \boldsymbol{v}_p)_\Omega \\
&\quad + (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(i_{\mathsf{SZ}}^k \boldsymbol{v}_p))_\Omega + \langle 2\mu \boldsymbol{\varepsilon}(i_{\mathsf{SZ}}^k \boldsymbol{v}_p) \cdot \boldsymbol{n}, \boldsymbol{u} - \boldsymbol{u}_h \rangle_{\partial\Omega} \\
&\lesssim h\mu^{-\frac{1}{2}} \|\nabla(p - p_h)\|_{0,\Omega} h^{-1} \mu^{\frac{1}{2}} \|\boldsymbol{v}_p - i_{\mathsf{SZ}}^k \boldsymbol{v}_p\|_{0,\Omega} \\
&\quad + 2\mu \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} \|\nabla i_{\mathsf{SZ}}^k \boldsymbol{v}_p\|_{0,\Omega} + 2\mu \|\nabla i_{\mathsf{SZ}}^k \boldsymbol{v}_p\|_{0,\Omega} h^{-\frac{1}{2}} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \\
&\lesssim \mu^{\frac{1}{2}} |||(\boldsymbol{u} - \boldsymbol{u}_h), (p - p_h)||| |\boldsymbol{v}_p|_{1,\Omega}.
\end{aligned}$$

We conclude by applying the stability $\|\boldsymbol{v}_p\|_{1,\Omega} \leq C_{\boldsymbol{v}_p} \|p - p_h\|_{0,\Omega}$. We observe that $C'_{u\mu} = \mathcal{O}(\mu)$ and $C'_{p\mu} = \mathcal{O}(1)$. □

## 2.4 Numerical results

In this section we will present some numerical experiments verifying the above theory. The package FreeFem++ [75] was used for the numerical study. In the first two sections we consider the domain $\Omega$ as the unit square $[0,1] \times [0,1]$. For compressible and incompressible elasticity we use a manufactured solution to test the precision of the method. In the third section we study the performance of the penalty-free Nitsche's method for the Cook's membrane problem.

### 2.4.1 Compressible elasticity

The two dimensional function below is a manufactured solution considered for the tests

$$\boldsymbol{u} = \begin{pmatrix} \left(x^5 - x^4\right)\left(y^3 - y^2\right) \\ \left(x^4 - x^3\right)\left(y^6 - y^5\right) \end{pmatrix}.$$

The nonsymmetric Nitsche's method given by equation (2.21) is used to compute approximations on a series of structured meshes. We consider first and second order polynomials and we study the convergence rates of the error in the $L^2$ and $H^1$-norms. We choose $\mu = 1$ and consider several values of $\lambda$ in order to see numerically the locking phenomena for large values of $\lambda$. The piecewise affine case (Figure 2.3) shows locking for $\lambda = 10^5$. When $\lambda$ becomes large, the error does not converge if $h$ is not small enough. When the piecewise quadratic approximation is used (Figure 2.4), the problem with large values of $\lambda$ only changes the value of the error constant and has negligible effect on the observed rates of convergence. The numerical results show that for both cases the rate of convergence of the $H^1$-error corresponds to what has been shown theoretically. For the $L^2$-error, we observe a convergence of order $\mathcal{O}\left(h^{k+1}\right)$, which is a super convergence

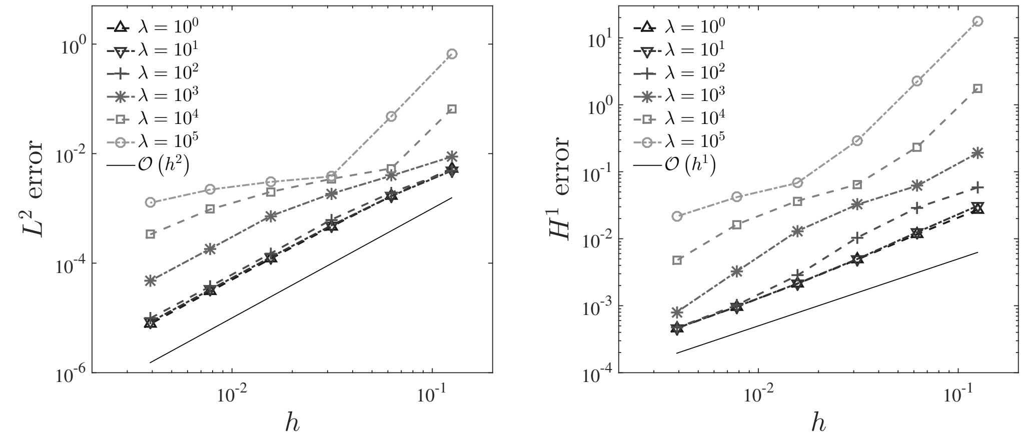



FIGURE 2.3: Compressible elasticity, $V_h^1$: error versus the maximal element diameter $h$. Left: $L^2$-error, right: $H^1$-error.

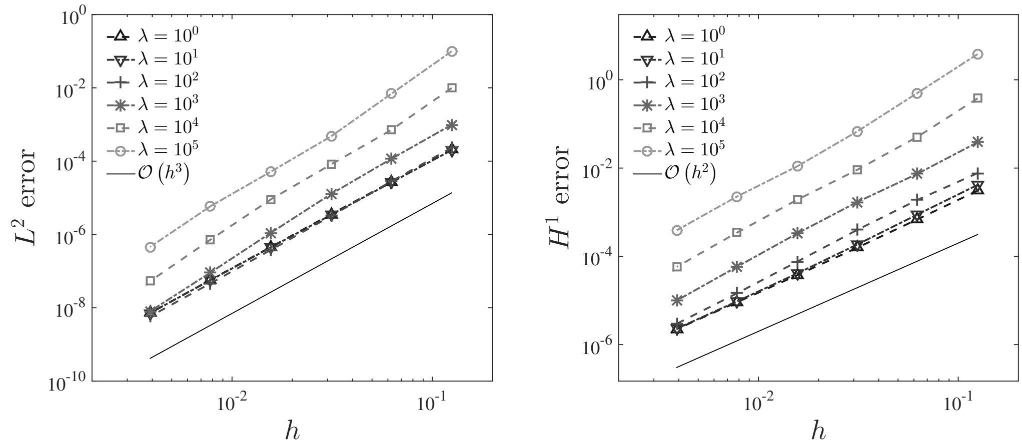



FIGURE 2.4: Compressible elasticity, $V_h^2$: error versus the maximal element diameter $h$. Left: $L^2$-error, right: $H^1$-error.

of order $\mathcal{O}(h^{\frac{1}{2}})$ compared to the theoretical result. In spite of numerous numerical experiments not reported here, we have not been able to find an example exhibiting the suboptimal $L^2$-convergence of Proposition 2.2.2.

### 2.4.2 Incompressible elasticity

The manufactured solution considered in this part defines the velocity and the pressure respectively such that

$$\boldsymbol{u} = \begin{pmatrix} \sin(4\pi x)\cos(4\pi y) \\ -\cos(4\pi x)\sin(4\pi y) \end{pmatrix}, \qquad p = \pi\cos(4\pi x)\cos(4\pi y).$$

The nonsymmetric Nitsche's method without penalty given by equation (2.36) is used to compute approximations on a series of structured meshes. We take $\mu = 1$, a range of values of $\gamma_p$ have been considered in the tests to study numerically the effect of the stabilisation parameter on the computational error. Figure 2.5 considers piecewise affine

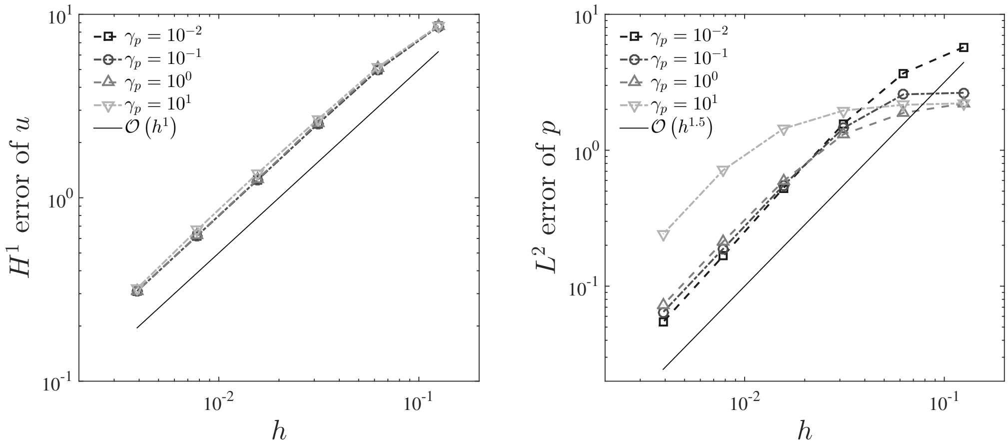



FIGURE 2.5: Incompressible elasticity, $V_h^1 \times Q_h^1$: errors for a range of value of $\gamma_p$ versus the maximal element diameter $h$. Left: $H^1$-error of the velocity, right : $L^2$-error of the pressure.

approximation. It shows that in this case the $H^1$-error of the velocity has an order of convergence $\mathcal{O}(h)$ for all the values of $\gamma_p$ tested. The convergence rates for the $L^2$-error of the pressure are close to $\mathcal{O}(h^{\frac{3}{2}})$ for all the values of $\gamma_p$ considered and for $h$ small enough.

### 2.4.3 Cook's membrane problem

The Cook's membrane problem is a bending dominated test case. Figure 2.6 represents the computational domain $\Omega$. On the face $(CD)$ the Dirichlet boundary condition $\boldsymbol{u} = 0$ is imposed. On the face $(AC)$ the Neumann boundary condition $\boldsymbol{\sigma}(\boldsymbol{u}) = (0, 100)$ is imposed. In this part we compare the results given by the strong and weak imposition of the Dirichlet boundary condition. The weak imposition is implemented using the non-symmetric Nitsche's method without penalty. We use first and second order polynomial approximations on unstructured meshes. For the first test $E = 10^5$ and $\nu = 0.3333$, we use compressible elasticity, note that $\mathcal{O}(\mu) = \mathcal{O}(\lambda)$ $(\mu = 37501,\ \lambda = 74979)$ . Figure 2.7 shows the deformed mesh obtained. We compute the vertical displacement of the point $A$ (top corner) versus the meshsize. Figure 2.8 shows the results for this case, by refining the mesh the approximation of the displacement of $A$ becomes more accurate. Both weak and strong imposition of the Dirichlet boundary are displayed. For first and second order approximation the weak imposition case converges faster than the strong imposition. For the second test we consider $E = 250$ and $\nu = 0.4999$, we expect to observe locking as $\mathcal{O}(\mu) \ll \mathcal{O}(\lambda)$ $(\mu = 83,\ \lambda = 416610)$. Using compressible elasticity we perform the

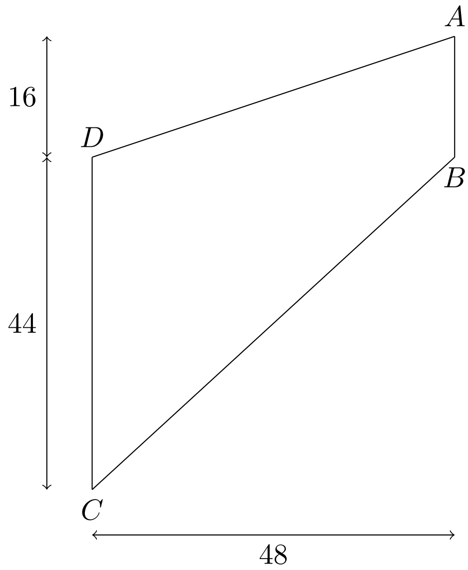



FIGURE 2.6: Cook's membrane, computational domain.

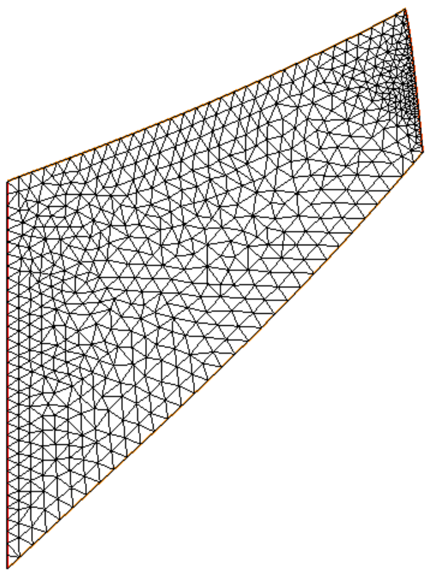


FIGURE 2.7: Deformed mesh, with a magnification factor of 10.

same tests as for the first study. Figure 2.9 represents the vertical displacement of the point $A$ (top corner) versus the meshsize. We observe locking for both methods for first order approximation. The second order approximation converges without locking even for the coarse meshes. Similarly as the previous case the convergence is faster for the weak imposition. In view of the observed locking, we use the nearly incompressible problem to perform the same computations. The nearly incompressible problem, is obtained considering (2.34) and replacing $\nabla \cdot \boldsymbol{u} = 0$ by $\nabla \cdot \boldsymbol{u} = p/\lambda$. Figure 2.10 displays the nearly incompressible elasticity for first and second order approximations for the weak

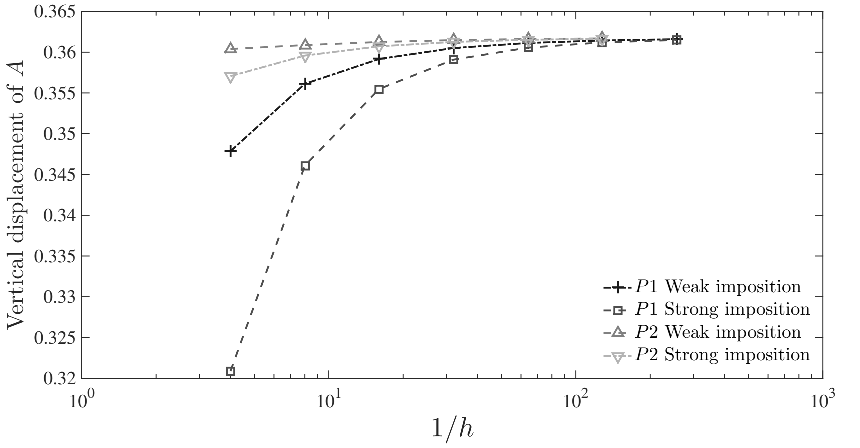



FIGURE 2.8: Convergence of the vertical displacement, $E = 10^5$ $\nu = 0.3333$.

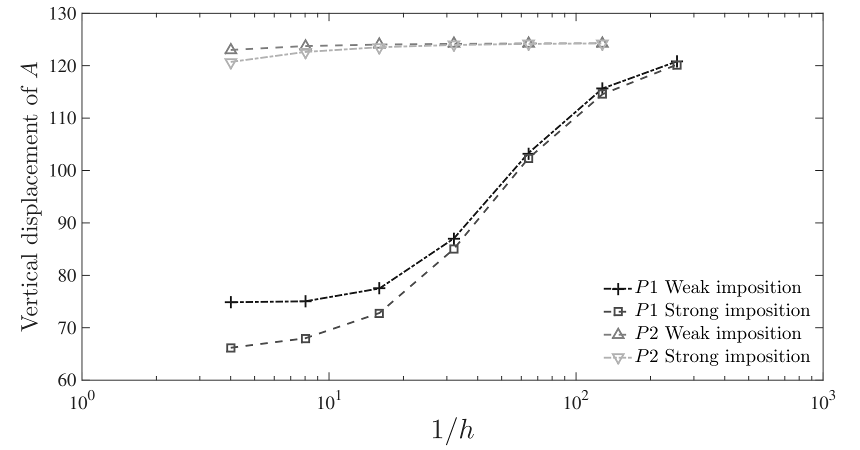



FIGURE 2.9: Convergence of the vertical displacement, $E = 250$ $\nu = 0.4999$.

and strong imposition but also the compressible elasticity with second order approximation. It shows that for nearly incompressible elasticity there is no locking for the method using first order polynomial approximation however for second order approximation the compressible elasticity converges faster than the nearly incompressible elasticity. Once again the weak imposition case converges faster than the strong imposition.

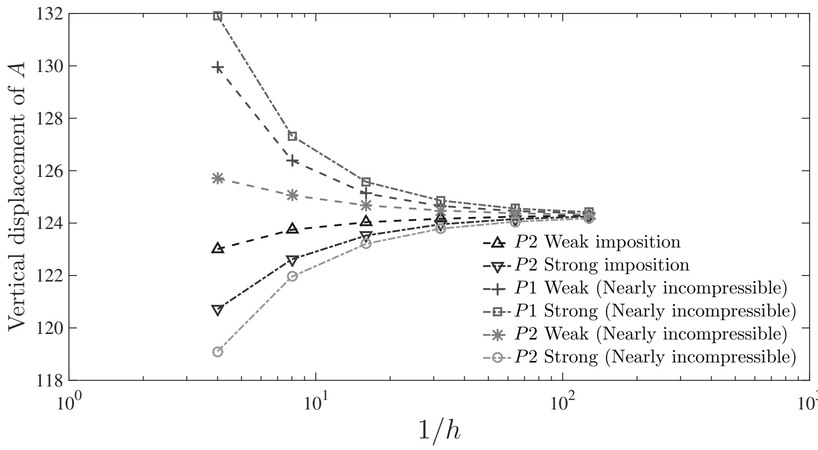



FIGURE 2.10: Convergence of the vertical displacement, $E = 250$ $\nu = 0.4999$.

# Chapter 3

# Fictitious domain

In the previous chapter the computational mesh was fitted to the physical domain, in this chapter we extend the study to the unfitted case when the physical domain is embedded in a background mesh. The physical boundary is allowed to cut elements of the mesh, the ghost penalty [25] is considered to ensure that the condition number is independent of the cuts.

## 3.1 Preliminaries

### 3.1.1 Unfitted framework

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\Gamma$. Let $\{\mathcal{T}_h\}_h$ be a family of quasi-uniform and shape regular triangulations (in the sense of (2.1)), $\Omega_{\mathcal{T}}$ is the domain covered by a mesh $\mathcal{T}_h$, the physical domain is embedded in this mesh, therefore $\Omega \subset \Omega_{\mathcal{T}}$. In a generic sense a node of the triangulation is designated by $x_i$, $K$ denotes a triangle of $\mathcal{T}_h$ and $F$ denotes a face of a triangle $K$. Figure 3.1 shows an example of configuration. Let us define the computational domain $\Omega^* = \{K \in \mathcal{T}_h \mid K \cap \Omega \neq \emptyset\}$, then we define

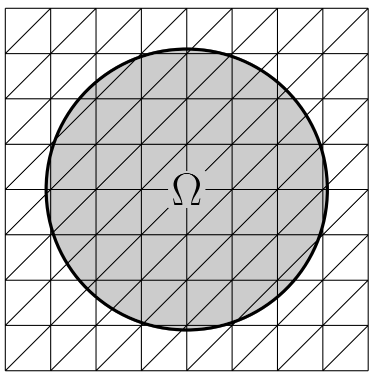



FIGURE 3.1: Fictitious domain, $\Omega$ is embedded in a background mesh.

$$V_h^k = \{v_h \in H^1(\Omega^*) : v_h|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\} \quad k \geq 1.$$

The trace inequality of Lemma 2.0.1 is extended such that for $w \in H^1(K)$

$$\|w\|_{0,K\cap\Gamma} \lesssim (h_K^{-\frac{1}{2}}\|w\|_{0,K} + h_K^{\frac{1}{2}}\|\nabla w\|_{0,K}) \qquad \forall K \in \mathcal{T}_h. \tag{3.1}$$

this result is shown in [67]. Let $\mathbb{E}$ be an $H^s$-extension on $\Omega^*$, $\mathbb{E} : H^s(\Omega) \to H^s(\Omega^*)$ such that $(\mathbb{E}w)|_\Omega = w$ and

$$\|\mathbb{E}w\|_{s,\Omega^*} \lesssim \|w\|_{s,\Omega} \qquad \forall w \in H^s(\Omega), s \geq 1. \tag{3.2}$$

For simplicity we will write $w$ instead of $\mathbb{E}w$ for any $w \in H^s(\Omega)$. Let $i_{\mathsf{SZ}} : H^s(\Omega^*) \to V_h^k$ be the Scott-Zhang interpolant [99], we construct the interpolation operator $\mathcal{I}_h$ such that

$$\mathcal{I}_h w = i_{\mathsf{SZ}}\mathbb{E}w. \tag{3.3}$$

Using the approximation property of the Scott-Zhang interpolant, we have the interpolation estimate for $0 \leq r \leq s \leq k+1$ and $v \in H^{k+1}(\Omega)$,

$$\|v - i_{\mathsf{SZ}}v\|_{r,K} \lesssim h^{s-r}|v|_{s,S_K} \qquad \forall K \in \mathcal{T}_h, \tag{3.4}$$

with $S_K = \text{interior}(\cup\{\overline{K}_i | \overline{K}_i \cap \overline{K} \neq \emptyset, K_i \in \mathcal{T}_h\})$. Using the estimate (3.2) with (3.4) we have

$$\Big(\sum_{K\in\Omega^*} \|v - \mathcal{I}_h v\|_{r,K}^2\Big)^{\frac{1}{2}} \lesssim h^{s-r}|v|_{s,\Omega}. \tag{3.5}$$

In order to follow the process used in the previous chapter to show the inf-sup condition in the different cases we need to adapt the structure of patches introduced in the Section 2.1.1. Let $G_h$ be the set of elements that intersect the boundary $\Gamma$

$$G_h = \{K \in \mathcal{T}_h \mid K \cap \Gamma \neq \emptyset\}.$$

For the sake of precision we make the following assumptions regarding the mesh $\mathcal{T}_h$ and the boundary $\Gamma$ :

- The boundary $\Gamma$ intersects each element boundary $\partial K$ exactly twice, and each (open) edge at most once for $K \in G_h$.

- Let $\Gamma_{K,h}$ be the straight line segment connecting the points of intersection between $\Gamma$ and $\partial K$. We assume that $\Gamma_K$ is a function of length on $\Gamma_{K,h}$ ; in local coordinates

$$\Gamma_{K,h} = \{(\xi,\eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = 0\}$$

  and

$$\Gamma_K = \{(\xi,\eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = \delta(\xi)\}.$$

- We assume that for all $K \in G_h$ there exists $K' \notin G_h$ and $K \cap K' \neq \emptyset$ and such that the measures of $K$ and $K'$ are comparable in the sense that there exists two positive constants $c_1$ and $c_2$ such that for $h_K = \text{diam}(K)$ we have

$$c_1 h \leq h_K \leq c_2 h, \qquad c_1 h \leq h_{K'} \leq c_2 h$$

and that the faces $F$ such that $K \cap F \neq \emptyset$ and $K' \cap F \neq \emptyset$ satisfy

$$h_{K'} \leq c_3 \text{meas}(F),$$

with $c_3$ a positive constant.

- We assume that in a triangle $K$ intersected by $\Gamma$, the normal $n_F$ of the face that does not intersects $\Gamma$ and the normal $n$ of $\Gamma$ verify

$$|n(\varsigma) \cdot n_F| \neq 0 \qquad \forall \varsigma \in \Gamma|_K. \tag{3.6}$$

### 3.1.2 An unfitted boundary mortaring

Let us split $G_h$ into $N_p$ smaller disjoint sets of elements $G_j$ with $j = 1, \ldots, N_p$. Let $I_{G_j}$ be the set of nodes belonging to $G_j$. We define the set of nodes $I_j$ such that

$$I_j = \{x_i \in I_{G_j} \mid x_i \in \Omega, \; x_i \notin I_{G_n} \; \forall n \neq j\},$$

we define $P_j$ for each $G_j$ such that

$$P_j = G_j \cup \{K \in \mathcal{T}_h \mid \exists x_i \in I_j \text{ such that } x_i \in K\}.$$

Each patch $P_j$ is constructed such that $I_j \neq \emptyset$. $\Gamma_j = \Gamma \cap G_j$ is the part of the boundary included in the patch $P_j$. For all $j$, the patch $P_j$ has the following properties

$$h \lesssim \text{meas}(\Gamma_j) \lesssim h \quad \text{ and } \quad h^2 \lesssim \text{meas}(P_j) \lesssim h^2. \tag{3.7}$$

Let $\chi_j \in V_h^1$ be defined for each node $x_i \in \mathcal{T}_h$ such that for each patch $P_j$

$$\chi_j(x_i) = \begin{cases} 0 & \text{for} \quad x_i \notin I_j \\ 1 & \text{for} \quad x_i \in I_j, \end{cases}$$

with $i = 1, \ldots, N_n$. Figure 3.2 shows an example of patch with the value of $\chi_j$ at each node.

**Lemma 3.1.1.** *For every patch $P_j$ with $1 \leq j \leq N_p$ ; $\forall r_j \in \mathbb{R}$ there exists $\varphi_r \in V_h^1$ such that*

$$\text{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla \varphi_r \cdot n \; \text{d}s = r_j, \tag{3.8}$$

*and the following property holds*

$$\|\nabla \varphi_r\|_{0,P_j} \lesssim h^{\frac{1}{2}} \|r_j\|_{0,\Gamma_j}. \tag{3.9}$$

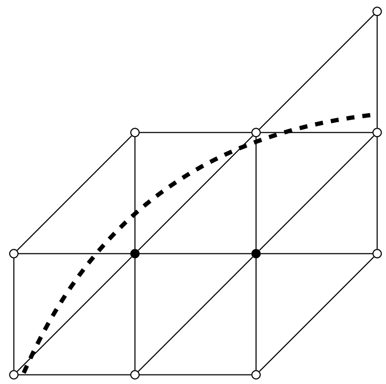


FIGURE 3.2: Example of $P_j$, the dashed line is $\Gamma_j$, $\chi_j$ is equal to 0 in the nonfilled nodes, 1 in the filled nodes.

*Proof.* Let

$$\Xi_j = \text{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla \chi_j \cdot n \, \mathrm{d}s.$$

The normalised function $\varphi_j$ is defined such that

$$\varphi_j = \Xi_j^{-1} \chi_j.$$

Let $K_1, \dots, K_m$ be the triangles crossed by the boundary within a patch $P_j$, considering (3.7) the number of triangles is small, then we have

$$|h\Xi_j| = \big|h\text{meas}(\Gamma_j)^{-1}\Big( \int_{\Gamma_j \cap K_1} \nabla \chi_j \cdot n \, \mathrm{d}s + \dots + \int_{\Gamma_j \cap K_m} \nabla \chi_j \cdot n \, \mathrm{d}s \Big)\big| \geq C_\Xi > 0,$$

where we used the fact that each integral is negative given by (3.6). This lower bound that holds uniformly in $j$ and $h$ tells us that $\varphi_j$ is well defined, the constant $C_\Xi$ depends only on the local geometry of the patches $P_j$. By definition there holds

$$\text{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla \varphi_j \cdot n \, \mathrm{d}s = 1, \tag{3.10}$$

using the inverse inequality of Lemma 2.0.2 we obtain

$$\|\nabla \varphi_j\|_{0,P_j} \lesssim |h^{-1}\Xi_j^{-1}| \|\chi_j\|_{0,P_j} \lesssim |h^{-1}\Xi_j^{-1}| \text{meas}(P_j)^{\frac{1}{2}} \lesssim C_\Xi^{-1} h. \tag{3.11}$$

Let $\varphi_r = r_j \varphi_j$, then condition (3.8) is verified considering (3.10). The upper bound (3.9) is obtained using (3.11), (3.7) and

$$\|\nabla \varphi_r\|_{0,P_j} = |r_j| \|\nabla \varphi_j\|_{0,P_j} \lesssim \text{meas}(\Gamma_j)^{\frac{1}{2}} |r_j| h^{\frac{1}{2}} = h^{\frac{1}{2}} \|r_j\|_{0,\Gamma_j}.$$

□

It is straightforward to observe that (2.11) and (2.12) still hold in this new framework,

$$h\|\nabla u_h \cdot \tau\|_{0,\Gamma_j} \gtrsim \|u_h - \overline{u_h}^{\Gamma_j}\|_{0,\Gamma_j}. \tag{3.12}$$

$$h^{-\frac{1}{2}}\|\overline{u_h}^{\Gamma_j}\|_{0,\Gamma_j} \geq h^{-\frac{1}{2}}\|u_h\|_{0,\Gamma_j} - C'\|\nabla u_h\|_{0,P_j}. \tag{3.13}$$

## 3.2 Poisson problem

The Poisson problem with Dirichlet boundary conditions is given by

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma. \end{aligned} \tag{3.14}$$

with $f \in L^2(\Omega)$ and $g \in H^{\frac{3}{2}}(\Gamma)$. The weak formulation can be expressed as: find $u \in H^1_g(\Omega)$ such that

$$a(u,v) = (f,v)_\Omega \qquad \forall v \in H^1_0(\Omega),$$

with

$$a(u,v) = (\nabla u, \nabla v)_\Omega.$$

The Lax-Milgram Lemma gives us the well-posedness of this problem. Under the assumptions on $\Omega$ we have the following regularity estimate [64]

$$\|u\|_{s+2,\Omega} \lesssim \|f\|_{s,\Omega} + \|g\|_{s+\frac{3}{2},\Gamma} \qquad \forall s \geq 0. \tag{3.15}$$

The finite element formulation using the penalty free Nitsche's method reads: find $u_h \in V_h^k$

$$A_h(u_h,v_h) + J_h(u_h,v_h) = L_h(v_h) \qquad \forall v_h \in V_h^k, \tag{3.16}$$

the term $J_h$ is the ghost penalty [25], the linear forms are defined as

$$\begin{aligned} A_h(u_h,v_h) &= (\nabla u_h, \nabla v_h)_\Omega - \langle \nabla u_h \cdot n, v_h\rangle_\Gamma + \langle \nabla v_h \cdot n, u_h\rangle_\Gamma, \\ J_h(u_h,v_h) &= \gamma_g \sum_{F\in\mathcal{F}_G} \sum_{l=1}^{k} h^{2l-1} \langle [\![D^l_{n_F} u_h]\!]_F, [\![D^l_{n_F} v_h]\!]_F\rangle_F, \\ L_h(v_h) &= (f, v_h)_\Omega + \langle \nabla v_h \cdot n, g\rangle_\Gamma, \end{aligned}$$

with $\mathcal{F}_G = \{F \in G_h \mid F \cap \Omega \neq \emptyset\}$, $\gamma_g = \mathcal{O}(1)$ the ghost penalty parameter and $n_F$ the unit normal to the face $F$ with fixed but arbitrary orientation. $D^l_{n_F}$ is the partial derivative of order $l$ in the direction of the normal $n_F$. $[\![w]\!]_F = w_F^+ - w_F^-$, with $w_F^\pm = \lim_{s\to 0^+} w(x \mp s n_F)$, is the jump of $w$ across the face $F$. The ghost penalty provides the control of the gradient for any function $v_h \in V_h^k$ on $\Omega$, it is characterised by the following property that has been proved in [89]

$$\|\nabla v_h\|^2_{0,\Omega^*} \lesssim \|\nabla v_h\|^2_{0,\Omega} + J_h(v_h,v_h) \lesssim \|\nabla v_h\|^2_{0,\Omega^*}. \tag{3.17}$$

Let us introduce the norms

$$|||w|||^2 = \|\nabla w\|_{0,\Omega}^2 + h^{-1}\|w\|_{0,\Gamma}^2 + J_h(w,w),$$
$$\|w\|_*^2 = |||w|||^2 + h\|\nabla w \cdot n\|_{0,\Gamma}^2.$$

### 3.2.1 Inf-sup stability

**Theorem 3.2.1.** *There exists $\beta > 0$ such that for all functions $u_h \in V_h^k$ the following inequality holds*

$$\beta |||u_h||| \leq \sup_{v_h \in V_h^k} \frac{A_h(u_h,v_h) + J_h(u_h,v_h)}{|||v_h|||}.$$

*Proof.* Let $v_h = u_h + \alpha \sum_{j=1}^{N_p} v_j$, such that $v_j = \nu_j \chi_j$, with $\nu_j \in \mathbb{R}$, each $v_j$ has the property

$$\text{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla v_j \cdot n \, \text{d}s = h^{-1} \overline{u_h}^{\Gamma_j}. \tag{3.18}$$

Replacing $v_h$ in the bilinear form

$$(A_h + J_h)(u_h, v_h) = (A_h + J_h)(u_h, u_h) + \alpha \sum_{j=1}^{N_p} [A_h(u_h, v_j) + J_h(u_h, v_j)].$$

Using the inequality (3.17) we have

$$(A_h + J_h)(u_h, u_h) = \|\nabla u_h\|_{0,\Omega}^2 + J_h(u_h, u_h) \gtrsim \|\nabla u_h\|_{0,\Omega^*}^2.$$

Applying (3.18) we get

$$\alpha \langle \nabla v_j \cdot n, u_h \rangle_{\Gamma_j} = \alpha h^{-1} \|\overline{u_h}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \alpha \langle \nabla v_j \cdot n, u_h - \overline{u_h}^{\Gamma_j} \rangle_{\Gamma_j},$$

the second term can be bounded using (3.12), the trace inequality (3.1) and the inverse inequality of Lemma 2.0.2,

$$\alpha \langle \nabla v_j \cdot n, u_h - \overline{u_h}^{\Gamma_j} \rangle_{\Gamma_j} \lesssim \alpha \|\nabla u_h\|_{0,P_j} \|\nabla v_j\|_{0,P_j} \leq \epsilon \|\nabla u_h\|_{0,P_j}^2 + \frac{C\alpha^2}{4\epsilon} \|\nabla v_j\|_{0,P_j}^2.$$

Using (3.17) and the Young's inequality we get

$$\begin{aligned} \alpha(\nabla u_h, \nabla v_j)_{P_j \cap \Omega} + \alpha J_h(u_h, v_j) &\lesssim \alpha \|\nabla u_h\|_{0,P_j \cap \Omega} \|\nabla v_j\|_{0,P_j \cap \Omega} + \alpha J_h(u_h, u_h)^{\frac{1}{2}} J_h(v_j, v_j)^{\frac{1}{2}} \\ &\leq \epsilon \|\nabla u_h\|_{0,P_j}^2 + \frac{C\alpha^2}{4\epsilon} \|\nabla v_j\|_{0,P_j}^2. \end{aligned}$$

The inequality (2.15) still holds in this context

$$\|v_j\|_{0,P_j} \lesssim h \|\nabla v_j\|_{0,P_j}, \tag{3.19}$$

using this result, the trace inequality (3.1) and the inverse inequality we get

$$\langle \nabla u_h \cdot n, v_j \rangle_{\Gamma_j} \lesssim \|\nabla u_h\|_{0,P_j} \|\nabla v_j\|_{0,P_j} \leq \epsilon \|\nabla u_h\|^2_{0,P_j} + \frac{C\alpha^2}{4\epsilon} \|\nabla v_j\|^2_{0,P_j}.$$

Then using Lemma 3.1.1

$$A_h(u_h, v_h) \geq (C - 3\epsilon)\|\nabla u_h\|^2_{0,\Omega^*} + \alpha\Big(1 - \frac{3C\alpha}{4\epsilon}\Big) \sum_{j=1}^{N_p} h^{-1} \|\overline{u_h}^{\Gamma_j}\|^2_{0,\Gamma_j}.$$

Applying (3.13) and (3.17)

$$(A_h + J_h)(u_h, v_h) \geq (C - \epsilon - C'\alpha)(\|\nabla u_h\|^2_{0,\Omega} + J_h(u_h, u_h)) + \frac{\alpha}{2}\Big(1 - \frac{3C\alpha}{4\epsilon}\Big) \sum_{j=1}^{N_p} h^{-1} \|u_h\|^2_{0,\Gamma_j}.$$

By choosing $\alpha$ and $\epsilon$ in the right way we obtain

$$A_h(u_h, v_h) \gtrsim |||u_h|||^2.$$

To complete the proof we need $|||v_h||| \lesssim |||u_h|||$, we know that

$$|||v_h|||^2 \lesssim |||u_h|||^2 + \alpha^2 \sum_{j=1}^{N_p} |||v_j|||^2 \quad \text{with} \quad |||v_j|||^2 = \|\nabla v_j\|^2_{0,P_j \cap \Omega} + h^{-1}\|v_j\|^2_{0,\Gamma_j} + J_h(v_j, v_j).$$

Using the trace and inverse inequalities, (3.19), (3.17) and the Lemma 3.1.1 we have

$$h^{-1}\|v_j\|^2_{0,\Gamma_j} \lesssim \|\nabla v_j\|^2_{0,P_j} \lesssim \|\nabla v_j\|^2_{0,P_j \cap \Omega} + J_h(v_j, v_j) \lesssim h^{-1} \|\overline{u_h}^{\Gamma_j}\|^2_{0,\Gamma_j} \lesssim h^{-1}\|u_h\|^2_{0,\Gamma_j}.$$

As a consequence we get $\sum_{j=1}^{N_p} |||v_j|||^2 \lesssim |||u_h|||^2$ which completes the proof. □

### 3.2.2 A priori error estimate

The consistency of the scheme is characterised by the following orthogonality relation.

**Lemma 3.2.1.** *Let $u_h \in V_h^k$ be the solution of* (3.16) *and $u \in H^2(\Omega)$ be the solution of* (3.14)*, then*

$$A_h(u - u_h, v_h) - J_h(u_h, v_h) = 0, \qquad \forall v_h \in V_h^k.$$

*Proof.* $A_h(u, v_h) = L_h(v_h) = A_h(u_h, v_h) + J_h(u_h, v_h), \ \forall v_h \in V_h^k.$ □

**Lemma 3.2.2.** *Let $w \in H^2(\Omega) + V_h^k$ and $v_h \in V_h^k$, there exists a positive constant $M$ such that*

$$A_h(w, v_h) \leq M \|w\|_* |||v_h|||.$$

*Proof.* Using the trace inequality, the inverse inequality and (3.17) we get

$$\begin{aligned}&(\nabla w, \nabla v_h)_\Omega - \langle \nabla w \cdot n, v_h\rangle_\Gamma + \langle \nabla v_h \cdot n, w\rangle_\Gamma \\ &\lesssim \|\nabla w\|_{0,\Omega}\|\nabla v_h\|_{0,\Omega} + \|\nabla w \cdot n\|_{0,\Gamma}\|v_h\|_{0,\Gamma} + (\|\nabla v_h\|_{0,\Omega} + J_h(v_h, v_h)^{\frac{1}{2}})h^{-\frac{1}{2}}\|w\|_{0,\Gamma}.\end{aligned}$$

□

**Theorem 3.2.2.** *Let $u \in H^{k+1}(\Omega)$ be the solution of* (3.14) *and $u_h \in V_h^k$ the solution of* (3.16)*, then there holds*

$$|||u - u_h||| \lesssim \inf_{w_h \in V_h^k} \|u - w_h\|_*.$$

*Proof.* Let $w_h \in V_h^k$, the triangle inequality gives us

$$|||u - u_h||| \leq |||u - w_h||| + |||w_h - u_h|||.$$

The regularity of $u$ gives $J_h(u, v_h) = 0$ then we have

$$J_h(w_h, v_h) \leq J_h(w_h - u, w_h - u)^{\frac{1}{2}} J_h(v_h, v_h)^{\frac{1}{2}}.$$

Using this result, the orthogonality of Lemma 3.2.1, the Theorem 3.2.1 and the Lemma 3.2.2 we can write

$$\beta|||u_h - w_h||| \leq \sup_{v_h \in V_h^k} \frac{A_h(u - w_h, v_h) - J_h(w_h, v_h)}{|||v_h|||} \leq (M + 1)\|u - w_h\|_*.$$

Note that $|||u - w_h||| \leq \|u - w_h\|_*$, using this result we get

$$|||u - u_h||| \leq \Big(1 + \frac{M+1}{\beta}\Big) \inf_{w_h \in V_h^k} \|u - w_h\|_*.$$

□

**Corollary 3.2.1.** *Let $u \in H^{k+1}(\Omega)$ be the solution of* (3.14) *and $u_h \in V_h^k$ the solution of* (3.16)*, then there holds*

$$|||u - u_h||| \lesssim h^k |u|_{k+1,\Omega}.$$

*Proof.* The proof follows the arguments used to prove Corollary 2.1.1 using the trace inequality (3.1) and (3.5). Additionally we extend the ghost penalty term to the full domain $\Omega^*$ to obtain

$$J_h(u - \mathcal{I}_h u, u - \mathcal{I}_h u)^{\frac{1}{2}} \lesssim \Big( \sum_{K \in \Omega^*} \sum_{l=k}^{k+1} h^{2(l-1)} \|D^l(u - \mathcal{I}_h u)\|_{0,K}^2 \Big)^{\frac{1}{2}} \lesssim h^k |u|_{k+1,\Omega}.$$

Then we have the following estimate

$$\|u - \mathcal{I}_h u\|_* \lesssim h^k |u|_{k+1,\Omega}. \tag{3.20}$$

We conclude using the Theorem 3.2.2 with $w_h = \mathcal{I}_h u$. □

**Proposition 3.2.1.** *Let $u \in H^{k+1}(\Omega)$ be the solution of* (3.14) *and $u_h \in V_h^k$ the solution of* (3.16)*, then there holds*

$$\|u - u_h\|_\Omega \lesssim h^{k+\frac{1}{2}} |u|_{k+1,\Omega}.$$

*Proof.* Let $z$ satisfy the adjoint problem

$$\begin{aligned} -\Delta z &= u - u_h && \text{in } \Omega, \\ z &= 0 && \text{on } \Gamma. \end{aligned}$$

Using integration by parts we can write

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 = (u - u_h, -\Delta z)_\Omega &= (\nabla(u - u_h), \nabla z)_\Omega - \langle u - u_h, \nabla z \cdot n \rangle_\Gamma \\ &= A_h(u - u_h, z) - 2\langle u - u_h, \nabla z \cdot n \rangle_\Gamma. \end{aligned}$$

Using the orthogonality proprety of Lemma 3.2.1 we have

$$A_h(u - u_h, z) = A_h(u - u_h, z - \mathcal{I}_h z) + J_h(u_h, \mathcal{I}_h z).$$

Note that using the trace inequality (3.1), the estimate (3.5) and the inverse inequality of Lemma 2.0.2 we have

$$\begin{aligned} h^{\frac{1}{2}} \|\nabla(u - u_h) \cdot n\|_{0,\Gamma} &\lesssim \|\nabla(u - u_h)\|_{0,\Omega^*} + h\Big(\sum_{K \in \Omega_*} \|D^2(u - u_h)\|_{0,K}^2\Big)^{\frac{1}{2}} \\ &\lesssim \|\nabla(u - \mathcal{I}_h u)\|_{0,\Omega^*} + h\Big(\sum_{K \in \Omega_*} \|D^2(u - \mathcal{I}_h u)\|_{0,K}^2\Big)^{\frac{1}{2}} \\ &\quad + \|\nabla(u_h - \mathcal{I}_h u)\|_{0,\Omega^*} + h\Big(\sum_{K \in \Omega_*} \|D^2(u_h - \mathcal{I}_h u)\|_{0,K}^2\Big)^{\frac{1}{2}} \\ &\lesssim h^k |u|_{k+1,\Omega} + |||u_h - \mathcal{I}_h u|||. \end{aligned} \tag{3.21}$$

Using this result, the Cauchy-Schwarz inequality, the Corollary 3.2.1 and (3.5) we obtain

$$\begin{aligned} &A_h(u - u_h, z - \mathcal{I}_h z) \\ &\quad = (\nabla(u - u_h), \nabla(z - \mathcal{I}_h z))_\Omega - \langle z - \mathcal{I}_h z, \nabla(u - u_h) \cdot n \rangle_\Gamma + \langle u - u_h, \nabla(z - \mathcal{I}_h z) \cdot n \rangle_\Gamma \\ &\quad \leq (|||u - u_h||| + h^{\frac{1}{2}} \|\nabla(u - u_h) \cdot n\|_{0,\Gamma}) \|z - \mathcal{I}_h z\|_* \\ &\quad \lesssim (h^k |u|_{k+1,\Omega} + |||u_h - \mathcal{I}_h u|||) h |z|_{2,\Omega}. \end{aligned}$$

The regularity of $z$ gives us $J_h(u_h, z) = 0$, using (3.5) we have

$$\begin{aligned} J_h(u_h, \mathcal{I}_h z) &\leq J_h(u_h, u_h)^{\frac{1}{2}} J_h(\mathcal{I}_h z - z, \mathcal{I}_h z - z)^{\frac{1}{2}} \\ &\leq (|||u_h - \mathcal{I}_h u||| + J_h(\mathcal{I}_h u, \mathcal{I}_h u)^{\frac{1}{2}}) J_h(\mathcal{I}_h z - z, \mathcal{I}_h z - z)^{\frac{1}{2}} \\ &\lesssim (|||u_h - \mathcal{I}_h u||| + h^k |u|_{k+1,\Omega}) h |z|_{2,\Omega}, \end{aligned}$$

where we also used $J_h(\mathcal{I}_h u, \mathcal{I}_h u)^{\frac{1}{2}} = J_h(\mathcal{I}_h u - u, \mathcal{I}_h u - u)^{\frac{1}{2}}$ which follows from the regularity of $u$. Using the global trace inequality $\|\nabla z \cdot n\|_{0,\Gamma} \lesssim \|z\|_{2,\Omega}$

$$\langle u - u_h, \nabla z \cdot n \rangle_\Gamma \lesssim h^{\frac{1}{2}} |||u - u_h||| \|z\|_{2,\Omega}.$$

In the proof of Theorem 3.2.2 we have shown that

$$|||u_h - \mathcal{I}_h u||| \lesssim \|u - \mathcal{I}_h u\|_* \lesssim h^k |u|_{k+1,\Omega},$$

where the second estimate is (3.20). Using this result and Corollary 3.2.1 once again we have

$$\|u - u_h\|_{0,\Omega}^2 \lesssim (h + h^{\frac{1}{2}}) h^k |u|_{k+1,\Omega} \|z\|_{2,\Omega},$$

we conclude using $\|z\|_{2,\Omega} \lesssim \|u - u_h\|_{0,\Omega}$ that follows from the regularity estimate (3.15) applied to the adjoint problem. □

## 3.3 Compressible elasticity

The compressible elasticity problem with Dirichlet boundary condition is given by

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \Gamma. \end{aligned} \tag{3.22}$$

with $\boldsymbol{f} \in [L^2(\Omega)]^2$, $\boldsymbol{g} \in [H^{\frac{3}{2}}(\Gamma)]^2$ and the stress tensor

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda(\nabla \cdot \boldsymbol{u})\mathbb{I}_{2\times 2},$$

with $\mu$ and $\lambda$ the Lamé coefficients. The weak formulation of this problem gives: find $\boldsymbol{u} \in [H_g^1(\Omega)]^2$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall \boldsymbol{v} \in [H_0^1(\Omega)]^2,$$

with

$$a(\boldsymbol{u}, \boldsymbol{v}) = (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_\Omega + (\lambda \nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_\Omega.$$

The well-posedness of this problem follows from the Lax-Milgram Lemma. The following elliptic regularity estimate [103] holds

$$\mu\|\boldsymbol{u}\|_{s+2,\Omega} + (\mu+\lambda)\|\nabla\cdot\boldsymbol{u}\|_{s+1,\Omega} \lesssim \|\boldsymbol{f}\|_{s,\Omega} + \|\boldsymbol{g}\|_{s+\frac{3}{2},\Gamma} \qquad \forall s \geq 0. \tag{3.23}$$

The finite element formulation using the penalty free Nitsche's method is given by: find $\boldsymbol{u}_h \in W_h^k$

$$A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + J_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = L_h(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in W_h^k, \tag{3.24}$$

with $W_h^k = [V_h^k]^2$, the linear forms $A_h$, $J_h$ and $L_h$ are defined as

$$\begin{aligned} A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) &= a(\boldsymbol{u}_h,\boldsymbol{v}_h) - b(\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}_h,\boldsymbol{u}_h), \\ J_h(\boldsymbol{u}_h,\boldsymbol{v}_h) &= \gamma_g \mu \sum_{F\in\mathcal{F}_G}\sum_{l=1}^{k} h^{2l-1}\langle [\![D_{n_F}^l \boldsymbol{u}_h]\!]_F, [\![D_{n_F}^l \boldsymbol{v}_h]\!]_F\rangle_F, \\ L_h(\boldsymbol{v}_h) &= (\boldsymbol{f},\boldsymbol{v}_h)_\Omega + b(\boldsymbol{v}_h,\boldsymbol{g}). \end{aligned}$$

The bilinear form $b$ is such that

$$b(\boldsymbol{u}_h,\boldsymbol{v}_h) = \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\cdot\boldsymbol{n},\boldsymbol{v}_h\rangle_\Gamma + \langle\lambda\nabla\cdot\boldsymbol{u}_h,\boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_\Gamma.$$

### 3.3.1 Inf-sup stability

In order to show an inf-sup condition for this case we define the average normal vector on $\Gamma_j$, $\overline{\boldsymbol{n}}^{\Gamma_j}$, in the same way $\overline{\boldsymbol{\tau}}^{\Gamma_j}$ is the corresponding average tangent vector. We define the rotated frame $(\xi,\eta)$ the direction of the $\eta$ axis is the same as $\overline{\boldsymbol{n}}^{\Gamma_j}$and the direction of the $\xi$ axis is the same as $\overline{\boldsymbol{\tau}}^{\Gamma_j}$. As in the previous chapter the hat denotes a quantity expressed in the frame $(\xi,\eta)$, a function $\boldsymbol{z} = (z_1,z_2)$ expressed in the two-dimentional rotated frame has the following properties

$$\hat{z}_1 = \boldsymbol{z}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}, \qquad \hat{z}_2 = \boldsymbol{z}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}.$$

By looking at the proof of Lemma 3.1.1 it is straightforward to observe that the following Lemma holds true.

**Lemma 3.3.1.** *For every patch $P_j$ with $1 \leq j \leq N_p$ ; $\forall r_j \in \mathbb{R}$ there exists $\varphi_r \in V_h^1$ such that*

$$\operatorname{meas}(\Gamma_j)^{-1}\int_{\Gamma_j} \nabla\varphi_r\cdot\overline{n}^{\Gamma_j}\ \mathrm{d}s = r_j,$$

*and the following property holds*

$$\|\nabla\varphi_r\|_{P_j} \lesssim h^{\frac{1}{2}}\|r_j\|_{\Gamma_j}.$$

Let $\boldsymbol{v}_j \in V_h^1$ be the two dimmentional function such that $\boldsymbol{v}_j = (\alpha_1 v_1, \alpha_2 v_2)^T$, we define $v_1 = \nu_{j1}\chi_j$ and $v_2 = \nu_{j2}\chi_j$ with $\nu_{j1},\nu_{j2} \in \mathbb{R}$ and $\chi_j$ as defined in Section 3.1.2.

Let

$$\text{meas}(\Gamma_j)^{-1}\int_{\Gamma_j}\frac{\partial \hat{v}_1}{\partial \eta}\,\mathrm{d}s = h^{-1}\overline{\hat{u}_1}^{\Gamma_j}, \qquad \text{meas}(\Gamma_j)^{-1}\int_{\Gamma_j}\frac{\partial \hat{v}_2}{\partial \eta}\,\mathrm{d}s = h^{-1}\overline{\hat{u}_2}^{\Gamma_j}, \tag{3.25}$$

applying the Lemma 3.3.1 we get

$$\|\hat{\nabla}\hat{v}_1\|_{0,P_j} \lesssim h^{-\frac{1}{2}}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}, \qquad \|\hat{\nabla}\hat{v}_2\|_{0,P_j} \lesssim h^{-\frac{1}{2}}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}. \tag{3.26}$$

We start by giving two technical Lemmas, proofs are provided in appendix D.

**Lemma 3.3.2.** *There exists $C > 0$ independent of $h$, $\mu$ and $\lambda$, but not of the mesh geometry, $\forall \boldsymbol{u}_h \in W_h^k$ and for $h < h_0$, on each patch $P_j$ for $\boldsymbol{v}_j \in W_h^1$ as defined above and $\forall \epsilon, \alpha_1, \alpha_2 \in \mathbb{R}_+^*$, such that*

$$\langle \lambda\nabla\cdot\boldsymbol{v}_j, \boldsymbol{u}_h\cdot\boldsymbol{n}\rangle_{\Gamma_j} \gtrsim \alpha_2\Big(1-\frac{C\alpha_2}{4\epsilon}\Big)\frac{\lambda}{h}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2 - \frac{C\alpha_1^2}{4\epsilon}\frac{\lambda}{h}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}^2 - 2\epsilon\lambda\|\nabla\boldsymbol{u}_h\|_{0,P_j}^2.$$

**Lemma 3.3.3.** *There exists $C > 0$ independent of $h$, $\mu$ and $\lambda$, but not of the mesh geometry, $\forall \boldsymbol{u}_h \in W_h^k$ and $h < h_0$, on each patch $P_j$ for $\boldsymbol{v}_j \in W_h^1$ as defined above and $\forall \epsilon, \alpha_1, \alpha_2 \in \mathbb{R}_+^*$, such that*

$$\begin{aligned}\langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_j)\cdot\boldsymbol{n}, \boldsymbol{u}_h\rangle_{\Gamma_j} \geq{}& \alpha_2\Big(2-\frac{5C\alpha_2}{4\epsilon}\Big)\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2 \\ &+ \alpha_1\Big(1-\frac{C\alpha_1}{4\epsilon}\Big)\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}^2 - 3\epsilon\mu\|\nabla\boldsymbol{u}_h\|_{0,P_j}^2.\end{aligned}$$

Let us introduce the following norms

$$\begin{aligned}|||\boldsymbol{w}|||^2 &= \mu(\|\nabla\boldsymbol{w}\|_{0,\Omega}^2 + h^{-1}\|\boldsymbol{w}\|_{0,\Gamma}^2) + \lambda(\|\nabla\cdot\boldsymbol{w}\|_{0,\Omega}^2 + h^{-1}\|\boldsymbol{w}\cdot\boldsymbol{n}\|_{0,\Gamma}^2) + J_h(\boldsymbol{w},\boldsymbol{w}), \\ \|\boldsymbol{w}\|_*^2 &= |||\boldsymbol{w}|||^2 + \mu h\|\nabla\boldsymbol{w}\cdot\boldsymbol{n}\|_{0,\Gamma}^2 + \lambda h\|\nabla\cdot\boldsymbol{w}\|_{0,\Gamma}^2.\end{aligned}$$

Observe that these are norms by the Poincaré inequality.

**Theorem 3.3.1.** *There exists positive constants $\beta$ and $h_0$ such that for all functions $\boldsymbol{u}_h \in W_h^k$ and for $h < h_0$, the following inequality holds*

$$\beta|||\boldsymbol{u}_h||| \leq \sup_{\boldsymbol{v}_h\in W_h^k}\frac{A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + J_h(\boldsymbol{u}_h,\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||}.$$

*Proof.* Let $\boldsymbol{v}_h = \boldsymbol{u}_h + \sum_{j=1}^{N_p}\boldsymbol{v}_j$, decomposing the bilinear form we can write the following

$$(A_h + J_h)(\boldsymbol{u}_h,\boldsymbol{v}_h) = (A_h + J_h)(\boldsymbol{u}_h,\boldsymbol{u}_h) + \sum_{j=1}^{N_p}[A_h(\boldsymbol{u}_h,\boldsymbol{v}_j) + J_h(\boldsymbol{u}_h,\boldsymbol{v}_j)].$$

Using (3.17) and the Theorem 2.2.1 we have

$$\begin{aligned}(A_h + J_h)(\boldsymbol{u}_h, \boldsymbol{u}_h) &= 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2 + \lambda\|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 + J_h(\boldsymbol{u}_h, \boldsymbol{u}_h),\\ &\geq 2\mu C\|\nabla\boldsymbol{u}_h\|_{0,\Omega^*}^2 + \lambda\|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 - 2\mu|\boldsymbol{u}_h|_\Gamma^2.\end{aligned}$$

Using (3.17), the trace inequality, the Young's inequality and (3.26) we obtain

$$\begin{aligned}&(2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_j))_{P_j\cap\Omega} + (\lambda\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}_j)_{P_j\cap\Omega} + J_h(\boldsymbol{u}_h, \boldsymbol{v}_j)\\ &\quad\lesssim (2\mu+\lambda)\|\nabla\boldsymbol{u}_h\|_{0,P_j\cap\Omega}\|\nabla\boldsymbol{v}_j\|_{0,P_j\cap\Omega} + J_h(\boldsymbol{u}_h, \boldsymbol{u}_h)^{\frac{1}{2}} J_h(\boldsymbol{v}_j, \boldsymbol{v}_j)^{\frac{1}{2}}\\ &\quad\leq \epsilon(2\mu+\lambda)\|\nabla\boldsymbol{u}_h\|_{0,P_j}^2 + \frac{C}{4\epsilon}(2\mu+\lambda)\|\nabla\boldsymbol{v}_j\|_{0,P_j}^2\\ &\quad\leq \epsilon(2\mu+\lambda)\|\nabla\boldsymbol{u}_h\|_{0,P_j}^2 + \frac{C}{4\epsilon}(2\mu+\lambda)h^{-1}(\alpha_1^2\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \alpha_2^2\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2).\end{aligned}$$

Using the trace inequality once again with (3.26) and (3.19)

$$\begin{aligned}&\langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\cdot\boldsymbol{n}, \boldsymbol{v}_j\rangle_{\Gamma_j} + \langle\lambda\nabla\cdot\boldsymbol{u}_h, \boldsymbol{v}_j\cdot\boldsymbol{n}\rangle_{\Gamma_j}\\ &\quad\leq 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\cdot\boldsymbol{n}\|_{0,\Gamma_j}\|\boldsymbol{v}_j\|_{0,\Gamma_j} + \lambda\|\nabla\boldsymbol{u}_h\|_{0,\Gamma_j}\|\boldsymbol{v}_j\cdot\boldsymbol{n}\|_{0,\Gamma_j}\\ &\quad\lesssim (2\mu+\lambda)\|\nabla\boldsymbol{u}_h\|_{0,P_j}\|\nabla\boldsymbol{v}_j\|_{0,P_j}\\ &\quad\leq \epsilon(2\mu+\lambda)\|\nabla\boldsymbol{u}_h\|_{0,P_j}^2 + \frac{C}{4\epsilon}(2\mu+\lambda)h^{-1}(\alpha_1^2\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \alpha_2^2\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2).\end{aligned}$$

Using these bounds as well as the Lemmas 3.3.2 and 3.3.3

$$\begin{aligned}(A_h + J_h)(\boldsymbol{u}_h, \boldsymbol{v}_h) \geq\ &(2\mu C - \epsilon(7\mu+4\lambda))\|\nabla\boldsymbol{u}_h\|_{0,\Omega^*}^2 + \lambda\|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2\\ &+ \Big(\alpha_1\Big(\mu - C\alpha_1\frac{5\mu+3\lambda}{4\epsilon}\Big) - 2\mu h\Big)\sum_{j=1}^{N_p} h^{-1}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}^2\\ &+ \Big(\alpha_2\Big(2\mu+\lambda - C\alpha_2\frac{9\mu+3\lambda}{4\epsilon}\Big) - 2\mu h\Big)\sum_{j=1}^{N_p} h^{-1}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2.\end{aligned}$$

Setting the constants,

$$\begin{aligned}C_a &= 2C - \epsilon\frac{7\mu+4\lambda}{\mu},\\ C_b &= \alpha_1\Big(1 - C\alpha_1\frac{5\mu+3\lambda}{4\epsilon\mu}\Big) - 2h,\\ C_c &= \alpha_2\Big(1 - C\alpha_2\frac{9\mu+3\lambda}{\epsilon(8\mu+4\lambda)}\Big) - \frac{2\mu h}{2\mu+\lambda},\end{aligned}$$

and using (3.13), (3.17) it becomes

$$\begin{aligned}
&A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad \geq (C_a - C'(C_b + C_c(2 + \lambda\mu^{-1})))(\mu\|\nabla \boldsymbol{u}_h\|_{0,\Omega}^2 + J_h(\boldsymbol{u_h}, \boldsymbol{u_h})) + \lambda\|\nabla \cdot \boldsymbol{u}_h\|_{0,\Omega}^2 \\
&\quad + \frac{C_b}{2}\sum_{j=1}^{N_p} \mu h^{-1}\|\boldsymbol{u}_h \cdot \overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \frac{C_c}{2}\sum_{j=1}^{N_p}(2\mu + \lambda)h^{-1}\|\boldsymbol{u}_h \cdot \overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2.
\end{aligned}$$

By choosing $\epsilon$, $\alpha_1$, $\alpha_2$ in the right way and $h_0$ small enough we get

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \gtrsim \beta_0 |||\boldsymbol{u}_h|||^2,$$

we can show $|||\boldsymbol{u}_h||| \gtrsim |||\boldsymbol{v}_h|||$ following the proofs of the Theorems 3.2.1 and 2.2.2, the claim follows and $\beta = \mathcal{O}(\frac{\mu}{\lambda+\mu})$, $h_0 = \mathcal{O}(\frac{\mu^2}{(\lambda+\mu)^2})$. □

### 3.3.2 A priori error estimate

**Lemma 3.3.4.** *If $\boldsymbol{u} \in [H^2(\Omega)]^2$ is the solution of* (3.22) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (3.24) *the following property holds*

$$A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) - J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in W_h^k.$$

**Lemma 3.3.5.** *Let $\boldsymbol{w} \in [H^2(\Omega)]^2 + W_h^k$ and $\boldsymbol{v}_h \in W_h^k$, there exists a positive constant $M$ such that*

$$A_h(\boldsymbol{w}, \boldsymbol{v}_h) \leq M\|\boldsymbol{w}\|_* |||\boldsymbol{v}_h|||.$$

*Proof.* See proof of Lemma 2.2.5. The ghost penalty term can be handled using (3.17) as in the proof of Lemma 3.2.2. □

**Theorem 3.3.2.** *If $\boldsymbol{u} \in [H^{k+1}(\Omega)]^2$ is the solution of* (3.22) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (3.24) *with $h < h_0$, then there holds*

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq C_\beta \inf_{\boldsymbol{w}_h \in W_h^k} \|\boldsymbol{u} - \boldsymbol{w}_h\|_*.$$

*where $C_\beta$ is a positive constant that depends on the mesh geometry.*

*Proof.* Same proof as Theorem 3.2.2 using Lemma 3.3.4, Theorem 3.3.1 and Lemma 3.3.5, $C_\beta = \mathcal{O}(\beta^{-1})$. □

**Corollary 3.3.1.** *If $\boldsymbol{u} \in [H^{k+1}(\Omega)]^2$ is the solution of* (3.22) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (3.24) *with $h < h_0$, then there holds*

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq C_{\mu\lambda} h^k |\boldsymbol{u}|_{k+1,\Omega},$$

*with $C_{\mu\lambda} = \mathcal{O}(\frac{\lambda+\mu}{\mu}(\lambda^{\frac{1}{2}} + \mu^{\frac{1}{2}}))$.*

*Proof.* Using the same arguments as in the proof of Corollary 2.2.1 with the trace inequality (3.1) and (3.5) and the estimate of the ghost penalty term from the proof of Corollary 3.2.1 we obtain the estimate

$$\|\boldsymbol{u} - \mathcal{I}_h \boldsymbol{u}\|_* \lesssim (\lambda^{\frac{1}{2}} + \mu^{\frac{1}{2}}) h^k |\boldsymbol{u}|_{k+1,\Omega}. \tag{3.27}$$

We conclude using Theorem 3.3.2 with $\boldsymbol{w}_h = \mathcal{I}_h \boldsymbol{u}$. □

**Proposition 3.3.1.** *Let* $\boldsymbol{u} \in [H^{k+1}(\Omega)]^2$ *be the solution of* (3.22) *and* $\boldsymbol{u}_h$ *the solution of* (3.24) *with* $h < h_0$*, then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq C'_{\mu\lambda} h^{k+\frac{1}{2}} |\boldsymbol{u}|_{k+1,\Omega}.$$

*with* $C'_{\mu\lambda} = \mathcal{O}((1 + \frac{\lambda}{\mu})^2)$.

*Proof.* Let $\boldsymbol{z}$ satisfy the adjoint problem

$$\begin{aligned} -2\mu \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{z}) - \lambda \nabla(\nabla \cdot \boldsymbol{z}) &= \boldsymbol{u} - \boldsymbol{u}_h && \text{in } \Omega, \\ \boldsymbol{z} &= 0 && \text{on } \Gamma. \end{aligned}$$

Then we can write using integration by parts

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}^2 &= (\boldsymbol{u} - \boldsymbol{u}_h, -2\mu \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{z}) - \lambda \nabla(\nabla \cdot \boldsymbol{z}))_\Omega \\ &= (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z}))_\Omega + (\lambda \nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h), \nabla \cdot \boldsymbol{z})_\Omega \\ &\quad - \langle 2\mu(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z}) \cdot \boldsymbol{n} \rangle_\Gamma - \langle \lambda(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}, \nabla \cdot \boldsymbol{z} \rangle_\Gamma \\ &= A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}) - 2\langle 2\mu(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z}) \cdot \boldsymbol{n} \rangle_\Gamma - 2\langle \lambda(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}, \nabla \cdot \boldsymbol{z} \rangle_\Gamma. \end{aligned}$$

The consistency relation of Lemma 3.3.4 allows us to write

$$A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}) = A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z} - \mathcal{I}_h \boldsymbol{z}) + J_h(\boldsymbol{u}_h, \mathcal{I}_h \boldsymbol{z}).$$

Note that using the trace inequality (3.1), the estimate (3.5), the stability of the extension operator (3.2) and the inverse inequality of Lemma 2.0.2 we have

$$\begin{aligned} & h^{\frac{1}{2}} (\mu^{\frac{1}{2}} \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}\|_{0,\Gamma} + \lambda^{\frac{1}{2}} \|\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Gamma}) \\ &\quad \lesssim (\mu^{\frac{1}{2}} + \lambda^{\frac{1}{2}}) \Big( \|\nabla(\boldsymbol{u} - \mathcal{I}_h \boldsymbol{u})\|_{0,\Omega^*} + h \Big( \sum_{K \in \Omega_*} \|D^2(\boldsymbol{u} - \mathcal{I}_h \boldsymbol{u})\|_{0,K}^2 \Big)^{\frac{1}{2}} \\ &\qquad + \|\nabla(\boldsymbol{u}_h - \mathcal{I}_h \boldsymbol{u})\|_{0,\Omega^*} + h \Big( \sum_{K \in \Omega_*} \|D^2(\boldsymbol{u}_h - \mathcal{I}_h \boldsymbol{u})\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big) \\ &\quad \lesssim (\mu^{\frac{1}{2}} + \lambda^{\frac{1}{2}}) h^k |\boldsymbol{u}|_{k+1,\Omega} + (1 + \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}}) |||\boldsymbol{u}_h - \mathcal{I}_h \boldsymbol{u}|||. \end{aligned}$$

Using this result, the Cauchy-Schwarz inequality, the Corollary 3.3.1 and (3.5) we obtain

$$\begin{aligned}
&A_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{z}-\mathcal{I}_h\boldsymbol{z}) \\
&\quad = (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z}))_\Omega + (\lambda\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h), \nabla\cdot(\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z}))_\Omega \\
&\qquad - \langle 2\mu(\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z}), \boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{n}\rangle_\Gamma + \langle\lambda(\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z})\cdot\boldsymbol{n}, \nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\rangle_\Gamma \\
&\qquad + \langle 2\mu(\boldsymbol{u}-\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z})\cdot\boldsymbol{n}\rangle_\Gamma + \langle\lambda(\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{n}, \nabla\cdot(\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z})\rangle_\Gamma \\
&\quad \lesssim (|||\boldsymbol{u}-\boldsymbol{u}_h||| + h^{\frac{1}{2}}(\mu^{\frac{1}{2}}\|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{n}\|_{0,\Gamma} + \lambda^{\frac{1}{2}}\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,\Gamma}))\|\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z}\|_* \\
&\quad \lesssim (\mu^{\frac{1}{2}}+\lambda^{\frac{1}{2}})^2(h^k|\boldsymbol{u}|_{k+1,\Omega} + \lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}}|||\boldsymbol{u}_h-\mathcal{I}_h\boldsymbol{u}|||)h|\boldsymbol{z}|_{2,\Omega}.
\end{aligned}$$

Using similar arguments as in the proof of Proposition 3.2.1 we get

$$\begin{aligned}
J_h(\boldsymbol{u}_h, \mathcal{I}_h\boldsymbol{z}) &\leq J_h(\boldsymbol{u}_h,\boldsymbol{u}_h)^{\frac{1}{2}} J_h(\mathcal{I}_h\boldsymbol{z}-\boldsymbol{z}, \mathcal{I}_h\boldsymbol{z}-\boldsymbol{z})^{\frac{1}{2}} \\
&\leq (|||\boldsymbol{u}_h-\mathcal{I}_h\boldsymbol{u}||| + J_h(\mathcal{I}_h\boldsymbol{u},\mathcal{I}_h\boldsymbol{u})^{\frac{1}{2}}) J_h(\mathcal{I}_h\boldsymbol{z}-\boldsymbol{z}, \mathcal{I}_h\boldsymbol{z}-\boldsymbol{z})^{\frac{1}{2}} \\
&\lesssim \mu^{\frac{1}{2}}(|||\boldsymbol{u}_h-\mathcal{I}_h\boldsymbol{u}||| + \mu^{\frac{1}{2}}h^k|\boldsymbol{u}|_{k+1,\Omega})h|\boldsymbol{z}|_{2,\Omega}.
\end{aligned}$$

The global trace inequalities $\|\boldsymbol{\varepsilon}(\boldsymbol{z})\cdot\boldsymbol{n}\|_{0,\Gamma} \lesssim \|\boldsymbol{z}\|_{2,\Omega}$ and $\|\nabla\cdot\boldsymbol{z}\|_{0,\Gamma} \lesssim \|\boldsymbol{z}\|_{2,\Omega}$, lead to

$$|\langle 2\mu(\boldsymbol{u}-\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{z})\cdot\boldsymbol{n}\rangle_\Gamma| + |\langle\lambda(\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{n}, \nabla\cdot\boldsymbol{z}\rangle_\Gamma| \lesssim (\lambda^{\frac{1}{2}}+\mu^{\frac{1}{2}})h^{\frac{1}{2}}|||\boldsymbol{u}-\boldsymbol{u}_h|||\|\boldsymbol{z}\|_{2,\Omega}.$$

In the proof of Theorem 3.3.2 we have shown that

$$|||\boldsymbol{u}_h-\mathcal{I}_h\boldsymbol{u}||| \lesssim \|\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u}\|_* \lesssim (\mu^{\frac{1}{2}}+\lambda^{\frac{1}{2}})h^k|\boldsymbol{u}|_{k+1,\Omega},$$

where the second estimate is (3.27). Using this result and Corollary 3.3.1

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}^2 \lesssim ((\lambda^{\frac{1}{2}}+\mu^{\frac{1}{2}})^2(1+\lambda^{\frac{1}{2}}\mu^{-\frac{1}{2}})h + (\lambda^{\frac{1}{2}}+\mu^{\frac{1}{2}})^2h^{\frac{1}{2}})h^k|\boldsymbol{u}|_{k+1,\Omega}\|\boldsymbol{z}\|_{2,\Omega}.$$

Using (3.23) we obtain $\mu\|\boldsymbol{z}\|_{2,\Omega} \lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$, the claim follows. □

## 3.4 Incompressible elasticity

The incompressible elasticity problem with Dirichlet boundary condition is given by

$$\begin{aligned}
-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p) &= \boldsymbol{f} \quad \text{in } \Omega, \\
\nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \Gamma.
\end{aligned} \tag{3.28}$$

with $\boldsymbol{f} \in [L^2(\Omega)]^2$, $\boldsymbol{g} \in [H^{\frac{3}{2}}(\Gamma)]^2$, the stress tensor is given by

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbb{I}_{2\times 2},$$

and $\int_\Omega p \, \mathrm{d}x = 0$. We have the following weak formulation: find $(\boldsymbol{u}, p) \in [H^1_g(\Omega)]^2 \times L^2(\Omega)$ such that

$$a[(\boldsymbol{u}, p), (\boldsymbol{v}, q)] = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall (\boldsymbol{v}, q) \in [H^1_0(\Omega)]^2 \times L^2(\Omega),$$

with

$$a[(\boldsymbol{u}, p), (\boldsymbol{v}, q)] = (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_\Omega - (p, \nabla \cdot \boldsymbol{v})_\Omega + (\nabla \cdot \boldsymbol{u}, q)_\Omega.$$

The well-posedness of this problem follows from the Lax-Milgram Lemma, we also have the regularity estimate [2]

$$\mu \|\boldsymbol{u}\|_{s+2,\Omega} + \|p\|_{s+1,\Omega} \lesssim \|\boldsymbol{f}\|_{s,\Omega} + \|\boldsymbol{g}\|_{s+\frac{3}{2},\Gamma} \qquad s \geq 0. \tag{3.29}$$

Let $Q^* = \{q \in L^2(\Omega^*), \ \int_\Omega q \, \mathrm{d}x = 0\}$, then we define the finite element space

$$Q^k_h = \{q_h \in Q^* : q_h|_K \in \mathbb{P}_k(K) \ \ \forall K \in \mathcal{T}_h\}, \quad k \geq 1.$$

The penalty free Nitsche's method leads to the following finite element formulation: find $\boldsymbol{u}_h \in W^k_h$ and $p_h \in Q^k_h$ such that

$$A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] + J_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = L_h(\boldsymbol{v}_h, q_h) \qquad \forall (\boldsymbol{v}_h, q_h) \in W^k_h \times Q^k_h, \tag{3.30}$$

where the linear forms $A_h$, $J_h$ and $L_h$ are defined as

$$\begin{aligned} A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] &= a[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] - b(\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) + b(\boldsymbol{v}_h, \boldsymbol{u}_h, q_h) \\ &\quad + S_h(\boldsymbol{u}_h, p_h, q_h), \\ J_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] &= J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + I_h(p_h, q_h), \\ L_h(\boldsymbol{v}_h, q_h) &= (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega + b(\boldsymbol{v}_h, \boldsymbol{g}, q_h) + \Lambda_h(\boldsymbol{f}, q_h). \end{aligned}$$

The bilinear form $b$ is defined as

$$b(\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) = \langle (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h \mathbb{I}_{2\times 2}) \cdot \boldsymbol{n}, \boldsymbol{v}_h \rangle_\Gamma.$$

The ghost penalty terms and the stabilisation terms are written as

$$\begin{aligned} J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \gamma_g \mu \sum_{F \in \mathcal{F}_G} \sum_{l=1}^{k} h^{2l-1} \langle [\![ D^l_{n_F} \boldsymbol{u}_h ]\!]_F, [\![ D^l_{n_F} \boldsymbol{v}_h ]\!]_F \rangle_F, \\ I_h(p_h, q_h) &= \frac{\gamma_g}{\mu} \sum_{F \in \mathcal{F}_G} \sum_{l=1}^{k} h^{2l+1} \langle [\![ D^l_{n_F} p_h ]\!]_F, [\![ D^l_{n_F} q_h ]\!]_F \rangle_F, \\ S_h(\boldsymbol{u}_h, p_h, q_h) &= \frac{\gamma_p}{\mu} \sum_{K \in \mathcal{T}_h} \int_K h^2 (-2\mu \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \nabla p_h) \nabla q_h \, \mathrm{d}x, \\ \Lambda_h(\boldsymbol{f}, q_h) &= \frac{\gamma_p}{\mu} \sum_{K \in \mathcal{T}_h} \int_K h^2 \boldsymbol{f} \nabla q_h \, \mathrm{d}x, \end{aligned}$$

the terms $J_h$ and $I_h$ are the ghost penalty terms, the pressure stabilisation term is necessary as we want to work with equal order interpolation. The following inequality has been shown in [89] for $q_h \in Q_h^k$

$$\mu^{-1}h^2\|\nabla q_h\|_{0,\Omega^*}^2 \lesssim h^2\mu^{-1}\|\nabla q_h\|_{0,\Omega}^2 + I_h(q_h,q_h) \lesssim \mu^{-1}h^2\|\nabla q_h\|_{0,\Omega^*}^2. \tag{3.31}$$

### 3.4.1 Inf-sup stability

Let us define the following norms

$$\begin{aligned}
|||(\boldsymbol{w},\varrho)|||^2 &= \mu(\|\nabla \boldsymbol{w}\|_{0,\Omega}^2 + h^{-1}\|\boldsymbol{w}\|_{0,\Gamma}^2) + h^2\mu^{-1}\|\nabla\varrho\|_{0,\Omega}^2 + J_h[(\boldsymbol{w},\varrho),(\boldsymbol{w},\varrho)],\\
\|(\boldsymbol{w},\varrho)\|_*^2 &= |||(\boldsymbol{w},\varrho)|||^2 + \mu h\|\nabla\boldsymbol{w}\cdot\boldsymbol{n}\|_{0,\Gamma}^2 + \mu^{-1}\|\varrho\|_{0,\Omega}^2 + \mu^{-1}h\|\varrho\|_{0,\Gamma}^2 + \mu h^{-2}\|\boldsymbol{w}\|_{0,\Omega}^2\\
&\quad + h^2\mu^{-1}\|\nabla\varrho\|_{0,\Omega^*}^2 + h^2\mu\sum_{K\in\Omega^*}\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{0,K}^2.
\end{aligned}$$

**Theorem 3.4.1.** *There exists positive constants $\beta$ and $h_0$ such that for all functions $(\boldsymbol{u}_h,p_h) \in W_h^k \times Q_h^k$ and for $h < h_0$, the following inequality holds*

$$\beta|||(\boldsymbol{u}_h,p_h)||| \leq \sup_{(\boldsymbol{v}_h,q_h)\in W_h^k\times Q_h^k} \frac{A_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)] + J_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)]}{|||(\boldsymbol{v}_h,q_h)|||}.$$

*Proof.* Let $q_h = p_h$ and $\boldsymbol{v}_h = \boldsymbol{u}_h + \sum_{j=1}^{N_p}\boldsymbol{v}_j$ with $\boldsymbol{v}_j$ as defined as in Section 3.3.1, then we have

$$\begin{aligned}
(A_h+J_h)[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)] &= (A_h+J_h)[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)]\\
&\quad + \sum_{j=1}^{N_p}\big[A_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_j,0)] + J_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_j,0)]\big].
\end{aligned}$$

Note that most of the terms have been bounded in the compressible case. Using the inverse inequality of Lemma 2.0.2, (3.17) and the Theorem 2.2.1 we have

$$\begin{aligned}
&(A_h+J_h)[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)]\\
&= (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h),\boldsymbol{\varepsilon}(\boldsymbol{u}_h))_\Omega + S_h(\boldsymbol{u}_h,p_h,q_h) + J_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)]\\
&\gtrsim 2\mu\|\nabla\boldsymbol{u}_h\|_{0,\Omega^*}^2 - 2\mu|\boldsymbol{u}_h|_\Gamma^2 - \frac{\gamma_p}{\mu}h^2\Big(2\mu\sum_{K\in\Omega_*}(\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,K}\|\nabla p_h\|_{0,K}) - \|\nabla p_h\|_{0,\Omega^*}^2\Big)\\
&\gtrsim 2\mu\|\nabla\boldsymbol{u}_h\|_{0,\Omega^*}^2 - 2\mu|\boldsymbol{u}_h|_\Gamma^2 - 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega^*}\frac{\gamma_p}{\mu}h\|\nabla p_h\|_{0,\Omega^*} + \frac{\gamma_p}{\mu}h^2\|\nabla p_h\|_{0,\Omega^*}^2\\
&\gtrsim 2(1-\epsilon')\mu\|\nabla\boldsymbol{u}_h\|_{0,\Omega^*}^2 - 2\mu|\boldsymbol{u}_h|_\Gamma^2 + \frac{\gamma_p}{\mu}\Big(1-\frac{\gamma_p}{4\epsilon'}\Big)h^2\|\nabla p_h\|_{0,\Omega^*}^2.
\end{aligned}$$

Using (3.26) and (3.19) we have the bound

$$\begin{aligned}
(\nabla p_h,\boldsymbol{v}_j)_{P_j\cap\Omega} &\leq \|\nabla p_h\|_{0,P_j\cap\Omega}\|\boldsymbol{v}_j\|_{0,P_j\cap\Omega} \leq \|\nabla p_h\|_{0,P_j}\|\boldsymbol{v}_j\|_{0,P_j} \lesssim h\|\nabla p_h\|_{0,P_j}\|\nabla\boldsymbol{v}_j\|_{0,P_j}\\
&\leq \epsilon\mu^{-1}h^2\|\nabla p_h\|_{0,P_j}^2 + \frac{C}{4\epsilon}\mu h^{-1}(\alpha_1^2\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \alpha_2^2\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2).
\end{aligned}$$

Then we obtain the bound

$$\begin{aligned} A_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)] \geq{}& (2-2\epsilon'-7\epsilon)\mu\|\nabla\boldsymbol{u}_h\|^2_{0,\Omega^*} + \frac{\gamma_p}{\mu}\Big(1-\frac{C\gamma_p}{4\epsilon'}-\frac{\epsilon}{\gamma_p}\Big)h^2\|\nabla p_h\|^2_{0,\Omega^*} \\ &+\Big(\alpha_1\Big(1-C\alpha_1\frac{3}{2\epsilon}\Big)-2h\Big)\sum_{j=1}^{N_p}\mu h^{-1}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|^2_{0,\Gamma_j} \\ &+\Big(\alpha_2\Big(2-C\alpha_2\frac{5}{2\epsilon}\Big)-2h\Big)\sum_{j=1}^{N_p}\mu h^{-1}\|\overline{\boldsymbol{u}_h}^{\Gamma_j}\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|^2_{0,\Gamma_j}. \end{aligned}$$

Let the constants

$$\begin{aligned} C_a &= (2-2\epsilon'-7\epsilon), \\ C_b &= \gamma_p\Big(1-\frac{C\gamma_p}{4\epsilon'}-\frac{\epsilon}{\gamma_p}\Big), \\ C_c &= \Big(\alpha_1\Big(1-C\alpha_1\frac{3}{2\epsilon}\Big)-2h\Big), \\ C_d &= \Big(\alpha_2\Big(2-C\alpha_2\frac{5}{2\epsilon}\Big)-2h\Big), \end{aligned}$$

and using (3.13), (3.17), (3.31) it becomes

$$\begin{aligned} (A_h+J_h)[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)] \geq{}& (C_a-C'C_c-C'C_d)(\mu\|\nabla\boldsymbol{u}_h\|^2_{0,\Omega}+J_h(\boldsymbol{u}_h,\boldsymbol{u}_h)) \\ &+C_b(\mu^{-1}h^2\|\nabla p_h\|^2_{0,\Omega}+I_h(p_h,p_h)) \\ &+\frac{C_c}{2}\sum_{j=1}^{N_p}\mu h^{-1}\|\boldsymbol{u}_h\cdot\overline{\boldsymbol{\tau}}^{\Gamma_j}\|^2_{0,\Gamma_j}+\frac{C_d}{2}\sum_{j=1}^{N_p}\mu h^{-1}\|\boldsymbol{u}_h\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\|^2_{0,\Gamma_j}. \end{aligned}$$

By choosing $\epsilon$, $\epsilon'$, $\alpha_1$ and $\alpha_2$ in the right way and $h_0$ small enough we get

$$A_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)]+J_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)] \gtrsim |||\boldsymbol{u}_h|||^2.$$

We conclude using $|||(\boldsymbol{v}_h,q_h)||| \lesssim |||(\boldsymbol{u}_h,p_h)|||$ which is shown following the proofs of the Theorems 3.2.1 and 2.3.1. Note that $\beta=\mathcal{O}(1)$, $h_0=\mathcal{O}(1)$. □

### 3.4.2 A priori error estimate

**Lemma 3.4.1.** *If* $(\boldsymbol{u},p)\in[H^2(\Omega)]^2\times H^1(\Omega)$ *is the solution of* (3.28) *and* $(\boldsymbol{u}_h,p_h)\in W_h^k\times Q_h^k$ *the solution of* (3.30) *the following property holds*

$$A_h[(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h),(\boldsymbol{v}_h,q_h)]-J_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)]=0, \qquad \forall(\boldsymbol{v}_h,q_h)\in W_h^k\times Q_h^k.$$

**Lemma 3.4.2.** *Let* $(\boldsymbol{w},\varrho)\in([H^2(\Omega)]^2+W_h^k)\times(H^1(\Omega)+Q_h^k)$ *and* $(\boldsymbol{v}_h,q_h)\in W_h^k\times Q_h^k$ *there exists a positive constant* $M$ *such that*

$$A_h[(\boldsymbol{w},\varrho),(\boldsymbol{v}_h,q_h)] \leq M\|(\boldsymbol{w},\varrho)\|_*|||(\boldsymbol{v}_h,q_h)|||.$$

*Proof.* We have

$$A_h[(\boldsymbol{w},\varrho),(\boldsymbol{v}_h,q_h)] = (2\mu\boldsymbol{\varepsilon}(\boldsymbol{w}),\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_\Omega - (\varrho,\nabla\cdot\boldsymbol{v}_h)_\Omega + (\nabla q_h,\boldsymbol{w})_\Omega + S_h(\boldsymbol{u}_h,p_h,q_h) - \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{w})\cdot\boldsymbol{n},\boldsymbol{v}_h\rangle_\Gamma + \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_\Gamma + \langle\varrho,\boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_\Gamma,$$

using the Cauchy Schwarz inequality we have

$$(2\mu\boldsymbol{\varepsilon}(\boldsymbol{w}),\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_\Omega - \langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{w})\cdot\boldsymbol{n},\boldsymbol{v}_h\rangle_\Gamma \lesssim \|(\boldsymbol{w},\varrho)\|_*|||(\boldsymbol{v}_h,q_h)|||,$$
$$(\nabla q_h,\boldsymbol{w})_\Omega - (\varrho,\nabla\cdot\boldsymbol{v}_h)_\Omega + \langle\varrho,\boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_\Gamma \lesssim \|(\boldsymbol{w},\varrho)\|_*|||(\boldsymbol{v}_h,q_h)|||.$$

Using the trace inequality and (3.17)

$$\langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\cdot\boldsymbol{n},\boldsymbol{w}\rangle_\Gamma \le 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\cdot\boldsymbol{n}\|_{0,\Gamma}\|\boldsymbol{w}\|_{0,\Gamma} \lesssim 2\mu\|\nabla\boldsymbol{v}_h\|_{0,\Omega^*}h^{-\frac{1}{2}}\|\boldsymbol{w}\|_{0,\Gamma} \lesssim \|(\boldsymbol{w},\varrho)\|_*|||(\boldsymbol{v}_h,q_h)|||.$$

Using (3.31) the stabilisation term can be bounded

$$\begin{aligned} S_h(\boldsymbol{w},\varrho,q_h) &\le \frac{\gamma_p}{\mu}h^2(2\mu\Big(\sum_{K\in\Omega_*}\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\|^2_{0,K}\Big)^{\frac{1}{2}} + \|\nabla\varrho\|_{0,\Omega^*})\|\nabla q_h\|_{0,\Omega^*} \\ &\lesssim \frac{\gamma_p}{\mu}(2\mu h\Big(\sum_{K\in\Omega_*}\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\|^2_{0,K}\Big)^{\frac{1}{2}} + h\|\nabla\varrho\|_{0,\Omega^*})(h\|\nabla q_h\|_{0,\Omega} + I_h(q_h,q_h)^{\frac{1}{2}}) \\ &\lesssim \|(\boldsymbol{w},\varrho)\|_*|||(\boldsymbol{v}_h,q_h)|||. \end{aligned}$$

□

**Theorem 3.4.2.** *If $(\boldsymbol{u},p)\in[H^{k+1}(\Omega)]^2\times H^k(\Omega)$ is the solution of* (3.28) *and $(\boldsymbol{u}_h,p_h)\in W_h^k\times Q_h^k$ the solution of* (3.30) *with $h<h_0$, then there holds*

$$|||(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h)||| \lesssim \inf_{(\boldsymbol{w}_h,\varrho_h)\in W_h^k\times Q_h^k}\|(\boldsymbol{u}-\boldsymbol{w}_h,p-\varrho_h)\|_*.$$

*Proof.* Let $(\boldsymbol{w}_h,\varrho_h)\in W_h^k\times Q_h^k$ using the triangle inequality we get

$$|||(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h)||| \le |||(\boldsymbol{u}-\boldsymbol{w}_h,p-\varrho_h)||| + |||(\boldsymbol{w}_h-\boldsymbol{u}_h,\varrho_h-p_h)|||.$$

Using the regularities of $\boldsymbol{u}$ and $p$ we have $J_h(\boldsymbol{u},\boldsymbol{v}_h)=0$ and $I_h(p,q_h)=0$

$$J_h[(\boldsymbol{w}_h,\varrho_h),(\boldsymbol{v}_h,q_h)] \le (J_h(\boldsymbol{w}_h-\boldsymbol{u},\boldsymbol{w}_h-\boldsymbol{u})^{\frac{1}{2}} + I_h(\varrho_h-p,\varrho_h-p)^{\frac{1}{2}})|||(\boldsymbol{v}_h,q_h)|||,$$

Using the orthogonality of Lemma 3.4.1, the Theorem 3.4.1 and the Lemma 3.4.2 we can write

$$
\begin{aligned}
&\beta |||(\boldsymbol{u}_h - \boldsymbol{w}_h, p_h - \varrho_h)||| \\
&\qquad \leq \sup_{(\boldsymbol{w}_h,\varrho_h)\in W_h^k\times Q_h^k} \frac{A_h[(\boldsymbol{u}-\boldsymbol{w}_h, p-\varrho_h),(\boldsymbol{v}_h,q_h)] - J_h[(\boldsymbol{w}_h,\varrho_h),(\boldsymbol{v}_h,q_h)]}{|||(\boldsymbol{v}_h,q_h)|||} \\
&\qquad \leq (M+1)\|(\boldsymbol{u}-\boldsymbol{w}_h, p-\varrho_h)\|_*.
\end{aligned}
$$

Using that $|||(\boldsymbol{u}-\boldsymbol{w}_h, p-\varrho_h)||| \leq \|(\boldsymbol{u}-\boldsymbol{w}_h, p-\varrho_h)\|_*$ we obtain

$$
|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leq \Big(1+\frac{M+1}{\beta}\Big) \inf_{(\boldsymbol{w}_h,\varrho_h)\in W_h^k\times Q_h^k} \|(\boldsymbol{u}-\boldsymbol{w}_h, p-\varrho_h)\|_*.
$$

□

**Corollary 3.4.1.** *If* $(\boldsymbol{u},p)\in[H^{k+1}(\Omega)]^2\times H^k(\Omega)$ *is the solution of* (3.28) *and* $(\boldsymbol{u}_h,p_h)\in W_h^k\times Q_h^k$ *the solution of* (3.30) *with* $h<h_0$*, then there holds*

$$
|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \lesssim h^k(\mu^{\frac{1}{2}}|\boldsymbol{u}|_{k+1,\Omega} + \mu^{-\frac{1}{2}}|p|_{k,\Omega}).
$$

*Proof.* We the same arguments as for the proof of Corollary 2.3.1 with (3.1) and (3.5). Also, the ghost penalty parameter $J_h$ is handled as it is done in the proof of Corollary 2.1.1. Extending the ghost penalty term $I_h$ to the full domain $\Omega^*$ we obtain

$$
I_h(p-\mathcal{I}_h p, p-\mathcal{I}_h p)^{\frac{1}{2}} \lesssim \Big(\sum_{K\in\Omega^*}\sum_{l=k}^{k+1} h^{2(l+1)}\|D^l(p-\mathcal{I}_h p)\|_{0,K}^2\Big)^{\frac{1}{2}} \lesssim h^k|p|_{k,\Omega}.
$$

Then we obtain

$$
\|(\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u}, p-\mathcal{I}_h p)\|_* \lesssim h^k(\mu^{\frac{1}{2}}|\boldsymbol{u}|_{k+1,\Omega} + \mu^{-\frac{1}{2}}|p|_{k,\Omega}).
$$

We conclude using Theorem 3.4.2 with $\boldsymbol{w}_h = \mathcal{I}_h\boldsymbol{u}$ and $\varrho_h = \mathcal{I}_h p$. □

## 3.5 Numerical results

In this section, for each computation a structured background mesh is defined such that $\Omega_{\mathcal{T}} = [0,1]\times[0,1]$ and the physical domain is the disc:

$$
\Omega = \{(x,y)\in\Omega_{\mathcal{T}} |\, |(0.5,0.5)-(x,y)| \leq 0.3\}.
$$

We want to verify numerically the convergence properties shown theoretically. In each case we use a manufactured solution and compute the convergence of the errors. The computations are done using the package FEniCS [88] together with the library CutFEM [28]. We consider piecewise affine approximations.

### 3.5.1 Poisson problem

For the Poisson case the manufactured solution used is defined as

$$u = [(x - 0.5)^2 + (y - 0.5)^2]^2.$$

The penalty free Nitsche's method is tested for a range of values for the ghost penalty parameter $\gamma_g$. Figure 3.3 shows the $L^2$ and $H^1$-convergence slopes for each value of

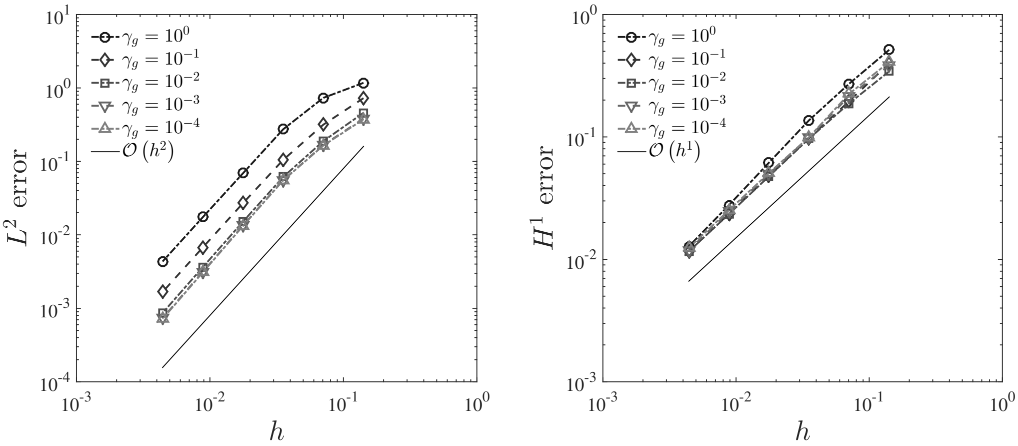



FIGURE 3.3: Poisson problem, $V_h^1$: error versus the maximal element diameter $h$. Left: $L^2$-error, right: $H^1$-error.

$\gamma_g$ considered. The convergence of the $L^2$-error is half an order better than what has been shown theoretically. Also, we observe that the constant involved in the convergence result of Proposition 3.2.1 grows as the $\gamma_g$ becomes bigger. The convergence observed for the $H^1$-error is optimal as shown theoretically.

### 3.5.2 Compressible elasticity

The two dimentional manufactured solution used for compressible elasticity is defined as

$$\boldsymbol{u} = \begin{pmatrix} (x^5 - x^4)(y^3 - y^2) \\ (x^4 - x^3)(y^6 - y^5) \end{pmatrix}.$$

The penalty free Nitsche's method is tested for a range of values for the ghost penalty parameter $\gamma_g$. Figure 3.4 shows the $L^2$ and $H^1$-convergence slopes. The same observations as for the Poisson problem can be seen. In order to observe locking, we set the ghost penalty parameter to $\gamma_g = 0.001$ and we consider a range of values for $\lambda$. Figure 3.5 shows that as $\lambda$ becomes large the convergence is lost if the mesh if not fine enough, it characterises locking as observed in the previous chapter.

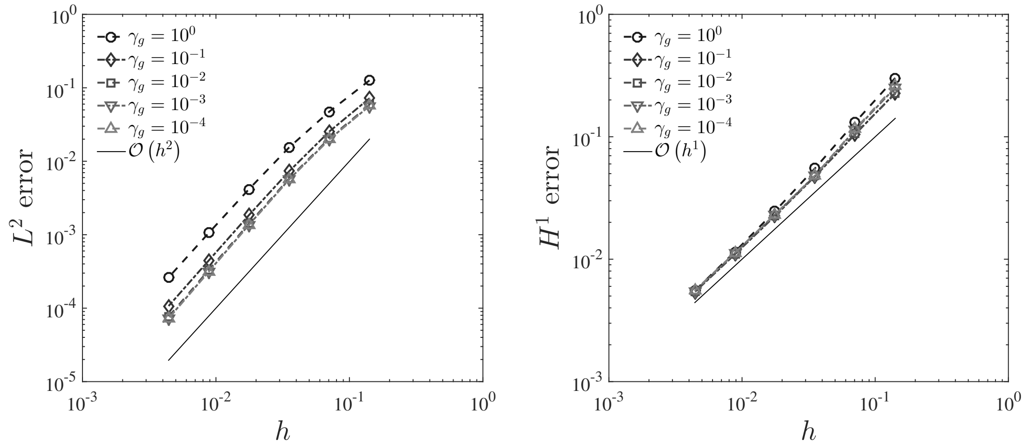



FIGURE 3.4: Compressible elasticity, $V_h^1$: error versus the maximal element diameter $h$. Left: $L^2$-error, right: $H^1$-error. $\mu = \lambda = 1.0$.

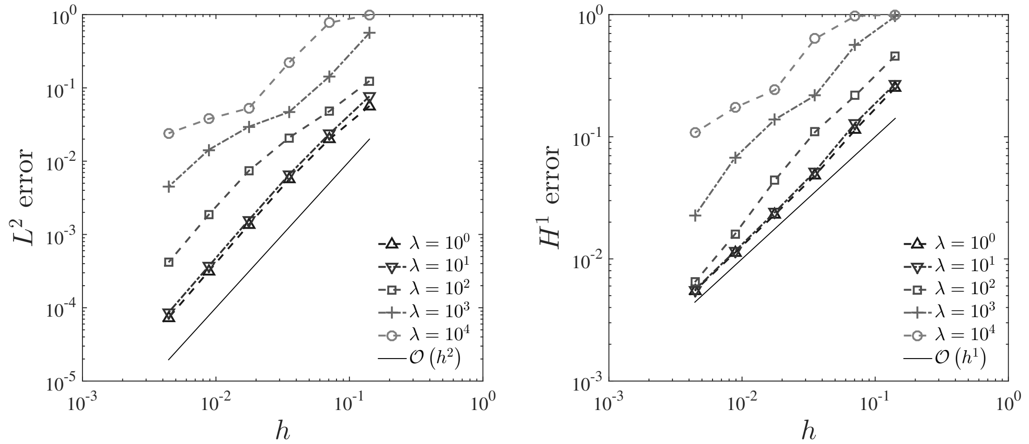



FIGURE 3.5: Compressible elasticity, $V_h^1$: error versus the maximal element diameter $h$. Left: $L^2$-error, right: $H^1$-error. $\gamma_g = 0.001$, $\mu = 1.0$.

### 3.5.3 Inompressible elasticity

The manufactured solution used for incompressible elasticity is defined as

$$\boldsymbol{u} = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{pmatrix}, \qquad p = 2\mu\cos(\pi x)\sin(\pi y).$$

The $H^1$-convergence of $\boldsymbol{u}$ and $L^2$-convergence of $p$ are obtained for a range of $\gamma_g$. Figure 3.6 shows that the convergence of the $H^1$-error of $\boldsymbol{u}$ is optimal as shown theoretically. The $L^2$-error of $p$ shows a convergence of order $\mathcal{O}(h^{\frac{3}{2}})$ the same behaviour has been observed for the weak imposition case in the previous chapter. The constant related to the $L^2$-convergence of the pressure becomes larger as the ghost penalty parameter becomes bigger.

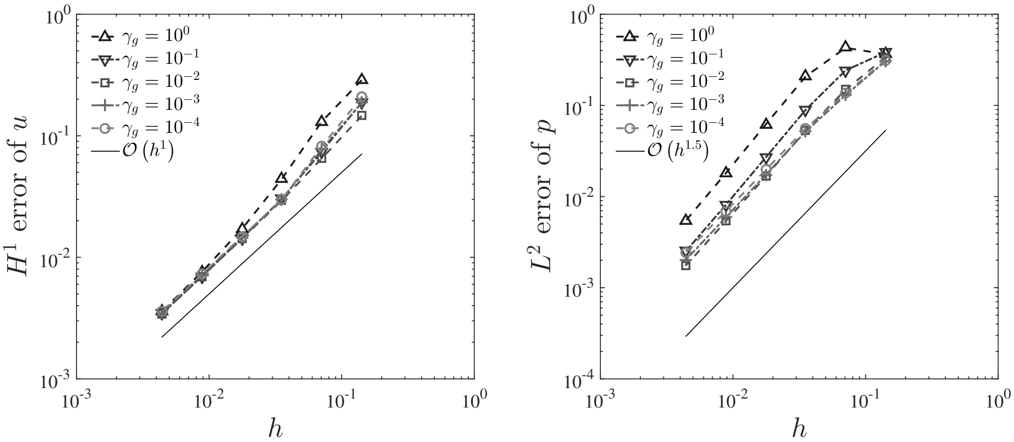



FIGURE 3.6: Incompressible elasticity, $V_h^1$: error versus the maximal element diameter $h$. Left: $H^1$-error of $\boldsymbol{u}$, right: $L^2$-error of $p$, $\mu = 1$, $\gamma_p = 0.1$.

# Chapter 4

# Domain decomposition

This chapter presents a study of the penalty-free Nitsche's method in the framework of nonconforming domain decomposition. We consider one computational domain divided into two subdomains, both subdomains are meshed independently, the coupling at the interface is done using the penalty-free Nitsche's method, each subdomain has its own material parameters.

## 4.1 Preliminaries

Let $\Omega_1$ and $\Omega_2$ be two convex bounded domain in $\mathbb{R}^2$ with polygonal boundary, these two domains share an interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, for simplicity $\Gamma$ is considered as plane. We define the domain $\Omega = \Omega_1 \cup \Omega_2$ with boundary $\partial\Omega$, an example of $\Omega$ is represented in Figure 4.1. The vector $n_i$ is the exterior unit normal to the boundary $\partial\Omega_i$. The set $\{\mathcal{T}_h^i\}_h$ defines the

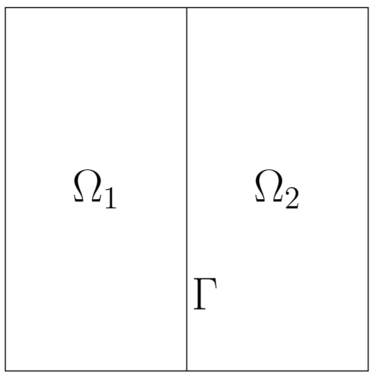



FIGURE 4.1: Example of computational domain $\Omega$.

family of quasi-uniform and shape regular triangulations fitted to $\Omega_i$. A generic triangle is denoted as $K$ and $h_K = \text{diam}(K)$. The mesh parameter for a given triangulation $\mathcal{T}_h^i$ is $h_i = \max_{K \in \mathcal{T}_h^i} h_K$ and we set $h = \max(h_1, h_2)$. Figure 4.2 gives an example of two subdomains of $\Omega$ meshed independently. Let $V_i = \{v \in H^1(\Omega_i) : v|_{\partial\Omega} = 0\}$ for $i = 1, 2$. On each domain $\Omega_i$ we define the space of continuous piecewise polynomial functions

$$V_i^k = \{v_h \in V_i : v_h|_K \in \mathbb{P}_k(K) \;\; \forall K \in \mathcal{T}_h^i\}, \quad k \geq 1,$$

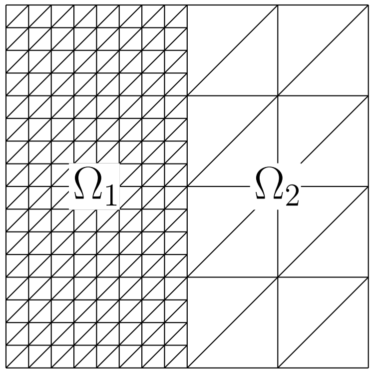



FIGURE 4.2: Example of meshes in $\Omega$.

$V_h^k = V_1^k \times V_2^k$, every function in $V_h^k$ has two components, $v_h = (v_h^1, v_h^2)$ such that $v_h^1 \in V_1^k$ and $v_h^2 \in V_2^k$. At the interface $\Gamma$ we use the notations

$$[\![w]\!] = w^1 - w^2,$$

for the jump, and

$$\{w\} = \omega_1 w^1 + \omega_2 w^2, \qquad \langle w \rangle = \omega_2 w^1 + \omega_1 w^2,$$

for the weighted averages with $\omega_1$ and $\omega_2$ the weights that will be specified later. At the interface $\Gamma$ let $n = n_1 = -n_2$, then we define

$$\{w \cdot n\} = \omega_1 w^1 \cdot n + \omega_2 w^2 \cdot n.$$

We now introduce a structure of patches that will be used in the upcoming inf-sup analysis similarly as in Chapter 2. Let the interface elements be the triangles with either a face or a vertex on the interface $\Gamma$. We regroup the interface elements of $\Omega_i$ in closed disjoint patches $P_j^i$ with boundary $\partial P_j^i$, $j = 1, ..., N_p^i$. $N_p^i$ defines the total number of patches in $\Omega_i$. Let $F_j^i = \partial P_j^i \cap \Gamma$, each $F_j^i$ has the property

$$h_i \lesssim \text{meas}(F_j^i) \lesssim h_i,$$

for $j = 1, \dots, N_p^i$. Let us focus on the patches $\{P_j^1\}_{1 \leq j \leq N_p^1}$ attached to the domain $\Omega_1$. Each patch $P_j^1$ is associated with a function $\chi_j \in V_1^1$ defined such that for each node $x_i \in \mathcal{T}_h^1$ we have

$$\chi_j(x_i) = \begin{cases} 0 & \text{for} \quad x_i \in \Omega_1 \backslash \mathring{F}_j^1 \\ 1 & \text{for} \quad x_i \in \mathring{F}_j^1, \end{cases} \tag{4.1}$$

with $i = 1, \dots, N_n$. $N_n$ is the number of nodes in the triangulation $\mathcal{T}_h^1$ .

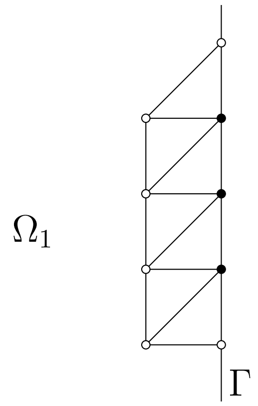



FIGURE 4.3: Example of $P_j^1$, the function $\chi_j$ is equal to 0 in the nonfilled nodes, 1 in the filled nodes.

We define the broken norm and semi-norm on $\Omega$ for any $v = (v^1, v^2) \in V_1 \times V_2$ such that $\|v\|_{s,\Omega}^2 = \|v^1\|_{s,\Omega_1}^2 + \|v^2\|_{s,\Omega_2}^2$ and $|v|_{s,\Omega}^2 = |v^1|_{s,\Omega_1}^2 + |v^2|_{s,\Omega_2}^2$.

## 4.2 Poisson problem

We consider the Poisson problem with discontinuous material parameters as

$$\begin{aligned} -\mu_i \Delta u^i &= f \quad && \text{in } \Omega_i, \quad i = 1, 2, \\ u^i &= 0 && \text{on } \partial\Omega \cap \Omega_i, \quad i = 1, 2, \\ [\![u]\!] &= 0 && \text{on } \Gamma, \\ [\![\mu \nabla u \cdot n]\!] &= 0 && \text{on } \Gamma, \end{aligned} \tag{4.2}$$

with $\mu_i$ the diffusivity of the domain $\Omega_i$ and $f \in L^2(\Omega)$, let $u = (u^1, u^2)$. The following regularity estimate holds [44]

$$\mu_1 \|D^2 u^1\|_{0,\Omega_1} + \mu_2 \|D^2 u^2\|_{0,\Omega_2} \lesssim \|f\|_{0,\Omega}.$$

We consider the following weights

$$\omega_1 = \frac{h_1 \mu_2}{h_1 \mu_2 + h_2 \mu_1}, \qquad \omega_2 = \frac{h_2 \mu_1}{h_1 \mu_2 + h_2 \mu_1}, \tag{4.3}$$

we note that $\omega_1 + \omega_2 = 1$. Considering the above problem we have

$$\{\mu \nabla u \cdot n\} = \mu_1 \nabla u^1 \cdot n_1 = -\mu_2 \nabla u^2 \cdot n_2.$$

To simplify the notations in the analysis we set

$$\gamma = \frac{\mu_1 \mu_2}{h_1 \mu_2 + h_2 \mu_1}.$$

In this chapter we assume that $\mu_2 h_1 \geq \mu_1 h_2$.

### 4.2.1 Finite element formulation

Classically for the problem (4.2) we obtain by integration by parts on each domain $\Omega_i$

$$(\mu_i \nabla u^i, \nabla v^i)_{\Omega_i} - \langle \mu_i \nabla u^i \cdot n_i, v^i \rangle_\Gamma = (f, v^i)_{\Omega_i}, \qquad \forall v^i \in V_i,$$

for $i = 1, 2$. By taking the sum of the interface terms and applying the identity $[\![ab]\!] = \{a\}[\![b]\!] + [\![a]\!]\langle b\rangle$, we obtain

$$\sum_{i=1}^{2} \langle \mu_i \nabla u^i \cdot n_i, v^i \rangle_\Gamma = \int_\Gamma [\![(\mu \nabla u \cdot n) v]\!] \mathrm{d}s = \langle \{\mu \nabla u \cdot n\}, [\![v]\!] \rangle_\Gamma + \langle [\![\mu \nabla u \cdot n]\!], \langle v \rangle \rangle_\Gamma,$$

the problem (4.2) tells us that $[\![\mu \nabla u \cdot n]\!] = 0$, then we obtain

$$\sum_{i=1}^{2} (\mu_i \nabla u^i, \nabla v^i)_{\Omega_i} - \langle \{\mu \nabla u \cdot n\}, [\![v]\!] \rangle_\Gamma = \sum_{i=1}^{2} (f, v^i)_{\Omega_i}, \qquad \forall v^i \in V_i. \tag{4.4}$$

Adding the corresponding antisymmetric Nitsche term, it leads to the following finite element formulation: find $u_h \in V_h^k$ such that

$$A_h(u_h, v_h) = L_h(v_h) \qquad \forall v_h \in V_h^k, \tag{4.5}$$

where

$$A_h(u_h, v_h) = \sum_{i=1}^{2} (\mu_i \nabla u_h^i, \nabla v_h^i)_{\Omega_i} - \langle \{\mu \nabla u_h \cdot n\}, [\![v_h]\!] \rangle_\Gamma + \langle \{\mu \nabla v_h \cdot n\}, [\![u_h]\!] \rangle_\Gamma,$$
$$L_h(v_h) = \sum_{i=1}^{2} (f, v_h^i)_{\Omega_i}.$$

### 4.2.2 Inf-sup stability

This section leads to the inf-sup stability of the penalty-free scheme previously introduced, we first prove an auxiliary Lemma that extends inequality (2.12) to the new framework.

**Lemma 4.2.1.** *Considering the patches $\{P_j^i\}_{1 \leq j \leq N_p^i}$ as defined above $\forall u_h \in V_h^k$ the following inequality holds*

$$\sum_{j=1}^{N_p^1} \gamma \| \overline{[\![u_h]\!]}^{F_j^1} \|_{0,F_j^1}^2 \geq \sum_{j=1}^{N_p^1} \frac{\gamma}{2} \| [\![u_h]\!] \|_{0,F_j^1}^2 - C\omega_1 \sum_{j=1}^{N_p^1} \mu_1 \| \nabla u_h^1 \|_{0,P_j^1}^2 - C\omega_2 \sum_{j=1}^{N_p^2} \mu_2 \| \nabla u_h^2 \|_{0,P_j^2}.$$

*Proof.* Considering the triangle inequality and the definition of the jump we can write

$$\begin{aligned}\frac{\gamma}{2}\|[\![u_h]\!]\|^2_{0,F_j^1} &\leq \gamma\|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} + \gamma\|[\![u_h]\!] - \overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} \\ &\leq \gamma\|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} + \gamma\|u_h^1 - u_h^2 - (\overline{u_h^1}^{F_j^1} - \overline{u_h^2}^{F_j^1})\|^2_{0,F_j^1}.\end{aligned}$$

Taking the sum over the whole interface and using the triangle inequality once again followed by inequality (2.11), the trace inequality of Lemma 2.0.1 and the inverse inequality of Lemma 2.0.2 we obtain

$$\begin{aligned}\sum_{j=1}^{N_p^1}\frac{\gamma}{2}\|[\![u_h]\!]\|^2_{0,F_j^1} &\leq \gamma\sum_{j=1}^{N_p^1}\left(\|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} + \|u_h^1 - \overline{u_h^1}^{F_j^1}\|^2_{0,F_j^1}\right) + \gamma\sum_{j=1}^{N_p^2}\|u_h^2 - \overline{u_h^2}^{F_j^2}\|^2_{0,F_j^2} \\ &\leq \gamma\sum_{j=1}^{N_p^1}\left(\|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} + Ch_1^2\|\nabla u_h^1\|^2_{0,F_j^1}\right) + C\gamma h_2^2\sum_{j=1}^{N_p^2}\|\nabla u_h^2\|^2_{0,F_j^2} \\ &\leq \sum_{j=1}^{N_p^1}\left(\gamma\|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} + C\omega_1\mu_1\|\nabla u_h^1\|^2_{0,P_j^1}\right) + C\omega_2\sum_{j=1}^{N_p^2}\mu_2\|\nabla u_h^2\|^2_{0,P_j^2}.\end{aligned} \tag{4.6}$$

□

We define the norms

$$\begin{aligned}|\!|\!|w|\!|\!|^2 &= \sum_{i=1}^{2}\mu_i\|\nabla w^i\|^2_{0,\Omega_i} + \gamma\|[\![w]\!]\|^2_{0,\Gamma}, \\ \|w\|_*^2 &= |\!|\!|w|\!|\!|^2 + \mu_1 h_1\|\nabla w^1\cdot n\|^2_{0,\Gamma} + \mu_2 h_2\|\nabla w^2\cdot n\|^2_{0,\Gamma}.\end{aligned}$$

**Theorem 4.2.1.** *There exists $\beta > 0$ such that for all functions $u_h \in V_h^k$ the following inequality holds*

$$\beta|\!|\!|u_h|\!|\!| \leq \sup_{v_h\in V_h^k}\frac{A_h(u_h,v_h)}{|\!|\!|v_h|\!|\!|}.$$

*Proof.* Let $v_h = u_h + \alpha\sum_{j=1}^{N_p^1}(v_j^1, 0)$, such that $v_j^1 = \nu_j\chi_j$, with $\nu_j \in \mathbb{R}$, each $v_j^1$ has the property

$$\text{meas}(F_j^1)^{-1}\int_{F_j^1}\nabla v_j^1\cdot n\ \mathrm{d}s = h_1^{-1}\overline{[\![u_h]\!]}^{F_j^1}. \tag{4.7}$$

Using Lemma 2.1.1 with $\varphi_r = v_j^1$ and $r_j = h_1^{-1}\overline{[\![u_h]\!]}^{F_j^1}$ we obtain the inequality

$$\|\nabla v_j^1\|_{0,P_j^1} \lesssim h_1^{-\frac{1}{2}}\|\overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1}. \tag{4.8}$$

Using the definition of $v_h$, we can write the following

$$A_h(u_h,v_h) = A_h(u_h,u_h) + \alpha\sum_{j=1}^{N_p^1}A_h(u_h,v_j^1).$$

Clearly we have

$$A_h(u_h, u_h) = \mu_1 \|\nabla u_h^1\|_{0,\Omega_1}^2 + \mu_2 \|\nabla u_h^2\|_{0,\Omega_2}^2,$$

and

$$\alpha A_h(u_h, v_j^1) = \alpha(\mu_1 \nabla u_h^1, \nabla v_j^1)_{P_j^1} - \alpha\langle\{\mu\nabla u_h \cdot n\}, v_j^1\rangle_{F_j^1} + \alpha\omega_1\langle\mu_1 \nabla v_j^1 \cdot n, [\![u_h]\!]\rangle_{F_j^1}.$$

Using Cauchy-Schwarz inequality, inequality (4.8) and the Young's inequality we obtain

$$\begin{aligned}(\mu_1 \nabla u_h^1, \alpha\nabla v_j^1)_{P_j^1} &\leq \mu_1^{\frac{1}{2}} \|\nabla u_h^1\|_{0,P_j^1} \alpha\mu_1^{\frac{1}{2}} \|\nabla v_j^1\|_{0,P_j^1} \\ &\leq \epsilon\mu_1 \|\nabla u_h^1\|_{0,P_j^1}^2 + \frac{C\alpha^2}{4\epsilon}\Big(1 + \frac{h_2\mu_1}{h_1\mu_2}\Big)\gamma\|\overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1}^2.\end{aligned}$$

Using the trace inequality of Lemma 2.0.1, the inverse inequality of Lemma 2.0.2, inequality (2.15) and (4.8) we can write

$$\begin{aligned}\langle\{\mu\nabla u_h \cdot n\}, \alpha v_j^1\rangle_{F_j^1} &= \langle\omega_1\mu_1\nabla u_h^1 \cdot n + \omega_2\mu_2\nabla u_h^2 \cdot n, \alpha v_j^1\rangle_{F_j^1} \\ &= \langle(\omega_1\mu_1 h_1)^{\frac{1}{2}}\nabla u_h^1 \cdot n + (\omega_2\mu_2 h_2)^{\frac{1}{2}}\nabla u_h^2 \cdot n, \alpha\gamma^{\frac{1}{2}} v_j^1\rangle_{F_j^1} \\ &\leq ((\omega_1\mu_1 h_1)^{\frac{1}{2}}\|\nabla u_h^1 \cdot n\|_{0,F_j^1} + (\omega_2\mu_2 h_2)^{\frac{1}{2}}\|\nabla u_h^2 \cdot n\|_{0,F_j^1})\alpha\gamma^{\frac{1}{2}}\|v_j^1\|_{0,F_j^1} \\ &\lesssim ((\omega_1\mu_1 h_1)^{\frac{1}{2}}\|\nabla u_h^1 \cdot n\|_{0,F_j^1} + (\omega_2\mu_2 h_2)^{\frac{1}{2}}\|\nabla u_h^2 \cdot n\|_{0,F_j^1})\alpha\gamma^{\frac{1}{2}}\|\overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1}.\end{aligned}$$

Taking the sum over the whole interface $\Gamma$ and using the Young's inequality, the trace inequality and the inverse inequality we obtain

$$\begin{aligned}&\sum_{j=1}^{N_p^1}\langle\{\mu\nabla u_h \cdot n\}, \alpha v_j^1\rangle_{F_j^1} \\ &\leq \frac{C\alpha^2}{2\epsilon}\sum_{j=1}^{N_p^1}\gamma\|\overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1}^2 + \epsilon(\omega_1\mu_1 h_1)\sum_{j=1}^{N_p^1}\|\nabla u_h^1 \cdot n\|_{0,F_j^1}^2 + \epsilon(\omega_2\mu_2 h_2)\sum_{j=1}^{N_p^2}\|\nabla u_h^2 \cdot n\|_{0,F_j^2}^2 \\ &\leq \frac{C\alpha^2}{2\epsilon}\sum_{j=1}^{N_p^1}\gamma\|\overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1}^2 + \epsilon\omega_1\mu_1\sum_{j=1}^{N_p^1}\|\nabla u_h^1\|_{0,P_j^1}^2 + \epsilon\omega_2\mu_2\sum_{j=1}^{N_p^2}\|\nabla u_h^2\|_{0,P_j^2}^2.\end{aligned}$$

Using the property (4.7) of $v_j^1$ we can write for each face $F_j^1$

$$\alpha\omega_1\langle\mu_1\nabla v_j^1 \cdot n, [\![u_h]\!]\rangle_{F_j^1} = \alpha\gamma\|\overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1}^2 + \alpha\omega_1\langle\mu_1\nabla v_j^1 \cdot n, [\![u_h]\!] - \overline{[\![u_h]\!]}^{F_j^1}\rangle_{F_j^1}.$$

Using the trace inequality and (4.8) we get

$$
\begin{aligned}
\omega_1\langle \mu_1 \nabla v_j^1 \cdot n, [\![u_h]\!] - \overline{[\![u_h]\!]}^{F_j^1}\rangle_{F_j^1} &\le \omega_1\mu_1 \|\nabla v_j^1 \cdot n\|_{0,F_j^1} \|[\![u_h]\!] - \overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1} \\
&\lesssim \omega_1\mu_1 h_1^{-\frac{1}{2}} \|\nabla v_j^1\|_{0,P_j^1} \|u_h^1 - u_h^2 - (\overline{u_h^1}^{F_j^1} - \overline{u_h^2}^{F_j^1})\|_{0,F_j^1} \\
&\lesssim \gamma \|\overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1} \big(\|u_h^1 - \overline{u_h^1}^{F_j^1}\|_{0,F_j^1} + \|u_h^2 - \overline{u_h^2}^{F_j^1}\|_{0,F_j^1}\big).
\end{aligned}
$$

Taking the sum over the whole interface $\Gamma$, using the Young's inequality and similar arguments as in (4.6) we get

$$
\begin{aligned}
&\sum_{j=1}^{N_p^1} \alpha\omega_1 \langle \mu_1 \nabla v_j^1 \cdot n, [\![u_h]\!] - \overline{[\![u_h]\!]}^{F_j^1}\rangle_{F_j^1} \\
&\qquad \le \sum_{j=1}^{N_p^1} \Big( \frac{C\alpha^2}{2\epsilon} \gamma \|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} + \epsilon\gamma \|u_h^1 - \overline{u_h^1}^{F_j^1}\|^2_{0,F_j^1} + \epsilon\gamma \|u_h^2 - \overline{u_h^2}^{F_j^1}\|^2_{0,F_j^1}\Big) \\
&\qquad \le \frac{C\alpha^2}{2\epsilon} \sum_{j=1}^{N_p^1} \gamma \|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} + \epsilon\omega_1 \sum_{j=1}^{N_p^1} \mu_1 \|\nabla u_h^1\|^2_{0,F_j^1} + \epsilon\omega_2 \sum_{j=1}^{N_p^2} \mu_2 \|\nabla u_h^2\|^2_{0,F_j^2}.
\end{aligned}
$$

It allows us to write

$$
\begin{aligned}
\sum_{j=1}^{N_p^1} \alpha\omega_1 \langle \mu_1 \nabla v_j^1 \cdot n, [\![u_h]\!]\rangle_{F_j^1} \ge \alpha\Big(1 - \frac{C\alpha}{2\epsilon}\Big) &\sum_{j=1}^{N_p^1} \gamma \|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1} \\
&- \epsilon\omega_1\mu_1 \sum_{j=1}^{N_p^1} \|\nabla u_h^1\|^2_{0,P_j^1} - \epsilon\omega_2\mu_2 \sum_{j=1}^{N_p^2} \|\nabla u_h^2\|^2_{0,P_j^2}.
\end{aligned}
$$

The full bilinear form $A_h$ now has the following lower bound

$$
A_h(u_h, v_h) \ge C_a\mu_1 \|\nabla u_h^1\|^2_{0,\Omega_1} + C_b\mu_2 \|\nabla u_h^2\|^2_{0,\Omega_2} + C_c \sum_{j=1}^{N_p^1} \gamma \|\overline{[\![u_h]\!]}^{F_j^1}\|^2_{0,F_j^1},
$$

with the constants

$$
C_a = 1 - \epsilon(2\omega_1 + 1), \qquad C_b = 1 - 2\epsilon\omega_2, \qquad C_c = \alpha\Big(1 - \alpha\frac{C}{4\epsilon}\Big(5 + \frac{h_2\mu_1}{h_1\mu_2}\Big)\Big).
$$

Using Lemma 4.2.1 it becomes

$$
A_h(u_h, v_h) \ge (C_a - \omega_1 C C_c)\mu_1 \|\nabla u_h^1\|^2_{0,\Omega_1} + (C_b - \omega_2 C C_c)\mu_2 \|\nabla u_h^2\|^2_{0,\Omega_2} + C_c \frac{\gamma}{2} \|[\![u_h]\!]\|^2_{0,\Gamma}.
$$

First let $\epsilon = \frac{1}{8}$. The constant $C_c$ will be positive for

$$
\alpha < \frac{1}{12C}.
$$

The terms $(C_a - \omega_1 C C_c)$ and $(C_b - \omega_2 C C_c)$ will be both positive for

$$\alpha < \frac{1}{2C}.$$

Then we get

$$A_h(u_h, v_h) \geq \beta_0 |||u_h|||^2.$$

with $\beta_0 = \mathcal{O}(1)$. To complete the proof we show $|||v_h||| \lesssim |||u_h|||$, using similar arguments as (2.16) we obtain

$$|||v_h|||^2 \lesssim |||u_h|||^2 + \alpha^2 \sum_{j=1}^{N_p^1} |||v_j^1|||^2,$$

with

$$|||v_j^1|||^2 = \mu_1 \|\nabla v_j^1\|^2_{0,P_j^1} + \gamma \|v_j^1\|^2_{0,F_j^1}.$$

Using (4.8) and

$$\|\overline{[\![u_h]\!]}^{F_j^1}\|_{0,F_j^1} \lesssim \|[\![u_h]\!]\|_{0,F_j^1},$$

it gives the appropriate upper bound

$$\sum_{j=1}^{N_p^1} \mu_1 \|\nabla v_j^1\|^2_{0,P_j^1} \lesssim \left(1 + \frac{h_2 \mu_1}{h_1 \mu_2}\right) |||u_h|||^2 \lesssim |||u_h|||^2.$$

Using the trace inequality of Lemma 2.0.1 and the inequality (2.15)

$$\sum_{j=1}^{N_p^1} \gamma \|v_j^1\|^2_{0,F_j^1} \lesssim \omega_1 \sum_{j=1}^{N_p^1} \mu_1 \|\nabla v_j^1\|^2_{0,P_j^1} \lesssim |||u_h|||^2.$$

Note that $\beta = \mathcal{O}(1)$. □

### 4.2.3 A priori error estimate

The proof of the stability done in the previous part leads to the study of the a priori error estimates. Let $H^k_\partial(\Omega_i) = \{v \in H^k(\Omega_i) : v|_{\partial\Omega} = 0\}$, the Galerkin orthogonality characterises the following consistency relation.

**Lemma 4.2.2.** *If $u \in H^2_\partial(\Omega_1) \times H^2_\partial(\Omega_2)$ is the solution of* (4.2) *and $u_h \in V_h^k$ the solution of* (4.5) *the following property holds*

$$A_h(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h^k.$$

*Proof.* Considering (4.4) and adding the consistent antisymmetric Nitsche term we have

$$\sum_{i=1}^{2} (\mu_i \nabla u^i, \nabla v_h^i)_{\Omega_i} - \langle \{\mu \nabla u \cdot n\}, [\![v_h]\!] \rangle_\Gamma + \langle \{\mu \nabla v_h \cdot n\}, [\![u]\!] \rangle_\Gamma = \sum_{i=1}^{2} (f, v_h^i)_{\Omega_i}, \qquad \forall v_h \in V_h^k.$$

Then $A_h(u, v_h) = \sum_{i=1}^{2}(f, v_h^i)_{\Omega_i} = L_h(v_h)$, owing the properties of the jump $[\![\cdot]\!]$ and average $\{\cdot\}$ we have

$$A_h(u - u_h, v_h) = A_h(u, v_h) - A_h(u_h, v_h) = L_h(v_h) - L_h(v_h) = 0 \qquad \forall v_h \in V_h^k.$$

$\square$

**Lemma 4.2.3.** *Let $w \in H^2_\partial(\Omega_1) \times H^2_\partial(\Omega_2) + V_h^k$ and $v_h \in V_h^k$, there exists a positive constant $M$ such that*

$$A_h(w, v_h) \leq M\|w\|_* |\!|\!| v_h |\!|\!|.$$

*Proof.* Using the Cauchy Schwarz inequality we have

$$\begin{aligned}\langle\{\mu\nabla w\cdot n\}, [\![v_h]\!]\rangle_\Gamma &= \langle(\omega_1\mu_1 h_1)^{\frac{1}{2}}\nabla w\cdot n + (\omega_2\mu_2 h_2)^{\frac{1}{2}}\nabla w\cdot n, \gamma^{\frac{1}{2}}[\![v_h]\!]\rangle_\Gamma \\ &\leq ((\omega_1\mu_1 h_1)^{\frac{1}{2}}\|\nabla w^1\cdot n\|_{0,\Gamma} + (\omega_2\mu_2 h_2)^{\frac{1}{2}}\|\nabla w^2\cdot n\|_{0,\Gamma})\gamma^{\frac{1}{2}}\|[\![v_h]\!]\|_{0,\Gamma}.\end{aligned}$$

The trace inequality of Lemma 2.0.1 gives us

$$\begin{aligned}\langle\{\mu\nabla v_h\cdot n\}, [\![w]\!]\rangle_\Gamma &\leq ((\omega_1\mu_1 h_1)^{\frac{1}{2}}\|\nabla v_h^1\cdot n\|_{0,\Gamma} + (\omega_2\mu_2 h_2)^{\frac{1}{2}}\|\nabla v_h^2\cdot n\|_{0,\Gamma})\gamma^{\frac{1}{2}}\|[\![w]\!]\|_{0,\Gamma} \\ &\lesssim ((\omega_1\mu_1)^{\frac{1}{2}}\|\nabla v_h^1\|_{0,\Omega_1} + (\omega_2\mu_2)^{\frac{1}{2}}\|\nabla v_h^2\|_{0,\Omega_2})\gamma^{\frac{1}{2}}\|[\![w]\!]\|_{0,\Gamma}.\end{aligned}$$

Using these two upper bounds it is straightforward to show that

$$\sum_{i=1}^{2}(\mu_i\nabla w^i, \nabla v_h^i)_{\Omega_i} - \langle\{\mu\nabla w\cdot n\}, [\![v_h]\!]\rangle_\Gamma + \langle\{\mu\nabla v_h\cdot n\}, [\![w]\!]\rangle_\Gamma \lesssim \|w\|_* |\!|\!| v_h |\!|\!|.$$

Note that $M = \mathcal{O}(\omega_1^{\frac{1}{2}} + \omega_2^{\frac{1}{2}}) = \mathcal{O}(1)$. $\square$

**Theorem 4.2.2.** *If $u \in H^{k+1}_\partial(\Omega_1) \times H^{k+1}_\partial(\Omega_2)$ is the solution of* (4.2) *and $u_h \in V_h^k$ the solution of* (4.5)*, then there holds*

$$|\!|\!| u - u_h |\!|\!| \leq C_\beta \inf_{w_h\in V_h^k}\|u - w_h\|_*,$$

*with $C_\beta$ a positive constant that depends on the mesh geometry.*

*Proof.* The triangle inequality provides

$$|\!|\!| u - u_h |\!|\!| \leq |\!|\!| u - w_h |\!|\!| + |\!|\!| w_h - u_h |\!|\!|.$$

Using the Galerkin orthogonality of Lemma 4.2.2, the Theorem 4.2.1 and the Lemma 4.2.3 we can write

$$\beta |\!|\!| u_h - w_h |\!|\!| \leq \sup_{v_h\in V_h^k}\frac{A_h(u - w_h, v_h)}{|\!|\!| v_h |\!|\!|} \leq M\|u - w_h\|_*.$$

Note that $|||u - w_h||| \leq \|u - w_h\|_*$, taking the inf over all $w_h$ we obtain

$$|||u - u_h||| \leq \left(1 + \frac{M}{\beta}\right) \inf_{w_h \in V_h^k} \|u - w_h\|_*,$$

and $C_\beta = \mathcal{O}(1)$. □

**Corollary 4.2.1.** *If $u \in H_\partial^{k+1}(\Omega_1) \times H_\partial^{k+1}(\Omega_2)$ is the solution of (4.2) and $u_h \in V_h^k$ the solution of (4.5), then there holds*

$$|||u - u_h||| \leq C_\mu h^k |u|_{k+1,\Omega},$$

*with $C_\mu$ a positive constant that depends on $\mu$ and the mesh geometry.*

*Proof.* The triangle inequality gives us

$$\gamma^{\frac{1}{2}} \|[\![u - \pi_h^k u]\!]\|_{0,\Gamma} \leq (\omega_1 \mu_1)^{\frac{1}{2}} h_1^{-\frac{1}{2}} \|u^1 - \pi_h^k u^1\|_{0,\Gamma} + (\omega_2 \mu_2)^{\frac{1}{2}} h_2^{-\frac{1}{2}} \|u^2 - \pi_h^k u^2\|_{0,\Gamma}.$$

Using the trace inequality and the approximation property of the nodal interpolant (2.17) we have

$$h_i^{-\frac{1}{2}} \|u^i - \pi_h^k u^i\|_{0,\Gamma} \lesssim h_i^{-1} \|u^i - \pi_h^k u^i\|_{0,\Omega_i} + \|\nabla(u^i - \pi_h^k u^i)\|_{0,\Omega_i} \lesssim h_i^k |u^i|_{k+1,\Omega_i},$$

and

$$\begin{aligned} h_i^{\frac{1}{2}} \|\nabla(u^i - \pi_h^k u^i) \cdot n\|_{0,\Gamma} \lesssim & \|\nabla(u^i - \pi_h^k u^i)\|_{0,\Omega_i} \\ & + h_i \Big( \sum_{K \in \mathcal{T}_h^i} \|D^2(u^i - \pi_h^k u^i)\|_{0,K}^2 \Big)^{\frac{1}{2}} \lesssim h_i^k |u^i|_{k+1,\Omega_i}. \end{aligned}$$

Then we deduce that

$$\|u - \pi_h^k u\|_* \lesssim \mu_1^{\frac{1}{2}} h_1^k |u^1|_{k+1,\Omega_1} + \mu_2^{\frac{1}{2}} h_2^k |u^2|_{k+1,\Omega_2}. \tag{4.9}$$

Applying the Theorem 4.2.2 with $w_h = \pi_h^k u$ the result follows and $C_\mu = \mathcal{O}(\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}})$. □

**Proposition 4.2.1.** *Let $u \in H_\partial^{k+1}(\Omega_1) \times H_\partial^{k+1}(\Omega_2)$ be the solution of (4.2) and $u_h \in V_h^k$ the solution of (4.5), then there holds*

$$\|u - u_h\|_\Omega \leq C'_\mu h^{k+\frac{1}{2}} |u|_{k+1,\Omega},$$

*with $C'_\mu$ is a positive constant that depends on $\mu$ and the mesh geometry.*

*Proof.* Let $z$ satisfy the adjoint problem

$$\begin{aligned} -\mu_i \Delta z^i &= u^i - u_h^i && \text{in } \Omega_i \ , \quad i = 1,2, \\ z^i &= 0 && \text{on } \partial\Omega \cap \Omega_i \ , \quad i = 1,2, \\ [\![ z ]\!] &= 0 && \text{on } \Gamma, \\ [\![ \mu \nabla z \cdot n ]\!] &= 0 && \text{on } \Gamma. \end{aligned} \tag{4.10}$$

We assume the following regularity estimate

$$\mu_1 \|z^1\|_{2,\Omega_1} + \mu_2 \|z^2\|_{2,\Omega_2} \lesssim \|u - u_h\|_{0,\Omega}. \tag{4.11}$$

By integration by parts and using that $(u - u_h)|_{\partial\Omega} = 0$ we obtain

$$\|u^i - u_h^i\|_{\Omega_i}^2 = (u^i - u_h^i, -\mu_i \Delta z^i)_{\Omega_i} = (\nabla(u^i - u_h^i), \mu_i \nabla z^i)_{\Omega_i} - \langle \mu_i \nabla z^i \cdot n_i, u^i - u_h^i \rangle_\Gamma.$$

Using $[\![ \mu \nabla z \cdot n ]\!] = 0$ and $[\![ z ]\!] = 0$ on the interface, the $L^2$-error can be upper bounded by

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 &= \|u^1 - u_h^1\|_{0,\Omega_1}^2 + \|u^2 - u_h^2\|_{0,\Omega_2}^2 \\ &= \sum_{i=1}^2 (\nabla(u^i - u_h^i), \mu_i \nabla z^i)_{\Omega_i} - \langle \{\mu \nabla z \cdot n\}, [\![ u - u_h ]\!] \rangle_\Gamma \\ &= A_h(u - u_h, z) - 2\langle \{\mu \nabla z \cdot n\}, [\![ u - u_h ]\!] \rangle_\Gamma. \end{aligned}$$

Using the global trace inequality $\|\nabla z^i \cdot n\|_{0,\Gamma} \lesssim \|z^i\|_{2,\Omega_i}$ for $i = 1,2$, we can write

$$\begin{aligned} &|\langle \{\mu \nabla z \cdot n\}, [\![ u - u_h ]\!] \rangle_\Gamma| \\ &\qquad \le ((\omega_1 \mu_1 h_1)^{\frac{1}{2}} \|\nabla z^1 \cdot n\|_{0,\Gamma} + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \|\nabla z^2 \cdot n\|_{0,\Gamma}) \gamma^{\frac{1}{2}} \|[\![ u - u_h ]\!]\|_{0,\Gamma} \\ &\qquad \lesssim ((\omega_1 \mu_1 h_1)^{\frac{1}{2}} \|z^1\|_{2,\Omega_1} + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \|z^2\|_{2,\Omega_2}) |\!|\!| u - u_h |\!|\!|. \end{aligned}$$

Using Lemma 4.2.2 we get

$$\begin{aligned} A_h(u - u_h, z) &= A_h(u - u_h, z - \pi_h^1 z) \\ &= \sum_{i=1}^2 (\nabla(u^i - u_h^i), \mu_i \nabla(z^i - \pi_h^1 z^i))_{\Omega_i} \\ &\quad - \langle \{\mu \nabla(u - u_h) \cdot n\}, [\![ z - \pi_h^1 z ]\!] \rangle_\Gamma + \langle \{\mu \nabla(z - \pi_h^1 z) \cdot n\}, [\![ u - u_h ]\!] \rangle_\Gamma \\ &\le \sum_{i=1}^2 \mu_i \|\nabla(u^i - u_h^i)\|_{0,\Omega_i} \|\nabla(z^i - \pi_h^1 z^i)\|_{0,\Omega_i} \\ &\quad + \sum_{i=1}^2 \big((\omega_i \mu_i h_i)^{\frac{1}{2}} \|\nabla(u^i - u_h^i) \cdot n\|_{0,\Gamma}\big) \gamma^{\frac{1}{2}} \|[\![ z - \pi_h^1 z ]\!]\|_{0,\Gamma} \\ &\quad + \sum_{i=1}^2 \big((\omega_i \mu_i h_i)^{\frac{1}{2}} \|\nabla(z^i - \pi_h^1 z^i) \cdot n\|_{0,\Gamma}\big) \gamma^{\frac{1}{2}} \|[\![ u - u_h ]\!]\|_{0,\Gamma}. \end{aligned}$$

Similarly as (3.21) using the trace inequality of Lemma 2.0.1 and the approximation property of the nodal interpolant we have

$$h_i^{\frac{1}{2}} \|\nabla(u^i - u_h^i) \cdot n\|_{0,\Gamma} \lesssim h_i^k |u^i|_{k+1,\Omega_i} + \|\nabla(u_h^i - \pi_h^k u^i)\|_{0,\Omega_i}.$$

Then using that $|||u_h - \pi_h^k u||| \lesssim \|u - \pi_h^k u\|_*$, estimate (4.9), Corollary 4.2.1 and the approximation property of the nodal interpolant we obtain

$$\begin{aligned} A_h(u - u_h, z) &\lesssim \|z - \pi_h^1 z\|_* \Big( |||u - u_h||| + |||u_h - \pi_h^k u||| + \sum_{i=1}^{2} (\omega_i \mu_i)^{\frac{1}{2}} h_i^k |u^i|_{k+1,\Omega_i} \Big) \\ &\lesssim \|z - \pi_h^1 z\|_* (\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}}) h^k |u|_{k+1,\Omega} \\ &\lesssim (\mu_1^{\frac{1}{2}} h_1 |z^1|_{2,\Omega_1} + \mu_2^{\frac{1}{2}} h_2 |z^2|_{2,\Omega_2})(\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}}) h^k |u|_{k+1,\Omega} \end{aligned}$$

Then using Corollary 4.2.1 once again we get

$$\begin{aligned} \|u - u_h\|_\Omega^2 &\lesssim ((h_1 + (\omega_1 h_1)^{\frac{1}{2}})\mu_1^{\frac{1}{2}} \|z^1\|_{2,\Omega_1} \\ &\qquad + (h_2 + (\omega_2 h_2)^{\frac{1}{2}})\mu_2^{\frac{1}{2}} \|z^2\|_{2,\Omega_2})(\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}}) h^k |u|_{k+1,\Omega} \\ &\lesssim (\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}}) h^{\frac{1}{2}} ((\omega_1 \mu_1)^{\frac{1}{2}} \|z^1\|_{2,\Omega_1} + (\omega_2 \mu_2)^{\frac{1}{2}} \|z^2\|_{2,\Omega_2}) h^k |u|_{k+1,\Omega}. \end{aligned}$$

We conclude by applying the regularity estimate (4.11), $C'_\mu = \mathcal{O}(1)$. □

## 4.3 Compressible elasticity

We consider the compressible elasticity problem with discontinuous material parameters as

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}^i) &= \boldsymbol{f} && \text{in } \Omega_i\ , \quad i = 1, 2, \\ \boldsymbol{u}^i &= 0 && \text{on } \partial\Omega \cap \Omega_i\ , \quad i = 1, 2, \\ [\![\boldsymbol{u}]\!] &= 0 && \text{on } \Gamma, \\ [\![\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}]\!] &= 0 && \text{on } \Gamma, \end{aligned} \tag{4.12}$$

with $\boldsymbol{f} \in [L^2(\Omega)]^2$, the stress tensor is expressed as

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda(\nabla \cdot \boldsymbol{u})\mathbb{I}_{2\times 2}.$$

In a subdomain $\Omega_i$ the Lamé coefficients are denoted as $\mu_i$ and $\lambda_i$. Since the displacement $\boldsymbol{u}$ is equal to zero on the boundary of $\Omega$ the following Korn inequality holds ([45] Theorem 4.2.4) for all $\boldsymbol{u}^i \in [H^1(\Omega_i)]^2$ and $\boldsymbol{u}^i|_{\partial\Omega} = 0$, then

$$C_K \|\boldsymbol{u}^i\|_{1,\Omega_i} \leq \|\boldsymbol{\varepsilon}(\boldsymbol{u}^i)\|_{0,\Omega_i}. \tag{4.13}$$

### 4.3.1 Finite element formulation

Classically by using integration by parts we have for each domain

$$(2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}^i))_{\Omega_i} + (\lambda_i \nabla \cdot \boldsymbol{u}^i, \nabla \cdot \boldsymbol{v}^i)_{\Omega_i} - \langle \boldsymbol{\sigma}(\boldsymbol{u}^i) \cdot \boldsymbol{n}_i, \boldsymbol{v}^i \rangle_\Gamma = (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i} \qquad \forall \boldsymbol{v}^i \in W_i$$

for $i = 1, 2$ with $W_i = [V_i]^2$. By summing the interface terms we obtain

$$\sum_{i=1}^{2} \langle \boldsymbol{\sigma}(\boldsymbol{u}^i) \cdot \boldsymbol{n}_i, \boldsymbol{v}^i \rangle_\Gamma = \int_\Gamma [\![ (\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}) \boldsymbol{v} ]\!] \mathrm{d}s = \langle \{ \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} \}, [\![ \boldsymbol{v} ]\!] \rangle_\Gamma + \langle [\![ \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} ]\!], \langle \boldsymbol{v} \rangle \rangle_\Gamma,$$

the problem (4.12) tells us that $[\![ \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} ]\!] = 0$, so we obtain

$$\sum_{i=1}^{2} \left( (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}^i))_{\Omega_i} + (\lambda_i \nabla \cdot \boldsymbol{u}^i, \nabla \cdot \boldsymbol{v}^i)_{\Omega_i} \right) - \langle \{ \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} \}, [\![ \boldsymbol{v} ]\!] \rangle_\Gamma = \sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i}. \quad (4.14)$$

By adding the corresponding Nitsche term we get the following penalty-free finite element formulation: find $\boldsymbol{u}_h \in W_h^k$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = L_h(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in W_h^k, \quad (4.15)$$

with $W_h^k = [V_h^k]^2$, and the linear forms

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \boldsymbol{u}_h),$$
$$L_h(v_h) = \sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i}.$$

The bilinear forms $a_h$ and $b_h$ are defined as

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{i=1}^{2} \left[ (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}_h^i), \boldsymbol{\varepsilon}(\boldsymbol{v}_h^i))_{\Omega_i} + (\lambda_i \nabla \cdot \boldsymbol{u}_h^i, \nabla \cdot \boldsymbol{v}_h^i)_{\Omega_i} \right],$$
$$b(\boldsymbol{u}_h, \boldsymbol{v}_h) = \langle \{ 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h) \cdot \boldsymbol{n} \}, [\![ \boldsymbol{v}_h ]\!] \rangle_\Gamma + \langle \{ \lambda \nabla \cdot \boldsymbol{u}_h \}, [\![ \boldsymbol{v}_h \cdot \boldsymbol{n} ]\!] \rangle_\Gamma.$$

As for the Poisson case we choose $\mu_2 h_1 \geq \mu_1 h_2$.

### 4.3.2 Inf-sup stability

As specified at the beginning of this chapter the interface $\Gamma$ is considered as plane. Any function $\boldsymbol{z} = (z_1, z_2) \in W_h^k$ is expressed in the two dimensional generic frame $(x, y)$ and $z_1, z_2 \in V_h^k$. The function $\boldsymbol{z}$ can also be decomposed such that $\boldsymbol{z}^1 \in W_1^k$ and $\boldsymbol{z}^2 \in W_2^k$ with $W_1^k = [V_1^k]^2$ and $W_2^k = [V_2^k]^2$. The interface $\Gamma$ is parallel to the $x$-axis then for $\boldsymbol{\tau}$ and $\boldsymbol{n}$ respectively the tangent and normal unit vectors to the plane interface $\Gamma$ we have $z_1 = \boldsymbol{z} \cdot \boldsymbol{\tau}$ and $z_2 = \boldsymbol{z} \cdot \boldsymbol{n}$. We introduce the function $\boldsymbol{v}_j^1$ such that $\boldsymbol{v}_j^1 = (\alpha_1 v_1^1, \alpha_2 v_2^1)^T$. We define $v_1^1 = \nu_1 \chi_j$ and $v_2^1 = \nu_2 \chi_j$ with $\nu_1, \nu_2 \in \mathbb{R}$ and $\chi_j$ as defined in (4.1). In order to be able to use Lemma 2.1.1, the function $\boldsymbol{v}_j^1$ has the properties

$$\text{meas}(F_j^1)^{-1}\int_{F_j^1}\frac{\partial v_1^1}{\partial y}\,\mathrm{d}s = h_1^{-1}\overline{[\![u_1]\!]}^{F_j^1}, \qquad \text{meas}(F_j^1)^{-1}\int_{F_j^1}\frac{\partial v_2^1}{\partial y}\,\mathrm{d}s = h_1^{-1}\overline{[\![u_2]\!]}^{F_j^1}, \quad (4.16)$$

with $\boldsymbol{u}_h^i = (u_1^i, u_2^i)^T$. Using Lemma 2.1.1 it is straightforward to show

$$\|\nabla v_1^1\|_{0,P_j^1} \lesssim h_1^{-\frac{1}{2}}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}, \qquad \|\nabla v_2^1\|_{0,P_j^1} \lesssim h_1^{-\frac{1}{2}}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}. \qquad (4.17)$$

Let the norms

$$\begin{aligned}
|\!|\!|\boldsymbol{w}|\!|\!|^2 &= \sum_{i=1}^{2}\left[\mu_i\|\nabla\boldsymbol{w}^i\|_{0,\Omega_i}^2 + \lambda_i\|\nabla\cdot\boldsymbol{w}^i\|_{0,\Omega_i}^2\right] + \gamma\|[\![\boldsymbol{w}]\!]\|_{0,\Gamma}^2,\\
\|\boldsymbol{w}\|_*^2 &= |\!|\!|\boldsymbol{w}|\!|\!|^2 + \sum_{i=1}^{2}\left[\mu_i h_i\|\nabla\boldsymbol{w}^i\cdot\boldsymbol{n}\|_{0,\Gamma}^2 + \mu_i h_i\|\nabla\cdot\boldsymbol{w}^i\|_{0,\Gamma}^2\right].
\end{aligned}$$

First we give two technical Lemmas, proofs are provided in appendix D.

**Lemma 4.3.1.** *There exists $C > 0$ independent of $h$, $\mu_i$ and $\lambda_i$ ($i = 1, 2$), but not of the mesh geometry, $\forall\boldsymbol{u}_h \in W_h^k$, for $\boldsymbol{v}_j^1 \in W_1^1$ as defined above and $\forall\epsilon, \alpha_1, \alpha_2 \in \mathbb{R}_+^*$, the following inequality holds*

$$\begin{aligned}
\sum_{j=1}^{N_p^1}\langle\omega_1\lambda_1\nabla\cdot\boldsymbol{v}_j^1, [\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\rangle_{F_j^1} \gtrsim{}& \alpha_2\gamma\Big(1-\frac{C\alpha_2}{2\epsilon}\Big)\frac{\lambda_1}{\mu_1}\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2\\
&-\gamma\frac{C\alpha_1^2}{2\epsilon}\frac{\lambda_1}{\mu_1}\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2\\
&-2\epsilon\omega_1\lambda_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 - 2\epsilon\omega_2\frac{\lambda_1\mu_2}{\mu_1\lambda_2}\lambda_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2.
\end{aligned}$$

**Lemma 4.3.2.** *There exists $C > 0$ independent of $h$, $\mu_i$ and $\lambda_i$ ($i = 1, 2$), but not of the mesh geometry, $\forall\boldsymbol{u}_h \in W_h^k$, for $\boldsymbol{v}_j^1 \in W_1^1$ as defined above and $\forall\epsilon, \alpha_1, \alpha_2 \in \mathbb{R}_+^*$, the following inequality holds*

$$\begin{aligned}
\sum_{j=1}^{N_p^1}\langle 2\omega_1\mu_1\boldsymbol{\varepsilon}(\boldsymbol{v}_j^1)\cdot\boldsymbol{n}, [\![\boldsymbol{u}_h]\!]\rangle_{F_j^1} \geq{}& \alpha_2\gamma\Big(2-\frac{3C\alpha_2}{2\epsilon}\Big)\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2\\
&+\alpha_1\gamma\Big(1-\frac{C\alpha_1}{2\epsilon}\Big)\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2\\
&-3\epsilon\omega_1\mu_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 - 3\epsilon\omega_2\mu_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2.
\end{aligned}$$

**Lemma 4.3.3.** *For $\boldsymbol{u}_h, \boldsymbol{v}_h \in W_h^k$ with $\boldsymbol{v}_h = \boldsymbol{u}_h + \sum_{j=1}^{N_p^1} (\boldsymbol{v}_j^1, 0)$, $\boldsymbol{v}_j^1$ defined by (4.16), there exist a positive constant $\beta_0$ such that the following inequality holds*

$$\beta_0 |||\boldsymbol{u}_h|||^2 \leq A_h(\boldsymbol{u}_h, \boldsymbol{v}_h).$$

*Proof.* Applying the definition of $\boldsymbol{v}_h$ we get

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + \sum_{j=1}^{N_p^1} A_h(\boldsymbol{u}_h, \boldsymbol{v}_j^1),$$

with

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{v}_j^1) = &(2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1))_{P_j^1} - \langle \{2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h) \cdot \boldsymbol{n}\}, \boldsymbol{v}_j^1 \rangle_{F_j^1} + \langle 2\omega_1 \mu_1 \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1) \cdot \boldsymbol{n}, [\![\boldsymbol{u}_h]\!] \rangle_{F_j^1} \\ &+ (\lambda_1 \nabla \cdot \boldsymbol{u}_h^1, \nabla \cdot \boldsymbol{v}_j^1)_{P_j^1} - \langle \{\lambda \nabla \cdot \boldsymbol{u}_h\}, \boldsymbol{v}_j^1 \cdot \boldsymbol{n} \rangle_{F_j^1} + \langle \omega_1 \lambda_1 \nabla \cdot \boldsymbol{v}_j^1, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!] \rangle_{F_j^1}, \end{aligned}$$

Classically we get

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) = \sum_{i=1}^{2} \big(2\mu_i \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^i)\|_{0,\Omega_i}^2 + \lambda_i \|\nabla \cdot \boldsymbol{u}_h^i\|_{0,\Omega_i}^2\big).$$

Using the Cauchy-Schwarz inequality, (4.17) and the Young's inequality we get

$$\begin{aligned} &(2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1))_{0,P_j^1} \\ &\quad \leq 2\mu_1 \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^1)\|_{0,P_j^1} \|\nabla \boldsymbol{v}_j^1\|_{0,P_j^1} \\ &\quad \lesssim 2\mu_1 \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^1)\|_{0,P_j^1} (\alpha_1 h_1^{-\frac{1}{2}} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1} + \alpha_2 h_1^{-\frac{1}{2}} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}) \\ &\quad \leq \epsilon \mu_1 \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^1)\|_{0,P_j^1}^2 + \gamma \Big(1 + \frac{h_2 \mu_1}{h_1 \mu_2}\Big) \Big(\frac{C\alpha_1^2}{\epsilon} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2 + \frac{C\alpha_2^2}{\epsilon} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2\Big). \end{aligned}$$

Similarly we have

$$\begin{aligned} &(\lambda_1 \nabla \cdot \boldsymbol{u}_h^1, \nabla \cdot \boldsymbol{v}_j^1)_{P_j^1} \\ &\quad \leq \epsilon \lambda_1 \|\nabla \cdot \boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \gamma \Big(1 + \frac{h_2 \mu_1}{h_1 \mu_2}\Big) \frac{\lambda_1}{\mu_1} \Big(\frac{C\alpha_1^2}{2\epsilon} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2 + \frac{C\alpha_2^2}{2\epsilon} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2\Big). \end{aligned}$$

Using the Cauchy-Schwarz inequality, the trace inequality and inequality (2.15) we have

$$\begin{aligned} &\langle \{2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h) \cdot \boldsymbol{n}\}, \boldsymbol{v}_j^1 \rangle_{F_j^1} \\ &\quad \leq 2\langle \omega_1 \mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1) \cdot \boldsymbol{n} + \omega_2 \mu_2 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^2) \cdot \boldsymbol{n}, \boldsymbol{v}_j^1 \rangle_{F_j^1} \\ &\quad \leq 2\langle (\omega_1 \mu_1 h_1)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1) \cdot \boldsymbol{n} + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\boldsymbol{u}_h^2) \cdot \boldsymbol{n}, \gamma^{\frac{1}{2}} \boldsymbol{v}_j^1 \rangle_{F_j^1} \\ &\quad \leq 2((\omega_1 \mu_1 h_1)^{\frac{1}{2}} \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^1) \cdot \boldsymbol{n}\|_{0,F_j^1} + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^2) \cdot \boldsymbol{n}\|_{0,F_j^1}) \gamma^{\frac{1}{2}} \|\boldsymbol{v}_j^1\|_{0,F_j^1} \\ &\quad \lesssim 2((\omega_1 \mu_1 h_1)^{\frac{1}{2}} \|\nabla \boldsymbol{u}_h^1 \cdot \boldsymbol{n}\|_{0,F_j^1} + (\omega_2 \mu_2 h_2)^{\frac{1}{2}} \|\nabla \boldsymbol{u}_h^2 \cdot \boldsymbol{n}\|_{0,F_j^1}) \gamma^{\frac{1}{2}} h_1^{\frac{1}{2}} \|\nabla \boldsymbol{v}_j^1\|_{0,P_j^1}, \end{aligned}$$

taking the sum over the whole interface and using (4.17) we have

$$\begin{aligned}\sum_{j=1}^{N_p^1}\langle\{2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\cdot\boldsymbol{n}\},\boldsymbol{v}_j^1\rangle_{F_j^1} &\le \epsilon\omega_1\mu_1 h_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\cdot\boldsymbol{n}\|_{0,F_j^1}^2+\epsilon\omega_2\mu_2 h_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\cdot\boldsymbol{n}\|_{0,F_j^2}^2\\ &\quad+\frac{C\alpha_1^2}{\epsilon}\gamma\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2+\frac{C\alpha_2^2}{\epsilon}\gamma\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2\\ &\le \epsilon\omega_1\mu_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2+\epsilon\omega_2\mu_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2\\ &\quad+\frac{C\alpha_1^2}{\epsilon}\gamma\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2+\frac{C\alpha_2^2}{\epsilon}\gamma\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2.\end{aligned}$$

Similarly we have

$$\begin{aligned}&\langle\{\lambda\nabla\cdot\boldsymbol{u}_h\},\boldsymbol{v}_j^1\cdot\boldsymbol{n}\rangle_{F_j^1}\\ &\le\langle\omega_1\lambda_1\nabla\cdot\boldsymbol{u}_h^1+\omega_2\lambda_2\nabla\cdot\boldsymbol{u}_h^2,\boldsymbol{v}_j^1\cdot\boldsymbol{n}\rangle_{F_j^1}\\ &\le\langle(\omega_1\lambda_1h_1)^{\frac12}\nabla\cdot\boldsymbol{u}_h^1+(\omega_2\lambda_2h_2)^{\frac12}\Big(\frac{\lambda_2\mu_1}{\mu_2\lambda_1}\Big)^{\frac12}\nabla\cdot\boldsymbol{u}_h^2,\gamma^{\frac12}\Big(\frac{\lambda_1}{\mu_1}\Big)^{\frac12}\boldsymbol{v}_j^1\cdot\boldsymbol{n}\rangle_{F_j^1},\\ &\lesssim((\omega_1\lambda_1h_1)^{\frac12}\|\nabla\cdot\boldsymbol{u}_h^1\|_{0,F_j^1}+(\omega_2\lambda_2h_2)^{\frac12}\Big(\frac{\lambda_2\mu_1}{\mu_2\lambda_1}\Big)^{\frac12}\|\nabla\cdot\boldsymbol{u}_h^2\|_{0,F_j^1})\gamma^{\frac12}\Big(\frac{\lambda_1}{\mu_1}\Big)^{\frac12}h_1^{\frac12}\|\nabla\boldsymbol{v}_j^1\|_{0,P_j^1},\end{aligned}$$

taking the sum over the whole interface and using (4.17) we have

$$\begin{aligned}&\sum_{j=1}^{N_p^1}\langle\{\lambda\nabla\cdot\boldsymbol{u}_h\},\boldsymbol{v}_j^1\cdot\boldsymbol{n}\rangle_{F_j^1}\\ &\le\epsilon\omega_1\lambda_1h_1\sum_{j=1}^{N_p^1}\|\nabla\cdot\boldsymbol{u}_h^1\|_{0,F_j^1}^2+\epsilon\omega_2\lambda_2h_2\frac{\lambda_2\mu_1}{\mu_2\lambda_1}\sum_{j=1}^{N_p^2}\|\nabla\cdot\boldsymbol{u}_h^2\|_{0,F_j^2}^2\\ &\quad+\frac{C\alpha_1^2}{2\epsilon}\gamma\frac{\lambda_1}{\mu_1}\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2+\frac{C\alpha_2^2}{2\epsilon}\gamma\frac{\lambda_1}{\mu_1}\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2\\ &\le\epsilon\omega_1\lambda_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2+\epsilon\omega_2\lambda_2\frac{\lambda_2\mu_1}{\mu_2\lambda_1}\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2\\ &\quad+\frac{C\alpha_1^2}{2\epsilon}\gamma\frac{\lambda_1}{\mu_1}\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2+\frac{C\alpha_2^2}{2\epsilon}\gamma\frac{\lambda_1}{\mu_1}\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2.\end{aligned}$$

Considering Lemmas 4.3.1, 4.3.2 all the terms are bounded, collecting the bounds we get

$$\begin{aligned}
&A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)\\
&\quad \geq \sum_{i=1}^{2} \left[2\mu_i \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^i)\|_{0,\Omega_i}^2 + \lambda_i \|\nabla \cdot \boldsymbol{u}_h^i\|_{0,\Omega_i}^2\right]\\
&\qquad - \sum_{j=1}^{N_p^1} \left[\epsilon\mu_1 \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^1)\|_{0,P_j^1}^2 + \epsilon\lambda_1 \|\nabla \cdot \boldsymbol{u}_h^1\|_{0,P_j^1}^2\right]\\
&\qquad - \epsilon\omega_1(4\mu_1 + 3\lambda_1) \sum_{j=1}^{N_p^1} \|\nabla \boldsymbol{u}_h^1\|_{0,P_j^1}^2\\
&\qquad - \epsilon\omega_2\Big(4\mu_2 + \lambda_2\Big(2\frac{\lambda_1\mu_2}{\mu_1\lambda_2} + \frac{\lambda_2\mu_1}{\lambda_1\mu_2}\Big)\Big) \sum_{j=1}^{N_p^2} \|\nabla \boldsymbol{u}_h^2\|_{0,P_j^2}^2\\
&\qquad + \alpha_1\Big(1 - \frac{C\alpha_1}{\epsilon}\Big(\frac{5}{2} + \frac{3\lambda_1}{2\mu_1} + \Big(1 + \frac{\lambda_1}{2\mu_1}\Big)\frac{h_2\mu_1}{h_1\mu_2}\Big)\Big)\gamma \sum_{j=1}^{N_p^1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2\\
&\qquad + \alpha_2\Big(2 + \frac{\lambda_1}{\mu_1} - \frac{C\alpha_2}{\epsilon}\Big(\frac{7}{2} + \frac{3\lambda_1}{2\mu_1} + \Big(1 + \frac{\lambda_1}{2\mu_1}\Big)\frac{h_2\mu_1}{h_1\mu_2}\Big)\Big)\gamma \sum_{j=1}^{N_p^1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2.
\end{aligned}$$

Using the Korn inequality (4.13) we get

$$\begin{aligned}
&A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)\\
&\quad \geq (2C_K\mu_1 - \epsilon(\mu_1 + \omega_1(4\mu_1 + 3\lambda_1)))\|\nabla \boldsymbol{u}_h^1\|_{0,\Omega_1}^2 + \lambda_1(1-\epsilon)\|\nabla \cdot \boldsymbol{u}_h^1\|_{0,\Omega_1}^2\\
&\qquad + \Big(2C_K\mu_2 - \epsilon\omega_2\Big(4\mu_2 + \lambda_2\Big(2\frac{\lambda_1\mu_2}{\mu_1\lambda_2} + \frac{\lambda_2\mu_1}{\lambda_1\mu_2}\Big)\Big)\Big)\|\nabla \boldsymbol{u}_h^2\|_{0,\Omega_2}^2 + \lambda_2\|\nabla \cdot \boldsymbol{u}_h^2\|_{0,\Omega_2}^2\\
&\qquad + \alpha_1\Big(1 - \frac{C\alpha_1}{\epsilon}\Big(\frac{5}{2} + \frac{3\lambda_1}{2\mu_1} + \Big(1 + \frac{\lambda_1}{2\mu_1}\Big)\frac{h_2\mu_1}{h_1\mu_2}\Big)\Big)\gamma \sum_{j=1}^{N_p^1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2\\
&\qquad + \alpha_2\Big(2 + \frac{\lambda_1}{\mu_1} - \frac{C\alpha_2}{\epsilon}\Big(\frac{7}{2} + \frac{3\lambda_1}{2\mu_1} + \Big(1 + \frac{\lambda_1}{2\mu_1}\Big)\frac{h_2\mu_1}{h_1\mu_2}\Big)\Big)\gamma \sum_{j=1}^{N_p^1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2.
\end{aligned}$$

Let us define the constants

$$\begin{aligned}
C_a &= 2C_K\mu_1 - \epsilon(\mu_1 + \omega_1(4\mu_1 + 3\lambda_1)),\\
C_b &= 2C_K\mu_2 - \epsilon\omega_2\Big(4\mu_2 + \lambda_2\Big(2\frac{\lambda_1\mu_2}{\mu_1\lambda_2} + \frac{\lambda_2\mu_1}{\lambda_1\mu_2}\Big)\Big),\\
C_c &= \alpha_1\Big(1 - \frac{C\alpha_1}{\epsilon}\Big(\frac{5}{2} + \frac{3\lambda_1}{2\mu_1} + \Big(1 + \frac{\lambda_1}{2\mu_1}\Big)\frac{h_2\mu_1}{h_1\mu_2}\Big)\Big),\\
C_d &= \alpha_2\Big(2 + \frac{\lambda_1}{\mu_1} - \frac{C\alpha_2}{\epsilon}\Big(\frac{7}{2} + \frac{3\lambda_1}{2\mu_1} + \Big(1 + \frac{\lambda_1}{2\mu_1}\Big)\frac{h_2\mu_1}{h_1\mu_2}\Big)\Big),
\end{aligned}$$

using Lemma 4.2.1 we get

$$\begin{aligned}A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) \geq (C_a - C(C_c+C_d)\omega_1\mu_1)\|\nabla \boldsymbol{u}_h^1\|_{0,\Omega_1}^2 + \lambda_1(1-\epsilon)\|\nabla\cdot \boldsymbol{u}_h^1\|_{0,\Omega_1}^2\\ + (C_b - C(C_c+C_d)\omega_2\mu_2)\|\nabla \boldsymbol{u}_h^2\|_{0,\Omega_2}^2 + \lambda_2\|\nabla\cdot \boldsymbol{u}_h^2\|_{0,\Omega_2}^2\\ + C_c\frac{\gamma}{2}\|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{\tau}\|_{0,\Gamma}^2 + C_d\frac{\gamma}{2}\|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\|_{0,\Gamma}^2.\end{aligned}$$

The constants $C_a$ and $C_b$ are positive by choosing $\epsilon$ such that

$$\epsilon < \min\Big(\frac{C_K\mu_1}{5\mu_1+3\lambda_1}, \frac{C_K\mu_2}{4\mu_2+\lambda_2(\frac{\lambda_1\mu_2}{\lambda_2\mu_1}+\frac{\lambda_2\mu_1}{\lambda_1\mu_2})}\Big).$$

The constants $C_c$ and $C_d$ are positive for

$$\frac{2\mu_1\epsilon}{C(7\mu_1+4\lambda_1)} > \alpha_1, \qquad \frac{(4\mu_1+2\lambda_1)\epsilon}{C(9\mu_1+4\lambda_1)} > \alpha_2.$$

The terms $(C_a - C(C_c+C_d)\omega_1\mu_1)$ and $(C_b - C(C_c+C_d)\omega_2\mu_2)$ are positive respectively for

$$\min\Big(\frac{C_K\mu_1}{2C(2\mu_1+\lambda_1)}, \frac{C_K}{6C}\Big) > \alpha_1, \alpha_2.$$

We note that the method is not robust for $\lambda_1$ or $\lambda_2$ too large, this corresponds to the locking phenomena. Observe that if $\mu_1 \gtrsim \lambda_1$ and $\mu_2 \gtrsim \lambda_2$, the method is robust and $\beta_0 = \mathcal{O}(1)$. □

**Theorem 4.3.1.** *There exists a positive constant $\beta$ such that for all functions $\boldsymbol{u}_h \in W_h^k$ the following inequality holds*

$$\beta|||\boldsymbol{u}_h||| \leq \sup_{\boldsymbol{v}_h\in W_h^k} \frac{A_h(\boldsymbol{u}_h,\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||}.$$

*Proof.* Considering the definition of the test function used in the previous Lemma we have

$$|||\boldsymbol{v}_h|||^2 \lesssim |||\boldsymbol{u}_h|||^2 + \sum_{j=1}^{N_p^1} |||\boldsymbol{v}_j^1|||^2,$$

with

$$|||\boldsymbol{v}_j^1|||^2 = \mu_1\|\nabla\boldsymbol{v}_j^1\|_{0,P_j^1}^2 + \lambda_1\|\nabla\cdot\boldsymbol{v}_j^1\|_{0,P_j^1}^2 + \gamma\|\boldsymbol{v}_j^1\|_{0,F_j^1}^2.$$

Using (4.17) and $\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\|_{0,F_j^1} \lesssim \|[\![\boldsymbol{u}_h]\!]\|_{0,F_j^1}$ we obtain

$$\sum_{j=1}^{N_p^1}\mu_1\|\nabla\boldsymbol{v}_j^1\|_{0,P_j^1}^2 \lesssim \Big(1+\frac{h_2\mu_1}{h_1\mu_2}\Big)|||\boldsymbol{u}_h|||^2.$$

Also, using this result we deduce

$$\sum_{j=1}^{N_p^1} \lambda_1 \|\nabla \cdot \boldsymbol{v}_j^1\|_{0,P_j^1}^2 \lesssim \sum_{j=1}^{N_p^1} \lambda_1 \|\nabla \boldsymbol{v}_j^1\|_{0,P_j^1}^2 \lesssim \Big(\frac{\lambda_1}{\mu_1}\Big)\Big(1+\frac{h_2\mu_1}{h_1\mu_2}\Big)|||\boldsymbol{u}_h|||^2.$$

Then using the trace inequality, and inequality (2.15) we obtain

$$\sum_{j=1}^{N_p^1} \gamma \|\boldsymbol{v}_j^1\|_{0,F_j^1}^2 \lesssim \omega_1 \sum_{j=1}^{N_p^1} \mu_1 \|\nabla \boldsymbol{v}_j^1\|_{0,P_j^1}^2 \lesssim |||\boldsymbol{u}_h|||^2,$$

then we get $|||\boldsymbol{v}_h||| \lesssim |||\boldsymbol{u}_h|||$. We conclude using the Lemma 4.3.3. Also, $\beta = \mathcal{O}(1)$ for $\mu_1 \gtrsim \lambda_1$ and $\mu_2 \gtrsim \lambda_2$. □

### 4.3.3 A priori error estimate

**Lemma 4.3.4.** *If $\boldsymbol{u} \in [H^2_\partial(\Omega_1)]^2 \times [H^2_\partial(\Omega_2)]^2$ is the solution of* (4.12) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (4.15) *the following property holds*

$$A_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in W_h^k.$$

*Proof.* Considering (4.14) we have $\forall \boldsymbol{v}_h \in W_h^k$

$$\sum_{i=1}^{2} \big((2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}_h^i))_{\Omega_i} + (\lambda_i \nabla\cdot \boldsymbol{u}^i, \nabla\cdot \boldsymbol{v}_h^i)_{\Omega_i}\big) - \langle\{\boldsymbol{\sigma}(\boldsymbol{u})\cdot \boldsymbol{n}\}, [\![\boldsymbol{v}_h]\!]\rangle_\Gamma = \sum_{i=1}^{2}(\boldsymbol{f}, \boldsymbol{v}_h^i)_{\Omega_i},$$

Adding the consistent antisymmetric Nitsche term $\langle\{\boldsymbol{\sigma}(\boldsymbol{v}_h)\cdot \boldsymbol{n}\}, [\![\boldsymbol{u}]\!]\rangle_\Gamma$ on the left hand side we get

$$A_h(\boldsymbol{u}, \boldsymbol{v}_h) = \sum_{i=1}^{2}(\boldsymbol{f}, \boldsymbol{v}_h^i)_{\Omega_i} = L_h(\boldsymbol{v}_h).$$

Then we obtain $A_h(\boldsymbol{u}, \boldsymbol{v}_h) = L_h(\boldsymbol{v}_h) = A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$, $\forall \boldsymbol{v}_h \in W_h^k$. □

**Lemma 4.3.5.** *Let $\boldsymbol{w} \in [H^2_\partial(\Omega_1)]^2 \times [H^2_\partial(\Omega_2)]^2 + W_h^k$ and $\boldsymbol{v}_h \in W_h^k$, there exists a positive constant $M$ such that*

$$A_h(\boldsymbol{w}, \boldsymbol{v}_h) \leq M\|\boldsymbol{w}\|_* |||\boldsymbol{v}_h|||.$$

*Proof.* Using Cauchy-Schwarz inequality we show

$$\sum_{i=1}^{2} \big[(2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{w}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}_h^i))_{\Omega_i} + (\lambda_i \nabla\cdot \boldsymbol{w}^i, \nabla\cdot \boldsymbol{v}_h^i)_{\Omega_i}\big] \lesssim \|\boldsymbol{w}\|_* |||\boldsymbol{v}_h|||.$$

and

$$\langle \{2\mu\boldsymbol{\varepsilon}(\boldsymbol{w})\cdot\boldsymbol{n}\}, [\![\boldsymbol{v}_h]\!]\rangle_\Gamma \leq 2((\omega_1\mu_1 h_1)^{\frac{1}{2}}\|\nabla\boldsymbol{w}^1\cdot\boldsymbol{n}\|_{0,\Gamma} + (\omega_2\mu_2 h_2)^{\frac{1}{2}}\|\nabla\boldsymbol{w}^2\cdot\boldsymbol{n}\|_{0,\Gamma})\gamma^{\frac{1}{2}}\|[\![\boldsymbol{v}_h]\!]\|_{0,\Gamma},$$

$$\langle \{\lambda\nabla\cdot\boldsymbol{w}\}, [\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\rangle_\Gamma \leq (\lambda_1^{\frac{1}{2}}\mu_1^{-\frac{1}{2}}(\omega_1\lambda_1 h_1)^{\frac{1}{2}}\|\nabla\cdot\boldsymbol{w}^1\|_{0,\Gamma} + \lambda_2^{\frac{1}{2}}\mu_2^{-\frac{1}{2}}(\omega_2\lambda_2 h_2)^{\frac{1}{2}}\|\nabla\cdot\boldsymbol{w}^2\|_{0,\Gamma})\gamma^{\frac{1}{2}}\|[\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\|_{0,\Gamma}.$$

Using the trace inequality we also have

$$\begin{aligned}\langle \{2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\cdot\boldsymbol{n}\}, [\![\boldsymbol{w}]\!]\rangle_\Gamma &= 2\langle(\omega_1\mu_1 h_1)^{\frac{1}{2}}\boldsymbol{\varepsilon}(\boldsymbol{v}_h^1)\cdot\boldsymbol{n} + (\omega_2\mu_2 h_2)^{\frac{1}{2}}\boldsymbol{\varepsilon}(\boldsymbol{v}_h^2)\cdot\boldsymbol{n}, \gamma^{\frac{1}{2}}[\![\boldsymbol{w}]\!]\rangle_\Gamma \\ &\leq 2((\omega_1\mu_1)^{\frac{1}{2}}\|\nabla\boldsymbol{v}_h^1\|_{0,\Omega_1} + (\omega_2\mu_2)^{\frac{1}{2}}\|\nabla\boldsymbol{v}_h^2\|_{0,\Omega_2})\gamma^{\frac{1}{2}}\|[\![\boldsymbol{w}]\!]\|_{0,\Gamma},\end{aligned}$$

$$\begin{aligned}\langle \{\lambda\nabla\cdot\boldsymbol{v}_h\}, [\![\boldsymbol{w}\cdot\boldsymbol{n}]\!]\rangle_\Gamma &= \langle\lambda_1^{\frac{1}{2}}\mu_1^{-\frac{1}{2}}(\omega_1\lambda_1 h_1)^{\frac{1}{2}}\nabla\cdot\boldsymbol{v}_h^1 + \lambda_2^{\frac{1}{2}}\mu_2^{-\frac{1}{2}}(\omega_2\lambda_2 h_2)^{\frac{1}{2}}\nabla\cdot\boldsymbol{v}_h^2, \gamma^{\frac{1}{2}}[\![\boldsymbol{w}\cdot\boldsymbol{n}]\!]\rangle_\Gamma \\ &\leq (\lambda_1^{\frac{1}{2}}\mu_1^{-\frac{1}{2}}(\omega_1\lambda_1)^{\frac{1}{2}}\|\nabla\boldsymbol{v}_h^1\|_{0,\Omega_1} \\ &\quad + \lambda_2^{\frac{1}{2}}\mu_2^{-\frac{1}{2}}(\omega_2\lambda_2)^{\frac{1}{2}}\|\nabla\boldsymbol{v}_h^2\|_{0,\Omega_2})\gamma^{\frac{1}{2}}\|[\![\boldsymbol{w}\cdot\boldsymbol{n}]\!]\|_{0,\Gamma}.\end{aligned}$$

The claim follows and $M = \mathcal{O}(1)$ if $\mu_1 \gtrsim \lambda_1$ and $\mu_2 \gtrsim \lambda_2$. □

**Theorem 4.3.2.** *If $\boldsymbol{u} \in [H_\partial^{k+1}(\Omega_1)]^2 \times [H_\partial^{k+1}(\Omega_2)]^2$ is the solution of (4.12) and $\boldsymbol{u}_h \in W_h^k$ the solution of (4.15), then there holds*

$$|\!|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!|\!| \leq C_\beta \inf_{\boldsymbol{w}_h\in W_h^k}\|\boldsymbol{u}-\boldsymbol{w}_h\|_*,$$

*where $C_\beta$ is a positive constant that depends on the mesh geometry.*

*Proof.* Same proof as Theorem 4.2.2 using the Galerkin orthogonality of Lemma 4.3.4, the Theorem 4.3.1 and the Lemma 4.3.5, $C_\beta = \mathcal{O}(M\beta^{-1})$. □

**Corollary 4.3.1.** *If $\boldsymbol{u} \in [H_\partial^{k+1}(\Omega_1)]^2 \times [H_\partial^{k+1}(\Omega_2)]^2$ is the solution of (4.12) and $\boldsymbol{u}_h \in W_h^k$ the solution of (4.15), then there holds*

$$|\!|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!|\!| \leq C_{\mu\lambda}h^k|\boldsymbol{u}|_{k+1,\Omega},$$

*where $C_{\mu\lambda}$ is a positive constant that depends on $\mu$, $\lambda$ and the mesh geometry.*

*Proof.* Combining arguments from the proofs of Corollaries 4.2.1 and 2.2.1 we obtain

$$\|\boldsymbol{u}-i_{\text{SZ}}^k\boldsymbol{u}\|_* \lesssim (\mu_1^{\frac{1}{2}}+\lambda_1^{\frac{1}{2}})h_1^k|\boldsymbol{u}^1|_{k+1,\Omega_1} + (\mu_2^{\frac{1}{2}}+\lambda_2^{\frac{1}{2}})h_2^k|\boldsymbol{u}^2|_{k+1,\Omega_2}. \tag{4.18}$$

Applying the Theorem 4.3.2 with $\boldsymbol{w}_h = i_{\text{SZ}}^k\boldsymbol{u}$ the result follows and $C_{\mu\lambda} = \mathcal{O}(C_\beta(\mu_1^{\frac{1}{2}} + \lambda_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}}))$. □

**Proposition 4.3.1.** *Let* $\boldsymbol{u} \in [H_\partial^{k+1}(\Omega_1)]^2 \times [H_\partial^{k+1}(\Omega_2)]^2$ *be the solution of* (4.12) *and* $\boldsymbol{u}_h$ *the solution of* (4.15), *then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq C'_{\mu\lambda} h^{p+\frac{1}{2}} |\boldsymbol{u}|_{k+1,\Omega},$$

*where* $C'_{\mu\lambda}$ *is a positive constant that depends on* $\mu$, $\lambda$ *and the mesh geometry.*

*Proof.* We follows the same arguments as the proof of Proposition 4.2.1. Let $\boldsymbol{z}$ satisfy the adjoint problem

$$\begin{aligned}
-2\mu_i \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{z}^i) - \lambda_i \nabla(\nabla \cdot \boldsymbol{z}^i) = \boldsymbol{u}^i - \boldsymbol{u}_h^i \qquad & \text{in } \Omega_i\ , \quad i = 1, 2, \\
\boldsymbol{z}^i = 0 \qquad & \text{on } \partial\Omega \cap \Omega_i\ , \quad i = 1, 2, \\
[\![\boldsymbol{z}]\!] = 0 \qquad & \text{on } \Gamma, \\
[\![\boldsymbol{\sigma}(\boldsymbol{z}) \cdot \boldsymbol{n}]\!] = 0 \qquad & \text{on } \Gamma,
\end{aligned}$$

We assume the following elliptic regularity [85] for this problem

$$\mu_1 \|\boldsymbol{z}^1\|_{2,\Omega_1} + \mu_2 \|\boldsymbol{z}^2\|_{2,\Omega_2} \lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}. \tag{4.19}$$

By integration by parts we obtain

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega_i}^2 &= (\boldsymbol{u}^i - \boldsymbol{u}_h^i, -2\mu \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{z}^i) - \lambda_i \nabla(\nabla \cdot \boldsymbol{z}^i))_{\Omega_i} \\
&= (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}^i - \boldsymbol{u}_h^i), \boldsymbol{\varepsilon}(\boldsymbol{z}^i))_{\Omega_i} + (\lambda_i \nabla \cdot (\boldsymbol{u}^i - \boldsymbol{u}_h^i), \nabla \cdot \boldsymbol{z}^i)_{\Omega_i} \\
&\quad - \langle 2\mu_i (\boldsymbol{u}^i - \boldsymbol{u}_h^i), \boldsymbol{\varepsilon}(\boldsymbol{z}_i) \cdot \boldsymbol{n}_i \rangle_\Gamma - \langle \lambda_i (\boldsymbol{u}^i - \boldsymbol{u}_h^i) \cdot \boldsymbol{n}_i, \nabla \cdot \boldsymbol{z}^i \rangle_\Gamma.
\end{aligned}$$

Then the $L^2$-error can be written as

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}^2 &= \|\boldsymbol{u}^1 - \boldsymbol{u}_h^1\|_{0,\Omega_1}^2 + \|\boldsymbol{u}^2 - \boldsymbol{u}_h^2\|_{0,\Omega_2}^2 \\
&= A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}) - 2\langle \{2\mu \boldsymbol{\varepsilon}(\boldsymbol{z}) \cdot \boldsymbol{n}\}, [\![\boldsymbol{u} - \boldsymbol{u}_h]\!] \rangle_\Gamma \\
&\qquad - 2\langle \{\lambda \nabla \cdot \boldsymbol{z}\}, [\![(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}]\!] \rangle_\Gamma.
\end{aligned}$$

By Lemma 4.3.4 and similar arguments as in the proof of Lemma 4.3.5 we deduce that

$$
\begin{aligned}
A_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z}) &= A_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z}-i^1_{\mathsf{SZ}}\boldsymbol{z}) \\
&= \sum_{i=1}^{2}\left[(2\mu_i\boldsymbol{\varepsilon}(\boldsymbol{u}^i-\boldsymbol{u}^i_h),\boldsymbol{\varepsilon}(\boldsymbol{z}^i-i^1_{\mathsf{SZ}}\boldsymbol{z}^i))_{\Omega_i}+(\lambda_i\nabla\cdot(\boldsymbol{u}^i-\boldsymbol{u}^i_h),\nabla\cdot(\boldsymbol{z}^i-i^1_{\mathsf{SZ}}\boldsymbol{z}^i))_{\Omega_i}\right] \\
&\quad -\langle 2\{\mu\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{n}\},[\![\boldsymbol{z}-i^1_{\mathsf{SZ}}\boldsymbol{z}]\!]\rangle_\Gamma-\langle\{\lambda\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\},[\![\boldsymbol{z}-i^1_{\mathsf{SZ}}\boldsymbol{z}]\!]\cdot\boldsymbol{n}\rangle_\Gamma \\
&\quad +\langle 2\{\mu\boldsymbol{\varepsilon}(\boldsymbol{z}-i^1_{\mathsf{SZ}}\boldsymbol{z})\cdot\boldsymbol{n}\},[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\rangle_\Gamma+\langle\{\lambda\nabla\cdot(\boldsymbol{z}-i^1_{\mathsf{SZ}}\boldsymbol{z})\},[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\rangle_\Gamma \\
&\le \sum_{i=1}^{2}(\mu_i^{\frac12}+\lambda_i^{\frac12})\|\nabla(\boldsymbol{u}^i-\boldsymbol{u}^i_h)\|_{0,\Omega_i}\|\nabla(\boldsymbol{z}^i-i^1_{\mathsf{SZ}}\boldsymbol{z}^i)\|_{0,\Omega_i} \\
&\quad +\sum_{i=1}^{2}\big((\omega_i\mu_ih_i)^{\frac12}\|\nabla(\boldsymbol{u}^i-\boldsymbol{u}^i_h)\cdot\boldsymbol{n}\|_{0,\Gamma}\big)\gamma^{\frac12}\|[\![\boldsymbol{z}-i^1_{\mathsf{SZ}}\boldsymbol{z}]\!]\|_{0,\Gamma} \\
&\quad +\sum_{i=1}^{2}\big(\lambda_i^{\frac12}\mu_i^{-\frac12}(\omega_i\lambda_ih_i)^{\frac12}\|\nabla\cdot(\boldsymbol{u}^i-\boldsymbol{u}^i_h)\|_{0,\Gamma}\big)\gamma^{\frac12}\|[\![\boldsymbol{z}-i^1_{\mathsf{SZ}}\boldsymbol{z}]\!]\|_{0,\Gamma} \\
&\quad +\sum_{i=1}^{2}\big((\omega_i\mu_ih_i)^{\frac12}\|\nabla(\boldsymbol{z}^i-i^1_{\mathsf{SZ}}\boldsymbol{z}^i)\cdot\boldsymbol{n}\|_{0,\Gamma}\big)\gamma^{\frac12}\|[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\|_{0,\Gamma} \\
&\quad +\sum_{i=1}^{2}\big(\lambda_i^{\frac12}\mu_i^{-\frac12}(\omega_i\lambda_ih_i)^{\frac12}\|\nabla\cdot(\boldsymbol{z}^i-i^1_{\mathsf{SZ}}\boldsymbol{z}^i)\|_{0,\Gamma}\big)\gamma^{\frac12}\|[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\|_{0,\Gamma},
\end{aligned}
$$

and using the approximation property of the Scott-Zhang interpolant

$$
\begin{aligned}
h_i^{\frac12}\|\nabla(\boldsymbol{u}^i-\boldsymbol{u}^i_h)\cdot\boldsymbol{n}\|_{0,\Gamma} &\lesssim h_i^k|\boldsymbol{u}^i|_{k+1,\Omega_i}+\|\nabla(\boldsymbol{u}^i_h-i^k_{\mathsf{SZ}}\boldsymbol{u}^i)\|_{0,\Omega_i}, \\
h_i^{\frac12}\|\nabla\cdot(\boldsymbol{u}^i-\boldsymbol{u}^i_h)\|_{0,\Gamma} &\lesssim h_i^k|\boldsymbol{u}^i|_{k+1,\Omega_i}+\|\nabla(\boldsymbol{u}^i_h-i^k_{\mathsf{SZ}}\boldsymbol{u}^i)\|_{0,\Omega_i}.
\end{aligned}
$$

then using Corollary 4.3.1 we have

$$
\begin{aligned}
&A_h(\boldsymbol{u}-\boldsymbol{u}_h,z) \\
&\quad\lesssim \|\boldsymbol{z}-i^1_{\mathsf{SZ}}\boldsymbol{z}\|_*C_{\mu\lambda}(1+\lambda_1^{\frac12}\mu_1^{-\frac12}+\lambda_2^{\frac12}\mu_2^{-\frac12})h^k|\boldsymbol{u}|_{k+1,\Omega} \\
&\quad\lesssim ((\mu_1^{\frac12}+\lambda_1^{\frac12})h_1|\boldsymbol{z}^1|_{2,\Omega_1}+(\mu_2^{\frac12}+\lambda_2^{\frac12})h_2|\boldsymbol{z}^2|_{2,\Omega_2})C_{\mu\lambda}(1+\lambda_1^{\frac12}\mu_1^{-\frac12}+\lambda_2^{\frac12}\mu_2^{-\frac12})h^k|\boldsymbol{u}|_{k+1,\Omega}.
\end{aligned}
$$

The global trace inequalities $\|\boldsymbol{\varepsilon}(\boldsymbol{z}^i)\cdot\boldsymbol{n}\|_{0,\Gamma}\lesssim\|\boldsymbol{z}^i\|_{2,\Omega_i}$ and $\|\nabla\cdot\boldsymbol{z}^i\|_{0,\Gamma}\lesssim\|\boldsymbol{z}^i\|_{2,\Omega_i}$, lead to

$$
\begin{aligned}
&|\langle 2\{\mu\boldsymbol{\varepsilon}(\boldsymbol{z})\cdot\boldsymbol{n}\},[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\rangle_\Gamma|+|\langle\{\lambda\nabla\cdot\boldsymbol{z}\},[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\rangle_\Gamma| \\
&\quad\lesssim ((\omega_1h_1)^{\frac12}(\mu_1^{\frac12}+\lambda_1^{\frac12})\|\boldsymbol{z}^1\|_{2,\Omega_1} \\
&\qquad\qquad +(\omega_2h_2)^{\frac12}(\mu_2^{\frac12}+\lambda_2^{\frac12})\|\boldsymbol{z}^2\|_{2,\Omega_2})(1+\lambda_1^{\frac12}\mu_1^{-\frac12}+\lambda_2^{\frac12}\mu_2^{-\frac12})|\!|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!|\!|.
\end{aligned}
$$

Collecting the estimates and applying Corollary 4.3.1 the proof follows by using the regularity estimate (4.19). Note that $C'_{\mu\lambda}=\mathcal{O}(1)$ if $\mu_1\gtrsim\lambda_1$ and $\mu_2\gtrsim\lambda_2$. □

## 4.4 Incompressible elasticity

The incompressible elasticity problem with discontinuous parameters considered is expressed as

$$\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}^i, p^i) &= \boldsymbol{f} && \text{in } \Omega_i \ , \quad i = 1, 2, \\
-\nabla \cdot \boldsymbol{u}^i &= 0 && \text{in } \Omega_i \ , \quad i = 1, 2, \\
\boldsymbol{u}^i &= 0 && \text{on } \partial\Omega \cap \Omega_i \ , \quad i = 1, 2, \\
[\![\boldsymbol{u}]\!] &= 0 && \text{on } \Gamma, \\
[\![\boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}]\!] &= 0 && \text{on } \Gamma,
\end{aligned} \tag{4.20}$$

with $\boldsymbol{f} \in [L^2(\Omega)]^2$ and $\int_{\Omega_i} p^i \ \mathrm{d}x = 0$ for $i = 1, 2$. The stress tensor is expressed as

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbb{I}_{2\times 2}.$$

In a subdomain $\Omega_i$ the viscosity is denoted as $\mu_i$. We note that the inequality (4.13) still holds for this problem.

### 4.4.1 Finite element formulation

Let the pressure space $Q_i = \{q \in L^2(\Omega_i), \ \int_{\Omega_i} q \ \mathrm{d}x = 0\}$. For each domain $\Omega_i$ we obtain using integration by parts

$$(2\mu_i\boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}^i))_{\Omega_i} - (p^i, \nabla \cdot \boldsymbol{v}^i)_{\Omega_i} + (\nabla \cdot \boldsymbol{u}^i, q^i)_{\Omega_i} - \langle \boldsymbol{\sigma}(\boldsymbol{u}^i, p^i) \cdot \boldsymbol{n}_i, \boldsymbol{v}^i \rangle_\Gamma = (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i} \quad \forall (\boldsymbol{v}^i \times q^i) \in W_i \times Q_i.$$

Summing the interface terms we obtain

$$\sum_{i=1}^{2} \langle \boldsymbol{\sigma}(\boldsymbol{u}^i, p^i) \cdot \boldsymbol{n}_i, \boldsymbol{v}^i \rangle_\Gamma = \int_\Gamma [\![(\boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n})\boldsymbol{v}]\!] \mathrm{d}s = \langle \{\boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}\}, [\![\boldsymbol{v}]\!] \rangle_\Gamma + \langle [\![\boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}]\!], \langle \boldsymbol{v} \rangle \rangle_\Gamma,$$

knowing that on the interface $[\![\boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}]\!] = 0$ from (4.20), it leads to

$$\sum_{i=1}^{2} \left[ (2\mu_i\boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}^i))_{\Omega_i} - (p^i, \nabla \cdot \boldsymbol{v}^i)_{\Omega_i} + (\nabla \cdot \boldsymbol{u}^i, q^i)_{\Omega_i} \right] - \langle \{\boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}\}, [\![\boldsymbol{v}]\!] \rangle_\Gamma = \sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i}. \tag{4.21}$$

In order to obtain the control of the terms related to the pressure in the analysis we choose a master/slave configuration, then the weights are chosen such that

$$\omega_1 = 1, \qquad \omega_2 = 0. \tag{4.22}$$

Note that we still consider that $\mu_2 h_1 \geq \mu_1 h_2$. Inserting these new weights in (4.21) we obtain

$$\sum_{i=1}^{2} \left[ (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}^i))_{\Omega_i} - (p^i, \nabla\cdot\boldsymbol{v}^i)_{\Omega_i} + (\nabla\cdot\boldsymbol{u}^i, q^i)_{\Omega_i} \right] - \langle \boldsymbol{\sigma}(\boldsymbol{u}^1, p^1)\cdot\boldsymbol{n}, [\![\boldsymbol{v}]\!] \rangle_\Gamma = \sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i}. \tag{4.23}$$

Let the discrete pressure space be $Q_h^k = Q_1^k \times Q_2^k$ such that

$$Q_i^k = \{ q_h \in Q_i : q_h|_K \in \mathbb{P}_k(K) \;\; \forall K \in \mathcal{T}_h^i \}, \quad k \geq 1.$$

Adding the Nitsche term, the penalty-free finite element formulation is written as: find $\boldsymbol{u}_h \in W_h^k$ and $p_h \in Q_h^k$ such that

$$A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = L_h(\boldsymbol{v}_h, q_h) \qquad \forall \boldsymbol{v}_h \in W_h^k \times Q_h^k. \tag{4.24}$$

The linear forms $A_h$ and $L_h$ are expressed as

$$\begin{aligned} A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] &= a[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] - b(\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) + b(\boldsymbol{v}_h, \boldsymbol{u}_h, q_h) \\ &\quad + S_h(\boldsymbol{u}_h, p_h, q_h), \\ L_h(v_h) &= \sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i} + \Lambda_h(\boldsymbol{f}, q_h), \end{aligned}$$

with $a$, $b$, $S_h$ and $\Lambda_h$ such that

$$\begin{aligned} a[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] &= \sum_{i=1}^{2} \left[ (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}_h^i), \boldsymbol{\varepsilon}(\boldsymbol{v}_h^i))_{\Omega_i} - (p_h^i, \nabla\cdot\boldsymbol{v}_h^i)_{\Omega_i} + (\nabla\cdot\boldsymbol{u}_h^i, q_h^i)_{\Omega_i} \right], \\ b(\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) &= \langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1)\cdot\boldsymbol{n}, [\![\boldsymbol{v}_h]\!] \rangle_\Gamma - \langle p_h^1, [\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!] \rangle_\Gamma, \\ S_h(\boldsymbol{u}_h, p_h, q_h) &= \sum_{i=1}^{2} \frac{\gamma_p}{\mu_i} \sum_{K\in\mathcal{T}_h^i} \int_K h_i^2 (-2\mu_i \nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h^i) + \nabla p_h^i) \nabla q_h^i \,\mathrm{d}x, \\ \Lambda_h(\boldsymbol{f}, q_h) &= \sum_{i=1}^{2} \frac{\gamma_p}{\mu_i} \sum_{K\in\mathcal{T}_h^i} \int_K h_i^2 \boldsymbol{f} \nabla q_h^i \,\mathrm{d}x, \end{aligned}$$

the stabilisation ($S_h$ and $\Lambda_h$) allows us to work with equal order interpolation.

### 4.4.2 Inf-sup stability

Let the norm

$$|\!|\!|(\boldsymbol{w}, \varrho)|\!|\!|^2 = \sum_{i=1}^{2} (\mu_i \|\nabla \boldsymbol{w}^i\|_{0,\Omega_i}^2 + h_i^2 \mu_i^{-1} \|\nabla \varrho^i\|_{0,\Omega_i}^2) + \mu_1 h_1^{-1} \|[\![\boldsymbol{w}]\!]\|_{0,\Gamma}^2.$$

**Lemma 4.4.1.** *For $\boldsymbol{u}_h, \boldsymbol{v}_h \in W_h^k$ with $\boldsymbol{v}_h = \boldsymbol{u}_h + \sum_{j=1}^{N_p^1}(\boldsymbol{v}_j^1, 0)$, $\boldsymbol{v}_j^1$ defined by (4.16), and $q_h = p_h$, there exist a positive constant $\beta_0$ such that*

$$\beta_0 |||(\boldsymbol{u}_h, p_h)|||^2 \leq A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)].$$

*Proof.* Applying the definition of $\boldsymbol{v}_h$ we obtain

$$A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] + \sum_{j=1}^{N_p^1} A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_j^1, 0)],$$

with

$$\begin{aligned} A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_j^1, 0)] = {} & (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1))_{P_j^1} + (\nabla p_h^1, \boldsymbol{v}_j^1)_{P_j^1} \\ & - \langle 2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1) \cdot \boldsymbol{n}, \boldsymbol{v}_j^1 \rangle_{F_j^1} + \langle 2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1) \cdot \boldsymbol{n}, [\![\boldsymbol{u}_h]\!] \rangle_{F_j^1}. \end{aligned}$$

Similarly as in the proof of Lemma 2.3.1 we get

$$A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] \geq \sum_{i=1}^{2} \left[ 2(1-\epsilon')\mu_i \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^i)\|_{0,\Omega_i}^2 + \frac{\gamma_p}{\mu_i}\Big(1 - \frac{C\gamma_p}{4\epsilon'}\Big) h_i^2 \|\nabla p_h^i\|_{0,\Omega_i}^2 \right],$$

Using similar arguments as in the proofs of Lemmas 2.2.3 and 2.3.1 we get the following bounds

$$\begin{aligned} (\nabla p_h^1, \boldsymbol{v}_j^1)_{P_j^1} \leq {} & \frac{\epsilon}{\mu_1} h_1^2 \|\nabla p_h^1\|_{0,P_j^1}^2 + \frac{C\alpha_1^2}{2\epsilon}\frac{\mu_1}{h_1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2 \\ & + \frac{C\alpha_2^2}{2\epsilon}\frac{\mu_1}{h_1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2, \\ (2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1))_{0,P_j^1} \leq {} & \epsilon\mu_1 \|\nabla \boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \frac{C\alpha_1^2}{\epsilon}\frac{\mu_1}{h_1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2 \\ & + \frac{C\alpha_2^2}{\epsilon}\frac{\mu_1}{h_1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2, \\ \langle 2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1) \cdot \boldsymbol{n}, \boldsymbol{v}_j^1 \rangle_{F_j^1} \leq {} & \epsilon\mu_1 \|\nabla \boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \frac{C\alpha_1^2}{\epsilon}\frac{\mu_1}{h_1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2 \\ & + \frac{C\alpha_2^2}{\epsilon}\frac{\mu_1}{h_1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2. \end{aligned}$$

From Lemma 4.3.2 considering the weights (4.22) and the fact that $\mu_2 h_1 \geq \mu_1 h_2$

$$\begin{aligned} \langle 2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1) \cdot \boldsymbol{n}, [\![\boldsymbol{u}_h]\!] \rangle_{F_j^1} \geq {} & \alpha_2\Big(1 - \frac{3C\alpha_2}{4\epsilon}\Big)\frac{\mu_1}{h_1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2 \\ & + \alpha_1\Big(\frac{1}{2} - \frac{C\alpha_1}{4\epsilon}\Big)\frac{\mu_1}{h_1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2 - 3\epsilon\mu_1 \|\nabla \boldsymbol{u}_h^1\|_{0,P_j^1}^2. \end{aligned}$$

Collecting the bounds for each term

$$
\begin{aligned}
A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] \geq & \sum_{i=1}^{2} \left[2(1-\epsilon')\mu_i \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^i)\|_{0,\Omega_i}^2 + \frac{\gamma_p}{\mu_i}\Big(1 - \frac{C\gamma_p}{4\epsilon'}\Big) h_i^2 \|\nabla p_h^i\|_{0,\Omega_i}^2\right] \\
& - 5\epsilon\mu_1 \sum_{j=1}^{N_p^1} \|\nabla \boldsymbol{u}_h^1\|_{0,P_j^1}^2 - \epsilon\mu_1^{-1} h_1^2 \sum_{j=1}^{N_p^1} \|\nabla p_h^1\|_{0,P_j^1}^2 \\
& + \alpha_1\Big(\frac{1}{2} - \frac{11C\alpha_1}{4\epsilon}\Big) \sum_{j=1}^{N_p^1} \mu_1 h_1^{-1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2 \\
& + \alpha_2\Big(1 - \frac{13C\alpha_2}{4\epsilon}\Big) \sum_{j=1}^{N_p^1} \mu_1 h_1^{-1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2 .
\end{aligned}
$$

Using the Korn inequality (4.13) we obtain

$$
\begin{aligned}
& A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] \\
& \quad \geq (2C_K(1-\epsilon') - 5\epsilon)\mu_1 \|\nabla \boldsymbol{u}_h^1\|_{0,\Omega_1}^2 + \frac{\gamma_p}{\mu_1}\Big(1 - \frac{C\gamma_p}{4\epsilon'} - \frac{\epsilon}{\gamma_p}\Big) h_1^2 \|\nabla p_h^1\|_{0,\Omega_1}^2 \\
& \quad + 2C_K(1-\epsilon') \sum_{j=1}^{N_p^1} \mu_2 \|\nabla \boldsymbol{u}_h^2\|_{0,\Omega_2}^2 + \frac{\gamma_p}{\mu_2}\Big(1 - \frac{C\gamma_p}{4\epsilon'}\Big) \sum_{j=1}^{N_p^1} h_2^2 \|\nabla p_h^2\|_{0,\Omega_2}^2 \\
& \quad + \alpha_1\Big(\frac{1}{2} - \frac{11C\alpha_1}{4\epsilon}\Big) \sum_{j=1}^{N_p^1} \mu_1 h_1^{-1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{\tau}\|_{0,F_j^1}^2 \\
& \quad + \alpha_2\Big(1 - \frac{13C\alpha_2}{4\epsilon}\Big) \sum_{j=1}^{N_p^1} \mu_1 h_1^{-1} \|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n}\|_{0,F_j^1}^2 .
\end{aligned}
$$

Let us define the constants

$$
\begin{aligned}
C_a &= 2C_K(1-\epsilon') - 5\epsilon, \\
C_b &= \gamma_p\Big(1 - \frac{C\gamma_p}{4\epsilon'} - \frac{\epsilon}{\gamma_p}\Big), \\
C_c &= 2C_K(1-\epsilon'), \\
C_d &= \gamma_p\Big(1 - \frac{C\gamma_p}{4\epsilon'}\Big), \\
C_e &= \alpha_1\Big(\frac{1}{2} - \frac{11C\alpha_1}{4\epsilon}\Big), \\
C_f &= \alpha_2\Big(1 - \frac{13C\alpha_2}{4\epsilon}\Big),
\end{aligned}
$$

using Lemma 4.2.1 we get

$$\begin{aligned} A_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)] \geq{}& (C_a - C(C_e+C_f))\mu_1\|\nabla \boldsymbol{u}_h^1\|_{0,\Omega_1}^2 + C_b h_1^2\mu_1^{-1}\|\nabla p_h^1\|_{0,\Omega_1}^2 \\ &+ (C_c - \frac{\mu_1 h_2}{\mu_2 h_1}C(C_e+C_f))\mu_2\|\nabla \boldsymbol{u}_h^2\|_{0,\Omega_2}^2 + C_d h_2^2\mu_2^{-1}\|\nabla p_h^2\|_{0,\Omega_2}^2 \\ &+ \frac{C_e}{2}\sum_{j=1}^{N_p^1}\mu_1 h_1^{-1}\|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2 + \frac{C_f}{2}\sum_{j=1}^{N_p^1}\mu_1 h_1^{-1}\|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\|_{0,F_j^1}^2. \end{aligned}$$

By taking $\epsilon = \frac{\gamma_p^2}{4}$, $\epsilon' = \frac{1}{4}$ and $\gamma_p < \min(\frac{1}{C+1}, \sqrt{\frac{2C_K}{5}})$, the constants $C_a$, $C_b$, $C_c$ and $C_d$ are positive, $C_e$ and $C_f$ are positive for

$$\frac{\gamma_p^2}{22C} > \alpha_1, \qquad \frac{\gamma_p^2}{13C} > \alpha_2.$$

The terms $(C_a - C(C_e+C_f))$ and $(C_c - \frac{\mu_1 h_2}{\mu_2 h_1}C(C_e+C_f))$ are positive for

$$\frac{C_K}{C} > \alpha_1, \qquad \frac{C_K}{2C} > \alpha_2.$$

Note that $\beta_0 = \mathcal{O}(1)$. □

**Theorem 4.4.1.** *There exist a positive constant $\beta$ such that for all functions $(\boldsymbol{u}_h,p_h) \in W_h^k \times Q_h^k$, the following inequality holds*

$$\beta|\!|\!|(\boldsymbol{u}_h,p_h)|\!|\!| \leq \sup_{(\boldsymbol{v}_h,q_h)\in W_h^k\times Q_h^k} \frac{A_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)]}{|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|}.$$

*Proof.* Using the definitions of $\boldsymbol{v}_h$ and $q_h$ from the previous proof we have

$$|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|^2 \lesssim |\!|\!|(\boldsymbol{u}_h,p_h)|\!|\!|^2 + \sum_{j=1}^{N_p^1}|\!|\!|(\boldsymbol{v}_j^1,0)|\!|\!|^2,$$

with

$$|\!|\!|(\boldsymbol{v}_j^1,0)|\!|\!|^2 = \mu_1\|\nabla \boldsymbol{v}_j^1\|_{0,\Omega_1}^2 + \mu_1 h_1^{-1}\|\boldsymbol{v}_j^1\|_{0,\Gamma}^2.$$

Using the same arguments as in the proof of Lemma 4.3.1 we get

$$|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!| \lesssim |\!|\!|(\boldsymbol{u}_h,p_h)|\!|\!|.$$

We conclude by combining this result and Lemma 4.4.1, $\beta = \mathcal{O}(1)$. □

### 4.4.3 A priori error estimate

**Lemma 4.4.2.** *If* $(\boldsymbol{u}, p) \in ([H^2_\partial(\Omega_1)]^2 \times [H^2_\partial(\Omega_2)]^2) \times (H^1(\Omega_1) \times H^1(\Omega_2))$ *is the solution of* (4.20) *and* $(\boldsymbol{u}_h, p_h) \in W^k_h \times Q^k_h$ *the solution of* (4.24) *the following property holds*

$$A_h[(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}_h, q_h)] = 0, \qquad \forall (\boldsymbol{v}_h, q_h) \in W^k_h \times Q^k_h.$$

*Proof.* Considering (4.23), $\forall (\boldsymbol{v}_h, q_h) \in W^k_h \times Q^k_h$

$$\sum_{i=1}^{2} \left[ (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}^i_h))_{\Omega_i} - (p^i, \nabla \cdot \boldsymbol{v}^i_h)_{\Omega_i} + (\nabla \cdot \boldsymbol{u}^i, q^i_h)_{\Omega_i} \right]$$
$$- \langle \boldsymbol{\sigma}(\boldsymbol{u}^1, p^1) \cdot \boldsymbol{n}, [\![\boldsymbol{v}_h]\!] \rangle_\Gamma = \sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{v}^i_h)_{\Omega_i}.$$

Adding $S_h$, $\Lambda_h$ and the antisymmetric Nitsche term $\langle \boldsymbol{\sigma}(\boldsymbol{v}^1, q^1) \cdot \boldsymbol{n}, [\![\boldsymbol{u}]\!] \rangle_\Gamma$ on the left hand side we obtain

$$A_h[(\boldsymbol{u}, p), (\boldsymbol{v}_h, q_h)] = L_h(\boldsymbol{v}_h, q_h).$$

Then we get $A_h[(\boldsymbol{u}, p), (\boldsymbol{v}_h, q_h)] = L_h(\boldsymbol{v}_h, q_h) = A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)]$, $\forall (\boldsymbol{v}_h, q_h) \in W^k_h \times Q^k_h$. □

We introduce an interpolation operator that will be useful for the convergence analysis. Let $\mathcal{I}^k_\vartheta$ such that $\mathcal{I}^k_\vartheta \boldsymbol{v} = (i_{\mathsf{SZ}} \boldsymbol{v}^1, \mathcal{I}^k_2 \boldsymbol{v}^2)$ and $\mathcal{I}^k_2 \boldsymbol{v}^2 = i^k_{\mathsf{SZ}} \boldsymbol{v}^2 + \sum_{j=1}^{N^2_p} \vartheta_j \boldsymbol{\chi}^2_j$, with $\vartheta_j \in \mathbb{R}$ and $\boldsymbol{\chi}^2_j = (\chi^2_j, \chi^2_j)^T \in W^1_2$ such that for each node $x_i \in \mathcal{T}^2_h$ we have

$$\chi^2_j(x_i) = \begin{cases} 0 & \text{for} \quad x_i \in \Omega_2 \backslash \mathring{F}^2_j \\ 1 & \text{for} \quad x_i \in \mathring{F}^2_j. \end{cases} \tag{4.25}$$

$\vartheta_j$ is chosen such that

$$\int_{F^2_j} (\boldsymbol{u}^2 - \mathcal{I}^k_2 \boldsymbol{u}^2) \cdot \boldsymbol{n} \, \mathrm{d}s = 0. \tag{4.26}$$

for $j = 1, \ldots, N^2_p$. Then we can write

$$\vartheta_j = \frac{\int_{F^2_j} (\boldsymbol{u}^2 - i^k_{\mathsf{SZ}} \boldsymbol{u}^2) \cdot \boldsymbol{n} \, \mathrm{d}s}{\int_{F^2_j} \boldsymbol{\chi}^2_j \cdot \boldsymbol{n} \, \mathrm{d}s}.$$

We note that $h_2 \lesssim |\int_{F_j^2} \boldsymbol{\chi}_j^2 \, \mathrm{d}s|$, then using the trace inequality and the approximation property (2.32) of the Scott-Zhang interpolant we obtain

$$\begin{aligned}\sum_{j=1}^{N_p^2} |\vartheta_j|^2 \lesssim h_2^{-2} \sum_{j=1}^{N_p^2} \Big| \int_{F_j^2} (\boldsymbol{u}^2 - i_{\mathsf{SZ}}^k \boldsymbol{u}^2) \cdot \boldsymbol{n} \, \mathrm{d}s \Big|^2 &\lesssim h_2^{-1} \sum_{j=1}^{N_p^2} \|(\boldsymbol{u}^2 - i_{\mathsf{SZ}}^k \boldsymbol{u}^2) \cdot \boldsymbol{n}\|_{0,F_j^2}^2 \\ &\lesssim \sum_{j=1}^{N_p^2} (h_2^{-2} \|\boldsymbol{u}^2 - i_{\mathsf{SZ}}^k \boldsymbol{u}^2\|_{0,P_j^2}^2 + \|\nabla(\boldsymbol{u}^2 - i_{\mathsf{SZ}}^k \boldsymbol{u}^2)\|_{0,P_j^2}^2) \\ &\lesssim h_2^{2k} |\boldsymbol{u}^2|_{k+1,\Omega_2}^2.\end{aligned} \tag{4.27}$$

Then we deduce

$$\begin{aligned}\|\boldsymbol{u}^2 - \mathcal{I}_2^k \boldsymbol{u}^2\|_{0,\Omega_2} &\le \|\boldsymbol{u}^2 - i_{\mathsf{SZ}}^k \boldsymbol{u}^2\|_{0,\Omega_2} + \Big( \sum_{j=1}^{N_p^2} \|\vartheta_j \boldsymbol{\chi}_j^2\|_{0,P_j^2}^2 \Big)^{\frac{1}{2}} \\ &\lesssim h_2^{k+1} |\boldsymbol{u}^2|_{k+1,\Omega_2} + \Big( h_2^2 \sum_{j=1}^{N_p^2} |\vartheta_j|^2 \Big)^{\frac{1}{2}}.\end{aligned}$$

Using a discrete inverse inequality and similar arguments as above we can show the estimates

$$\begin{aligned}\|\nabla(\boldsymbol{u}^2 - \mathcal{I}_2^k \boldsymbol{u}^2)\|_{0,\Omega_2} &\lesssim h_2^k |\boldsymbol{u}^2|_{k+1,\Omega_2}, \\ \Big( \sum_{K \in \mathcal{T}_h^2} \|D^2(\boldsymbol{u}^2 - \mathcal{I}_2^k \boldsymbol{u}^2)\|_{0,K}^2 \Big)^{\frac{1}{2}} &\lesssim h_2^{k-1} |\boldsymbol{u}^2|_{k+1,\Omega_2},\end{aligned}$$

then we obtain

$$\begin{aligned}\|\boldsymbol{u}^2 - \mathcal{I}_2^k \boldsymbol{u}^2\|_{0,\Omega_2} + h_2 \|\nabla(\boldsymbol{u}^2 - \mathcal{I}_2^k \boldsymbol{u}^2)\|_{0,\Omega_2} & \\ + h_2^2 \Big( \sum_{K \in \mathcal{T}_h^2} \|D^2(\boldsymbol{u}^2 - \mathcal{I}_2^k \boldsymbol{u}^2)\|_{0,K}^2 \Big)^{\frac{1}{2}} &\lesssim h_2^{k+1} |\boldsymbol{u}^2|_{k+1,\Omega_2}.\end{aligned} \tag{4.28}$$

**Theorem 4.4.2.** *If* $(\boldsymbol{u}, p) \in ([H_\partial^{k+1}(\Omega_1)]^2 \times [H_\partial^{k+1}(\Omega_2)]^2) \times (H^k(\Omega_1) \times H^k(\Omega_2))$ *is the solution of* (4.20) *and* $(\boldsymbol{u}_h, p_h) \in W_h^k \times Q_h^k$ *the solution of* (4.24), *then there holds*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \le h^k (C_{u\mu} |\boldsymbol{u}|_{k+1,\Omega} + C_{p\mu} |p|_{k,\Omega}),$$

*where* $C_{u\mu}$ *and* $C_{p\mu}$ *are positive constants that depends on* $\mu$ *and the mesh geometry.*

*Proof.* Let $\mathcal{I}_\vartheta^k$ the interpolant as defined above and $i_{\mathsf{SZ}}^k$ the Scott-Zhang interpolant, the triangle inequality gives us

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \le |||(\boldsymbol{u} - \mathcal{I}_\vartheta^k \boldsymbol{u}, p - i_{\mathsf{SZ}}^k p)||| + |||(\mathcal{I}_\vartheta^k \boldsymbol{u} - \boldsymbol{u}_h, i_{\mathsf{SZ}}^k p - p_h)|||$$

Using Theorem 4.4.1 and Lemma 4.4.2 we obtain

$$|||(\mathcal{I}_\vartheta^k \boldsymbol{u} - \boldsymbol{u}_h, i_{\mathsf{SZ}}^k p - p_h)||| \leq \sup_{(\boldsymbol{v}_h, q_h)\in W_h^k\times Q_h^k} \frac{A_h[(\mathcal{I}_\vartheta^k \boldsymbol{u} - \boldsymbol{u}, i_{\mathsf{SZ}}^k p - p), (\boldsymbol{v}_h, q_h)]}{|||(\boldsymbol{v}_h, q_h)|||}. \tag{4.29}$$

We want to show

$$A_h[(\mathcal{I}_\vartheta^k \boldsymbol{u} - \boldsymbol{u}, i_{\mathsf{SZ}}^k p - p), (\boldsymbol{v}_h, q_h)] \lesssim |||(\boldsymbol{v}_h, q_h)||| \sum_{i=1}^{2} (\mu_i^{\frac{1}{2}} h_i^k |\boldsymbol{u}^i|_{k+1,\Omega_i} + \mu_i^{-\frac{1}{2}} h_i^k |p^i|_{k,\Omega_i}) \tag{4.30}$$

Using the identity $[\![ab]\!] = \{a\}[\![b]\!] + [\![a]\!]\langle b\rangle$ we have

$$\sum_{i=1}^{2} (\nabla\cdot(\mathcal{I}_\vartheta^k \boldsymbol{u}^i - \boldsymbol{u}^i), q_h^i)_{\Omega_i} - \langle q_h^1, [\![(\mathcal{I}_\vartheta^k \boldsymbol{u} - \boldsymbol{u})\cdot\boldsymbol{n}]\!]\rangle_\Gamma$$
$$= \langle [\![q_h]\!], (\mathcal{I}_\vartheta^k \boldsymbol{u}^2 - \boldsymbol{u}^2)\cdot\boldsymbol{n}\rangle_\Gamma - \sum_{i=1}^{2} (\nabla q_h^i, \mathcal{I}_\vartheta^k \boldsymbol{u}^i - \boldsymbol{u}^i)_{\Omega_i}.$$

Due to the orthogonality property (4.26), the Poincaré type inequality $h_i\|\nabla q_h^i\|_{0,F_j^i} \gtrsim \|q_h^i - \overline{q_h^i}^{F_j^i}\|_{0,F_j^i}$, the approximation property (4.28) and the trace inequality we can write

$$\begin{aligned}
\langle [\![q_h]\!], (\mathcal{I}_\vartheta^k \boldsymbol{u}^2 - \boldsymbol{u}^2)\cdot\boldsymbol{n}\rangle_\Gamma &= \sum_{j=1}^{N_p^2} \langle [\![q_h]\!] - \overline{[\![q_h]\!]}^{F_j^2}, (\mathcal{I}_\vartheta^k \boldsymbol{u}^2 - \boldsymbol{u}^2)\cdot\boldsymbol{n}\rangle_{F_j^2} \\
&\leq \sum_{j=1}^{N_p^2} \|[\![q_h]\!] - \overline{[\![q_h]\!]}^{F_j^2}\|_{F_j^2} \|(\mathcal{I}_\vartheta^k \boldsymbol{u}^2 - \boldsymbol{u}^2)\cdot\boldsymbol{n}\|_{F_j^2} \\
&\leq \Big(\sum_{j=1}^{N_p^2} \|[\![q_h]\!] - \overline{[\![q_h]\!]}^{F_j^2}\|_{F_j^2}^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{N_p^2} \|(\mathcal{I}_\vartheta^k \boldsymbol{u}^2 - \boldsymbol{u}^2)\cdot\boldsymbol{n}\|_{F_j^2}^2\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\Big(\sum_{j=1}^{N_p^1} \|q_h^1 - \overline{q_h^1}^{F_j^1}\|_{0,F_j^1}^2\Big)^{\frac{1}{2}} + \Big(\sum_{j=1}^{N_p^2} \|q_h^2 - \overline{q_h^2}^{F_j^2}\|_{0,F_j^2}^2\Big)^{\frac{1}{2}}\Big) \|(\mathcal{I}_\vartheta^k \boldsymbol{u}^2 - \boldsymbol{u}^2)\cdot\boldsymbol{n}\|_{0,\Gamma} \\
&\lesssim \Big(\Big(\sum_{j=1}^{N_p^1} h_1\|\nabla q_h^1\|_{0,F_j^1}^2\Big)^{\frac{1}{2}} + \Big(\sum_{j=1}^{N_p^2} h_2\|\nabla q_h^2\|_{0,F_j^2}^2\Big)^{\frac{1}{2}}\Big) \|(\mathcal{I}_\vartheta^k \boldsymbol{u}^2 - \boldsymbol{u}^2)\cdot\boldsymbol{n}\|_{0,\Gamma} \\
&\lesssim \mu_2^{-\frac{1}{2}} h_2^{\frac{1}{2}} \Big(h_1^{\frac{1}{2}}\|\nabla q_h^1\|_{0,\Omega_1} + h_2^{\frac{1}{2}}\|\nabla q_h^2\|_{0,\Omega_2}\Big) \mu_2^{\frac{1}{2}} h_2^k |\boldsymbol{u}^2|_{k+1,\Omega_2} \\
&\lesssim |||(\boldsymbol{v}_h, q_h)||| \mu_2^{\frac{1}{2}} h_2^k |\boldsymbol{u}^2|_{k+1,\Omega_2}.
\end{aligned}$$

Where we also used $\mu_2 h_1 \geq \mu_1 h_2$. Using Cauchy-Schwarz inequality and (2.37) we get

$$\begin{aligned}
(2\mu_i \boldsymbol{\varepsilon}(\mathcal{I}_\vartheta^k \boldsymbol{u}^i - \boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}_h^i))_{\Omega_i} &\lesssim |||(\boldsymbol{v}_h, q_h)||| \mu_i^{\frac{1}{2}} h_i^k |\boldsymbol{u}^i|_{k+1,\Omega_i}, \\
(i_{\mathsf{SZ}}^k p^i - p^i, \nabla\cdot\boldsymbol{v}_h^i)_{\Omega_i} &\lesssim |||(\boldsymbol{v}_h, q_h)||| \mu_i^{-\frac{1}{2}} h_i^k |p^i|_{k,\Omega_i}, \\
(\nabla q_h^i, \boldsymbol{w}^i)_{\Omega_i} &\lesssim |||(\boldsymbol{v}_h, q_h)||| \mu_i^{-\frac{1}{2}} h_i^k |p^i|_{k,\Omega_i}.
\end{aligned}$$

Similarly using the trace inequality of Lemma 2.0.1

$$\begin{aligned}\langle i^k_{\mathsf{SZ}} p^1 - p^1, [\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!]\rangle_\Gamma &\leq \|i^k_{\mathsf{SZ}} p^1 - p^1\|_{0,\Gamma} \|[\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!]\|_{0,\Gamma} \lesssim |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!| \mu_1^{-\frac{1}{2}} h_1^k |p_1|_{k,\Omega_1},\\ \langle 2\mu\boldsymbol{\varepsilon}(i^k_{\mathsf{SZ}}\boldsymbol{u}^1 - \boldsymbol{u}^1)\cdot \boldsymbol{n}, [\![\boldsymbol{v}_h]\!]\rangle_\Gamma &\leq 2\mu_1 \|\nabla(i^k_{\mathsf{SZ}}\boldsymbol{u}^1 - \boldsymbol{u}^1)\|_{0,\Gamma}\|[\![\boldsymbol{v}_h]\!]\|_{0,\Gamma}\\ &\lesssim |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!| \mu_1^{\frac{1}{2}} h_1^k |\boldsymbol{u}^1|_{k+1,\Omega_1}.\end{aligned}$$

Using $\mu_2 h_1 \geq \mu_1 h_2$, (2.32) and (4.28) one more time

$$\begin{aligned}\langle 2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_h^1)\cdot \boldsymbol{n}, [\![\mathcal{I}^k_\vartheta \boldsymbol{u} - \boldsymbol{u}]\!]\rangle_\Gamma &\lesssim 2\mu_1 h_1^{-\frac{1}{2}} \|\nabla \boldsymbol{v}_h^1\|_{0,\Omega_1} \|[\![\mathcal{I}^k_\vartheta \boldsymbol{u} - \boldsymbol{u}]\!]\|_{0,\Gamma}\\ &\lesssim |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!| \mu_1^{\frac{1}{2}} h_1^{-\frac{1}{2}} (\|i^k_{\mathsf{SZ}}\boldsymbol{u}^1 - \boldsymbol{u}^1\|_{0,\Gamma} + \|\mathcal{I}^k_2 \boldsymbol{u}^2 - \boldsymbol{u}^2\|_{0,\Gamma})\\ &\lesssim |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!| (\mu_1^{\frac{1}{2}} h_1^k |\boldsymbol{u}^1|_{k+1,\Omega_1} + \mu_2^{\frac{1}{2}} h_2^k |\boldsymbol{u}^2|_{k+1,\Omega_2}).\end{aligned}$$

Finally the pressure stabilisation can be bounded using similar arguments as in the proofs of Lemma 2.3.3 and Corollary 2.3.1 such that

$$\begin{aligned}&S_h(\mathcal{I}^k_\vartheta \boldsymbol{u} - \boldsymbol{u}, i^k_{\mathsf{SZ}} p - p, q_h)\\ &\quad\lesssim |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!| (h_1^k (\mu_1^{\frac{1}{2}} |\boldsymbol{u}^1|_{k+1,\Omega_1} + \mu_1^{-\frac{1}{2}} |p^1|_{k,\Omega_1}) + h_2^k (\mu_2^{\frac{1}{2}} |\boldsymbol{u}^2|_{k+1,\Omega_2} + \mu_2^{-\frac{1}{2}} |p^2|_{k,\Omega_2})).\end{aligned}$$

Then (4.30) is shown, getting back to (4.29) we can write

$$|\!|\!|(\mathcal{I}^k_\vartheta \boldsymbol{u} - \boldsymbol{u}_h, i^k_{\mathsf{SZ}} p - p_h)|\!|\!| \lesssim h_1^k (\mu_1^{\frac{1}{2}} |\boldsymbol{u}^1|_{k+1,\Omega_1} + \mu_1^{-\frac{1}{2}} |p^1|_{k,\Omega_1}) + h_2^k (\mu_2^{\frac{1}{2}} |\boldsymbol{u}^2|_{k+1,\Omega_2} + \mu_2^{-\frac{1}{2}} |p^2|_{k,\Omega_2}).$$

Combining arguments from the proofs of corollaries 4.2.1 and 2.3.1 we obtain

$$|\!|\!|(\boldsymbol{u} - \mathcal{I}^k_\vartheta \boldsymbol{u}, p - i^k_{\mathsf{SZ}} p)|\!|\!| \lesssim h_1^k (\mu_1^{\frac{1}{2}} |\boldsymbol{u}^1|_{k+1,\Omega_1} + \mu_1^{-\frac{1}{2}} |p^1|_{k,\Omega_1}) + h_2^k (\mu_2^{\frac{1}{2}} |\boldsymbol{u}^2|_{k+1,\Omega_2} + \mu_2^{-\frac{1}{2}} |p^2|_{k,\Omega_2}).$$

The claim is shown and $C_{u\mu} = \mathcal{O}(\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}})$ and $C_{p\mu} = \mathcal{O}(\mu_1^{-\frac{1}{2}} + \mu_2^{-\frac{1}{2}})$. □

## 4.5 Numerical results

The aim of this section is to verify numerically the convergence results proved in this chapter. The package FreeFem++ [75] is used for the computations, structured meshes are considered. The computational domain $\Omega$ is the unit square separated in two subdomains $[0, 0.5] \times [0, 1]$ and $[0.5, 1] \times [0, 1]$ meshed independently. For each case we use a manufactured solution to test the precision of the penalty-free Nitsche's method for nonconforming domain decomposition.

### 4.5.1 Poisson problem

The $L^2$ and $H^1$-errors are investigated for various values of the ratio $h_1/h_2$, for each case we choose $\mu_1 = 1$, and, a range of values for $\mu_2$ is investigated. Piecewise affine approximation is used. We consider the following manufactured solution

$$u = \exp(xy)\sin(\pi x)\sin(\pi y).$$

Figures 4.4 and 4.5 display the convergence of the $L^2$-error for a range of values of the ratio $h_1/h_2$. For each case the convergence rate is slightly bigger than what has been proved in the theory. We also see that, as the ratio $h_1/h_2$ becomes smaller, the constant $C'_\mu$ from Proposition 4.2.1 becomes very slightly bigger as $\mu_2$ grows.

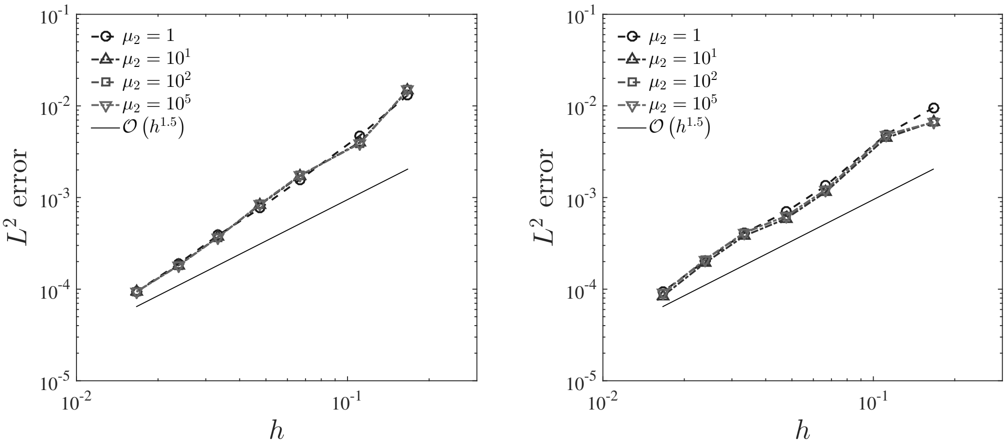



FIGURE 4.4: $\mathbb{P}_1$, $\mu_1 = 1$, left $\frac{h_1}{h_2} = 1$, right $\frac{h_1}{h_2} = \frac{3}{5}$.

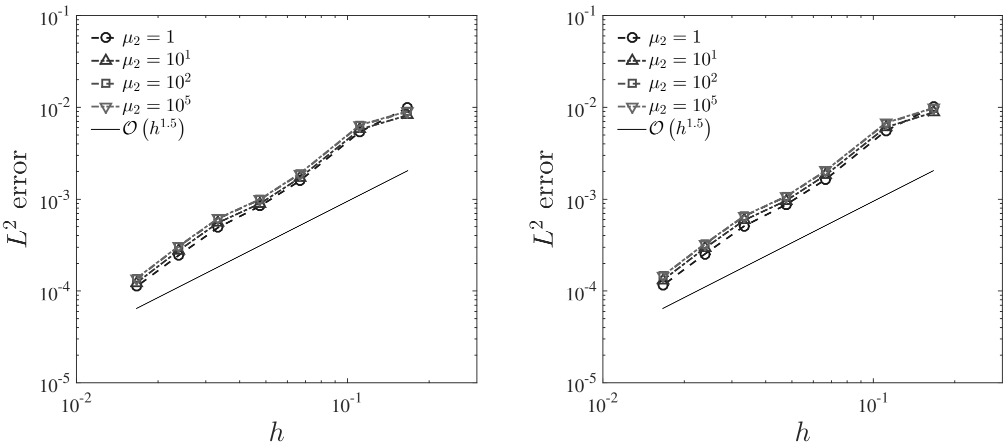



FIGURE 4.5: $\mathbb{P}_1$, $\mu_1 = 1$, left $\frac{h_1}{h_2} = \frac{3}{10}$, right $\frac{h_1}{h_2} = \frac{1}{5}$.

Figures 4.6 and 4.7 display the convergence of the $H^1$-error. For each case, the convergence rate observed corresponds to the convergence rate that has been shown theoretically. For each value of $h_1/h_2$ the constant $C_\mu$ from Corollary 4.2.1 is the same for any value of $\mu_2$ considered. As the ratio $h_1/h_2$ becomes smaller the constant $C_\mu$ become slightly smaller.

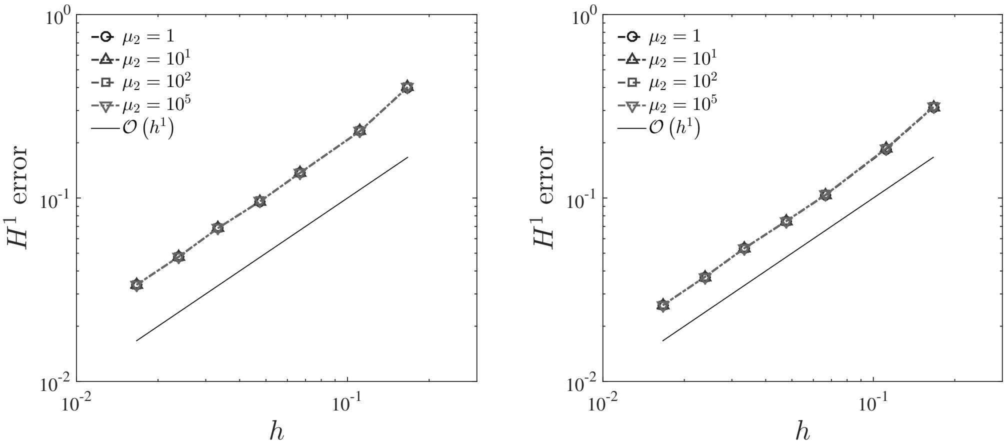



FIGURE 4.6: $\mathbb{P}_1$, $\mu_1 = 1$, left $\frac{h_1}{h_2} = 1$, right $\frac{h_1}{h_2} = \frac{3}{5}$.

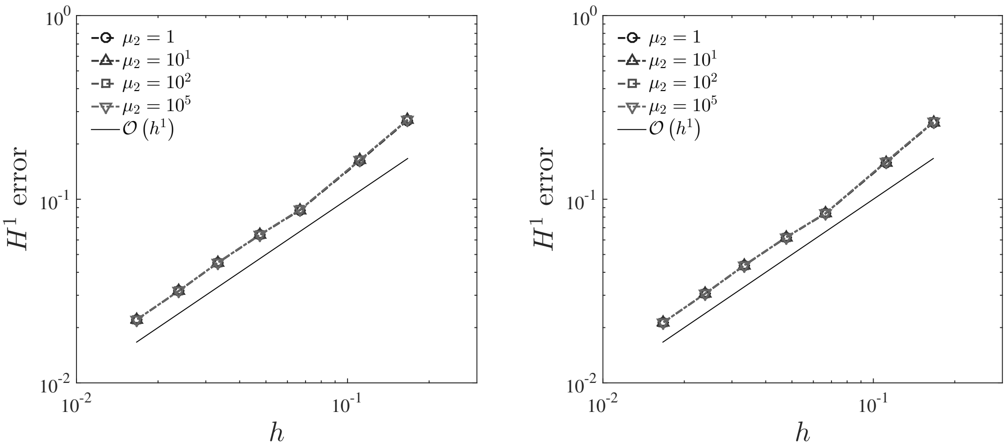



FIGURE 4.7: $\mathbb{P}_1$, $\mu_1 = 1$, left $\frac{h_1}{h_2} = \frac{3}{10}$, right $\frac{h_1}{h_2} = \frac{1}{5}$.

### 4.5.2 Compressible elasticity

The $L^2$ and $H^1$-errors are investigated for $h_1/h_2 = 1/5$ and $\mu_1 = \lambda_1 = 1$, for ranges of values for $\mu_2$ and $\lambda_2$. First and second order approximations are investigated. We consider the following manufactured solution

$$\boldsymbol{u} = \begin{pmatrix} \left(x^5 - x^4\right)\left(y^3 - y^2\right) \\ \left(x^4 - x^3\right)\left(y^6 - y^5\right) \end{pmatrix}.$$

Figure 4.8 shows the convergence slopes of the $L^2$-error for first order approximation, it shows a convergence rate slightly bigger than in the theory as observed for the Poisson case. As $\mu_2$ grows the constant $C'_\mu$ from Proposition 4.3.1 becomes very slightly bigger. As $\lambda_2$ grows the constant $C'_{\mu\lambda}$ grows this characterises locking.

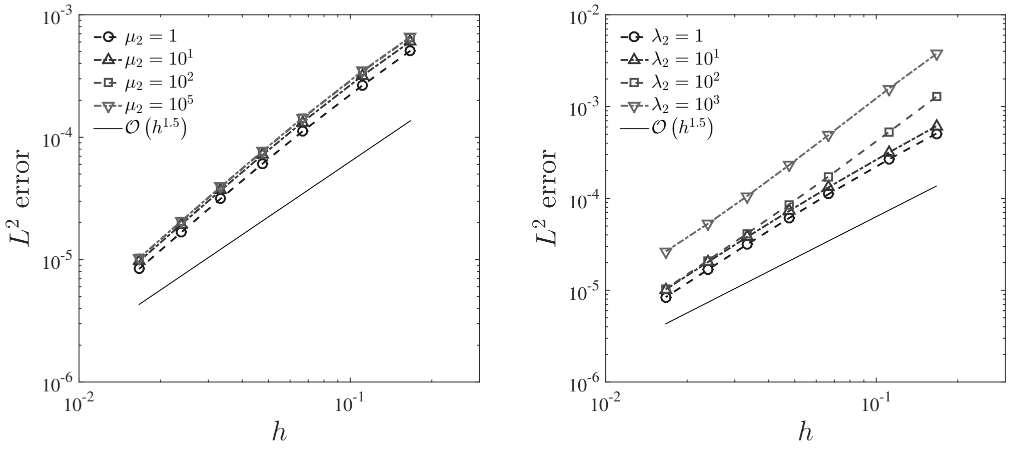



FIGURE 4.8: $L^2$-error, $\mathbb{P}_1$, $\mu_1 = \lambda_1 = 1$, left $\lambda_2 = 1$, right $\mu_2 = 1$.

Figure 4.9 shows the convergence slopes of the $H^1$-error for first order approximation, the slopes observed corresponds to the rate of convergence that has been shown theoretically. We see that the influence of $\mu_2$ on the error is negligible. However as $\lambda_2$ grows the mesh needs to be fine enough to recover the expected convergence rate, this is due to locking.

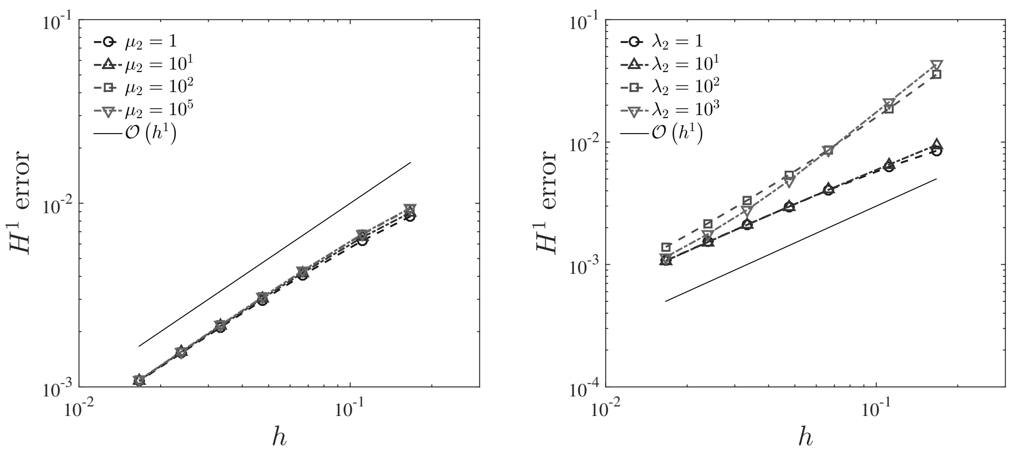



FIGURE 4.9: $H^1$-error, $\mathbb{P}_1$, $\mu_1 = \lambda_1 = 1$, left $\lambda_2 = 1$, right $\mu_2 = 1$.

Figure 4.10 shows the convergence slopes of the $L^2$-error for second order approximation. The slopes of convergence are once again slightly bigger than what has been shown theoretically. We see that $\mu_2$ has a very small impact on the slope of convergence whereas for $\lambda_2$ large enough we observe once again locking.

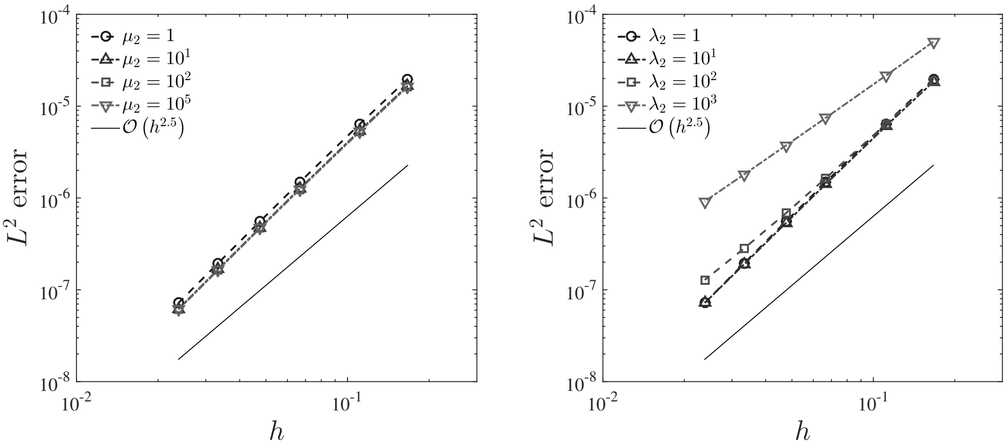



FIGURE 4.10: $L^2$-error, $\mathbb{P}_2$, $\mu_1 = \lambda_1 = 1$, left $\lambda_2 = 1$, right $\mu_2 = 1$.

Figure 4.11 shows the convergence slopes of the $H^1$-error for second order approximation. The slopes corresponds to what has been shown theoretically, $\mu_2$ has a negligible impact on the convergence. For $\lambda_2 \leq 10^2$ locking is not observed, for $\lambda_2 = 10^3$ locking generates a small rise of the constant $C_{\mu\lambda}$ of Corollary 4.3.1.

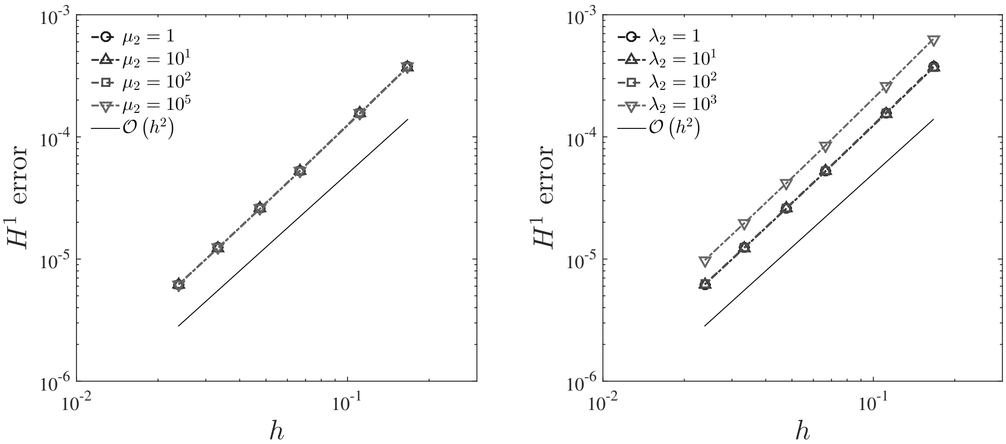



FIGURE 4.11: $H^1$-error, $\mathbb{P}_2$, $\mu_1 = \lambda_1 = 1$, left $\lambda_2 = 1$, right $\mu_2 = 1$.

### 4.5.3 Incompressible elasticity

The $H^1$-error of $\boldsymbol{u}$ and $L^2$-error of $p$ are investigated for $h_1/h_2 = 3/5$, and $\mu_1 = 1$ for a range of values for $\mu_2$. First and second order approximations are considered. We consider the following manufactured solution

$$\boldsymbol{u} = \begin{pmatrix} \sin(4\pi x)\cos(4\pi y) \\ -\cos(4\pi x)\sin(4\pi y) \end{pmatrix}, \qquad p = \pi\cos(4\pi x)\cos(4\pi y).$$

Figure 4.12 shows the convergence slopes of the $H^1$-error of the displacment and $L^2$-error of the pressure for first order approximation. The slopes of convergence observed for the $H^1$-error of $\boldsymbol{u}$ corresponds to the theoretical result, the influence of $\mu_2$ is negligible. The slopes of convergence observed for the $L^2$-error of $p$ is of order $\mathcal{O}(h^{1.5})$, the constant is multiplied by a factor $\sqrt{\mu_2}$ as $\mu_2$ becomes bigger. Figure 4.13 shows the convergence

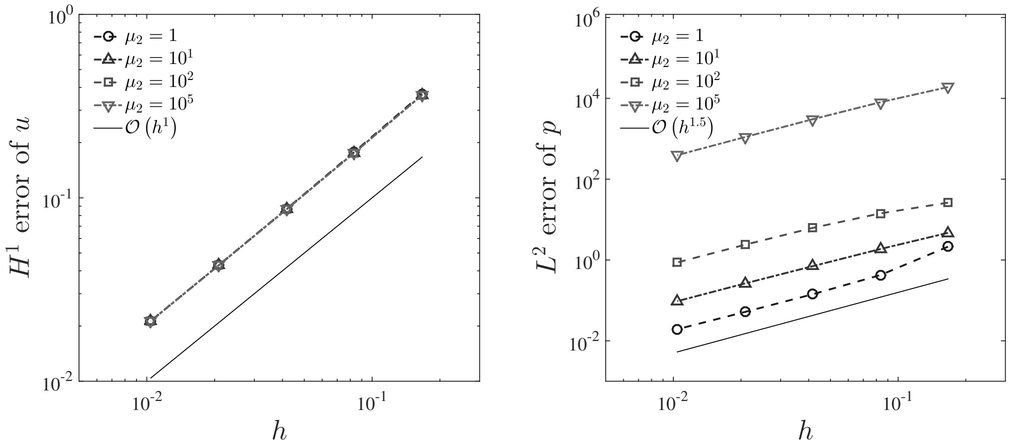



FIGURE 4.12: $\mathbb{P}_1$, $\mu_1 = 1$, $\gamma_p = 10^{-1}$, left $H^1$-error of $\boldsymbol{u}$, right, $L^2$-error of $p$.

slopes of the $H^1$-error of the displacement and $L^2$-error of the pressure for second order approximation. The slopes of convergence observed for the $H^1$-error of $\boldsymbol{u}$ show optimal convergence which corresponds to the theoretical result, once again the influence of $\mu_2$ is negligible. The slopes of convergence observed for the $L^2$-error of $p$ are slightly bigger than $\mathcal{O}(h^{2.5})$ for $\mu_2 \leq 10^2$, for $\mu_2 = 10^5$ the convergence is of order $\mathcal{O}(h^2)$.

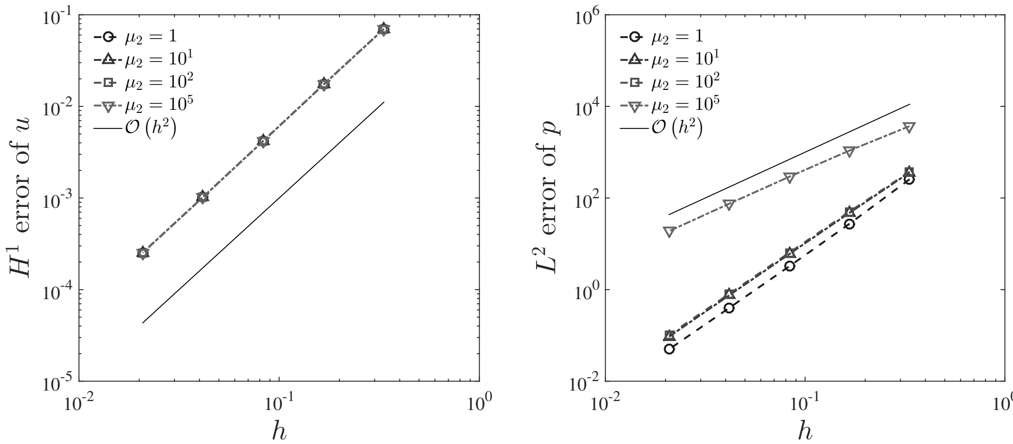



FIGURE 4.13: $\mathbb{P}_2$, $\mu_1 = 1$, $\gamma_p = 10^{-4}$, left $H^1$-error of $\boldsymbol{u}$, right $L^2$-error of $p$.

# Chapter 5

# Unfitted domain decomposition

In this chapter, we consider unfitted domain decomposition with the penalty-free Nitsche's method as a coupling tool. The computational domain is divided into two subdomains that can have different material parameters, however the computational domain is meshed with only one triangulation. A consequence is that the interface between the two subdomains does not fit with the triangulation i.e. some simplices are crossed by the interface.

## 5.1 Preliminaries

Let $\Omega_1$ and $\Omega_2$ be two convex bounded domain in $\mathbb{R}^2$ with polygonal boundary, these two domains share an interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, for simplicity $\Gamma$ is considered as plane. We define the domain $\Omega = \Omega_1 \cup \Omega_2$ with boundary $\partial\Omega$, an example of $\Omega$ is represented in Figure 5.1. Let $\{\mathcal{T}_h\}_h$ be a family of quasi-uniform and shape regular triangulation

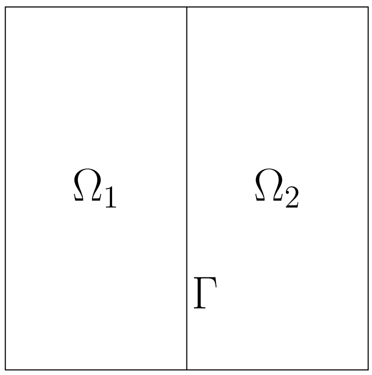



FIGURE 5.1: Example of computational domain $\Omega$.

fitted to $\Omega$, the mesh size is defined as $h = \max_{K \in \mathcal{T}_h} h_K$. Figure 5.2 shows an example of configuration, the mesh do not fit with the interface $\Gamma$. For $i = 1, 2$, let

$$\Omega_i^* = \{K \in \mathcal{T}_h \mid K \cap \Omega_i \neq \emptyset\},$$

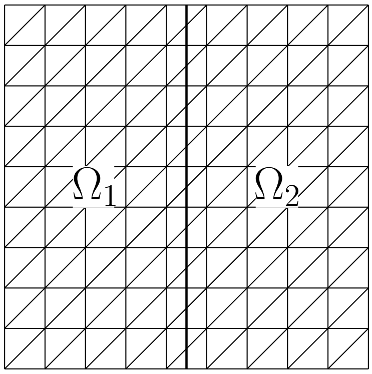



FIGURE 5.2: Example of mesh in $\Omega$.

we note that $\Omega_1^* \cap \Omega_2^* \neq \emptyset$. Let us define the spaces

$$V_i^* = \{v \in H^1(\Omega_i^*) : v|_{\partial\Omega} = 0\},$$
$$V_i^k = \{v_h \in V_i^* : v_h|_K \in \mathbb{P}_k(K) \ \ \forall K \in \mathcal{T}_h\} \qquad \forall k \geq 1,$$

then $V_h^k = V_1^k \times V_2^k$, for any $w_h \in V_h^k$ we have $w_h = (w_h^1, w_h^2)$ with $w_h^1 \in V_1^k$ and $w_h^2 \in V_2^k$. Similarly as in Section 3.1.1 we introduce an extension operator. Let $\mathbb{E}_i$ be an $H^s$-extension on $\Omega_i^*$, $\mathbb{E}_i : H^s(\Omega_i) \to H^s(\Omega_i^*)$ such that $(\mathbb{E}_i w^i)|_{\Omega_i} = w^i$ and

$$\|\mathbb{E}_i w^i\|_{s,\Omega_i^*} \lesssim \|w^i\|_{s,\Omega_i} \qquad \forall w^i \in H^s(\Omega_i), s \geq 0. \tag{5.1}$$

For simplicity we will write $w^i$ instead of $\mathbb{E}_i w^i$. Let $i_{\mathsf{SZ}} : H^s(\Omega_i^*) \to V_i^k$ be the Scott-Zhang interpolant, we construct the interpolation operator $\mathcal{I}_h$ such that $\mathcal{I}_h w = (\mathcal{I}_h^1 w^1, \mathcal{I}_h^2 w^2)$ with

$$\mathcal{I}_h^i w^i = i_{\mathsf{SZ}} \mathbb{E}_i w^i. \tag{5.2}$$

Using the estimate (5.1) together with (3.4), then for $v^i \in H^{k+1}(\Omega_i^*)$ and $0 \leq r \leq s \leq k+1$ we have

$$\Big( \sum_{K \in \Omega_i^*} \|v^i - \mathcal{I}_h^i v^i\|_{r,K}^2 \Big)^{\frac{1}{2}} \lesssim h^{s-r} |v^i|_{s,\Omega_i}. \tag{5.3}$$

We define the set of elements that intersect the interface

$$G_h = \{K \in \mathcal{T}_h \mid K \cap \Gamma \neq \emptyset\}.$$

Let us split the set $G_h$ into $N_p$ smaller disjoint sets of elements $\{G_j\}_{1 \leq j \leq N_p}$. Let $I_{G_j}$ be the set of nodes belonging to $G_j$, we define the sets of nodes $I_j^1$ and $I_j^2$ such that

$$I_j^1 = \left\{x_i \in I_{G_j} \mid x_i \in \Omega_1, \ x_i \notin I_{G_n} \ \forall n \neq j\right\},$$
$$I_j^2 = \left\{x_i \in I_{G_j} \mid x_i \in \Omega_2, \ x_i \notin I_{G_n} \ \forall n \neq j\right\},$$

now we define $P_j^1$ and $P_j^2$ for each $G_j$ such that

$$P_j^1 = G_j \cup \left\{K \in \mathcal{T}_h \mid \exists x_i \in I_j^1 \text{ such that } x_i \in K\right\},$$
$$P_j^2 = G_j \cup \left\{K \in \mathcal{T}_h \mid \exists x_i \in I_j^2 \text{ such that } x_i \in K\right\}.$$

Each patch $P_j^i$ is constructed such that $I_j^i \neq \emptyset$ for $i = 1, 2$. Figure 5.3 shows an example of two patches $P_j^1$ and $P_j^2$. $\Gamma_j = \Gamma \cap G_j$ is the part of the boundary included in the patches $P_j^1$ and $P_j^2$. For all $j$ and $i = 1, 2$, the patch $P_j^i$ has the following properties

$$h \lesssim \text{meas}(\Gamma_j) \lesssim h, \qquad h^2 \lesssim \text{meas}(P_j^i) \lesssim h^2. \tag{5.4}$$

The function $\chi_j \in V_1^1$ attached to $P_j^1$ is such that

$$\chi_j(x_i) = \begin{cases} 0 & \text{for} \quad x_i \notin I_j^1 \\ 1 & \text{for} \quad x_i \in I_j^1, \end{cases} \tag{5.5}$$

with $i = 1, \ldots, N_n$. $N_n$ is the number of nodes in the triangulation $\mathcal{T}_h$. The broken norms are defined in the same way as in Section 4.1.

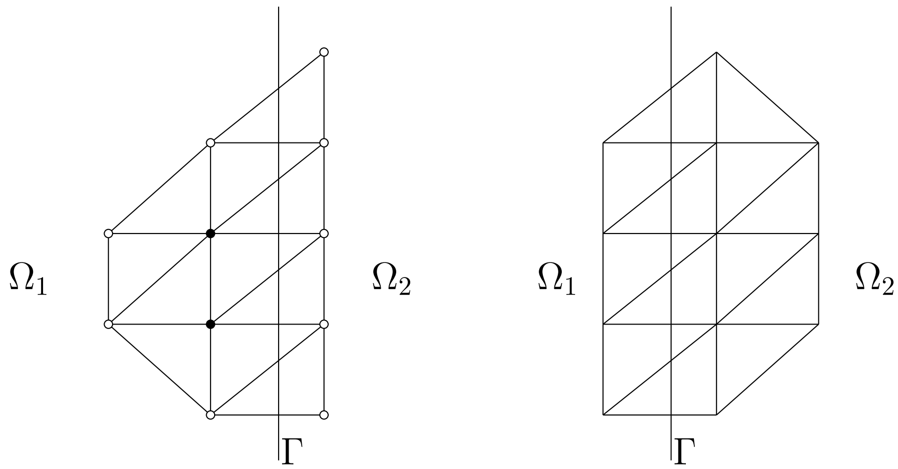



FIGURE 5.3: Left : example of $P_j^1$, the function $\chi_j$ is equal to 0 in the nonfilled nodes, 1 in the filled nodes ; right : example of $P_j^2$.

## 5.2 Poisson problem

We consider the Poisson problem with discontinuous material parameters as

$$\begin{aligned} -\mu_i \Delta u^i &= f \quad && \text{in } \Omega_i \ , \quad i = 1, 2, \\ u^i &= 0 && \text{on } \partial\Omega \cap \Omega_i \ , \quad i = 1, 2, \\ [\![u]\!] &= 0 && \text{on } \Gamma, \\ [\![\mu \nabla u \cdot n]\!] &= 0 && \text{on } \Gamma, \end{aligned} \tag{5.6}$$

with $\mu_i$ the diffusivity of the domain $\Omega_i$ and $f \in L^2(\Omega)$, we have $u = (u^1, u^2)$. Similarly as in the previous chapter, the following regularity estimate holds [44]

$$\mu_1 \|D^2 u^1\|_{0,\Omega_1} + \mu_2 \|D^2 u^2\|_{0,\Omega_2} \lesssim \|f\|_{0,\Omega}.$$

We consider the following weights

$$\omega_1 = \frac{\mu_2}{\mu_1 + \mu_2}, \qquad \omega_2 = \frac{\mu_1}{\mu_1 + \mu_2}, \tag{5.7}$$

also, in order to simplify the notations

$$\gamma = \frac{\mu_1 \mu_2}{h(\mu_1 + \mu_2)}.$$

We assume that $\mu_1 \leq \mu_2$. The formulation is obtained similarly as for the fitted domain decomposition case (see Section 4.2.1). Using the penalty free Nitsche's method, the finite element formulation for the problem (5.6) is written as: find $u_h \in V_h^k$ such that

$$A_h(u_h, v_h) + J_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h^k, \tag{5.8}$$

where

$$A_h(u_h, v_h) = \sum_{i=1}^{2} (\mu_i \nabla u_h^i, \nabla v_h^i)_{\Omega_i} - \langle \{\mu \nabla u_h \cdot n\}, [\![v_h]\!] \rangle_\Gamma + \langle \{\mu \nabla v_h \cdot n\}, [\![u_h]\!] \rangle_\Gamma,$$
$$L_h(v_h) = \sum_{i=1}^{2} (f, v_h^i)_{\Omega_i}.$$

The operator $J_h$ is the ghost penalty [25], defined such that $J_h(u_h, v_h) = J_h^1(u_h^1, v_h^1) + J_h^2(u_h^2, v_h^2)$ with

$$J_h^i(u_h^i, v_h^i) = \gamma_g \sum_{F \in \mathcal{F}_G^i} \sum_{l=1}^{k} \langle \mu_i h^{2l-1} [\![D_{n_F}^l u_h^i]\!]_F, [\![D_{n_F}^l v_h^i]\!]_F \rangle_F.$$

This penalisation ensures that the condition number is independent of how the interface cut the elements of the mesh. We recall that in a generic sense $F$ is a face of a triangle $K \in \mathcal{T}_h$. The sets $\mathcal{F}_G^i$ for $i = 1, 2$ are defined as

$$\mathcal{F}_G^1 = \{F \in G_h \mid F \cap \Omega_1 \neq \emptyset\}, \qquad \mathcal{F}_G^2 = \{F \in G_h \mid F \cap \Omega_2 \neq \emptyset\}.$$

$D_{n_F}^l$ is the partial derivative of order $l$ in the direction $n_F$. The estimate (3.17) still holds in this framework for $v_h \in V_h^k$

$$\mu_i \|\nabla v_h^i\|_{0,\Omega_i^*}^2 \lesssim \mu_i \|\nabla v_h^i\|_{0,\Omega_i}^2 + J_h^i(v_h^i, v_h^i) \lesssim \mu_i \|\nabla v_h^i\|_{0,\Omega_i^*}^2, \tag{5.9}$$

here we assume that $\gamma_g = \mathcal{O}(1)$. Also, the following inequality is true for any $u_h \in V_h^k$

$$\sum_{j=1}^{N_p} \gamma \| \overline{[\![u_h]\!]}^{\Gamma_j} \|_{0,\Gamma_j}^2 \geq \sum_{j=1}^{N_p} \frac{\gamma}{2} \| [\![u_h]\!] \|_{0,\Gamma_j}^2 - C\omega_1 \sum_{j=1}^{N_p} \mu_1 \| \nabla u_h^1 \|_{0,P_j^1}^2 - C\omega_2 \sum_{j=1}^{N_p} \mu_2 \| \nabla u_h^2 \|_{0,P_j^2}^2, \tag{5.10}$$

this result can be shown using similar arguments as in the proof of Lemma 4.2.1.

### 5.2.1 Inf-sup stability

We define the norms

$$|\!|\!| w |\!|\!|^2 = \sum_{i=1}^{2} \mu_i \| \nabla w^i \|_{0,\Omega_i}^2 + \gamma \| [\![w]\!] \|_{0,\Gamma}^2 + J_h(w,w),$$
$$\| w \|_*^2 = |\!|\!| w |\!|\!|^2 + \mu_1 h \| \nabla w^1 \cdot n \|_{0,\Gamma}^2 + \mu_2 h \| \nabla w^2 \cdot n \|_{0,\Gamma}^2.$$

**Theorem 5.2.1.** *There exists $\beta > 0$ such that for all functions $u_h \in V_h^k$ the following inequality holds*

$$\beta |\!|\!| u_h |\!|\!| \leq \sup_{v_h \in V_h^k} \frac{A_h(u_h,v_h) + J_h(u_h,v_h)}{|\!|\!| v_h |\!|\!|}.$$

*Proof.* Let $v_h = u_h + \alpha \sum_{j=1}^{N_p} (v_j^1, 0)$, such that $v_j^1 = \nu_j \chi_j$, with $\nu_j \in \mathbb{R}$ and $\chi_j$ defined by (5.5), each $v_j^1$ has the property

$$\text{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \nabla v_j^1 \cdot n \, \text{d}s = h^{-1} \overline{[\![u_h]\!]}^{\Gamma_j}. \tag{5.11}$$

Then using Lemma 3.1.1 with $\varphi_r = v_j^1$ and $r_j = h^{-1} \overline{[\![u_h]\!]}^{\Gamma_j}$ we obtain the inequality

$$\| \nabla v_j^1 \|_{0,P_j^1} \lesssim h^{-\frac{1}{2}} \| \overline{[\![u_h]\!]}^{\Gamma_j} \|_{0,\Gamma_j}. \tag{5.12}$$

We can write the following

$$(A_h + J_h)(u_h, v_h) = (A_h + J_h)(u_h, u_h) + \alpha \sum_{j=1}^{N_p} [A_h(u_h, v_j^1) + J_h(u_h, v_j^1)].$$

with

$$\alpha (A_h + J_h)(u_h, v_j^1) = \alpha (\mu_1 \nabla u_h^1, \nabla v_j^1)_{P_j^1 \cap \Omega_1} - \alpha \langle \{\mu \nabla u_h \cdot n\}, v_j^1 \rangle_{\Gamma_j}$$
$$+ \alpha \omega_1 \langle \mu_1 \nabla v_j^1 \cdot n, [\![u_h]\!] \rangle_{\Gamma_j} + \alpha J_h(u_h, v_j^1).$$

Using inequality (5.9) we have

$$(A_h + J_h)(u_h, u_h) = \sum_{i=1}^{2} (\mu_i \nabla u_h^i, \nabla u_h^i)_{\Omega_i} + J_h(u_h, u_h) \gtrsim \mu_1 \| \nabla u_h^1 \|_{0,\Omega_1^*}^2 + \mu_2 \| \nabla u_h^2 \|_{0,\Omega_2^*}^2.$$

Using inequality (3.1), Lemma 2.0.2 and (5.12)

$$\begin{aligned}(\mu_1\nabla u_h^1, \alpha\nabla v_j^1)_{P_j^1\cap\Omega_1} &+ \alpha J_h(u_h, v_j^1)\\ &\le \mu_1^{\frac{1}{2}}\|\nabla u_h^1\|_{0,P_j^1\cap\Omega_1}\alpha\mu_1^{\frac{1}{2}}\|\nabla v_j^1\|_{0,P_j^1\cap\Omega_1} + J_h(u_h,u_h)^{\frac{1}{2}}\alpha J_h(v_j^1,v_j^1)^{\frac{1}{2}}\\ &\lesssim \epsilon\mu_1\|\nabla u_h^1\|_{0,P_j^1}^2 + \frac{C\alpha^2}{4\epsilon}\mu_1\|\nabla v_j^1\|_{0,P_j^1}^2\\ &\lesssim \epsilon\mu_1\|\nabla u_h^1\|_{0,P_j^1}^2 + \frac{C\alpha^2}{4\epsilon}\Big(1+\frac{\mu_1}{\mu_2}\Big)\gamma\|\overline{[\![u_h]\!]}^{\Gamma_j}\|_{0,\Gamma_j}^2.\end{aligned}$$

Similarly as in the proof of Theorem 4.2.1, using the trace and inverse inequalities, (3.12), (3.19), (5.11) and (5.12) we have

$$\begin{aligned}\langle\{\mu\nabla u_h\cdot n\},\alpha v_j^1\rangle_{\Gamma_j} &\le \frac{C\alpha^2}{2\epsilon}\gamma\|\overline{[\![u_h]\!]}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \epsilon\omega_1\mu_1\|\nabla u_h^1\|_{0,P_j^1}^2 + \epsilon\omega_2\mu_2\|\nabla u_h^2\|_{0,P_j^2}^2,\\ \alpha\omega_1\langle\mu_1\nabla v_j^1\cdot n,[\![u_h]\!]\rangle_{\Gamma_j} &\ge \alpha\Big(1-\frac{C\alpha}{2\epsilon}\Big)\gamma\|\overline{[\![u_h]\!]}^{\Gamma_j}\|_{0,\Gamma_j}^2 - \epsilon\omega_1\mu_1\|\nabla u_h^1\|_{0,P_j^1}^2\\ &\qquad - \epsilon\omega_2\mu_2\|\nabla u_h^2\|_{0,P_j^2}^2.\end{aligned}$$

Collecting the bounds and using (5.10), we have the lower bound

$$A_h(u_h,v_h) \ge (C_a-\omega_1 CC_c)\mu_1\|\nabla u_h^1\|_{0,\Omega_1^*}^2 + (C_b-\omega_2 CC_c)\mu_2\|\nabla u_h^2\|_{0,\Omega_2^*}^2 + C_c\frac{\gamma}{2}\|[\![u_h]\!]\|_{0,\Gamma}^2,$$

with the constants

$$C_a = C-\epsilon(2\omega_1+1),\qquad C_b = C-2\epsilon\omega_2,\qquad C_c = \alpha\Big(1-\alpha\frac{C}{4\epsilon}\Big(5+\frac{\mu_1}{\mu_2}\Big)\Big).$$

Let $\epsilon=\frac{1}{8}$, $C_c$ is positive for $\alpha<\frac{1}{12C}$. The terms $(C_a-\omega_1 CC_c)$ and $(C_b-\omega_2 CC_c)$ are positive for $\alpha<\frac{C}{2}$. Using (5.9) we get

$$\beta_0|\!|\!|u_h|\!|\!|^2 \le A_h(u_h,v_h) + J_h(u_h,v_h).$$

Similarly as (2.16) we have

$$|\!|\!|v_h|\!|\!|^2 \lesssim |\!|\!|u_h|\!|\!|^2 + \alpha^2\sum_{j=1}^{N_p}|\!|\!|v_j^1|\!|\!|^2 \quad\text{with}\quad |\!|\!|v_j^1|\!|\!|^2 = \mu_1\|\nabla v_j^1\|_{0,P_j^1\cap\Omega_1}^2 + \gamma\|v_j^1\|_{0,\Gamma_j}^2 + J_h(v_j^1,v_j^1).$$

Using (5.12), (5.9) and $\|\overline{[\![u_h]\!]}^{\Gamma_j}\|_{\Gamma_j}\lesssim\|[\![u_h]\!]\|_{\Gamma_j}$ it gives the upper bound

$$\sum_{j=1}^{N_p}\left[\mu_1\|\nabla v_j^1\|_{0,P_j^1\cap\Omega_1}^2 + J_h(v_j^1,v_j^1)\right] \lesssim \Big(1+\frac{\mu_1}{\mu_2}\Big)|\!|\!|u_h|\!|\!|^2.$$

Using the trace inequality and the inequality (3.19) we get

$$\gamma \sum_{j=1}^{N_p} \|v_j^1\|_{0,\Gamma_j}^2 \lesssim \omega_1\mu_1 \sum_{j=1}^{N_p} \|\nabla v_j^1\|_{0,P_j^1}^2 \lesssim |||u_h|||^2.$$

Then we get $|||v_h||| \lesssim |||u_h|||$. □

### 5.2.2 A priori error estimate

**Lemma 5.2.1.** *If $u \in H^2_\partial(\Omega_1)\times H^2_\partial(\Omega_2)$ is the solution of (5.6) and $u_h \in V_h^k$ the solution of (5.8) the following property holds*

$$A_h(u-u_h, v_h) - J_h(u_h, v_h) = 0 \quad , \qquad \forall v_h \in V_h^k.$$

*Proof.* The proof is done using similar arguments as for Lemma 4.2.2 and using that $A_h(u_h, v_h) + J_h(u_h, v_h) = L_h(v_h), \forall v_h \in V_h^k$. □

**Lemma 5.2.2.** *Let $w \in H^2_\partial(\Omega_1) \times H^2_\partial(\Omega_2) + V_h^k$ and $v_h \in V_h^k$, there exists a positive constant $M$ such that*

$$A_h(w, v_h) \leq M\|w\|_* |||v_h|||.$$

*Proof.* Using the trace inequality (3.1) and the Cauchy Schwarz inequality we have

$$\langle\{\mu\nabla w\cdot n\}, [\![v_h]\!]\rangle_\Gamma \leq h^{\frac{1}{2}}((\omega_1\mu_1)^{\frac{1}{2}}\|\nabla w^1\cdot n\|_{0,\Gamma} + (\omega_2\mu_2)^{\frac{1}{2}}\|\nabla w^2\cdot n\|_{0,\Gamma})\gamma^{\frac{1}{2}}\|[\![v_h]\!]\|_{0,\Gamma},$$
$$\langle\{\mu\nabla v_h\cdot n\}, [\![w]\!]\rangle_\Gamma \lesssim ((\omega_1\mu_1)^{\frac{1}{2}}\|\nabla v_h^1\|_{0,\Omega_1} + (\omega_2\mu_2)^{\frac{1}{2}}\|\nabla v_h^2\|_{0,\Omega_2})\gamma^{\frac{1}{2}}\|[\![w]\!]\|_{0,\Gamma}.$$

Using these two upper bound it is straightforward to conclude that

$$\sum_{i=1}^{2}(\mu_i\nabla w^i, \nabla v_h^i)_{\Omega_i} - \langle\{\mu\nabla w\cdot n\}, [\![v_h]\!]\rangle_\Gamma + \langle\{\mu\nabla v_h\cdot n\}, [\![w]\!]\rangle_\Gamma \lesssim \|w\|_*|||v_h|||.$$

□

**Theorem 5.2.2.** *If $u \in H^{k+1}_\partial(\Omega_1) \times H^{k+1}_\partial(\Omega_2)$ is the solution of (5.6) and $u_h \in V_h^k$ the solution of (5.8), then there holds*

$$|||u-u_h||| \lesssim \inf_{w_h\in V_h^k}\|u-w_h\|_*.$$

*Proof.* Same as Theorem 3.2.2 using Lemma 5.2.1, Theorem 5.2.1 and Lemma 5.2.2. □

**Corollary 5.2.1.** *If $u \in H^{k+1}_\partial(\Omega_1) \times H^{k+1}_\partial(\Omega_2)$ is the solution of (5.6) and $u_h \in V_h^k$ the solution of (5.8), then there holds*

$$|||u-u_h||| \leq C_\mu h^k|u|_{k+1,\Omega},$$

*and $C_\mu = \mathcal{O}(\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}})$.*

*Proof.* Combining arguments from the proofs of corollaries 3.2.1, 4.2.1 and using (5.1), (5.3) we obtain the estimate

$$\|u - \mathcal{I}_h u\|_* \lesssim \mu_1^{\frac{1}{2}} h^k |u^1|_{k+1,\Omega_1} + \mu_2^{\frac{1}{2}} h^k |u^2|_{k+1,\Omega_2}. \tag{5.13}$$

We conclude by applying Theorem 5.2.2 with $w_h = \mathcal{I}_h u$. □

**Proposition 5.2.1.** *Let $u \in H_\partial^{k+1}(\Omega_1) \times H_\partial^{k+1}(\Omega_2)$ be the solution of* (5.6) *and $u_h \in V_h^k$ the solution of* (5.8)*, then there holds*

$$\|u - u_h\|_\Omega \le C'_\mu h^{k+\frac{1}{2}} |u|_{k+1,\Omega},$$

*with $C'_\mu = \mathcal{O}(1)$.*

*Proof.* Let $z$ satisfy the adjoint problem

$$\begin{aligned} -\mu_i \Delta z^i &= u^i - u_h^i \quad \text{in } \Omega_i \ , \quad i = 1,2, \\ z^i &= 0 \qquad\qquad \text{on } \partial\Omega \cap \Omega_i \ , \quad i = 1,2, \\ [\![z]\!] &= 0 \qquad\qquad \text{on } \Gamma, \\ [\![\mu \nabla z \cdot n]\!] &= 0 \qquad\qquad \text{on } \Gamma. \end{aligned} \tag{5.14}$$

We assume the following regularity estimate

$$\mu_1 \|z^1\|_{2,\Omega_1} + \mu_2 \|z^2\|_{2,\Omega_2} \lesssim \|u - u_h\|_{0,\Omega}. \tag{5.15}$$

Similarly as in the proof of Proposition 4.2.1, the $L^2$-error can be written as

$$\|u - u_h\|_{0,\Omega}^2 = A_h(u - u_h, z) - 2\langle \{\mu \nabla z \cdot n\} \, , [\![u - u_h]\!] \rangle_\Gamma.$$

Using the global trace inequality $\|\nabla z^i \cdot n\|_{0,\Gamma} \lesssim \|z^i\|_{2,\Omega_i}$ for $i = 1,2$, we can write

$$\begin{aligned} &|\langle \{\mu \nabla z \cdot n\}, [\![u - u_h]\!] \rangle_\Gamma| \\ &\qquad \le ((\omega_1 \mu_1)^{\frac{1}{2}} \|\nabla z^1 \cdot n\|_{0,\Gamma} + (\omega_2 \mu_2)^{\frac{1}{2}} \|\nabla z^2 \cdot n\|_{0,\Gamma}) h^{\frac{1}{2}} \gamma^{\frac{1}{2}} \|[\![u - u_h]\!]\|_{0,\Gamma} \\ &\qquad \lesssim ((\omega_1 \mu_1)^{\frac{1}{2}} \|z^1\|_{2,\Omega_1} + (\omega_2 \mu_2)^{\frac{1}{2}} \|z^2\|_{2,\Omega_2}) h^{\frac{1}{2}} |\!|\!| u - u_h |\!|\!|. \end{aligned}$$

The consistency of Lemma 5.2.1 gives

$$A_h(u - u_h, z) = A_h(u - u_h, z - \mathcal{I}_h z) + J_h(u_h, \mathcal{I}_h z).$$

As a consequence from $z^1|_\Gamma = z^2|_\Gamma$ we have $\mathcal{I}_h z^1|_\Gamma = \mathcal{I}_h z^2|_\Gamma$ then $[\![z - \mathcal{I}_h z]\!]|_\Gamma = 0$. Using this result and (5.3), we obtain

$$\begin{aligned}
&A_h(u-u_h, z-\mathcal{I}_h z) \\
&\qquad = \sum_{i=1}^{2} (\nabla(u^i - u_h^i), \mu_i \nabla(z^i - \mathcal{I}_h z^i))_{\Omega_i} + \langle \{\mu \nabla(z - \mathcal{I}_h z)\cdot n\}, [\![u-u_h]\!]\rangle_\Gamma \\
&\qquad \lesssim (((1+\omega_1)\mu_1)^{\frac{1}{2}} |z^1|_{2,\Omega_1} + ((1+\omega_2)\mu_2)^{\frac{1}{2}} |z^2|_{2,\Omega_2}) h |\!|\!| u-u_h |\!|\!|.
\end{aligned}$$

Similarly as in the proof of Proposition 3.2.1 and using the proof of Theorem 5.2.2 we have

$$J_h(u_h, \mathcal{I}_h z) \lesssim \|u - \mathcal{I}_h u\|_* h(\mu_1^{\frac{1}{2}} |z^1|_{2,\Omega_1} + \mu_2^{\frac{1}{2}} |z^2|_{2,\Omega_2}).$$

Then using Corollary 5.2.1, (5.13) and (5.3) we obtain

$$\begin{aligned}
\|u-u_h\|_{0,\Omega}^2 &\lesssim (\mu_1^{\frac{1}{2}} \|z^1\|_{2,\Omega_1} + \mu_2^{\frac{1}{2}} \|z^2\|_{2,\Omega_2})((h + h^{\frac{1}{2}}) |\!|\!| u-u_h |\!|\!| + h\|u - \mathcal{I}_h u\|_*) \\
&\lesssim C_\mu (\mu_1^{\frac{1}{2}} \|z^1\|_{2,\Omega_1} + \mu_2^{\frac{1}{2}} \|z^2\|_{2,\Omega_2}) h^{k+\frac{1}{2}} |u|_{k+1,\Omega}.
\end{aligned}$$

We conclude by applying the regularity estimate (5.15). □

## 5.3 Compressible elasticity

The compressible elasticity problem with discontinuous material parameters is considered as

$$\begin{aligned}
-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}^i) &= \boldsymbol{f} && \text{in } \Omega_i\ ,\quad i=1,2, \\
\boldsymbol{u}^i &= 0 && \text{on } \partial\Omega\cap\Omega_i\ ,\quad i=1,2, \\
[\![\boldsymbol{u}]\!] &= 0 && \text{on } \Gamma, \\
[\![\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}]\!] &= 0 && \text{on } \Gamma,
\end{aligned} \tag{5.16}$$

with $\boldsymbol{f} \in [L^2(\Omega)]^2$, the stress tensor is expressed as

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda(\nabla\cdot\boldsymbol{u})\mathbb{I}_{2\times 2}.$$

In a subdomain $\Omega_i$ the Lamé coefficients are denoted as $\mu_i$ and $\lambda_i$, we assume that $\mu_1 \leq \mu_2$. The finite element formulation of this problem is obtained similarly as in Section 4.3.1: find $\boldsymbol{u}_h \in W_h^k$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = L_h(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in W_h^k, \tag{5.17}$$

with $W_h^k = [V_h^k]^2$. The linear forms $A_h$ and $L_h$ are defined as

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \boldsymbol{u}_h),$$
$$L_h(v_h) = \sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i}.$$

The term $J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = J_h^1(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + J_h^2(\boldsymbol{u}_h^2, \boldsymbol{v}_h^2)$ is the ghost penalty

$$J_h^i(\boldsymbol{u}_h^i, \boldsymbol{v}_h^i) = \gamma_g \sum_{F \in \mathcal{F}_G^i} \sum_{l=1}^{k} \langle \mu_i h^{2l-1} [\![ D_{n_F}^l \boldsymbol{u}_h^i ]\!]_F, [\![ D_{n_F}^l \boldsymbol{v}_h^i ]\!]_F \rangle_F.$$

The bilinear forms $a$ and $b$ are defined as

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{i=1}^{2} \left[ (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}_h^i), \boldsymbol{\varepsilon}(\boldsymbol{v}_h^i))_{\Omega_i} + (\lambda_i \nabla \cdot \boldsymbol{u}_h^i, \nabla \cdot \boldsymbol{v}_h^i)_{\Omega_i} \right],$$
$$b(\boldsymbol{u}_h, \boldsymbol{v}_h) = \langle \{2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h) \cdot \boldsymbol{n}\}, [\![ \boldsymbol{v}_h ]\!] \rangle_\Gamma + \langle \{\lambda \nabla \cdot \boldsymbol{u}_h\}, [\![ \boldsymbol{v}_h \cdot \boldsymbol{n} ]\!] \rangle_\Gamma.$$

### 5.3.1 Inf-sup stability

As specified at the beginning of this chapter the interface $\Gamma$ is considered as plane. Any function $\boldsymbol{z} = (z_1, z_2) \in W_h^k$ is expressed in the two dimensional generic frame $(x, y)$ and $z_1, z_2 \in V_h^k$. The function $\boldsymbol{z}$ can also be decomposed such that $\boldsymbol{z}^1 \in W_1^k$ and $\boldsymbol{z}^2 \in W_2^k$ with $W_1^k = [V_1^k]^2$ and $W_2^k = [V_2^k]^2$. The interface $\Gamma$ is parallel to the $x$-axis then for $\boldsymbol{\tau}$ and $\boldsymbol{n}$ respectively the tangent and normal unit vectors to the plane interface $\Gamma$ we have $z_1 = \boldsymbol{z} \cdot \boldsymbol{\tau}$ and $z_2 = \boldsymbol{z} \cdot \boldsymbol{n}$. We introduce the function $\boldsymbol{v}_j^1$ such that $\boldsymbol{v}_j^1 = (\alpha_1 v_1^1, \alpha_2 v_2^1)^T$. We define $v_1^1 = \nu_1 \chi_j$ and $v_2^1 = \nu_2 \chi_j$ with $\nu_1, \nu_2 \in \mathbb{R}$ and $\chi_j$ as defined in (5.5). In order to be able to use Lemma 3.1.1, the function $\boldsymbol{v}_j^1$ has the properties

$$\operatorname{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \frac{\partial v_1^1}{\partial y} \, \mathrm{d}s = h^{-1} \overline{[\![ u_1 ]\!]}^{\Gamma_j}, \qquad \operatorname{meas}(\Gamma_j)^{-1} \int_{\Gamma_j} \frac{\partial v_2^1}{\partial y} \, \mathrm{d}s = h^{-1} \overline{[\![ u_2 ]\!]}^{\Gamma_j}, \quad (5.18)$$

with $\boldsymbol{u}_h^i = (u_1^i, u_2^i)^T$. Using Lemma 3.1.1 it is straightforward to show

$$\|\nabla v_1^1\|_{0,P_j^1} \lesssim h^{-\frac{1}{2}} \| \overline{[\![ \boldsymbol{u}_h ]\!]}^{\Gamma_j} \cdot \boldsymbol{\tau} \|_{0,\Gamma_j}, \qquad \|\nabla v_2^1\|_{0,P_j^1} \lesssim h^{-\frac{1}{2}} \| \overline{[\![ \boldsymbol{u}_h ]\!]}^{\Gamma_j} \cdot \boldsymbol{n} \|_{0,\Gamma_j}. \quad (5.19)$$

Let the norms

$$|\!|\!| \boldsymbol{w} |\!|\!|^2 = \sum_{i=1}^{2} \left[ \mu_i \|\nabla \boldsymbol{w}^i\|_{0,\Omega_i}^2 + \lambda_i \|\nabla \cdot \boldsymbol{w}^i\|_{0,\Omega_i}^2 \right] + \gamma \| [\![ \boldsymbol{w} ]\!] \|_{0,\Gamma}^2 + J_h(\boldsymbol{w}, \boldsymbol{w}),$$
$$\|\boldsymbol{w}\|_*^2 = |\!|\!| \boldsymbol{w} |\!|\!|^2 + \sum_{i=1}^{2} \left[ \mu_i h \|\nabla \boldsymbol{w}^i \cdot \boldsymbol{n}\|_{0,\Gamma}^2 + \mu_i h \|\nabla \cdot \boldsymbol{w}^i\|_{0,\Gamma}^2 \right].$$

First we give two technical Lemmas, proof are provided in appendix D.

**Lemma 5.3.1.** *There exists $C > 0$ independent of $h$, $\mu_i$ and $\lambda_i$ $(i = 1, 2)$, but not of the mesh geometry, $\forall \boldsymbol{u}_h \in W_h^k$, for $\boldsymbol{v}_j^1 \in W_1^1$ as defined above and $\forall \epsilon, \alpha_1, \alpha_2 \in \mathbb{R}_+^*$, the following inequality holds*

$$\langle \omega_1 \lambda_1 \nabla \cdot \boldsymbol{v}_j^1, [\![\boldsymbol{u}_h]\!] \cdot \boldsymbol{n} \rangle_{\Gamma_j} \gtrsim \alpha_2 \gamma \Big(1 - \frac{C\alpha_2}{2\epsilon}\Big) \frac{\lambda_1}{\mu_1} \| \overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j} \cdot \boldsymbol{n} \|_{0,\Gamma_j}^2 - \gamma \frac{C\alpha_1^2}{2\epsilon} \frac{\lambda_1}{\mu_1} \| \overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j} \cdot \boldsymbol{\tau} \|_{0,\Gamma_j}^2 - 2\epsilon \omega_1 \lambda_1 \|\nabla \boldsymbol{u}_h^1\|_{0,P_j^1}^2 - 2\epsilon \omega_2 \frac{\lambda_1 \mu_2}{\mu_1 \lambda_2} \lambda_2 \|\nabla \boldsymbol{u}_h^2\|_{0,P_j^2}^2 .$$

**Lemma 5.3.2.** *There exists $C > 0$ independent of $h$, $\mu_i$ and $\lambda_i$ $(i = 1, 2)$, but not of the mesh geometry, $\forall \boldsymbol{u}_h \in W_h^k$, for $\boldsymbol{v}_j^1 \in W_1^1$ as defined above and $\forall \epsilon, \alpha_1, \alpha_2 \in \mathbb{R}_+^*$, the following inequality holds*

$$\langle 2\omega_1 \mu_1 \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1) \cdot \boldsymbol{n}, [\![\boldsymbol{u}_h]\!] \rangle_{\Gamma_j} \geq \alpha_2 \gamma \Big(2 - \frac{3C\alpha_2}{2\epsilon}\Big) \| \overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j} \cdot \boldsymbol{n} \|_{0,\Gamma_j}^2 - 3\epsilon \omega_1 \mu_1 \|\nabla \boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \alpha_1 \gamma \Big(1 - \frac{C\alpha_1}{2\epsilon}\Big) \| \overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j} \cdot \boldsymbol{\tau} \|_{0,\Gamma_j}^2 - 3\epsilon \omega_2 \mu_2 \|\nabla \boldsymbol{u}_h^2\|_{0,P_j^2}^2 .$$

**Lemma 5.3.3.** *For $\boldsymbol{u}_h, \boldsymbol{v}_h \in W_h^k$ with $\boldsymbol{v}_h = \boldsymbol{u}_h + \sum_{j=1}^{N_p} (\boldsymbol{v}_j^1, 0)$, $\boldsymbol{v}_j^1$ defined by (5.18), there exist a positive constant $\beta_0$ such that*

$$\beta_0 |\!|\!| \boldsymbol{u}_h |\!|\!|^2 \leq A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + J_h(\boldsymbol{u}_h, \boldsymbol{v}_h).$$

*Proof.* Substituting the function $\boldsymbol{v}_h$ we get

$$(A_h + J_h)(\boldsymbol{u}_h, \boldsymbol{v}_h) = (A_h + J_h)(\boldsymbol{u}_h, \boldsymbol{u}_h) + \sum_{j=1}^{N_p} [A_h(\boldsymbol{u}_h, \boldsymbol{v}_j^1) + J_h(\boldsymbol{u}_h, \boldsymbol{v}_j^1)],$$

with

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{v}_j^1) = {} & (2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1))_{P_j^1 \cap \Omega_1} + (\lambda_1 \nabla \cdot \boldsymbol{u}_h^1, \nabla \cdot \boldsymbol{v}_j^1)_{P_j^1 \cap \Omega_1} \\ & - \langle \{2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h) \cdot \boldsymbol{n}\}, \boldsymbol{v}_j^1 \rangle_{\Gamma_j} - \langle \{\lambda \nabla \cdot \boldsymbol{u}_h\}, \boldsymbol{v}_j^1 \cdot \boldsymbol{n} \rangle_{\Gamma_j} \\ & + \langle 2\omega_1 \mu_1 \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1) \cdot \boldsymbol{n}, [\![\boldsymbol{u}_h]\!] \rangle_{\Gamma_j} + \langle \omega_1 \lambda_1 \nabla \cdot \boldsymbol{v}_j^1, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!] \rangle_{\Gamma_j} . \end{aligned}$$

Using the estimate (5.9) and Korn inequality (4.13) we obtain

$$\begin{aligned} (A_h + J_h)(\boldsymbol{u}_h, \boldsymbol{u}_h) &= \sum_{i=1}^{2} \big[ 2\mu_i \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^i)\|_{0,\Omega_i}^2 + \lambda_i \|\nabla \cdot \boldsymbol{u}_h^i\|_{0,\Omega_i}^2 \big] + J_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \\ &\geq \sum_{i=1}^{2} \big[ 2C\mu_i \|\nabla \boldsymbol{u}_h^i\|_{0,\Omega_i^*}^2 + \lambda_i \|\nabla \cdot \boldsymbol{u}_h^i\|_{0,\Omega_i}^2 \big]. \end{aligned}$$

Using the Cauchy-Schwarz inequality, (5.9) and (5.19) we get

$$
\begin{aligned}
&(2\mu_1\boldsymbol{\varepsilon}(\boldsymbol{u}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1))_{0,P_j^1\cap\Omega_1} + J_h(\boldsymbol{u}_h, \boldsymbol{v}_j^1) \\
&\qquad \leq \epsilon\mu_1\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \frac{\mu_1}{\epsilon}\|\nabla\boldsymbol{v}_j^1\|_{0,P_j^1}^2 \\
&\qquad \leq \epsilon\mu_1\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \gamma\Big(1+\frac{\mu_1}{\mu_2}\Big)\Big(\frac{C\alpha_1^2}{\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2 + \frac{C\alpha_2^2}{\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}^2\Big).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&(\lambda_1\nabla\cdot\boldsymbol{u}_h^1, \nabla\cdot\boldsymbol{v}_j^1)_{P_j^1\cap\Omega_1} \\
&\leq \epsilon\lambda_1\|\nabla\cdot\boldsymbol{u}_h^1\|_{0,P_j^1\cap\Omega_1}^2 + \Big(1+\frac{\mu_1}{\mu_2}\Big)\frac{\lambda_1}{\mu_1}\Big(\frac{C\alpha_1^2}{2\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2 + \frac{C\alpha_2^2}{2\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}^2\Big).
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, (3.19) and (5.19) we have

$$
\begin{aligned}
\langle\{2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\cdot\boldsymbol{n}\}, \boldsymbol{v}_j^1\rangle_{\Gamma_j} &\leq \epsilon\omega_1\mu_1\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \epsilon\omega_2\mu_2\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2 \\
&\quad + \frac{C\alpha_1^2}{\epsilon}\gamma\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2 + \frac{C\alpha_2^2}{\epsilon}\gamma\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}^2.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\langle\{\lambda\nabla\cdot\boldsymbol{u}_h\}, \boldsymbol{v}_j^1\cdot\boldsymbol{n}\rangle_{\Gamma_j} &\leq \epsilon\omega_1\lambda_1\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \epsilon\omega_2\lambda_2\frac{\lambda_2\mu_1}{\mu_2\lambda_1}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2 \\
&\quad + \frac{C\alpha_1^2}{2\epsilon}\gamma\frac{\lambda_1}{\mu_1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2 + \frac{C\alpha_2^2}{2\epsilon}\gamma\frac{\lambda_1}{\mu_1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}^2.
\end{aligned}
$$

Using Lemmas 5.3.1, 5.3.2 and collecting all the terms we get

$$
\begin{aligned}
A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \geq &\sum_{i=1}^{2}\big[2C\mu_i\|\nabla\boldsymbol{u}_h^i\|_{0,\Omega_i^*}^2 + \lambda_i\|\nabla\cdot\boldsymbol{u}_h^i\|_{0,\Omega_i}^2\big] \\
&- \sum_{j=1}^{N_p}\big[\epsilon\mu_1\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \epsilon\lambda_1\|\nabla\cdot\boldsymbol{u}_h^1\|_{0,P_j^1\cap\Omega_1}^2\big] \\
&- \epsilon\omega_1(4\mu_1+3\lambda_1)\sum_{j=1}^{N_p}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 \\
&- \epsilon\omega_2\Big(4\mu_2+\lambda_2\Big(2\frac{\lambda_1\mu_2}{\mu_1\lambda_2}+\frac{\lambda_2\mu_1}{\lambda_1\mu_2}\Big)\Big)\sum_{j=1}^{N_p}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2 \\
&+ \alpha_1\Big(1-\frac{C\alpha_1}{\epsilon}\Big(\frac{5}{2}+\frac{3\lambda_1}{2\mu_1}+\Big(1+\frac{\lambda_1}{2\mu_1}\Big)\frac{\mu_1}{\mu_2}\Big)\Big)\gamma\sum_{j=1}^{N_p}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2 \\
&+ \alpha_2\Big(2+\frac{\lambda_1}{\mu_1}-\frac{C\alpha_2}{\epsilon}\Big(\frac{7}{2}+\frac{3\lambda_1}{2\mu_1}+\Big(1+\frac{\lambda_1}{2\mu_1}\Big)\frac{\mu_1}{\mu_2}\Big)\Big)\gamma\sum_{j=1}^{N_p}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}^2.
\end{aligned}
$$

Let us define the constants

$$
\begin{aligned}
C_a &= 2C\mu_1 - \epsilon(\mu_1 + \omega_1(4\mu_1 + 3\lambda_1)),\\
C_b &= 2C\mu_2 - \epsilon\omega_2\Big(4\mu_2 + \lambda_2\Big(2\frac{\lambda_1\mu_2}{\mu_1\lambda_2} + \frac{\lambda_2\mu_1}{\lambda_1\mu_2}\Big)\Big),\\
C_c &= \alpha_1\Big(1 - \frac{C\alpha_1}{\epsilon}\Big(\frac{5}{2} + \frac{3\lambda_1}{2\mu_1} + \Big(1 + \frac{\lambda_1}{2\mu_1}\Big)\frac{\mu_1}{\mu_2}\Big)\Big),\\
C_d &= \alpha_2\Big(2 + \frac{\lambda_1}{\mu_1} - \frac{C\alpha_2}{\epsilon}\Big(\frac{7}{2} + \frac{3\lambda_1}{2\mu_1} + \Big(1 + \frac{\lambda_1}{2\mu_1}\Big)\frac{\mu_1}{\mu_2}\Big)\Big).
\end{aligned}
$$

Using (5.10) and (5.9) we obtain

$$
\begin{aligned}
A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \geq{}& (C_a - C(C_c + C_d)\omega_1\mu_1)(\|\nabla \boldsymbol{u}_h^1\|^2_{0,\Omega_1} + J_h^1(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1)) + \lambda_1(1-\epsilon)\|\nabla\cdot \boldsymbol{u}_h^1\|^2_{0,\Omega_1}\\
&+ (C_b - C(C_c + C_d)\omega_2\mu_2)(\|\nabla \boldsymbol{u}_h^2\|^2_{0,\Omega_2} + J_h^2(\boldsymbol{u}_h^2, \boldsymbol{u}_h^2)) + \lambda_2\|\nabla\cdot \boldsymbol{u}_h^2\|^2_{0,\Omega_2}\\
&+ \frac{C_c\gamma}{2}\sum_{j=1}^{N_p}\|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{\tau}\|^2_{0,\Gamma_j} + \frac{C_d\gamma}{2}\sum_{j=1}^{N_p}\|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\|^2_{0,\Gamma_j}.
\end{aligned}
$$

Following the proof of Lemma 4.3.3 with $h_1 = h_2 = h$, the parameters $\epsilon$, $\alpha_1$ and $\alpha_2$ can be chosen in such a way that all the terms of this expression are positive. □

**Theorem 5.3.1.** *There exists a positive constant $\beta$ such that for all functions $\boldsymbol{u}_h \in W_h^k$ and for $h < h_0$, the following inequality holds*

$$
\beta|||\boldsymbol{u}_h||| \leq \sup_{\boldsymbol{v}_h\in W_h^k}\frac{A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + J_h(\boldsymbol{u}_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||}.
$$

*Proof.* The proof is obtained by using similar arguments as in the proofs of Theorems 4.3.1 and 5.2.1. □

### 5.3.2 A priori error estimate

**Lemma 5.3.4.** *If $\boldsymbol{u} \in [H^2_\partial(\Omega_1)]^2 \times [H^2_\partial(\Omega_2)]^2$ is the solution of* (5.16) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (5.17) *the following property holds*

$$
A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in W_h^k.
$$

*Proof.* The proof is done using similar arguments as for Lemma 4.3.4 and using that $A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = L_h(\boldsymbol{v}_h), \forall \boldsymbol{v}_h \in W_h^k$. □

**Lemma 5.3.5.** *Let $\boldsymbol{w} \in [H^2_\partial(\Omega_1)]^2 \times [H^2_\partial(\Omega_2)]^2 + W_h^k$ and $\boldsymbol{v}_h \in W_h^k$, there exists a positive constant $M$ such that*

$$
A_h(\boldsymbol{w}, \boldsymbol{v}_h) \leq M\|\boldsymbol{w}\|_*|||\boldsymbol{v}_h|||.
$$

*Proof.* The proof follows the same arguments as the proof of Lemma 4.3.5 with $h_1 = h_2 = h$ and (5.9) to handle the terms over $\Omega_1^*$ and $\Omega_2^*$. □

**Theorem 5.3.2.** *If $\boldsymbol{u} \in [H_\partial^{k+1}(\Omega_1)]^2 \times [H_\partial^{k+1}(\Omega_2)]^2$ is the solution of* (5.16) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (5.17) *then there holds*

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \lesssim \inf_{\boldsymbol{w}_h \in W_h^k} \|\boldsymbol{u} - \boldsymbol{w}_h\|_*.$$

*Proof.* Same as Theorem 3.3.2 using Lemma 5.3.4, Theorem 5.3.1 and Lemma 5.3.5. □

**Corollary 5.3.1.** *If $\boldsymbol{u} \in [H_\partial^{k+1}(\Omega_1)]^2 \times [H_\partial^{k+1}(\Omega_2)]^2$ is the solution of* (5.16) *and $\boldsymbol{u}_h \in W_h^k$ the solution of* (5.17)*, then there holds*

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq C_{\mu\lambda} h^k |\boldsymbol{u}|_{k+1,\Omega},$$

*where $C_{\mu\lambda}$ is a positive constant that depends on $\mu$, $\lambda$ and the mesh geometry.*

*Proof.* Combining arguments from the proofs of corollaries 3.3.1, 4.3.1 and using (5.1), (5.3) we obtain the estimate

$$\|\boldsymbol{u} - \mathcal{I}_h \boldsymbol{u}\|_* \lesssim (\mu_1^{\frac{1}{2}} + \lambda_1^{\frac{1}{2}}) h^k |\boldsymbol{u}^1|_{k+1,\Omega_1} + (\mu_2^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}}) h^k |\boldsymbol{u}^2|_{k+1,\Omega_2}. \tag{5.20}$$

Then we use Theorem 5.3.2 with $\boldsymbol{w}_h = \mathcal{I}_h \boldsymbol{u}$ to conclude. □

**Proposition 5.3.1.** *Let $\boldsymbol{u} \in [H_\partial^{k+1}(\Omega_1)]^2 \times [H_\partial^{k+1}(\Omega_2)]^2$ be the solution of* (5.16) *and $\boldsymbol{u}_h$ the solution of* (5.17)*, then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq C'_{\mu\lambda} h^{k+\frac{1}{2}} |\boldsymbol{u}|_{k+1,\Omega},$$

*where $C'_{\mu\lambda}$ is a positive constant that depends on $\mu$, $\lambda$ and the mesh geometry.*

*Proof.* Let $\boldsymbol{z}$ satisfy the adjoint problem

$$\begin{aligned} -2\mu_i \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{z}^i) - \lambda_i \nabla(\nabla \cdot \boldsymbol{z}^i) &= \boldsymbol{u}^i - \boldsymbol{u}_h^i && \text{in } \Omega_i \ , \quad i = 1,2, \\ \boldsymbol{z}^i &= 0 && \text{on } \partial\Omega \cap \Omega_i \ , \quad i = 1,2, \\ [\![\boldsymbol{z}]\!] &= 0 && \text{on } \Gamma, \\ [\![\boldsymbol{\sigma}(\boldsymbol{z}) \cdot \boldsymbol{n}]\!] &= 0 && \text{on } \Gamma, \end{aligned}$$

We assume the following elliptic regularity [85] for this problem

$$\mu_1 \|\boldsymbol{z}^1\|_{2,\Omega_1} + \mu_2 \|\boldsymbol{z}^2\|_{2,\Omega_2} \lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}. \tag{5.21}$$

Similarly as in the proof of Proposition 4.3.1, the $L^2$-error can be written as

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}^2 = A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}) &- 2\langle \{2\mu \boldsymbol{\varepsilon}(\boldsymbol{z}) \cdot \boldsymbol{n}\}, [\![\boldsymbol{u} - \boldsymbol{u}_h]\!] \rangle_\Gamma \\ &- 2\langle \{\lambda \nabla \cdot \boldsymbol{z}\}, [\![(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}]\!] \rangle_\Gamma. \end{aligned}$$

The orthogonality relation of Lemma 5.3.4 gives

$$A_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z}) = A_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z}) + J_h(\boldsymbol{u}_h,\mathcal{I}_h\boldsymbol{z}).$$

Similarly as in the proof of Proposition 5.2.1

$$J_h(\boldsymbol{u}_h,\mathcal{I}_h\boldsymbol{z}) \lesssim \|\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u}\|_* h(\mu_1^{\frac{1}{2}}|\boldsymbol{z}^1|_{2,\Omega_1} + \mu_2^{\frac{1}{2}}|\boldsymbol{z}^2|_{2,\Omega_2}).$$

Using $[\![\boldsymbol{z}-\mathcal{I}_h\boldsymbol{z}]\!]|_\Gamma \equiv 0$ we deduce that

$$\begin{aligned}
&A_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z}-i^1_{\mathrm{SZ}}\boldsymbol{z})\\
&\quad= \sum_{i=1}^{2}\left[(2\mu_i\boldsymbol{\varepsilon}(\boldsymbol{u}^i-\boldsymbol{u}^i_h),\boldsymbol{\varepsilon}(\boldsymbol{z}^i-i^1_{\mathrm{SZ}}\boldsymbol{z}^i))_{\Omega_i} + (\lambda_i\nabla\cdot(\boldsymbol{u}^i-\boldsymbol{u}^i_h),\nabla\cdot(\boldsymbol{z}^i-i^1_{\mathrm{SZ}}\boldsymbol{z}^i))_{\Omega_i}\right]\\
&\qquad + \langle 2\{\mu\boldsymbol{\varepsilon}(\boldsymbol{z}-i^1_{\mathrm{SZ}}\boldsymbol{z})\cdot\boldsymbol{n}\},[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\rangle_\Gamma + \langle\{\lambda\nabla\cdot(\boldsymbol{z}-i^1_{\mathrm{SZ}}\boldsymbol{z}\},[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\rangle_\Gamma\\
&\quad\lesssim ((1+\omega_1)^{\frac{1}{2}}(\mu_1^{\frac{1}{2}}+\lambda_1^{\frac{1}{2}})h|\boldsymbol{z}^1|_{2,\Omega_1}\\
&\qquad\qquad + (1+\omega_2)^{\frac{1}{2}}(\mu_2^{\frac{1}{2}}+\lambda_2^{\frac{1}{2}})h|\boldsymbol{z}^2|_{2,\Omega_2})(1+\lambda_1^{\frac{1}{2}}\mu_1^{-\frac{1}{2}}+\lambda_2^{\frac{1}{2}}\mu_2^{-\frac{1}{2}})|\!|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!|\!|.
\end{aligned}$$

The global trace inequalities $\|\boldsymbol{\varepsilon}(\boldsymbol{z}^i)\cdot\boldsymbol{n}\|_{0,\Gamma} \lesssim \|\boldsymbol{z}^i\|_{2,\Omega_i}$ and $\|\nabla\cdot\boldsymbol{z}^i\|_{0,\Gamma} \lesssim \|\boldsymbol{z}^i\|_{2,\Omega_i}$, lead to

$$\begin{aligned}
&|\langle 2\{\mu\boldsymbol{\varepsilon}(\boldsymbol{z})\cdot\boldsymbol{n}\},[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\rangle_\Gamma| + |\langle\{\lambda\nabla\cdot\boldsymbol{z}\},[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\rangle_\Gamma|\\
&\qquad\lesssim h^{\frac{1}{2}}(\omega_1^{\frac{1}{2}}(\mu_1^{\frac{1}{2}}+\lambda_1^{\frac{1}{2}})\|\boldsymbol{z}^1\|_{2,\Omega_1}\\
&\qquad\qquad + \omega_2^{\frac{1}{2}}(\mu_2^{\frac{1}{2}}+\lambda_2^{\frac{1}{2}})\|\boldsymbol{z}^2\|_{2,\Omega_2})(1+\lambda_1^{\frac{1}{2}}\mu_1^{-\frac{1}{2}}+\lambda_2^{\frac{1}{2}}\mu_2^{-\frac{1}{2}})|\!|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!|\!|.
\end{aligned}$$

Collecting the estimates and applying Corollary 5.3.1 and (5.20) the proof follows by using the regularity estimates (5.21). □

**Remark 5.3.1.** *The order of the constants $C_{\mu\lambda}$ and $C'_{\mu\lambda}$ are the same as in the previous chapter for the fitted domain decomposition case.*

## 5.4 Incompressible elasticity

The incompressible elasticity problem with discontinuous parameters considered is expressed as

$$\begin{aligned}
-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}^i,p^i) &= \boldsymbol{f} && \text{in } \Omega_i\ ,\quad i=1,2,\\
-\nabla\cdot\boldsymbol{u}^i &= 0 && \text{in } \Omega_i\ ,\quad i=1,2,\\
\boldsymbol{u}^i &= 0 && \text{on } \partial\Omega\cap\Omega_i\ ,\quad i=1,2,\\
[\![\boldsymbol{u}]\!] &= 0 && \text{on } \Gamma,\\
[\![\boldsymbol{\sigma}(\boldsymbol{u},p)\cdot\boldsymbol{n}]\!] &= 0 && \text{on } \Gamma,
\end{aligned} \tag{5.22}$$

with $\boldsymbol{f} \in [L^2(\Omega)]^2$ and $\int_{\Omega_i} p^i \, \mathrm{d}x = 0$ for $i = 1, 2$, the stress tensor is expressed as

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbb{I}_{2\times 2}.$$

In a subdomain $\Omega_i$ the viscosity is denoted as $\mu_i$ (and $\mu_1 \leq \mu_2$). The finite element formulation of this problem is obtained similarly as in Section 4.4.1: find $\boldsymbol{u}_h \in W_h^k$ and $p_h \in Q_h^k$ such that

$$A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] + J_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = L_h(\boldsymbol{v}_h, q_h) \qquad \forall \boldsymbol{v}_h \in W_h^k \times Q_h^k, \quad (5.23)$$

with $Q_h^k = Q_1^k \times Q_2^k$ such that

$$Q_i^k = \{q_h \in Q_i^* : q_h|_K \in \mathbb{P}_k(K) \ \ \forall K \in \mathcal{T}_h\}, \quad k \geq 1,$$

and $Q_i^* = \{q \in L^2(\Omega_i^*), \ \int_{\Omega_i} q \, \mathrm{d}x = 0\}$. The linear forms $A_h$ and $L_h$ are expressed as

$$\begin{aligned} A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] &= a[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] - b(\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) + b(\boldsymbol{v}_h, \boldsymbol{u}_h, q_h) \\ &\quad + S_h(\boldsymbol{u}_h, p_h, q_h), \\ L_h(v_h) &= \sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{v}^i)_{\Omega_i} + \Lambda_h(\boldsymbol{f}, q_h). \end{aligned}$$

The bilinear forms $a$, $b$ are such that

$$\begin{aligned} a[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] &= \sum_{i=1}^{2} \left[ (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}_h^i), \boldsymbol{\varepsilon}(\boldsymbol{v}_h^i))_{\Omega_i} - (p_h^i, \nabla \cdot \boldsymbol{v}_h^i)_{\Omega_i} + (\nabla \cdot \boldsymbol{u}_h^i, q_h^i)_{\Omega_i} \right], \\ b(\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) &= \langle 2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1) \cdot \boldsymbol{n}, [\![\boldsymbol{v}_h]\!] \rangle_\Gamma - \langle p_h^1, [\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!] \rangle_\Gamma. \end{aligned}$$

As we work with equal order interpolation we stabilise the problem with $S_h$ and $\Lambda_h$

$$\begin{aligned} S_h(\boldsymbol{u}_h, p_h, q_h) &= \sum_{i=1}^{2} \frac{\gamma}{\mu_i} \sum_{K \in \Omega_i^*} \int_K h^2 (-2\mu_i \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_h^i) + \nabla p_h^i) \nabla q_h^i \, \mathrm{d}x, \\ \Lambda_h(\boldsymbol{f}, q_h) &= \sum_{i=1}^{2} \frac{\gamma}{\mu_i} \sum_{K \in \Omega_i^*} \int_K h^2 \boldsymbol{f} \nabla q_h^i \, \mathrm{d}x. \end{aligned}$$

The ghost penalty term $J_h$ is defined such that $J_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + I_h(p_h, q_h)$ with $J_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = J_h^1(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + J_h^2(\boldsymbol{u}_h^2, \boldsymbol{v}_h^2)$ and $I_h(p_h, q_h) = I_h^1(p_h^1, q_h^1) +$

$I_h^2(p_h^2, q_h^2)$ and

$$J_h^i(\boldsymbol{u}_h^i, \boldsymbol{v}_h^i) = \gamma_g \mu_i \sum_{F \in \mathcal{F}_G^i} \sum_{l=1}^{k} h^{2l-1} \langle [\![ D_{n_F}^l \boldsymbol{u}_h^i ]\!]_F, [\![ D_{n_F}^l \boldsymbol{v}_h^i ]\!]_F \rangle_F,$$

$$I_h^i(p_h^i, q_h^i) = \frac{\gamma_g}{\mu_i} \sum_{F \in \mathcal{F}_G^i} \sum_{l=1}^{p} h^{2l+1} \langle [\![ D_{n_F}^l p_h^i ]\!]_F, [\![ D_{n_F}^l q_h^i ]\!]_F \rangle_F.$$

Note that as in Section 4.4 the weights are considered as

$$\omega_1 = 1, \qquad \omega_2 = 0. \tag{5.24}$$

The following estimate is proven in [89]

$$\mu_i^{-1} h^2 \|\nabla q_h^i\|_{0,\Omega_i^*}^2 \lesssim h^2 \mu_i^{-1} \|\nabla q_h^i\|_{0,\Omega_i}^2 + I_h^i(q_h^i, q_h^i) \lesssim \mu_i^{-1} h^2 \|\nabla q_h^i\|_{0,\Omega_i^*}^2. \tag{5.25}$$

### 5.4.1 Inf-sup stability

Let the norm

$$|\!|\!|(\boldsymbol{w}, \varrho)|\!|\!|^2 = \sum_{i=1}^{2} (\mu_i \|\nabla \boldsymbol{w}^i\|_{0,\Omega_i}^2 + h^2 \mu_i^{-1} \|\nabla \varrho^i\|_{0,\Omega_i}^2) + \mu_1 h^{-1} \|[\![\boldsymbol{w}]\!]\|_{0,\Gamma}^2 + J_h[(\boldsymbol{w}, \varrho), (\boldsymbol{w}, \varrho)].$$

**Lemma 5.4.1.** *For $\boldsymbol{u}_h, \boldsymbol{v}_h \in W_h^k$ with $\boldsymbol{v}_h = \boldsymbol{u}_h + \sum_{j=1}^{N_p} (\boldsymbol{v}_j^1, 0)$, $\boldsymbol{v}_j^1$ defined by (5.18), and $q_h = p_h$, there exist a positive constant $\beta_0$ such that the following inequality holds*

$$\beta_0 |\!|\!|(\boldsymbol{u}_h, p_h)|\!|\!|^2 \leq A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] + J_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)].$$

*Proof.* Applying the definition of $\boldsymbol{v}_h$ we obtain

$$\begin{aligned}(A_h + J_h)[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] &= (A_h + J_h)[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] \\ &\quad + \sum_{j=1}^{N_p} \left[ A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_j^1, 0)] + J_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_j^1, 0)] \right]\end{aligned}$$

with

$$\begin{aligned}A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_j^1, 0)] &= (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1))_{P_j^1 \cap \Omega_1} + (\nabla p_h^1, \boldsymbol{v}_j^1)_{P_j^1 \cap \Omega_1} \\ &\quad - \langle 2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_h^1) \cdot \boldsymbol{n}, \boldsymbol{v}_j^1 \rangle_{\Gamma_j} + \langle 2\mu_1 \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1) \cdot \boldsymbol{n}, [\![\boldsymbol{u}_h]\!] \rangle_{\Gamma_j}.\end{aligned}$$

Combining the proofs of Lemma 4.3.3 and Theorem 3.3.1 we obtain the following bounds

$$(A_h + J_h)[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] \geq \sum_{i=1}^{2} \left[ 2C(1-\epsilon')\mu_i \|\nabla \boldsymbol{u}_h^i\|_{0,\Omega_i^*}^2 + \frac{\gamma}{\mu_i} \Big( 1 - \frac{C\gamma}{4\epsilon'} \Big) h^2 \|\nabla p_h^i\|_{0,\Omega_i^*}^2 \right],$$

$$(\nabla p_h^1, \boldsymbol{v}_j^1)_{P_j^1} \leq \frac{\epsilon}{\mu_1} h^2 \|\nabla p_h^1\|^2_{0,P_j^1} + \mu_1 h^{-1}\Big(\frac{C\alpha_1^2}{2\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|^2_{0,\Gamma_j} + \frac{C\alpha_2^2}{2\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|^2_{0,\Gamma_j}\Big),$$

$$\begin{aligned}(2\mu_1\boldsymbol{\varepsilon}(\boldsymbol{u}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{v}_j^1))_{0,P_j^1\cap\Omega_1} &+ J_h(\boldsymbol{u}_h,\boldsymbol{v}_j)\\ &\leq \epsilon\mu_1\|\nabla \boldsymbol{u}_h^1\|^2_{0,P_j^1} + \mu_1 h^{-1}\Big(\frac{C\alpha_1^2}{\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|^2_{0,\Gamma_j} + \frac{C\alpha_2^2}{\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|^2_{0,\Gamma_j}\Big),\end{aligned}$$

$$\begin{aligned}\langle 2\mu_1\boldsymbol{\varepsilon}(\boldsymbol{u}_h^1)\cdot\boldsymbol{n}, \boldsymbol{v}_j^1\rangle_{\Gamma_j} \leq\ & \epsilon\mu_1\|\nabla \boldsymbol{u}_h^1\|^2_{0,P_j^1}\\ &+ \mu_1 h^{-1}\Big(\frac{C\alpha_1^2}{\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|^2_{0,\Gamma_j} + \frac{C\alpha_2^2}{\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|^2_{0,\Gamma_j}\Big).\end{aligned}$$

From Lemma 5.3.2 considering the new weights (5.24) and using the assumption $\mu_2 \geq \mu_1$

$$\begin{aligned}\langle 2\omega_1\mu_1\boldsymbol{\varepsilon}(\boldsymbol{v}_j^1)\cdot\boldsymbol{n}, [\![\boldsymbol{u}_h]\!]\rangle_{\Gamma_j} \geq\ & \alpha_2\Big(1-\frac{3C\alpha_2}{4\epsilon}\Big)\mu_1 h^{-1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|^2_{0,\Gamma_j}\\ &+ \alpha_1\Big(\frac{1}{2}-\frac{C\alpha_1}{4\epsilon}\Big)\mu_1 h^{-1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|^2_{0,\Gamma_j} - 3\epsilon\mu_1\|\nabla\boldsymbol{u}_h^1\|^2_{0,P_j^1}.\end{aligned}$$

Collecting all the terms, using (5.10), (5.25) and (5.9) we get

$$\begin{aligned}A_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)] \geq\ & (C_a - C(C_e+C_f))(\mu_1\|\nabla\boldsymbol{u}_h^1\|^2_{0,\Omega_1} + J_h^1(\boldsymbol{u}_h^1,\boldsymbol{u}_h^1))\\ &+ C_b h^2(\mu_1^{-1}\|\nabla p_h^1\|^2_{0,\Omega_1} + I_h^1(p_h^1,p_h^1))\\ &+ (C_c + \frac{\mu_1}{\mu_2}C(C_e+C_f))(\mu_2\|\nabla\boldsymbol{u}_h^2\|^2_{0,\Omega_2} + J_h^2(\boldsymbol{u}_h^2,\boldsymbol{u}_h^2))\\ &+ C_d h^2(\mu_2^{-1}\|\nabla p_h^2\|^2_{0,\Omega_2} + I_h^2(p_h^2,p_h^2))\\ &+ \frac{C_e}{2}\sum_{j=1}^{N_p}\mu_1 h^{-1}\|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{\tau}\|^2_{0,\Gamma_j} + \frac{C_f}{2}\sum_{j=1}^{N_p}\mu_1 h^{-1}\|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}\|^2_{0,\Gamma_j},\end{aligned}$$

with the constants

$$\begin{aligned}C_a &= 2C(1-\epsilon') - 5\epsilon,\\ C_b &= \gamma\Big(1-\frac{C\gamma}{4\epsilon'}-\frac{\epsilon}{\gamma}\Big),\\ C_c &= 2C(1-\epsilon'),\\ C_d &= \gamma\Big(1-\frac{C\gamma}{4\epsilon'}\Big),\\ C_e &= \alpha_1\Big(\frac{1}{2}-\frac{11C\alpha_1}{4\epsilon}\Big),\\ C_f &= \alpha_2\Big(1-\frac{13C\alpha_2}{4\epsilon}\Big).\end{aligned}$$

Following the proof of Lemma 4.3.3 with $h_1 = h_2 = h$, the parameters $\epsilon$, $\alpha_1$ and $\alpha_2$ can be chosen in such a way that all the terms of this expression are positive. □

**Theorem 5.4.1.** *There exists a positive constant $\beta$ such that for all functions $(\boldsymbol{u}_h,p_h) \in W_h^k \times Q_h^k$, the following inequality holds*

$$\beta|\!|\!|(\boldsymbol{u}_h,p_h)|\!|\!| \leq \sup_{(\boldsymbol{v}_h,q_h)\in W_h^k\times Q_h^k}\frac{A_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)] + J_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)]}{|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|}.$$

*Proof.* Using the definitions of $\boldsymbol{v}_h$ and $q_h$ from in the previous proof we have

$$|||(\boldsymbol{v}_h, q_h)|||^2 \lesssim |||(\boldsymbol{u}_h, p_h)|||^2 + \sum_{j=1}^{N_p} |||(\boldsymbol{v}_j^1, 0)|||^2,$$

with

$$|||(\boldsymbol{v}_j^1, 0)|||^2 = \mu_1 \|\nabla \boldsymbol{v}_j^1\|^2_{0,P_j^1\cap\Omega_1} + \mu_1 h^{-1} \|\boldsymbol{v}_j^1\|^2_{0,\Gamma_j} + J_h(\boldsymbol{v}_j, \boldsymbol{v}_j)$$

we get $|||(\boldsymbol{v}_h, q_h)||| \lesssim |||(\boldsymbol{u}_h, p_h)|||$ using similar arguments as in the proof of Theorem 5.3.1. □

### 5.4.2 A priori error estimate

**Lemma 5.4.2.** *If* $(\boldsymbol{u}, p) \in ([H^2_\partial(\Omega_1)]^2 \times [H^2_\partial(\Omega_2)]^2) \times H^1(\Omega)$ *is the solution of* (5.22) *and* $(\boldsymbol{u}_h, p_h) \in W_h^k \times Q^{ph}$ *the solution of* (5.23) *the following property holds*

$$A_h[(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}_h, q_h)] - J_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = 0, \qquad \forall (\boldsymbol{v}_h, q_h) \in W_h^k \times Q_h^k.$$

*Proof.* The proof is done using similar arguments as for Lemma 4.4.2 and using that $A_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] + J_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = L_h(\boldsymbol{v}_h), \forall \boldsymbol{v}_h \in W_h^k$. □

Similarly as in Section 4.4.3 we introduce an interpolant such that $\mathcal{I}_\vartheta \boldsymbol{v} = (\mathcal{I}_h^1 \boldsymbol{v}^1, \mathcal{I}_\vartheta^2 \boldsymbol{v}^2)$ and $\mathcal{I}_\vartheta^2 \boldsymbol{v}^2 = \mathcal{I}_h^2 \boldsymbol{v}^2 + \sum_{j=1}^{N_p} \vartheta_j \boldsymbol{\chi}_j^2$, with $\vartheta_j \in \mathbb{R}$ and $\boldsymbol{\chi}_j^2 = (\chi_j^2, \chi_j^2)^T \in W_2^1$ such that for each node $x_i \in \mathcal{T}_h$ we have

$$\chi_j^2(x_i) = \begin{cases} 0 & \text{for} \quad x_i \notin I_j^2 \\ 1 & \text{for} \quad x_i \in I_j^2, \end{cases} \tag{5.26}$$

$\vartheta_j$ is chosen such that

$$\int_{\Gamma_j} (\boldsymbol{u}^2 - \mathcal{I}_\vartheta^2 \boldsymbol{u}^2) \cdot \boldsymbol{n} \, \mathrm{d}s = 0, \tag{5.27}$$

for $j = 1, \dots, N_p$. Then we can write

$$\vartheta_j = \frac{\int_{\Gamma_j} (\boldsymbol{u}^2 - \mathcal{I}_h^2 \boldsymbol{u}^2) \cdot \boldsymbol{n} \, \mathrm{d}s}{\int_{\Gamma_j} \boldsymbol{\chi}_j^2 \cdot \boldsymbol{n} \, \mathrm{d}s}.$$

We note that $h \lesssim |\int_{\Gamma_j} \boldsymbol{\chi}_j^2 \, \mathrm{d}s|$, then using the trace inequality (3.1) and the approximation property (5.3), we obtain $\sum_{j=1}^{N_p} |\vartheta_j|^2 \lesssim h^{2k} |\boldsymbol{u}^2|^2_{k+1,\Omega_2}$ where we used similar arguments as in (4.27). Then we deduce

$$\begin{aligned} \|\boldsymbol{u}^2 - \mathcal{I}_\vartheta^2 \boldsymbol{u}^2\|_{0,\Omega_2} &+ h \|\nabla (\boldsymbol{u}^2 - \mathcal{I}_\vartheta^2 \boldsymbol{u}^2)\|_{0,\Omega_2} \\ &+ h^2 \Big( \sum_{K \in \mathcal{T}_h} \|D^2 (\boldsymbol{u}^2 - \mathcal{I}_\vartheta^2 \boldsymbol{u}^2)\|^2_{0,K} \Big)^{\frac{1}{2}} \lesssim h^{k+1} |\boldsymbol{u}^2|_{k+1,\Omega_2}, \end{aligned} \tag{5.28}$$

where we used the same arguments as in Section 4.4.3.

**Theorem 5.4.2.** *If* $(\boldsymbol{u}, p) \in ([H_\partial^{k+1}(\Omega_1)]^2 \times [H_\partial^{k+1}(\Omega_2)]^2) \times H^k(\Omega)$ *is the solution of* (5.22) *and* $(\boldsymbol{u}_h, p_h) \in W_h^k \times Q_h^k$ *the solution of* (5.23)*, then there holds*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leq h^k (C_{u\mu} |\boldsymbol{u}|_{k+1,\Omega} + C_{p\mu} |p|_{k,\Omega}),$$

*where* $C_{u\mu}$ *and* $C_{p\mu}$ *are positive constants that depends on* $\mu$ *and the mesh geometry.*

*Proof.* Let $\mathcal{I}_\vartheta$, and $\mathcal{I}_h$ be the interpolation operators as defined previously in this chapter, the triangle inequality provides

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leq |||(\boldsymbol{u} - \mathcal{I}_\vartheta \boldsymbol{u}, p - \mathcal{I}_h p)||| + |||(\mathcal{I}_\vartheta \boldsymbol{u} - \boldsymbol{u}_h, \mathcal{I}_h p - p_h)|||.$$

Using Theorem 5.4.1 and Lemma 5.4.2 we obtain

$$|||(\boldsymbol{u}_h - \mathcal{I}_\vartheta \boldsymbol{u}, p_h - \mathcal{I}_h p)||| \leq \sup_{(\boldsymbol{v}_h, q_h) \in W_h^k \times Q_h^k} \frac{A_h[(\boldsymbol{u} - \mathcal{I}_\vartheta \boldsymbol{u}, p - \mathcal{I}_h p), (\boldsymbol{v}_h, q_h)] - J_h[(\mathcal{I}_\vartheta \boldsymbol{u}, \mathcal{I}_h p), (\boldsymbol{v}_h, q_h)]}{|||(\boldsymbol{v}_h, q_h)|||}. \tag{5.29}$$

Using similar arguments as in the proof of Theorem 4.4.2, with the approximations properties (5.3), (5.28) we obtain

$$A_h[(\boldsymbol{u} - \mathcal{I}_\vartheta \boldsymbol{u}, p - \mathcal{I}_h p), (\boldsymbol{v}_h, q_h)] \lesssim |||(\boldsymbol{v}_h, q_h)||| h^k \sum_{i=1}^{2} (\mu_i^{\frac{1}{2}} |\boldsymbol{u}^i|_{k+1,\Omega_i} + \mu_i^{-\frac{1}{2}} |p^i|_{k,\Omega_i}).$$

Using arguments from Theorem 3.4.2 and Corollary 3.4.1 we have

$$J_h[(\mathcal{I}_\vartheta \boldsymbol{u}, \mathcal{I}_h p), (\boldsymbol{v}_h, q_h)] \lesssim |||(\boldsymbol{v}_h, q_h)||| h^k \sum_{i=1}^{2} (\mu_i^{\frac{1}{2}} |\boldsymbol{u}^i|_{k+1,\Omega_i} + \mu_i^{-\frac{1}{2}} |p^i|_{k,\Omega_i}).$$

Using these two results we obtain

$$|||(\mathcal{I}_\vartheta \boldsymbol{u} - \boldsymbol{u}_h, \mathcal{I}_h p - p_h)||| \lesssim h^k (\mu_1^{\frac{1}{2}} |\boldsymbol{u}^1|_{k+1,\Omega_1} + \mu_1^{-\frac{1}{2}} |p^1|_{k,\Omega_1} + \mu_2^{\frac{1}{2}} |\boldsymbol{u}^2|_{k+1,\Omega_2} + \mu_2^{-\frac{1}{2}} |p^2|_{k,\Omega_2}).$$

Combining arguments from the proofs of corollaries 5.2.1 and 3.4.1 we obtain

$$|||(\boldsymbol{u} - \mathcal{I}_\vartheta \boldsymbol{u}, p - \mathcal{I}_h p)||| \lesssim h^k (\mu_1^{\frac{1}{2}} |\boldsymbol{u}^1|_{k+1,\Omega_1} + \mu_1^{-\frac{1}{2}} |p^1|_{k,\Omega_1} + \mu_2^{\frac{1}{2}} |\boldsymbol{u}^2|_{k+1,\Omega_2} + \mu_2^{-\frac{1}{2}} |p^2|_{k,\Omega_2}).$$

□

**Remark 5.4.1.** *The order of the constants* $C_{u\mu}$ *and* $C'_{p\mu}$ *are the same as in the previous chapter for the fitted domain decomposition case.*

## 5.5 Numerical results

In this section we verify numerically the results that has been proven theoretically in this chapter. The computations are done using the package FEniCS [88] together with the library CutFEM [28]. The computational domain considered is the unit square $\Omega = [0,1] \times [0,1]$, the subdomain $\Omega_1$ is defined as

$$\Omega_1 = \{(x,y) \in \Omega |\, |(0.5, 0.5) - (x,y)| \leq 0.3\},$$

and $\Omega_2 = \Omega \backslash \Omega_1$. For each configuration we test the precision of the penalty-free Nitsche's method for unfitted domain decomposition. For each case we consider a manufactured solution to perform the computations, we consider piecewise affine approximations.

### 5.5.1 Poisson problem

For this problem the manufactured solution is considered as

$$u = [(x - 0.5)^2 + (y - 0.5)^2]^2.$$

The $L^2$ and $H^1$-errors are investigated, we set $\mu_1 = 1$ and we consider a range of values for $\mu_2$. Figure 5.4 shows that the slope of the $L^2$-error corresponds to the theory

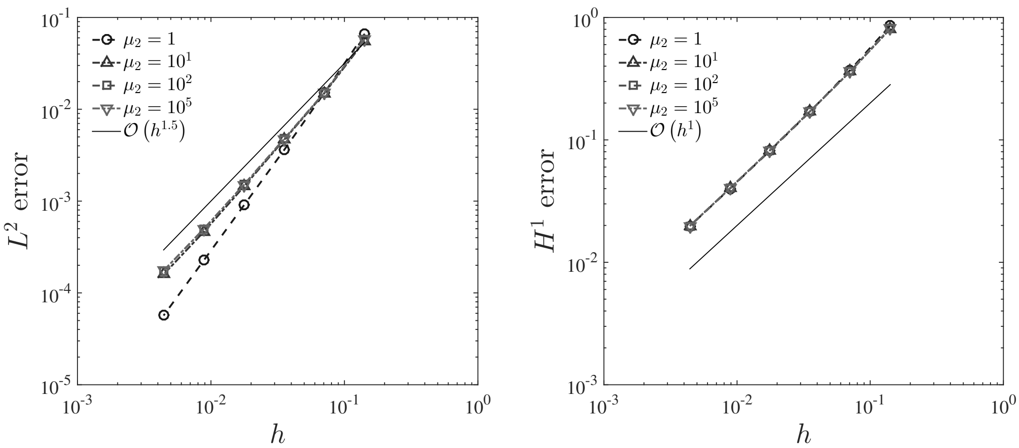



FIGURE 5.4: Poisson problem, $\mathbb{P}_1$, $\mu_1 = 1$.

(Proposition 5.2.1) for the cases $\mu_1 \neq \mu_2$ as we observe a rate of convergence of order $\mathcal{O}(h^{1.5})$. For $\mu_1 = \mu_2$ a super convergence of order $\mathcal{O}(h^{0.5})$ is observed. The $H^1$-error shows optimal convergence, no difference is observed as $\mu_2$ becomes bigger.

### 5.5.2 Compressible elasticity

For this problem the manufactured solution is considered as

$$\boldsymbol{u} = \begin{pmatrix} (x^5 - x^4)(y^3 - y^2) \\ (x^4 - x^3)(y^6 - y^5) \end{pmatrix}.$$

The $L^2$ and $H^1$-errors are investigated, we set $\mu_1 = \lambda_1 = 1$ and we consider ranges of values for $\mu_2$ and $\lambda_2$. Figure 5.5 shows that the $L^2$-error converges with a rate

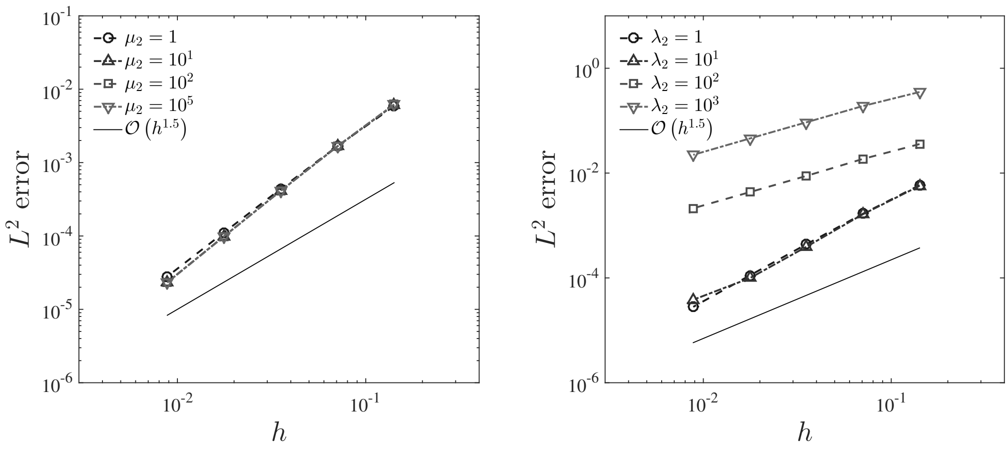



FIGURE 5.5: Compressible elasticity, $L^2$-error, $\mathbb{P}_1$, $\mu_1 = \lambda_1 = 1$.

slightly larger than $\mathcal{O}(h^{1.5})$. The parameter $\mu_2$ has a very small influence on the rate of convergence whereas if $\lambda_2$ is too large we observe locking. Figure 5.6 shows that the

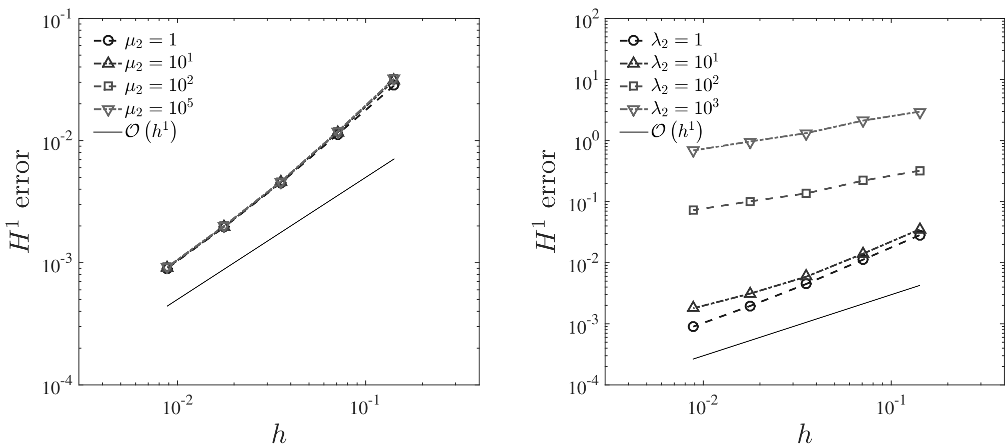



FIGURE 5.6: Compressible elasticity, $H^1$-error, $\mathbb{P}_1$, $\mu_1 = \lambda_1 = 1$.

$H^1$-error converges optimally. The parameter $\mu_2$ has a negligible influence on the rate of convergence, as for the $L^2$-error, for $\lambda_2$ too large locking is observed.

### 5.5.3 Incompressible elasticity

For this problem the manufactured solution is considered as

$$\boldsymbol{u} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}, \qquad p = \pi\cos(\pi x)\sin(\pi y).$$

The $H^1$-error of $\boldsymbol{u}$ and the $L^2$-error of $p$ are investigated, we set $\mu_1 = 1$ and we consider a range of values for $\mu_2$. Figure 5.7 shows optimal convergence for the $H^1$-error of $\boldsymbol{u}$.

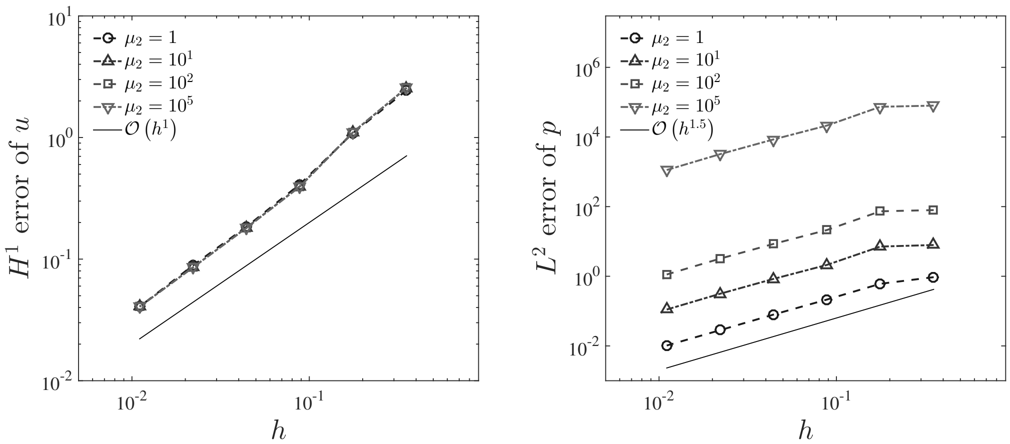



FIGURE 5.7: Incompressible elasticity, $\mathbb{P}_1$, $\mu_1 = 1$.

The slopes of convergence observed for the $L^2$-error of $p$ is of order $\mathcal{O}(h^{1.5})$, the error is multiplied by a factor $\sqrt{\mu_2}$ as $\mu_2$ becomes bigger.

# Chapter 6

# Fluid-structure interaction

In this chapter we propose a Nitsche based implicit time dependent scheme for fluid structure interaction. The penalty-free Nitsche's method is used at the interface to implement the coupling between the solid domain and the fluid domain. As done for incompressible elasticity in Chapters 4 and 5 we consider a master/slave configuration. The fluid domain is considered as the master and the solid domain the slave, as a consequence the Nitsche mortaring is taken only from the fluid side.

## 6.1 Linear model problem

The physical domain consists of $\Omega = \Omega_f \cup \Omega_s \cup \Sigma \in \mathbb{R}^2$, $\Omega_f$ and $\Omega_s$ are respectively the fluid and solid subdomains and $\Sigma = \overline{\Omega_f} \cap \overline{\Omega_s} \in \mathbb{R}$ is the interface between the fluid and the solid considered as plane. Let $\partial\Omega_f$ and $\partial\Omega_s$ be the boundaries of the domains $\Omega_f$ and $\Omega_s$ then $\Gamma^f = \partial\Omega_f \backslash \Sigma$ and $\Gamma^s = \partial\Omega_s \backslash \Sigma$. The exterior unit normal to $\partial\Omega_f$ and $\partial\Omega_s$ are denoted $\boldsymbol{n}$ and $\boldsymbol{n}_s$. The fluid is described by the Stokes equations in the polyhedral domain $\Omega_f \in \mathbb{R}^2$. The solid is described by the elastodynamics equations in the polyhedral domain $\Omega_s \in \mathbb{R}^2$. Figure 6.1 shows an example of configuration. The

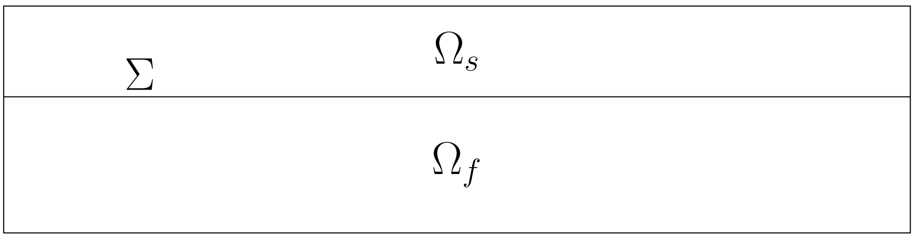



FIGURE 6.1: Example of computational domain $\Omega = \Omega_f \cup \Omega_s \cup \Sigma$.

coupled problem is considered as follows: find the fluid velocity $\boldsymbol{u} : \Omega_f \times \mathbb{R}^+ \to \mathbb{R}^2$ and the fluid pressure $p : \Omega_f \times \mathbb{R}^+ \to \mathbb{R}$, the solid displacement $\boldsymbol{d} : \Omega_s \times \mathbb{R}^+ \to \mathbb{R}^2$ and the solid velocity $\dot{\boldsymbol{d}} : \Omega_s \times \mathbb{R}^+ \to \mathbb{R}^2$ such that

$$\begin{cases} \rho_f \partial_t \boldsymbol{u} - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) = 0 & \text{in } \Omega_f, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega_f, \\ \boldsymbol{u} = 0 & \text{on } \Gamma^f, \end{cases} \tag{6.1}$$

$$\begin{cases} \dot{\boldsymbol{d}} = \partial_t \boldsymbol{d} & \text{in } \Omega_s \\ \rho_s \partial_t \dot{\boldsymbol{d}} - \nabla \cdot \boldsymbol{\sigma}_s(\boldsymbol{d}) = 0 & \text{in } \Omega_s, \\ \boldsymbol{d} = 0 & \text{on } \Gamma^s, \end{cases} \tag{6.2}$$

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n} = -\boldsymbol{\sigma}_s(\boldsymbol{d}) \cdot \boldsymbol{n}_s & \text{on } \Sigma, \\ \boldsymbol{u} = \dot{\boldsymbol{d}} & \text{on } \Sigma. \end{cases} \tag{6.3}$$

Where $\rho_f$ and $\rho_s$ denote the fluid and solid densities. The fluid and solid stress tensors are respectively defined as

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbb{I}_{2\times 2}, \qquad \boldsymbol{\sigma}_s(\boldsymbol{d}) = 2L_1\boldsymbol{\varepsilon}(\boldsymbol{d}) + L_2(\nabla \cdot \boldsymbol{d})\mathbb{I}_{2\times 2},$$

where $\mu$ denotes the dynamic viscosity of the fluid and $L_1$, $L_2$ the Lamé coefficients of the solid, we assume $L_1 \gtrsim L_2$ to avoid locking. We set the following initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{d}(0) = \boldsymbol{d}_0$ and $\dot{\boldsymbol{d}}(0) = \dot{\boldsymbol{d}}_0$.

## 6.2 Spatial semi-discrete formulation

Let $\{\mathcal{T}_h\}_h$ be a family of quasi-uniform and shape regular triangulations fitted to $\Omega$, with mesh parameter $h = \max_{K \in \mathcal{T}_h} h_K$. We define $K$ as a generic triangle in a triangulation $\mathcal{T}_h$ and $h_K = \text{diam}(K)$ the diameter of $K$. The set $\mathcal{F}_h$ denotes the faces of a triangulation $\mathcal{T}_h$ and $\boldsymbol{n}_F$ is the unit normal to the face $F$ with fixed but arbitrary orientation. $[\![w]\!]_F = w_F^+ - w_F^-$, with $w_F^\pm = \lim_{s\to 0^+} w(x \mp s n_F)$, is the jump of $w$ across the face $F$. A triangulation $\mathcal{T}_h$ covers $\Omega_f$ and $\Omega_s$ and is fitted to the Dirichlet boundaries $\Gamma_f$ and $\Gamma_s$. In the upcoming analysis the interface $\Sigma$ between the fluid and the solid is fitted to the mesh i.e. the plane interface $\Sigma$ coincides exactly with faces of the triangulation $\mathcal{T}_h$. We introduce the following spaces of admissible displacements for the solid problem (6.2)

$$W^s = [H^1_{\Gamma_s}(\Omega_s)]^2,$$

with $H^1_{\Gamma_s}(\Omega_s) = \{w \in H^1(\Omega_s) \ : \ w|_{\Gamma_s} = 0\}$. For the fluid problem (6.1), the velocity and pressure spaces are defined as

$$W^f = [H^1_{\Gamma_f}(\Omega_f)]^2, \qquad Q^f = L^2(\Omega_f),$$

with $H^1_{\Gamma_f}(\Omega_s) = \{v \in H^1(\Omega_f) \ : \ v|_{\Gamma_f} = 0\}$. We introduce the corresponding spaces of continuous piecewise affine functions,

$$\begin{aligned} W_h^s &= \{\boldsymbol{w}_h \in W^s : \boldsymbol{w}_h|_K \in [\mathbb{P}_1(K)]^2 \ \ \forall K \in \mathcal{T}_h\}, \\ W_h^f &= \{\boldsymbol{v}_h \in W^f : \boldsymbol{v}_h|_K \in [\mathbb{P}_1(K)]^2 \ \ \forall K \in \mathcal{T}_h\}, \\ Q_h^f &= \{q_h \in Q^f : q_h|_K \in \mathbb{P}_1(K) \ \ \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Since the choice of velocity and pressure spaces does not satisfy the inf-sup condition, we consider a pressure stabilisation. As we use piecewise affine approximations we consider the following Brezzi-Pitkäranta pressure stabilisation [24]

$$S_h(p_h, q_h) = \gamma_p \int_{\Omega_f} \frac{h^2}{\mu} \nabla p_h \nabla q_h \, \mathrm{d}x. \tag{6.4}$$

Alternatively the interior penalty stabilisation [35] could be used

$$S_h'(p_h, q_h) = \gamma_p \sum_{F \in \mathcal{F}_h^f} \int_F \frac{h^3}{\mu} [\![\nabla p_h \cdot \boldsymbol{n}_F]\!]_F [\![\nabla q_h \cdot \boldsymbol{n}_F]\!]_F \, \mathrm{d}s, \tag{6.5}$$

with $\mathcal{F}_h^f = \{F \in \mathcal{F}_h \mid F \cap \mathring{\Omega}_f \neq \emptyset\}$. The spatial semi-discrete formulation of the system (6.1)-(6.3) is considered such that: for $t > 0$, find

$$(\boldsymbol{u}_h(t), p_h(t), \boldsymbol{d}_h(t), \dot{\boldsymbol{d}}_h(t)) \in W_h^f \times Q_h^f \times W_h^s \times W_h^s,$$

such that $\dot{\boldsymbol{d}}_h(t) = \partial_t \boldsymbol{d}_h(t)$ and

$$\begin{cases} \rho_f(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h)_{\Omega_f} + A((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) + \rho_s(\partial_t \dot{\boldsymbol{d}}_h, \boldsymbol{w}_h)_{\Omega_s} + a^s(\boldsymbol{d}_h, \boldsymbol{w}_h) \\ \quad - \langle \boldsymbol{\sigma}(\boldsymbol{u}_h, p_h) \cdot \boldsymbol{n}, \boldsymbol{v}_h - \boldsymbol{w}_h \rangle_\Sigma + \langle \boldsymbol{u}_h - \dot{\boldsymbol{d}}_h, \boldsymbol{\sigma}(\boldsymbol{v}_h, q_h) \cdot \boldsymbol{n} \rangle_\Sigma + S_h(p_h, q_h) = 0 \end{cases} \tag{6.6}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{w}_h) \in W_h^f \times Q_h^f \times W_h^s$. Where we have

$$\begin{aligned} A((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) &= 2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_{\Omega_f} - (p_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_f} + (q_h, \nabla \cdot \boldsymbol{u}_h)_{\Omega_f}, \\ a^s(\boldsymbol{d}_h, \boldsymbol{w}_h) &= 2L_1(\boldsymbol{\varepsilon}(\boldsymbol{d}_h), \boldsymbol{\varepsilon}(\boldsymbol{w}_h))_{\Omega_s} + L_2(\nabla \cdot \boldsymbol{d}_h, \nabla \cdot \boldsymbol{w}_h)_{\Omega_s}. \end{aligned}$$

The elastic bilinear form $a^s$ is associated with the elastic energy norm $\|\boldsymbol{w}\|_s = a^s(\boldsymbol{w}, \boldsymbol{w})^{\frac{1}{2}}$.

**Remark 6.2.1.** *The semi-discrete formulation* (6.6) *is penalty-free in the sense that the coupling at the interface is done using the Nitsche term* $\langle \boldsymbol{u}_h - \dot{\boldsymbol{d}}_h, \boldsymbol{\sigma}(\boldsymbol{v}_h, q_h) \cdot \boldsymbol{n} \rangle_\Sigma$*, therefore no Nitsche penalisation in involved. The penalised version of this scheme is proposed in [34] where an additional penalty term* $\frac{\gamma\mu}{h} \langle (\boldsymbol{u}_h - \dot{\boldsymbol{d}}_h), (\boldsymbol{v}_h - \boldsymbol{w}_h) \rangle_\Sigma$ *is considered.*

## 6.3 Fully discrete scheme

In this section we introduce a fully discrete formulation of the spatial semi discrete scheme introduced in the previous section. The spatial semi-discrete formulation (6.6) is discretised in time using the first order backward difference. In what follows, $\tau > 0$ denotes the time-step, at step $n \in \mathbb{N}$ we have $t_n = n\tau$ and

$$\partial_\tau x^n = \frac{x^n - x^{n-1}}{\tau}.$$

Considering the semi-discrete scheme (6.6) and replacing the time derivatives $\partial_t$ by the backward difference $\partial_\tau$, it yields to the following time-advancing formulation: for $n \geq 1$, find $(\boldsymbol{u}_h^n(t), p_h^n(t), \boldsymbol{d}_h^n(t), \dot{\boldsymbol{d}}_h^n(t)) \in W_h^f \times Q_h^f \times W_h^s \times W_h^s$, such that $\dot{\boldsymbol{d}}_h^n = \partial_\tau \boldsymbol{d}_h^n$ and

$$\begin{cases} \rho_f(\partial_\tau \boldsymbol{u}_h^n, \boldsymbol{v}_h)_{\Omega_f} + A((\boldsymbol{u}_h^n, p_h^n), (\boldsymbol{v}_h, q_h)) + \rho_s(\partial_\tau \dot{\boldsymbol{d}}_h^n, \boldsymbol{w}_h)_{\Omega_s} + a^s(\boldsymbol{d}_h^n, \boldsymbol{w}_h) \\ \quad - \langle \boldsymbol{\sigma}(\boldsymbol{u}_h^n, p_h^n) \cdot \boldsymbol{n}, \boldsymbol{v}_h - \boldsymbol{w}_h \rangle_\Sigma + \langle \boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n, \boldsymbol{\sigma}(\boldsymbol{v}_h, q_h) \cdot \boldsymbol{n} \rangle_\Sigma + S_h(p_h^n, q_h^n) = 0 \end{cases} \tag{6.7}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{w}_h) \in W_h^f \times Q_h^f \times W_h^s$. In order to study the consistency of the scheme (6.7), we multiply (6.1) by $\boldsymbol{v}_h, q_h \in W_h^f \times Q_h^f$ and (6.2) by $\boldsymbol{w}_h \in W_h^s$, integrate by parts both systems and use that $\boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n} = -\boldsymbol{\sigma}_s(\boldsymbol{d}) \cdot \boldsymbol{n}_s$ on $\Sigma$, then we obtain

$$\rho_f(\partial_t \boldsymbol{u}, \boldsymbol{v}_h)_{\Omega_f} + A((\boldsymbol{u}, p), (\boldsymbol{v}_h, q_h)) + \rho_s(\partial_t \dot{\boldsymbol{d}}, \boldsymbol{w}_h)_{\Omega_s} + a^s(\boldsymbol{d}, \boldsymbol{w}_h) \\ - \langle \boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}, \boldsymbol{v}_h - \boldsymbol{w}_h \rangle_\Sigma = 0.$$

Using the condition $\boldsymbol{u} = \dot{\boldsymbol{d}}$ on $\Sigma$, for all $(\boldsymbol{v}_h, q_h, \boldsymbol{w}_h) \in W_h^f \times Q_h^f \times W_h^s$ we may write

$$\rho_f(\partial_t \boldsymbol{u}, \boldsymbol{v}_h)_{\Omega_f} + A((\boldsymbol{u}, p), (\boldsymbol{v}_h, q_h)) + \rho_s(\partial_t \dot{\boldsymbol{d}}, \boldsymbol{w}_h)_{\Omega_s} + a^s(\boldsymbol{d}, \boldsymbol{w}_h) \\ - \langle \boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}, \boldsymbol{v}_h - \boldsymbol{w}_h \rangle_\Sigma + \langle \boldsymbol{u} - \dot{\boldsymbol{d}}, \boldsymbol{\sigma}(\boldsymbol{v}_h, q_h) \cdot \boldsymbol{n} \rangle_\Sigma = 0. \tag{6.8}$$

Subtracting (6.7) and (6.8) we get the following consistency relation.

**Lemma 6.3.1.** *Let* $(\boldsymbol{u}, p, \boldsymbol{d}, \dot{\boldsymbol{d}})$ *be the solution of* (6.1), (6.2), (6.3) *and* $(\boldsymbol{u}_h^n, p_h^n, \boldsymbol{d}_h^n, \dot{\boldsymbol{d}}_h^n)$ *the solution of* (6.7), *at time* $t = t_n$ *the following Galerkin orthogonality holds*

$$\begin{aligned} &\rho_f(\partial_\tau(\boldsymbol{u}^n - \boldsymbol{u}_h^n), \boldsymbol{v}_h)_{\Omega_f} + A((\boldsymbol{u}^n - \boldsymbol{u}_h^n, p_h - p_h^n), (\boldsymbol{v}_h, q_h)) + \rho_s(\partial_\tau(\dot{\boldsymbol{d}}^n - \dot{\boldsymbol{d}}_h^n), \boldsymbol{w}_h)_{\Omega_s} \\ &+ a^s(\boldsymbol{d}^n - \boldsymbol{d}_h^n, \boldsymbol{w}_h) - \langle \boldsymbol{\sigma}(\boldsymbol{u}^n - \boldsymbol{u}_h^n, p^n - p_h^n) \cdot \boldsymbol{n}, \boldsymbol{v}_h - \boldsymbol{w}_h \rangle_\Sigma \\ &\qquad + \langle (\boldsymbol{u}^n - \boldsymbol{u}_h^n) - (\dot{\boldsymbol{d}}^n - \dot{\boldsymbol{d}}_h^n), \boldsymbol{\sigma}(\boldsymbol{v}_h, q_h) \cdot \boldsymbol{n} \rangle_\Sigma \\ &\qquad\qquad = -\rho_f((\partial_t - \partial_\tau)\boldsymbol{u}^n, \boldsymbol{v}_h)_{\Omega_f} - \rho_s((\partial_t - \partial_\tau)\dot{\boldsymbol{d}}^n, \boldsymbol{w}_h)_{\Omega_s} + S_h(p_h^n, q_h) \end{aligned}$$

*for all* $(\boldsymbol{v}_h, q_h, \boldsymbol{w}_h) \in W_h^f \times Q_h^f \times W_h^s$.

## 6.4 Stability analysis

At time $t_n$, we introduce the total discrete energy $E_h^n$ and dissipation $D_h^n$ such that

$$\begin{aligned} E_h^n &= \rho_f \|\boldsymbol{u}_h^n\|_{0,\Omega_f}^2 + \rho_s \|\dot{\boldsymbol{d}}_h^n\|_{0,\Omega_s}^2 + \|\boldsymbol{d}_h^n\|_s^2, \\ D_h^n &= \rho_f \tau^{-1} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{0,\Omega_f}^2 + \rho_s \tau^{-1} \|\dot{\boldsymbol{d}}_h^n - \dot{\boldsymbol{d}}_h^{n-1}\|_{0,\Omega_s}^2 + \tau^{-1} \|\boldsymbol{d}_h^n - \boldsymbol{d}_h^{n-1}\|_s^2 \\ &\quad + \mu \|\nabla \boldsymbol{u}_h^n\|_{0,\Omega_f}^2 + \mu h^{-1} \|(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n) \cdot \boldsymbol{n}\|_{0,\Sigma}^2 + \gamma_p h^2 \mu^{-1} \|\nabla p_h^n\|_{0,\Omega_f}^2. \end{aligned}$$

**Remark 6.4.1.** *If* $\Omega_f$ *and* $\Omega_s$ *are meshed with two nonconforming triangulations the interface term becomes* $\mu h^{-1} \|(\boldsymbol{u}_h^n - \mathcal{I}\dot{\boldsymbol{d}}_h^n) \cdot \boldsymbol{n}\|_{0,\Sigma}^2$ *with* $\mathcal{I}$ *an interpolant.*

**Theorem 6.4.1.** *Let* $\{(\boldsymbol{u}_h^n, p_h^n, \boldsymbol{d}_h^n, \dot{\boldsymbol{d}}_h^n)\}_{1\leq n\leq N}$ *be the solution of* (6.7) *at time* $t_n$ *and* $(\boldsymbol{u}_h^0, p_h^0, \boldsymbol{d}_h^0, \dot{\boldsymbol{d}}_h^0)$ *the initial state, the following stability holds for* $N > 0$

$$E_h^N + \tau \sum_{n=1}^{N} D_h^n \lesssim E_h^0.$$

*Proof.* Let us choose $\boldsymbol{v}_h = \tau \boldsymbol{u}_h^n$, $q_h = \tau p_h^n + \kappa_h$ and $\boldsymbol{w}_h = \tau \dot{\boldsymbol{d}}_h^n$, then (6.7) becomes

$$\begin{aligned}\tau\rho_f(\partial_\tau \boldsymbol{u}_h^n, \boldsymbol{u}_h^n)_{\Omega_f} &+ \tau(2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h^n), \boldsymbol{\varepsilon}(\boldsymbol{u}_h^n))_{\Omega_f} + \tau\rho_s(\partial_\tau \dot{\boldsymbol{d}}_h^n, \dot{\boldsymbol{d}}_h^n)_{\Omega_s} + \tau a^s(\boldsymbol{d}_h^n, \dot{\boldsymbol{d}}_h^n)\\ &+ \tau S_h(p_h^n, p_h^n) - \langle(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n)\cdot \boldsymbol{n}, \kappa_h\rangle_\Sigma + (\kappa_h, \nabla\cdot \boldsymbol{u}_h^n)_{\Omega_f} + S_h(p_h^n, \kappa_h) = 0.\end{aligned}$$

Observe that

$$\begin{aligned}&\tau\rho_f(\partial_\tau \boldsymbol{u}_h^n, \boldsymbol{u}_h^n)_{\Omega_f}\\ &\quad = \frac{\rho_f}{2}\big[(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)_{\Omega_f} + (\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1})_{\Omega_f} + (\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^{n-1})_{\Omega_f}\big]\\ &\quad = \frac{\rho_f}{2}\big[\|\boldsymbol{u}_h^n\|_{0,\Omega_f}^2 - \|\boldsymbol{u}_h^{n-1}\|_{0,\Omega_f}^2 + \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{0,\Omega_f}^2\big],\end{aligned}$$

similarly we have

$$\begin{aligned}\tau\rho_s(\partial_\tau \dot{\boldsymbol{d}}_h^n, \dot{\boldsymbol{d}}_h^n)_{\Omega_f} &= \frac{\rho_s}{2}\big[\tau\partial_\tau\|\dot{\boldsymbol{d}}_h^n\|_{0,\Omega_s}^2 + \|\dot{\boldsymbol{d}}_h^n - \dot{\boldsymbol{d}}_h^{n-1}\|_{0,\Omega_s}^2\big],\\ \tau a^s(\boldsymbol{d}_h^n, \dot{\boldsymbol{d}}_h^n) &= \frac{1}{2}(\tau\partial_\tau\|\boldsymbol{d}_h^n\|_s^2 + \|\boldsymbol{d}_h^n - \boldsymbol{d}_h^{n-1}\|_s^2).\end{aligned}$$

Now we define the perturbation $\kappa_h \in Q_h^f$, let $P_\Sigma = \{K \in \mathcal{T}_h \mid K\cap\Sigma \neq \emptyset \text{ and } K \in \overline{\Omega_f}\}$, and for any node $x_i \in \mathcal{T}_h$ we have $\chi_h \in Q_h^f$ such that

$$\chi_h(x_i) = \begin{cases} 0 & \text{for} \quad x_i \in \Omega\backslash\mathring{\Sigma}\\ 1 & \text{for} \quad x_i \in \mathring{\Sigma},\end{cases}$$

then $\kappa_h(x_i) = \nu(x_i)\chi_h(x_i)$ with $\nu(x_i) \in \mathbb{R}$ such that $\kappa_h|_\Sigma = \alpha\frac{\tau\mu}{h}(\dot{\boldsymbol{d}}_h^n - \boldsymbol{u}_h^n)\cdot\boldsymbol{n}$. Using this and the Korn's inequality we obtain

$$\begin{aligned}&\frac{\rho_f}{2}(\tau\partial_\tau\|\boldsymbol{u}_h^n\|_{0,\Omega_f}^2 + \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{0,\Omega_f}^2) + \frac{\rho_s}{2}(\tau\partial_\tau\|\dot{\boldsymbol{d}}_h^n\|_{0,\Omega_s}^2 + \|\dot{\boldsymbol{d}}_h^n - \dot{\boldsymbol{d}}_h^{n-1}\|_{0,\Omega_s}^2)\\ &\quad + 2C_K\tau\mu\|\nabla \boldsymbol{u}_h^n\|_{0,\Omega_f}^2 + \frac{1}{2}(\tau\partial_\tau\|\boldsymbol{d}_h^n\|_s^2 + \|\boldsymbol{d}_h^n - \boldsymbol{d}_h^{n-1}\|_s^2) + \tau\gamma_p h^2\mu^{-1}\|\nabla p_h^n\|_{0,\Omega_f}^2\\ &\quad + \alpha\frac{\tau\mu}{h}\|(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2 + (\kappa_h, \nabla\cdot\boldsymbol{u}_h^n)_{\Omega_f} + S_h(p_h^n, \kappa_h) \leq 0.\end{aligned}$$

For piecewise affine approximations we have the estimate

$$\|\kappa_h\|_{0,P_\Sigma} \lesssim \alpha\tau\mu h^{-\frac{1}{2}}\|(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}, \tag{6.9}$$

using this result, the Cauchy-Schwarz and the Young's inequality we obtain

$$(\kappa_h, \nabla \cdot \boldsymbol{u}_h^n)_{\Omega_f} \leq \|\nabla \boldsymbol{u}_h^n\|_{0,\Omega_f} \|\kappa_h\|_{0,P_\Sigma} \leq \epsilon\tau\mu \|\nabla \boldsymbol{u}_h^n\|_{0,\Omega_f}^2 + \alpha^2 C \frac{\tau\mu}{4\epsilon h} \|(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n) \cdot \boldsymbol{n}\|_{0,\Sigma}^2.$$

Using the inverse inequality of Lemma 2.0.2 and (6.9) we have

$$\begin{aligned} S_h(p_h^n, \kappa_h) &\leq S_h(p_h^n, p_h^n)^{\frac{1}{2}} S_h(\kappa_h, \kappa_h)^{\frac{1}{2}} \\ &\leq \epsilon\tau\gamma_p h^2 \mu^{-1} \|\nabla p_h^n\|_{0,\Omega_f}^2 + \alpha^2 C \gamma_p \frac{\tau\mu}{4\epsilon h} \|(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n) \cdot \boldsymbol{n}\|_{0,\Sigma}^2. \end{aligned}$$

Then we obtain

$$\begin{aligned} &\frac{\rho_f}{2}(\tau\partial_\tau \|\boldsymbol{u}_h^n\|_{0,\Omega_f}^2 + \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{0,\Omega_f}^2) + \frac{\rho_s}{2}(\tau\partial_\tau \|\dot{\boldsymbol{d}}_h^n\|_{0,\Omega_s}^2 + \|\dot{\boldsymbol{d}}_h^n - \dot{\boldsymbol{d}}_h^{n-1}\|_{0,\Omega_s}^2) \\ &\quad + \tau\mu(2C_K - \epsilon)\|\nabla \boldsymbol{u}_h^n\|_{0,\Omega_f}^2 + \frac{1}{2}(\tau\partial_\tau \|\boldsymbol{d}_h^n\|_s^2 + \|\boldsymbol{d}_h^n - \boldsymbol{d}_h^{n-1}\|_s^2) \\ &\quad + \tau(1-\epsilon)\gamma_p h^2 \mu^{-1} \|\nabla p_h^n\|_{0,\Omega_f}^2 + \alpha\Big(1 - \alpha \frac{C(1+\gamma_p)}{4\epsilon}\Big) \frac{\tau\mu}{h} \|(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n) \cdot \boldsymbol{n}\|_{0,\Sigma}^2 \leq 0. \end{aligned}$$

We choose $\epsilon$ and $\alpha$ such that all the terms are positive, then taking the sum from $t_1$ to $t_N$ we obtain

$$\begin{aligned} &\rho_f \|\boldsymbol{u}_h^N\|_{0,\Omega_f}^2 + \rho_s \|\dot{\boldsymbol{d}}_h^N\|_{0,\Omega_s}^2 + \|\boldsymbol{d}_h^N\|_s^2 + \tau \sum_{n=1}^{N} \Big[ \frac{\rho_f}{\tau} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{0,\Omega_f}^2 + \frac{\rho_s}{\tau} \|\dot{\boldsymbol{d}}_h^n - \dot{\boldsymbol{d}}_h^{n-1}\|_{0,\Omega_s}^2 \\ &\quad + 2\mu(2C_K - \epsilon)\|\nabla \boldsymbol{u}_h^n\|_{0,\Omega_f}^2 + \frac{1}{\tau}\|\boldsymbol{d}_h^n - \boldsymbol{d}_h^{n-1}\|_s^2 + 2(1-\epsilon)\gamma_p h^2 \mu^{-1} \|\nabla p_h^n\|_{0,\Omega_f}^2 \\ &\quad + 2\alpha\Big(1 - \alpha \frac{C(1+\gamma_p)}{4\epsilon}\Big) \frac{\mu}{h} \|(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n) \cdot \boldsymbol{n}\|_{0,\Sigma}^2 \Big] \leq \rho_f \|\boldsymbol{u}_h^0\|_{0,\Omega_f}^2 + \rho_s \|\dot{\boldsymbol{d}}_h^0\|_{0,\Omega_s}^2 + \|\boldsymbol{d}_h^0\|_s^2. \end{aligned}$$

□

## 6.5 Convergence analysis

We introduce the following result from [76] (Lemma 5.1), this Lemma is a discrete version of the well known Gronwall inequality.

**Lemma 6.5.1.** *Let $\tau$, $B$ and $a_n$, $b_n$, $c_n$, $\eta_n$ (for integers $n \geq 1$) be nonnegative numbers such that*

$$a_N + \tau \sum_{n=1}^{N} b_n \leq \tau \sum_{n=1}^{N} \eta_n a_n + \tau \sum_{n=1}^{N} c_n + B$$

*for $N \geq 1$. Suppose that $\tau\eta_n < 1$ for all $n \geq 1$. Then there holds*

$$a_N + \tau \sum_{n=1}^{N} b_n \leq \exp\left(\tau \sum_{n=1}^{N} \frac{\eta_n}{1 - \tau\eta_n}\right) \left(\tau \sum_{n=1}^{N} c_n + B\right)$$

*for $n \geq 1$.*

Similarly as in Chapter 2 and 4 we introduce a structure of patch at the interface $\Sigma$. We split the set of elements $P_\Sigma$ into $N_p^f$ smaller disjoint patches $P_j^f$ with $F_j^f = \partial P_j^f \cap \Sigma$, we consider that $F_j^f$ has at least one inner node and

$$h \lesssim \text{meas}(F_j^f) \lesssim h, \qquad h^2 \lesssim \text{meas}(P_j^f) \lesssim h^2.$$

For any node $x_i \in \mathcal{T}_h$ we have

$$\chi_h^f(x_i) = \begin{cases} 0 & \text{for} \quad x_i \in \Omega \backslash \mathring{F}_j^f \\ 1 & \text{for} \quad x_i \in \mathring{F}_j^f. \end{cases}$$

Let $P_\Sigma^s = \{K \in \mathcal{T}_h \mid K \cap \Sigma \neq \emptyset \text{ and } K \in \overline{\Omega_s}\}$, we define similarly the patches $P_j^s$ by splitting $P_\Sigma^s$ into $N_p^s$ smaller disjoint patches with $F_j^s = \partial P_j^s \cap \Sigma$ such that

$$h \lesssim \text{meas}(F_j^s) \lesssim h, \qquad h^2 \lesssim \text{meas}(P_j^s) \lesssim h^2,$$

and we assume that $F_j^s$ has at least one inner node. For any node $x_i \in \mathcal{T}_h$ we have

$$\chi_h^s(x_i) = \begin{cases} 0 & \text{for} \quad x_i \in \Omega \backslash \mathring{F}_j^s \\ 1 & \text{for} \quad x_i \in \mathring{F}_j^s. \end{cases}$$

We introduce the following quantities relative to the errors of the fluid velocity and pressure

$$\begin{aligned} \boldsymbol{\theta}_h &= \boldsymbol{u}_h - \mathcal{I}_\alpha \boldsymbol{u}, \\ \boldsymbol{\theta}_\pi &= \boldsymbol{u} - \mathcal{I}_\alpha \boldsymbol{u}, \\ y_h &= p_h - \mathcal{I}_\beta p, \\ y_\pi &= p - \mathcal{I}_\beta p. \end{aligned}$$

The interpolants $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta$ are such that

$$\mathcal{I}_\alpha \boldsymbol{u} = i_{\mathsf{SZ}} \boldsymbol{u} + \sum_{j=1}^{N_p^f} \alpha_j \boldsymbol{\phi}_j, \qquad \mathcal{I}_\beta p = i_{\mathsf{SZ}} p + \sum_{j=1}^{N_p^f} \beta_j \psi_j,$$

with $i_{\mathsf{SZ}}$ the Scott-Zhang interpolant and $\boldsymbol{\phi}_j = (\chi_j^f, \chi_j^f)^T$, $\psi_j = \chi_j^f$, the scalars $\alpha_j \in \mathbb{R}$ and $\beta_j \in \mathbb{R}$ are chosen such that

$$\int_{F_j^f} \boldsymbol{\varepsilon}(\boldsymbol{u} - \mathcal{I}_\alpha \boldsymbol{u}) \cdot \boldsymbol{n} = 0, \qquad \int_{F_j^f} (p - \mathcal{I}_\beta p) \mathbb{I}_{2\times 2} \cdot \boldsymbol{n} = 0,$$

as a consequence we have

$$\int_{F_j^f} \boldsymbol{\sigma}(\boldsymbol{\theta}_\pi, y_\pi) \cdot \boldsymbol{n} = 0. \tag{6.10}$$

We introduce the following quantities relative to the errors of the solid displacement and velocity

$$\begin{aligned}
\boldsymbol{\xi}_h &= \boldsymbol{d}_h - \pi_h^s \boldsymbol{d}, \\
\boldsymbol{\xi}_\pi &= \boldsymbol{d} - \pi_h^s \boldsymbol{d}, \\
\dot{\boldsymbol{\xi}}_h &= \dot{\boldsymbol{d}}_h - \mathcal{I}_\gamma \dot{\boldsymbol{d}}, \\
\dot{\boldsymbol{\xi}}_\pi &= \dot{\boldsymbol{d}} - \mathcal{I}_\gamma \dot{\boldsymbol{d}},
\end{aligned}$$

where $\pi_h^s$ denotes the elastic Ritz-projection of the solid [102, 31]. The interpolant $\mathcal{I}_\gamma$ is defined as

$$\mathcal{I}_\gamma \dot{\boldsymbol{d}} = \pi_h^s \dot{\boldsymbol{d}} + \sum_{j=1}^{N_p^s} \gamma_j \boldsymbol{\varphi}_j,$$

with $\boldsymbol{\varphi}_j = (\chi_j^s, \chi_j^s)^T$ and $\gamma_j \in \mathbb{R}$ chosen such that

$$\int_{F_j^s} \dot{\boldsymbol{\xi}}_\pi \cdot \boldsymbol{n} = 0. \tag{6.11}$$

Note that $\dot{\boldsymbol{\xi}}_h \neq \partial_\tau \boldsymbol{\xi}_h$, therefore we define $\boldsymbol{z}_h = \pi_h^s \partial_\tau \boldsymbol{d} - \mathcal{I}_\gamma \dot{\boldsymbol{d}}$, and we have

$$\dot{\boldsymbol{\xi}}_h = \partial_\tau \boldsymbol{\xi}_h + \boldsymbol{z}_h. \tag{6.12}$$

**Lemma 6.5.2.** *Let $\boldsymbol{u} \in [H^2(\Omega_f)]^2$, $p \in H^1(\Omega_f)$, $\boldsymbol{d} \in [H^2(\Omega_s)]^2$ and $\dot{\boldsymbol{d}} \in [H^2(\Omega_s)]^2$ the following approximation estimates holds*

$$\begin{aligned}
\|\boldsymbol{u} - \mathcal{I}_\alpha \boldsymbol{u}\|_{0,\Omega_f} + h\|\nabla(\boldsymbol{u} - \mathcal{I}_\alpha \boldsymbol{u})\|_{0,\Omega_f} &\lesssim h^2 |\boldsymbol{u}|_{2,\Omega_f}, \\
\|p - \mathcal{I}_\beta p\|_{0,\Omega_f} + h\|\nabla(p - \mathcal{I}_\beta p)\|_{0,\Omega_f} &\lesssim h|p|_{1,\Omega_f}, \\
\|\boldsymbol{d} - \pi_h^s \boldsymbol{d}\|_{0,\Omega_s} + h\|\nabla(\boldsymbol{d} - \pi_h^s \boldsymbol{d})\|_{0,\Omega_s} &\lesssim h^2 |\boldsymbol{d}|_{2,\Omega_s}, \\
\|\dot{\boldsymbol{d}} - \mathcal{I}_\gamma \dot{\boldsymbol{d}}\|_{0,\Omega_s} + h\|\nabla(\dot{\boldsymbol{d}} - \mathcal{I}_\gamma \dot{\boldsymbol{d}})\|_{0,\Omega_s} &\lesssim h^2 |\dot{\boldsymbol{d}}|_{2,\Omega_s}.
\end{aligned}$$

*Proof.* The third estimate follows directly from the approximation property of the Ritz-projection. Using the construction of $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta$ we have

$$\alpha_j = \frac{\int_{F_j^f} \boldsymbol{\varepsilon}(\boldsymbol{u} - i_{\mathrm{SZ}} \boldsymbol{u}) \cdot \boldsymbol{n}}{\int_{F_j^f} \boldsymbol{\varepsilon}(\boldsymbol{\phi}_j) \cdot \boldsymbol{n}}, \qquad \beta_j = \frac{\int_{F_j^f} (p - i_{\mathrm{SZ}} p) \mathbb{I}_{2\times 2} \cdot \boldsymbol{n}}{\int_{F_j^f} \psi_j \mathbb{I}_{2\times 2} \cdot \boldsymbol{n}}.$$

Since $1 \lesssim |\int_{F_j^f} \boldsymbol{\varepsilon}(\boldsymbol{\phi}_j) \cdot \boldsymbol{n}|$ and $\mathrm{meas}(F_j^f) = \mathcal{O}(h)$, using the standard approximation property of the Scott-Zhang interpolant we have

$$\sum_{j=1}^{N_p^f} |\alpha_j|^2 \lesssim \sum_{j=1}^{N_p^f} \Big| \int_{F_j^f} \boldsymbol{\varepsilon}(\boldsymbol{u} - i_{\mathrm{SZ}} \boldsymbol{u}) \cdot \boldsymbol{n} \Big|^2 \lesssim h \sum_{j=1}^{N_p^f} \|\nabla(\boldsymbol{u} - i_{\mathrm{SZ}} \boldsymbol{u})\|_{0,F_j^f}^2 \lesssim h^2 |\boldsymbol{u}|_{2,\Omega_f}^2.$$

Also $h \lesssim |\int_{F_j^f} \psi_j \mathbb{I}_{2\times 2}\cdot \boldsymbol{n}|$, then we have

$$\sum_{j=1}^{N_p^f} |\beta_j|^2 \lesssim h^{-2} \sum_{j=1}^{N_p^f} \Big| \int_{F_j^f} (p - i_{\mathrm{SZ}} p)\mathbb{I}_{2\times 2}\cdot \boldsymbol{n}\Big|^2 \lesssim h^{-1} \sum_{j=1}^{N_p^f} \|p - i_{\mathrm{SZ}} p\|^2_{0,F_j^f} \lesssim |p|^2_{1,\Omega_f}.$$

Then we obtain using the approximation property of the Scott-Zhang interpolant

$$\|\boldsymbol{u} - \mathcal{I}_\alpha \boldsymbol{u}\|_{0,\Omega_f} \leq \|\boldsymbol{u} - i_{\mathrm{SZ}}\boldsymbol{u}\|_{0,\Omega_f} + \Big(\sum_{j=1}^{N_p^f} \|\alpha_j \boldsymbol{\phi}_j\|^2_{0,P_j^f}\Big)^{\frac{1}{2}} \lesssim h^2 |\boldsymbol{u}|_{2,\Omega_f} + \Big(h^2 \sum_{j=1}^{N_p^f} |\alpha_j|^2\Big)^{\frac{1}{2}},$$

$$\|p - \mathcal{I}_\beta p\|_{0,\Omega_f} \leq \|p - i_{\mathrm{SZ}} p\|_{0,\Omega_f} + \Big(\sum_{j=1}^{N_p^f} \|\beta_j \psi_j\|^2_{0,P_j^f}\Big)^{\frac{1}{2}} \lesssim h|p|_{1,\Omega_f} + \Big(h^2 \sum_{j=1}^{N_p^f} |\beta_j|^2\Big)^{\frac{1}{2}}.$$

Using the construction of $\mathcal{I}_\gamma$ we have

$$\gamma_j = \frac{\int_{F_j^s} (\dot{\boldsymbol{d}} - \pi_h^s \dot{\boldsymbol{d}})\cdot \boldsymbol{n}}{\int_{F_j^s} \boldsymbol{\varphi}_j \cdot \boldsymbol{n}}.$$

Since $h \lesssim |\int_{F_j^s} \boldsymbol{\varphi}_j \cdot \boldsymbol{n}|$ and using the standard approximation property of the Ritz projection we have

$$\sum_{j=1}^{N_p^s} |\gamma_j|^2 \lesssim h^{-2} \sum_{j=1}^{N_p^s} \Big| \int_{F_j^s} (\dot{\boldsymbol{d}} - \pi_h^s \dot{\boldsymbol{d}})\cdot \boldsymbol{n}\Big|^2 \lesssim h^{-1} \sum_{j=1}^{N_p^s} \|\dot{\boldsymbol{d}} - \pi_h^s \dot{\boldsymbol{d}}\|^2_{0,F_j^s} \lesssim h^2 |\dot{\boldsymbol{d}}|^2_{2,\Omega_s}. \quad (6.13)$$

Then using once again the approximation property of the Ritz projection

$$\|\dot{\boldsymbol{d}} - \mathcal{I}_\gamma \dot{\boldsymbol{d}}\|_{0,\Omega_s} \leq \|\dot{\boldsymbol{d}} - \pi_h^s \dot{\boldsymbol{d}}\|_{0,\Omega_s} + \Big(\sum_{j=1}^{N_p^s} \|\gamma_j \boldsymbol{\varphi}_j\|^2_{0,P_j^s}\Big)^{\frac{1}{2}} \lesssim h^2 |\dot{\boldsymbol{d}}|_{2,\Omega_s} + \Big(h^2 \sum_{j=1}^{N_p^s} |\gamma_j|^2\Big)^{\frac{1}{2}}.$$

The estimates for $\|\nabla(\boldsymbol{u} - \mathcal{I}_\alpha \boldsymbol{u})\|_{0,\Omega_f}$, $\|\nabla(p - \mathcal{I}_\beta p)\|_{0,\Omega_f}$ and $\|\nabla(\dot{\boldsymbol{d}} - \mathcal{I}_\gamma \dot{\boldsymbol{d}})\|_{0,\Omega_s}$ follow similarly using a discrete inverse inequality on $P_j^f$ or $P_j^s$. □

For further use we show the following stability of the interpolant $\mathcal{I}_\beta$ using similar arguments as in the previous proof and the stability of the Scott-Zhang interpolant

$$\|\nabla \mathcal{I}_\beta p\|^2_{0,\Omega_f} \lesssim \|\nabla i_{\mathrm{SZ}} p\|^2_{0,\Omega_f} + \sum_{j=1}^{N_p^f} \|\nabla(\beta_j \psi_j)\|^2_{0,P_j^f} \lesssim \|\nabla p\|^2_{0,\Omega_f}. \quad (6.14)$$

We define the two quantities

$$\mathcal{E}_h^n = \rho_f \|\boldsymbol{\theta}_h^n\|^2_{0,\Omega_f} + \rho_s \|\dot{\boldsymbol{\xi}}_h^n\|^2_{0,\Omega_s} + \|\boldsymbol{\xi}_h^n\|^2_s,$$
$$\mathcal{D}_h^n = \mu \|\nabla \boldsymbol{\theta}_h^n\|^2_{0,\Omega_f} + h^2 \mu^{-1} \|\nabla y_h^n\|^2_{0,\Omega_f} + \mu h^{-1} \|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot \boldsymbol{n}\|^2_{0,\Sigma}.$$

We assume that the exact solution of (6.1)-(6.3) has the following regularity for a given final time $T \geq 0$

$$\begin{aligned}
\boldsymbol{u} &\in [H^1(0,T;H^2(\Omega_f))]^2,\\
\partial_{tt}\boldsymbol{u} &\in [L^2(0,T;L^2(\Omega_f))]^2,\\
p &\in C^0(0,T;H^1(\Omega_f)),\\
\dot{\boldsymbol{d}} &\in [H^1(0,T;H^2(\Omega_s))]^2,\\
\partial_{tt}\dot{\boldsymbol{d}} &\in [L^2(0,T;L^2(\Omega_s))]^2.
\end{aligned} \tag{6.15}$$

**Theorem 6.5.1.** *Let* $(\boldsymbol{u}, p, \boldsymbol{d}, \dot{\boldsymbol{d}})$ *be the solution of the coupled problem* (6.1)-(6.3)*, let* $\{(\boldsymbol{u}_h^n, p_h^n, \boldsymbol{d}_h^n, \dot{\boldsymbol{d}}_h^n)\}_{1\leq n\leq N}$ *be the solution of* (6.7) *at time* $t_n$ *with initialisation* $(\boldsymbol{u}_h^0, \boldsymbol{d}_h^0, \dot{\boldsymbol{d}}_h^0) = (\mathcal{I}_\alpha \boldsymbol{u}^0, \pi_h^s \boldsymbol{d}^0, \mathcal{I}_\gamma \dot{\boldsymbol{d}}^0)$*. Suppose that* $(\boldsymbol{u}, p, \boldsymbol{d}, \dot{\boldsymbol{d}})$ *has the regularity* (6.15)*, then the following error estimate holds for* $N > 0$

$$\mathcal{E}_h^N + \sum_{n=1}^{N} \mathcal{D}_h^n \lesssim \mathcal{E}_h^0 + c_1\tau^2 + c_2 h^2,$$

*where* $c_1, c_2$ *denote positive constants that depends on the physical parameters and the regularity of* $(\boldsymbol{u}, p, \boldsymbol{d}, \dot{\boldsymbol{d}})$ *but are independent of* $h$ *and* $\tau$*.*

*Proof.* Considering the Galerkin orthogonality of Lemma 6.3.1 and introducing the discrete errors $\boldsymbol{\theta}_h^n$, $y_h^n$, $\boldsymbol{\xi}_h^n$ and $\dot{\boldsymbol{\xi}}_h^n$

$$\begin{aligned}
&\rho_f(\partial_\tau \boldsymbol{\theta}_h^n, \boldsymbol{v}_h)_{\Omega_f} + A((\boldsymbol{\theta}_h^n, y_h^n), (\boldsymbol{v}_h, q_h)) + \rho_s(\partial_\tau \dot{\boldsymbol{\xi}}_h^n, \boldsymbol{w}_h)_{\Omega_s} + a^s(\boldsymbol{\xi}_h^n, \boldsymbol{w}_h)\\
&- \langle \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, y_h^n)\cdot \boldsymbol{n}, \boldsymbol{v}_h - \boldsymbol{w}_h\rangle_\Sigma + \langle \boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n, \boldsymbol{\sigma}(\boldsymbol{v}_h, q_h)\cdot \boldsymbol{n}\rangle_\Sigma + S_h(y_h^n, q_h)\\
&\quad = \rho_f((\partial_t - \partial_\tau)\boldsymbol{u}^n, \boldsymbol{v}_h)_{\Omega_f} + \rho_s((\partial_t - \partial_\tau)\dot{\boldsymbol{d}}^n, \boldsymbol{w}_h)_{\Omega_s} + \rho_f(\partial_\tau \boldsymbol{\theta}_\pi^n, \boldsymbol{v}_h)_{\Omega_f}\\
&\qquad + A((\boldsymbol{\theta}_\pi^n, y_\pi^n), (\boldsymbol{v}_h, q_h)) + \rho_s(\partial_\tau \dot{\boldsymbol{\xi}}_\pi^n, \boldsymbol{w}_h)_{\Omega_s} + a^s(\boldsymbol{\xi}_\pi^n, \boldsymbol{w}_h) - S_h(\mathcal{I}_\beta p^n, q_h)\\
&\qquad - \langle \boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot \boldsymbol{n}, \boldsymbol{v}_h - \boldsymbol{w}_h\rangle_\Sigma + \langle \boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^n, \boldsymbol{\sigma}(\boldsymbol{v}_h, q_h)\cdot \boldsymbol{n}\rangle_\Sigma.
\end{aligned}$$

We choose the test functions such that $\boldsymbol{v}_h = \tau\boldsymbol{\theta}_h^n$, $q_h = \tau y_h^n + \zeta_h$, $\boldsymbol{w}_h = \tau\dot{\boldsymbol{\xi}}_h^n$, with $\zeta_h|_\Sigma = \varsigma\frac{\tau\mu}{h}(\dot{\boldsymbol{\xi}}_h^n - \boldsymbol{\theta}_h^n)\cdot\boldsymbol{n}$ ($\zeta_h \in Q_h^f$ is constructed in the same way as $\kappa_h$ in the proof of Theorem 6.4.1). By definition of the Ritz-projection we have $a^s(\boldsymbol{\xi}_\pi^n, \boldsymbol{w}_h) = 0$, then rearranging similarly as in the proof of Theorem 6.4.1 and using (6.12) we obtain

$$
\begin{aligned}
&\frac{\rho_f}{2}(\tau\partial_\tau\|\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2 + \|\boldsymbol{\theta}_h^n - \boldsymbol{\theta}_h^{n-1}\|_{0,\Omega_f}^2) + 2C_K\tau\mu\|\nabla\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2 + \tau\gamma_p h^2\mu^{-1}\|\nabla y_h^n\|_{0,\Omega_f}^2 \\
&+ \frac{\rho_s}{2}(\tau\partial_\tau\|\dot{\boldsymbol{\xi}}_h^n\|_{0,\Omega_s}^2 + \|\dot{\boldsymbol{\xi}}_h^n - \dot{\boldsymbol{\xi}}_h^{n-1}\|_{0,\Omega_s}^2) + \frac{1}{2}(\tau\partial_\tau\|\boldsymbol{\xi}_h^n\|_s^2 + \|\boldsymbol{\xi}_h^n - \boldsymbol{\xi}_h^{n-1}\|_s^2) \\
&+ \varsigma\frac{\tau\mu}{h}\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2 + (\zeta_h, \nabla\cdot\boldsymbol{\theta}_h^n)_{\Omega_f} + S_h(y_h^n, \zeta_h) \\
&\qquad\leq \underbrace{\rho_f\tau((\partial_t - \partial_\tau)\boldsymbol{u}^n, \boldsymbol{\theta}_h^n)_{\Omega_f} + \rho_f\tau(\partial_\tau\boldsymbol{\theta}_\pi^n, \boldsymbol{\theta}_h^n)_{\Omega_f}}_{T_1} \\
&\qquad\quad \underbrace{+\rho_s\tau(\partial_\tau\dot{\boldsymbol{\xi}}_\pi^n, \dot{\boldsymbol{\xi}}_h^n)_{\Omega_s} + \rho_s\tau((\partial_t - \partial_\tau)\dot{\boldsymbol{d}}^n, \dot{\boldsymbol{\xi}}_h^n)_{\Omega_s}}_{T_2} \underbrace{-\tau a^s(\boldsymbol{\xi}_h^n, z_h^n)}_{T_3} \\
&\qquad\quad \underbrace{+\tau\langle\boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, y_h^n)\cdot\boldsymbol{n}\rangle_\Sigma + \tau A((\boldsymbol{\theta}_\pi^n, y_\pi^n), (\boldsymbol{\theta}_h^n, y_h^n))}_{T_4} \\
&\qquad\quad \underbrace{+\varsigma\frac{\tau\mu}{h}\langle(\boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^n)\cdot\boldsymbol{n}, (\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\rangle_\Sigma}_{T_5} \underbrace{+(\zeta_h, \nabla\cdot\boldsymbol{\theta}_\pi^n)_{\Omega_f}}_{T_6} \\
&\qquad\quad \underbrace{-\tau\langle\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n\rangle_\Sigma}_{T_7} \underbrace{-\tau S_h(\mathcal{I}_\beta p^n, y_h^n)}_{T_8} \underbrace{-S_h(\mathcal{I}_\beta p^n, \zeta_h)}_{T_9}.
\end{aligned}
$$

First we consider the last two terms of the left hand side. The terms $(\zeta_h, \nabla\cdot\boldsymbol{\theta}_h^n)_\Omega$ and $S_h(y_h^n, \zeta_h)$ are handled in the same way as in the proof of stability

$$
\begin{aligned}
(\zeta_h, \nabla\cdot\boldsymbol{\theta}_h^n)_\Omega &\leq \epsilon\tau\mu\|\nabla\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2 + \varsigma^2 C\frac{\tau\mu}{4\epsilon h}\|(\dot{\boldsymbol{\xi}}_h^n - \boldsymbol{\theta}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2, \\
S_h(y_h^n, \zeta_h) &\leq \epsilon\tau\gamma_p h^2\mu^{-1}\|\nabla y_h^n\|_{0,\Omega_f}^2 + \varsigma^2 C\gamma_p\frac{\tau\mu}{4\epsilon h}\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2.
\end{aligned}
$$

In order to handle the term $T_1$ we need the following result. Considering a second order Taylor expansion and its remainder we see that

$$
\partial_\tau\boldsymbol{u}^n = \tau^{-1}(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}) = \partial_t\boldsymbol{u}^n - \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\partial_{tt}\boldsymbol{u}(t)(t - (t_n - \tau))\mathrm{d}t,
$$

then using Cauchy-Schwarz inequality we get

$$
(\partial_t\boldsymbol{u}^n - \partial_\tau\boldsymbol{u}^n)^2 \leq \frac{1}{\tau^2}\int_{t_{n-1}}^{t_n}(t - (t_n - \tau))^2\mathrm{d}t\int_{t_{n-1}}^{t_n}(\partial_{tt}\boldsymbol{u}(t))^2\mathrm{d}t = \frac{\tau}{3}\int_{t_{n-1}}^{t_n}(\partial_{tt}\boldsymbol{u}(t))^2\mathrm{d}t. \tag{6.16}
$$

Using this result, a first order Taylor expansion, the Young's inequality and Lemma 6.5.2 we get

$$
\begin{aligned}
T_1 &\leq \rho_f\tau(\|\partial_t\boldsymbol{u}^n - \partial_\tau\boldsymbol{u}^n\|_{0,\Omega_f} + \|\partial_\tau\boldsymbol{\theta}_\pi^n\|_{0,\Omega_f})\|\boldsymbol{\theta}_h^n\|_{0,\Omega_f} \\
&\leq \rho_f T(\tau\|\partial_t\boldsymbol{u}^n - \partial_\tau\boldsymbol{u}^n\|_{0,\Omega_f}^2 + \tau\|\partial_\tau\boldsymbol{\theta}_\pi^n\|_{0,\Omega_f}^2) + \frac{\rho_f\tau}{2T}\|\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2 \\
&\lesssim \rho_f T(\tau^2\|\partial_{tt}\boldsymbol{u}\|_{L^2(t_{n-1},t_n;L^2(\Omega_f))}^2 + \|\partial_t\boldsymbol{\theta}_\pi\|_{L^2(t_{n-1},t_n;L^2(\Omega_f))}^2) + \frac{\rho_f\tau}{2T}\|\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2 \\
&\lesssim \rho_f T(\tau^2\|\partial_{tt}\boldsymbol{u}\|_{L^2(t_{n-1},t_n;L^2(\Omega_f))}^2 + h^2\|\partial_t\boldsymbol{u}\|_{L^2(t_{n-1},t_n;H^1(\Omega_f))}^2) + \frac{\rho_f\tau}{2T}\|\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2.
\end{aligned}
$$

with $\|v\|^2_{L^2(t_{n-1},t_n;L^2(\omega))} = \int_{t_{n-1}}^{t_n} \|v\|^2_{0,\omega}\mathrm{d}t$, $\|v\|^2_{L^2(t_{n-1},t_n;H^s(\omega))} = \int_{t_{n-1}}^{t_n} \|v\|^2_{s,\omega}\mathrm{d}t$, and $t_N = T$ the final time. Using similar arguments the term $T_2$ can be bounded in the following way

$$\begin{aligned}
T_2 &\leq \rho_s\tau(\|\partial_t\dot{\boldsymbol{d}}^n - \partial_\tau\dot{\boldsymbol{d}}^n\|_{0,\Omega_s} + \|\partial_\tau\dot{\boldsymbol{\xi}}^n_\pi\|_{0,\Omega_s})\|\dot{\boldsymbol{\xi}}^n_h\|_{0,\Omega_s} \\
&\leq \rho_s T(\tau\|\partial_t\dot{\boldsymbol{d}}^n - \partial_\tau\dot{\boldsymbol{d}}^n\|^2_{0,\Omega_s} + \tau\|\partial_\tau\dot{\boldsymbol{\xi}}^n_\pi\|^2_{0,\Omega_s}) + \frac{\tau\rho_s}{2T}\|\dot{\boldsymbol{\xi}}^n_h\|^2_{0,\Omega_s} \\
&\lesssim \rho_s T(\tau^2\|\partial_{tt}\dot{\boldsymbol{d}}\|^2_{L^2(t_{n-1},t_n;L^2(\Omega_s))} + \|\partial_t\dot{\boldsymbol{\xi}}_\pi\|^2_{L^2(t_{n-1},t_n;L^2(\Omega_s))}) + \frac{\tau\rho_s}{2T}\|\dot{\boldsymbol{\xi}}^n_h\|^2_{0,\Omega_s} \\
&\lesssim \rho_s T(\tau^2\|\partial_{tt}\dot{\boldsymbol{d}}\|^2_{L^2(t_{n-1},t_n;L^2(\Omega_s))} + h^2\|\partial_t\dot{\boldsymbol{d}}\|^2_{L^2(t_{n-1},t_n;H^1(\Omega_s))}) + \frac{\tau\rho_s}{2T}\|\dot{\boldsymbol{\xi}}^n_h\|^2_{0,\Omega_s}.
\end{aligned}$$

Using that $a^s(\boldsymbol{\xi}^n_h, \boldsymbol{\xi}^n_\pi) = 0$, a triangle inequality, Lemma 6.5.2 and (6.16) we obtain

$$\begin{aligned}
T_3 &= \tau a^s(\boldsymbol{\xi}^n_h, \mathcal{I}_\gamma\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n) \leq \tau\|\boldsymbol{\xi}^n_h\|_s\|\mathcal{I}_\gamma\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n\|_s \\
&\leq \tau\|\boldsymbol{\xi}^n_h\|_s(L_1^{\frac{1}{2}} + L_2^{\frac{1}{2}})\|\nabla(\mathcal{I}_\gamma\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n)\|_{0,\Omega_s} \\
&\leq \tau\|\boldsymbol{\xi}^n_h\|_s(L_1^{\frac{1}{2}} + L_2^{\frac{1}{2}})(\|\nabla(\mathcal{I}_\gamma\dot{\boldsymbol{d}}^n - \dot{\boldsymbol{d}}^n)\|_{0,\Omega_s} + \|\nabla(\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n)\|_{0,\Omega_s}) \\
&\leq \frac{\tau}{2T}\|\boldsymbol{\xi}^n_h\|^2_s + \tau T(L_1+L_2)(\|\nabla(\mathcal{I}_\gamma\dot{\boldsymbol{d}}^n - \dot{\boldsymbol{d}}^n)\|^2_{0,\Omega_s} + \|\nabla(\partial_t\boldsymbol{d}^n - \partial_\tau\boldsymbol{d}^n)\|^2_{0,\Omega_s}) \\
&\lesssim \frac{\tau}{2T}\|\boldsymbol{\xi}^n_h\|^2_s + T(L_1+L_2)(\tau h^2\|\dot{\boldsymbol{d}}^n\|^2_{2,\Omega_s} + \tau^2\|\partial_t\dot{\boldsymbol{d}}\|^2_{L^2(t_{n-1},t_n;H^1(\Omega_s))}).
\end{aligned}$$

Using integration by parts, the term $T_4$ can be written as

$$\begin{aligned}
T_4 = &\underbrace{\tau(2\mu\boldsymbol{\varepsilon}(\boldsymbol{\theta}^n_\pi), \boldsymbol{\varepsilon}(\boldsymbol{\theta}^n_h))_{\Omega_f} + \tau\langle\boldsymbol{\theta}^n_\pi - \dot{\boldsymbol{\xi}}^n_\pi, 2\mu\boldsymbol{\varepsilon}(\boldsymbol{\theta}^n_h)\cdot\boldsymbol{n}\rangle_\Sigma}_{T_{41}} \\
&\underbrace{-\tau(y^n_\pi, \nabla\cdot\boldsymbol{\theta}^n_h)_{\Omega_f}}_{T_{42}}\underbrace{-\tau(\nabla y^n_h, \boldsymbol{\theta}^n_\pi)_{\Omega_f}}_{T_{43}}\underbrace{+\tau\langle\dot{\boldsymbol{\xi}}^n_\pi\cdot\boldsymbol{n}, y^n_h\rangle_\Sigma}_{T_{44}},
\end{aligned}$$

using the trace and inverse inequalities and Lemma 6.5.2 we can write

$$\begin{aligned}
T_{41} &\leq 2\mu\tau(\|\boldsymbol{\varepsilon}(\boldsymbol{\theta}^n_\pi)\|_{0,\Omega_f}\|\boldsymbol{\varepsilon}(\boldsymbol{\theta}^n_h)\|_{0,\Omega_f} + \|\boldsymbol{\theta}^n_\pi - \dot{\boldsymbol{\xi}}^n_\pi\|_{0,\Sigma}\|\boldsymbol{\varepsilon}(\boldsymbol{\theta}^n_h)\cdot\boldsymbol{n}\|_{0,\Sigma}) \\
&\lesssim 2\mu\tau\|\nabla\boldsymbol{\theta}^n_h\|_{0,\Omega_f}(\|\nabla\boldsymbol{\theta}^n_\pi\|_{0,\Omega_f} + h^{-\frac{1}{2}}(\|\boldsymbol{\theta}^n_\pi\|_{0,\Sigma} + \|\dot{\boldsymbol{\xi}}^n_\pi\|_{0,\Sigma})) \\
&\lesssim 2C\epsilon\mu\tau\|\nabla\boldsymbol{\theta}^n_h\|^2_{0,\Omega_f} + \frac{2\mu}{\epsilon}\tau h^2\|\boldsymbol{u}^n\|^2_{2,\Omega_f} + \frac{\mu}{\epsilon}\tau h^2\|\dot{\boldsymbol{d}}^n\|^2_{2,\Omega_s}.
\end{aligned}$$

also using Lemma 6.5.2

$$\begin{aligned}
T_{42} &\leq \tau\|y^n_\pi\|_{0,\Omega_f}\|\nabla\cdot\boldsymbol{\theta}^n_h\|_{0,\Omega_f} \leq \frac{\tau}{4\epsilon\mu}h^2\|p^n\|^2_{1,\Omega} + C\epsilon\tau\mu\|\nabla\boldsymbol{\theta}^n_h\|^2_{0,\Omega_f}, \\
T_{43} &\leq \tau\|\nabla y^n_h\|_{0,\Omega_f}\|\boldsymbol{\theta}^n_\pi\|_{0,\Omega_f} \leq C\frac{\epsilon}{\mu}\tau h^2\|\nabla y^n_h\|^2_{0,\Omega_f} + \frac{\mu}{4\epsilon}\tau h^2\|\boldsymbol{u}^n\|^2_{2,\Omega_f}.
\end{aligned}$$

Let $\overline{y_h^n}^{F_j^s}$ is the average of $y_h^n$ on the face $F_j^s$, then $\|y_h^n - \overline{y_h^n}^{F_j^s}\|_{0,F_j^s} \lesssim h\|\nabla y_h^n\|_{0,F_j^s}$, using this property and (6.11) we can write

$$\begin{aligned} T_{44} &= \sum_{j=1}^{N_p^s} \tau\langle \dot{\boldsymbol{\xi}}_\pi^n \cdot \boldsymbol{n}, y_h^n - \overline{y_h^n}^{F_j^s}\rangle_{F_j^s} \leq \sum_{j=1}^{N_p^s} \tau\|\dot{\boldsymbol{\xi}}_\pi^n \cdot \boldsymbol{n}\|_{0,F_j^s} h\|\nabla y_h^n\|_{0,F_j^s} \\ &\leq \tau h^2 \|\dot{\boldsymbol{d}}^n\|_{2,\Omega_s}\|\nabla y_h^n\|_{0,\Omega_f} \leq \frac{\mu}{4\epsilon}\tau h^2\|\dot{\boldsymbol{d}}^n\|_{2,\Omega_s}^2 + C\frac{\epsilon}{\mu}\tau h^2\|\nabla y_h^n\|_{0,\Omega_f}^2. \end{aligned}$$

The term $T_5$ is controlled using Lemma 6.5.2 such that

$$\begin{aligned} T_5 &\leq \varsigma\frac{\tau\mu}{h}\|(\boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^n)\cdot\boldsymbol{n}\|_{0,\Sigma}\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma} \\ &\leq \epsilon\tau\mu h^2(\|\boldsymbol{u}^n\|_{2,\Omega_f}^2 + \|\dot{\boldsymbol{d}}^n\|_{2,\Omega_s}^2) + \varsigma^2 C\frac{\tau\mu}{2\epsilon h}\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2. \end{aligned}$$

Using that $\|\zeta_h\|_{0,P_\Sigma} \lesssim h^{\frac{1}{2}}\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}$ we have

$$\begin{aligned} T_6 &\leq \varsigma\frac{\tau\mu}{h}\|\zeta_h\|_{0,P_\Sigma}\|\nabla\boldsymbol{\theta}_\pi^n\|_{0,\Omega_f} \\ &\leq \varsigma^2 C\frac{\tau\mu}{4\epsilon h}\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2 + \epsilon\tau\mu\|\nabla\boldsymbol{\theta}_\pi^n\|_{0,\Omega_f}^2 \\ &\leq \varsigma^2 C\frac{\tau\mu}{4\epsilon h}\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2 + \epsilon\tau\mu h^2\|\boldsymbol{u}^n\|_{2,\Omega_f}^2. \end{aligned}$$

The term $T_7$ can be expressed such that

$$\begin{aligned} T_7 = &\underbrace{-\tau\langle\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \boldsymbol{\theta}_h^n\rangle_\Sigma}_{T_{71}} \underbrace{+\tau\langle\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \partial_\tau\boldsymbol{\xi}_h^n\rangle_\Sigma}_{T_{72}} \\ &\underbrace{-\tau\langle\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \pi_h^s(\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n)\rangle_\Sigma}_{T_{73}} \underbrace{-\tau\langle\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \sum_{j=1}^{N_p^s}\gamma_j\boldsymbol{\varphi}_j\rangle_\Sigma}_{T_{74}} \end{aligned}$$

Let $\overline{\boldsymbol{\theta}_h^n}^{F_j^f}$ is the average of $\boldsymbol{\theta}_h^n$ on the interval $F_j^f$, then $\|\boldsymbol{\theta}_h^n - \overline{\boldsymbol{\theta}_h^n}^{F_j^f}\|_{0,F_j^f} \lesssim h\|\nabla\boldsymbol{\theta}_h^n\|_{0,F_j^f}$, using this property and (6.10) we get

$$\begin{aligned} T_{71} &= -\sum_{j=1}^{N_p^f}\tau\langle\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \boldsymbol{\theta}_h^n - \overline{\boldsymbol{\theta}_h^n}^{F_j^f}\rangle_{F_j^f} \leq \sum_{j=1}^{N_p^f}\tau\|\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}\|_{0,F_j^f} h\|\nabla\boldsymbol{\theta}_h^n\|_{0,F_j^f} \\ &\leq \frac{\tau}{2\epsilon}h^2(\mu\|\boldsymbol{u}^n\|_{2,\Omega_f}^2 + \mu^{-1}\|p^n\|_{1,\Omega_f}^2) + C\epsilon\tau\mu\|\nabla\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2. \end{aligned}$$

The term $T_{72}$ is handled using a summation by parts

$$\sum_{n=1}^{N} T_{72} = \langle\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^N, y_\pi^N)\cdot\boldsymbol{n}, \boldsymbol{\xi}_h^N\rangle_\Sigma - (\boldsymbol{\sigma}\langle\boldsymbol{\theta}_\pi^1, y_\pi^1)\cdot\boldsymbol{n}, \boldsymbol{\xi}_h^0\rangle_\Sigma - \sum_{n=2}^{N}\tau\langle\partial_\tau\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \boldsymbol{\xi}_h^{n-1}\rangle_\Sigma.$$

Using (6.10) and the average on $F_j^f$ once again and using Korn's inequality we have

$$\begin{aligned}\langle \boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^N, y_\pi^N)\cdot\boldsymbol{n}, \boldsymbol{\xi}_h^N\rangle_\Sigma &= \sum_{j=1}^{N_p^f}\langle \boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^N, y_\pi^N)\cdot\boldsymbol{n}, \boldsymbol{\xi}_h^N - \overline{\boldsymbol{\xi}_h^N}^{F_j^f}\rangle_{F_j^f}\\ &\lesssim \|\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^N, y_\pi^N)\cdot\boldsymbol{n}\|_{0,\Sigma} h\|\nabla\boldsymbol{\xi}_h^N\|_{0,\Sigma}\\ &\le \frac{1}{\epsilon}h^2(\mu\|\boldsymbol{u}^N\|_{2,\Omega_f}^2 + \mu^{-1}\|p^N\|_{1,\Omega_f}^2) + C\frac{\epsilon\mu}{2L_1}\|\boldsymbol{\xi}_h^N\|_s^2,\end{aligned}$$

similarly

$$\langle \boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^1, y_\pi^1)\cdot\boldsymbol{n}, \boldsymbol{\xi}_h^0\rangle_\Sigma \le \frac{1}{\epsilon}h^2(\mu\|\boldsymbol{u}^1\|_{2,\Omega_f}^2 + \mu^{-1}\|p^1\|_{1,\Omega_f}^2) + C\frac{\epsilon\mu}{2L_1}\|\boldsymbol{\xi}_h^0\|_s^2,$$

and by linearity of the operator $\partial_\tau$ we obtain

$$\begin{aligned}\tau\langle \partial_\tau\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \boldsymbol{\xi}_h^{n-1}\rangle_\Sigma &= \sum_{j=1}^{N_p^f}\langle \partial_\tau\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \boldsymbol{\xi}_h^{n-1} - \overline{\boldsymbol{\xi}_h^{n-1}}^{F_j^f}\rangle_{F_j^f}\\ &\lesssim \tau\|\boldsymbol{\sigma}(\partial_\tau\boldsymbol{\theta}_\pi^n, \partial_\tau y_\pi^n)\cdot\boldsymbol{n}\|_{0,\Sigma} h\|\nabla\boldsymbol{\xi}_h^{n-1}\|_{0,\Sigma}\\ &\le \frac{T\tau}{\epsilon}h^2(\mu\|\partial_\tau\boldsymbol{u}^n\|_{2,\Omega_f}^2 + \mu^{-1}\|\partial_\tau p^n\|_{1,\Omega_f}^2) + C\frac{\epsilon\tau\mu}{2TL_1}\|\boldsymbol{\xi}_h^{n-1}\|_s^2.\end{aligned}$$

Using the stability of the Ritz projection, a Taylor expansion and (6.10) one more time we have

$$\begin{aligned}T_{73} &= -\sum_{j=1}^{N_p^f}\tau\langle \boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}, \pi_h^s(\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n) - \overline{\pi_h^s(\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n)}^{F_j^f}\rangle_{F_j^f}\\ &\le \tau\|\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}\|_{0,\Sigma} h\|\nabla\pi_h^s(\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n)\|_{0,\Sigma}\\ &\lesssim \frac{\tau}{2T}h^2(\mu\|\boldsymbol{u}^n\|_{2,\Omega_f}^2 + \mu^{-1}\|p^n\|_{1,\Omega_f}^2) + T\mu\tau\|\nabla(\dot{\boldsymbol{d}}^n - \partial_\tau\boldsymbol{d}^n)\|_{0,\Omega_s}^2\\ &\lesssim \frac{\tau}{2T}h^2(\mu\|\boldsymbol{u}^n\|_{2,\Omega_f}^2 + \mu^{-1}\|p^n\|_{1,\Omega_f}^2) + T\mu\tau^2\|\partial_t\dot{\boldsymbol{d}}\|_{L^2(t_{n-1},t_n;H^1(\Omega_s))}^2.\end{aligned}$$

Finally using (6.13) and classical inequalities

$$\begin{aligned}T_{74} &\le \tau h^{\frac{1}{2}}\|\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, y_\pi^n)\cdot\boldsymbol{n}\|_{0,\Sigma} h^{-\frac{1}{2}}\|\sum_{j=1}^{N_p^s}\gamma_j\boldsymbol{\varphi}_j\|_{0,\Sigma}\\ &\lesssim \frac{\tau}{2}h^2(\mu\|\boldsymbol{u}^n\|_{2,\Omega_f}^2 + \mu^{-1}\|p^n\|_{1,\Omega_f}^2) + \mu\tau h^{-2}\sum_{j=1}^{N_p^s}\|\gamma_j\boldsymbol{\varphi}_j\|_{0,P_j^s}^2\\ &\lesssim \frac{\tau}{2}h^2(\mu\|\boldsymbol{u}^n\|_{2,\Omega_f}^2 + \mu^{-1}\|p^n\|_{1,\Omega_f}^2) + \mu\tau h^2\|\dot{\boldsymbol{d}}^n\|_{2,\Omega_s}^2.\end{aligned}$$

Using the stability of the interpolation operator $\mathcal{I}_\beta$ (6.14) the term $T_8$ is bounded such that

$$\begin{aligned} T_8 &\le \tau S_h(\mathcal{I}_\beta p^n, \mathcal{I}_\beta p^n)^{\frac{1}{2}} S_h(y_h^n, y_h^n)^{\frac{1}{2}} \\ &\le \tau\gamma_p h^2 \mu^{-1} \|\nabla \mathcal{I}_\beta p^n\|_{0,\Omega_f} \|\nabla y_h^n\|_{0,\Omega_f} \\ &\le \frac{\tau\gamma_p}{4\epsilon\mu} h^2 \|\nabla p^n\|_{0,\Omega_f}^2 + \epsilon\tau\gamma_p h^2\mu^{-1}\|\nabla y_h^n\|_{0,\Omega_f}^2 \\ &\le \frac{\tau\gamma_p}{4\epsilon\mu} h^2 \|p^n\|_{1,\Omega_f}^2 + C\epsilon\tau\gamma_p h^2\mu^{-1}\|\nabla y_h^n\|_{0,\Omega_f}^2. \end{aligned}$$

Using similar arguments, the inverse inequality and the properties of $\zeta_h$ we obtain

$$\begin{aligned} T_9 &\le S_h(\mathcal{I}_\beta p^n, \mathcal{I}_\beta p^n)^{\frac{1}{2}} S_h(\zeta_h, \zeta_h)^{\frac{1}{2}} \\ &\le \gamma_p h^2 \mu^{-1} \|\nabla \mathcal{I}_\beta p^n\|_{0,\Omega_f} \|\nabla \zeta_h\|_{0,P_\Sigma} \\ &\le C\gamma_p h^{\frac{1}{2}} \|\nabla p^n\|_{0,\Omega_f} \varsigma\tau \|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma} \\ &\le \epsilon\frac{\tau\gamma_p}{\mu} h^2\|p^n\|_{1,\Omega_f}^2 + C\varsigma^2\gamma_p\frac{\tau\mu}{4\epsilon h}\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2 \end{aligned}$$

Collecting the terms $T_1$-$T_9$ and taking the sum from $t_1$ to $t_N$ we get

$$\begin{aligned} &\frac{\rho_f}{2}\|\boldsymbol{\theta}_h^N\|_{0,\Omega_f}^2 + \frac{\rho_s}{2}\|\dot{\boldsymbol{\xi}}_h^N\|_{0,\Omega_s}^2 + \frac{1}{2}\Big(1-\epsilon C\frac{\mu}{L_1}\Big)\|\boldsymbol{\xi}_h^N\|_s^2 + \tau\sum_{n=1}^N\Big[\frac{\rho_f}{2\tau}\|\boldsymbol{\theta}_h^n - \boldsymbol{\theta}_h^{n-1}\|_{0,\Omega_f}^2 \\ &+ \frac{\rho_s}{2\tau}\|\dot{\boldsymbol{\xi}}_h^n - \dot{\boldsymbol{\xi}}_h^{n-1}\|_{0,\Omega_s}^2 + \frac{1}{2\tau}\|\boldsymbol{\xi}_h^n - \boldsymbol{\xi}_h^{n-1}\|_s^2 + \mu(2C_K - 5\epsilon C)\|\nabla\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2 \\ &+ h^2\mu^{-1}(\gamma_p - 2\epsilon C(1+\gamma_p))\|\nabla y_h^n\|_{0,\Omega_f}^2 + \varsigma\frac{\mu}{h}\Big(1-\varsigma C\frac{2+\gamma_p}{2\epsilon}\Big)\|(\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)\cdot\boldsymbol{n}\|_{0,\Sigma}^2\Big] \\ &\quad\le \frac{\rho_f}{2}\|\boldsymbol{\theta}_h^0\|_{0,\Omega_f}^2 + \frac{\rho_s}{2}\|\dot{\boldsymbol{\xi}}_h^0\|_{0,\Omega_s}^2 + \frac{1}{2}\Big(1+\epsilon\frac{\mu}{L_1}\Big)\|\boldsymbol{\xi}_h^0\|_s^2 \\ &\qquad + CT\tau^2(\rho_f\|\partial_{tt}\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \rho_s\|\partial_{tt}\dot{\boldsymbol{d}}\|_{L^2(0,T;L^2(\Omega_s))}^2 \\ &\qquad\quad + (L_1+L_2+\mu)\|\partial_t\dot{\boldsymbol{d}}\|_{L^2(0,T;H^1(\Omega_s))}^2) \\ &\qquad + CTh^2(\rho_f\|\partial_t\boldsymbol{u}\|_{L^2(0,T;H^1(\Omega_f))}^2 + \rho_s\|\partial_t\dot{\boldsymbol{d}}\|_{L^2(0,T;H^1(\Omega_s))}^2) \\ &\qquad + C\tau h^2\sum_{n=1}^N\Big[T(L_1+L_2)\|\dot{\boldsymbol{d}}^n\|_{2,\Omega_s}^2 + \mu(1+T^{-1})\|\boldsymbol{u}^n\|_{2,\Omega_f}^2 + \frac{1+T^{-1}+\gamma_p}{\mu}\|p^n\|_{1,\Omega}^2 \\ &\qquad\quad + \mu\|\dot{\boldsymbol{d}}^n\|_{2,\Omega_s}^2 + T\mu\|\partial_\tau\boldsymbol{u}^n\|_{2,\Omega_f}^2 + T\mu^{-1}\|\partial_\tau p^n\|_{1,\Omega_f}^2\Big] \\ &\qquad + Ch^2(\mu\|\boldsymbol{u}^1\|_{2,\Omega_f}^2 + \mu^{-1}\|p^1\|_{1,\Omega_f}^2 + \mu\|\boldsymbol{u}^N\|_{2,\Omega_f}^2 + \mu^{-1}\|p^N\|_{1,\Omega_f}^2) \\ &\qquad + C\frac{\tau}{T}\sum_{n=1}^N\Big[\frac{\rho_f}{2}\|\boldsymbol{\theta}_h^n\|_{0,\Omega_f}^2 + \frac{\rho_s}{2}\|\dot{\boldsymbol{\xi}}_h^n\|_{0,\Omega_s}^2 + \frac{1}{2}\Big(1+\epsilon\frac{\mu}{L_1}\Big)\|\boldsymbol{\xi}_h^n\|_s^2\Big], \end{aligned}$$

with $\varsigma$ and $\epsilon$ chosen in the right way. Then we conclude applying Lemma 6.5.1 with

$$a_n = \frac{\rho_f}{2}\|\boldsymbol{\theta}_h^N\|_{0,\Omega_f}^2 + \frac{\rho_s}{2}\|\dot{\boldsymbol{\xi}}_h^N\|_{0,\Omega_s}^2 + \frac{1}{2}\Big(1-\epsilon C\frac{\mu}{L_1}\Big)\|\boldsymbol{\xi}_h^N\|_s^2, \quad \eta_n = \max\Big(\frac{1}{T}, \frac{1}{T}\frac{L_1+\epsilon\mu}{L_1-\epsilon C\mu}\Big).$$

□

We define the energy and dissipation errors such that

$$\mathcal{E}^n = \rho_f \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|^2_{0,\Omega_f} + \rho_s \|\dot{\boldsymbol{d}}^n - \dot{\boldsymbol{d}}_h^n\|^2_{0,\Omega_s} + \|\boldsymbol{d}^n - \boldsymbol{d}_h^n\|^2_s,$$
$$\mathcal{D}^n = \mu \|\nabla(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|^2_{0,\Omega_f} + h^2\mu^{-1}\|\nabla(p^h - p_h^n)\|^2_{0,\Omega_f} + \mu h^{-1}\|(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^n)\cdot \boldsymbol{n}\|^2_{0,\Sigma}.$$

**Corollary 6.5.1.** *Under the assumptions of Theorem 6.5.1 the following error estimate holds for $N > 0$*

$$\mathcal{E}^N + \sum_{n=1}^{N} \mathcal{D}^n \lesssim \mathcal{E}^0 + \tilde{c}_1 \tau^2 + \tilde{c}_2 h^2,$$

*where $\tilde{c}_1, \tilde{c}_2$ denote positive constants that depends on the physical parameters and the regularity of $(\boldsymbol{u}, p, \boldsymbol{d}, \dot{\boldsymbol{d}})$ but independent of $h$ and $\tau$.*

*Proof.* Applying triangle inequality we get

$$\mathcal{E}^n \leq \mathcal{E}_h^n + \rho_f \|\boldsymbol{u}^n - \mathcal{I}_\alpha \boldsymbol{u}^n\|^2_{0,\Omega_f} + \rho_s \|\dot{\boldsymbol{d}}^n - \pi_h^s \dot{\boldsymbol{d}}^n\|^2_{0,\Omega_s} + \|\boldsymbol{d}^n - \mathcal{I}_\gamma \boldsymbol{d}^n\|^2_s,$$
$$\mathcal{D}^n \leq \mathcal{D}_h^n + \mu \|\nabla(\boldsymbol{u}^n - \mathcal{I}_\alpha \boldsymbol{u}^n)\|^2_{0,\Omega_f} + h^2\mu^{-1}\|\nabla(p^n - \mathcal{I}_\beta p^n)\|^2_{0,\Omega_f}.$$

The claim follows directly by applying the interpolation results of Lemma 6.5.2 and Lemma 6.5.2. □

## 6.6 Extension to the unfitted case

In the previous sections the fluid-solid interface $\Sigma$ coincides exactly with faces of the triangulation $\mathcal{T}_h$. In this section we present the tools needed to extend the stability and the convergence to the unfitted case when $\Sigma$ does not coincide with $\mathcal{T}_h$. In this case we let $\Omega_f^* = \{K \in \mathcal{T}_h \mid K \cap \Omega_f \neq \emptyset\}$, $\Omega_s^* = \{K \in \mathcal{T}_h \mid K \cap \Omega_s \neq \emptyset\}$ and $G_h = \{K \in \mathcal{T}_h \mid K \cap \Sigma \neq \emptyset\}$. First, the pressure stabilisation (6.4) (or (6.5)) has to be considered on the full domain $\Omega_f^*$ such that

$$S_h(p_h, q_h) = \gamma_p \int_{\Omega_f^*} \frac{h^2}{\mu} \nabla p_h \nabla q_h \, \mathrm{d}x.$$

As in the fitted case this term ensures the stabilisation of the pressure over the physical domain, in the unfitted framework it also ensures the stability of the coupling over the unfitted boundary playing the role of the pressure ghost penalty (see Chapters 3 and 5) at the interface. The fluid velocity needs to be stabilised at the interface for $K \cap \Omega_f$ small, once again we use the ghost penalty [25]

$$J_h^f(\boldsymbol{u}_h, \boldsymbol{v}_h) = \gamma_g \sum_{F \in \mathcal{F}_G^f} \int_F \mu h [\![\nabla \boldsymbol{u}_h \cdot \boldsymbol{n}_F]\!]_F [\![\nabla \boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!]_F \, \mathrm{d}s,$$

with $\mathcal{F}_G^f = \{F \in G_h \mid F \cap \Omega_f \neq \emptyset\}$. This ghost penalty operator brings additional control of the fluid velocity in the domain $\Omega_f^*$, such that

$$\mu \|\nabla \boldsymbol{v}_h\|^2_{0,\Omega_j^*} \lesssim \mu \|\nabla \boldsymbol{v}_h\|^2_{0,\Omega_j} + J_h^f(\boldsymbol{v}_h, \boldsymbol{v}_h).$$

In the same way the solid displacement is stabilised at the interface such that

$$J_h^s(\boldsymbol{d}_h, \boldsymbol{w}_h) = \gamma_g \sum_{F \in \mathcal{F}_G^s} \int_F (L_1 + L_2) h [\![\nabla \boldsymbol{d}_h \cdot \boldsymbol{n}_F]\!]_F [\![\nabla \boldsymbol{w}_h \cdot \boldsymbol{n}_F]\!]_F \, \mathrm{d}s,$$

with $\mathcal{F}_G^s = \{F \in G_h \mid F \cap \Omega_s \neq \emptyset\}$, and

$$\mu \|\nabla \boldsymbol{w}_h\|^2_{0,\Omega_j^*} \lesssim \mu \|\nabla \boldsymbol{w}_h\|^2_{0,\Omega_j} + J_h^f(\boldsymbol{w}_h, \boldsymbol{w}_h).$$

The interface patches introduced in the analysis need to be extended, their structure is identical to the patches introduced in Chapters 3 and 5. In another hand extension operators $\mathbb{E}_2^f : H^2(\Omega_f) \to H^2(\Omega_f^*)$, $\mathbb{E}_1^f : H^1(\Omega_f) \to H^1(\Omega_f^*)$ need to be used for the fluid velocity and pressure such that $\|\mathbb{E}_2^f \boldsymbol{v}\|_{2,\Omega_f^*} \lesssim \|\boldsymbol{v}\|_{2,\Omega_f}$ and $\|\mathbb{E}_1^f q\|_{1,\Omega_f^*} \lesssim \|q\|_{1,\Omega_f}$. For the solid displacement and velocity $\mathbb{E}_2^s : H^2(\Omega_s) \to H^2(\Omega_s^*)$ such that $\|\mathbb{E}_2^s \boldsymbol{v}\|_{2,\Omega_s^*} \lesssim \|\boldsymbol{v}\|_{2,\Omega_s}$. For the fluid velocity and pressure, these extensions operators brings control on $\Omega_f^* \backslash \Omega_f$ and similarly for the solid velocity and displacement on $\Omega_s^* \backslash \Omega_s$. In Chapters 3 and 5 the tools presented here are introduced to show the stability and convergence of unfitted schemes, considering these analyses we claim that the analysis done in this chapter is straightforward to extend to the unfitted case.

## 6.7 Numerical results

In order to verify numerically the accuracy of the implicit scheme (6.7) we consider the two dimensional pressure wave propagation benchmark [54, 49, 34] for the coupling of a two dimensional fluid with a one dimensional elastic solid. The analysis above covers the case where the fluid and solid domains are both two dimensional, however, [34] presents an analysis for the case of a one dimensional solid for a penalised version of (6.6) which is directly applicable to the penalty-free scheme considered here. The fluid domain is defined as $\Omega_f = [0, L] \times [0, R]$, the solid domain is considered as one dimensional $\Sigma = [0, L] \times \{R\}$. In the system of equations (6.1)-(6.3) the relations (6.2) and (6.3) are replaced by

$$\begin{cases} \rho_s \partial_t \dot{\boldsymbol{d}} + \boldsymbol{L}^e \boldsymbol{d} = -\boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n} & \text{in } \Sigma, \\ \dot{\boldsymbol{d}} = \partial_t \boldsymbol{d} & \text{in } \Sigma, \\ \boldsymbol{u} = \dot{\boldsymbol{d}} & \text{on } \Sigma, \\ \boldsymbol{d} = 0 & \text{on } \Gamma^s. \end{cases}$$

The generalised one dimensional elastic string equation is considered then $\boldsymbol{d} = (0, d_2)$ and $\boldsymbol{L}^e\boldsymbol{d} = (0, -\lambda_1\partial_{xx}d_y + \lambda_0 d_y)^T$ with $\lambda_1 = \frac{E\epsilon}{2(1+\nu)}$ and $\lambda_0 = \frac{E\epsilon}{R^2(1-\nu^2)}$ where $\epsilon$ is the thickness of the solid, $E$ is the Young modulus and $\nu$ denotes the Poisson ratio. The unfitted case is considered, Figure 6.2 shows the configuration of the mesh. At the beginning of the

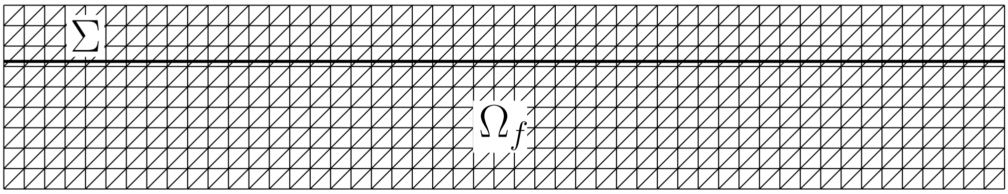



FIGURE 6.2: Pressure wave mesh, the solid domain is the bold line.

simulation, we impose half a sinusoid of pressure with maximal amplitude $2 \times 10^4$ on the side $[0, R] \times 0$ between time $t = 0.0$ and $t = 0.005$. The pressure is fixed to zero on the side $[0, R] \times L$ and a symmetry condition is imposed on the lower boundary $0 \times [0, L]$. The stabilisation parameters are considered such that $\gamma_g = 1.0$, $\gamma_p = 10^{-3}$. The fluid is characterised by the following material parameters $\rho_f = 1.0$, $\mu = 0.035$, the parameters of the solid are taken such that $\rho_s = 1.1$, $\epsilon = 0.1$, $E = 0.75 \times 10^6$ and $\nu = 0.5$. The computations are done using the package FreeFem++ [75], Figure 6.4 shows the pressure profiles and the displacement of the solid at different times during the simulation. In order to show the convergence of the scheme as $h$ and $\tau$ are refined we generate a reference solution with a fitted configuration where the solid $\Sigma$ fits exactly with the upper boundary of the mesh. For this reference solution $h = 3.125 \times 10^{-3}$ and $\tau = 10^{-6}$. To generate a convergence curve, we refine simultaneously the mesh size $h$ and the time step $\tau$ with $\tau = \mathcal{O}(h)$. We also compare the slope of convergence obtained with the scheme considered in [34] that considers the classical Nitsche's method (symmetric with penalty parameter $10^3$). Figure 6.3 shows that the convergence rate observed corresponds to what has been shown in Corollary 6.5.1. The penalty-free scheme performs very slightly better than the classical Nitsche scheme in terms of absolute error, with the advantage that for the penalty-free scheme, no Nitsche penalty parameter had to be investigated in order to get the expected convergence of the scheme.

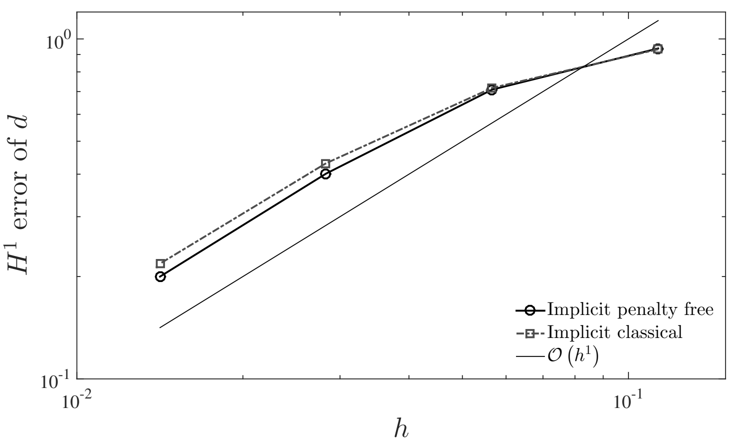
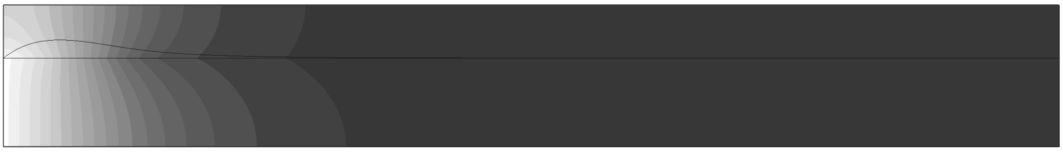
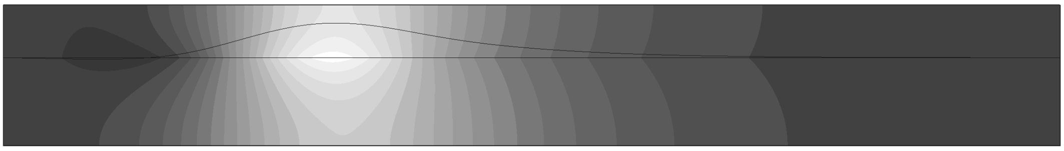



FIGURE 6.3: Classical Nitsche's method versus penalty-free Nitsche's method, time convergence history of the solid displacement at $t = 0.015$, $\tau = \mathcal{O}(h)$.

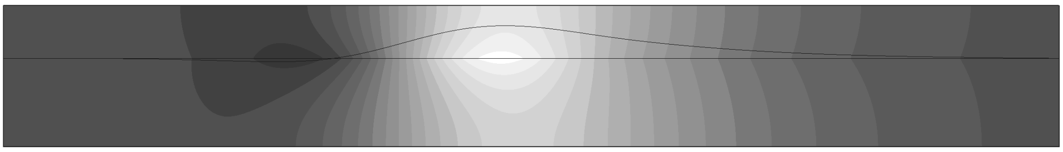


(a) $t = 0.0025$

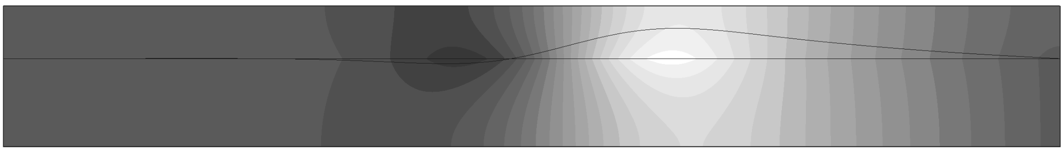
(b) $t = 0.005$



(c) $t = 0.0075$



(d) $t = 0.01$



(e) $t = 0.0125$

(f) $t = 0.015$

FIGURE 6.4: Pressure profiles, the black straight line is the profile of $\boldsymbol{d}$, the second black line is the exaggerated displacement is $5 \times \boldsymbol{d}$.

# Chapter 7

# Conclusions & Further work

In this thesis, we have investigated several frameworks for the penalty-free Nitsche's method, in this final chapter we highlight the main results of the thesis and we present the future directions that will be investigated. These future investigations are either precise ideas that are already ongoing work, or future projects that are mostly ideas that needs to be refined. In the thesis, several penalisations has been introduced to ensure the stability of the schemes proposed (i.e. pressure stabilisation & ghost penalty), we call the method penalty-free in the sense that we have removed the penalty term relative to the Nitsche's method (see Section 1.2.2). We considered the two dimensional case, the extension to the three dimensional case would need some arguments of the proofs to be refined.

## 7.1 Fictitious domain method

The fictitious domain method as investigated in Chapter 3 is often called Cut Finite Element Method. The boundary is allowed to cut arbitrarily elements of the mesh. We have introduced the ghost penalty in order to regain stability which is lost because of the possible small elements pieces that result from the cut.

### 7.1.1 Results

In this thesis we have shown theoretically the optimal convergence of the $H^1$-error and the convergence of the $L^2$-error with suboptimality of order $\mathcal{O}(h^{\frac{1}{2}})$

$$\|u - u_h\|_{1,\Omega} \lesssim h^k |u|_{k+1,\Omega}, \qquad \|u - u_h\|_{0,\Omega} \lesssim h^{k+\frac{1}{2}} |u|_{k+1,\Omega}.$$

The theory has been extended to compressible and incompressible elasticity. Numerical investigations have been performed and illustrate the convergence for first order approximations. Higher order approximations have not been considered due to the fact that the library CutFEM [28] considers yet only a piecewise affine approximation of the boundary.

### 7.1.2 Boundary value correction

In order to handle the loss of accuracy generated by the piecewise affine approximation of the boundary, a boundary value correction method has been introduced recentely in [39]. The principle of the method involves a Taylor expansion in the formulation in order to approximate the solution at the boundary. In this method, the Nitsche's method is used to enforce the boundary conditions, in a paper in preparation, we extend this method to the penalty-free Nitsche's method [T. Boiveau, E. Burman, S. Claus and M. G. Larson, *Fictitious domain method with boundary value correction using penalty-free Nitsche method*]. The domain $\Omega_h$ with boundary $\Gamma_h$ is the approximation of the domain $\Omega$ with piecewise affine approximation of the boundary, and $T_k(u)$ is the Taylor expansion of order $k$ of $u$ at the boundary. For $V_h$ a $H^1$-conforming finite element space attached to a mesh that covers but not fits with $\Omega$ and $\Omega_h$. For the Poisson problem (3.14) the finite element formulation reads: find $u_h \in V_h$ for all $v_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h)_{\Omega_h} - \langle \nabla u_h \cdot n_h, v_h \rangle_{\Gamma_h} + \langle \nabla v_h \cdot n_h, T_k(u_h) \rangle_{\Gamma_h} = (f, v_h)_{\Omega_h} + \langle \nabla v_h \cdot n_h, g \circ p_h \rangle_{\Gamma_h},$$

with $n_h$ the unit normal vector to $\Gamma_h$ and $g \circ p_h$ the projection of the boundary condition $g$ on $\Gamma_h$. The ghost penalty has to be added similarly as in Chapter 3.

## 7.2 Domain decomposition

In Chapters 4 and 5 domain decomposition is considered, the fictitious domain method from Chapters 3 is the key that makes the analysis of the unfitted case very close to the fitted case. In both chapters we have shown a theoretical convergence in the broken norms such that

$$\|u - u_h\|_{1,\Omega} \lesssim h^k |u|_{k+1,\Omega}, \qquad \|u - u_h\|_{0,\Omega} \lesssim h^{k+\frac{1}{2}} |u|_{k+1,\Omega}.$$

The theory has been extended to compressible and incompressible elasticity. Numerical results are provided for first and second order piecewise approximation in the fitted case, and only first order approximation in the unfitted case, because of the piecewise affine description of the interface as explained in the previous section. The study of fitted and unfitted domain decomposition for the Poisson case presented in this thesis will be reported in an article that is currently in preparation [T. Boiveau, *Fitted and unfitted domain decomposition using penalty-free Nitsche method for the Poisson problem with discontinuous material parameters*].

### 7.2.1 Fitted domain decomposition

An important result to highlight for the fitted domain decomposition study is that the analysis is robust regardless of the material parameters discontinuity. Another interesting result is that the analysis is robust regardless of the ratio $h_1/h_2$ with $h_1$ and $h_2$ the mesh parameters of the subdomains $\Omega_1$ and $\Omega_2$ as defined in Chapter 4.

### 7.2.2 Unfitted domain decomposition

In the unfitted configuration the interface is allowed to cross the elements of the mesh, however, as for the fitted case the analysis for unfitted domain decomposition is robust regardless of the material parameters discontinuity. An extension of the boundary value correction as presented in Section 7.1.2 could be designed for this case in order to handle the piecewise affine approximation of the interface.

## 7.3 Fluid-structure interaction

In Chapter 6 we have introduced an implicit scheme for time dependent fluid structure interaction, the scheme considers a master/slave configuration for the coupling with the fluid as the master and the solid as the slave. At the interface the penalty-free Nitsche's method is used in order to enforce the coupling.

### 7.3.1 Results

Stability and optimal convergence have been shown theoretically for the scheme, the convergence result reads:

$$\mathcal{E}^N + \sum_{n=1}^{N} \mathcal{D}^n \lesssim \mathcal{E}^0 + c_1\tau^2 + c_2 h^2,$$

with $\mathcal{E}^n$ and $\mathcal{D}^n$ respectively the energy and dissipation errors at time $\{t_n\}_{1\leq n\leq N}$ as defined in Section 6.5. The positive constants $c_1, c_2$ depends on the physical parameters and the regularity of the exact solution. $h$ and $\tau$ are respectively the mesh parameter and the time step.

### 7.3.2 Further inversigations

In a paper in preparation [T. Boiveau, E. Burman, B. Fabrèges, M. A. Fernández, M. Landajuela, *Penalty-free immersed boundary methods for incompressible fluid-structure interaction*] we will study the case of thin structures for several penalty-free schemes.

- An implicit scheme as presented in Section 6.7.
- An explicit scheme as presented in [27].
- A semi-implicit scheme inspired from [50, 51, 98, 7].
- An explicit scheme inspired from the immersed boundary method [95, 19].

In this article we will present various numerical examples, we will also show the stability and optimal convergence for all the schemes except the explicit scheme from [27] for which we have not been able to find a proof. The proofs of stability and convergence for this explicit scheme remains an open problem, investigating the convergence for this scheme would be interesting given its numerical convergence properties observed in Figure 1.8. Another extension of this work is to investigate the time dependant fluid-fluid interaction [52], the analysis would be an extension of the results from Chapter 6.

## 7.4 General remarks

In this final section we highlight some advantages and limitations of the new method introduced in this thesis. An important point is the nonsymmetric property of the method, this leads to a lack of adjoint consistency and a suboptimality of order $\mathcal{O}(h^{\frac{1}{2}})$ for the $L^2$-error. Also, the coercivity of the problem is lost by removing the penalty term, however inf-sup stability can be proven using a boundary mortaring. The great advantage of the method is that no arbitrary parameter has to be chosen in order to get stability, optimal convergence of the error in the $H^1$-norm can be proven. The application to the fictitious domain framework is straightforward, the domain decomposition case shows interesting properties such as convergence independent of the physical parameters of the subdomains. As mentioned in the introduction, for fluid-structure interaction the explicit coupling shows optimal convergence when the penalty-free Nitsche coupling is used which is not the case when the classical Nitsche's method is used.

# Appendix A

# Functional analysis

**Definition A.0.1.** *A Banach space is a normed vector space whose associated metric is complete.*

**Definition A.0.2.** *A Hilbert space is an inner product space whose associated norm defines a complete metric.*

## A.1 Lebesgue spaces

**Definition A.1.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ and assume $1 < p < +\infty$, $L^p(\Omega)$ is a space of integrable functions*

$$L^p(\Omega) = \left\{ v : \Omega \to \mathbb{R} \mid \int_\Omega |v(x)|^p \ dx < +\infty \right\}$$

The space $L^p(\Omega)$ is a Banach space, its norm is defined as

$$\|v\|_{L^p(\Omega)} = \left( \int_\Omega |v(x)|^p \ \mathrm{d}x \right)^{\frac{1}{p}}$$

The space $L^2(\Omega)$ is a Hilbert space with inner product $(\cdot, \cdot)_\Omega$ defined as

$$(u, v)_\Omega = \int_\Omega u(x) v(x) \ \mathrm{d}x,$$

associated with the norm $\|v\|_{0,\Omega} = \|v\|_{L^2(\Omega)}$.

## A.2 Sobolev spaces

**Definition A.2.1.** *Let $k$ be a nonnegative integer and $1 < p < +\infty$. A Sobolev space $W^{k,p}(\Omega)$ is defined as*

$$W^{k,p}(\Omega) = \{ v \in L^p(\Omega) \mid D^{\boldsymbol{\alpha}} v \in L^p(\Omega) \ \forall |\boldsymbol{\alpha}| \leq k \}.$$

The space $W^{k,p}(\Omega)$ is a Banach space, its norm is defined as

$$\|v\|_{W^{k,p}(\Omega)} = \left( \sum_{|\boldsymbol{\alpha}|\leq k} \|D^{\boldsymbol{\alpha}} v\|_{0,\Omega}^p \right)^{\frac{1}{p}}.$$

The space $H^k(\Omega)$ is a Hilbert space defined as $H^k(\Omega) = W^{k,2}(\Omega)$, with the inner product

$$(u,v)_{H^k(\Omega)} = \sum_{|\boldsymbol{\alpha}|\leq k} (D^{\boldsymbol{\alpha}} u, D^{\boldsymbol{\alpha}} v)_{L^2(\Omega)},$$

associated with the norm $\|v\|_{k,\Omega} = \|v\|_{W^{k,2}(\Omega)}$. The $H^k$-semi-norm is defined as

$$|v|_{k,\Omega} = \left( \sum_{|\boldsymbol{\alpha}|=k} \|D^{\boldsymbol{\alpha}} v\|_{0,\Omega}^2 \right)^{\frac{1}{2}}.$$

## A.3 Standard inequalities

**Lemma A.3.1.** *The Poincaré inequality is defined for all $v \in W_0^{1,p}(\Omega)$ with $1 \leq p < +\infty$ and $\Omega$ a bounded open set such that*

$$\|v\|_{L^p(\Omega)} \lesssim \|\nabla v\|_{L^p(\Omega)},$$

*with $W_0^{1,p}(\Omega) = \{v \in W_0^{1,p}(\Omega) \mid v|_{\partial\Omega} = 0\}$.*

**Lemma A.3.2.** *The Minkowski's inequality is defined for $u, v \in L^p(\Omega)$ with $1 < p < +\infty$ by*

$$\|u+v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

The Minkowski's inequality is the trace inequality in $L^p(\Omega)$.

**Lemma A.3.3.** *The Hölder inequality is defined for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, $1 \leq p$, $q \leq +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$*

$$|(u,v)_\Omega| \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Taking $p = q = 2$ the Hölder inequality leads to the Cauchy-Schwarz inequality.

**Lemma A.3.4.** *The Cauchy-Schwarz inequality is defined for $u, v \in L^2(\Omega)$ by*

$$|(u,v)_\Omega| \leq \|u\|_{0,\Omega} \|v\|_{0,\Omega}.$$

**Lemma A.3.5.** *The Korn's inequality is defined for all $\boldsymbol{v} \in [H^1(\Omega)]^2$ with $\Omega \in \mathbb{R}^2$ such that*

$$\|\boldsymbol{v}\|_{1,\Omega} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega}.$$

**Lemma A.3.6.** *The second Korn's inequality is defined for all* $\boldsymbol{v} \in [H_0^1(\Omega)]^2$ *with* $\Omega \in \mathbb{R}^2$ *such that*

$$\|\boldsymbol{v}\|_{1,\Omega} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}.$$

# Appendix B

# Main notations

## Basic notations

| | |
|---|---|
| $C$ | Constant of order $\mathcal{O}(1)$, may change at each occurence |
| $\text{meas}\,(E)$ | Lebesgue measure of $E \subset \mathbb{R}^n$ |
| $v\|_E$ | Restriction of the function $v$ to the set $E$ |
| $a \lesssim b$ | $a \leq Cb$ |
| $\partial\Omega$ | is the boundary of the domain $\Omega$ |
| $\Omega^*$ | $\{K \in \mathcal{T}_h \mid K \cap \Omega \neq \emptyset\}$ |
| $(a, b)_\Omega$ | $\int_\Omega ab \, \mathrm{d}x$ |
| $\langle a, b\rangle_{\partial\Omega}$ | $\int_{\partial\Omega} ab \, \mathrm{d}s$ |
| $\\|\cdot\\|_{s,\omega}$ | Usual Sobolev norm in $H^s(\omega)$ with $s \geq 0$ |
| $\|\cdot\|_{s,\omega}$ | Usual Sobolev semi-norm in $H^s(\omega)$ with $s \geq 0$ |
| $n, \boldsymbol{n}$ | Unit normal vector |
| $\tau, \boldsymbol{\tau}$ | Unit tangent vector |
| $n_F$ | Unit normal vector to a face $F$ |
| $[\![v]\!]_F$ | $v_F^+ - v_F^-$, with $v_F^\pm = \lim_{s\to 0^+} v(x \mp s n_F)$, jump across $F$ |
| $\overline{v}^I$ | $\text{meas}(I)^{-1} \int_I v \, \mathrm{d}s$ |

## Vectors and matrices

| | |
|---|---|
| $u$ | One dimensional variable of $\mathbb{R}$ |
| $\boldsymbol{u}$ | Two-dimensional variable of $\mathbb{R}^2$ |
| $\boldsymbol{u} \cdot \boldsymbol{v}$ | Dot product |
| $\boldsymbol{u}^T$ | Transpose of $u$ |
| $\mathbb{I}_{n\times n}$ | Identity matrix with $n$ rows and columns |

## Interface related symbols

| | |
|---|---|
| $v^i$ | Variable related to $\Omega_i$ |
| $[\![v]\!]$ | $v^1 - v^2$ |
| $\{v\}$ | $\omega_1 v^1 + \omega_2 v^2$ |
| $\langle v\rangle$ | $\omega_2 v^1 + \omega_1 v^2$ |
| $\omega_1, \omega_2$ | Weights related to $\Omega_1$ and $\Omega_2$ |

## Differential operators

| | |
|---|---|
| $(x_1, \dots, x_d)$ | Coordinates in $\mathbb{R}^d$ |
| $\partial_t u$ | Time derivative of $u$ |
| $\partial_i u$ | Partial derivative of $u$ with respect to $x_i$ |
| $\partial_{ij} u$ | Second-order derivative of $u$ with respect to $x_i$ and $x_j$ |
| $D^{\boldsymbol{\alpha}}$ | $\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} u$ where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ is a multi index |
| $D_n^{\boldsymbol{\alpha}}$ | $D^{\boldsymbol{\alpha}} \cdot n$ |
| $\lvert\boldsymbol{\alpha}\rvert$ | Length of $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $\alpha_1 + \dots + \alpha_d$ |
| $\nabla u$ | Gradient of $u$, $(\partial_1 u, \dots, \partial_d u)^T \in \mathbb{R}^d$ |
| $\nabla \boldsymbol{u}$ | Gradient of $u \in \mathbb{R}^d$, $(\partial_j u_i)_{1 \le i,j \le d} \in \mathbb{R}^{d \times d}$ |
| $\nabla \cdot \boldsymbol{u}$ | Divergence of $\boldsymbol{u} \in \mathbb{R}$, $\sum_{i=1}^d \partial_i u_i \in \mathbb{R}$, if $\boldsymbol{u} \in \mathbb{R}^\times$, $\left(\sum_{j=1}^d \partial_j u_{ij}\right)^T_{1 \le i \le d} \in \mathbb{R}^d$ |
| $\Delta u$ | Laplacian of $u \in \mathbb{R}$, $\sum_{i=1}^d \partial_{ii} u \in \mathbb{R}$ |
| $\Delta \boldsymbol{u}$ | Laplacian of $\boldsymbol{u} \in \mathbb{R}^d$, $\left(\sum_{j=1}^d \partial_{jj} u_i\right)_{1 \le i \le d} \in \mathbb{R}^d$ |
| $\boldsymbol{\varepsilon}(\boldsymbol{u})$ | $\frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ |

## Mesh-related notations

| | |
|---|---|
| $\mathcal{T}_h$ | Triangulation of the domain $\Omega$ |
| $\mathcal{T}_h^i$ | Triangulation of the domain $\Omega_i$ (Chapter 4) |
| $K$ | Generic triangle of a triangulation |
| $F$ | Generic face of a triangulation |
| $h_K$ | $\mathrm{diam}\,(K)$ |
| $h$ | $\max_{K \in \mathcal{T}_h} h_K$ (Chapters 2,3,5,6) |
| $h_i$ | $\max_{K \in \mathcal{T}_h^i} h_K$ |
| $h$ | $\max(h_1, h_2)$ (Chapter 4) |
| $\mathbb{P}_k(K)$ | Polynomial of order less than or equal to $k$ on $K$ |
| $x_i$ | Generic node of a triangulation |

## Spaces

| | |
|---|---|
| $H_g^1(\Omega)$ | $\{v \in H^1(\Omega) : v\vert_{\partial\Omega} = g\}$ |

#### Weak imposition of boundary conditions

| | |
|---|---|
| $V_h^k$ | $\{v_h \in H^1(\Omega) : v_h\vert_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\} \quad k \ge 1$ |
| $W_h^k$ | $\{\boldsymbol{v}_h \in [H^1(\Omega)]^2 : \boldsymbol{v}_h\vert_K \in [\mathbb{P}_k(K)]^2, \forall K \in \mathcal{T}_h\} \quad k \ge 1$ |
| $Q_h^k$ | $\{q_h \in L^2(\Omega) : q_h\vert_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h; \int_\Omega q_h \,\mathrm{d}x = 0\} \quad k \ge 1$ |

**Fictitious domain**

| | |
|---|---|
| $V_h^k$ | $\{v_h \in H^1(\Omega^*) : v_h\|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\} \quad k \geq 1$ |
| $W_h^k$ | $\{\boldsymbol{v}_h \in [H^1(\Omega^*)]^2 : \boldsymbol{v}_h\|_K \in [\mathbb{P}_k(K)]^2, \forall K \in \mathcal{T}_h\} \quad k \geq 1$ |
| $Q_h^k$ | $\{q_h \in L^2(\Omega^*) : q_h\|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h; \int_\Omega q_h \, \mathrm{d}x = 0\} \quad k \geq 1$ |

**Domain decomposition**

| | |
|---|---|
| $V_i$ | $\{v \in H^1(\Omega_i) : v\|_{\partial\Omega} = 0\}$ |
| $V_i^k$ | $\{v_h \in V_i : v_h\|_K \in \mathbb{P}_k(K) \;\; \forall K \in \mathcal{T}_h^i\}, \quad k \geq 1$ |
| $V_h^k$ | $V_1^k \times V_2^k$ |
| $H_\partial^k(\Omega_i)$ | $\{v \in H^k(\Omega_i) : v\|_{\partial\Omega} = 0\}$ |
| $W_i$ | $[V_i]^2$ |
| $W_i^k$ | $[V_i^k]^2$ |
| $W_h^k$ | $W_1^k \times W_2^k$ |
| $Q_i$ | $\{q \in L^2(\Omega_i), \; \int_{\Omega_i} q \, \mathrm{d}x = 0\}$ |
| $Q_i^k$ | $\{q_h \in Q_i : q_h\|_K \in \mathbb{P}_k(K) \;\; \forall K \in \mathcal{T}_h^i\}, \quad k \geq 1$ |
| $Q_h^k$ | $Q_1^k \times Q_2^k$ |

**Unfitted domain decomposition**

| | |
|---|---|
| $V_i^*$ | $\{v \in H^1(\Omega_i^*) : v\|_{\partial\Omega} = 0\}$ |
| $V_i^k$ | $\{v_h \in V_i^* : v_h\|_K \in \mathbb{P}_k(K) \;\; \forall K \in \mathcal{T}_h\}, \qquad \forall k \geq 1$ |
| $V_h^k$ | $V_1^k \times V_2^k$ |
| $H_\partial^k(\Omega_i)$ | $\{v \in H^k(\Omega_i) : v\|_{\partial\Omega} = 0\}$ |
| $W_i^k$ | $[V_i^k]^2$ |
| $W_h^k$ | $W_1^k \times W_2^k$ |
| $Q_i^*$ | $\{q \in L^2(\Omega_i^*), \; \int_{\Omega_i} q \, \mathrm{d}x = 0\}$ |
| $Q_i^k$ | $\{q_h \in Q_i^* : q_h\|_K \in \mathbb{P}_k(K) \;\; \forall K \in \mathcal{T}_h^i\}, \quad k \geq 1$ |
| $Q_h^k$ | $Q_1^k \times Q_2^k$ |

# Appendix C

# General concepts

In this section the well-posedness is defined in the sense introduced by Hadamard [66]. Let us define an abstract problem. Let $W$ and $V$ be Hilbert spaces equipped respectively with the norms $\|\cdot\|_W$ and $\|\cdot\|_V$. Let $\mathcal{L}(E,F)$ the vector space of the bounded linear operators from $E$ to $F$. Let $f \in V' = \mathcal{L}(V,\mathbb{R})$ be a continuous linear form, we write $f(v)$ instead of $\langle f, v\rangle_{V',V}$ for simplicity, with $\langle \cdot,\cdot\rangle_{V',V}$ the duality pairing. Let $a \in \mathcal{L}(W \times V, \mathbb{R})$ be a continuous bilinear form. We consider the following abstract problem.

$$\text{Find } u \in W \text{ such that } a(u,v) = f(v) \;\; \forall v \in V. \tag{C.1}$$

Taking $W = V$ we get the following abstract problem

$$\text{Find } u \in V \text{ such that } a(u,v) = f(v) \;\; \forall v \in V. \tag{C.2}$$

**Theorem C.0.1.** ***(Lax-Milgram)*** *Let $a \in \mathcal{L}(V \times V, \mathbb{R})$ and $f \in V'$. Assume that the bilinear form $a(\cdot,\cdot)$ is coercive, i.e.,*

$$\exists \alpha > 0 \textit{ such that } a(v,v) \geq \alpha \|v\|_V^2 \;\; \forall v \in V.$$

*Then the problem (C.2) is well-posed with a priori estimate*

$$\forall f \in V', \;\; \|v\|_V \leq \frac{1}{\alpha} \|f\|_{V'}.$$

Let us define a discrete abstract problem. Let $V_h$ and $W_h$ be two finite-dimentional spaces equipped with the norms $\|\cdot\|_{V_h}$ and $\|\cdot\|_{W_h}$. Using Galerkin method we construct the approximation of the abstract problem (C.1)

$$\text{Find } v_h \in V_h \text{ such that } a_h(v_h, w_h) = f_h(w_h) \;\; \forall w_h \in W_h. \tag{C.3}$$

Note that $a_h$ is an approximation of the bilinear form $a$ and $f_h$ an approximation of the linear form $f$. The discrete inf-sup condition condition is written as

$$\exists \beta > 0, \quad \inf_{w_h \in W_h} \sup_{v_h \in V_h} \frac{a_h(v_h, w_h)}{\|v_h\|_{V_h} \|w_h\|_{W_h}} \geq \beta. \tag{C.4}$$

**Theorem C.0.2.** *Assume that*

- $a_h$ *is bounded on* $V_h \times W_h$ *and* $f_h$ *is continuous on* $W_h$.
- *The discrete inf-sup condition* (C.4) *is fulfiled.*
- $V_h$ *and* $W_h$ *have the same dimension.*

*Then the approximate problem* (C.3) *is well-posed, and the a priori estimate* $\|u_h\|_{V_h} \leq \frac{1}{\beta}\|f_h\|_{W_h}$ *holds.*

# Appendix D

# Technical proofs

**Lemma 2.2.1**

*Proof.* In the rotated frame $(\xi, \eta)$ by applying (2.27) we can write

$$\begin{aligned}\langle \lambda \hat{\nabla} \cdot \hat{\boldsymbol{v}}_j, \hat{\boldsymbol{u}}_h \cdot \hat{\boldsymbol{n}} \rangle_{F_j} &= \lambda \int_{F_j} \Big( \alpha_1 \frac{\partial \hat{v}_1}{\partial \xi} + \alpha_2 \frac{\partial \hat{v}_2}{\partial \eta} \Big) \hat{u}_2 \ \mathrm{d}s \\ &= \lambda \int_{F_j} \alpha_1 \frac{\partial \hat{v}_1}{\partial \xi} \hat{u}_2 + \alpha_2 h^{-1} (\overline{\hat{u}_2}^{F_j})^2 \ \mathrm{d}s + \lambda \int_{F_j} \alpha_2 \frac{\partial \hat{v}_2}{\partial \eta} (\hat{u}_2 - \overline{\hat{u}_2}^{F_j}) \ \mathrm{d}s.\end{aligned}$$

We observe that $\frac{\partial \hat{v}_1}{\partial \xi} = \hat{\nabla} \cdot (\hat{v}_1, 0)^{\mathrm{T}}$. Using the trace inequality, the inverse inequality and (2.28), we can show

$$\| \frac{\partial \hat{v}_1}{\partial \xi} \|_{0,F_j} \lesssim h^{-1} \| \overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{\tau} \|_{0,F_j},$$

we also note that $\int_{F_j} \frac{\partial \hat{v}_1}{\partial \xi} \ \mathrm{d}s = 0$, using these properties and the inequality (2.11), it follows that

$$\begin{aligned}\lambda \int_{F_j} \alpha_1 \frac{\partial \hat{v}_1}{\partial \xi} \hat{u}_2 \ \mathrm{d}s &= \lambda \int_{F_j} \alpha_1 \frac{\partial \hat{v}_1}{\partial \xi} (\hat{u}_2 - \overline{\hat{u}_2}^{F_j}) \ \mathrm{d}s \\ &\leq C \alpha_1 h^{-1} \lambda \| \overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{\tau} \|_{0,F_j} \| (\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}^{F_j}) \cdot \boldsymbol{n} \|_{0,F_j} \\ &\leq \frac{C \alpha_1^2}{4\epsilon} \frac{\lambda}{h} \| \overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{\tau} \|_{0,F_j}^2 + \epsilon \lambda \| \nabla \boldsymbol{u}_h \|_{0,P_j}^2.\end{aligned}$$

We also have $\frac{\partial \hat{v}_2}{\partial \eta} = \hat{\nabla} \cdot (0, \hat{v}_2)^{\mathrm{T}}$, using (2.27) and (2.28) we obtain similarly

$$\begin{aligned}\lambda \int_{F_j} \alpha_2 \frac{\partial \hat{v}_2}{\partial \eta} (\hat{u}_2 - \overline{\hat{u}_2}^{F_j}) \ \mathrm{d}s &\leq \frac{C \alpha_2^2}{4\epsilon} \frac{\lambda}{h} \| \overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{n} \|_{0,F_j}^2 + \epsilon \lambda \| \nabla \boldsymbol{u}_h \|_{0,P_j}^2, \\ \lambda \int_{F_j} \alpha_2 h^{-1} (\overline{\hat{u}_2}^{F_j})^2 \ \mathrm{d}s &= \alpha_2 \frac{\lambda}{h} \| \overline{\boldsymbol{u}_h}^{F_j} \cdot \boldsymbol{n} \|_{0,F_j}^2.\end{aligned}$$

□

**Lemma 2.2.2**

*Proof.* In the rotated frame $(\xi, \eta)$, applying (2.27), we can write similarly as in the previous proof

$$\begin{aligned}\langle 2\mu\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}_j)\cdot\hat{\boldsymbol{n}}, \hat{\boldsymbol{u}}_h\rangle_{F_j} &= \mu\int_{F_j}\alpha_1\frac{\partial\hat{v}_1}{\partial\eta}\hat{u}_1 + \alpha_2\frac{\partial\hat{v}_2}{\partial\xi}\hat{u}_1 + 2\alpha_2\frac{\partial\hat{v}_2}{\partial\eta}\hat{u}_2\ \mathrm{d}s\\ &= \mu\int_{F_j}\alpha_1 h^{-1}(\overline{\hat{u}_1}^{F_j})^2 + \alpha_2\frac{\partial\hat{v}_2}{\partial\xi}\hat{u}_1 + 2\alpha_2 h^{-1}(\overline{\hat{u}_2}^{F_j})^2\ \mathrm{d}s\\ &\quad + \mu\int_{F_j}\alpha_1\frac{\partial\hat{v}_1}{\partial\eta}(\hat{u}_1 - \overline{\hat{u}_1}^{F_j})\ \mathrm{d}s + 2\mu\int_{F_j}\alpha_2\frac{\partial\hat{v}_2}{\partial\eta}(\hat{u}_2 - \overline{\hat{u}_2}^{F_j})\ \mathrm{d}s.\end{aligned}$$

Term by term we obtain

$$\begin{aligned}\mu\int_{F_j}\alpha_1 h^{-1}(\overline{\hat{u}_1}^{F_j})^2\ \mathrm{d}s &= \alpha_1\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|_{0,F_j}^2,\\ \mu\int_{F_j}2\alpha_2 h^{-1}(\overline{\hat{u}_2}^{F_j})^2\ \mathrm{d}s &= 2\alpha_2\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|_{0,F_j}^2,\\ \mu\int_{F_j}\alpha_1\frac{\partial\hat{v}_1}{\partial\eta}(\hat{u}_1 - \overline{\hat{u}_1}^{F_j})\ \mathrm{d}s &\le \frac{C\alpha_1^2}{4\epsilon}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{\tau}\|_{0,F_j}^2 + \epsilon\mu\|\nabla\boldsymbol{u}_h\|_{0,P_j}^2,\\ 2\mu\int_{F_j}\alpha_2\frac{\partial\hat{v}_2}{\partial\eta}(\hat{u}_2 - \overline{\hat{u}_2}^{F_j})\ \mathrm{d}s &\le \frac{C\alpha_2^2}{\epsilon}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|_{0,F_j}^2 + \epsilon\mu\|\nabla\boldsymbol{u}_h\|_{0,P_j}^2.\end{aligned}$$

We observe that $\frac{\partial\hat{v}_2}{\partial\xi} = \hat{\nabla}(0,\hat{v}_2)^{\mathrm{T}}\cdot\boldsymbol{\tau}$. Using the trace inequality, the inverse inequality and (2.27), we can show the stability

$$\|\frac{\partial\hat{v}_2}{\partial\xi}\|_{0,F_j} \lesssim h^{-1}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|_{0,F_j}.$$

Note that since $\int_{F_j}\frac{\partial\hat{v}_2}{\partial\xi}\ \mathrm{d}s = 0$, we have

$$\mu\int_{F_j}\alpha_2\frac{\partial\hat{v}_2}{\partial\xi}\hat{u}_1\ \mathrm{d}s = \mu\int_{F_j}\alpha_2\frac{\partial\hat{v}_2}{\partial\xi}(\hat{u}_1 - \overline{\hat{u}_1}^{F_j})\ \mathrm{d}s \le \frac{C\alpha_2^2}{4\epsilon}\frac{\mu}{h}\|\overline{\boldsymbol{u}_h}^{F_j}\cdot\boldsymbol{n}\|_{0,F_j}^2 + \epsilon\mu\|\nabla\boldsymbol{u}_h\|_{0,P_j}^2.$$

□

**Lemma 3.3.2**

*Proof.* The bilinear form can be written as

$$\langle\lambda\nabla\cdot\boldsymbol{v}_j, \boldsymbol{u}_h\cdot\boldsymbol{n}\rangle_{\Gamma_j} = \langle\lambda\nabla\cdot\boldsymbol{v}_j, \boldsymbol{u}_h\cdot(\boldsymbol{n} - \overline{\boldsymbol{n}}^{\Gamma_j})\rangle_{\Gamma_j} + \langle\lambda\nabla\cdot\boldsymbol{v}_j, \boldsymbol{u}_h\cdot\overline{\boldsymbol{n}}^{\Gamma_j}\rangle_{\Gamma_j}.$$

Using the bound $\|\boldsymbol{n} - \overline{\boldsymbol{n}}^{\Gamma_j}\|_{L^\infty(\Gamma_j)} \lesssim h$ and the trace inequality we obtain

$$\begin{aligned}\langle \lambda \nabla \cdot \boldsymbol{v}_j, \boldsymbol{u}_h \cdot (\boldsymbol{n} - \overline{\boldsymbol{n}}^{\Gamma_j}) \rangle_{\Gamma_j} &\leq \lambda \|\nabla \cdot \boldsymbol{v}_j\|_{0,\Gamma_j} \|\boldsymbol{u}_h \cdot (\boldsymbol{n} - \overline{\boldsymbol{n}}^{\Gamma_j})\|_{0,\Gamma_j} \\ &\lesssim \lambda h \|\nabla \cdot \boldsymbol{v}_j\|_{0,\Gamma_j} \|\boldsymbol{u}_h\|_{0,\Gamma_j} \\ &\lesssim \lambda h \|\nabla \boldsymbol{v}_j\|_{0,P_j} h^{-\frac{1}{2}} \|\boldsymbol{u}_h\|_{0,\Gamma_j}.\end{aligned}$$

Similarly as in the proof of lemma 2.2.1,

$$\langle \lambda \hat{\nabla} \cdot \hat{\boldsymbol{v}}_j, \hat{\boldsymbol{u}}_h \cdot \overline{\hat{\boldsymbol{n}}}^{\Gamma_j} \rangle_{\Gamma_j} = \lambda \int_{\Gamma_j} \alpha_1 \frac{\partial \hat{v}_1}{\partial \xi} \hat{u}_2 + \alpha_2 h^{-1} (\overline{\hat{u}_2}^{\Gamma_j})^2 \, \mathrm{d}s + \lambda \int_{\Gamma_j} \alpha_2 \frac{\partial \hat{v}_2}{\partial \eta} (\hat{u}_2 - \overline{\hat{u}_2}^{\Gamma_j}) \, \mathrm{d}s.$$

Term by term we have

$$\begin{aligned}\lambda \int_{\Gamma_j} \alpha_1 \frac{\partial \hat{v}_1}{\partial \xi} \hat{u}_2 \, \mathrm{d}s &\leq \frac{C\alpha_1^2}{4\epsilon} \frac{\lambda}{h} \|\overline{\boldsymbol{u}_h}^{\Gamma_j} \cdot \overline{\boldsymbol{\tau}}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \epsilon \lambda \|\nabla \boldsymbol{u}_h\|_{0,P_j}^2. \\ \lambda \int_{\Gamma_j} \alpha_2 \frac{\partial \hat{v}_2}{\partial \eta} (\hat{u}_2 - \overline{\hat{u}_2}^{\Gamma_j}) \, \mathrm{d}s &\leq \frac{C\alpha_2^2}{4\epsilon} \frac{\lambda}{h} \|\overline{\boldsymbol{u}_h}^{\Gamma_j} \cdot \overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \epsilon \lambda \|\nabla \boldsymbol{u}_h\|_{0,P_j}^2, \\ \lambda \int_{\Gamma_j} \alpha_2 h^{-1} (\overline{\hat{u}_2}^{\Gamma_j})^2 \, \mathrm{d}s &= \alpha_2 \frac{\lambda}{h} \|\overline{\boldsymbol{u}_h}^{\Gamma_j} \cdot \overline{\boldsymbol{n}}^{\Gamma_j}\|_{0,\Gamma_j}^2.\end{aligned}$$

The claim holds under the condition $h_0 \leq \mathcal{O}(1)$. □

**Lemma 3.3.3**

*Proof.* We can write

$$\langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{v}_j) \cdot \boldsymbol{n}, \boldsymbol{u}_h \rangle_{\Gamma_j} = \langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{v}_j) \cdot (\boldsymbol{n} - \overline{\boldsymbol{n}}^{\Gamma_j}), \boldsymbol{u}_h \rangle_{\Gamma_j} + \langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{v}_j) \cdot \overline{\boldsymbol{n}}^{\Gamma_j}, \boldsymbol{u}_h \rangle_{\Gamma_j}.$$

Using the bound $\|\boldsymbol{n} - \overline{\boldsymbol{n}}^{\Gamma_j}\|_{L^\infty(\Gamma_j)} \lesssim h$ and the trace inequality

$$\begin{aligned}\langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{v}_j) \cdot (\boldsymbol{n} - \overline{\boldsymbol{n}}^{\Gamma_j}), \boldsymbol{u}_h \rangle_{\Gamma_j} &\leq 2\mu \|\boldsymbol{\varepsilon}(\boldsymbol{v}_j) \cdot (\boldsymbol{n} - \overline{\boldsymbol{n}}^{\Gamma_j})\|_{0,\Gamma_j} \|\boldsymbol{u}_h\|_{0,\Gamma_j} \\ &\lesssim 2\mu h \|\nabla \boldsymbol{v}_j\|_{0,\Gamma_j} \|\boldsymbol{u}_h\|_{0,\Gamma_j} \\ &\lesssim 2\mu h \|\nabla \boldsymbol{v}_j\|_{0,P_j} h^{-\frac{1}{2}} \|\boldsymbol{u}_h\|_{0,\Gamma_j}.\end{aligned}$$

Then similarly as lemma 2.2.2

$$\begin{aligned}\langle 2\mu \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}_j) \cdot \overline{\hat{\boldsymbol{n}}}^{\Gamma_j}, \hat{\boldsymbol{u}}_h \rangle_{\Gamma_j} = \mu \int_{\Gamma_j} & \alpha_1 h^{-1} (\overline{\hat{u}_1}^{\Gamma_j})^2 + \alpha_2 \frac{\partial \hat{v}_2}{\partial \xi} \hat{u}_1 + 2\alpha_2 h^{-1} (\overline{\hat{u}_2}^{\Gamma_j})^2 \, \mathrm{d}s \\ &+ \mu \int_{\Gamma_j} \alpha_1 \frac{\partial \hat{v}_1}{\partial \eta} (\hat{u}_1 - \overline{\hat{u}_1}^{\Gamma_j}) \, \mathrm{d}s + 2\mu \int_{\Gamma_j} \alpha_2 \frac{\partial \hat{v}_2}{\partial \eta} (\hat{u}_2 - \overline{\hat{u}_2}^{\Gamma_j}) \, \mathrm{d}s.\end{aligned}$$

Term by term we have

$$\mu \int_{\Gamma_j} \alpha_1 h^{-1} (\overline{\hat{u}_1}^{\Gamma_j})^2 \, \mathrm{d}s = \alpha_1 \frac{\mu}{h} \| \overline{\boldsymbol{u}_h}^{\Gamma_j} \cdot \overline{\boldsymbol{\tau}}^{\Gamma_j} \|_{0,\Gamma_j}^2,$$

$$\mu \int_{\Gamma_j} 2\alpha_2 h^{-1} (\overline{\hat{u}_2}^{\Gamma_j})^2 \, \mathrm{d}s = 2\alpha_2 \frac{\mu}{h} \| \overline{\boldsymbol{u}_h}^{\Gamma_j} \cdot \overline{\boldsymbol{n}}^{\Gamma_j} \|_{0,\Gamma_j}^2,$$

$$\mu \int_{\Gamma_j} \alpha_1 \frac{\partial \hat{v}_1}{\partial \eta} (\hat{u}_1 - \overline{\hat{u}_1}^{\Gamma_j}) \, \mathrm{d}s \leq \frac{C\alpha_1^2}{4\epsilon} \frac{\mu}{h} \| \overline{\boldsymbol{u}_h}^{\Gamma_j} \cdot \overline{\boldsymbol{\tau}}^{\Gamma_j} \|_{0,\Gamma_j}^2 + \epsilon\mu \| \nabla \boldsymbol{u}_h \|_{0,P_j}^2,$$

$$2\mu \int_{\Gamma_j} \alpha_2 \frac{\partial \hat{v}_2}{\partial \eta} (\hat{u}_2 - \overline{\hat{u}_2}^{\Gamma_j}) \, \mathrm{d}s \leq \frac{C\alpha_2^2}{\epsilon} \frac{\mu}{h} \| \overline{\boldsymbol{u}_h}^{\Gamma_j} \cdot \overline{\boldsymbol{n}}^{\Gamma_j} \|_{0,\Gamma_j}^2 + \epsilon\mu \| \nabla \boldsymbol{u}_h \|_{0,P_j}^2,$$

$$\mu \int_{\Gamma_j} \alpha_2 \frac{\partial \hat{v}_2}{\partial \xi} \hat{u}_1 \, \mathrm{d}s \leq \frac{C\alpha_2^2}{4\epsilon} \frac{\mu}{h} \| \overline{\boldsymbol{u}_h}^{\Gamma_j} \cdot \overline{\boldsymbol{n}}^{\Gamma_j} \|_{0,\Gamma_j}^2 + \epsilon\mu \| \nabla \boldsymbol{u}_h \|_{0,P_j}^2.$$

The claim holds under the condition $h_0 \leq \mathcal{O}(1)$. □

**Lemma 4.3.1**

*Proof.* Using the property (4.16) and the fact that $\int_{F_j^1} \frac{\partial v_1^1}{\partial x} \, \mathrm{d}s = 0$ we get

$$\begin{aligned}
\langle \omega_1 \lambda_1 \nabla \cdot \boldsymbol{v}_j^1, [\![\boldsymbol{u}_h]\!] \cdot \boldsymbol{n} \rangle_{F_j^1} = \int_{F_j^1} & \alpha_1 \omega_1 \lambda_1 \frac{\partial v_1^1}{\partial x} ([\![u_2]\!] - \overline{[\![u_2]\!]}^{F_j^1}) \\
& + \alpha_2 \omega_1 \lambda_1 \frac{\partial v_2^1}{\partial y} ([\![u_2]\!] - \overline{[\![u_2]\!]}^{F_j^1}) + \alpha_2 \omega_1 \lambda_1 h_1^{-1} (\overline{[\![u_2]\!]}^{F_j^1})^2 \, \mathrm{d}s.
\end{aligned}$$

Where we can write the last term as

$$\alpha_2 \omega_1 \lambda_1 h_1^{-1} \int_{F_j^1} (\overline{[\![u_2]\!]}^{F_j^1})^2 \, \mathrm{d}s = \alpha_2 \gamma \frac{\lambda_1}{\mu_1} \| \overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n} \|_{0,F_j^1}^2.$$

We observe that $\frac{\partial v_2^1}{\partial y} = \nabla \cdot (0, v_2^1)^{\mathrm{T}}$, then using (4.17) we obtain

$$\| \frac{\partial v_2^1}{\partial y} \|_{0,F_j^1} \lesssim h_1^{-1} \| \overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1} \cdot \boldsymbol{n} \|_{0,F_j^1}.$$

Using this result with inequality (2.11) and the trace inequality

$$\begin{aligned}
&\sum_{j=1}^{N_p^1}\int_{F_j^1}\alpha_2\omega_1\lambda_1\frac{\partial v_2^1}{\partial y}(\llbracket u_2\rrbracket-\overline{\llbracket u_2\rrbracket}^{F_j^1})\,\mathrm{d}s\\
&\lesssim\sum_{j=1}^{N_p^1}\alpha_2\omega_1\lambda_1 h_1^{-1}\|\overline{\llbracket \boldsymbol{u}_h\rrbracket}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}\|(\llbracket \boldsymbol{u}_h\rrbracket-\overline{\llbracket \boldsymbol{u}_h\rrbracket}^{F_j^1})\cdot\boldsymbol{n}\|_{0,F_j^1}\\
&\lesssim\sum_{j=1}^{N_p^1}\alpha_2\gamma\frac{\lambda_1}{\mu_1}\|\overline{\llbracket \boldsymbol{u}_h\rrbracket}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}(\|\boldsymbol{u}_h^1-\overline{\boldsymbol{u}_h^1}^{F_j^1}\|_{0,F_j^1}+\|\boldsymbol{u}_h^2-\overline{\boldsymbol{u}_h^2}^{F_j^1}\|_{0,F_j^1})\\
&\le\gamma\frac{\lambda_1}{\mu_1}\sum_{j=1}^{N_p^1}\Big(\frac{C\alpha_2^2}{2\epsilon}\|\overline{\llbracket \boldsymbol{u}_h\rrbracket}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2+\epsilon\|\boldsymbol{u}_h^1-\overline{\boldsymbol{u}_h^1}^{F_j^1}\|_{0,F_j^1}^2\Big)+\epsilon\gamma\frac{\lambda_1}{\mu_1}\sum_{j=1}^{N_p^2}\|\boldsymbol{u}_h^2-\overline{\boldsymbol{u}_h^2}^{F_j^2}\|_{0,F_j^2}^2\\
&\le\gamma\frac{\lambda_1}{\mu_1}\frac{C\alpha_2^2}{2\epsilon}\sum_{j=1}^{N_p^1}\|\overline{\llbracket \boldsymbol{u}_h\rrbracket}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2+\epsilon\omega_1\lambda_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2+\epsilon\omega_2\frac{\lambda_1\mu_2}{\mu_1\lambda_2}\lambda_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2.
\end{aligned}$$

Also $\frac{\partial v_1^1}{\partial x}=\nabla\cdot(v_1^1,0)^{\mathrm{T}}$, using (4.17) we get

$$\|\frac{\partial v_1^1}{\partial x}\|_{0,F_j^1}\lesssim h_1^{-1}\|\overline{\llbracket \boldsymbol{u}_h\rrbracket}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1},$$

we obtain similarly the bound

$$\begin{aligned}
&\sum_{j=1}^{N_p^1}\int_{F_j^1}\alpha_1\omega_1\lambda_1\frac{\partial v_1^1}{\partial x}(\llbracket u_2\rrbracket-\overline{\llbracket u_2\rrbracket}^{F_j^1})\\
&\le\gamma\frac{\lambda_1}{\mu_1}\frac{C\alpha_1^2}{2\epsilon}\sum_{j=1}^{N_p^1}\|\overline{\llbracket \boldsymbol{u}_h\rrbracket}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2+\epsilon\omega_1\lambda_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2+\epsilon\omega_2\frac{\lambda_1\mu_2}{\mu_1\lambda_2}\lambda_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2.
\end{aligned}$$

We conclude by collecting the bounds for each term. □

**Lemma 4.3.2**

*Proof.* Using (4.16) and the fact $\int_{F_j^1}\frac{\partial v_2^1}{\partial x}\,\mathrm{d}s=0$

$$\begin{aligned}
\langle 2\omega_1\mu_1\boldsymbol{\varepsilon}(\boldsymbol{v}_j^1)\cdot\boldsymbol{n},\llbracket \boldsymbol{u}_h\rrbracket\rangle_{F_j^1}=&\int_{F_j^1}\alpha_1\omega_1\mu_1\frac{\partial v_1^1}{\partial y}(\llbracket u_1\rrbracket-\overline{\llbracket u_1\rrbracket}^{F_j^1})+\alpha_1\omega_1\mu_1 h_1^{-1}(\overline{\llbracket u_1\rrbracket}^{F_j^1})^2\\
&+2\alpha_2\omega_1\mu_1\frac{\partial v_2^1}{\partial y}(\llbracket u_2\rrbracket-\overline{\llbracket u_2\rrbracket}^{F_j^1})+2\alpha_2\omega_1\mu_1 h_1^{-1}(\overline{\llbracket u_2\rrbracket}^{F_j^1})^2\\
&+\alpha_2\omega_1\mu_1\frac{\partial v_2^1}{\partial x}(\llbracket u_2\rrbracket-\overline{\llbracket u_2\rrbracket}^{F_j^1})\,\mathrm{d}s.
\end{aligned}$$

Where we have

$$\alpha_1\omega_1\mu_1 h_1^{-1}\int_{F_j^1}(\overline{[\![u_1]\!]}^{F_j^1})^2\ \mathrm{d}s = \alpha_1\gamma\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2,$$
$$2\alpha_2\omega_1\mu_1 h_1^{-1}\int_{F_j^1}(\overline{[\![u_2]\!]}^{F_j^1})^2\ \mathrm{d}s = 2\alpha_2\gamma\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2.$$

Since $\frac{\partial v_1^1}{\partial y} = \nabla(v_1^1,0)^{\mathrm{T}}\cdot\boldsymbol{n}$, using (4.17) we have

$$\|\frac{\partial v_1^1}{\partial y}\|_{0,F_j^1} \lesssim h_1^{-1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}.$$

Similarly as in the proof of 4.3.1

$$\begin{aligned}
&\sum_{j=1}^{N_p^1}\int_{F_j^1}\alpha_1\omega_1\mu_1\frac{\partial v_1^1}{\partial y}([\![u_1]\!]-\overline{[\![u_1]\!]}^{F_j^1})\ \mathrm{d}s\\
&\quad\lesssim\sum_{j=1}^{N_p^1}\alpha_1\gamma\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}(\|\boldsymbol{u}_h^1-\overline{\boldsymbol{u}_h^1}^{F_j^1}\|_{0,F_j^1}+\|\boldsymbol{u}_h^2-\overline{\boldsymbol{u}_h^2}^{F_j^1}\|_{0,F_j^1})\\
&\quad\le\sum_{j=1}^{N_p^1}\gamma\frac{C\alpha_1^2}{2\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2+\epsilon\gamma\sum_{j=1}^{N_p^1}\|\boldsymbol{u}_h^1-\overline{\boldsymbol{u}_h^1}^{F_j^1}\|_{0,F_j^1}^2+\epsilon\gamma\sum_{j=1}^{N_p^2}\|\boldsymbol{u}_h^2-\overline{\boldsymbol{u}_h^2}^{F_j^2}\|_{0,F_j^2}^2\\
&\quad\le\gamma\frac{C\alpha_1^2}{2\epsilon}\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{\tau}\|_{0,F_j^1}^2+\epsilon\omega_1\mu_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2+\epsilon\omega_2\mu_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2.
\end{aligned}$$

Also $\frac{\partial v_2^1}{\partial x} = \nabla(0,v_2^1)^{\mathrm{T}}\cdot\boldsymbol{\tau}$, then using (4.17) we have

$$\|\frac{\partial v_2^1}{\partial x}\|_{0,F_j^1} \lesssim h_1^{-1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1},$$

we obtain similarly the bound

$$\begin{aligned}
&\sum_{j=1}^{N_p^1}\int_{F_j^1}\alpha_2\omega_1\mu_1\frac{\partial v_2^1}{\partial x}([\![u_2]\!]-\overline{[\![u_2]\!]}^{F_j^1})\ \mathrm{d}s\\
&\quad\le\gamma\frac{C\alpha_2^2}{2\epsilon}\sum_{j=1}^{N_p^1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{F_j^1}\cdot\boldsymbol{n}\|_{0,F_j^1}^2+\epsilon\omega_1\mu_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2+\epsilon\omega_2\mu_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2.
\end{aligned}$$

Similarly as in the proof of 4.3.1 we have

$$\sum_{j=1}^{N_p^1}\int_{F_j^1} 2\alpha_2\omega_1\mu_1\frac{\partial v_2^1}{\partial y}(\llbracket u_2\rrbracket - \overline{\llbracket u_2\rrbracket}^{F_j^1})\ \mathrm{d}s$$
$$\leq \gamma\frac{C\alpha_2^2}{\epsilon}\sum_{j=1}^{N_p^1}\|\overline{\llbracket \boldsymbol{u}_h\rrbracket}^{F_j^1}\cdot\boldsymbol{n}\|^2_{0,F_j^1} + \epsilon\omega_1\mu_1\sum_{j=1}^{N_p^1}\|\nabla\boldsymbol{u}_h^1\|^2_{0,P_j^1} + \epsilon\omega_2\mu_2\sum_{j=1}^{N_p^2}\|\nabla\boldsymbol{u}_h^2\|^2_{0,P_j^2}.$$

We conclude by collecting all the terms. □

**Lemma 5.3.1**

*Proof.* Using the property (5.18) and the fact that $\int_{\Gamma_j}\frac{\partial v_1^1}{\partial x}\ \mathrm{d}s = 0$ we get

$$\langle\omega_1\lambda_1\nabla\cdot\boldsymbol{v}_j^1, \llbracket\boldsymbol{u}_h\rrbracket\cdot\boldsymbol{n}\rangle_{\Gamma_j} = \int_{\Gamma_j}\alpha_1\omega_1\lambda_1\frac{\partial v_1^1}{\partial x}(\llbracket u_2\rrbracket - \overline{\llbracket u_2\rrbracket}^{\Gamma_j})$$
$$+ \alpha_2\omega_1\lambda_1\frac{\partial v_2^1}{\partial y}(\llbracket u_2\rrbracket - \overline{\llbracket u_2\rrbracket}^{\Gamma_j}) + \alpha_2\omega_1\lambda_1 h^{-1}(\overline{\llbracket u_2\rrbracket}^{\Gamma_j})^2\ \mathrm{d}s.$$

Where we can write the last term as

$$\alpha_2\omega_1\lambda_1 h^{-1}\int_{\Gamma_j}(\overline{\llbracket u_2\rrbracket}^{\Gamma_j})^2\ \mathrm{d}s = \alpha_2\gamma\frac{\lambda_1}{\mu_1}\|\overline{\llbracket\boldsymbol{u}_h\rrbracket}^{\Gamma_j}\cdot\boldsymbol{n}\|^2_{0,\Gamma_j}.$$

We observe that $\frac{\partial v_2^1}{\partial y} = \nabla\cdot(0, v_2^1)^{\mathrm{T}}$, then using (5.19) we obtain

$$\|\frac{\partial v_2^1}{\partial y}\|_{0,\Gamma_j} \lesssim h^{-1}\|\overline{\llbracket\boldsymbol{u}_h\rrbracket}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}.$$

Using this result with inequality (3.12) and the trace inequality

$$\int_{\Gamma_j}\alpha_2\omega_1\lambda_1\frac{\partial v_2^1}{\partial y}(\llbracket u_2\rrbracket - \overline{\llbracket u_2\rrbracket}^{\Gamma_j})\ \mathrm{d}s$$
$$\lesssim \alpha_2\omega_1\lambda_1 h^{-1}\|\overline{\llbracket\boldsymbol{u}_h\rrbracket}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}\|(\llbracket\boldsymbol{u}_h\rrbracket - \overline{\llbracket\boldsymbol{u}_h\rrbracket}^{\Gamma_j})\cdot\boldsymbol{n}\|_{0,\Gamma_j}$$
$$\lesssim \alpha_2\gamma\frac{\lambda_1}{\mu_1}\|\overline{\llbracket\boldsymbol{u}_h\rrbracket}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}(\|\boldsymbol{u}_h^1 - \overline{\boldsymbol{u}_h^1}^{\Gamma_j}\|_{0,\Gamma_j} + \|\boldsymbol{u}_h^2 - \overline{\boldsymbol{u}_h^2}^{\Gamma_j}\|_{0,\Gamma_j})$$
$$\leq \gamma\frac{\lambda_1}{\mu_1}\frac{C\alpha_2^2}{2\epsilon}\|\overline{\llbracket\boldsymbol{u}_h\rrbracket}^{\Gamma_j}\cdot\boldsymbol{n}\|^2_{0,\Gamma_j} + \epsilon\gamma\frac{\lambda_1}{\mu_1}(\|\boldsymbol{u}_h^1 - \overline{\boldsymbol{u}_h^1}^{\Gamma_j}\|^2_{0,\Gamma_j} + \|\boldsymbol{u}_h^2 - \overline{\boldsymbol{u}_h^2}^{\Gamma_j}\|^2_{0,\Gamma_j})$$
$$\leq \gamma\frac{\lambda_1}{\mu_1}\frac{C\alpha_2^2}{2\epsilon}\|\overline{\llbracket\boldsymbol{u}_h\rrbracket}^{\Gamma_j}\cdot\boldsymbol{n}\|^2_{0,\Gamma_j} + \epsilon\omega_1\lambda_1\|\nabla\boldsymbol{u}_h^1\|^2_{0,P_j^1} + \epsilon\omega_2\frac{\lambda_1\mu_2}{\mu_1\lambda_2}\lambda_2\|\nabla\boldsymbol{u}_h^2\|^2_{0,P_j^2}.$$

Also $\frac{\partial v_1^1}{\partial x} = \nabla\cdot(v_1^1, 0)^{\mathrm{T}}$, using (5.19) we get

$$\|\frac{\partial v_1^1}{\partial x}\|_{0,\Gamma_j} \lesssim h^{-1}\|\overline{\llbracket\boldsymbol{u}_h\rrbracket}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j},$$

we obtain similarly the bound

$$\int_{\Gamma_j} \alpha_1\omega_1\lambda_1 \frac{\partial v_1^1}{\partial x}([\![u_2]\!] - \overline{[\![u_2]\!]}^{\Gamma_j})\ \mathrm{d}s$$
$$\leq \gamma\frac{\lambda_1}{\mu_1}\frac{C\alpha_1^2}{2\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2 + \epsilon\omega_1\lambda_1\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \epsilon\omega_2\frac{\lambda_1\mu_2}{\mu_1\lambda_2}\lambda_2\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2.$$

We conclude by collecting the bounds for each term. □

**Lemma 5.3.2**

*Proof.* Using (5.18) and the fact $\int_{\Gamma_j}\frac{\partial v_2^1}{\partial x}\ \mathrm{d}s = 0$

$$\begin{aligned}\langle 2\omega_1\mu_1\boldsymbol{\varepsilon}(\boldsymbol{v}_j^1)\cdot\boldsymbol{n}, [\![\boldsymbol{u}_h]\!]\rangle_{\Gamma_j} = \int_{\Gamma_j} &\alpha_1\omega_1\mu_1\frac{\partial v_1^1}{\partial y}([\![u_1]\!] - \overline{[\![u_1]\!]}^{\Gamma_j}) + \alpha_1\omega_1\mu_1 h^{-1}(\overline{[\![u_1]\!]}^{\Gamma_j})^2\\ &+ 2\alpha_2\omega_1\mu_1\frac{\partial v_2^1}{\partial y}([\![u_2]\!] - \overline{[\![u_2]\!]}^{\Gamma_j}) + 2\alpha_2\omega_1\mu_1 h^{-1}(\overline{[\![u_2]\!]}^{\Gamma_j})^2\\ &+ \alpha_2\omega_1\mu_1\frac{\partial v_2^1}{\partial x}([\![u_2]\!] - \overline{[\![u_2]\!]}^{\Gamma_j})\ \mathrm{d}s.\end{aligned}$$

Where we have

$$\alpha_1\omega_1\mu_1 h^{-1}\int_{\Gamma_j}(\overline{[\![u_1]\!]}^{\Gamma_j})^2\ \mathrm{d}s = \alpha_1\gamma\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2,$$
$$2\alpha_2\omega_1\mu_1 h^{-1}\int_{\Gamma_j}(\overline{[\![u_2]\!]}^{\Gamma_j})^2\ \mathrm{d}s = 2\alpha_2\gamma\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j}^2.$$

Since $\frac{\partial v_1^1}{\partial y} = \nabla(v_1^1, 0)^{\mathrm{T}}\cdot\boldsymbol{n}$, using (5.19) we have

$$\|\frac{\partial v_1^1}{\partial y}\|_{0,\Gamma_j} \lesssim h^{-1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}.$$

Similarly as in the proof of 5.3.1

$$\begin{aligned}\int_{\Gamma_j} &\alpha_1\omega_1\mu_1\frac{\partial v_1^1}{\partial y}([\![u_1]\!] - \overline{[\![u_1]\!]}^{\Gamma_j})\ \mathrm{d}s\\ &\lesssim \alpha_1\gamma\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}(\|\boldsymbol{u}_h^1 - \overline{\boldsymbol{u}_h^1}^{\Gamma_j}\|_{0,\Gamma_j} + \|\boldsymbol{u}_h^2 - \overline{\boldsymbol{u}_h^2}^{\Gamma_j}\|_{0,\Gamma_j})\\ &\leq \gamma\frac{C\alpha_1^2}{2\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2 + \epsilon\gamma\|\boldsymbol{u}_h^1 - \overline{\boldsymbol{u}_h^1}^{\Gamma_j}\|_{0,\Gamma_j}^2 + \epsilon\gamma\|\boldsymbol{u}_h^2 - \overline{\boldsymbol{u}_h^2}^{\Gamma_j}\|_{0,\Gamma_j}^2\\ &\leq \gamma\frac{C\alpha_1^2}{2\epsilon}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{\tau}\|_{0,\Gamma_j}^2 + \epsilon\omega_1\mu_1\|\nabla\boldsymbol{u}_h^1\|_{0,P_j^1}^2 + \epsilon\omega_2\mu_2\|\nabla\boldsymbol{u}_h^2\|_{0,P_j^2}^2.\end{aligned}$$

Also $\frac{\partial v_2^1}{\partial x} = \nabla(0, v_2^1)^{\mathrm{T}}\cdot\boldsymbol{\tau}$, then using (5.19) we have

$$\|\frac{\partial v_2^1}{\partial x}\|_{0,\Gamma_j} \lesssim h^{-1}\|\overline{[\![\boldsymbol{u}_h]\!]}^{\Gamma_j}\cdot\boldsymbol{n}\|_{0,\Gamma_j},$$

we obtain similarly the bound

$$\int_{\Gamma_j} \alpha_2 \omega_1 \mu_1 \frac{\partial v_2^1}{\partial x} (\llbracket u_2 \rrbracket - \overline{\llbracket u_2 \rrbracket}^{\Gamma_j}) \, \mathrm{d}s$$
$$\leq \gamma \frac{C\alpha_2^2}{2\epsilon} \| \overline{\llbracket \boldsymbol{u}_h \rrbracket}^{\Gamma_j} \cdot \boldsymbol{n} \|_{0,\Gamma_j}^2 + \epsilon \omega_1 \mu_1 \| \nabla \boldsymbol{u}_h^1 \|_{0,P_j^1}^2 + \epsilon \omega_2 \mu_2 \| \nabla \boldsymbol{u}_h^2 \|_{0,P_j^2}^2.$$

Similarly as in the proof of 5.3.1 we have

$$\int_{\Gamma_j} 2\alpha_2 \omega_1 \mu_1 \frac{\partial v_2^1}{\partial y} (\llbracket u_2 \rrbracket - \overline{\llbracket u_2 \rrbracket}^{\Gamma_j}) \, \mathrm{d}s$$
$$\leq \gamma \frac{C\alpha_2^2}{\epsilon} \| \overline{\llbracket \boldsymbol{u}_h \rrbracket}^{\Gamma_j} \cdot \boldsymbol{n} \|_{0,\Gamma_j}^2 + \epsilon \omega_1 \mu_1 \| \nabla \boldsymbol{u}_h^1 \|_{0,P_j^1}^2 + \epsilon \omega_2 \mu_2 \| \nabla \boldsymbol{u}_h^2 \|_{0,P_j^2}^2.$$

We conclude by collecting all the terms. □

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