Syzygies and diagonal resolutions for dihedral groups

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Abstract

Let G be a finite group with integral group ring $\Lambda = \mathbf{Z}[G]$. The syzygies $\Omega_r(\mathbf{Z})$ are the stable classes of the intermediate modules in a free Λ-resolution of the trivial module. They are of significance in the cohomology theory of G via the 'co-represention theorem' $H^r(G, N) = \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_r(\mathbf{Z}), N)$. We describe the $\Omega_r(\mathbf{Z})$ explicitly for the dihedral groups D_{4n+2}, so allowing the construction of free resolutions whose differentials are diagonal matrices over Λ.

Keywords: Syzygy; free resolution; diagonal resolution; dihedral group.

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Let Λ denote the integral group ring $\Lambda = \mathbf{Z}[G]$ of a finite group G. We say that Λ-modules M, M' are *stably equivalent* (written $M \sim M'$) when $M \oplus \Lambda^a \cong M' \oplus \Lambda^b$ for some integers a, $b \geq 0$. Let

$$(\mathcal{F}) \qquad \ldots \stackrel{\partial_{n+2}}{\to} F_{n+1} \stackrel{\partial_{n+1}}{\to} F_n \stackrel{\partial_n}{\longrightarrow} \ldots\ldots \stackrel{\partial_2}{\to} F_1 \stackrel{\partial_1}{\longrightarrow} F_0 \stackrel{\partial_0}{\to} \mathbf{Z} \to 0$$

be a resolution over Λ of the trivial module $\mathbf{Z}$ in which each F_r is a finitely generated free module. The *syzygy modules* $(J_r)_{1\leq r}$ of $\mathcal{F}$ are the intermediate modules

$$J_r \quad = \quad \mathrm{Im}(\partial_r) \quad = \quad \mathrm{Ker}(\partial_{r-1}).$$

The stable syzygy $\Omega_r(\mathbf{Z})$ is then defined to be the stable class $[J_r]$ of any such J_r. It is a standard consequence of Schanuel's Lemma (cf [6] pp. 121-122) that $\Omega_r(\mathbf{Z})$ is independent of the particular choice of $(\mathcal{F})$. These stable modules are of significant interest in the cohomology theory of finite groups G; for example, they 'co-represent' cohomology in the sense that

$$H^r(G, N) \;=\; \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_r(\mathbf{Z}), N)$$

where $\mathcal{D}\mathrm{er}$ denotes the derived module category of Λ (cf [5] Chap 4).

In this paper we give an explicit description of the stable syzygies $\Omega_r(\mathbf{Z})$ for the dihedral groups

$$D_{4n+2} \;=\; \langle x,\, y \;|\; x^{2n+1} = 1,\, y^2 \;=\; 1,\, yxy^{-1} \;=\; x^{2n}\rangle.$$

Taking $\Sigma_x \;=\; \sum_{r=0}^{2n} x^s$, we shall show:

$$
\textbf{(A)} \qquad \Omega_r(\mathbf{Z}) \sim \begin{cases} [\Sigma_x, y-1] \oplus [y+1] & r \equiv 0 \mod 4 \\ [(x-1)(y-1)] \oplus [y-1] & r \equiv 1 \mod 4 \\ [\Sigma_x, y+1] \oplus [y+1] & r \equiv 2 \mod 4 \\ [(x-1)(y+1)] \oplus [y-1] & r \equiv 3 \mod 4 \end{cases}
$$

where for $\alpha_1, \ldots, \alpha_m \in \Lambda$, $[\alpha_1, \ldots, \alpha_m]$ denotes the stable class of the right ideal

$$
[\alpha_1, \ldots, \alpha_m) = \{\sum_{r=1}^{m} \alpha_i \lambda_i \mid \lambda_1, \ldots, \lambda_m \in \Lambda\}.
$$

Repetition with period four is to be expected for, as is well known ([5], [9]), the dihedral groups D_{4n+2} have cohomological period four. By contrast, periodicity is not shared by the dihedral groups of order $4n$ and for these groups (cf [7]) the task of describing the syzygies is far more difficult and increases steadily with r

Taken in conjunction with periodicity the above description allows for the construction of free resolutions of an especially simple type. Thus we shall show that D_{4n+2} admits a 'diagonalised' free resolution of period four;

$$
\textbf{(B)} \qquad 0 \to \mathbf{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{\begin{pmatrix} \partial_3^+ \\ y-1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} \partial_2^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{(\partial_1^+, y-1)} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \to 0.
$$

We may contrast this with the rather more complicated resolutions considered in [4]. The possibility of constructing diagonal resolutions for more general groups than cyclic groups was first raised in the thesis of Strouthos [8] who gave a diagonal resolution for the smallest non-abelian group, namely the dihedral group of order six.

§1 : Basis calculations :

In what follows Λ will denote the integral group ring $\mathbf{Z}[D_{4n+2}]$, and $\mathcal{I}$ the two-sided ideal $\mathcal{I} = \mathcal{I}(D_{4n+2}) = \mathrm{Ker}(\epsilon)$ where $\epsilon : \mathbf{Z}[D_{4n+2}] \to \mathbf{Z}$ is the augmentation homomorphism $\epsilon(g) = 1$ for $g \in D_{4n+2}$. Throughout we work only with *right modules* which are also *lattices* over Λ; that is, Λ-modules whose underlying additive group is free abelian of finite rank. Such a right Λ-lattice M determines a representation $\rho_M : G \to GL_{\mathbf{Z}}(M)$ by $\rho_M(g)(m) = m \cdot g^{-1}$.

For any finite group G the operation of taking inverses induces a canonical involution on $\mathbf{Z}[G]$

$$
\overline{} : \mathbf{Z}[G] \to \mathbf{Z}[G] \quad ; \quad \overline{\sum a_g g} = \sum a_g g^{-1}.
$$

We note that Λ contains the group ring $\mathbf{Z}[C_{2n+1}]$ where C_{2n+1} is the cyclic group of order $2n+1$ having generator x. This subring contains some distinguished elements which play a special role in our calculations. On defining

$$
\Sigma_x = \sum_{r=0}^{2n} x^r \ ; \qquad \theta = \sum_{r=0}^{n-1} x^r
$$

we note that

(1.1) $\overline{\theta} \;=\; \theta x^{n+2}\,;$

(1.2) Σ_x is central in Λ;

Given $\alpha, \beta \in \Lambda$ we denote by $[\alpha)$, $[\alpha, \beta)$ the right ideals

$$\begin{array}{rcl} [\alpha) & = & \{\,\alpha\lambda \mid \lambda \in \Lambda\} \\ [\alpha,\beta) & = & \{\,\alpha\lambda + \beta\mu \mid \lambda, \mu \in \Lambda\}. \end{array}$$

We stress that any ideal in Λ is a Λ-lattice. In what follows we shall frequently use:

Proposition 1.3 : Let $\{E_\psi\}_{\psi\in\Psi}$ be a $\mathbf{Z}$-basis for the free abelian group A and let $B \subset A$ be an additive subgroup such that $\mathrm{rk}_{\mathbf{Z}}(B) \le m$. Suppose also that there exists a subset $\Phi \subset \Psi$ such that $|\Phi| = m$ and $E_\phi \in B$ for each $\phi \in \Phi$; then

i) $\mathrm{rk}_{\mathbf{Z}}(B) \;=\; m$;

ii) $\{E_\phi\}_{\phi\in\Phi}$ is a $\mathbf{Z}$-basis for B;

iii) A/B is torsion free.

We define elements $E_r \in \Lambda$ by

$$\left\{ \begin{array}{lcll} E_r & = & x^r - 1 & (1 \le r \le 2n) \\ E_{4n+1} & = & y - 1 & \end{array} \right. \qquad \begin{array}{lcll} E_{2n+r} & = & (y-1)(x^r-1) & (1 \le r \le 2n) \\ E_{4n+2} & = & 1 & \end{array}$$

Λ has the canonical $\mathbf{Z}$=basis $\{y^a x^b \mid 0 \le a \le 1, 0 \le b \le 2n\}$, starting from which we proceed by elementary basis transformations to the following conclusions:

(1.4) $\{E_r\}_{1\le r\le 4n+2}$ is a $\mathbf{Z}$-basis for Λ.

(1.5) $\{E_r\}_{1\le r\le 4n+1}$ is a $\mathbf{Z}$-basis for $\mathcal{I}$.

Proposition 1.6 : $\{E_r\}_{1\le r\le 4n}$ is a $\mathbf{Z}$-basis for $[x-1)$.

Proof : We may regard $[x-1)$ as the induced module $[x-1) \;=\; \mathcal{I}(C_{2n+1}) \otimes_{\mathbf{Z}[C_{2n+1}]} \Lambda$. As Λ is a free module of rank 2 over $\mathbf{Z}[C_{2n+1}]$ we see that

(1.7) $\mathrm{rk}_{\mathbf{Z}}([x-1)) \;=\; 2\mathrm{rk}_{\mathbf{Z}}(\mathcal{I}(C_{2n+1})) \;=\; 4n.$

Clearly $E_r \in [x-1)$ for $1 \le r \le 2n$ whilst

$$E_{2n+r} \;=\; (x^{2n+1-r} - 1)y \;-\; E_r \;=\; (x-1)(\sum_{s=0}^{2n-r} x^s)y \;-\; E_r.$$

Either way, $E_r \in [x-1)$ for $1 \le r \le 4n$ so the result follows from (1.7) and (1.1). □

Taking $C_2 = \langle y \,|\, y^2 = 1\rangle$ then a similar argument to the above using the fact that $[y-1) \cong \mathcal{I}(C_2) \otimes_{\mathbf{Z}[C_2]} \Lambda$ shows that:

(1.8) $\{E_{2n+r}\}_{1 \le r \le 2n+1}$ is a **Z**-basis for $[y-1)$.

From the identities $x^r - 1 = (x-1)\sum_{s=0}^{r-1} x^s$; $yx^r - 1 = (x^r - 1) + (y-1)x^r$; we observe that

(1.9) $\mathcal{I} = [x-1) + [y-1)$.

As we shall see, the sum in (1.9) is far from being direct.

Proposition 1.10 $\{E_{2n+r}\}_{1 \le r \le 2n}$ is a **Z**-basis for $[x-1) \cap [y-1)$.

Proof: From (1.9) we obtain an exact sequence

$$0 \to [x-1) \cap [y-1) \to [x-1) \oplus [y-1) \to \mathcal{I} \to 0$$

from which, using (1.5), (1.6) and (1.8) we calculate that $\mathrm{rk}_{\mathbf{Z}}([x-1) \cap [y-1)) = 2n$. However, from (1.6) and (1.8) we see that $E_{2n+r} \in [x-1) \cap [y-1)$ for $1 \le r \le 2n$. The result now follows from (1.3). □

§2 : Decomposing the augmentation ideal :

We define elements $\pi,\ \rho,\ \widetilde{\rho}\ \in \Lambda$ as follows:

$$\textbf{(2.1)} \quad \begin{cases} \pi = (x^n - 1)(y-1) \\ \rho = (y-1)(x^{n+1} - x^n) = (x^n - x^{n+1})(y+1) \\ \widetilde{\rho} = (y-1)(x-1) \end{cases}$$

Clearly $\widetilde{\rho} = \rho \cdot x^{n+1}$ and $\rho = \widetilde{\rho} \cdot x^n$ so that $[\rho) = [\widetilde{\rho})$. We define

$$P = [\pi) \quad ; \quad R = [\rho) = [\widetilde{\rho}).$$

Evidently $\pi = (x-1)\{\sum_{s=0}^{n-1} x^s\}(y-1) \in [x-1)$ so that:

(2.2) $P \subset [x-1)$.

Proposition 2.3: $R = [x-1) \cap [y-1)$.

Proof : Clearly $\widetilde{\rho} \in [y-1)$ so that $R \subset [y-1)$. However, $\rho = (x-1)\{-x^n(y+1)\}$ so that $R \subset [x-1)$. Hence $R \subset [x-1) \cap [y-1)$. To show the opposite inclusion note that $E_{2n+1} = \widetilde{\rho} \in R$ and $E_{2n+r+1} = E_{2n+1} \cdot \{1 + x + \ldots + x^r\}$ so that $E_{2n+r} \in R$ for $1 \le r \le 2n$. Hence $[x-1) \cap [y-1) \subset R$. □

Theorem 2.4 : The ideal $[x-1)$ decomposes as a direct sum

$$[x-1) \ = \ P \dot{+} R.$$

Proof : Put $Q = [x-1)/R$ and consider the canonical exact sequence

(*) $$0 \to R \hookrightarrow [x-1) \xrightarrow{\natural} Q \to 0.$$

It suffices to show that (*) splits over Λ; in turn, it suffices then to show that

(**) the natural map $\natural$ restricts to an isomorphism $\natural : P \xrightarrow{\simeq} Q$.

As R has the **Z**-basis $\{E_{2n+r}\}_{1\leq r\leq 2n}$ which extends to a basis for Λ then Q is torsion free. Furthermore, it follows from (1.6), (2.8) that:

(***) $\{\natural(E_r)\}_{1\leq r\leq 2n}$ is a **Z**-basis for Q.

Recall that $\pi = (x^n-1)(y-1)$. Define $\widetilde{\pi} = \pi x^{n+1}$ so that $\pi = \widetilde{\pi}x^n$ and $[\widetilde{\pi}) = [\pi) = P$. A straightforward calculation shows that

$$\widetilde{\pi} \ = \ (x-1)+(y-1)(x-1)-(y-1)(x^{n+1}-1).$$

Hence $\natural(\widetilde{\pi}) = \natural(E_1)$ and hence $\natural(\widetilde{\pi}\cdot x^r) = \natural(E_1\cdot x^r)$. However

$$E_r \ = \ E_1\cdot\{\sum_{s=0}^{r-1} x^s\}$$

so that $$\natural(E_r) \ = \ \natural(\widetilde{\pi}\cdot\{\textstyle\sum_{s=0}^{r-1} x^s\}).$$

Thus $\natural : P \to Q$ is surjective and $\mathrm{rk}_{\mathbf{Z}}(P) \geq 2n$. However $\pi\cdot y = -\pi$ so that $P = \mathrm{span}_{\mathbf{Z}}\{\pi\cdot x^r \mid 0\leq r\leq 2n\}$. Moreover $\pi\cdot\Sigma_x = 0$ so that

$$P \ = \ \mathrm{span}_{\mathbf{Z}}\{\pi\cdot x^r \mid 1\leq r\leq 2n\}$$

and so $\mathrm{rk}_{\mathbf{Z}}(P)\leq 2n$. Thus $\mathrm{rk}_{\mathbf{Z}}(P) = 2n = \mathrm{rk}_{\mathbf{Z}}(Q)$. As $\natural : P\to Q$ is surjective then $\natural : P \to Q$ is an isomorphism as required. □

We note that in the course of the above proof we established:

(2.5) $\{\pi\cdot x^r\}_{1\leq r\leq 2n}$ is a **Z**-basis for P.

As a consequence of (2.5) we have:

(2.6) $P\cap R \ = \ \{0\}$.

Corollary 2.7 : The augmentation ideal $\mathcal{I}$ decomposes as an internal direct sum

$$\mathcal{I} \ = \ [\pi) \dot{+} [y-1).$$

Proof : By (1.9) we have $\mathcal{I} = [x-1) + [y-1)$ so that, by (2.4),

$$\mathcal{I} \ = \ P + R + [y-1).$$

However $R = [x-1) \cap [y-1) \subset [y-1)$ so that

$$\mathcal{I} \;=\; P \;+\; [y-1).$$

Now $P \subset [x-1)$ so that $P \cap [y-1) \subset P \cap [x-1) \cap [y-1) \subset P \cap R$. As $P \cap R \;=\; \{0\}$ then $P \cap [y-1) \;=\; \{0\}$ and $\mathcal{I} \;=\; [\pi) \;\dot{+}\; [y-1)$. □

It follows from (2.7) that $\mathcal{I}/[\pi)$ is torsion free. As $\Lambda/\mathcal{I} \cong \mathbf{Z}$ then :

(2.8) $\Lambda/[\pi)$ is torsion free.

§3 : Characterising the modules P and R.

We have defined the module P using a quite specific description, namely :

$$P \;=\; [(x^n-1)(y-1)).$$

In practice, it is useful to be able to recognise when a given Λ-module is isomorphic to P without being identical to the above model. Thus consider the following properties of a Λ-lattice M:

$\mathcal{M}(-)$: there exists $\widehat{\varphi}_- \in M$ such that $\{\varphi_- \cdot x^r \;\mid 1 \le r \le 2n\}$ is a $\mathbf{Z}$-basis for M and for which $\widehat{\varphi}_- \cdot y \;=\; -\widehat{\varphi}_-$;

$\mathcal{M}(\Sigma)$: the identity $m \cdot \Sigma_x \;=\; 0$ holds for each $m \in M$;

We recall that $P \;=\; [\pi)$ where $\pi \;=\; (x^n-1)(y-1)$. As Σ_x is central and $(x^n-1)\Sigma_x = 0$ then P satisfies $\mathcal{M}(\Sigma)$. Furthermore $\pi = (x^n-1)(y-1)$ satisfies $\pi \cdot y \;=\; -\pi$ and, by (2.5), $\{\pi \cdot x^r\}_{1 \le r \le 2n}$ is a $\mathbf{Z}$-basis for P. Thus P also satisfies $\mathcal{M}(-)$. These two properties characterize P up to Λ-isomorphism, as if M satisfies $\mathcal{M}(-)$ and $\mathcal{M}(\Sigma)$ then $\{\widehat{\varphi}_- \cdot x^r \mid 1 \le r \le 2n\}$ is a $\mathbf{Z}$-basis for M and the correspondence

$$\widehat{\varphi}_- \mapsto \pi \quad ; \qquad \sum_{r=1}^{2n} \widehat{\varphi}_+ \cdot x^r \;\; \mapsto \;\; \sum_{r=1}^{2n} \pi \cdot x^r$$

gives an isomorphism of Λ-modules $M \xrightarrow{\simeq} P$; that is:

(3.1) $M \cong P$ if and only if M satisfies $\mathcal{M}(-)$ and $\mathcal{M}(\Sigma)$.

There is a corresponding characterisation of R in terms of the following property:

$\mathcal{M}(+)$: there exists $\widehat{\varphi}_+ \in M$ such that $\{\varphi_+ \cdot x^r \;\mid 1 \le r \le 2n\}$ is a $\mathbf{Z}$-basis for M and for which $\widehat{\varphi}_+ \cdot y \;=\; \widehat{\varphi}_+$.

We recall that $R \;=\; [\rho)$ where $\rho \;=\; (x^n - x^{n+1})(y+1)$. The module R evidently satisfies $\mathcal{M}(+)$ and $\mathcal{M}(\Sigma)$. Moreover, $\rho \in R$ satisfies $\rho \cdot y \;=\; \rho$ in consequence of which $R \;=\; \mathrm{span}_{\mathbf{Z}}\{\rho \cdot x^r \;:\; 0 \le r \le 2n-1\}$. A similar argument to the above shows:

(3.2) $M \cong R$ if and only if M satisfies $\mathcal{M}(+)$ and $\mathcal{M}(\Sigma)$.

These criteria enable us to recognise non-obvious isomorphs of P, R; for example:

Proposition 3.3 : Let $a, b \in \mathbf{Z}$ be such that $a - b$ is coprime to $2n + 1$; then $[(x^a - x^b)(y-1)) \cong P$.

Proof : If $k \in \mathbf{Z}$ put $\pi(k) = (x^k - 1)(y - 1)$, so that $P = [\pi(n))$. Consider the Λ-module automorphism $\lambda : \Lambda \to \Lambda$ given by

$$\Lambda(z) = x^b \cdot z.$$

Then $\lambda : [\pi(a-b)) \xrightarrow{\simeq} [(x^a - x^b)(y-1)$ is a Λ-isomorphism. As $[\pi(k))$ clearly satisfies $\mathcal{M}(\Sigma)$ it suffices to show that $\pi(k))$ satisfies $\mathcal{M}(-)$ when k is coprime to $2n + 1$.

Thus suppose that k is coprime to $2n + 1$ so that, in particular, x^k generates C_{2n+1}. As n is also coprime to $2n + 1$ then x^n also generates C_{2n+1}. Hence there is an automorphism $\alpha : D_{4n+2} \to D_{4n+2}$ with the properties that $\alpha(x^n) = x^k$ and $\alpha(y) = y$. Let $\alpha_* : \Lambda \to \Lambda$ be the ring automorphism induced by α. Then $\alpha_*(\pi) = \pi(k)$ so that $\alpha_*(P) = [\pi(k))$. Hence $\mathrm{rk}_{\mathbf{Z}}([\pi(k)) = 2n$. However as

$$\pi(k) \cdot y = -\pi(k)$$

then $[\pi(k)) = \mathrm{span}_{\mathbf{Z}}\{\pi(k) \cdot x^s \mid 0 \leq s \leq 2n\}$. However $\pi(k) \cdot \Sigma_x = 0$ so that, as $\mathrm{rk}_{\mathbf{Z}}([\pi(k)) = 2n$ then $\{\pi(k) \cdot x^s \mid 1 \leq s \leq 2n\}$ is a $\mathbf{Z}$-basis for $[\pi(k))$. Thus $[\pi(k))$ satisfies $\mathcal{M}(-)$ as required. □

A similar argument yields the corresponding statement for R:

Proposition 3.4: Let $a, b \in \mathbf{Z}$ be such that $a - b$ is coprime to $2n + 1$; then $[(x^a - x^b)(y+1)) \cong R$.

§4: The modules K, L:

We define $K = [\Sigma_x, y - 1)$ and $L = [\Sigma_x, y + 1)$; we claim

Proposition 4.1 : $\mathrm{rk}_{\mathbf{Z}}(K) = 2n + 2$ and Λ/K is torsion free.

Proof : Put $K_0 = \{(1 - y)a(x) \mid a(x) \in \mathbf{Z}[C_{2n+1}]\} \subset K$. We note that

$$(y-1)x^s \cdot y = -(y-1)x^{2n+1-s}$$

from which it follows that K_0 is a Λ-submodule of K. Moreover, as

$$\Sigma_x y = \sum_{s=0}^{2n}(y-1)x^s + \Sigma_x$$

it follows that K is spanned over $\mathbf{Z}$ by $\{\, (y-1)x^s \mid 0 \leq s \leq 2n\} \cup \{\Sigma_x\}$. However, starting from the canonical basis for Λ and proceeding by elementary basis transformations, it is easy to see that $\{(y-1)x^r \mid 0 \leq r \leq 2n\} \cup \{\Sigma_x\} \cup \{x^s \mid 1 \leq s \leq 2n\}$

is a **Z**-basis for Λ. It follows from (1.3) that $\{(y-1)x^s \mid 0 \le s \le 2n\} \cup \{\Sigma_x\}$ is a **Z**-basis for K and Λ/K is torsion free □

For future reference we note that we have also shown:

(4.2) $\{(y-1)x^s \mid 0 \le s \le 2n\} \cup \{\Sigma_x\}$ is a **Z**-basis for K.

Proposition 4.3 K is monogenic, generated by $(1-y)\theta + \Sigma_x y$;

Proof : It is clear that $(1-y)\theta + \Sigma_x y \in [\Sigma_x, y-1)$. However, the identity

$$\{(1-y)\theta + \Sigma_x y\} \cdot x^{n+1}(1-y) \quad = \quad (y-1)$$

shows that $(y-1) \in [(1-y)\theta + \Sigma_x y)$. Thus $(y-1)\theta \in [(1-y)\theta + \Sigma_x y)$. Hence $\Sigma_x y \in [(1-y)\theta + \Sigma_x y)$ and so $\Sigma_x \;=\; \{\Sigma_x y\} \cdot y \;\in\; [(1-y)\theta + \Sigma_x y)$. □

Now put $L_0 \;=\; \{(y+1)a(x) \;\mid\; a(x) \in \mathbf{Z}[C_{2n+1}]\} \subset L$; similarly to (4.1) we have:

Proposition 4.4 : $\mathrm{rk}_{\mathbf{Z}}(L) \;=\; 2n+2$ and Λ/L is torsion free.

Furthermore:

(4.5) $\{(y+1)x^s \mid 0 \le s \le 2n\} \cup \{\Sigma_x\}$ is a **Z**-basis for L.

Noting that $\{(1+y)\theta - \Sigma_x y\} \cdot x^{n+1}(y+1) \;=\; -(y+1)$ an analogous argument to (4.3) then shows that:

Proposition 4.6 L is monogenic, generated by $(1+y)\theta - \Sigma_x y$.

§5 : Two diagonal resolutions :
Define elements in Λ as follows

$$\partial_0^+ \;=\; (1-y)\theta \;+\; \Sigma_x \cdot y\;;$$

$$\partial_1^+ \;=\; (x^{n+1} - x)(y-1)\;;$$

$$\partial_2^+ \;=\; (1+y)\theta \;-\; \Sigma_x \cdot y\;;$$

$$\partial_3^+ \;=\; (x^{n+1} - x)(y+1)$$

and put $\partial_4^+ = \partial_0^+$; we have a sequence repeating with period four infinitely in both directions:

$$(\mathcal{S}^+) \qquad \dots \stackrel{\partial_1^+}{\to} \Lambda \stackrel{\partial_0^+}{\to} \Lambda \stackrel{\partial_3^+}{\to} \Lambda \stackrel{\partial_2^+}{\to} \Lambda \stackrel{\partial_1^+}{\to} \Lambda \stackrel{\partial_0^+}{\to} \Lambda \stackrel{\partial_3^+}{\to} \dots$$

We shall show that $(\mathcal{S}^+)$ is exact. To do this first observe that:

Proposition 5.1 : $\partial_0^+\partial_1^+ = 0.$

Proof : From the above definitions we see that $\partial_0^+\partial_1^+ = A + B$ where

$$A = (1-y)\theta(x^{n+1}-x)(y-1) \quad ; \quad B = \Sigma_x y(x^{n+1}-x)(y-1).$$

As Σ_x is central in Λ and $\Sigma_x(x^{n+1}-x) = 0$ it follows that $B = 0$.
To show that $A = 0$ we first note that

$$\begin{aligned} A &= (1-y)y\{\overline{\theta x^{n+1}} - \overline{\theta x}\} + (y-1)\{\theta x^{n+1} - \theta x\} \\ &= (y-1)\{(\overline{\theta x^{n+1}} - \theta x) + (\theta x^{n+1} - \overline{\theta x})\}. \end{aligned}$$

However, by (1.1), $\overline{\theta} = \theta x^{n+2}$ so that $\overline{\theta x^{n+1}} = \theta x^{n+2}x^n = \theta x$ and likewise $\theta x^{n+1} = \overline{\theta x}$. As required we have $A = 0$. □

A similar proof shows that

(5.2) $\partial_2^+\partial_3^+ = 0.$

From the fact that $y^2 = 1$ and that Σ_x is central in Λ it follows that:

(5.3) $\partial_1^+\partial_2^+ = 0;$

(5.4) $\partial_3^+\partial_0^+ = 0.$

Proposition 5.5: $\Lambda/[\partial_r^+)$ is torsion free for each r.

Proof : Let $\tau : \Lambda \to \Lambda$ be the Λ-isomorphism $\tau(\lambda) = x\lambda$. Then τ induces an isomorphism $\Lambda/[\pi) \xrightarrow{\simeq} \Lambda/[\tau(\pi))$. However $\tau(\pi) = \partial_1^+$ and $\Lambda/[\pi)$ is torsion free, by (2.8). Thus $\Lambda/[\partial_1^+)$ is torsion free.

To show that $\Lambda/[\partial_3^+)$ is torsion free, put $\upsilon = (x-1)(y+1)$. Then $\tau^n(\upsilon) = -\rho$ so that $\Lambda/[\upsilon)$ is torsion free by (1.3), (1.10) and (2.3). Observing that n is coprime to $2n+1$, let $\alpha : D_{4n+2} \to D_{4n+2}$ be the automorphism $\alpha(x) = x^n$; $\alpha(y) = y$ and let $\alpha_* : \Lambda \to \Lambda$ be the ring automorphism induced from α. Then $\tau \circ \alpha_*(\upsilon) = \partial_3^+$ so that $\Lambda/[\partial_3^+)$ is torsion free.

For the remaining two cases, observe that $[\partial_0^+) = K$ and $[\partial_2^+) = L$. However, Λ/K is torsion free by (4.1) and Λ/L is torsion free by (4.4). □

Proposition 5.6 : $\mathrm{rk}_{\mathbf{Z}}[\mathrm{Ker}(\partial_r^+)] = \mathrm{rk}_{\mathbf{Z}}[\mathrm{Im}(\partial_{r+1}^+)]$ for $0 \leq r \leq 3$.

Proof : Observe that

(*) $$\mathrm{rk}_{\mathbf{Z}}[\mathrm{Im}(\partial_r^+)] = \begin{cases} 2n+2 & r \text{ even} \\ 2n & r \text{ odd.} \end{cases}$$

However $\mathrm{rk}_{\mathbf{Z}}(\mathrm{Ker}(\partial_r^+) \; = \; 4n+2-\mathrm{rk}_{\mathbf{Z}}(\mathrm{Im}(\partial_r^+)$; on applying this to (*) we see that

$$\textbf{(**)} \qquad \mathrm{rk}_{\mathbf{Z}}[\mathrm{Ker}(\partial_r^+)] \; = \; \begin{cases} 2n & r \text{ even} \\ \\ 2n+2 & r \text{ odd.} \end{cases}$$

On re-expressing (*) in the following form

$$\textbf{(***)} \qquad \mathrm{rk}_{\mathbf{Z}}[\mathrm{Im}(\partial_{r+1}^+)] \; = \; \begin{cases} 2n & r \text{ even} \\ \\ 2n+2 & r \text{ odd} \end{cases}$$

the result follows immediately. □

Theorem 5.7: The sequence $(\mathcal{S}^+)$ is exact.

Proof : From (5.1) - (5.4) we see that $\mathrm{Im}(\partial_{r+1}^+) \subset \mathrm{Ker}(\partial_r^+)$ for $0 \le r \le 4$. From (5.6) it follows that each $\mathrm{Ker}(\partial_r^+)/\mathrm{Im}(\partial_{r+1}^+)$ is finite. However, by (5.1) each $\mathrm{Ker}(\partial_r^+)/\mathrm{Im}(\partial_{r+1}^+)$ is also torsion free and so $\mathrm{Ker}(\partial_r^+) \; = \; \mathrm{Im}(\partial_{r+1}^+)$ □

In addition to $(\mathcal{S}^+)$ we have another long exact sequence $(\mathcal{S}^-)$ of period two;

$$(\mathcal{S}^-) \qquad \dots \stackrel{y+1}{\to} \Lambda \stackrel{y-1}{\to} \Lambda \stackrel{y+1}{\to} \Lambda \stackrel{y-1}{\to} \Lambda \stackrel{y+1}{\to} \Lambda \stackrel{y-1}{\to} \Lambda \stackrel{y+1}{\to} \dots$$

From $(\mathcal{S}^+)$ and $(\mathcal{S}^-)$ we may form another exact sequence, again repeating with period four infinitely in both directions:

$$(\mathcal{S})_\infty \quad \Lambda\oplus\Lambda \xrightarrow{\begin{pmatrix} \partial_0^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda\oplus\Lambda \xrightarrow{\begin{pmatrix} \partial_3^+ & 0 \\ 0 & y-1 \end{pmatrix}} \Lambda\oplus\Lambda \xrightarrow{\begin{pmatrix} \partial_2^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda\oplus\Lambda \xrightarrow{\begin{pmatrix} \partial_1^+ & 0 \\ 0 & y-1 \end{pmatrix}} \Lambda\oplus\Lambda \xrightarrow{\begin{pmatrix} \partial_0^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda\oplus\Lambda.$$

The sequence $(\mathcal{S})$ is a complete resolution of D_{4n+2} in the sense of Tate [1]. We proceed to modify $(\mathcal{S})$ in a number of ways. Taking $\epsilon : \Lambda \to \mathbf{Z}$ to be the augmentation map and defining

$$\partial_1 : \Lambda \oplus \Lambda \to \Lambda \quad ; \quad \partial_1 = (\partial_1^+, y-1)$$

$$\partial_2 : \Lambda \oplus \Lambda \to \Lambda \oplus \Lambda \quad ; \quad \partial_2 \; = \; \begin{pmatrix} \partial_2^+ & 0 \\ 0 & y+1 \end{pmatrix}$$

we have the following sequence

$$(\mathcal{S}) \quad .. \to \Lambda\oplus\Lambda \xrightarrow{\begin{pmatrix} \partial_1^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda\oplus\Lambda \xrightarrow{\begin{pmatrix} \partial_0^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda\oplus\Lambda \xrightarrow{\begin{pmatrix} \partial_3^+ & 0 \\ 0 & y-1 \end{pmatrix}} \Lambda\oplus\Lambda \stackrel{\partial_2}{\to} \Lambda\oplus\Lambda \stackrel{\partial_1}{\to} \Lambda \stackrel{\epsilon}{\to} \mathbf{Z} \to 0$$

which continues thereafter infinitely to the left with the same differentials as $(\mathcal{S})_\infty$. Noting that $\mathrm{Ker}(\epsilon) \; = \; \mathcal{I}$ and that, by (2.7),

$$\mathcal{I} \; = \; [\pi) \; \dot{+} \; [y-1) \; = \; [\partial_1^+) \; \dot{+} \; \mathrm{Im}(y-1) \; = \; \mathrm{Im}(\partial_1)$$

we see that:

(5.8) $\quad \mathrm{Ker}(\epsilon) = \mathrm{Im}(\partial_1).$

To proceed we note the following identity.

(5.9) $\quad [(1-y)\theta + \Sigma_x y](y-1) = (y-1).$

Proposition 5.10 : $\quad \mathrm{Ker}(\partial_1) = \mathrm{Im}(\partial_2).$

Proof : It is straightforward to check that $\partial_1\partial_2 = 0$ so it suffices to show that $\mathrm{Ker}(\partial_1) \subset \mathrm{Im}(\partial_2)$. Thus suppose that

$$\partial_1 \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

Then $\partial_1^+(a) = -(y-1)b$. However, $\partial_0^+\partial_1^+ = 0$ so that, by the identity of (5.9),

$$\partial_0^+(y-1)b = \{(1-y)\theta + \Sigma_x y\}(y-1)b = (y-1)b = 0$$

Thus $b = (y+1)d$ for some $d \in \Lambda$ and $\partial_1^+(a) = -(y-1)(y+1)d = 0$. Hence $a = \partial_2^+(c)$ for some $c \in \Lambda$ and

$$\begin{pmatrix} a \\ \\ b \end{pmatrix} = \begin{pmatrix} \partial_2^+ & 0 \\ \\ 0 & y+1 \end{pmatrix} \begin{pmatrix} c \\ \\ d \end{pmatrix} = \partial_2 \begin{pmatrix} c \\ \\ d \end{pmatrix}. \qquad \square$$

From the foregoing we see that:

(5.11) The sequence $(\mathcal{S})$ is exact.

The sequence $(\mathcal{S})$ is a free diagonal resolution of D_{4n+2} of period four. There is, however, an even simpler free resolution to be obtained. Thus if we now define

$$\partial_3 : \Lambda \to \Lambda \oplus \Lambda \quad ; \quad \partial_3 = \begin{pmatrix} \partial_3^+ \\ y-1 \end{pmatrix}$$

$$\epsilon^* : \mathbf{Z} \to \Lambda \quad ; \quad \epsilon^*(1) = \Sigma_x(1+y)$$

we obtain the following finite sequence

$$(\mathcal{D})_{\mathrm{fin}} \qquad 0 \to \mathbf{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{\partial_3} \Lambda \oplus \Lambda \xrightarrow{\partial_2} \Lambda \oplus \Lambda \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \to 0.$$

From the definition of ∂_3 and the exactness of $(\mathcal{S})$ it follows immediately that

(5.12) $\mathrm{Ker}(\partial_2) = \mathrm{Im}(\partial_3)$.

Proposition 5.13 : $\mathrm{Ker}(\partial_3) = \mathrm{Im}(\epsilon^*)$.

Proof : It is straightforward to see that $\mathrm{Im}(\epsilon^*) \subset \mathrm{Ker}(\partial_3)$. To establish the reverse inclusion, suppose $e \in \Lambda$ satisfies $\partial_3(e) = \binom{0}{0}$; then

$$(x^{n+1} - x^n)(y+1)e \;=\; (y-1)e \;=\; 0.$$

In particular, $e = (y+1)f$ and so $(x^{n+1} - x^n)(y+1)(y+1)f = 0$; that is $2x^n(x-1)(y+1)f = 0$ or equivalently

$$(x-1)(y+1)f \;=\; 0.$$

Write $f = g(x) + h(x)y$ where $g(x), h(x) \in \mathbf{Z}[C_{2n+1}]$ so that

$$e \;=\; (1+y)f \;=\; \alpha(x)(1+y)$$

where $\alpha(x) = g(x) + \overline{h(x)}$. As $(x-1)e = 0$ then $(x-1)\alpha(x) = 0$ so that $\alpha(x) = \lambda\Sigma_x$ and so $e = \lambda\Sigma_x(1+y) = \epsilon^*(\lambda)$. Thus $\mathrm{Ker}(\partial_3) \subset \mathrm{Im}(\epsilon^*)$. □

In consequence of the foregoing we obtain the following, which is statement **(B)** of the Introduction:

(5.14) The sequence $(\mathcal{D})_{\mathrm{fin}}$ is exact.

Observing that $\epsilon\epsilon^* = \Sigma_x(1+y)$ we may repeat $(\mathcal{D})_{\mathrm{fin}}$ infinitely to the left to obtain another free resolution of D_{4n+2} with period four thus:

$$(\mathcal{D})_\infty \qquad \dots \xrightarrow{\partial_2} \Lambda\oplus\Lambda \xrightarrow{\partial_1} \Lambda \xrightarrow{\Sigma_x(1+y)} \Lambda \xrightarrow{\partial_3} \Lambda\oplus\Lambda \xrightarrow{\partial_2} \Lambda\oplus\Lambda \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \to 0.$$

§6 : The syzygies $\Omega_r^{D_{4n+2}}(\mathbf{Z})$:

Let Λ denote the integral group ring $\Lambda = \mathbf{Z}[G]$ of a finite group G. If M is a Λ-module and

$$(\mathcal{F}) \qquad \dots \xrightarrow{\partial_{n+2}} F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} \dots\dots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

is a free resolution of M of finite type over Λ the *syzygy modules* $(J_r)_{1\le r}$ of $\mathcal{F}$ are the intermediate modules $J_r = \mathrm{Im}(\partial_r) = \mathrm{Ker}(\partial_{r-1})$. The stable class $[\mathrm{Im}(\partial_r]$ is independent of $(\mathcal{F})$ and is written

$$\Omega_r^G(M) \;=\; [\mathrm{Im}(\partial_r].$$

From the resolution $(\mathcal{D})_\infty$ we can read off the syzygies $\Omega_r(\mathbf{Z})\ (= \Omega_r^{D_{4n+2}}(\mathbf{Z}))$ directly:

$$\textbf{(6.1)} \qquad \Omega_r(\mathbf{Z}) \ \sim \begin{cases} [\mathbf{Z}] & r \equiv 0 \mod 4 \\ [(x^{n+1}-x)(y-1)] \oplus [y-1] & r \equiv 1 \mod 4 \\ [(1+y)\theta - \Sigma_x y] \oplus [y+1] & r \equiv 2 \mod 4 \\ [(x^{n+1}-x)(y+1)] \oplus [y-1] & r \equiv 3 \mod 4. \end{cases}$$

This description can be simplified; as n is coprime to $2n+1$ then by (3.3) and (3.4)

$$[(x^{n+1}-x)(y-1)) \cong P \quad ; \quad [(x^{n+1}-x)(y+1)) \cong R$$

whilst from (4.3) and (4.6) we have $[(1+y)\theta - \Sigma_x y) \ \cong \ L$. Thus

$$\textbf{(6.2)} \qquad \Omega_r(\mathbf{Z}) \ \sim \begin{cases} [\mathbf{Z}] & r \equiv 0 \mod 4 \\ [P] \oplus [y-1] & r \equiv 1 \mod 4 \\ [L] \oplus [y+1] & r \equiv 2 \mod 4 \\ [R] \oplus [y-1] & r \equiv 3 \mod 4. \end{cases}$$

Reading off the syzygies from the resolution $(\mathcal{S})$ gives a slightly different expression for $\Omega_4(\mathbf{Z})$; recalling from (4.3) that $[(1-y)\theta + \Sigma_x y) \ \cong \ K$, then

$$\textbf{(6.3)} \qquad \Omega_4(\mathbf{Z}) \ \sim \ [K] \oplus [y+1].$$

Comparing the expressions for $\Omega_4(\mathbf{Z})$ in (6.1) and (6.3) we find that :

$$\textbf{(6.4)} \qquad [\mathbf{Z}] \ = \ [K] \oplus [y+1].$$

Together with (6.4), the isomorphisms $[(1-y)\theta + \Sigma_x y) \cong K$, $[(x-1)(y-1)) \cong P$, $[(1+y)\theta - \Sigma_x y) \cong L$, $[(x-1)(y+1)) \cong R$ show that (6.2) is equivalent to the statement **(A)** of the Introduction.

The decomposition (6.4) illustrates a somewhat paradoxical aspect of the theory of stable modules, namely that whilst a module (in this case the trivial module $\mathbf{Z}$) may be indecomposable, its stable class may decompose non-trivially. This phenomenon seems first to have been pointed out, though without an explicit example, in the paper of Gruenberg and Roggenkamp ([3] Proposition 1). They attribute the original observation to E.C. Dade ([3] p. 153).

§7 : Relations between the modules :

If M, N are Λ-lattices the tensor product $M \otimes N$ is defined by imposing the group action $(m \otimes n) \cdot g \ = \ m \cdot g \otimes n \cdot g$ on the abelian group $M \otimes_{\mathbf{Z}} N$. Extending this in an obvious way to stable modules it is well known and straightforward to show that

(7.1) $$\Omega_k(\mathbf{Z}) \otimes \Omega_l(\mathbf{Z}) \ = \ \Omega_{k+l}(\mathbf{Z}).$$

This suggests corresponding relations between the modules K, P, L, R. For example, the relation $\Omega_1(\mathbf{Z}) \otimes \Omega_1(\mathbf{Z}) \ = \ \Omega_2(\mathbf{Z})$ suggests a stable equivalence $P \otimes P \sim L$. This is indeed the case. More precisely, the author's student John Evans has shown that (cf [2]), under tensor product, the relations amongst the modules K, P, L, R are given by the following table.

(7.2)

$\otimes$	K	P	L	R
K	$K \oplus \Lambda^{n+1}$	$P \oplus \Lambda^{n}$	$L \oplus \Lambda^{n+1}$	$R \oplus \Lambda^{n}$
P	$P \oplus \Lambda^{n}$	$L \oplus \Lambda^{n-1}$	$R \oplus \Lambda^{n}$	$K \oplus \Lambda^{n-1}$
L	$L \oplus \Lambda^{n+1}$	$R \oplus \Lambda^{n}$	$K \oplus \Lambda^{n+1}$	$P \oplus \Lambda^{n}$
R	$R \oplus \Lambda^{n}$	$K \oplus \Lambda^{n-1}$	$P \oplus \Lambda^{n}$	$L \oplus \Lambda^{n-1}$

Thus under the operation of tensor product one may view the stable modules $[K]$,$[P]$,$[L]$,$[R]$ as a cyclic group of order 4 generated either by $[P]$ or $[R]$, with $[K]$ as identity.

There are corresponding duality statements. Over an arbitrary finite group one has $\Omega_r(\mathbf{Z})^* \ = \ \Omega_{-r}(\mathbf{Z})$. However in the special case $G = D_{4n+2}$ the syzygies have period four, $\Omega_r(\mathbf{Z}) \ = \ \Omega_{r+4}(\mathbf{Z})$ so that

(7.3) $$\Omega_r^*(\mathbf{Z}) \ = \ \Omega_{4-r}(\mathbf{Z}).$$

In fact the corresponding relations already hold at the level of modules, namely

(7.4) $K^* \cong K$;

(7.5) $L^* \cong L$;

(7.6) $P^* \cong R$;

(7.7) $R^* \cong P$.

One should perhaps stress that no two of K, P, L, R are isomorphic. In fact, given that D_{4n+2} has cohomological period four, no two of K, P, L, R are stably isomorphic.

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