All-loop correlators of integrable λ-deformed σ-models

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Abstract

We compute the 2- and 3-point functions of currents and primary fields of λ-deformed integrable σ-models characterized also by an integer k. Our results apply for any semisimple group G, for all values of the deformation parameter λ and up to order $1/k$. We deduce the OPEs and equal-time commutators of all currents and primaries. We derive the currents' Poisson brackets which assume Rajeev's deformation of the canonical structure of the isotropic PCM, the underlying structure of the integrable λ-deformed σ-models. We also present analogous results in two limiting cases of special interest, namely for the non-Abelian T-dual of the PCM and for the pseudodual model.
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1. Introduction and motivation

One of the most intriguing conjectures in modern theoretical physics is the AdS/CFT correspondence [1] which, in its initial form, states the equivalence between type-IIB superstring theory on the $AdS_5 \times S^5$ background and the maximally supersymmetric field theory in four

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dimensions, i.e. $\mathcal{N} = 4$ SYM. In recent years, a huge progress has been made in calculating physical observables employing both sides of the duality. These calculations managed to probe the strongly coupled regime of the gauge theory which is practically unaccessible by other means. The key feature that allowed this progress is integrability. $\mathcal{N} = 4$ SYM from one side and the two-dimensional σ-model from the other, are believed to be integrable order by order in perturbation theory. It is clear that one way to construct generalizations of the original AdS/CFT scenario is to try to maintain the key property of integrability.

The aim of this work is to study the structure of a class of two-dimensional σ-models, the so-called λ-deformed models constructed in [2]. For isotropic couplings the deformation is integrable in the group case and in the symmetric and semi-symmetric coset cases [2–5] (for the $su(2)$ group case integrability is preserved for anisotropic, albeit diagonal couplings [6]). They are also closely related [7–12] to the so-called η-deformed models for group and coset spaces introduced in [7,8] and in [13–15], respectively. This relation is via Poisson–Lie T-duality and an analytic continuation of coordinates and of the parameters of the σ-models [10–12]. There are also embeddings of the λ-deformed models as solutions of supergravity [16–18].

In particular, we shed light into the structure of the λ-deformed models by computing the two- and three-point functions of all currents and operators exactly in the deformation parameter and up to order $1/k$. This work is based and further extends symmetry ideas and techniques originated in our previous work in [19]. The results of this work are summarized in section 7.

Our starting point is the WZW action

$$S_{\text{WZW},k}(g) = -\frac{k}{4\pi} \int d^2\sigma \, \text{Tr}(g^{-1}\partial_+ g \, g^{-1}\partial_- g) + \frac{k}{24\pi} \int_B \text{Tr}(g^{-1}dg)^3 \,, \tag{1.1}$$

for a generic semisimple group G, with $g \in G$ parametrized by X^μ, $\mu = 1, 2, \ldots, \dim G$. We will use the representation matrices t_a which obey the commutation relations $[t_a, t_b] = f_{abc} t_c$ and are normalized as $\text{Tr}(t_a t_b) = \delta_{ab}$. These matrices are taken to be Hermitian and therefore the Lie-algebra structure constants f_{abc} are purely imaginary. The chiral and anti-chiral currents are defined as

$$J_+^a = -i\,\text{Tr}(t_a \partial_+ g \, g^{-1}) = R_\mu^a \partial_+ X^\mu \,, \qquad J_-^a = -i\,\text{Tr}(t_a g^{-1} \partial_- g) = L_\mu^a \partial_- X^\mu \,. \tag{1.2}$$

The left and right invariant forms $L^a = L_\mu^a dX^\mu$ and $R^a = R_\mu^a dX^\mu$ are related as

$$R^a = D_{ab} L^b \,, \qquad D_{ab} = \text{Tr}(t_a g t_b g^{-1}) \,. \tag{1.3}$$

We are interested in the non-Abelian Thirring model action (for a general discussion, see [20,21]), namely the WZW two-dimensional conformal field theory (CFT) perturbed by a set of classically marginal operators which are bilinear in the currents

$$S = S_{\text{WZW},k}(g) + \frac{k}{2\pi} \sum_{a,b=1}^{\dim G} \lambda_{ab} \int d^2\sigma \, J_+^a J_-^b \,, \tag{1.4}$$

where the couplings are denoted by the constants λ_{ab}. An action having the same global symmetries as (1.4), and to which reduces for small values of λ_{ab} has been derived in [2] (see also [22] for the $SU(2)$ case), by gauging a common symmetry subgroup of an action involving the PCM model and the WZW actions. It reads [2]

$$S_{k,\lambda}(g) = S_{\text{WZW},k}(g) + \frac{k}{2\pi} \int d^2\sigma \, J_+^a (\lambda^{-1} - D^T)_{ab}^{-1} J_-^b \,, \tag{1.5}$$

where we have assembled in a general real matrix λ the coupling constants λ_{ab}. In addition, this action, as well as (1.4), is invariant under the generalized parity transformation

$$\sigma^\pm \mapsto \sigma^\mp , \qquad g \mapsto g^{-1} , \qquad \lambda \mapsto \lambda^T . \tag{1.6}$$

The β-functions for the running of couplings under the Renormanization Group (RG) flow using (1.5) were computed in [23,24] and completely agree with the computation of the same RG-flow equations using CFT techniques based on (1.4) in [25] for a single (isotropic) coupling, i.e. when $\lambda_{ab} = \lambda \delta_{ab}$ and in [26] for symmetric λ_{ab}. Based on that it was conjectured in [23,24] that (1.5) is the effective action for (1.4) valid to all orders in λ and up to order $1/k$. In the same works it was realized that (1.5) has the remarkable symmetry

$$S_{-k,\lambda^{-1}}(g^{-1}) = S_{k,\lambda}(g) . \tag{1.7}$$

This has been instrumental in computing the anomalous dimensions of currents for the isotropic case exactly in λ and up to order in $1/k$ [19] and will be central in the present work as well. We should stress that this is not a symmetry of the non-Abelian Thirring model action (1.4). However, using path integral techniques and special properties of the WZW model action, it was argued in [27] that the effective action of the non-Abelian Thirring model (not known at the time) should be invariant under the above duality-type symmetry $(\lambda, k) \mapsto (\lambda^{-1}, -k)$ (for $k \gg 1$).

2. The set up

2.1. OPE's at the conformal point

In what follows, we shall need the operator product expansion (OPE) of the currents in the Euclidean regime with complex coordinates $z = \frac{1}{2}(\tau + i\sigma)$ and $\bar{z}$. For the holomorphic ones the singular part of their OPE reads [28,29]

$$J^a(z) J^b(w) = \frac{f_{abc}}{\sqrt{k}} \frac{J^c(w)}{z - w} + \frac{\delta_{ab}}{(z - w)^2} \tag{2.1}$$

and similarly for the OPE between the anti-holomorphic currents $\bar{J}^a(\bar{z})$. Of course the OPE $J^a(z)\bar{J}^b(w)$ is regular. The difference from the more conventional form of these OPE's arises because we have rescaled the currents as $J^a \mapsto J^a/\sqrt{k}$ which suits our purposes since in that way, as will shall see, we keep easily track of the contributions of various terms to the correlators of the perturbed theory.

The CFT contains affine primary fields $\Phi_{i,i'}(z, \bar{z})$ transforming in the irreducible representations R and R', with matrices t_a and $\tilde{t}_a$, under the action of the currents J^a and $\bar{J}^a$, so that $i = 1, 2, \ldots, \dim R$ and $i' = 1, 2, \ldots, \dim R'$. Specifically,

$$\begin{aligned}
J_a(z) \Phi_{i,i'}(w, \bar{w}) &= -\frac{1}{\sqrt{k}} \frac{(t_a)_i{}^j \Phi_{j,i'}(w, \bar{w})}{z - w} , \\
\bar{J}_a(z) \Phi_{i,i'}(w, \bar{w}) &= \frac{1}{\sqrt{k}} \frac{(\tilde{t}_a)^{j'}{}_{i'} \Phi_{i,j'}(w, \bar{w})}{\bar{z} - \bar{w}} .
\end{aligned} \tag{2.2}$$

These fields are also Virasoro primaries with holomorphic and anti-holomorphic dimensions [29]

$$\Delta_R = \frac{c_R}{2k + c_G} , \qquad \bar{\Delta}_{R'} = \frac{c_{R'}}{2k + c_G} , \tag{2.3}$$

where c_R, $c_{R'}$ and c_G are the quadratic Casimir operators, all non-negative, in the representations R, R' and the adjoint representation for which $(t_a)_{bc} = f_{abc}$. They are defined as

$$(t_a t_a)_i{}^j = c_R \delta_i{}^j , \qquad (\tilde{t}_a \tilde{t}_a)_{i'}{}^{j'} = c_{R'} \delta_{i'}{}^{j'} , \qquad f_{acd} f_{bcd} = -c_G \delta_{ab} . \tag{2.4}$$

In our calculations we will need the basic two- and three-point functions for these fields. For the currents they are given by

$$\langle J_a(z_1) J_b(z_2) \rangle = \frac{\delta_{ab}}{z_{12}^2} , \qquad \langle J_a(z_1) J_b(z_2) J_c(z_3) \rangle = \frac{1}{\sqrt{k}} \frac{f_{abc}}{z_{12} z_{13} z_{23}} , \tag{2.5}$$

where we employ the general notation $z_{ij} = z_i - z_j$. We will also use the four-point function

$$\langle J^a(x_1) J^{a_1}(z_1) J^{a_2}(z_2) J^{a_3}(z_3) \rangle = \frac{1}{k} \frac{f_{a_1 a c} f_{c a_2 a_3}}{(z_1 - x_1)(x_1 - z_2)(x_1 - z_3)(z_1 - z_3)}$$
$$+ \frac{\delta_{a a_1} \delta_{a_2 a_3}}{(x_1 - z_1)^2 (z_2 - z_3)^2} + \text{cyclic in } 1, 2, 3 . \tag{2.6}$$

Similar expressions hold for the anti-holomorphic currents as well. Correlators involving both holomorphic and anti-holomorphic currents vanish at the conformal point. However, as we shall see, this will not be the case in the deformed theory.

The corresponding correlators for the affine primaries are

$$\langle \Phi_{i,i'}^{(1)}(z_1, \bar{z}_1) \Phi_{j,j'}^{(2)}(z_2, \bar{z}_2) \rangle = \frac{\delta_{ij} \, \delta_{i'j'}}{z_{12}^{2\Delta_R} \, \bar{z}_{12}^{2\bar{\Delta}_{R'}}} , \tag{2.7}$$

where the superscripts signify the fact that the representations for the different primaries in correlation functions could be, in general, different. However, for the two-point functions the two representations should in fact be conjugate to each other for the holomorphic and anti-holomorphic sectors separately. As such, they have the same conformal dimensions. Recalling that the matrices t_a and $\tilde{t}_a$ are Hermitian and after removing the superscripts by relabeling the representation matrices we have that

$$\text{Reps (1) and (2) conjugate: } t_a^{(1)} = t_a , \quad \tilde{t}_a^{(1)} = \tilde{t}_a , \quad t_a^{(2)} = -t_a^* , \quad \tilde{t}_a^{(2)} = -\tilde{t}_a^* . \tag{2.8}$$

The minus sign in the definition of the conjugate representation is very important for the matrices to obey the same Lie-algebra. It will turn out that, in the deformed theory, for correlation functions involving two primaries to be non-vanishing, their corresponding representations must be conjugate to each other, as well.

Next, consider three affine primaries transforming in the representations (R_i, R_i'), $i = 1, 2, 3$. Then the three-point function for them is given by

$$\langle \Phi_{i,i'}^{(1)}(z_1, \bar{z}_1) \Phi_{j,j'}^{(2)}(z_2, \bar{z}_2) \Phi_{k,k'}^{(3)}(z_3, \bar{z}_3) \rangle = \frac{C_{ii',jj',kk'}}{z_{12}^{\Delta_{12;3}} z_{13}^{\Delta_{13;2}} z_{23}^{\Delta_{23;1}} \bar{z}_{12}^{\bar{\Delta}_{12;3}} \bar{z}_{13}^{\bar{\Delta}_{13;2}} \bar{z}_{23}^{\bar{\Delta}_{23;1}}} , \tag{2.9}$$

where

$$\Delta_{12;3} = \Delta_{R_1} + \Delta_{R_2} - \Delta_{R_3} , \qquad \bar{\Delta}_{12;3} = \bar{\Delta}_{R_1'} + \bar{\Delta}_{R_2'} - \bar{\Delta}_{R_3'} , \tag{2.10}$$

and cyclic permutations of $1, 2$ and 3 for the rest. The structure constants $C_{ii',jj',kk'}$ depend on the representations and implicitly also on k. They obey various properties arising mainly from the global group invariance of the correlation functions, which will be mentioned below in the computation of the three-point functions involving only affine primaries.

Finally, we have the three-point functions with one current and two primaries. They are given by

$$\langle J_a(z)\Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle = -\frac{1}{\sqrt{k}}\frac{(t_a\otimes\mathbb{I}_{R'})_{ij,i'j'}}{x_{12}^{2\Delta_R}\bar{x}_{12}^{2\bar{\Delta}_{R'}}}\left(\frac{1}{z-x_1}-\frac{1}{z-x_2}\right) \qquad (2.11)$$

and

$$\langle \bar{J}_a(\bar{z})\Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle = \frac{1}{\sqrt{k}}\frac{(\mathbb{I}_R\otimes\tilde{t}^*_a)_{ij,i'j'}}{x_{12}^{2\Delta_R}\bar{x}_{12}^{2\bar{\Delta}_{R'}}}\left(\frac{1}{\bar{z}-\bar{x}_1}-\frac{1}{\bar{z}-\bar{x}_2}\right) \,, \qquad (2.12)$$

where we have used the fact that, for a non-vanishing result, the representations in which the primaries transform have to be conjugate to each other for the holomorphic and the anti-holomorphic sectors, separately. Also $\mathbb{I}_R$ and $\mathbb{I}_{R'}$ are the identity elements for the corresponding representations.

Correlators with two currents and one affine primary field are zero at the conformal point and will remain zero in the deformed theory as well.

2.2. Symmetry and correlation functions

In order to compute the correlation functions of currents and of primary fields we will heavily use the symmetry of the effective action for the non-Abelian Thirring model (1.7). First let's consider correlation functions for currents only. At the conformal point when $\lambda = 0$ the currents are given in terms of the group element by (1.2) and are, of course, chirally and anti-chirally conserved on shell. Obviously, in the deformed theory these currents will be dressed and will receive λ-corrections. One expects that since their definition contains derivatives there will be operator ambiguities at the quantum level. We propose that these dressed currents are given by

$$J^a_+(g)_{k,\lambda} = -\frac{i}{1+\lambda}(\mathbb{I}-\lambda D)^{-1}_{ab}\,\mathrm{Tr}(t^b\partial_+ gg^{-1})\,,$$
$$J^a_-(g)_{k,\lambda} = \frac{i}{1+\lambda}(\mathbb{I}-\lambda D^T)^{-1}_{ab}\,\mathrm{Tr}(t^b g^{-1}\partial_- g)\,. \qquad (2.13)$$

These become the correct chiral and anti-chiral currents when $\lambda = 0$ (up to a minus sign for J_-). Also, they are components of an on shell conserved current. The attentive reader will notice that the dressed current components in (2.13) are nothing, but, up to a factor of λ, the gauge fields evaluated on-shell in the original construction of (1.5) in [2] by a gauging procedure. Hence, it is natural to consider correlation functions of the $J^a_\pm$'s as defined above. In addition, we have that

$$J^a_\pm(g^{-1})_{-k,\lambda^{-1}} = \lambda^2 J^a_\pm(g)_{k,\lambda}\,. \qquad (2.14)$$

Passing to the Euclidean regime we have for the two-point function of the holomorphic component of the currents that

$$\langle J^a(x_1)J^b(x_2)\rangle_{k,\lambda} = \frac{1}{Z_{k,\lambda}}\int \mathcal{D}[g]J^a(g(x_1))_{k,\lambda}J^b(g(x_2))_{k,\lambda}e^{-S_{k,\lambda}(g)}\,, \qquad (2.15)$$

with the partition function being

$$Z_{k,\lambda} = \int \mathcal{D}[g]e^{-S_{k,\lambda}(g)} = \int \mathcal{D}[g^{-1}]e^{-S_{k,\lambda}(g^{-1})} = \int \mathcal{D}[g]e^{-S_{-k,\lambda^{-1}}(g)} = Z_{-k,\lambda^{-1}}\,, \quad (2.16)$$

where we have used the symmetry of the action (1.7) and the fact that the measure of integration is invariant under $g \mapsto g^{-1}$, i.e. $\mathcal{D}[g^{-1}] = \mathcal{D}[g]$.[1] Hence, the partition function of the deformed theory is invariant under the duality-type symmetry. In addition

$$
\begin{aligned}
\int \mathcal{D}[g] J^a(g(x_1))_{k,\lambda} & J^b(g(x_2))_{k,\lambda} e^{-S_{k,\lambda}(g)} \\
&= \int \mathcal{D}[g^{-1}] J^a(g^{-1}(x_1))_{k,\lambda} J^b(g^{-1}(x_2))_{k,\lambda} e^{-S_{k,\lambda}(g^{-1})} \\
&= \frac{1}{\lambda^4} \int \mathcal{D}[g] J^a(g(x_1))_{-k,\lambda^{-1}} J^b(g(x_2))_{-k,\lambda^{-1}} e^{-S_{-k,\lambda^{-1}}(g)} \,,
\end{aligned}
\tag{2.17}
$$

where we have also employed (2.14). Hence, we obtain that the correlation function should obey the non-trivial identity

$$
\lambda^2 \langle J^a(x_1) J^b(x_2) \rangle_{k,\lambda} = \lambda^{-2} \langle J^a(x_1) J^b(x_2) \rangle_{-k,\lambda^{-1}}.
\tag{2.18}
$$

This identity between current correlators is straightforwardly extendable to higher order correlators involving currents with any type of currents, J^a's or $\bar{J}^a$'s,

$$
\lambda^{n+m} \langle J^{a_1} \dots J^{a_n} \bar{J}^{b_1} \dots \bar{J}^{b_m} \rangle_{k,\lambda} = \lambda^{-n-m} \langle J^{a_1} \dots J^{a_n} \bar{J}^{b_1} \dots \bar{J}^{b_m} \rangle_{-k,\lambda^{-1}}.
\tag{2.19}
$$

The overall factors of λ can be absorbed by redefining the currents in (2.13) by a factor of λ. In the following we assume that this is the case which implies also the absence of the factor of λ^2 in the r.h.s. of (2.14).

The above conclusion for the current correlators is in full agreement with [27] who reached the same conclusion using the non-Abelian Thirring model action and certain special properties of the WZW action path integral. The advantage of employing the effective action is that one can employ the duality-type symmetry on correlation functions involving primary fields in the deformed theory which has not been considered before. For these fields we have that, under the inversion of the group element the primary field $\Phi^{(1)}$ transforms to its conjugate $\Phi^{(2)}$. Explicitly, we have that

$$
\Phi^{(1)}_{i,i'}(g^{-1}) = \Phi^{(2)}_{i',i}(g) \,,
\tag{2.20}
$$

which means that for the representation matrices we have

$$
t^{(1)} \leftrightarrow \tilde{t}^{(2)} \,, \qquad t^{(2)} \leftrightarrow \tilde{t}^{(1)} \,.
\tag{2.21}
$$

Note that if the inversion of g is followed by the $\sigma \mapsto -\sigma$, i.e. the parity transformation (1.6), then

$$
t^{(1)} \leftrightarrow -\tilde{t}^{(2)} \,, \qquad t^{(2)} \leftrightarrow -\tilde{t}^{(1)} \,,
\tag{2.22}
$$

and in addition the J_a's and $\bar{J}_a$'s are interchanged.

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[1] The measure of integration contains the Haar measure for the semisimple group G which certainly invariant under $g \mapsto g^{-1}$, but also the factor $\det(\lambda^{-1} - D^T)$ arising from integrating out the gauge fields in the path integral [2]. This can be easily seen to transform under $g \mapsto g^{-1}$ and $\lambda \mapsto \lambda^{-1}$ as (for a general matrix λ): $\det(\lambda^{-1} - D^T) \mapsto (-1)^n \det \lambda \times \det(\lambda^{-1} - D^T)$, with $n = \dim G$ and where we have used the property $D(g^{-1}) = D^T(g)$. This extra constant overall factor cancels out by the same factor arising from the partition function in the denominator in all correlation functions.

2.3. The non-Abelian and pseudodual chiral limits

Besides the small λ_{ab} limit, leading to (1.4), there are two other interesting limits of the action (1.5). They will be instrumental in our computation of correlation functions.

In the first limit [2] one expands the matrix and group elements near the identity as

$$\lambda_{ab} = \delta_{ab} - \frac{E_{ab}}{k} + \mathcal{O}\left(\frac{1}{k^2}\right) , \quad g = \mathbb{I} + i\frac{v_a t^a}{k} + \mathcal{O}\left(\frac{1}{k^2}\right) , \tag{2.23}$$

where E is a general dim$G$ square matrix. This leads to

$$J^a_{\pm} = \frac{\partial_{\pm} v^a}{k} + \mathcal{O}\left(\frac{1}{k^2}\right) , \quad D_{ab} = \delta_{ab} + \frac{f_{ab}}{k} + \mathcal{O}\left(\frac{1}{k^2}\right) , \quad f_{ab} = -i f_{abc} v^c . \tag{2.24}$$

Note that our structure constants are purely imaginary so that f_{ab} are indeed real. In this limit the action (1.5) becomes

$$S_{\text{non-Abel}}(v) = \frac{1}{2\pi} \int d^2\sigma\, \partial_+ v^a (E + f)^{-1}_{ab} \partial_- v^b , \tag{2.25}$$

which is the non-Abelian T-dual with respect to the G_L action of the σ-model given by the PCM action with general coupling matrix E_{ab}. We note that in this limit the WZW term in (1.5) does not contribute at all.

To discuss the second new limit, we first recall that the original derivation of the action (1.5) leads for compact groups to the restriction $0 < \lambda < 1$. However, once we have the action we may allow λ to take values beyond this range. For instance, the symmetry (1.7) clearly requires that. Here in order to take a new limit we will extend the range of λ to negative values. We will also need the following equivalent form of the action (1.5) given, after some manipulations needed to combine the quadratic part of the WZW action and the deformation term in (1.5), by

$$\begin{aligned} S_{k,\lambda}(g) = \frac{k}{4\pi} \int d^2\sigma\, J^a_+ \left[(\lambda^{-1} - D^T)^{-1}(\lambda^{-1} + D^T)D\right]_{ab} J^b_- \\ - \frac{ik}{48\pi} \int_B f_{abc} L^a \wedge L^b \wedge L^c , \end{aligned} \tag{2.26}$$

where we remind the reader that our structure constants are purely imaginary.

Then we take the limit

$$\lambda_{ab} = -\delta_{ab} + \frac{E_{ab}}{k^{1/3}} , \quad g = \mathbb{I} + i\frac{v^a t^a}{k^{1/3}} + \dots , \quad k \to \infty , \tag{2.27}$$

where again E is a general dim$G$ square matrix. The various quantities expand as in (2.24) with k replaced by $k^{1/3}$. Then the action (2.26) becomes

$$S_{\text{pseudodual}} = \frac{1}{8\pi} \int d^2\sigma\, \partial_+ v^a \partial_- v^b \left(E_{ab} + \frac{1}{3} f_{ab}\right) . \tag{2.28}$$

We see that E can be taken to be symmetric since any antisymmetric piece leads to a total derivative. This action for $E_{ab} = \delta_{ab}/b^{2/3}$ is nothing by the pseudodual model action [30]. Note that the quadratic part of the WZW action and the deformation term in (1.5) are equally important for the limit (2.27) to exist since each term separately diverges when this limit is taken.

Since the above non-Abelian and pseudodual limits exist at the action level, we expect that physical quantities such as the β-function and the anomalous dimensions of various operators should have a well defined limit as well. This will be an important ingredient in our method of computation.

2.4. The regularization method and useful integrals

In the Euclidean path integral the action appears as e^{-S}. The action we will be using is that of the non-Abelian Thrirring model action and will be expanding around the WZW CFT part of it. This is not in contrast with the approach of the last subsection where (1.5) was used, the reason being that the latter is the effective action of the non-Abelian Thrirring model. Hence, it contains all λ-corrections and can be considered as a starting point to find at the quantum level corrections in $1/k$. Schematically, to $\mathcal{O}(\lambda^n)$, the correlation function for a number of some generic fields F_i, $i = 1, 2, \ldots$, involves the sum of expressions of the type

$$\langle F_1(x_1, \bar{x}_1) F_2(x_2, \bar{x}_2) \ldots \rangle_\lambda^{(n)} = \frac{1}{n!} \left(-\frac{\lambda}{\pi} \right)^n \int d^2 z_{1\ldots n} \langle J^{a_1}(z_1) \ldots J^{a_n}(z_n)$$
$$\bar{J}^{a_1}(\bar{z}_1) \ldots \bar{J}^{a_n}(\bar{z}_n) F_1(x_1, \bar{x}_1) F_2(x_2, \bar{x}_2) \ldots \rangle , \qquad (2.29)$$

where $d^2 z_{1\ldots n} := d^2 z_1 \ldots d^2 z_n$ and for convenience we have dropped k from our notation in the correlation functions $\langle \cdots \rangle_{k, \lambda}$ of the deformed theory.

That way one encounters multiple integrals which need to be regularized. Our prescription to do so consists of two steps:

• We choose the order of integration from left to right $d^2 z_{1\ldots n}$ and never permute this order. This is due to the fact that due to the divergences appearing, the various integrations are not necessarily commuting.

• Internal points cannot coincide with external ones. This means that the domain of integration is

$$D_n = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}_n : |z_i - x_j| > \varepsilon, \varepsilon > 0\} , \quad \forall \, i, j . \qquad (2.30)$$

However, internal points can coincide. Also contact terms, arising from coincident external points will be allowed. The latter is a choice we make and not a part of the regularization scheme.[2] We shall need the very basic integral given by

$$\int \frac{d^2 z}{(x_1 - z)(\bar{z} - \bar{x}_2)} = \pi \ln |x_{12}|^2 . \qquad (2.31)$$

Clearly, if the domain of integration allows, the integral diverges for large distances. The above result is valid provided that the integration is performed in a domain of characteristic size R, e.g. a disc of radius R, with the external points x_1 and x_2 excluded and in addition obeying $R \gg |x_1|, |x_2|$. The latter conditions are responsible for the translational invariance and the reality of the result. Even then we have to make the replacement $|x_{12}|^2 \to |x_{12}|^2/R^2$ on the right hand

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[2] All these imply that we will have for the δ-functions arising in performing the various integrations that

$$\delta^{(2)}(z_i - x_j) \to 0 , \quad \delta^{(2)}(z_i - z_j) \text{ (kept)} , \quad \delta^{(2)}(x_i - x_j) \text{ (kept)} , \quad \forall i, j .$$

Note also that in the regularization of [31] no two points, internal or external, can coincide and therefore all δ-functions arising in integrations are set to zero. In contrast in [32] all such δ-functions are kept. The advantage of our regularization is that the symmetry of the correlation functions under $k \to -k$ and $\lambda \to \lambda^{-1}$ is manifest whereas for the others it is hidden.

side of (2.31). However, in our computations there will be integrals of the same kind but with opposite sign and x_1 equal to x_2 and which will have a small distance regulator ε. Hence the factor R will drop out at the end, leaving the ratio $\ln \dfrac{\varepsilon^2}{|x_{12}|^2}$. This means that in practice the domain of integration is $\mathbb{R}^2$ except for the points $x_{1,2}$ which are excluded. By appropriately taking derivatives we also have the useful integrals

$$\int \frac{d^2 z}{(x_1 - z)^2 (\bar{z} - \bar{x}_2)} = -\frac{\pi}{x_{12}} \,, \qquad \int \frac{d^2 z}{(x_1 - z)(\bar{z} - \bar{x}_2)^2} = -\frac{\pi}{\bar{x}_{12}} \tag{2.32}$$

and

$$\int \frac{d^2 z}{(x_1 - z)^2 (\bar{z} - \bar{x}_2)^2} = \pi^2 \delta^{(2)}(x_{12}) \,. \tag{2.33}$$

In Appendix A we have collected results for some useful to this work integrals. We single out

$$\int \frac{d^2 z}{(z - x_1)(z - x_2)(\bar{z} - \bar{x}_1)} = -\frac{\pi}{x_{12}} \ln \frac{\varepsilon^2}{|x_{12}|^2} \,,$$
$$\int \frac{d^2 z}{(z - x_1)(\bar{z} - \bar{x}_1)(\bar{z} - \bar{x}_2)} = -\frac{\pi}{\bar{x}_{12}} \ln \frac{\varepsilon^2}{|x_{12}|^2} \tag{2.34}$$

and

$$\int \frac{d^2 z}{(z - x_1)(\bar{z} - \bar{x}_2)} \ln |z - x_1|^2 = -\frac{\pi}{2} \ln^2 |x_1 - x_2|^2 \,, \tag{2.35}$$

which are valid under the assumptions spelled out below (2.31).

3. Current correlators

In this section, we will focus on the two- and three-point functions involving purely currents. These will be computed up to order $1/k$ and exactly in the deformation parameter λ. To establish our method, employed already in [19], as clearly as possible we first start with the computation of the two-point functions which enables to compute the β-function and the anomalous dimensions for the currents known already from using CFT methods in [19,25,26] and from gravitational computations [23,24]. Then we proceed to correlators involving three currents.

3.1. Two-point functions

On general grounds the correlator of J^a and J^b takes the form

$$\langle J^a(x_1) J^b(x_2) \rangle_\lambda = \delta^{ab} \frac{G_0(k, \lambda)}{x_{12}^2} \left(1 + \gamma^{(J)} \ln \frac{\varepsilon^2}{|x_{12}|^2} \right) + \cdots . \tag{3.1}$$

The result to $\mathcal{O}(1/k)$ and $\mathcal{O}(\lambda^3)$ was computed in sec. 2 of [19] and reads

$$\langle J^a(x_1) J^b(x_2) \rangle = \frac{\delta_{ab}}{x_{12}^2} \left(1 - 2 \frac{c_G}{k} \lambda^3 + \frac{c_G}{k} (\lambda^2 - 2\lambda^3) \ln \frac{\varepsilon^2}{|x_{12}|^2} + \frac{1}{k} \mathcal{O}(\lambda^4) \right) . \tag{3.2}$$

Comparing with the general form of the two-point function (3.1) we have that

$$G_0(k, \lambda) = 1 - 2 \frac{c_G}{k} \left(\lambda^3 + \mathcal{O}(\lambda^4) \right) \tag{3.3}$$

and

$$\gamma^{(J)} = \frac{c_G}{k} \left(\lambda^2 - 2\lambda^3 + \mathcal{O}(\lambda^4) \right) . \tag{3.4}$$

Similarly the correlator of J^a and $\bar{J}^b$ should assume the form

$$\langle J^a(x_1) \bar{J}^b(x_2) \rangle_\lambda = \delta^{ab} \frac{\tilde{G}_0(k, \lambda)}{|x_{12}|^2} \left(1 + \gamma^{(J)} \ln \frac{\varepsilon^2}{|x_{12}|^2} \right)$$
$$+ \delta_{ab} \delta^{(2)}(x_{12}) \left(A(k, \lambda) + B(k, \lambda) \ln \frac{\varepsilon^2}{|x_{12}|^2} \right) . \tag{3.5}$$

At the conformal point this correlator should vanish. We have also allowed for contact terms proportional to the δ-function since these are allowed by symmetry. The coupling functions A and B have to be computed.

After a long computation, all details are given in the [Appendix B](#), we found the result

$$\langle J^a(x_1) \bar{J}^b(\bar{x}_2) \rangle_\lambda = -\pi \lambda \delta^{ab} \delta^{(2)}(x_{12})$$
$$- \frac{\lambda^2 c_G}{k} \delta^{ab} \left[\frac{1}{|x_{12}|^2} + \pi \delta^{(2)}(x_{12}) \left(1 - \frac{1}{2} \ln \frac{\varepsilon^2}{|x_{12}|^2} \right) \right] \tag{3.6}$$
$$+ 2 \frac{\lambda^3 c_G}{k} \delta^{ab} \left[\frac{1}{|x_{12}|^2} + \pi \delta^{(2)}(x_{12}) \left(1 - \ln \frac{\varepsilon^2}{|x_{12}|^2} \right) \right] + \frac{1}{k} \mathcal{O}(\lambda^4) ,$$

which, keeping in mind that we are interested to terms up to $\mathcal{O}(1/k)$, is easily seen to be of the form (3.5). Note that this correlator takes the form

$$\langle J^a(x_1) \bar{J}^b(\bar{x}_2) \rangle = -\gamma^{(J)} \frac{\delta_{ab}}{|x_{12}|^2} + \text{contact terms} , \tag{3.7}$$

where $\gamma^{(J)}$ is the current anomalous dimension given perturbatively by (3.4).

3.1.1. The exact β-function and anomalous dimensions

To compute the wave function renormalization and that for the parameter λ we use the two-point functions $\langle J^a J^b \rangle$ and $\langle J^a \bar{J}^b \rangle$. In particular we need the most singular part of these correlation functions. For the purpose of this section let's denote the bare currents by J_0^a and $\bar{J}_0^a$ and similarly for the parameter λ_0.

We need the most singular part of the bare two-point functions up to order $1/k$. From (3.2) we have that

$$\langle J_0^a(x_1) J_0^b(x_2) \rangle = \frac{\delta_{ab}}{x_{12}^2} \left[1 - \frac{c_G}{k} \lambda_0^2 \left(2\lambda_0 + (1 - 2\lambda_0) \ln(|x_{12}|^2/\varepsilon^2) \right) \right] + \dots . \tag{3.8}$$

Also from (3.6) we have that

$$\langle J_0^a(x_1) \bar{J}_0^b(\bar{x}_2) \rangle = -\pi \lambda_0 \delta^{ab} \delta^{(2)}(x_{12}) \left[1 + \lambda_0 \frac{c_G}{k} \left(1 - \frac{1}{2} \ln \frac{\varepsilon^2}{|x_{12}|^2} \right. \right.$$
$$\left. \left. - 2\lambda_0 \left(1 - \ln \frac{\varepsilon^2}{|x_{12}|^2} \right) \right) \right] + \cdots , \tag{3.9}$$

where we have kept only the coefficient of the most singular term, i.e. of $\delta^{(2)}(x_{12})$.

The bare quantities and the renormalized ones are related as

$$J_0^a = Z^{1/2} J^a , \qquad \bar{J}_0^a = Z^{1/2} \bar{J}^a , \qquad \lambda_0 = Z_1 \lambda . \tag{3.10}$$

We make the following ansatz valid to order $1/k$ in the large k-expansion

$$Z^{-1} = 1 + 2\frac{c_G}{k}\lambda^3 - \frac{c_G}{k}\left(c_1\lambda^2 + c_2\lambda^3 + \mathcal{O}(\lambda^4)\right)\ln(\varepsilon^2\mu^2) ,$$
$$Z_1 = 1 - \frac{c_G}{k}\left(c_3\lambda + c_4\lambda^2 + \mathcal{O}(\lambda^3)\right)\ln(\varepsilon^2\mu^2) , \tag{3.11}$$

where the logarithm-independent term in Z^{-1} has been chosen so that the renormalized two-point function for the J_a's is normalized to one. The pure number coefficients c_i are computed so that the renormalized two-point functions

$$\langle J^a(x_1) J^b(x_2)\rangle = Z^{-1}\langle J_0^a(x_1) J_0^b(x_2)\rangle , \qquad \langle J^a(x_1)\bar{J}^b(x_2)\rangle = Z^{-1}\langle J_0^a(x_1)\bar{J}_0^b(x_2)\rangle , \tag{3.12}$$

are independent of the cutoff ε. We find that the unique choice is given by

$$c_1 = 1 , \qquad c_2 = -2 , \qquad c_3 = -\frac{1}{2} , \qquad c_4 = 1 . \tag{3.13}$$

The β-function is by definition

$$\beta_\lambda = \frac{1}{2}\mu\frac{d\lambda}{d\mu} = \frac{1}{2}\lambda Z_1\mu\frac{dZ_1^{-1}}{d\mu} = -\frac{c_G}{2k}\left(\lambda^2 - 2\lambda^3 + \mathcal{O}(\lambda^4)\right) , \tag{3.14}$$

where the bare coupling λ_0 is kept fixed. Next we compute the anomalous dimension of the current

$$\gamma^{(J)} = \mu\frac{d\ln Z^{1/2}}{d\mu} = \frac{c_G}{k}\left(\lambda^2 - 2\lambda^3 + \mathcal{O}(\lambda^4)\right) , \tag{3.15}$$

in agreement of course with (3.4).

The above perturbative expressions are enough to determine the exact in λ dependence of the β-function and of the anomalous dimensions up to order $1/k$. As explained, the exact β-function and anomalous dimensions should have a well defined behavior in the two limiting cases described by the non-Abelian and pseudodual model limits (2.23) and (2.27), respectively. In the isotropic case, which is the case of interest in this work, it implies regularity under the following independent limits

$$\lambda = 1 - \frac{\kappa^2}{k} , \qquad \lambda = -1 + \frac{1}{b^{2/3}k^{1/3}} , \qquad k \to \infty . \tag{3.16}$$

Regularity under (3.16) of the exact β-function and the anomalous dimensions implies an ansatz of the form

$$\beta_\lambda = -\frac{c_G}{2k}\frac{f(\lambda)}{(1+\lambda)^2} , \qquad \gamma^{(J)} = \frac{c_G}{k}\frac{g(\lambda)}{(1-\lambda)(1+\lambda)^3} , \tag{3.17}$$

where $f(\lambda)$ and $g(\lambda)$ are two analytic functions of λ. The assumed pole structure does not exclude the possibility that one of the poles reduces its degree or even ceases to exist. This can happen if the functions in the numerator are zero at $\lambda = 1$ or/and $\lambda = -1$. In addition, due to the symmetry under $(k,\lambda) \mapsto (-k,\lambda^{-1})$ we have that

$$\lambda^4 f(1/\lambda) = f(\lambda) , \qquad \lambda^4 g(1/\lambda) = g(\lambda) . \tag{3.18}$$

All these imply that these functions are in fact polynomials of, at most, degree four

$$
\begin{aligned}
f(\lambda) &= a_0 + a_1\lambda + a_2\lambda^2 + a_1\lambda^3 + a_0\lambda^4 \,, \\
g(\lambda) &= b_0 + b_1\lambda + b_2\lambda^2 + b_1\lambda^3 + b_0\lambda^4 \,.
\end{aligned}
\tag{3.19}
$$

Demanding agreement with the perturbative expressions (3.14) and (3.15) to $\mathcal{O}(\lambda^2)$ we obtain $a_0 = a_1 = b_0 = b_1 = 0$ and $a_2 = b_2 = 1$ which completely determines the exact β-function and anomalous dimensions to be

$$
\beta_\lambda = -\frac{c_G}{2k}\frac{\lambda^2}{(1+\lambda)^2} \leqslant 0
\tag{3.20}
$$

and

$$
\gamma^{(J)} = \frac{c_G}{k}\frac{\lambda^2}{(1-\lambda)(1+\lambda)^3} \geqslant 0.
\tag{3.21}
$$

It is also easily seen that the coefficient of the $\mathcal{O}(\lambda^3)$ term is in agreement with the perturbative results as well. The above expressions are in full agreement with the results found in [23,24,33] for the β-function and in [19] for the anomalous dimensions.

Note that the β-function and anomalous dimensions of the non-Abelian T-dual limit are

$$
\beta_{\kappa^2} = \frac{c_G}{8} \,, \quad \gamma^{(J)} = \frac{c_G}{8\kappa^2} \,,
\tag{3.22}
$$

which are valid for large κ^2. The anomalous dimensions correspond to

$$
J_\pm^a = \pm\frac{1}{2}(\kappa^2\mathbb{I} \mp f)_{ab}^{-1}\partial_\pm v^b \,,
\tag{3.23}
$$

which are obtained by taking this limit in (2.13).

The corresponding expressions for the pseudodual model are

$$
\beta_b = \frac{3}{4}c_G b^3 \,, \qquad \gamma^{(J)} = \frac{1}{2}c_G b^2 \,.
\tag{3.24}
$$

These are in agreement with the expressions derived in [30] (see above Fig. 2) and are valid for small b. The anomalous dimensions correspond to

$$
J_\pm^a = \pm b^{2/3}\partial_\pm v^a \,,
\tag{3.25}
$$

which as before are obtained by taking the appropriate limit in (2.13).

3.2. Three-point functions

We consider the $\langle JJJ\rangle$ and $\langle JJ\bar{J}\rangle$ correlators. The remaining correlators $\langle\bar{J}\bar{J}\bar{J}\rangle$ and $\langle\bar{J}\bar{J}J\rangle$ can be easily obtained by applying the parity transformation to the first two. The results of this subsection match those obtained in [32], where current–current perturbations of the WZW model on supergroups were studied with a different regularization scheme. Before moving to our analysis, let us note that analogue perturbations of the WZW models on supergroups were studied in [34], but the perturbation consists of the term $J_+^a D_{ab} J_-^b$ added to the action; effectively the non-critical WZW model.

3.2.1. The $\langle JJJ \rangle$ correlator

From Appendix C we have that the, up to $\mathcal{O}(\lambda^3)$, correlator reads

$$\langle J^a(x_1) J^b(x_2) J^c(x_3) \rangle_\lambda = \frac{1}{\sqrt{k}} \left(1 + \frac{3}{2}\lambda^2 - \lambda^3 \right) \frac{f_{abc}}{x_{12}x_{13}x_{23}} + \frac{1}{\sqrt{k}}\mathcal{O}(\lambda^4) \,. \tag{3.26}$$

The ansatz for the all-loop expression takes the form

$$\langle J^a(x_1) J^b(x_2) J^c(x_3) \rangle = \frac{f(\lambda)}{\sqrt{k(1-\lambda)(1+\lambda)^3}} \frac{f_{abc}}{x_{12}x_{13}x_{23}} \,, \tag{3.27}$$

where $f(\lambda)$ is everywhere analytic and obviously $f(0) = 1$ to agree with the CFT result. As before this form takes into account that under the limit (3.16) the correlator is well behaved. Invariance of the above expression under the duality-type symmetry $(k, \lambda) \mapsto (-k, \lambda^{-1})$ yields

$$\lambda^2 f(\lambda^{-1}) = f(\lambda) \quad \Longrightarrow \quad f(\lambda) = 1 + c\lambda + \lambda^2 \,. \tag{3.28}$$

Consistency with the perturbative expression up to $\mathcal{O}(\lambda)$ (3.27) gives $c = 1$. Therefore, the all-loop correlator reads

$$\boxed{\langle J^a(x_1) J^b(x_2) J^c(x_3) \rangle = \frac{1 + \lambda + \lambda^2}{\sqrt{k(1-\lambda)(1+\lambda)^3}} \frac{f_{abc}}{x_{12}x_{13}x_{23}}} \,. \tag{3.29}$$

As a check we see that this expression reproduces the $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^3)$ terms in the perturbative expression (3.26).

3.2.2. The $\langle JJ\bar{J} \rangle$ correlator

The perturbative calculation of this correlator is performed in Appendix D. The result up to order $\mathcal{O}(\lambda^2)$ reads

$$\langle J^a(x_1) J^b(x_2) \bar{J}^c(\bar{x}_3) \rangle = \frac{\lambda(1-\lambda)}{\sqrt{k}} \frac{\bar{x}_{12} f_{abc}}{x_{12}^2 \bar{x}_{23} \bar{x}_{13}} + \frac{1}{\sqrt{k}}\mathcal{O}(\lambda^3) \,. \tag{3.30}$$

We now make a similar to (3.27) ansatz for the all-loop expression

$$\langle J^a(x_1) J^b(x_2) \bar{J}^c(x_3) \rangle = \frac{\lambda f(\lambda)}{\sqrt{k(1-\lambda)(1+\lambda)^3}} \frac{\bar{x}_{12} f_{abc}}{x_{12}^2 \bar{x}_{23} \bar{x}_{13}} \,, \tag{3.31}$$

where $f(\lambda)$ is everywhere analytic and $f(0) = 1$. Invariance of the above expression under the duality-type symmetry yields

$$f(\lambda^{-1}) = f(\lambda) \quad \Longrightarrow \quad f(\lambda) = 1 \,. \tag{3.32}$$

Hence, we find the all-loop expression

$$\boxed{\langle J^a(x_1) J^b(x_2) \bar{J}^c(\bar{x}_3) \rangle = \frac{\lambda}{\sqrt{k(1-\lambda)(1+\lambda)^3}} \frac{f_{abc}\bar{x}_{12}}{x_{12}^2 \bar{x}_{13} \bar{x}_{23}}} \,, \tag{3.33}$$

whose expansion around $\lambda = 0$ agrees with (3.30).

Note that implementing the non-Abelian and pseudodual limits both lead to finite (non-zero) expressions for all of the above three-point functions. In these limiting cases the results are valid

for large κ^2 and small b where we refer to (3.16) for the definition of these parameters. We mention also that, our results for these correlators agree with those done for supergroups in [32] after an appropriate rescaling of the currents that presumably takes into account the different regularization schemes used in that work.

4. Primary field correlators

The purpose of this section is to compute two- and three-point functions of arbitrary primary fields. This will allow us to extract their anomalous dimensions and the deformed structure constants in the OPEs.

4.1. Two-point functions

After a long computation, all details of which are given in Appendix E, we found that a perturbative computation up to $\mathcal{O}(\lambda^3)$ and to order $1/k$, gives for the two-point function of primary fields the result

$$\langle \Phi_{i,i'}^{(1)}(x_1, \bar{x}_1) \Phi_{j,j'}^{(2)}(x_2, \bar{x}_2) \rangle_\lambda$$

$$= \frac{1}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \left[\left(1 + \frac{\lambda^2}{k}(c_R + c_{R'}) \ln \frac{\varepsilon^2}{|x_{12}|^2} \right) (\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii', jj'} \right.$$

$$\left. - 2\lambda \frac{1 + \lambda^2}{k} \ln \frac{\varepsilon^2}{|x_{12}|^2} (t_a \otimes t_a^*)_{ii', jj'} \right] + \frac{1}{k}\mathcal{O}(\lambda^3) . \tag{4.1}$$

We see that due to the deformation there is an operator mixing so that one should proceed by choosing an appropriate basis in which the dimension matrix is diagonal. For convenience we will adopt the double index notation $I = (ii')$. Then there is a matrix U chosen such that

$$(t_a \otimes t_a^*)_{IJ} = U_{IK} N_{KL} (U^{-1})_{LJ} , \qquad N_{IJ} = N_I \delta_{IJ} , \tag{4.2}$$

where N_I are the eigenvalues of the matrix $t_a \otimes t_a^*$. Note also that U is λ-independent as well as k-independent. Then in the rotated basis

$$\widetilde{\Phi}_I^{(1)} = (U^{-1})_I{}^J \Phi_J^{(1)} , \qquad \widetilde{\Phi}_I^{(2)} = U_I{}^J \Phi_J^{(2)} , \tag{4.3}$$

the correlator (4.1) becomes diagonal, i.e.

$$\langle \widetilde{\Phi}_I^{(1)}(x_1, \bar{x}_1) \widetilde{\Phi}_J^{(2)}(x_2, \bar{x}_2) \rangle_\lambda = \frac{\delta_{IJ}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \left(1 + \delta_I^{(\Phi)} \ln \frac{\varepsilon^2}{|x_{12}|^2} \right) , \tag{4.4}$$

where perturbatively

$$\delta_I^{(\Phi)} = \frac{1}{k} \left(-2\lambda(1 + \lambda^2) N_I + \lambda^2(c_R + c_{R'}) + \mathcal{O}(\lambda^4) \right) . \tag{4.5}$$

To determine the exact anomalous dimension of the general primary field we first realize that we should include in the above expression the k-dependent part coming from the CFT dimensions of Δ_R and $\bar{\Delta}_{R'}$ in (2.3) up to order $1/k$. Hence the anomalous dimension is given by

$$\gamma_{R,R'}^{(I)}(k, \lambda)\big|_{\text{pert}} = \frac{c_R}{2k} + \frac{\delta_I^{(\Phi)}}{2} =$$

$$= \frac{1}{2k} \left[c_R - 2N_I \lambda(1 + \lambda^2) + \lambda^2(c_R + c_{R'}) + \mathcal{O}(\lambda^4) \right] . \tag{4.6}$$

As in the case of currents we make the following ansatz for the exact anomalous dimensions

$$\gamma_{R,R'}^{(I)}(k,\lambda) = -\frac{1}{2k(1-\lambda)(1+\lambda)^3}\left[f(\lambda)N_I + f_1(\lambda)c_R + f_2(\lambda)c_{R'}\right],\tag{4.7}$$

where the yet unknown function should be analytic in λ. Using the symmetry (1.7) and the transformation of the primary fields under this symmetry (2.20), we have that

$$\gamma_{R,R'}^{(I)}(-k,\lambda^{-1}) = \gamma_{R',R}^{(I)}(k,\lambda),\tag{4.8}$$

which implies the following relations between the various unknown functions

$$\lambda^4 f(1/\lambda) = f(\lambda),\qquad \lambda^4 f_1(1/\lambda) = f_2(\lambda),\qquad \lambda^4 f_2(1/\lambda) = f_1(\lambda).\tag{4.9}$$

Hence, these functions should be fourth order polynomials in λ with related coefficients. It turns out that comparing with the perturbative expression (4.6) up to $\mathcal{O}(\lambda^2)$ we determine all these functions to be

$$f(\lambda) = 2\lambda(1+\lambda)^2,\qquad f_1(\lambda) = -(1+\lambda)^2,\qquad f_2(\lambda) = -\lambda^2(1+\lambda)^2.\tag{4.10}$$

Therefore, the exact in λ anomalous dimension is

$$\boxed{\gamma_{R,R'}^{(I)}(k,\lambda) = -\frac{1}{2k(1-\lambda^2)}(2\lambda N_I - c_R - \lambda^2 c_{R'})}.\tag{4.11}$$

It is easily checked that this expression is in agreement with the $\mathcal{O}(\lambda^3/k)$ term in (4.6). Note also that in the non-Abelian limit the above anomalous dimensions have a well defined and different than zero limit. In contrast the limit is zero in the pseudodual limit. This expression also applies for current–current perturbations of the WZW model on supergroups with vanishing Killing form [31].

Finally, the two point functions take the form

$$\boxed{\langle \Phi_I^{(1)}(x_1,\bar{x}_1)\Phi_J^{(2)}(x_2,\bar{x}_2)\rangle = \frac{\delta_{IJ}}{x_{12}^{\gamma_{R,R'}^{(I)}(k,\lambda)}\,\bar{x}_{12}^{\gamma_{R',R}^{(I)}(k,\lambda)}}}.\tag{4.12}$$

4.2. Three-point functions

To leading order in the λ-expansion after a straightforward computation this correlator is found to be

$$\begin{aligned}
\langle \Phi_{i,i'}^{(1)}(x_1)\Phi_{j,j'}^{(2)}(x_1)\Phi_{k,k'}^{(3)}(x_3)\rangle_\lambda^{(1)} = -\frac{\lambda}{k}\frac{1}{x_{12}^{\Delta_{12;3}}x_{13}^{\Delta_{13;2}}x_{23}^{\Delta_{23;1}}\bar{x}_{12}^{\bar{\Delta}_{12;3}}\bar{x}_{13}^{\bar{\Delta}_{13;2}}\bar{x}_{23}^{\bar{\Delta}_{23;1}}}\\
\Bigl[\ln\varepsilon^2\left((t_a^{(1)})_i{}^\ell(\tilde{t}_a^{(1)})^{\ell'}{}_{i'}C_{\ell\ell',jj',kk'} + (t_a^{(2)})_j{}^\ell(\tilde{t}_a^{(2)})^{\ell'}{}_{j'}C_{ii',\ell\ell',kk'}\right.\\
\left. + (t_a^{(3)})_k{}^\ell(\tilde{t}_a^{(3)})^{\ell'}{}_{k'}C_{ii',jj',\ell\ell'}\right)\\
+\ln|x_{12}|^2\left((t_a^{(1)})_i{}^\ell(\tilde{t}_a^{(2)})^{\ell'}{}_{j'}C_{\ell i',j\ell',kk'} + (t_a^{(2)})_j{}^\ell(\tilde{t}^{(1)})^{\ell'}{}_{i'}C_{i\ell'\ell j',kk'}\right)\\
+\ln|x_{13}|^2\left((t_a^{(1)})_i{}^\ell(\tilde{t}_a^{(3)})^{\ell'}{}_{k'}C_{\ell i',jj',k\ell'} + (t_a^{(3)})_k{}^\ell(\tilde{t}^{(1)})^{\ell'}{}_{i'}C_{i\ell',jj',\ell k'}\right)\\
+\ln|x_{23}|^2\left((t_a^{(2)})_j{}^\ell(\tilde{t}_a^{(3)})^{\ell'}{}_{k'}C_{ii',\ell i',k\ell'} + (t_a^{(3)})_k{}^\ell(\tilde{t}^{(2)})^{\ell'}{}_{j'}C_{ii',j\ell',\ell k'}\right)\Bigr].
\end{aligned}\tag{4.13}$$

Even for dimensional reasons we should be able to cast the above expression in a form in which all space dependence is in terms of ratios $\varepsilon^2/|x_{ij}|^2$. In order to do that we first recall that the structure constants $C_{ii',jj',kk'}$ are factorized according to their holomorphic and anti-holomorphic content as

$$C_{ii',jj',kk'} = C_{i,j,k}\tilde{C}_{i',j',k'} \, . \tag{4.14}$$

An important constraint, arises by making use of the global Ward identity. It reads

$$
\begin{aligned}
(t_a^{(1)})_i{}^\ell C_{\ell jk} + (t_a^{(2)})_j{}^\ell C_{i\ell k} + (t_a^{(3)})_k{}^\ell C_{ij\ell} = 0 \, , \\
(\tilde{t}_a^{(1)})^{\ell'}{}_{i'}\tilde{C}_{\ell'j'k'} + (\tilde{t}_a^{(2)})^{\ell'}{}_{j'}\tilde{C}_{i'\ell'k'} + (\tilde{t}_a^{(3)})^{\ell'}{}_{k'}\tilde{C}_{i'j'\ell'} = 0 \, .
\end{aligned}
\tag{4.15}
$$

From (4.15) it is straightforward to obtain the following relations

$$
\begin{aligned}
(t_a^{(3)})_k{}^\ell (\tilde{t}_a^{(3)})^{\ell'}{}_{k'} C_{ii',jj',\ell\ell'} &= (t_a^{(1)})_i{}^\ell (\tilde{t}_a^{(1)})^{\ell'}{}_{i'} C_{\ell\ell',jj',kk'} + (t_a^{(2)})_j{}^\ell (\tilde{t}_a^{(2)})^{\ell'}{}_{j'} C_{ii',\ell\ell',kk'} \\
&\quad + (t_a^{(1)})_i{}^\ell (\tilde{t}_a^{(2)})^{\ell'}{}_{j'} C_{\ell i',j\ell',kk'} + (t_a^{(2)})_j{}^\ell (\tilde{t}_a^{(1)})^{\ell'}{}_{i'} C_{i\ell',\ell j',kk'} \, , \\
(t_a^{(2)})_j{}^\ell (\tilde{t}_a^{(2)})^{\ell'}{}_{j'} C_{ii',\ell\ell',kk'} &= (t_a^{(1)})_i{}^\ell (\tilde{t}_a^{(1)})^{\ell'}{}_{i'} C_{\ell\ell',jj',kk'} + (t_a^{(3)})_k{}^\ell (\tilde{t}_a^{(3)})^{\ell'}{}_{k'} C_{ii',jj',\ell\ell'} \\
&\quad + (t_a^{(1)})_i{}^\ell (\tilde{t}_a^{(3)})^{\ell'}{}_{k'} C_{\ell i',jj',k\ell'} + (t_a^{(3)})_k{}^\ell (\tilde{t}_a^{(1)})^{\ell'}{}_{i'} C_{i\ell',jj',\ell k'} \, , \\
(t_a^{(1)})_i{}^\ell (\tilde{t}_a^{(1)})^{\ell'}{}_{i'} C_{\ell\ell',jj',kk'} &= (t_a^{(2)})_j{}^\ell (\tilde{t}_a^{(2)})^{\ell'}{}_{j'} C_{ii',\ell\ell',kk'} + (t_a^{(3)})_k{}^\ell (\tilde{t}_a^{(3)})^{\ell'}{}_{k'} C_{ii',jj',\ell\ell'} \, , \\
&\quad + (t_a^{(2)})_j{}^\ell (\tilde{t}_a^{(3)})^{\ell'}{}_{k'} C_{ii',\ell j',k\ell'} + (t_a^{(3)})_k{}^\ell (\tilde{t}_a^{(2)})^{\ell'}{}_{j'} C_{ii',j\ell',\ell k'} \, .
\end{aligned}
$$

Using the above relations we can rewrite the three-point function as

$$
\begin{aligned}
\langle \Phi_{i,i'}^{(1)}(x_1)\Phi_{j,j'}^{(2)}(x_1)\Phi_{k,k'}^{(3)}(x_3)\rangle_\lambda^{(1)} = -\frac{\lambda}{k}\frac{1}{x_{12}^{\Delta_{12;3}}x_{13}^{\Delta_{13;2}}x_{23}^{\Delta_{23;1}}\bar{x}_{12}^{\bar{\Delta}_{12;3}}\bar{x}_{13}^{\bar{\Delta}_{13;2}}\bar{x}_{23}^{\bar{\Delta}_{23;1}}} \\
\Bigg[\ln\left(\frac{\varepsilon^2}{|x_{12}|^2}\right)\Big[(t_a^{(1)})_i{}^\ell (\tilde{t}_a^{(1)})^{\ell'}{}_{i'} C_{\ell\ell',jj',kk'} + (t_a^{(2)})_j{}^\ell (\tilde{t}_a^{(2)})^{\ell'}{}_{j'} C_{ii',\ell\ell',kk'} \\
- (t_a^{(3)})_k{}^\ell (\tilde{t}_a^{(3)})^{\ell'}{}_{k'} C_{ii',jj',\ell\ell'}\Big] \\
\ln\left(\frac{\varepsilon^2}{|x_{13}|^2}\right)\Big[(t_a^{(1)})_i{}^\ell (\tilde{t}_a^{(1)})^{\ell'}{}_{i'} C_{\ell\ell',jj',kk'} + (t_a^{(3)})_k{}^\ell (\tilde{t}_a^{(3)})^{\ell'}{}_{k'} C_{ii',jj',kk'} \\
- (t_a^{(2)})_j{}^\ell (\tilde{t}_a^{(2)})^{\ell'}{}_{j'} C_{ii',\ell\ell',kk'}\Big] \\
\ln\left(\frac{\varepsilon^2}{|x_{23}|^2}\right)\Big[(t_a^{(2)})_j{}^\ell (\tilde{t}_a^{(2)})^{\ell'}{}_{j'} C_{ii',\ell\ell',kk'} + (t_a^{(3)})_k{}^\ell (\tilde{t}_a^{(3)})^{\ell'}{}_{k'} C_{ii',jj',kk'} \\
- (t_a^{(1)})_i{}^\ell (\tilde{t}_a^{(1)})^{\ell'}{}_{i'} C_{\ell\ell',jj',kk'}\Big]\Bigg] \, .
\end{aligned}
$$

The next step is to pass to the rotated basis. By using the double index notation we introduced before we have that

$$
\begin{aligned}
\tilde{\Phi}_I^{(q)} = (U^{(q)})_I{}^{-1J}\Phi_J^{(q)} \, , \quad (t_a^{(q)} \otimes t_a^{(q)*})_I{}^J = (U^{(q)})_I{}^K (N^{(q)})_K{}^L (U^{(q)})_L^{-1J} \, , \\
N_{IJ}^{(q)} = N_I^{(q)}\delta_{IJ} \, , \quad q = 1,2,3 \, ,
\end{aligned}
\tag{4.16}
$$

where in the new basis the structure constants read

$$\tilde{C}_{IJK} = (U^{(1)})_I^{-1M}(U^{(2)})_J^{-1N}(U^{(3)})_K^{-1L}C_{MNL} \, , \tag{4.17}$$

while the result for the correlator at $\mathcal{O}(\lambda)$ is given by

$$\langle \tilde{\Phi}_I^{(1)}(x_1) \tilde{\Phi}_J^{(2)}(x_1) \tilde{\Phi}_K^{(3)}(x_3) \rangle_\lambda^{(1)} = -\frac{\lambda}{k} \frac{\tilde{C}_{IJK}}{x_{12}^{\Delta_{12;3}} x_{13}^{\Delta_{13;2}} x_{23}^{\Delta_{23;1}} \bar{x}_{12}^{\bar{\Delta}_{12;3}} \bar{x}_{13}^{\bar{\Delta}_{13;2}} \bar{x}_{23}^{\bar{\Delta}_{23;1}}}$$

$$\left((N_I^{(1)} + N_I^{(2)} - N_I^{(3)}) \ln \frac{\varepsilon^2}{|x_{12}|^2} + \text{cyclic in } 1, 2, 3 \right) . \tag{4.18}$$

From this result we can write down the exact expression in λ for the three-point function. It is given by

$$\boxed{\langle \tilde{\Phi}_I^{(1)}(x_1) \tilde{\Phi}_J^{(2)}(x_1) \tilde{\Phi}_K^{(3)}(x_3) \rangle_\lambda = \frac{\tilde{C}_{IJK}(k, \lambda)}{x_{12}^{\gamma_{12;3}/2} x_{13}^{\gamma_{13;2}/2} x_{23}^{\gamma_{23;1}/2} \bar{x}_{12}^{\bar{\gamma}_{12;3}/2} \bar{x}_{13}^{\bar{\gamma}_{13;2}/2} \bar{x}_{23}^{\bar{\gamma}_{23;1}/2}} ,} \tag{4.19}$$

where $\gamma_{12;3}$ is given by

$$\gamma_{12;3} = -\frac{1}{2k(1-\lambda^2)} \left(2\lambda (N_I^{(1)} + N_I^{(2)} - N_I^{(3)}) \right.$$
$$\left. - c_{R_1} - c_{R_2} + c_{R_3} - \lambda^2 (c_{R_1'} + c_{R_2'} - c_{R_3'}) \right) , \tag{4.20}$$

and

$$\bar{\gamma}_{12;3} = -\frac{1}{2k(1-\lambda^2)} \left(2\lambda (N_I^{(1)} + N_I^{(2)} - N_I^{(3)}) \right.$$
$$\left. - c_{R_1'} - c_{R_2'} + c_{R_3'} - \lambda^2 (c_{R_1} + c_{R_2} - c_{R_3}) \right) . \tag{4.21}$$

The other differences of dimensions $\gamma_{23;1}, \bar{\gamma}_{23;1}$ and $\gamma_{13;2}, \bar{\gamma}_{13;2}$ are obtained by performing cyclic permutations in the indices $1, 2$ and 3.

We now turn our attention to the three-point function coefficients $\tilde{C}_{IJK}(k, \lambda)$. At $\lambda = 0$ these coefficients are considered as known since they are in principle fully determined from the WZW CFT data. On general grounds the following perturbative expansion holds

$$\tilde{C}_{IJK}(k, \lambda) = \tilde{C}_{IJK}^{(0)} + \frac{1}{k} \tilde{C}_{IJK}^{(1)}(\lambda) + \mathcal{O}\left(\frac{1}{k^2}\right) , \tag{4.22}$$

where note the leading coefficient $\tilde{C}_{IJK}^{(0)}$ in $1/k$ expansion does not depend on λ. This is so because such a term being k-independent and simultaneously having possible poles only at $\lambda = \pm 1$ and preserving the symmetry $k \mapsto -k, \lambda \mapsto \lambda^{-1}$ cannot be finite either in the non-Abelian T-dual or in the pseudodual limit. Using the same line of reasoning as in the rest of this paper we conclude that the first correction to the three-point function should be of the form[3]

$$\tilde{C}_{IJK}^{(1)}(\lambda) = \frac{f_{IJK}(\lambda)}{(1-\lambda)(1+\lambda)^3} , \tag{4.23}$$

with

$$\lambda^4 f_{IJK}(\lambda^{-1}) = f_{IJK}(\lambda) \implies f_{IJK}(\lambda) = \tilde{C}_{IJK}^{(1)}(0)(1+\lambda^4) + a_{IJK}^{(1)}(\lambda + \lambda^3) + a_{IJK}^{(2)}\lambda^2 . \tag{4.24}$$

---

[3] Notice that here we are using the duality (1.7) followed by parity. Under this combined symmetry $\Phi_I^{(i)}(x_i, \bar{x}_i) \mapsto \Phi_I^{(i)}(\bar{x}_i, x_i)$ and $\tilde{C}_{IJK}(\lambda^{-1}, -k) = \tilde{C}_{IJK}(\lambda, k)$.

We saw from the $\mathcal{O}(\lambda)$ calculation that $a^{(1)}_{IJK} = 0$. Furthermore, it is not difficult to see that $a^{(2)}_{IJK} = 0$ too. Indeed, by inspecting the $\mathcal{O}(\lambda^2)$ calculation one can see that in order to remain to order $1/k$ either the two holomorphic or the two anti-holomorphic currents should be contracted through the Abelian part of their OPE. Then the resulting integrals will be of the form $\int \dfrac{\mathrm{d}^2 z_{12}}{(z_1 - x_1)(z_2 - x_2)\bar{z}^2_{12}}$ which can only produce logarithms. But the logarithms have to be combined and exponentiated to give the differences of the anomalous dimensions. Thus, no finite part will be present at this order and as a result $a^{(2)}_{IJK} = 0$, as well. Thus, we conclude that

$$\tilde{C}^{(1)}_{IJK}(\lambda) = \frac{\tilde{C}^{(1)}_{IJK}(0)(1+\lambda^4)}{(1-\lambda)(1+\lambda)^3} , \tag{4.25}$$

where as explained, the constant $\tilde{C}^{(1)}_{IJK}(0)$ is fully determined from the WZW CFT initial data. As a result we have determined the exact in λ three-point function coefficient of three-primary fields up to order $1/k$.

5. Mixed $\langle J \Phi \Phi \rangle$ and $\langle \bar{J} \Phi \Phi \rangle$ correlators

In this section we focus on the mixed correlators involving two primary fields and one current. From Appendix F one can read off the $\mathcal{O}(\lambda^3)$ result which is given by

$$\langle J^a(x_3)\Phi^{(1)}_{i,i'}(x_1)\Phi^{(2)}_{j,j'}(x_2)\rangle_\lambda =$$
$$\left(1 + \frac{\lambda^2}{2}\right)\frac{(t_a \otimes \mathbb{I}_{R'})_{ii',jj'} - \lambda\,(\mathbb{I}_R \otimes \tilde{t}^*_a)_{ii',jj'}}{\sqrt{k}\,x^{2\Delta_R}_{12}\bar{x}^{2\bar{\Delta}_{R'}}_{12}}\left(\frac{1}{x_{13}} - \frac{1}{x_{23}}\right) . \tag{5.1}$$

The similar expression for the correlator $\bar{J}^a$ reads

$$\langle \bar{J}^a(\bar{x}_3)\Phi^{(1)}_{i,i'}(x_1)\Phi^{(2)}_{j,j'}(x_2)\rangle_\lambda =$$
$$-\left(1 + \frac{\lambda^2}{2}\right)\frac{(\mathbb{I}_R \otimes \tilde{t}^*_a)_{ii',jj'} - \lambda\,(t_a \otimes \mathbb{I}_{R'})_{ii',jj'}}{\sqrt{k}\,x^{2\Delta_R}_{12}\bar{x}^{2\bar{\Delta}_{R'}}_{12}}\left(\frac{1}{\bar{x}_{13}} - \frac{1}{\bar{x}_{23}}\right) . \tag{5.2}$$

Getting inspired by the previous computations and by the expression in (5.1) we conclude that the all-loop mixed correlators should assume the following form

$$\langle J_a(x_3)\Phi^{(1)}_{i,i'}(x_1)\Phi^{(2)}_{j,j'}(x_2)\rangle_\lambda =$$
$$\frac{f_1(\lambda)(t_a \otimes \mathbb{I}_{R'})_{ii',jj'} - \lambda f_2(\lambda)(\mathbb{I}_R \otimes \tilde{t}^*_a)_{ii',jj'}}{\sqrt{k(1-\lambda)(1+\lambda)^3}\,x^{2\Delta_R}_{12}\bar{x}^{2\bar{\Delta}_{R'}}_{12}}\left(\frac{1}{x_{13}} - \frac{1}{x_{23}}\right) , \tag{5.3}$$

where the functions $f_1(\lambda)$ and $f_2(\lambda)$ are everywhere analytic and $f_1(0) = f_2(0) = 1$. As usual, the denominator of (5.3) is written in such a way that the correlator has well-defined non-Abelian and pseudodual limits.

Applying the duality (1.7), as well as the corresponding transformation rules for the currents (2.14) and primary fields (2.20) we obtain that

$$\langle J_a(x_3)\Phi^{(2)}_{i',i}(x_1)\Phi^{(1)}_{j',j}(x_2)\rangle_\lambda =$$
$$\frac{\lambda^2 f_1(\lambda^{-1})(\mathbb{I}_{R'} \otimes t_a)_{i'i,j'j} - \lambda f_2(\lambda^{-1})(\tilde{t}^*_a \otimes \mathbb{I}_R)_{i'i,j'j}}{\sqrt{k(1-\lambda)(1+\lambda)^3}\,x^{2\Delta_R}_{12}\bar{x}^{2\bar{\Delta}_{R'}}_{12}}\left(\frac{1}{x_{13}} - \frac{1}{x_{23}}\right) , \tag{5.4}$$

where on the right hand side of the last equation we have changed the order of the indices for convenience. Subsequently, the left hand side of the above can be rewritten using appropriately (5.3). We have that

$$\langle J_a(x_3)\Phi^{(1)}_{i',i}(x_1)\Phi^{(2)}_{j',j}(x_2)\rangle_\lambda$$

$$= \frac{f_1(\lambda)(\tilde{t}^{(2)}_a \otimes \mathbb{I}_R)_{i'i,j'j} - \lambda f_2(\lambda)(\mathbb{I}_{R'} \otimes t^{(2)*}_a)_{i'i,j'j}}{\sqrt{k(1-\lambda)(1+\lambda)^3}\, x_{12}^{2\Delta_R}\bar{x}_{12}^{2\bar\Delta_{R'}}}\left(\frac{1}{x_{13}} - \frac{1}{x_{23}}\right)$$

$$= \frac{-f_1(\lambda)(\tilde{t}^*_a \otimes \mathbb{I}_R)_{i'i,j'j} + \lambda f_2(\lambda)(\mathbb{I}_{R'} \otimes t_a)_{i'i,j'j}}{\sqrt{k(1-\lambda)(1+\lambda)^3}\, x_{12}^{2\Delta_R}\bar{x}_{12}^{2\bar\Delta_{R'}}}\left(\frac{1}{x_{13}} - \frac{1}{x_{23}}\right). \tag{5.5}$$

Hence, comparing (5.4) with (5.5) we have the two equivalent conditions

$$\lambda f_1(\lambda^{-1}) = f_2(\lambda), \quad \lambda f_2(\lambda^{-1}) = f_1(\lambda) \quad \Longrightarrow \quad f_1(\lambda) = f_2(\lambda) = 1 + \lambda. \tag{5.6}$$

Plugging the latter into (5.3) we find after some rearrangement that

$$\boxed{\langle J^a(x_3)\Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle_\lambda = -\frac{(t_a \otimes \mathbb{I}_{R'})_{ii',jj'} - \lambda(\mathbb{I}_R \otimes \tilde{t}^*_a)_{ii',jj'}}{\sqrt{k(1-\lambda^2)}\, x_{12}^{2\Delta_R-1}\bar{x}_{12}^{2\bar\Delta_{R'}} x_{13}x_{23}}.} \tag{5.7}$$

Similar reasoning leads to

$$\boxed{\langle \bar{J}^a(\bar{x}_3)\Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle_\lambda = \frac{(\mathbb{I}_R \otimes \tilde{t}^*_a)_{ii',jj'} - \lambda(t_a \otimes \mathbb{I}_{R'})_{ii',jj'}}{\sqrt{k(1-\lambda^2)}\, x_{12}^{2\Delta_R}\bar{x}_{12}^{2\bar\Delta_{R'}-1}\bar{x}_{13}\bar{x}_{23}},} \tag{5.8}$$

whose expansions around $\lambda = 0$ agree with (5.1) and (5.2). We stress that the one- and two-loop calculations in conjunction with the symmetry (1.7) are enough to fully determine the all-loop expressions for the correlators under consideration. Thus, the $\mathcal{O}(\lambda^3)$ terms in (5.1) and (5.2) provide perturbative checks of the all-loop results. Note that the deformation mixes the left and right representations. It can be easily checked that the λ-deformed direct products in the numerators in the above correlators form representations of the algebra as well.

6. OPEs and equal-time commutators

In this section we use the two-point and three-point functions of the currents and primary fields to find their OPE algebra up to order $1/k$ reads and exact in the deformation parameter λ. The result is

$$J^a(x_1)J^b(x_2) = \frac{\delta_{ab}}{x_{12}^{2+\gamma^{(J)}}\bar{x}_{12}^{\gamma^{(J)}}} + c(\lambda)\frac{f_{abc}J^c(x_2)}{x_{12}} + d(\lambda)\frac{f_{abc}\bar{J}^c(\bar{x}_2)\bar{x}_{12}}{x_{12}^2} + \dots,$$

$$J^a(x_1)\bar{J}^b(\bar{x}_2) = -\gamma^{(J)}\frac{\delta_{ab}}{|x_{12}|^2} + d(\lambda)\frac{f_{abc}\bar{J}^c(\bar{x}_2)}{x_{12}} + d(\lambda)\frac{f_{abc}J^c(x_2)}{\bar{x}_{12}} + \dots,$$

$$J^a(x_1)\Phi_{i,i'}^{(1)}(x_2,\bar{x}_2) = -\frac{(t_a)_i{}^m\Phi_{m,i'}^{(1)}(x_2,\bar{x}_2) - \lambda(\tilde{t}_a^*)_{i'}{}^{m'}\Phi_{i,m'}^{(1)}(x_2,\bar{x}_2)}{x_{12}\sqrt{k(1-\lambda^2)}} + \dots, \qquad (6.1)$$

$$\bar{J}^a(\bar{x}_1)\Phi_{i,i'}^{(1)}(x_2,\bar{x}_2) = \frac{(\tilde{t}_a^*)_{i'}{}^{m'}\Phi_{i,m'}^{(1)}(x_2,\bar{x}_2) - \lambda(t_a)_i{}^m\Phi_{m,i'}^{(1)}(x_2,\bar{x}_2)}{\bar{x}_{12}\sqrt{k(1-\lambda^2)}} + \dots,$$

$$\tilde{\Phi}_I^{(1)}(x_1,\bar{x}_1)\tilde{\Phi}_J^{(2)}(x_2,\bar{x}_2) = \frac{\tilde{C}_{IJK}\,\tilde{\Phi}_K^{(3)}(x_2,\bar{x}_2)}{x_{12}^{\gamma_{12;3}}\bar{x}_{12}^{\bar{\gamma}_{12;3}}} + \dots,$$

where $\tilde{C}_{IJK}$ was given in (4.22), $\gamma^{(J)}$ is the anomalous dimension of the current given in (3.21), the $\gamma_{12;3}$ and $\bar{\gamma}_{12;3}$ are given by (4.20) and (4.21) and

$$c(\lambda) = \sqrt{\frac{(1-\lambda^3)^2}{k(1-\lambda^2)^3}}, \qquad d(\lambda) = \sqrt{\frac{\lambda^2(1-\lambda)^2}{k(1-\lambda^2)^3}}. \qquad (6.2)$$

Having the OPEs at our disposal, we can easily compute the equal-time commutator of the currents and primaries through a time-ordered limiting procedure

$$[f(\sigma_1,\tau), g(\sigma_2,\tau)] = \lim_{\varepsilon\to 0}(f(\sigma_1,\tau+i\varepsilon)g(\sigma_2,\tau) - g(\sigma_2,\tau+i\varepsilon)f(\sigma_1,\tau)) \qquad (6.3)$$

and the following representations of Dirac delta-function

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\sigma-i\varepsilon} - \frac{1}{\sigma+i\varepsilon}\right) = 2\pi i\,\delta(\sigma),$$

$$\lim_{\varepsilon\to 0}\left(\frac{1}{(\sigma-i\varepsilon)^2} - \frac{1}{(\sigma+i\varepsilon)^2}\right) = -2\pi i\,\delta'(\sigma), \qquad (6.4)$$

$$\lim_{\varepsilon\to 0}\left(\frac{\sigma+i\varepsilon}{(\sigma-i\varepsilon)^2} - \frac{\sigma-i\varepsilon}{(\sigma+i\varepsilon)^2}\right) = 2\pi i\,\delta(\sigma).$$

Employing Eqs. (6.1), (6.3) and (6.4), we find to order $1/\sqrt{k}$ that[4]

$$[J^a(\sigma_1), J^b(\sigma_2)] = 2\pi i\,\delta_{ab}\delta'(\sigma_{12}) + 2\pi f_{abc}\left(c(\lambda)J^c(\sigma_2) - d(\lambda)\bar{J}^c(\sigma_2)\right)\delta(\sigma_{12}),$$

$$[\bar{J}^a(\sigma_1), \bar{J}^b(\sigma_2)] = -2\pi i\,\delta_{ab}\delta'(\sigma_{12}) + 2\pi f_{abc}\left(c(\lambda)\bar{J}^c(\sigma_2) - d(\lambda)J^c(\sigma_2)\right)\delta(\sigma_{12}), \quad (6.5)$$

$$[J^a(\sigma_1), \bar{J}^b(\sigma_2)] = 2\pi d(\lambda)f_{abc}\left(J^c(\sigma_2) + \bar{J}^c(\sigma_2)\right)\delta(\sigma_{12}),$$

and

$$[\tilde{\Phi}_{i,i'}^{(1)}(\sigma_1), \tilde{\Phi}_{j,j'}^{(2)}(\sigma_2)] = 0, \qquad (6.6)$$

$$[J^a(\sigma_1), \Phi_{i,i'}^{(1)}(\sigma_2)] = -\frac{2\pi}{\sqrt{k(1-\lambda^2)}}\left((t_a)_i{}^m\Phi_{m,i'}^{(1)}(\sigma_2) - \lambda(\tilde{t}_a^*)_{i'}{}^{m'}\Phi_{i,m'}^{(1)}(\sigma_2)\right)\delta(\sigma_{12}),$$

$$[\bar{J}^a(\sigma_1), \Phi_{i,i'}^{(1)}(\sigma_2)] = \frac{2\pi}{\sqrt{k(1-\lambda^2)}}\left((\tilde{t}_a^*)_{i'}{}^{m'}\Phi_{i,m'}^{(1)}(\sigma_2) - \lambda(t_a)_i{}^m\Phi_{m,i'}^{(1)}(\sigma_2)\right)\delta(\sigma_{12}).$$

---

[4] The OPEs and the equal-time commutators for the currents are in agreement with those obtained in [32], for current–current perturbations of the WZW model on supergroups.

These equal-time commutators turn out to be isomorphic to two commuting copies of current algebras with opposite levels

$$[S^a(\sigma_1), S^b(\sigma_2)] = \frac{i\,k}{2\pi}\,\delta_{ab}\delta'(\sigma_{12}) + f_{abc}\,S^c(\sigma_2)\,\delta(\sigma_{12})\,,$$

$$[\bar{S}^a(\sigma_1), \bar{S}^b(\sigma_2)] = -\frac{i\,k}{2\pi}\,\delta_{ab}\delta'(\sigma_{12}) + f_{abc}\,\bar{S}^c(\sigma_2)\,\delta(\sigma_{12})\,,$$

$$[S^a(\sigma_1), \bar{S}^b(\sigma_2)] = 0\,,$$

$$[S^a(\sigma_1), \Phi^{(1)}_{i,i'}(\sigma_2)] = -(t_a)_i{}^m \Phi^{(1)}_{m,i'}(\sigma_2)\,\delta(\sigma_{12})\,,$$

$$[\bar{S}^a(\sigma_1), \Phi^{(1)}_{i,i'}(\sigma_2)] = (\tilde{t}^*_a)_{i'}{}^{m'} \Phi^{(1)}_{i,m'}(\sigma_2)\,\delta(\sigma_{12})\,,$$

(6.7)

where

$$S^a = \frac{1}{2\pi}\sqrt{\frac{k}{1-\lambda^2}}\left(J^a - \lambda \bar{J}^a\right)\,, \quad \bar{S}^a = \frac{1}{2\pi}\sqrt{\frac{k}{1-\lambda^2}}\left(\bar{J}^a - \lambda J^a\right)\,. \tag{6.8}$$

The parameter λ does not appear in this algebra but it does in the time evolution of the system due to the fact that, as it turns out, the Hamiltonian in terms of S^a and $\bar{S}^a$ is λ-dependent (cf. eq. (2.11) of [2]). Also, the λ-dependence still appears in the OPEs of the S^a and $\bar{S}^a$ among them. The reasons is that the OPEs, unlike the commutators (6.7), are not computed at equal times.

Finally we take the classical limit of (6.5) and appropriately rescaling the currents, we find Rajeev's *deformation of the canonical structure of the isotropic PCM* [35] (recall that, in our conventions the group structure constants f_{abc} are taken to be imaginary)

$$\{I^a_\pm, I^b_\pm\}_{\text{P.B.}} = -i\,e^2 f_{abc}\left(I^c_\mp(\sigma_2) - (1+2x)I^c_\pm(\sigma_2)\right)\delta(\sigma_{12}) \pm 2e^2\delta_{ab}\,\delta'(\sigma_{12})\,,$$

$$\{I^a_\pm, I^b_\mp\}_{\text{P.B.}} = i\,e^2 f_{abc}\left(I^c_+(\sigma_2) + I^c_-(\sigma_2)\right)\delta(\sigma_{12})\,,$$

(6.9)

realized through the action (1.5) of [2]

$$e = 2d(\lambda) = \frac{1}{\sqrt{k(1-\lambda^2)}}\frac{2\lambda}{1+\lambda}\,, \quad \frac{c(\lambda)}{d(\lambda)} = 1+2x\,, \quad x = \frac{1+\lambda^2}{2\lambda}\,. \tag{6.10}$$

That the deformed brackets (6.9) follow as the classical limit of the OPEs provides actually, for the isotropic case, the mathematical proof that the action (1.5) is in fact the effective action of the non-Abelian Thirring model action (1.4). The reason is that (1.5) provides, as was shown in [2], a realization of (6.9) which in turn was derived by using (1.4) as the starting point.

7. Conclusions

In this work we have computed all possible two- and three-point functions of current and primary field operators for the λ-deformed integrable σ-models. These models are characterized by the deformation parameter λ, as well as by the integer level k of the WZW model. Our results are valid for any semisimple group G, for all values of the deformation parameter λ and up to order $1/k$ in the large k expansion. We achieved this goal by combining the first few orders in perturbation theory with analyticity arguments as well as with a non-trivial duality-type symmetry shared by these models. The two- and three-point correlators allowed us to deduce the exact in λ OPEs of all currents and primary operators. Furthermore, based on our results we derived the anomalous dimensions and correlation functions for the operators in two important limits of

the aforementioned λ-deformed σ-models, namely the non-Abelian T-dual of the PCM and the pseudodual model.

Our results are summarized as follows:

1. In section 3.1 we presented the results for the two-point correlator of two currents. From these correlators and in conjunction with the aforementioned symmetry we derived the all-loop β-function of the theory as well as their anomalous dimension.
2. In section 3.2 we derived the all-loop expressions for the three current correlators.
3. In section 4.1 we provide the reader with the exact two-point functions of all primary operators of the theory, as well as with their exact in λ anomalous dimensions. In this case, the role of the symmetry is instrumental since it is realized in a non-trivial way.
4. In section 4.2 we provide the reader with the exact three-point functions of all primary operators of the theory.
5. In section 5 we calculated the exact, in λ, three-point correlators $\langle J \Phi \Phi \rangle$ and $\langle \bar{J} \Phi \Phi \rangle$.
6. In section 6 we deduced all relevant OPEs between currents and/or primary fields that are consistent with the exact results for the two- and three-point functions given in previous sections. We also derive the currents' Poisson brackets which assume Rajeev's *deformation of the canonical structure of the isotropic PCM*, the underlying structure of the integrable λ-deformed σ-models. This essentially proves in a mathematical sense that the action (1.5) for an isotropic deformation is indeed the effective action of the non-Abelian Thirring model action.

One direction for extending our work would be to consider cases beyond isotropy, i.e. when the matrix λ is not proportional to the identity. In particular, we believe that the equal-time commutators of the currents and primaries will take the form of (6.7), under an analogue to (6.8) relation. Another direction would be to calculate the subleading, in the $1/k$ expansion, terms of all physical quantities such as the β-function, the anomalous dimension matrix and the fusion coefficients. These line of research would, hopefully, be culminated by finding the exact in both λ and k expressions for these physical quantities as well as the underlying effective action.

Acknowledgements

Appendix A. Various integrals

In this appendix we assemble all the integrals that will be needed in our perturbative calculations. In all integrals considered below the integration domain is a disc of radius R in which the various external points labeled by x's are excluded. This can be done by encircling them with circles having arbitrarily vanishing radius. One way to prove the expressions below is to use Stokes' theorem in two-dimensions for appropriately chosen vectors and contours.

The first set of integrals is the exact version of the integrals in (2.31) and (2.33)

$$
\int \frac{\mathrm{d}^2 z}{(z - x_1)(\bar{z} - \bar{x}_2)} = -\pi \, \ln \frac{|x_1 - x_2|^2}{R^2 - x_1 \bar{x}_2} ,
$$

$$
\int \frac{\mathrm{d}^2 z}{(z - x_1)^2 (\bar{z} - \bar{x}_2)^2} = \pi^2 \delta^{(2)}(x_1 - x_2) - \frac{\pi R^2}{(R^2 - x_1 \bar{x}_2)^2} ,
$$

(A.1)

where the $R > |x_{1,2}|$. By taking derivatives we may compute the exact analog of the integrals in (2.32).

A generalization of the first of the above integrals is given by

$$
\int \frac{\mathrm{d}^2 z}{\prod\limits_{i=1}^{M}(z - x_i) \prod\limits_{i=1}^{N}(\bar{z} - \bar{y}_j)} = -\pi \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{1}{A_i B_j} \, \ln \frac{|x_i - y_j|^2}{R^2 - x_i \bar{y}_j} ,
$$

(A.2)

with $R > \{|x_i|, |y_j|\}$ and $A_i = \prod\limits_{\substack{j=1 \\ j \neq i}}^{M} (x_i - x_j)$, $B_i = \prod\limits_{\substack{j=1 \\ j \neq i}}^{N} (\bar{y}_i - \bar{y}_j)$. This relation can be proved by first performing a partial fraction decomposition and then use (A.1). A special case of this is when the denominators are cubic polynomials. Namely,

$$
\int \frac{\mathrm{d}^2 z}{(z - x_1)(z - x_2)(\bar{z} - \bar{x}_1)} = -\frac{\pi}{x_{12}} \left(\ln \frac{\varepsilon^2}{|x_{12}|^2} + \ln \frac{R^2 - x_2 \bar{x}_1}{R^2 - |x_1|^2} \right) ,
$$

$$
\int \frac{\mathrm{d}^2 z}{(z - x_1)(\bar{z} - \bar{x}_1)(\bar{z} - \bar{x}_2)} = -\frac{\pi}{\bar{x}_{12}} \left(\ln \frac{\varepsilon^2}{|x_{12}|^2} + \ln \frac{R^2 - x_1 \bar{x}_2}{R^2 - |x_1|^2} \right) ,
$$

(A.3)

which is the exact analogs of the integrals in (2.34).

Another important integral is given by

$$
\int \frac{\mathrm{d}^2 z}{(z - x_1)(\bar{z} - \bar{x}_2)} \ln \frac{|z - x_1|^2}{R^2 - x_1 \bar{z}} = -\frac{\pi}{2} \ln^2 \frac{|x_1 - x_2|^2}{R^2 - x_1 \bar{x}_2} .
$$

(A.4)

The large R limit of this is given in (2.35) and is necessary for the derivation of the two-loop contribution in (3.6).

Appendix B. Perturbative computation of the $\langle J \bar{J} \rangle$ correlator

In this appendix we present the perturbative calculation of the $\langle J \bar{J} \rangle$ two-point function. At the conformal point it vanishes.

Order $\mathcal{O}(\lambda)$. To that order we have that

$$
\langle J^a(x_1) J^b(x_2) \rangle_\lambda^{(1)} = -\frac{\lambda}{\pi} \int \mathrm{d}^2 z \langle J^a(x_1) J^c(z) \rangle \langle \bar{J}^c(\bar{z}) \bar{J}^b(\bar{x}_2) \rangle = -\pi \lambda \delta^{ab} \delta^{(2)}(x_{12}) .
$$

(B.1)

Order $\mathcal{O}(\lambda^2)$. To that order we have that

$$
\langle J^a(x_1) J^b(x_2) \rangle_\lambda^{(2)} = \frac{\lambda^2}{2\pi^2} \int \mathrm{d}^2 z_{12} \langle J^a(x_1) J^{a_1}(z_1) J^{a_2}(z_2) \rangle \langle \bar{J}^{a_1}(\bar{z}_1) \bar{J}^{a_2}(\bar{z}_2) \bar{J}^b(\bar{x}_2) \rangle
$$

$$
= -\frac{\lambda^2}{2\pi^2} \delta^{ab} \frac{c_G}{k} J(x_1, x_2) ,
$$

(B.2)

where

$$J(x_1, x_2) = \int \frac{d^2 z_{12}}{(x_1 - z_1)(x_1 - z_2)(z_1 - z_2)(\bar{x}_2 - \bar{z}_1)(\bar{x}_2 - \bar{z}_2)(\bar{z}_1 - \bar{z}_2)}$$

$$= \int \frac{d^2 z_{12}}{(x_1 - z_1)^2 (\bar{x}_2 - \bar{z}_2)^2} \left(\frac{1}{z_1 - z_2} - \frac{1}{x_1 - z_2} \right) \left(\frac{1}{\bar{z}_1 - \bar{z}_2} + \frac{1}{\bar{x}_2 - \bar{z}_1} \right) \quad \text{(B.3)}$$

$$= J_{11} + J_{12} + J_{21} + J_{22} \,,$$

where we have broken the integral J into the four integrals J_{ij}, resulting from multiplying out the terms in the parenthesis, in a rather self-explanatory notation. We have that

$$J_{11} = \partial_{x_1} \partial_{\bar{x}_2} \int \frac{d^2 z_{12}}{(x_1 - z_1)(\bar{x}_2 - \bar{z}_2)(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}$$

$$= -\pi \partial_{x_1} \partial_{\bar{x}_2} \int \frac{d^2 z_2}{(x_1 - z_2)(\bar{x}_2 - \bar{z}_2)} \ln \frac{\varepsilon^2}{|z_2 - x_1|^2}$$

$$= \pi^2 \partial_{x_1} \partial_{\bar{x}_2} \left(\ln \varepsilon^2 \ln |x_{12}|^2 - \frac{1}{2} \ln^2 |x_{12}|^2 \right) \quad \text{(B.4)}$$

$$= -\pi^3 \delta^{(2)}(x_{12}) \ln \frac{\varepsilon^2}{|x_{12}|^2} + \frac{\pi^2}{|x_{12}|^2} \,,$$

where we have used (2.35). Also

$$J_{12} = \pi \int \frac{d^2 z_1}{(x_1 - z_1)^2 (\bar{x}_2 - \bar{z}_1)^2} = \pi^3 \delta^{(2)}(x_{12}) \,,$$

$$J_{21} = \pi \int \frac{d^2 z_2}{(\bar{x}_2 - \bar{z}_2)^2 (x_1 - z_2)^2} = \pi^3 \delta^{(2)}(x_{12}) \,. \quad \text{(B.5)}$$

Note that for J_{12} we have first performed the z_2-integration which is not in accordance with our regularization prescription. However, we now show that the same result follows if we do first the z_1 according to our regularization. We easily find that

$$J_{12} = \partial_{x_1} \int \frac{d^2 z_{12}}{(x_1 - z_2)(\bar{x}_2 - \bar{z}_2)^2} \left(\frac{1}{(z_1 - x_1)(\bar{x}_2 - \bar{z}_1)} - \frac{1}{(z_1 - z_2)(\bar{x}_2 - \bar{z}_2)} \right)$$

$$= \pi \partial_{x_1} \left(\ln |x_{12}|^2 \int \frac{d^2 z_2}{(\bar{x}_2 - \bar{z}_2)^2 (x_1 - z_2)} \right) + \pi \int \frac{d^2 z_2 \ln |z_2 - x_2|^2}{(z_2 - x_1)^2 (\bar{z}_2 - \bar{x}_2)^2} \quad \text{(B.6)}$$

$$= -\pi^2 \partial_{x_1} \left(\frac{\ln |x_{12}|^2}{\bar{x}_{12}} \right) + \pi^3 (1 + \ln |x_{12}|^2) \delta^{(2)}(x_{12}) + \frac{\pi^2}{|x_{12}|^2} \,,$$

where we have used the fact that the second integral in the second line above can be obtained from (B.4) (with $\varepsilon = 1$). A simple algebra gives the same expression as in (B.5). Finally

$$J_{22} = -\frac{\pi}{x_{12}} \int \frac{d^2 z_2}{(\bar{x}_2 - \bar{z}_2)^2 (x_1 - z_2)} = \frac{\pi^2}{|x_{12}|^2} \,. \quad \text{(B.7)}$$

Therefore collecting all contributions we find that

$$J(x_1, x_2) = 2\pi^4 \left(1 - \frac{1}{2} \ln \frac{\varepsilon^2}{|x_{12}|^2} \right) \delta^{(2)}(x_{12}) + \frac{\pi^3}{|x_{12}|^2} \,. \quad \text{(B.8)}$$

Order $\mathcal{O}(\lambda^3)$. To that order we have that

$$\langle J^a(x_1)J^b(x_2)\rangle^{(3)}_\lambda = -\frac{\lambda^3}{6\pi^3}\int d^2z_{123}\langle J^a(x_1)J^{a_1}(z_1)J^{a_2}(z_2)J^{a_3}(z_3)\rangle\times$$
$$\langle\bar{J}^{a_1}(\bar{z}_1)\bar{J}^{a_2}(\bar{z}_2)\bar{J}^{a_3}(\bar{z}_3)\bar{J}^b(\bar{x}_2)\rangle \tag{B.9}$$

The four-point function is given by (2.6) and is multiplied by the analogous four-point function for anti-holomorphic currents. Keeping terms up to $\mathcal{O}(1/k)$, disregarding terms giving rise to bubbles and taking into account the above permutation symmetry we arrive at

$$\langle J^a(x_1)J^b(x_2)\rangle^{(3)}_\lambda = 2\delta^{ab}\frac{c_G}{k}\frac{\lambda^3}{\pi^3}K(x_1,x_2)\,, \tag{B.10}$$

where

$$K(x_1,x_2) = \int\frac{d^2z_{123}}{(z_1-x_2)(x_1-z_2)(x_1-z_3)(z_2-z_3)(\bar{z}_1-\bar{z}_2)^2(\bar{z}_3-\bar{x}_2)^2}\,. \tag{B.11}$$

Performing the integrations first over z_1 and then over z_2 we obtain that

$$K(x_1,x_2) = -\pi^2\int\frac{d^2z_3}{(z_3-x_1)^2(\bar{z}_3-\bar{x}_2)^2}\ln\frac{\varepsilon^2}{|z_3-x_1|^2} = -\pi^4\ln\varepsilon^2\delta^{(2)}(x_{12})$$
$$+\pi^2\left(\partial_{x_1}\partial_{\bar{x}_2}\int\frac{d^2z_3}{(z_3-x_1)(\bar{z}_3-\bar{x}_2)}\ln|z_3-x_1|^2 + \int\frac{d^2z_3}{(z_3-x_1)^2(\bar{z}_3-\bar{x}_2)^2}\right)$$
$$= -\pi^4\ln\varepsilon^2\delta^{(2)}(x_{12}) + \pi^4\delta^{(2)}(x_{12}) - \frac{\pi^3}{2}\partial_{x_1}\partial_{\bar{x}_2}\ln^2|x_{12}|^2 \tag{B.12}$$
$$= \pi^4\left(1 - \ln\frac{\varepsilon^2}{|x_{12}|^2}\right)\delta^{(2)}(x_{12}) + \frac{\pi^3}{|x_{12}|^2}\,.$$

In conclusion (B.1), (B.2) and (B.10) combine to (3.6) in the main text.

Appendix C. Perturbative computation of the $\langle JJJ\rangle$ correlator

In this appendix we present the perturbative calculation of the $\langle JJJ\rangle$ three-point correlator. The $\mathcal{O}(\lambda)$ contribution to this correlators vanishes since $\langle\bar{J}\rangle = 0$. Proceeding to higher orders in the λ-expansion we have:

Order $\mathcal{O}(\lambda^2)$. This contribution is immediately seen to be equal to

$$\langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle^{(2)}_\lambda = \frac{\lambda^2}{2!\pi^2}\int\frac{d^2z_{12}}{\bar{z}_{12}^2}\langle J^a(x_1)J^b(x_2)J^c(x_3)J^{a_1}(z_1)J^{a_1}(z_2)\rangle\,. \tag{C.1}$$

To proceed with the contractions we single out J^{a_1} to perform them. Disregarding the disconnected and bubble pieces and also noting that the Abelian contractions, i.e. contractions leading to second poles, of J^{a_1} with the external currents vanish in our regularization scheme, we have that

$$\langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle^{(2)}_\lambda = \frac{\lambda^2}{2!\pi^2}\frac{1}{\sqrt{k}}\int\frac{d^2z_{12}}{\bar{z}_{12}^2}\left(\frac{f_{a_1ad}}{z_1-x_1}\langle J_d(x_1)J_b(x_2)J_c(x_3)J_{a_1}(z_2)\rangle\right.$$
$$\left.+\frac{f_{a_1bd}}{z_1-x_2}\langle J_a(x_1)J_d(x_2)J_c(x_3)J_{a_1}(z_2)\rangle + \frac{f_{a_1cd}}{z_1-x_3}\langle J_a(x_1)J_b(x_2)J_d(x_3)J_{a_1}(z_2)\rangle\right)$$

$$= \frac{\lambda^2}{2\pi^2} \frac{1}{\sqrt{k}} \int d^2 z_2 \left(\frac{f_{a_1 a d}}{\bar{x}_1 - \bar{z}_2} \langle J_d(x_1) J_b(x_2) J_c(x_3) J_{a_1}(z_2) \rangle \right. \tag{C.2}$$

$$\left. + \frac{f_{a_1 b d}}{\bar{x}_2 - \bar{z}_2} \langle J_a(x_1) J_d(x_2) J_c(x_3) J_{a_1}(z_2) \rangle + \frac{f_{a_1 c d}}{\bar{x}_3 - \bar{z}_2} \langle J_a(x_1) J_b(x_2) J_d(x_3) J_{a_1}(z_2) \rangle \right) .$$

Computing this to $\mathcal{O}\left(1/\sqrt{k}\right)$ gives

$$\langle J^a(x_1) J^b(x_2) J^c(x_3) \rangle^{(2)} = \frac{3\lambda^2}{2\sqrt{k}} \frac{f_{abc}}{x_{12} x_{13} x_{23}} . \tag{C.3}$$

Order $\mathcal{O}(\lambda^3)$. The contribution is immediately seen to be

$$\langle J^a(x_1) J^b(x_2) J^c(x_3) \rangle^{(3)} = -\frac{\lambda^3}{3! \pi^3} \frac{f_{a_1 a_2 a_3}}{\sqrt{k}} \int \frac{d^2 z_{123}}{\bar{z}_{12} \bar{z}_{13} \bar{z}_{23}} \times$$

$$\langle J_a(x_1) J_b(x_2) J_c(x_3) J_{a_1}(z_1) J_{a_2}(z_2) J_{a_3}(z_3) \rangle . \tag{C.4}$$

As we have already saturated the $\mathcal{O}(1/\sqrt{k})$, we perform only Abelian contractions in this six-point function, yielding to

$$\langle J^a(x_1) J^b(x_2) J^c(x_3) \rangle_\lambda^{(3)} = -\frac{\lambda^3 f_{abc}}{\pi^3 \sqrt{k}} \int \frac{d^2 z_{123}}{\bar{z}_{12} \bar{z}_{13} \bar{z}_{23} (z_1 - x_1)^2 (z_2 - x_2)^2 (z_3 - x_3)^2} . \tag{C.5}$$

Using the identity

$$\frac{1}{\bar{z}_{12} \bar{z}_{13}} = \frac{1}{\bar{z}_{23}} \left(\frac{1}{\bar{z}_{12}} - \frac{1}{\bar{z}_{13}} \right) , \tag{C.6}$$

integrating over z_1

$$\langle J^a(x_1) J^b(x_2) J^c(x_3) \rangle_\lambda^{(3)}$$

$$= -\frac{\lambda^3 f_{abc}}{\pi^2 \sqrt{k}} \partial_{x_2} \partial_{x_3} \int \frac{d^2 z_{23}}{\bar{z}_{23}^2 (z_2 - x_2)(z_3 - x_3)} \left(\frac{1}{z_2 - x_1} - \frac{1}{z_3 - x_1} \right) \tag{C.7}$$

and employing an analogue of the identity (C.6) we get that

$$\langle J^a(x_1) J^b(x_2) J^c(x_3) \rangle_\lambda^{(3)} = -\frac{\lambda^3 f_{abc}}{\sqrt{k}} \partial_{x_2} \partial_{x_3} \left(\frac{1}{x_{12}} \ln \frac{|x_{13}|^2}{|x_{23}|^2} - \frac{1}{x_{13}} \ln \frac{|x_{12}|^2}{|x_{23}|^2} \right)$$

$$= -\frac{\lambda^3}{\sqrt{k}} \frac{f_{abc}}{x_{12} x_{13} x_{23}} . \tag{C.8}$$

Appendix D. Perturbative computation of the $\langle J J \bar{J} \rangle$ correlator

In this appendix we present the perturbative calculation of the $\langle J J \bar{J} \rangle$ three-point correlator. Of course at the conformal point this correlation function vanishes. Proceeding to higher orders in the λ-expansions we have that:

Order $\mathcal{O}(\lambda)$. The contribution to the one-loop equals

$$\langle J^a(x_1) J^b(x_2) \bar{J}^c(\bar{x}_3) \rangle_\lambda^{(1)} = -\frac{\lambda}{\pi} \int \mathrm{d}^2 z \langle J^a(x_1) J^b(x_2) J^{a_1}(z) \rangle \langle \bar{J}^{a_1}(\bar{z}) \bar{J}^c(\bar{x}_3) \rangle ,$$

$$= -\frac{\lambda f_{abc}}{\sqrt{k}\, x_{12}} \int \frac{\mathrm{d}^2 z}{(x_1 - z)(x_2 - z)(\bar{x}_3 - \bar{z})^2} .$$

(D.1)

Employing an analogue of the identity (C.6) we get that

$$\langle J^a(x_1) J^b(x_2) \bar{J}^c(\bar{x}_3) \rangle_\lambda^{(1)} = \frac{\lambda}{\sqrt{k}} \frac{f_{abc} \bar{x}_{12}}{x_{12}^2 \bar{x}_{23} \bar{x}_{13}} .$$

(D.2)

Order $\mathcal{O}(\lambda^2)$. The contribution to the two-loop is equal to

$$\langle J^a(x_1) J^b(x_2) \bar{J}^c(\bar{x}_3) \rangle_{abc}^{(2)} = \frac{\lambda^2 f_{a_1 a_2 c}}{2! \pi^2 \sqrt{k}} \int \mathrm{d}^2 z_{12} \frac{\langle J_a(x_1) J_b(x_2) J_{a_1}(z_1) J_{a_2}(z_2) \rangle}{\bar{z}_{12}(\bar{z}_1 - \bar{x}_3)(\bar{z}_2 - \bar{x}_3)}$$

$$= \frac{\lambda^2 f_{abc}}{\pi^2 \sqrt{k}} \int \frac{\mathrm{d}^2 z_{12}}{\bar{z}_{12}(\bar{z}_1 - \bar{x}_3)(\bar{z}_2 - \bar{x}_3)(x_1 - z_1)^2(x_2 - z_2)^2} .$$

(D.3)

Employing again an analogue of the identity (C.6) we get that

$$\langle J^a(x_1) J^b(x_2) \bar{J}^c(\bar{x}_3) \rangle_{abc}^{(2)}$$

$$= \frac{\lambda^2 f_{abc}}{\pi^2 \sqrt{k}} \int \frac{\mathrm{d}^2 z_{12}}{(\bar{z}_2 - \bar{x}_3)^2(x_1 - z_1)^2(x_2 - z_2)^2} \left(\frac{1}{\bar{z}_1 - \bar{z}_2} - \frac{1}{\bar{z}_1 - \bar{x}_3} \right)$$

$$= -\frac{\lambda^2 f_{abc}}{\pi \sqrt{k}} \int \frac{\mathrm{d}^2 z_2}{(\bar{z}_2 - \bar{x}_3)^2(x_2 - z_2)^2} \left(\frac{1}{x_1 - z_2} - \frac{1}{x_{13}} \right)$$

$$= \frac{\lambda^2 f_{abc}}{\pi \sqrt{k}} \partial_{x_2} \partial_{\bar{x}_3} \int \frac{\mathrm{d}^2 z_2}{(x_1 - z_2)(\bar{z}_2 - \bar{x}_3)(x_2 - z_2)}$$

$$= \frac{\lambda^2 f_{abc}}{\pi \sqrt{k}} \partial_{x_2} \partial_{\bar{x}_3} \frac{1}{x_{12}} \int \frac{\mathrm{d}^2 z_2}{\bar{z}_2 - \bar{x}_3} \left(\frac{1}{x_2 - z_2} - \frac{1}{x_1 - z_2} \right)$$

(D.4)

$$= \frac{\lambda^2 f_{abc}}{\sqrt{k}} \partial_{x_2} \partial_{\bar{x}_3} \frac{1}{x_{12}} \ln \frac{|x_{23}|^2}{|x_{13}|^2}$$

(D.5)

$$= -\frac{\lambda^2}{\sqrt{k}} f_{abc} \left(\frac{\bar{x}_{12}}{x_{12}^2 \bar{x}_{23} \bar{x}_{13}} + \pi \frac{\delta^{(2)}(x_{23})}{x_{12}} \right) ,$$

where we have included the contact term involving external points. This will be neglected in the main text.

Appendix E. Perturbative computation of the $\langle \Phi \Phi \rangle$ correlator

In this appendix we present the perturbative calculation of the $\langle \Phi \Phi \rangle$ correlator.

Order $\mathcal{O}(\lambda)$. To that order we have that

$$
\langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle^{(1)}_\lambda = -\frac{\lambda}{\pi}\int d^2 z \langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)J^c(z)\bar{J}^c(\bar{z})\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle =
$$

$$
= \frac{\lambda}{\pi\sqrt{k}}\int d^2 z \Big[\frac{(t^{(1)}_a)_i{}^\ell}{z-x_1}\langle \Phi^{(1)}_{\ell,i'}(x_1,\bar{x}_1)\bar{J}^c(\bar{z})\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle
$$

$$
+ \frac{(t^{(2)}_a)_j{}^\ell}{z-x_2}\langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)\bar{J}^c(\bar{z})\Phi^{(2)}_{\ell,j'}(x_2,\bar{x}_2)\rangle \Big] \tag{E.1}
$$

$$
= -\frac{\lambda/k}{x_{12}^{2\Delta_R}\bar{x}_{12}^{2\bar{\Delta}_{R'}}}\Big[\ln\varepsilon^2 \left(t^{(1)}_a \otimes \tilde{t}^{(1)T}_a + t^{(2)T}_a \otimes \tilde{t}^{(2)}_a \right)
$$

$$
+ \ln|x_{12}|^2 \left(t^{(1)}_a \otimes \tilde{t}^{(2)}_a + t^{(1)T}_a \otimes \tilde{t}^{(2)T}_a \right) \Big]_{ii',jj'} ,
$$

where we have used (2.11) and (2.12) and wrote the result as a direct product of matrices. The representations involved are in fact conjugate to each other. Therefore, using (2.8) we find that

$$
\langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle^{(1)}_\lambda = -2\frac{\lambda}{k}\frac{(t_a\otimes t^*_a)_{ii',jj'}}{x_{12}^{2\Delta_R}\bar{x}_{12}^{2\bar{\Delta}_{R'}}}\ln\frac{\varepsilon^2}{|x_{12}|^2} . \tag{E.2}
$$

Order $\mathcal{O}(\lambda^2)$. To that order we find that

$$
\langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1)\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle^{(2)}_\lambda = \frac{\lambda^2}{2\pi^2}\left[\frac{1}{\sqrt{k}}\left(I^{(1)}_{ii',jj'} + I^{(2)}_{ii',jj'} + I^{(3)}_{ii',jj'} \right) + I^{(4)}_{ii',jj'} \right], \tag{E.3}
$$

where the four different terms are computed below and arise by using the current Ward identity with respect to the current $J_a(z_1)$:

- The first term is

$$
I^{(1)}_{ii',jj'} = -\int d^2 z_{12}\frac{(t^{(1)}_a)_i{}^\ell}{z_1-x_1}\langle \Phi^{(1)}_{\ell,i'}(x_1.\bar{x}_1)\bar{J}_a(\bar{z}_1)J_b(z_2)\bar{J}_b(\bar{z}_2)\Phi^{(2)}_{j,j'}(x_2.\bar{x}_2)\rangle
$$

$$
= -\int d^2 z_{12}\frac{(t^{(1)}_a)_i{}^\ell}{(z_1-x_1)\bar{z}_{12}^2}\langle \Phi^{(1)}_{\ell,i'}(x_1,\bar{x}_1)J_a(z_2)\Phi^{(2)}_{j,j'}(x_2,\bar{x}_2)\rangle \tag{E.4}
$$

$$
= \pi^2\frac{C_R}{\sqrt{k}}\frac{(\mathbb{I}_R\otimes\mathbb{I}_{R'})_{ii',jj'}}{x_{12}^{2\Delta_R}\bar{x}_{12}^{2\bar{\Delta}_{R'}}}\ln\frac{\varepsilon^2}{|x_{12}|^2} ,
$$

where we have kept only contributions which will give terms of $\mathcal{O}(1/k)$ to the final result. In addition we used the integral

$$
\int\frac{d^2 z_{12}}{(z_1-x_1)(z_2-x_2)\bar{z}_{12}^2} = \pi\int\frac{d^2 z_2}{(\bar{x}_1-\bar{z}_2)(z_2-x_2)} = \pi^2\ln|x_{12}|^2 , \tag{E.5}
$$

as well as the same with $x_2\to x_1$ in which case $|x_{12}|^2\to\varepsilon^2$ in the result above.

- The second term is

$$
\begin{aligned}
I^{(2)}_{ii',jj'} &= -\int \mathrm{d}^2 z_{12} \frac{(t_a^{(2)})_j{}^\ell}{z_1 - x_2} \langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1) \bar{J}_a(\bar{z}_1) J_b(z_2) \bar{J}_b(\bar{z}_2) \Phi^{(2)}_{\ell,j'}(x_2,\bar{x}_2) \rangle \\
&= -\int \mathrm{d}^2 z_{12} \frac{(t_a^{(2)})_j{}^\ell}{(z_1 - x_2)\bar{z}_{12}^2} \langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1) J_a(z_2) \Phi^{(2)}_{\ell,j'}(x_2,\bar{x}_2) \rangle \\
&= \pi^2 \frac{C_R}{\sqrt{k}} \frac{(\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2} \,,
\end{aligned}
\tag{E.6}
$$

where as before we have kept only contributions providing at most $\mathcal{O}(1/k)$ terms in the final result.

- The third term is

$$
I^{(3)}_{ii',jj'} = f_{abc} \int \frac{\mathrm{d}^2 z_{12}}{z_{12}} \langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1) \bar{J}_a(\bar{z}_1) J_c(z_2) \bar{J}_b(\bar{z}_2) \Phi^{(2)}_{\ell,j'}(x_2,\bar{x}_2) \rangle = 0 \,,
\tag{E.7}
$$

since to $\mathcal{O}(1/\sqrt{k})$ we get a result proportional to $f_{abc}\,\delta_{ab} = 0$.

- The fourth term is more involved to compute. The result is

$$
\begin{aligned}
I^{(4)}_{ii',jj'} &= \int \frac{\mathrm{d}^2 z_{12}}{z_{12}^2} \langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1) \bar{J}_a(\bar{z}_1) \bar{J}_a(\bar{z}_2) \Phi^{(2)}_{\ell,j'}(x_2,\bar{x}_2) \rangle \\
&= 2\pi^2 \frac{C_{R'}}{k} \frac{(\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2} \,.
\end{aligned}
\tag{E.8}
$$

Note that this is expected since it is just the sum of the other two non-vanishing terms with the representations R and R' exchanged.

Order $\mathcal{O}(\lambda^3)$. To that order we have that

$$
\begin{aligned}
&\langle \Phi^{(1)}_{i,i'}(x_1,\bar{x}_1) \Phi^{(2)}_{j,j'}(x_2,\bar{x}_2) \rangle^{(3)}_\lambda \\
&= -\frac{\lambda^3}{6\pi^3} \left[J^{(1)}_{ii',jj'} + J^{(2)}_{ii',jj'} + J^{(3)}_{ii',jj'} + J^{(4)}_{ii',jj'} + J^{(5)}_{ii',jj'} + J^{(6)}_{ii',jj'} \right] \,,
\end{aligned}
\tag{E.9}
$$

where the six different terms are obtained by applying the Ward identity for the current $J_a(z_1)$:

- The first term originates from the contraction of the current $J_a(z_1)$ with the primary field $\Phi^{(1)}$ and leads to

$$
\begin{aligned}
J^{(1)}_{ii',jj'} &= -\frac{1}{\sqrt{k}} \int \mathrm{d}^2 z_{123} \frac{(t_a^{(1)})_i{}^k}{z_1 - x_1} \\
&\quad \langle \Phi^{(1)}_{k,i'}(x_1,\bar{x}_1) \bar{J}_a(\bar{z}_1) J_b(z_2) \bar{J}_b(\bar{z}_2) J_c(z_3) \bar{J}_c(\bar{z}_3) \Phi^{(2)}_{j,j'}(x_2,\bar{x}_2) \rangle.
\end{aligned}
\tag{E.10}
$$

The next step is to contract one of the remaining holomorphic currents, lets say $J_b(z_2)$. This current can not be contracted with any of the external primaries because in that case the last holomorphic current should also be contracted with an external field too and as a result this contribution will be of order $1/k^{3/2}$. Since in this calculation we keep terms of order $\mathcal{O}(1/k)$ this contribution can be ignored. For the same reason the holomorphic currents $J_b(z_2)$ and $J_c(z_3)$ can not be contracted through the non-Abelian part of their OPE but only via the

Abelian part. Once we have contracted all the holomorphic currents we start treating the anti-holomorphic ones by choosing $\bar{J}_a(\bar{z}_1)$ to use in the Ward identity. As above, this current cannot be contracted with any of the external primaries since in this case the remaining anti-holomorphic currents at $\bar{z}_2$ and at $\bar{z}_3$ should be contracted through a δ-Kronecker term and resulting into the term $\frac{1}{\bar{z}_{23}^2 \bar{z}_{23}^2}$ which indicates that it is disconnected and should be ignored. Thus, the current at $\bar{z}_1$ can be contracted only with the anti-holomorphic currents at $\bar{z}_2$ and at $\bar{z}_3$. Notice, however, that this contraction can not be non-Abelian because in that case the result will be proportional to $f_{abc}\delta_{bc} = 0$. We have thus concluded that the only non-vanishing terms up to order $\mathcal{O}(1/k)$ will come from the Abelian contractions of $\bar{J}_a(\bar{z}_1)$ with the other anti-holomorphic currents. The resulting integral is

$$J_{ii',jj'}^{(1)} = -\frac{2}{k} \frac{1}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\Delta_{R'}}} \int d^2 z_{123} \Big(\frac{\big(t_a^{(1)} \otimes \tilde{t}_a^{(1)T}\big)_{ii',jj'}}{z_{23}^2 \bar{z}_{12}^2 (z_1 - x_1)(\bar{z}_3 - \bar{x}_1)}$$
$$+ \frac{\big(t_a^{(1)} \otimes \tilde{t}_a^{(2)}\big)_{ii',jj'}}{z_{23}^2 \bar{z}_{12}^2 (z_1 - x_1)(\bar{z}_3 - \bar{x}_2)} + (z_2 \leftrightarrow z_3) \Big), \tag{E.11}$$

where the z_2, z_3 exchange term applies only in the integrand and not in the measure of integration in accordance with our regularization prescription. It turns out that this term doubles the result of the term explicitly written. The integrals can now be performed from the left to the right, the z_1 integration first then z_2 and the z_3 last. Using (2.8) the result can be written as follows

$$J_{ii',jj'}^{(1)} = 2\frac{\pi^3}{k} \frac{(t_a \otimes t_a^*)_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2}. \tag{E.12}$$

We should mention that the $\ln \varepsilon^2$ term originates from the first triple integral of (E.11) while the $\ln |x_{12}|^2$ term originates the second triple integral of (E.11).

• The second term originates from the contraction of the current $J_a(z_1)$ with $J_b(z_2)$ through the non-Abelian term of their OPE. It reads

$$J_{ii',jj'}^{(2)} = \frac{f_{abd}}{\sqrt{k}} \int \frac{d^2 z_{123}}{z_{12}} \langle \Phi_{i,i'}^{(1)}(x_1, \bar{x}_1) \bar{J}_a(\bar{z}_1) J_d(z_2) \bar{J}_b(\bar{z}_2) J_c(z_3) \bar{J}_c(\bar{z}_3) \Phi_{j,j'}^{(2)}(x_2, \bar{x}_2) \rangle. \tag{E.13}$$

Since we want to keep terms up to $\mathcal{O}(1/k)$ the holomorphic currents at points z_2 and z_3 must be contracted only through the Abelian term of their OPE. The resulting correlators will involve the two primary fields and the three anti-holomorphic currents. Next we employ the Ward identity for the current at the point $\bar{z}_1$. This current can not be contracted with the other anti-holomorphic currents through a δ-Kronecker term because in such a case this term will be proportional either to $f_{abc}\delta_{ab} = 0$ or to $f_{abc}\delta_{ac} = 0$. Also $\bar{J}_a(\bar{z}_1)$ can not be contracted with the external primary fields because in such a case the corresponding diagram will disconnected, thus it will be the product of a bubble involving the points z_2 and z_3 times the rest of the diagram. Consequently, the only contribution that remains comes from the non-Abelian contraction of either $\bar{J}_a(\bar{z}_1)$ with either $\bar{J}_b(\bar{z}_2)$ or $\bar{J}_c(\bar{z}_3)$. In both cases the resulting diagrams will be disconnected, i.e. they will be the product of the tree-level $\langle \Phi\Phi \rangle$ correlator times a bubble involving all interactions points z_i, $i = 1, 2, 3$. Therefore, we conclude that

$$J_{ii',jj'}^{(2)} = 0. \tag{E.14}$$

- The third term originates from the contraction of the current $J_a(z_1)$ with $J_b(z_2)$ through the Abelian term of their OPE

$$J_{ii',jj'}^{(3)} = \int \frac{d^2 z_{123}}{z_{12}^2} \langle \Phi_{i,i'}^{(1)}(x_1, \bar{x}_1) \bar{J}_a(\bar{z}_1) \bar{J}_a(\bar{z}_2) J_c(z_3) \bar{J}_c(\bar{z}_3) \Phi_{j,j'}^{(2)}(x_2, \bar{x}_2) \rangle . \tag{E.15}$$

The last holomorphic current, i.e. the one at point z_3 should necessarily be contracted with each of the external primaries giving a factor of $1/\sqrt{k}$ and leaving us with a sum of two correlators involving two primaries and the three anti-holomorphic currents. Choosing the current at z_1 to be the one for which we will apply the Ward identity we obtain the following terms:

i) the term when $\bar{J}_a(\bar{z}_1)$ is contracted with $\bar{J}_a(\bar{z}_2)$. This diagram will have a factor of $\frac{1}{z_{12}^2 \bar{z}_{12}^2}$ indicating that it is disconnected and should, thus be ignored.

ii) the term arising from the contraction of $\bar{J}_a(\bar{z}_1)$ with $\bar{J}_c(\bar{z}_3)$ through the non-Abelian term of their OPE will also be zero since we have saturated the powers of $1/k$ and all remaining contractions should be Abelian resulting to the factor of $f_{acd}\delta_{ad} = 0$.

iii) the term arising form the contraction of $\bar{J}_a(\bar{z}_1)$ with $\bar{J}_c(\bar{z}_3)$ through the Abelian term of their OPE contributes that

$$-\frac{1}{k} \frac{1}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \int \frac{d^2 z_{123}}{z_{12}^2 \bar{z}_{13}^2} \Bigg(\frac{(t_a^{(1)} \otimes \tilde{t}_a^{(1)T})_{ii',jj'}}{(z_3 - x_1)(\bar{z}_2 - \bar{x}_1)} + \frac{(t_a^{(1)} \otimes \tilde{t}_a^{(2)})_{ii',jj'}}{(z_3 - x_1)(\bar{z}_2 - \bar{x}_2)} +$$

$$+ \frac{(t_a^{(2)T} \otimes \tilde{t}_a^{(1)T})_{ii',jj'}}{(z_3 - x_2)(\bar{z}_2 - \bar{x}_1)} + \frac{(t_a^{(2)T} \otimes \tilde{t}_a^{(2)})_{ii',jj'}}{(z_3 - x_2)(\bar{z}_2 - \bar{x}_2)} \Bigg) = 2\frac{\pi^3}{k} \frac{(t_a \otimes t_a^*)_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2} . \tag{E.16}$$

Notice that as always we keep the order of integrations. Furthermore, the first integral over z_1 gives a $\delta^{(2)}(z_2 - z_3)$ which make the second integration over z_2 trivial. The third integral over z_3 is one of our basic ubiquitous ones.

iv) the last contribution arises when $\bar{J}_a(\bar{z}_1)$ is contracted with the external primaries. The corresponding integrals are

$$-\frac{1}{k} \frac{1}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \int \frac{d^2 z_{123}}{z_{12}^2 \bar{z}_{23}^2} \Bigg(\frac{(t_a^{(1)} \otimes \tilde{t}_a^{(1)T})_{ii',jj'}}{(z_3 - x_1)(\bar{z}_1 - \bar{x}_1)} + \frac{(t_a^{(1)} \otimes \tilde{t}_a^{(2)})_{ii',jj'}}{(z_3 - x_1)(\bar{z}_1 - \bar{x}_2)} +$$

$$+ \frac{(t_a^{(2)T} \otimes \tilde{t}_a^{(1)T})_{ii',jj'}}{(z_3 - x_2)(\bar{z}_1 - \bar{x}_1)} + \frac{(t_a^{(2)T} \otimes \tilde{t}_a^{(2)})_{ii',jj'}}{(z_3 - x_2)(\bar{z}_1 - \bar{x}_2)} \Bigg) = 2\frac{\pi^3}{k} \frac{(t_a \otimes t_a^*)_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2} . \tag{E.17}$$

Adding the contributions from (E.16) and (E.17) we get for $J_{ii',jj'}^{(3)}$ that

$$J_{ii',jj'}^{(3)} = 4\frac{\pi^3}{k} \frac{(t_a \otimes t_a^*)_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2} . \tag{E.18}$$

- The fourth term originates from the contraction of the current $J_a(z_1)$ with $J_c(z_3)$ through the non-Abelian term of their OPE. It reads

$$J^{(4)}_{ii',jj'} = \frac{f_{acd}}{\sqrt{k}} \int d^2 z_{123} \langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) \bar{J}_a(\bar{z}_1) J_b(z_2) \bar{J}_b(\bar{z}_2) J_d(z_3) \bar{J}_c(\bar{z}_3) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle. \tag{E.19}$$

Following the same steps as in the second contribution above one can show that

$$J^{(4)}_{ii',jj'} = 0. \tag{E.20}$$

- The fifth term originates from the contraction of the current $J_a(z_1)$ with $J_c(z_3)$ through the Abelian term of their OPE

$$J^{(5)}_{ii',jj'} = \int \frac{d^2 z_{123}}{z_{13}^2} \langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) \bar{J}_a(\bar{z}_1) J_b(z_2) \bar{J}_b(\bar{z}_2) \bar{J}_a(\bar{z}_3) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle. \tag{E.21}$$

Working as in the case of the third contribution we get that

$$J^{(5)}_{ii',jj'} = 4 \frac{\pi^3}{k} \frac{(t_a \otimes t_a^*)_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2}. \tag{E.22}$$

- Finally, the last term originates from the contraction of the current $J_a(z_1)$ with the primary field $\Phi^{(2)}$

$$\begin{aligned} J^{(6)}_{ii',jj'} = -\frac{1}{\sqrt{k}} \int d^2 z_{123} \frac{(t_a^{(2)})_j{}^k}{z - x_2} \\ \times \langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) \bar{J}_a(\bar{z}_1) J_b(z_2) \bar{J}_b(\bar{z}_2) J_c(z_3) \bar{J}_c(\bar{z}_3) \Phi^{(2)}_{k,j'}(x_2, \bar{x}_2) \rangle. \end{aligned} \tag{E.23}$$

Following the same steps as in the first contribution one can show that

$$J^{(6)}_{ii',jj'} = 2 \frac{\pi^3}{k} \frac{(t_a \otimes t_a^*)_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2}. \tag{E.24}$$

Summing up all six terms one obtains the final result at three-loops. It reads

$$\langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle^{(3)}_\lambda = -2 \frac{\lambda^3}{k} \frac{(t_a \otimes t_a^*)_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2}. \tag{E.25}$$

Appendix F. Perturbative computation of the $\langle J\Phi\Phi \rangle$ correlator

Finally, in this last appendix, we present the perturbative calculation of the $\langle J\Phi\Phi \rangle$ three-point correlator.

Order $\mathcal{O}(\lambda)$. This contribution is equal to

$$\begin{aligned} \langle \Phi J \Phi \rangle^{(1)} &= -\frac{\lambda}{\pi} \int d^2 z \, \langle \Phi^{(1)}_{i,i'}(x_1) J_a(x_3) J_{a_1}(z) \bar{J}_{a_1}(\bar{z}) \Phi^{(2)}_{j,j'}(x_2) \rangle \\ &= -\frac{\lambda}{\pi} \int d^2 z \, \frac{\langle \Phi^{(1)}_{i,i'}(x_1) \bar{J}_a(\bar{z}) \Phi^{(2)}_{j,j'}(x_2) \rangle}{(x_3 - z)^2} \\ &= -\frac{\lambda (\mathbb{I}_{R'} \otimes \tilde{t}_a^*)_{ii',jj'}}{\sqrt{k} \, x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \left(\frac{1}{x_{13}} - \frac{1}{x_{23}} \right), \end{aligned} \tag{F.1}$$

where we have used (2.12).

Order $\mathcal{O}(\lambda^2)$. This contribution is equal to

$$
\begin{aligned}
\langle \Phi J \Phi \rangle^{(2)} &= \frac{\lambda^2}{2! \pi^2} \int d^2 z_{12} \, \langle \Phi^{(1)}_{i,i'}(x_1) J_a(x_3) J_{a_1}(z_1) J_{a_2}(z_2) \bar{J}_{a_1}(\bar{z}_1) \bar{J}_{a_2}(\bar{z}_2) \Phi^{(2)}_{j,j'}(x_2) \rangle \\
&= \frac{\lambda^2}{2\pi^2} \int d^2 z_{12} \left(\frac{\langle \Phi^{(1)}_{i,i'}(x_1) J_a(z_1) \Phi^{(2)}_{j,j'}(x_2) \rangle}{(x_3 - z_2)^2 \bar{z}_{12}^2} + \frac{\langle \Phi^{(1)}_{i,i'}(x_1) J_a(z_2) \Phi^{(2)}_{j,j'}(x_2) \rangle}{(x_3 - z_1)^2 \bar{z}_{12}^2} \right) \\
&= \frac{\lambda^2 (t_a \otimes \mathbb{I}_R)_{ii',jj'}}{2\sqrt{k} \, x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \left(\frac{1}{x_{13}} - \frac{1}{x_{23}} \right),
\end{aligned}
\tag{F.2}
$$

where we have used (2.11) disregarding bubble diagrams. Notice that the second term in the second line of (F.2) vanish, since the z_1 integration will give a $\delta^{(2)}(x_3 - z_2)$ which is set to zero in our regularization scheme. Furthermore, notice that the order of integration is important. Had we changed this order the result of the vanishing term would have been non-zero doubling the contribution of the first term in the second line of (F.2).

Order $\mathcal{O}(\lambda^3)$. This contribution is equal to

$$
\begin{aligned}
\langle \Phi J \Phi \rangle^{(3)} &= \\
&- \frac{\lambda^3}{3! \pi^3} \int d^2 z_{123} \, \langle \Phi^{(1)}_{i,i'}(x_1) J_a(x_3) J_{a_1}(z_1) J_{a_2}(z_2) J_{a_2}(z_3) \bar{J}_{a_1}(\bar{z}_1) \bar{J}_{a_2}(\bar{z}_2) \bar{J}_{a_3}(\bar{z}_3) \Phi^{(2)}_{j,j'}(x_2) \rangle \\
&= - \frac{\lambda^3}{2\pi^3} \int d^2 z_{123} \frac{\langle \Phi^{(1)}_{i,i'}(x_1) \bar{J}_a(\bar{z}_3) \Phi^{(2)}_{j,j'}(x_2) \rangle}{(x_3 - z_3)^2 z_{12}^2 \bar{z}_{13}^2} = - \frac{\lambda^3 (\mathbb{I}_{R'} \otimes \tilde{t}_a^*)_{ii',jj'}}{2\sqrt{k} \, x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \left(\frac{1}{x_{13}} - \frac{1}{x_{23}} \right),
\end{aligned}
\tag{F.3}
$$

where we have used (2.12).

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