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Status of Chiral-Scale Perturbation Theory

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Chiral-scale perturbation theory $\chi\mathrm{PT}_\sigma$ has been proposed as an alternative to chiral $SU(3)_L \times SU(3)_R$ perturbation theory which explains the $\Delta I = 1/2$ rule for kaon decays. It is based on a low-energy expansion about an infrared fixed point in three-flavor QCD. In $\chi\mathrm{PT}_\sigma$, quark condensation $\langle \bar{q}q \rangle_{\mathrm{vac}} \neq 0$ induces nine Nambu-Goldstone bosons: π, K, η and a QCD dilaton σ which we identify with the $f_0(500)$ resonance. Partial conservation of the dilatation and chiral currents constrains low-energy constants which enter the effective Lagrangian of $\chi\mathrm{PT}_\sigma$. These constraints allow us to obtain new phenomenological bounds on the dilaton decay constant via the coupling of σ/f_0 to pions, whose value is known precisely from dispersive analyses of $\pi\pi$ scattering. Improved predictions for $\sigma \to \gamma\gamma$ and the σNN coupling are also noted. To test $\chi\mathrm{PT}_\sigma$ for kaon decays, we revive a 1985 proposal for lattice methods to be applied to $K \to \pi$ *on-shell*.

The 8th International Workshop on Chiral Dynamics, CD2015
29 June 2015 - 03 July 2015
Pisa, Italy

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PoS(CD15)132

1. Approximate Scale Invariance in Low-Energy QCD

In the low-energy regime of QCD with heavy quarks t,b,c decoupled, the relevance of scale (dilatation) invariance is determined by the trace anomaly [1]–[4] of the resulting 3-flavor theory:[1]

$$\theta^\mu_\mu = \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu} G^{a\mu\nu} + \big(1+\gamma_m(\alpha_s)\big) \sum_{q=u,d,s} m_q \bar{q} q \,. \tag{1.1}$$

Depending on the infrared behaviour of β, there are only two realistic scenarios (Fig. 1 (A)):

1. If β remains negative and non-zero, possibly diverging linearly at large α_s, scale invariance is explicitly broken by θ^μ_μ being large *as an operator*. There is *no hint* of approximate scale invariance: quantities such as the nucleon mass $M_N = \langle N|\theta^\mu_\mu|N\rangle$ are generated almost entirely by the gluonic term in (1.1). Then conventional chiral $SU(3)_L \times SU(3)_R$ perturbation theory χPT_3 is the appropriate low-energy effective theory for QCD amplitudes expanded in powers of $O(m_K)$ external momenta and light quark masses $m_{u,d,s} = O(m_K^2)$.

2. If β vanishes when α_s runs non-perturbatively to an infrared fixed point α_{IR}, the gluonic term $\sim G^a_{\mu\nu} G^{a\mu\nu}$ in (1.1) is absent and the dilatation current $D_\mu = x^\nu \theta_{\mu\nu}$ becomes conserved in the limit of vanishing quark masses:

$$\partial^\mu D_\mu\big|_{\alpha_s=\alpha_{\text{IR}}} = \theta^\mu_\mu\big|_{\alpha_s=\alpha_{\text{IR}}} = \big(1+\gamma_m(\alpha_{\text{IR}})\big) \sum_{q=u,d,s} m_q \bar{q} q$$
$$\to 0\,,\ SU(3)_L \times SU(3)_R \text{ limit}\,. \tag{1.2}$$

   Although the Hamiltonian preserves dilatations in this limit, *the vacuum state is not scale invariant* due to the formation of a quark condensate $\langle \bar{q}q\rangle_{\text{vac}} \neq 0$. As a result, both chiral $SU(3)_L \times SU(3)_R$ and scale symmetry are realized in the Nambu-Goldstone (NG) mode and the spectrum contains nine massless bosons: π, K, η and a 0^{++} QCD dilaton σ. Non-NG bosons remain massive *despite the vanishing of* θ^μ_μ and have their scale set by $\langle \bar{q}q\rangle_{\text{vac}}$. The relevant low-energy expansion involves a combined limit

$$m_{u,d,s} \sim 0 \quad \text{and} \quad \alpha_s \lesssim \alpha_{\text{IR}}\,, \tag{1.3}$$

   and leads to a new effective theory χPT_σ of approximate chiral-scale symmetry [5, 6]. In this scenario, the dilaton mass is set by m_s, so the natural candidate for σ is the $f_0(500)$ resonance, a broad 0^{++} state whose complex pole mass has real part $\lesssim m_K$ [7, 8, 9].

Until now, scenario 1 has been the generally accepted view, but we have observed [5, 6] that χPT_σ offers several advantages over χPT_3: it explains the mass and width of $f_0(500)$, produces convergent chiral expansions as a result of σ/f_0 being promoted to the NG sector, and most importantly, explains the $\Delta I = 1/2$ rule for non-leptonic K decays (Fig. 1 (B)).

Because approximate scale symmetry is included, the effective Lagrangian for χPT_σ (Sec. 2) contains several new low-energy constants (LECs) yet to be determined precisely from data. Of particular interest is the dilaton decay constant F_σ given by $m_\sigma^2 F_\sigma = -\langle\sigma|\theta^\mu_\mu|\text{vac}\rangle$. If F_σ is roughly 100

[1] Here, $G^a_{\mu\nu}$ is the gluon field strength, $\alpha_s = g_s^2/4\pi$ is the strong running coupling, and $\beta = \mu^2\partial\alpha_s/\partial\mu^2$ and $\gamma_m = \mu^2\partial \ln m_q/\partial\mu^2$ refer to a mass-independent renormalization scheme with scale μ.

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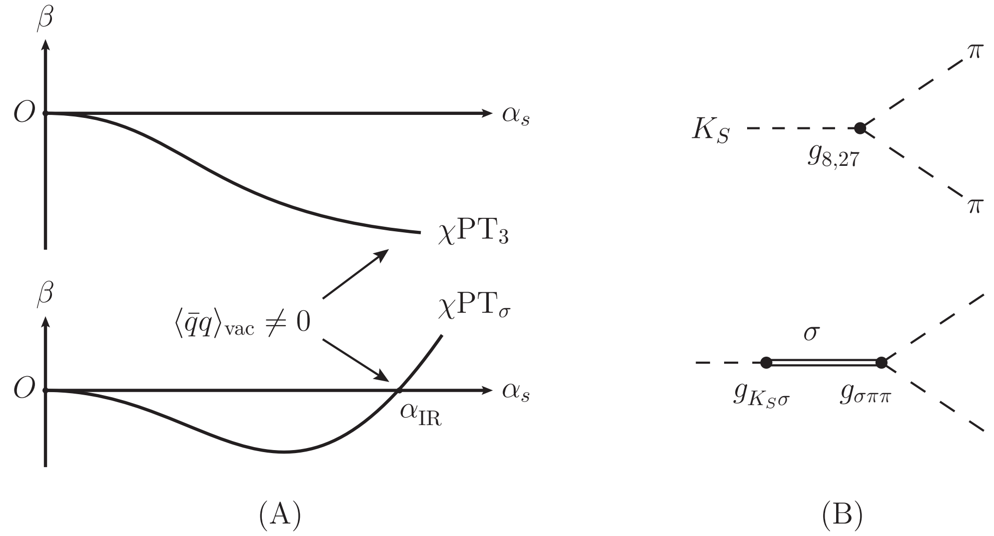

Figure 1: (A) Scenarios for the β function in three-flavor QCD, with corresponding low-energy expansions. In the absence of an infrared fixed point $\alpha_{\rm IR}$ (top diagram), there is no approximate scale invariance and chiral $SU(3)_L \times SU(3)_R$ perturbation theory χPT$_3$ is relevant at low-energies. If $\alpha_{\rm IR}$ exists (bottom diagram), quark condensation $\langle \bar{q}q \rangle_{\rm vac} \neq 0$ implies that the NG spectrum contains a QCD dilaton σ, and χPT$_3$ *must* be replaced by chiral-scale perturbation theory χPT$_\sigma$. (B) Diagrams for $K \to \pi\pi$ decay in lowest-order χPT$_\sigma$. The dilaton pole diagram is responsible for the dominant $\Delta I = 1/2$ amplitude.

MeV, scale breaking by the vacuum can generate large masses such as $m_N \approx F_\sigma g_{\sigma NN}$ (Goldberger-Treiman relation for dilatons [10]) for m_σ small. The imprecise value of F_σ in our previous work [5, 6] arose from large uncertainties in the phenomenological value of $g_{\sigma NN}$ [11, 12].

We circumvent this difficulty in Secs. 3 and 4. First, we find new constraints on LECs in the χPT$_\sigma$ effective Lagrangian by requiring full consistency with the dilatation and chiral currents being conserved in the limit (1.2). These constraints allow us to determine F_σ from the $\sigma\pi\pi$ coupling, whose value is known to remarkable precision from dispersive analyses [7, 8, 9] of $\pi\pi$ scattering. Then we obtain improved predictions for the non-perturbative Drell-Yan ratio

$$R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-) \quad \text{at } \alpha_{\rm IR}\,, \tag{1.4}$$

as well as the σNN coupling.

In Sec. 5, we resurrect an old proposal [13] to apply lattice QCD for $K \to \pi$ *on-shell* to determine the couplings $g_{8,27}$ in Fig. 1 (B). Comments on the validity of χPT$_\sigma$ are reviewed in Sec. 6.

2. Chiral-Scale Lagrangian

For strong interactions, the most general effective Lagrangian of χPT$_\sigma$ is of the form

$$\mathscr{L}_{\chi{\rm PT}_\sigma} = :\mathscr{L}^{d=4}_{\rm inv} + \mathscr{L}^{d>4}_{\rm anom} + \mathscr{L}^{d<4}_{\rm mass}:\,, \tag{2.1}$$

where

$$d_{\rm anom} = 4 + \gamma_{G^2}(\alpha_s) \qquad \text{and} \qquad d_{\rm mass} = 3 - \gamma_m(\alpha_s) \tag{2.2}$$

are the respective scaling dimensions of $G^a_{\mu\nu}G^{a\mu\nu}$ and $\bar{q}q$. In lowest order (LO) of the chiral-scale expansion, we have $\gamma_m = \gamma_m(\alpha_{\rm IR})$ and

$$\gamma_{G^2}(\alpha_s) \equiv \beta'(\alpha_s) - \beta(\alpha_s)/\alpha_s = \beta'(\alpha_{\rm IR}) + O(\alpha_s - \alpha_{\rm IR})\,, \tag{2.3}$$

so the resulting terms in (2.1) are

$$\begin{aligned}
\mathscr{L}^{d=4}_{\rm inv,LO} &= \{c_1\mathscr{K} + c_2\mathscr{K}_\sigma + c_3 e^{2\sigma/F_\sigma}\}e^{2\sigma/F_\sigma}\,, \\
\mathscr{L}^{d>4}_{\rm anom,LO} &= \{(1-c_1)\mathscr{K} + (1-c_2)\mathscr{K}_\sigma + c_4 e^{2\sigma/F_\sigma}\}e^{(2+\beta')\sigma/F_\sigma}\,, \\
\mathscr{L}^{d<4}_{\rm mass,LO} &= \mathrm{Tr}(MU^\dagger + UM^\dagger)e^{(3-\gamma_m)\sigma/F_\sigma}\,,
\end{aligned} \tag{2.4}$$

where

$$\mathscr{K} = \tfrac{1}{4}F_\pi^2\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger) \qquad \text{and} \qquad \mathscr{K}_\sigma = \tfrac{1}{2}(\partial_\mu\sigma)^2\,. \tag{2.5}$$

As $\alpha_s \to \alpha_{\rm IR}$, the gluonic anomaly vanishes, so $\mathscr{L}_{\rm anom} = O(\partial^2, M)$ and we must set $c_4 = O(M)$. Vacuum stability in the σ direction about $\sigma = 0$ (no tadpoles) implies

$$\begin{aligned}
4c_3 + (4+\beta')c_4 &= -(3-\gamma_m)\big\langle\mathrm{Tr}(MU^\dagger + UM^\dagger)\big\rangle_{\rm vac} \\
&= -(3-\gamma_m)F_\pi^2\big(m_K^2 + \tfrac{1}{2}m_\pi^2\big)\,,
\end{aligned} \tag{2.6}$$

so c_3 is also $O(M)$. Expanding (2.4) about $\sigma = 0$ and $U = I$ yields the $\sigma\pi\pi$ coupling

$$\mathscr{L}_{\sigma\pi\pi} = \big\{\big[2 + (1-c_1)\beta'\big]|\partial\boldsymbol{\pi}|^2 - (3-\gamma_m)m_\pi^2|\boldsymbol{\pi}|^2\big\}\sigma/(2F_\sigma)\,, \tag{2.7}$$

while the corresponding $\sigma\pi\pi$ vertex for an on-shell dilaton is

$$g_{\sigma\pi\pi} = -\frac{1}{2F_\sigma}\Big\{\big[2 + (1-c_1)\beta'\big]m_\sigma^2 + 2\big[1 - \gamma_m - (1-c_1)\beta'\big]m_\pi^2\Big\}\,. \tag{2.8}$$

3. Effective Energy-Momentum Tensor and its Trace

In any field theory, the energy-momentum tensor can be identified by adding a gravitational source field $g_{\mu\nu}(x)$ coupled to matter fields in a generally covariant fashion. In χPT$_\sigma$, this amounts to the substitution

$$\mathscr{L}_{\chi\mathrm{PT}_\sigma}[U, U^\dagger, \sigma] \to \mathscr{L}_{\chi\mathrm{PT}_\sigma}[U, U^\dagger, \sigma, g_{\mu\nu}]\,, \tag{3.1}$$

where the new effective Lagrangian must be constructed in terms of generally covariant operators. Then the energy-momentum tensor is defined via the variation

$$\theta_{\mu\nu}(x) = 2\left[\frac{\delta}{\delta g^{\mu\nu}(x)}\sqrt{-g}\mathscr{L}[U, U^\dagger, \sigma, g_{\mu\nu}]\right]_{g_{\mu\nu}=\eta_{\mu\nu}}\,, \tag{3.2}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor and $\eta_{\mu\nu}$ is the flat Minkowski metric. Generalising Donoghue and Leutwyler [14], we obtain the lowest order result

$$\begin{aligned}
\theta_{\mu\nu} = &\big[\tfrac{1}{2}F_\pi^2\mathrm{Tr}\big(\partial_\mu U\partial_\nu U^\dagger\big) - g_{\mu\nu}\mathscr{K}\big]\big[c_1 e^{2\sigma/F_\sigma} + (1-c_1)e^{(2+\beta')\sigma/F_\sigma}\big] \\
&+ \big(\partial_\mu\sigma\partial_\nu\sigma - g_{\mu\nu}\mathscr{K}_\sigma\big)\big[c_2 e^{2\sigma/F_\sigma} + (1-c_2)e^{(2+\beta')\sigma/F_\sigma}\big] \\
&- g_{\mu\nu}\mathrm{Tr}\big(MU^\dagger + UM^\dagger\big)e^{(3-\gamma_m)\sigma/F_\sigma} - g_{\mu\nu}e^{4\sigma/F_\sigma}\big(c_3 + c_4 e^{\beta'\sigma/F_\sigma}\big)\,.
\end{aligned} \tag{3.3}$$

The trace of (3.3) involves *scale invariant* operators like $\mathrm{Tr}\big(\partial_\mu U \partial^\mu U^\dagger\big)e^{2\sigma/F_\sigma}$ which obscure the connection between the scale invariance and a conserved dilatation current D_μ. To remedy this, we "improve" $\theta_{\mu\nu}$ [15] by adding a term

$$I_{\mu\nu} = \frac{F_\sigma^2}{6}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\left[c_2 e^{2\sigma/F_\sigma} + \frac{2(1-c_2)}{2+\beta'}e^{(2+\beta')\sigma/F_\sigma}\right], \tag{3.4}$$

such that the trace of

$$\theta_{\mu\nu}\big|_{\mathrm{eff}} = \theta_{\mu\nu} + I_{\mu\nu}, \tag{3.5}$$

is given entirely in terms of explicit scale-breaking operators $\mathscr{L}_d$ of scale dimension d:

$$\partial^\mu D_\mu|_{\mathrm{eff}} = \theta^\mu_\mu\big|_{\mathrm{eff}} = \sum_d (d-4)\mathscr{L}_d\,. \tag{3.6}$$

Explicitly, the improved trace is

$$\begin{aligned}\theta^\mu_\mu\big|_{\mathrm{eff}} &= \beta'\mathscr{L}^{d>4}_{\mathrm{anom}} - (1+\gamma_m)\mathscr{L}^{d<4}_{\mathrm{mass}} \\ &= \beta'\big\{(1-c_1)\mathscr{K} + (1-c_2)\mathscr{K}_\sigma + c_4 e^{2\sigma/F_\sigma}\big\}e^{(2+\beta')\sigma/F_\sigma} \\ &\quad -(1+\gamma_m)\mathrm{Tr}(MU^\dagger + UM^\dagger)e^{(3-\gamma_m)\sigma/F_\sigma}\,. \end{aligned}\tag{3.7}$$

It vanishes in the chiral-scale limit (1.2) only if the low-energy constants associated with $d > 4$ operators satisfy

$$c_1 = c_2 = 1, \qquad \text{for } m_{u,d,s} \to 0 \text{ and } \alpha_s \to \alpha_{\mathrm{IR}}, \tag{3.8}$$

in addition to the condition $c_4 = O(M)$ required by tadpole cancellation (2.6). Note that the condition $c_1 \to 1$ in (3.8) ensures that chiral currents have vanishing anomalous dimensions. We can summarise these LO conditions by writing

$$c_i = 1 + O(M), \qquad i = 1,2, \tag{3.9}$$

where the $O(M)$ term is a linear superposition of $O(p^2, M)$ operators and associated LECs.

4. Improved Predictions

An immediate consequence of the constraint (3.9) is that the $\sigma\pi\pi$ coupling for an on-shell dilaton (2.8) takes a particularly simple form

$$g_{\sigma\pi\pi} = -\frac{1}{F_\sigma}\big[m_\sigma^2 + (1-\gamma_m)m_\pi^2\big], \qquad \text{where } -1 \le 1-\gamma_m < 2\,. \tag{4.1}$$

Since the narrow-width approximation is valid in lowest order $\chi\mathrm{PT}_\sigma$ [6], we have

$$\Gamma_{\sigma\pi\pi} = \frac{|g_{\sigma\pi\pi}|^2}{16\pi m_\sigma}\sqrt{1 - 4m_\pi^2/m_\sigma^2}\,, \tag{4.2}$$

and this allows us to obtain bounds on F_σ from dispersive analyses of $\pi\pi$ scattering based on the Roy equations. For example, the f_0/σ's mass and width from [7]

$$m_\sigma = 441^{+16}_{-8}\ \mathrm{MeV}, \qquad \Gamma_{\sigma\pi\pi} = 544^{+18}_{-25}\ \mathrm{MeV}, \tag{4.3}$$

constrain F_σ to lie within the interval 44 MeV $\leq F_\sigma \leq$ 61 MeV, where we have allowed $1-\gamma_m$ to vary according to (4.1). For the moment, we assume that NLO corrections are not a problem.

With F_σ fixed in this manner, we can now use the Golberger-Treiman relation for dilatons [10] to *predict* the value for the σNN coupling. We find $16 \leq g_{\sigma NN} \leq 21$, which is somewhat larger than previous phenomenological determinations [11, 12]. Another important application concerns $\sigma \to \gamma\gamma$, where an analysis [5, 6] of the electromagnetic trace anomaly in $\chi\mathrm{PT}_\sigma$ relates the $\sigma\gamma\gamma$ coupling to (1.4):

$$g_{\sigma\gamma\gamma} = \frac{2\alpha}{3\pi F_\sigma}\Big(R_{\mathrm{IR}} - \tfrac{1}{2}\Big)\,. \tag{4.4}$$

By fixing $g_{\sigma\gamma\gamma}$ from the di-photon width $\Gamma_{\sigma\gamma\gamma} = 2.0 \pm 0.2$ keV [16], we find $2.4 \leq R_{\mathrm{IR}} \leq 3.1$, which is to be compared with our previous estimate $R_{\mathrm{IR}} \approx 5$ [5, 6].

5. Proposal to test $K \to \pi$ on the Lattice

The key idea [13] is to keep both K and π on shell and allow $O(m_K)$ momentum transfers.

The lowest-order diagrams for the decay $K \to \pi\pi$ in Fig. 1 (B) are derived from an effective weak $\chi\mathrm{PT}_\sigma$ Lagrangian [5, 6]

$$\mathscr{L}_{\mathrm{weak}} = Q_8 \sum_n g_{8n} e^{(2-\gamma_{8n})\sigma/F_\sigma} + g_{27} Q_{27} e^{(2-\gamma_{27})\sigma/F_\sigma} + Q_{mw} e^{(3-\gamma_{mw})\sigma/F_\sigma} + \mathrm{h.c.} \tag{5.1}$$

which reduces to the standard $\chi\mathrm{PT}_3$ Lagrangian

$$\mathscr{L}_{\mathrm{weak}}\big|_{\sigma=0} = g_8 Q_8 + g_{27} Q_{27} + Q_{mw} + \mathrm{h.c.} \tag{5.2}$$

in the limit $\sigma \to 0$. Eqs. (5.1) and (5.2) contain an octet operator [17]

$$Q_8 = J^\mu_{13} J_{\mu 21} - J^\mu_{23} J_{\mu 11}\ , \quad J^\mu_{ij} = (U\partial^\mu U^\dagger)_{ij} \tag{5.3}$$

the U-spin triplet component [13, 18] of a **27** operator

$$Q_{27} = J^\mu_{13} J_{\mu 21} + \tfrac{3}{2} J^\mu_{23} J_{\mu 11} \tag{5.4}$$

and a weak mass operator [19]

$$Q_{mw} = \mathrm{Tr}(\lambda_6 - i\lambda_7)\big(g_M M U^\dagger + \bar{g}_M U M^\dagger\big)\,. \tag{5.5}$$

Powers of e^{σ/F_σ} are used to adjust the operator dimensions of Q_8, Q_{27}, and Q_{mw} in (5.1), with octet quark-gluon operators allowed to have differing dimensions at α_{IR}.

In 1985, it was observed [13] that the isospin-$\frac{1}{2}$ term Q_{mw} in Eq. (5.2), when combined with the strong mass term, would be removed by vacuum realignment and therefore could not help solve the $\Delta I = 1/2$ puzzle. In $\chi\mathrm{PT}_\sigma$, the outcome is different [5, 6] due to the σ dependence of the Q_{mw} term in Eq. (5.1). Provided there is a mismatch between the weak mass operator's dimension $(3-\gamma_{mw})$ and the dimension $(3-\gamma_m)$ of $\mathscr{L}_{\mathrm{mass}}$, the σ dependence of $Q_{mw} e^{(3-\gamma_{mw})/F_\sigma}$ cannot be eliminated by a chiral rotation. As a result, there is a residual interaction $\mathscr{L}_{K_S\sigma} = g_{K_S\sigma} K_S \sigma$ which mixes K_S and σ in *lowest* $O(p^2)$ order[2]

$$g_{K_S\sigma} = (\gamma_m - \gamma_{mw})\mathrm{Re}\{(2m_K^2 - m_\pi^2)\bar{g}_M - m_\pi^2 g_M\} F_\pi / F_\sigma \tag{5.6}$$

[2]We have corrected a factor of 2 in the formula for the $K_S\sigma$ coupling in our original papers [5, 6].

and produces the $\Delta I = 1/2$ σ-pole amplitude of Fig. 1 (B).

The χPT$_3$ analysis of 1985 [13] included a suggestion that kaon decays be tested by applying lattice QCD to the weak process $K \to \pi$, with *both* K and π on shell. It was made at a time when low-lying scalar resonances ($\varepsilon(700)$ before 1974, $f_0(500)$ since 1996) were thought not to exist.

This proposal now needs to be taken seriously because:

- Lattice calculations are much easier with only two particles on shell instead of the three in $K \to \pi\pi$ (all on shell) being analysed by the RBC/UKQCD collaborations [20, 21].

- The 1985 analysis is easily extended to χPT$_\sigma$ by including σ/f_0 pole amplitudes in chiral Ward identities connecting on-shell $K \to \pi\pi$ to $K \to \pi$ on shell. The no-tadpoles theorem

$$\langle K|\mathscr{H}_{\text{weak}}|\text{vac}\rangle = O\big(m_s^2 - m_d^2\big)\,,\ K \text{ on shell}\,, \tag{5.7}$$

  remains valid.

- The lattice result for $K \to \pi\pi$ on-shell will not distinguish $\Delta I = 1/2$ contributions from the g_8 contact diagram and the σ/f_0 pole diagram in Fig. 1 (B). A lattice calculation of $K \to \pi$ on shell would measure g_8 (and g_{27}) directly, with no interference from σ/f_0 poles. Then we would *finally* learn whether g_8 is unnaturally large or not.

A key feature of the proposal is that the operator in the on-shell amplitude $\langle\pi|[F_5,\mathscr{H}_{\text{weak}}]|K\rangle$ necessarily carries *non-zero* momentum $q^\mu = O(m_K)$. For either χPT$_\sigma$ or χPT$_3$, the $K \to \pi$ amplitude can be evaluated in the range

$$-m_K^2 \lesssim q^2 \leqslant \big(m_K - m_\pi\big)^2\,. \tag{5.8}$$

We highlight the point $q^\mu \neq 0$ because since 1985, there has been a widespread misconception in the literature[3] that the analysis [13] involved setting $q^\mu = 0$ as in [19], with the pion in $K \to \pi$ sent off shell via an interpolating operator. There was and is no reason for this. For example, when writing a soft meson theorem for $\Sigma \to p\pi$, it is not necessary to force one of the baryons off shell.

6. Issues

When considering the validity of χPT$_\sigma$, it is important to avoid any presumption that dimensional transmutation necessarily implies that θ_μ^μ is large and $\neq 0$. Implicit in this intuition is a prejudice that scale invariance cannot be strongly broken via the vacuum when $\theta_\mu^\mu \to 0$. If the dilaton is a true NG boson, i.e. $m_\sigma \to 0$ with $F_\sigma \neq 0$ for $\theta_\mu^\mu \to 0$, it can couple to mass insertion terms in Callan-Symanzik equations and cause them to be *non-zero* in the zero-mass limit. Then Green's functions do not exhibit the power-law scaling expected for manifestly scale-invariant field theories.

This point is illustrated for the quark condensate in Fig. 1 (A). In scenario 1 (top diagram), the running of α_s is driven by the presence of quantities like $\langle\bar{q}q\rangle_{\text{vac}}$ (a mechanism often cited in papers on walking gauge theories [22]). In scenario 2 (bottom diagram), the running coupling freezes at α_{IR}, where the condensate is a *scale-breaking property of the vacuum*.

[3] We thank the final referee of our long paper [6] for drawing our attention to this.

Lattice investigations of IR fixed points inside the conformal window $8 \lesssim N_f \leqslant 16$ all depend on naive scaling of Green's functions [22], so they correspond to *scale-invariant vacua*. A recent lattice study [23] of the running of α_s for two flavors with *no* naive scaling suggests that it freezes: the fixed point realises scale invariance in NG mode, i.e. with a scale-breaking vacuum. That is what χPT_σ assumes for three flavors.

The term "dilaton" often refers to a spin-0 particle or resonance which couples to $\theta_{\mu\nu}$ and acquires its mass "spontaneously" due to self interactions. Originally, this idea concerned a scalar component of gravity [24], but now it is a key ingredient of dynamical electroweak symmetry breaking (pp. 198 and 1622-3, PDG tables [9]). This approximates theories with *scale-invariant vacua*, as is evident in walking technicolor. Therefore it has *nothing* to do with our dilaton [25].

It is well known that a resonance cannot be represented by a local interpolating operator, so is the fact that $\sigma/f_0(500)$ has a finite width a problem for χPT_σ? The answer is "no" because χPT_σ is an expansion in powers and logarithms of $m_{\pi,K,\eta,\sigma}$ with coefficients determined in the *exact* chiral-scale limit (1.2) where σ has zero width [6]. In any perturbation theory, decay rates are calculated that way.

A related remark concerns what is current best practice for scenario 1. The resonance $f_0(500)$ is treated as a member of the non-NG sector with an accidentally small mass. It causes χPT_3 to produce divergent expansions for amplitudes involving $f_0(500)$ poles: the radius of convergence is too small. Instead, these amplitudes are approximated dispersively via contributions from the dominant $f_0(500)$ poles with corrections from nearby thresholds, subject to exact chiral $SU(3) \times SU(3)$ constraints such as Adler zeros. One would certainly not use local fields in this framework.

However χPT_σ is a more ambitious theory. Having promoted σ/f_0 to the NG sector, we expect convergent asymptotic expansions for *all* mesonic amplitudes (scenario 2). The NLO corrections are still being worked out, but a first guess is to set all multi-dilaton vertices to zero. That is equivalent to adding the simplest dilaton diagrams to all χPT_3 diagrams. It seems to produce amplitudes very similar to those of the dispersive approximations of scenario 1.

7. Acknowledgements

We thank Claude Bernard, Gilberto Colangelo, Gerhard Ecker, Maarten Golterman, Martin Hoferichter, Heiri Leutwyler, and Daniel Phillips for useful comments about χPT_σ and the work we presented at CD2015. We also thank Nicolas Garron for informative discussions regarding RBC/UKQCD's analyses of $K \to \pi\pi$. LCT is supported by the Swiss National Science Foundation.

References

[1] S. L. Adler, J. C. Collins and A. Duncan, *Energy-Momentum-Tensor Trace Anomaly in Spin 1/2 Quantum Electrodynamics*, Phys. Rev. D **15**, 1712 (1977).

[2] P. Minkowski, *On the Anomalous Divergence of the Dilatation Current in Gauge Theories*, Berne PRINT-76-0813, September 1976.

[3] N. K. Nielsen, *The Energy Momentum Tensor in a Nonabelian Quark Gluon Theory*, Nucl. Phys. **B120**, 212 (1977).

[4] J. C. Collins, A. Duncan and S. D. Joglekar, *Trace and Dilatation Anomalies in Gauge Theories*, Phys. Rev. D **16**, 438 (1977).

[5] R. J. Crewther and L. C. Tunstall, *Origin of $\Delta I = 1/2$ Rule for Kaon Decays: QCD Infrared Fixed Point*, arXiv:1203.1321.

[6] R. J. Crewther and L. C. Tunstall, $\Delta I = 1/2$ *rule for kaon decays derived from QCD infrared fixed point*, Phys. Rev. D **91**, 034016 (2015) [arXiv:1312.3319].

[7] I. Caprini, G. Colangelo and H. Leutwyler, *Mass and width of the lowest resonance in QCD*, Phys. Rev. Lett. **96**, 132001 (2006) [arXiv:hep-ph/0512364].

[8] R. García-Martín, R. Kamiński, J. R. Peláez and J. R. de Elvira, *Precise determination of the $f_0(600)$ and $f_0(980)$ pole parameters from a dispersive data analysis*, Phys. Rev. Lett. **107**, 072001 (2011) [arXiv:1107.1635].

[9] K. A. Olive *et al.* [Particle Data Group Collaboration], *Review of Particle Physics*, Chin. Phys. C **38**, 090001 (2014).

[10] M. Gell-Mann, *Symmetries of Baryons and Mesons*, Phys. Rev. **125**, 1067 (1962), footnote 38.

[11] A. Calle Cordon and E. Ruiz Arriola, *Scalar meson mass from renormalized One Boson Exchange Potential*, AIP Conf. Proc. **1030**, 334 (2008) [arXiv:0804.2350].

[12] A. Calle Cordon and E. Ruiz Arriola, *Renormalization vs Strong Form Factors for One Boson Exchange Potentials*, Phys. Rev. C **81**, 044002 (2010) [arXiv:0905.4933].

[13] R. J. Crewther, *Chiral Reduction of $K \to 2\pi$ Amplitudes*, Nucl. Phys. B **264**, 277 (1986).

[14] J. F. Donoghue and H. Leutwyler, *Energy and momentum in chiral theories*, Z. Phys. C **52**, 343 (1991).

[15] C. G. Callan, Jr., S. R. Coleman and R. Jackiw, *A new improved energy-momentum tensor*, Annals Phys. **59**, 42 (1970).

[16] M. Hoferichter, D. R. Phillips and C. Schat, *Roy-Steiner equations for $\gamma\gamma \to \pi\pi$*, Eur. Phys. J. C **71**, 1743 (2011) [arXiv:1106.4147].

[17] J. A. Cronin, *Phenomenological Model of Strong and Weak Interactions in Chiral $U(3) \times U(3)$*, Phys. Rev. **161**, 1483 (1967).

[18] M. K. Gaillard and B. W. Lee, $\Delta I = 1/2$ *Rule for Nonleptonic Decays in Asymptotically Free Field Theories*, Phys. Rev. Lett. **33**, 108 (1974).

[19] C. Bernard, T. Draper, A. Soni, H. D. Politzer and M. B. Wise, *Application of chiral perturbation theory to $K \to 2\pi$ decays*, Phys. Rev. D **32**, 2343 (1985).

[20] P. A. Boyle *et al.* [RBC/UKQCD Collaboration], *Emerging understanding of the $\Delta I = 1/2$ Rule from Lattice QCD*, Phys. Rev. Lett. **110**, 152001 (2013) [arXiv:1212.1474].

[21] Z. Bai *et al.* [RBC/UKQCD Collaboration], *Standard-model prediction for direct CP violation in $K \to \pi\pi$ decay*, arXiv:1505.07863.

[22] L. Del Debbio, *The conformal window on the lattice*, Proc. Sci. LATTICE2010 (2010) 004 [arXiv:1102.4066].

[23] R. Horsley, H. Perlt, P. E. L. Rakow, G. Schierholz and A. Schiller, *The $SU(3)$ beta function from numerical stochastic perturbation theory*, Phys. Lett. B **728**, 1 (2014) [arXiv:1309.4311].

[24] Y. Fujii, *Dilaton and Possible Non-Newtonian Gravity*, Nat. Phys. Sci. **234**, 5 (1971).

[25] P. Carruthers, *Broken scale invariance in particle physics*, Phys. Rep. C **1**, 1 (1971).

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