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Free vibration analysis of stiffened panels with lumped mass and stiffness attachments

Dae Seung Cho ^{a,*}, Byung Hee Kim ^b, Jin-Hyeong Kim ^c, Tae Muk Choi ^c, Nikola Vladimir ^d^a Department of Naval Architecture and Ocean Engineering, Pusan National University, 63 beon-gil 2, Busandaehak-ro, Geumjeong-gu, Busan, 46241 Republic of Korea^b Samsung Heavy Industries Co. Ltd., Marine Research Institute, Geoje, Gyeongsangnam-do, Republic of Korea^c Createch Co. Ltd., Busan, Republic of Korea^d University of Zagreb, Faculty of Mechanical Engineering and Naval Architecture, Zagreb, Croatia

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ABSTRACT

Stiffened panels are basic constitutive members of ships and offshore structures, and in practice they often have different mass and stiffness attachments, which significantly influence their dynamic response. In this paper, a numerical procedure is presented for the free vibration analysis of stiffened panels with arbitrary sets of boundary conditions and carrying multiple lumped mass and stiffness attachments. It is based on the assumed mode method, where characteristic orthogonal polynomials having the properties of Timoshenko beam functions and satisfying the specified edge constraints are used as approximation functions. The Mindlin theory is applied for plate and the Timoshenko beam theory for stiffeners. The total potential and kinetic energies of the system are formulated in a convenient manner and further applied to derive an eigenvalue problem by means of Lagrange's equation of motion. Based on the developed numerical procedure, an in-house code is developed and is applied to a free vibration analysis of bare plates and stiffened panels carrying lumped masses and locally supported by pillars or springs. Comparisons of the results with those available in the literature and FEA solutions confirm the high accuracy and practical applicability of the presented procedure.

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1. Introduction

The vibration analysis of stiffened panels is an important issue in almost all aspects of practical engineering. This is especially pronounced in naval architecture and ocean engineering, where stiffened panels are used as primary constitutive elements of almost all structures. In many practical applications, one can find stiffened panels carrying multiple mass elements or locally supported with concentrated supports (such as, for instance, pillars or elastic springs) which significantly change their response to dynamic loading and make vibration analysis rather complicated. In the case of masses elastically connected to the plate structure, the system exhibits even more complex behaviour than in the case of a rigidly attached mass (Avalos et al., 1993).

Vibrations of rectangular plates carrying concentrated masses have been a topic of research from the beginning of the last century, as is obvious from the paper of Gershgorin (1933) where the effect of rotary inertia was ignored, or from some later references

(Stokey and Zorowski, 1959; Shah and Datta, 1969), where the mentioned rotary inertia effect is taken into account. Nicholson and Bergman (1985) studied the vibration of simply supported Mindlin plates (Mindlin et al., 1956) carrying concentrated masses which retained the effects of transverse shear and the rotary inertia of each mass. They also investigated the limit case of thin plate. Singal and Gorman (1992) presented a comparison of the analytical and experimental results for rectangular aluminium plates on rigid point supports and carrying one central mass or two masses of different weights. Their results show that rotary inertia reduces natural frequencies, while no additional modes due to the rotary inertia of masses were detected. According to Amabili et al. (2006), this is probably the consequence of relatively small rotary inertia of the masses used in calculations and experiments with respect to their weight. A Rayleigh-energy method with single-term trigonometric functions was used to study the vibration of plate with concentrated mass by Boay (1993) and comparisons with the experimental results for S-C-S-C rectangular plate were provided. Later, the same problem was investigated by multi-term trigonometric series expansion (Boay, 1995). Most of the references in this field actually deal with either plates carrying lumped masses or springs and point reinforcement, but only a few papers consider both additional inertia and stiffness attachments.

* Corresponding author.

E-mail addresses: daecho@pusan.ac.kr (D.S. Cho), bh48.kim@samsung.com (B.H. Kim), mzfg@createch.co.kr (J.-H. Kim), taemuk@createch.co.kr (T.M. Choi), nikola.vladimir@fsb.hr (N. Vladimir).

Wu and Luo (1995) analysed the natural frequencies and corresponding mode shapes of a uniform rectangular thin plate carrying any number of point masses and translational springs by means of the so-called analytical-and-numerical-combined method (ANCM). Further, Ostachowicz et al. (2002) presented a method for the identification of the location of a concentrated mass on isotropic plates by means of a genetic algorithm search technique based on changes in natural frequencies. The location and size of the mass are determined by the minimisation of an error function, which expresses the difference between calculated and measured natural frequencies. Amabili et al. (2006) studied the effect of concentrated masses on the free vibrations of rectangular plates by considering the rotary inertia of concentrated masses and the geometric imperfections of the plate both experimentally and numerically, by using a Rayleigh–Ritz based computer code. The results showed that large rotary inertia of concentrated masses introduces additional modes, which do not appear if rotary inertia is neglected. Watkins and Barton (2010) computed normalised frequencies for a rectangular, isotropic plate resting on elastic supports using an eigensensitivity analysis, which approximates the eigenparameters in a Maclaurin series, yielding an approximate closed-form expression. Also, Watkins et al. (2010) experimentally determined the natural frequencies and mode shapes for an elastically point-supported plate with attached masses under impulsive loading. The results are compared to frequencies and to mode shapes determined from the Rayleigh–Ritz method and a finite element analysis. It was found that the Rayleigh–Ritz analysis was able to suitably capture the first three rigid body mode natural frequencies for the majority of experimental configurations, while there were intriguing differences between the experimentally obtained mode shapes and the Rayleigh–Ritz analysis mode shapes. According to Watkins et al. (2010), the selection of basis functions has a significant influence on the accuracy of results obtained from the Ritz analysis. However, both the Rayleigh–Ritz analysis and the FE model were able to effectively capture the experimentally observed high frequency plate bending modes and were in good agreement with each other, although they were slightly higher than the experimental results.

Beside the above reviewed references to rectangular plates, circular and elliptical plates with attached masses have also been the subject of investigation (Laura et al., 1984; Bambill et al., 2004; Maiz et al., 2009). The vibration analysis of circular plates carrying a concentrated mass reported by Bambill et al. (2004) was motivated by the practical need to place a centrifugal pump rigidly attached to the thin, circular cover plate of a water tank in a medium-sized ocean vessel. Namely, due to a lack of space, it was necessary to place the system off-centre of the circular configuration and to calculate the fundamental frequency of the coupled system, where the Rayleigh–Ritz method was applied. In the case of stiffened panels, it was rather difficult to find a paper dealing with the vibration analysis of such structures carrying lumped mass attachments or having additional point supports.

During the last few decades, the finite element method (FEM) has become the most powerful tool for strength and vibration analysis, widely used in practical engineering. It is applied to very complex structures, and is also used to determine the static and dynamic response of different simple beam-like and plate-like structures. Various finite elements have even been developed and incorporated in general FE software. However, among other drawbacks, such software still requires rather lengthy model preparation, and modifications of the models when different topologies are being investigated are time consuming. Therefore, the application of simplified energy-based solutions, such as the one proposed in this paper, may have some advantages in the initial design stage.

In this paper, a simple numerical procedure is presented for the

free vibration analysis of stiffened panels carrying multiple lumped mass attachments and/or having point reinforcements arbitrarily placed within the plate area. The procedure is based on the assumed mode method, which was recently applied by the authors to both the dry and wet vibration analysis of simpler plate structures (Cho et al., 2015a,b,c). The effect of lumped attachments is expressed by adding their potential and kinetic energies to the corresponding panel energies, and the ordinary procedure by utilising Lagrange's equation is applied to calculate the natural frequencies and their corresponding mode shapes of the structures. The procedure is validated through several numerical examples dealing with the free vibration of both bare plates and stiffened panels with different sets of boundary conditions and lumped attachments. The results are compared with numerical solutions from open literature, and the FEA solutions obtained by the general FE tool, where the high accuracy and practical applicability of the method are confirmed.

2. Mathematical formulation

The dynamic response of stiffened panels with attachments is obtained by the assumed mode method which is actually an energy method (Kim et al., 2012; Cho et al., 2015a,b,c). Hence, a calculation of the total system potential and kinetic energies is required, which are further used in Lagrange's equation to formulate an eigenvalue problem. A stiffened panel of length a and width b with an arbitrary number of point masses (arbitrarily placed within the panel area) and having spring or pillar supports is considered, Fig. 1.

The plate model used here is based on the Mindlin thick first-order shear deformation plate theory which operates with three potential functions, i.e. plate deflection w , and angles of cross-section rotation about the x and y axes, ψ_x and ψ_y , respectively (Mindlin et al., 1956). All combinations of classical (simply supported, clamped, free) and non-classical (elastically supported in translation and rotation) boundary conditions are considered, Fig. 2. The equations of motion yield:

$$\frac{\rho h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2} - D \left(\frac{\partial^2 \psi_x}{\partial x^2} + \frac{1}{2}(1-\nu) \frac{\partial^2 \psi_x}{\partial y^2} + \frac{1}{2}(1+\nu) \frac{\partial^2 \psi_y}{\partial x \partial y} \right) - kGh \left(\frac{\partial w}{\partial x} - \psi_x \right) = 0, \quad (1)$$

$$\frac{\rho h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2} - D \left(\frac{\partial^2 \psi_y}{\partial y^2} + \frac{1}{2}(1-\nu) \frac{\partial^2 \psi_y}{\partial x^2} + \frac{1}{2}(1+\nu) \frac{\partial^2 \psi_x}{\partial x \partial y} \right) - kGh \left(\frac{\partial w}{\partial y} - \psi_y \right) = 0, \quad (2)$$

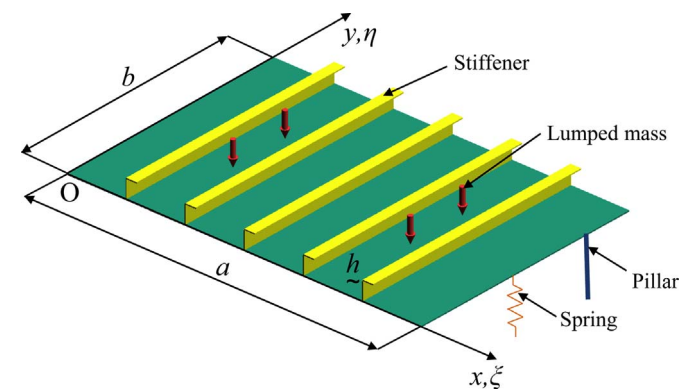


Fig. 1. Schematic presentation of a stiffened panel with attachments.

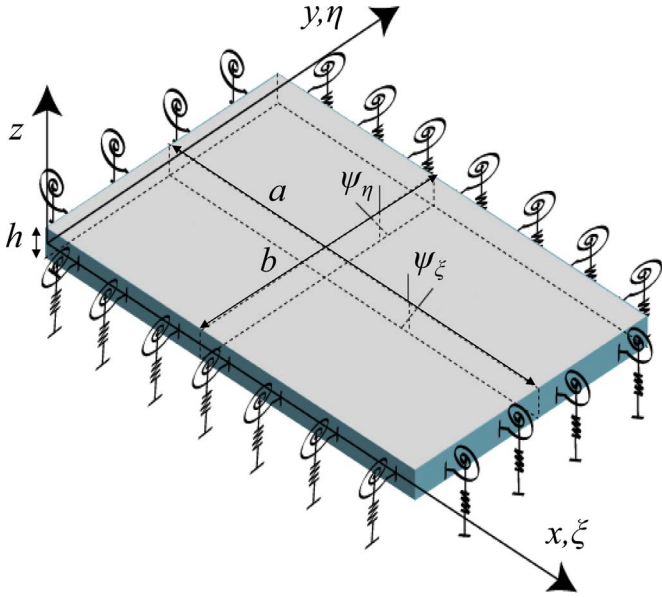


Fig. 2. Arbitrarily supported thick rectangular plate.

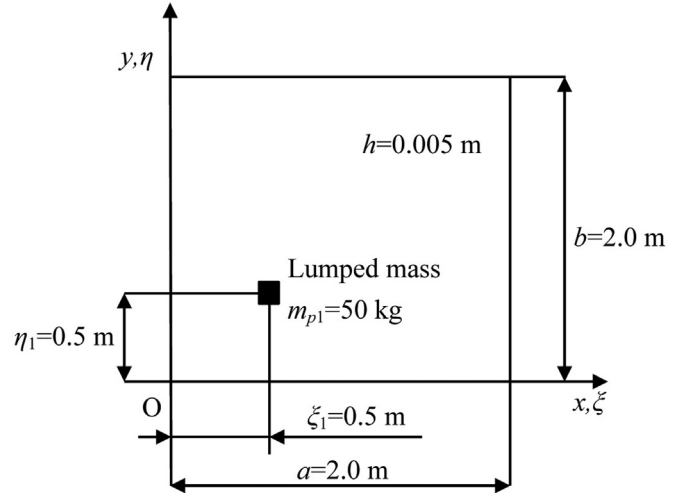


Fig. 3. Square plate with single lumped mass.

$$\frac{\rho}{kG} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} = 0, \tag{3}$$

where ρ represents plate density, h is plate thickness, k is shear coefficient, while ν is Poisson's ratio. Further, D represents plate flexural rigidity $D = Eh^3/(12(1 - \nu^2))$, while E and $G = E/(2(1 + \nu))$ are Young's modulus and shear modulus, respectively.

The total system potential and kinetic energies are obtained by adding the corresponding energies of stiffeners and lumped attachments to the bare plate potential and kinetic energies, respectively. In this sense, one can formally write:

$$V = V_p + V_s + V_l, \tag{4}$$

$$T = T_p + T_s + T_l, \tag{5}$$

where V_p is the plate strain energy, V_s represents the strain energy of the stiffeners, while V_l is the potential energy of the lumped stiffness such as pillars or springs. Similarly, T_p is the plate kinetic energy, while T_s and T_l are the kinetic energies of the stiffeners and lumped inertia, respectively.

If one introduces the non-dimensional parameters $\xi = x/a$, $\eta = y/b$, $\alpha = a/b$, and $S = kGh/D$ for a rectangular plate, the potential and kinetic energy of V and T , respectively, yield:

$$\begin{aligned} V_p = & \frac{D}{2\alpha} \int_0^1 \int_0^1 \left[\left(\frac{\partial \psi_\xi}{\partial \xi} \right)^2 + \alpha^2 \left(\frac{\partial \psi_\eta}{\partial \eta} \right)^2 + 2\nu\alpha \frac{\partial \psi_\xi}{\partial \xi} \frac{\partial \psi_\eta}{\partial \eta} \right. \\ & + \frac{1-\nu}{2} \left(\alpha \frac{\partial \psi_\xi}{\partial \eta} + \frac{\partial \psi_\eta}{\partial \xi} \right)^2 + S \left(\left(\frac{\partial w}{\partial \xi} - a\psi_\xi \right)^2 \right. \\ & + \alpha^2 \left(\frac{\partial w}{\partial \eta} - b\psi_\eta \right)^2 \left. \right] d\xi d\eta + \int_0^1 [K_{R_{x1}} \psi_\xi^2(0, \eta) + SK_{T_{x1}} w^2(0, \eta)] d\eta \\ & + \alpha^2 \int_0^1 [K_{R_{y1}} \psi_\eta^2(\xi, 0) + SK_{T_{y1}} w^2(\xi, 0)] d\xi \\ & + \int_0^1 [K_{R_{x2}} \psi_\xi^2(1, \eta) + SK_{T_{x2}} w^2(1, \eta)] d\eta \\ & + \alpha^2 \int_0^1 [K_{R_{y2}} \psi_\eta^2(\xi, 1) + SK_{T_{y2}} w^2(\xi, 1)] d\xi, \end{aligned} \tag{6}$$

Table 1
Natural frequencies of simply supported square plate with single lumped mass.

Method	Natural frequency (Hz)				
	1	2	3	4	5
Exact (Wu and Luo, 1995)	5.07	10.08	15.19	20.33	28.76
ANCM (Wu and Luo, 1995)	5.06	10.06	15.19	20.31	28.74
FEM (Wu and Luo, 1995)	5.17 (1.97)	10.17	15.46	20.70	29.12
FEM (NASTRAN)	5.03	9.98	15.16	20.85	29.18
AMM	5.06	10.06	15.40	20.44	29.10
	(-0.20)	(-0.20)	(1.38)	(0.54)	(1.18)

$$T_p = \frac{\rho ab}{2} \int_0^1 \int_0^1 \left[h \left(\frac{\partial w}{\partial t} \right)^2 + \frac{h^3}{12} \left(\frac{\partial \psi_\xi}{\partial t} \right)^2 + \frac{h^3}{12} \left(\frac{\partial \psi_\eta}{\partial t} \right)^2 \right] d\xi d\eta, \tag{7}$$

where

$$K_{T_{x1}} = (k_{T_{x1}}a/kGh), \quad K_{T_{x2}} = (k_{T_{x2}}a/kGh), \quad K_{T_{y1}} = (k_{T_{y1}}b/kGh) \text{ and } K_{T_{y2}} = (k_{T_{y2}}b/kGh)$$

are non-dimensional stiffness at $x = 0$, $x = a$, $y = 0$ and $y = b$, respectively, and correspond to the translational spring constants per unit length $k_{T_{x1}}$, $k_{T_{x2}}$, $k_{T_{y1}}$ and $k_{T_{y2}}$ (Cho et al., 2015a,b,c). In the same way, $K_{R_{x1}} = (k_{R_{x1}}a/D)$, $K_{R_{x2}} = (k_{R_{x2}}a/D)$, $K_{R_{y1}} = (k_{R_{y1}}b/D)$ and $K_{R_{y2}} = (k_{R_{y2}}b/D)$ are for the rotational spring constants per unit length $k_{R_{x1}}$, $k_{R_{x2}}$, $k_{R_{y1}}$ and $k_{R_{y2}}$ at the boundaries, respectively. The following quantities are related to the x direction: a , $k_{T_{x1}}$, $k_{T_{x2}}$, $k_{R_{x1}}$ and $k_{R_{x2}}$. Other quantities (b , $k_{T_{y1}}$, $k_{T_{y2}}$, $k_{R_{y1}}$ and $k_{R_{y2}}$) are relevant for the y direction.

The potential and kinetic energy of stiffeners placed in longitudinal and transverse directions yield:

$$\begin{aligned} V_s = & \frac{a}{2} \sum_{r=1}^{n_x} \left[\int_0^1 \left(\frac{EI_{x_r}}{a^2} \left(\frac{\partial \psi_\xi}{\partial \xi} \right)^2 + k_{x_r} A_{wx_r} G \left(\frac{\partial w}{\partial \xi} - \psi_\xi \right)^2 + \frac{GJ_{x_r}}{a^2} \left(\frac{\partial \psi_\eta}{\partial \xi} \right)^2 \right) \right]_{\eta=\eta_r} d\xi \\ & + \frac{b}{2} \sum_{r=1}^{n_y} \left[\int_0^1 \left(\frac{EI_{y_r}}{b^2} \left(\frac{\partial \psi_\eta}{\partial \eta} \right)^2 + k_{y_r} A_{wy_r} G \left(\frac{\partial w}{\partial \eta} - \psi_\eta \right)^2 + \frac{GJ_{y_r}}{b^2} \left(\frac{\partial \psi_\xi}{\partial \eta} \right)^2 \right) \right]_{\xi=\xi_r} d\eta. \end{aligned} \tag{8}$$

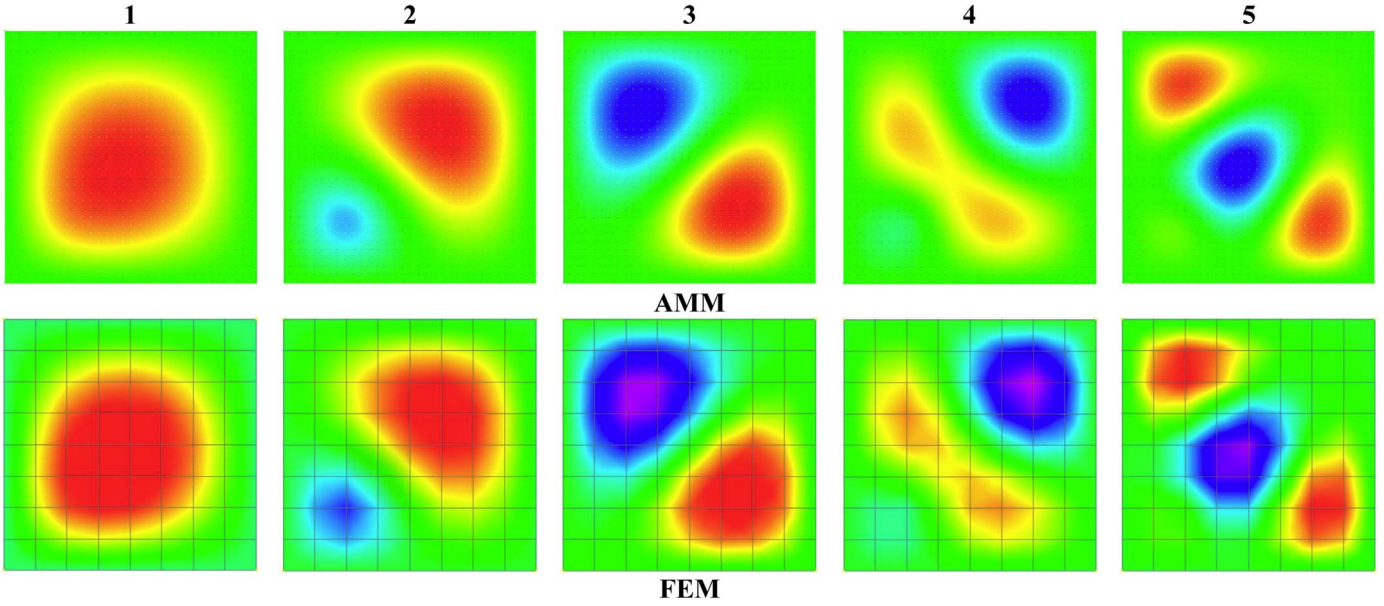


Fig. 4. Mode shapes of simply supported square plate with single lumped mass.

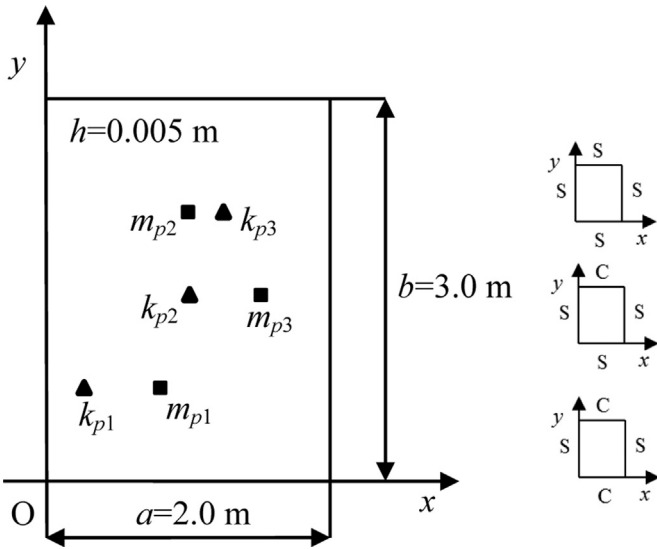


Fig. 5. Rectangular plate with three lumped masses and three translational springs.

$$T_s = \frac{a}{2} \sum_{r=1}^{n_x} \left[\int_0^1 \left\{ \rho I_{R_{x_r}} \left(\frac{\partial w_\xi}{\partial t} \right)^2 + \rho A_{w_{x_r}} \left(\frac{\partial w}{\partial t} \right)^2 \right\} d\xi \right]_{\eta=\eta_r} + \frac{b}{2} \sum_{r=1}^{n_y} \left[\int_0^1 \left\{ \rho I_{R_{y_r}} \left(\frac{\partial w_\eta}{\partial t} \right)^2 + \rho A_{w_{y_r}} \left(\frac{\partial w}{\partial t} \right)^2 \right\} d\eta \right]_{\xi=\xi_r} \quad (9)$$

where n_x and n_y represent the number of stiffeners in the x and y direction, respectively. k is the shear coefficient, and A_w and I_R are the cross-sectional area and the rotary moment of inertia of the stiffener cross-section, respectively. Further, GJ is torsional rigidity while EI is the bending rigidity of the stiffener, considering the plate flange contribution of effective width (Cho et al., 2015a). The potential and kinetic energy of lumped attachments yield:

$$V_l = \frac{1}{2} \sum_{r=1}^{l_s} \left(k_{p_r} w^2 + k_{p_{x_r}} \psi_\xi^2 + k_{p_{y_r}} \psi_\eta^2 \right)_{(\xi=\xi_r, \eta=\eta_r)}, \quad (10)$$

Table 2 Properties of lumped mass and spring attachments of a rectangular plate.

Attachment	Symbol	Value	Position
Lumped mass	m_{p1}	70 kg	$x=0.750$ m, $y=0.750$ m
	m_{p2}	50 kg	$x=1.000$ m, $y=1.875$ m
	m_{p3}	60 kg	$x=1.500$ m, $y=1.500$ m
Spring	k_{p1}	10^6 N/m	$x=0.250$ m, $y=0.750$ m
	k_{p2}	10^4 N/m	$x=1.000$ m, $y=1.500$ m
	k_{p3}	10^5 N/m	$x=1.250$ m, $y=1.875$ m

Table 3 Natural frequencies of rectangular plate with three lumped masses and three translational springs.

Boundary conditions	Method	Natural frequency (Hz)				
		1	2	3	4	5
SSSS	ANCM (Wu and Luo, 1995)	4.56	6.27	7.65	12.99	16.56
	FEM (Wu and Luo, 1995)	4.59	6.33	7.51	13.19	16.77
	AMM	4.46	6.29	7.86	13.66	16.45
SSCS	ANCM (Wu and Luo, 1995)	4.63	6.63	7.62	15.89	16.78
	FEM (Wu and Luo, 1995)	4.64	6.69	7.53	16.21	17.10
	AMM	4.54	6.55	8.00	16.21	16.66
CSCS	ANCM (Wu and Luo, 1995)	5.00	6.92	7.84	16.04	18.86
	FEM (Wu and Luo, 1995)	5.01	6.98	7.75	16.29	19.23
	AMM	4.88	7.01	8.08	16.40	18.77

$$T_l = \frac{1}{2} \sum_{r=1}^{l_m} \left(m_{p_r} \dot{w}^2 + J_{p_{x_r}} \dot{\psi}_\xi^2 + k_{p_{y_r}} \dot{\psi}_\eta^2 \right)_{(\xi=\xi_r, \eta=\eta_r)}, \quad (11)$$

where l_s and l_m represent the number of lumped inertias and stiffness, respectively. k_p is the lumped translational stiffness and k_{p_x} and k_{p_y} represent the lumped rotational stiffness in the x and y direction, respectively. m_p is the lumped mass and J_{p_x} and J_{p_y} represent the lumped mass moment of inertia in the x and y direction, respectively.

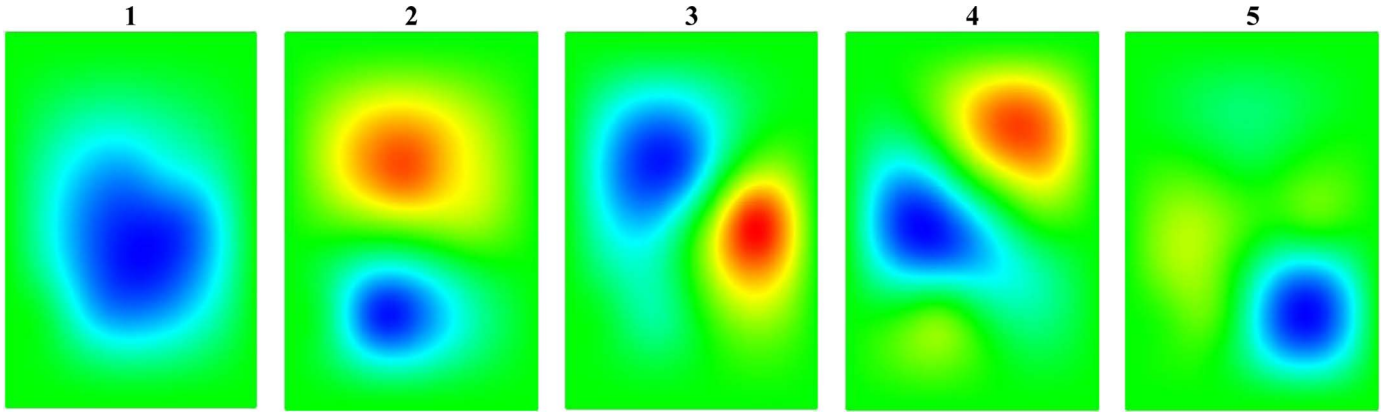


Fig. 6. Mode shapes of rectangular plate with three lumped masses and three translational springs, CSCS.

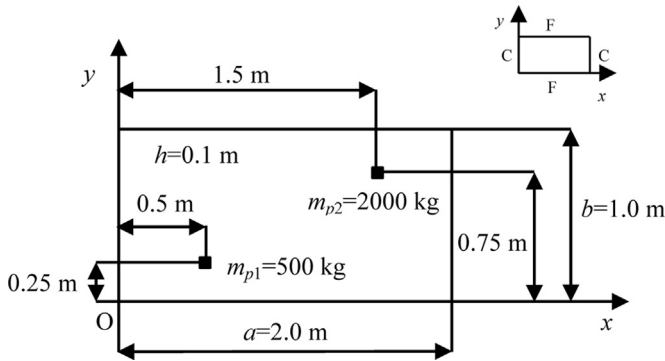


Fig. 7. Relatively thick rectangular plate with two lumped masses.

Table 4
Natural frequencies of relatively thick rectangular plate with two lumped masses.

Mode no.	Natural frequency (Hz)		
	AMM	FEM	Diff. (%)
1	81.93	81.08	1.05
2	142.97	139.99	2.13
3	263.56	258.23	2.06
4	323.48	317.68	1.83
5	602.59	593.03	1.61

In the assumed mode method, lateral displacement and rotational angles are expressed by superposing the products of the orthogonal polynomials having the characteristics of Timoshenko beam functions, whose derivation is introduced in (Kim et al., 2012):

$$w(\xi, \eta, t) = \sum_{m=1}^M \sum_{n=1}^N a_{mn}(t) X_m(\xi) Y_n(\eta), \quad (12)$$

$$\psi_{\xi}(\xi, \eta, t) = \sum_{m=1}^M \sum_{n=1}^N b_{mn}(t) \Theta_m(\xi) Y_n(\eta), \quad (13)$$

$$\psi_{\eta}(\xi, \eta, t) = \sum_{m=1}^M \sum_{n=1}^N c_{mn}(t) X_m(\xi) \Phi_n(\eta), \quad (14)$$

where $X_m(\xi)$, $Y_n(\eta)$, $\Theta_m(\xi)$ and $\Phi_n(\eta)$ are the orthogonal polynomials satisfying the specified elastic edge constraints with respect to ξ

and η . Furthermore, $a_{mn}(t)$, $b_{mn}(t)$ and $c_{mn}(t)$ are the influence coefficients of the orthogonal polynomials. Also, M and N are the number of orthogonal polynomials used for the approximate solution in the ξ and η direction, respectively. Alternatively, one can write Eqs. (12)–(14) in matrix form:

$$\{z(\xi, \eta, t)\} = [H(\xi, \eta)]\{q(t)\}, \quad (15)$$

where

$$\{z(\xi, \eta, t)\} = \{w(\xi, \eta, t), \psi_{\xi}(\xi, \eta, t), \psi_{\eta}(\xi, \eta, t)\}^T, \quad (16)$$

$$[H(\xi, \eta)] = \begin{bmatrix} X_1 Y_1 \dots X_M Y_N & 0 \dots 0 & 0 \dots 0 \\ 0 \dots 0 & \Theta_1 Y_1 \dots \Theta_M Y_N & 0 \dots 0 \\ 0 \dots 0 & 0 \dots 0 & X_1 \Phi_1 \dots X_M \Phi_N \end{bmatrix}, \quad (17)$$

$$\{q(t)\} = \{a_{11} \dots a_{MN} \ b_{11} \dots b_{MN} \ c_{11} \dots c_{MN}\}^T. \quad (18)$$

If Eqs. (4) and (5) are substituted into Lagrange's equation of motion below

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \quad (19)$$

the discrete matrix equation with $3 \times M \times N$ degrees of freedom is obtained as the following equation:

$$[M] \left\{ \frac{\partial^2 q(t)}{\partial t^2} \right\} + [K] \{q(t)\} = 0, \quad (20)$$

where $[M]$ and $[K]$ are the mass and the stiffness matrix, respectively. Their constitution for plate is described in detail in (Kim et al., 2012), while the constitution of the mass and stiffness matrix elements for the attached stiffeners is given in (Cho et al., 2015a). The constitution of the mass and stiffness matrix elements for the lumped attachments is given in the Appendix.

If we assume harmonic vibrations, i.e.

$$\begin{aligned} \{q(t)\} &= \{Q\} e^{j\omega t}, \quad w(\xi, \eta, t) = W(\xi, \eta) e^{j\omega t}, \psi_{\xi}(\xi, \eta, t) \\ &= \Psi_{\xi}(\xi, \eta) e^{j\omega t}, \quad \psi_{\eta}(\xi, \eta, t) = \Psi_{\eta}(\xi, \eta) e^{j\omega t}, \end{aligned} \quad (21)$$

Eq. (20) leads to an eigenvalue problem which gives natural frequencies and eigenvectors of the system, where j is the unit imaginary and ω represents the angular frequency. The mode shapes corresponding to each natural frequency are obtained from the following expression:

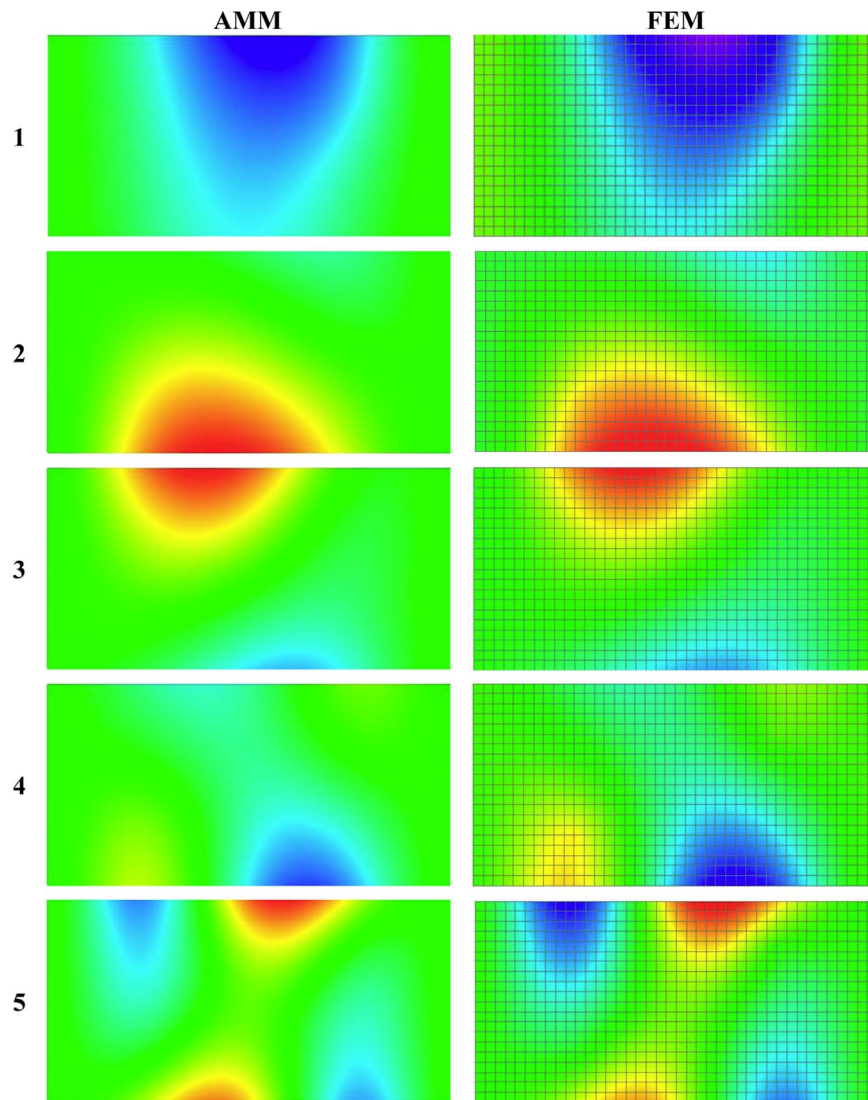


Fig. 8. Mode shapes of relatively thick rectangular plate with two lumped masses.

$$\{W(\xi, \eta), \Psi_{\xi}(\xi, \eta), \Psi_{\eta}(\xi, \eta)\}_l^T = [H(\xi, \eta)]\{Q\}_l, \quad (22)$$

where l represents the mode order.

3. Numerical examples – validation

The developed numerical procedure is validated through several numerical examples dealing with both bare plates and stiffened panels having different attachments and combinations of boundary conditions. In the case of bare plates, there are some results available in the open literature for comparison, while the results for stiffened panels, as a more complicated case, are compared to the FE solutions obtained by NASTRAN (MSC Software, 2010). All FE solutions are obtained for so called coupled formulation of mass matrix (MSC Software, 2010). The number of admissible functions (polynomials) is set at 13 for the longitudinal and transverse direction after performing convergence tests and achieving a numerically stable solution. The results obtained using a developed in-house code are denoted by AMM (Assumed Mode Method) in all numerical examples.

The first example is related to the free vibration of a square plate with a single lumped mass, $m_p=50$ kg, simply supported

along all four edges (SSSS), Fig. 3. The material density is set at $\rho=7850$ kg/m³, while the Young's modulus and Poisson ratio yield $E=2.051 \cdot 10^{11}$ N/m² and $\nu=0.3$, respectively. The first five natural frequencies are presented in Table 1, together with the results obtained by different methods and presented by Wu and Luo (1995), as well as the FE results obtained by NASTRAN. It should be mentioned that the FE results presented by Wu and Luo (1995) slightly differ from the NASTRAN results. However, the same mesh density is used (8×8) and, as indicated above, the NASTRAN values are calculated here for the coupled mass matrix formulation of the mass matrix and ordinary finite element incorporated in the code, while such data are not explicitly informed in (Wu and Luo, 1995). The natural frequencies presented in that paper are given in rad/s, and are recalculated here to Hz. Further, as mentioned in the introduction, Wu and Luo (1995) obtained their results by means of the analytical-and-numerical-combined method (ANCM) by taking into account several modes, and then comparing them to the closed-form (Exact) solution from Gershgorin (1933), which is also included here and considered as the reference one in this example. The differences from the Exact solutions in percentages (%) are given in parentheses. The assumed mode method values agree well with the other results, and very similar mode shape patterns are achieved for AMM and FE solutions obtained by

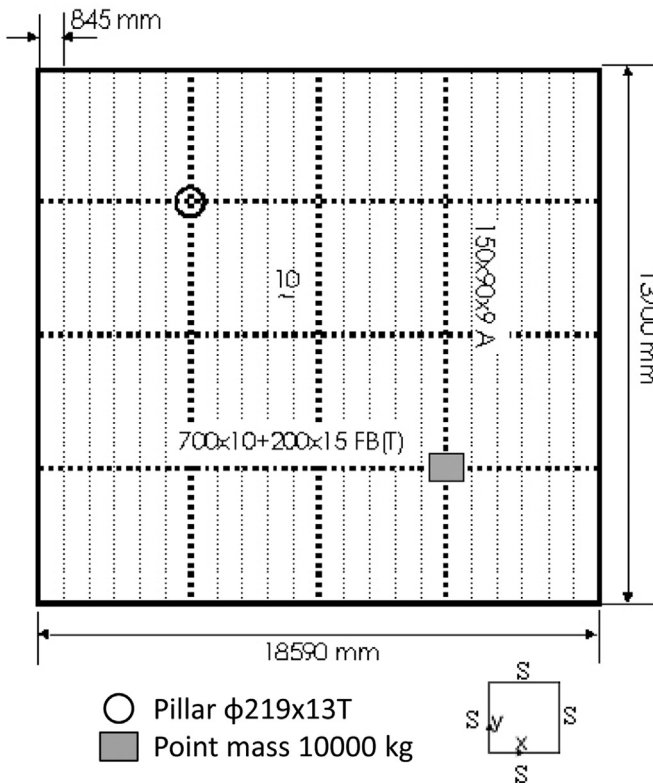


Fig. 9. Stiffened panel with lumped mass attachment and pillar.

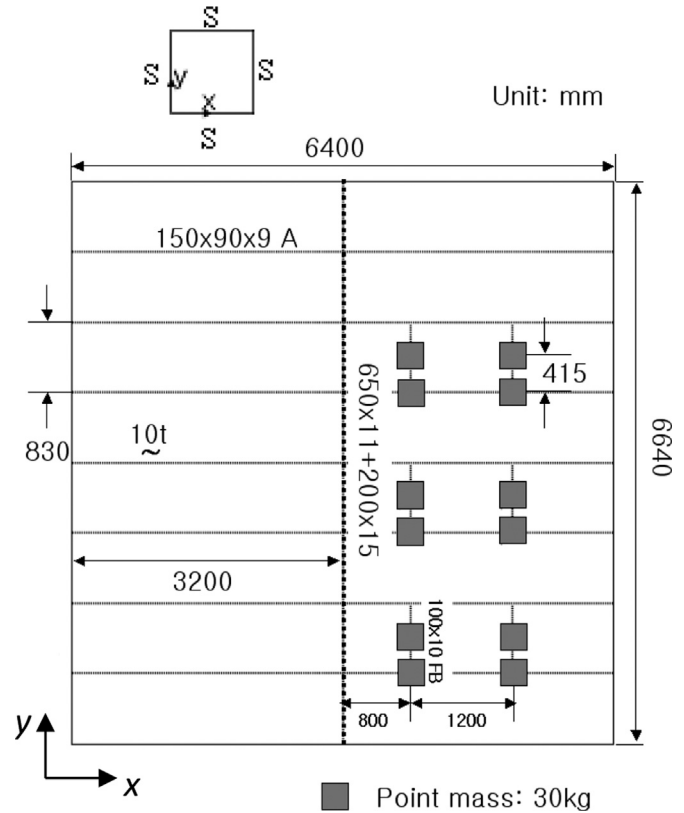


Fig. 10. Stiffened panel with multiple lumped mass attachments.

NASTRAN, Fig. 4.

The assumed mode method is further applied to calculate the natural frequencies of a uniform rectangular plate carrying three lumped masses and three translational springs for different combinations of boundary conditions, Fig. 5. The plate has the following dimensions: $a=2.0$ m, $b=3.0$ m and $h=0.005$ m. The material properties of the plate are the same as in the previous examples, while the lumped mass and stiffness properties and their coordinates are given in Table 2. The natural frequencies obtained by AMM are given in Table 3, together with the ANCM and FEM values calculated by Wu and Luo (1995). In the case of the first mode, the natural frequencies are slightly lower than the other solutions, but generally the results are in good agreement from an engineering point of view.

For illustration, the mode shapes of rectangular plate carrying three lumped masses and supported with three concentrated springs, in the case of the CSCS boundary condition, are shown in Fig. 6.

The above examples are related to relatively thin plates, and the developed theory can also consider relatively thick ones, as confirmed by the free vibration analysis of rectangular plates with two attached lumped masses and FCFC boundary conditions, Fig. 7. In this example, and all further examples, the Young's modulus is set at $E=2.1 \cdot 10^{11}$ N/m², while other material properties remain the same as in the previous examples. The FEM solutions are obtained with the FE mesh having 40 and 20 finite elements in longitudinal and transverse direction, respectively. The natural frequencies, Table 4, and the mode shapes, Fig. 8, obtained by AMM and FEM (MSC Software, 2010), respectively, are in very good agreement.

Further, numerical examples related to the free vibration analysis of stiffened panels with lumped attachments are considered. In this sense, the free vibration of stiffened panel having one lumped mass and a pillar and stiffened panel with

multiple lumped mass attachments is analysed, respectively, Figs. 9 and 10. Both stiffened panels are simply supported along all boundaries (SSSS). The FE model of stiffened panel with a lumped mass and pillar, Fig. 9, consists of 7040 plate elements (mesh density 88×80), and 1944 beam elements to account for the effect of stiffeners. The latter FE model, Fig. 10, consists of 1024 plate elements (mesh density 32×32) and 224 beam elements. The results, including the natural frequencies and mode shapes, are summarised in Tables 5 and 6, where excellent agreement is clear.

Finally, a free stiffened panel having four equidistant stiffeners ($150 \times 90 \times 9$ mm) and four equal lumped mass attachments (50 kg), and resting on four equal elastic springs (100 kN/m), is analysed, Fig. 11. This example is very interesting for practical engineering, because one can often find some engines, pumps or other devices mounted on plate structures with elastic supports. In the FE analysis, the mesh density is 18 elements in x - and 30 elements in y -direction, respectively. Four stiffeners are modelled with 72 beam finite elements in total. The natural frequencies and mode shapes are in excellent agreement with the FEM results calculated by NASTRAN, Table 7.

The above extensive numerical examples, with different structural topologies and various combinations of boundary conditions, confirm that the developed method is highly accurate. At the same time, it is very simple, and therefore recommended for practical application in the vibration assessment of plate structures.

4. Concluding remarks

The free vibration analysis of arbitrarily constrained plate structures carrying different lumped attachments is performed by

Table 5
Natural frequencies and mode shapes of a stiffened panel with lumped mass attachment and pillar.

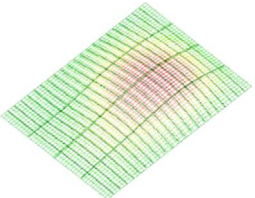
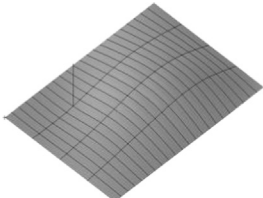
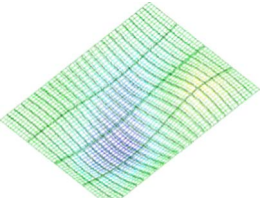
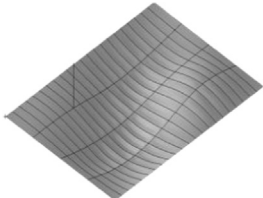
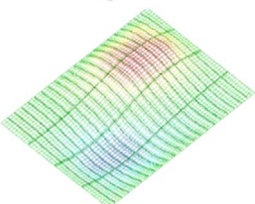
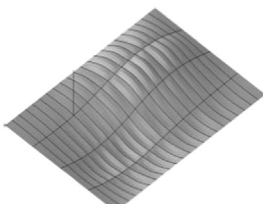
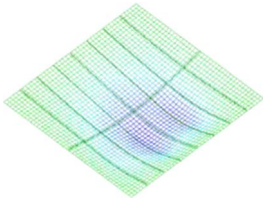
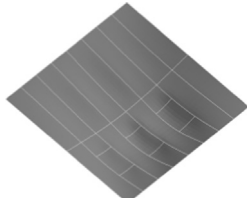
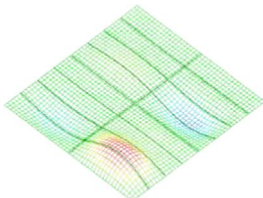
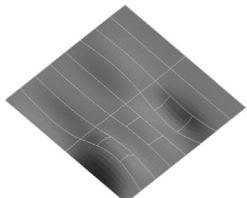
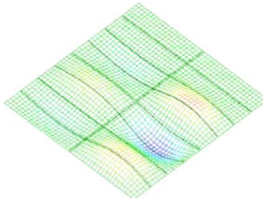
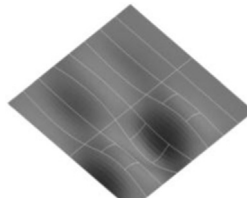
Mode no.	AMM		FEM		Diff. (%)
	Natural frequency (Hz)	Mode shape	Natural frequency (Hz)	Mode shape	
1	7.94		8.03		-1.12
2	16.74		16.89		-0.89
3	21.72		21.66		0.28

Table 6
Natural frequencies and mode shapes of a stiffened panel with multiple lumped mass attachments.

Mode no.	AMM		FEM		Diff. (%)
	Natural frequency (Hz)	Mode shape	Natural frequency (Hz)	Mode shape	
1	12.53		12.51		0.16
2	19.05		18.83		1.17
3	19.43		19.20		1.20

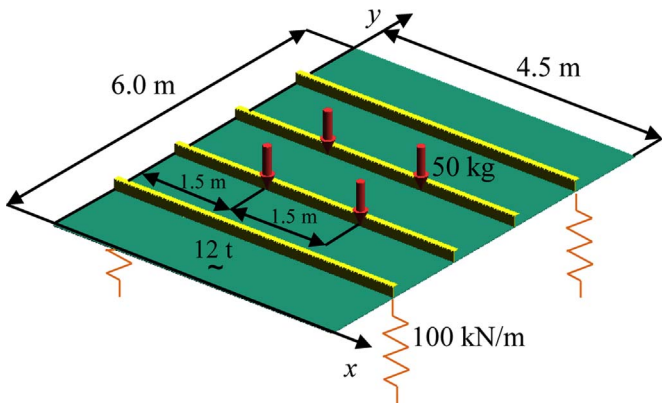


Fig. 11. Stiffened panel with lumped mass attachments resting on elastic springs.

the assumed mode method. The effects of stiffeners and lumped attachments are taken into account by adding their potential and kinetic energies to the corresponding plate energies. Extensive numerical calculations showed very good agreement of the results with those available in the open literature and general purpose FE software results. Moreover, in spite of the simplicity of the mathematical model, there are no limitations in the applicability of the presented method concerning plate thickness bounds or reinforcement dimensions and orientations. Thus, the proposed procedure can be used as a quick and reliable alternative to the widely used FEM, particularly in the definition of principal dimensions and topology. Another advantage of the proposed procedure compared to existing energy-based approaches is that it can be applied not only to bare plates, but also to stiffened panels with attachments, which are actually the main constitutive elements in naval architecture and ocean engineering.

Acknowledgement

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Appendix. mass and stiffness matrix elements for lumped attachments

The equations to calculate elements of the mass and stiffness matrix of plate and stiffeners whose dimensions are $(3 \times M \times N)$ by $(3 \times M \times N)$, with arbitrary edge constraints using the assumed mode method, are introduced by Kim et al. (2012) and Cho et al. (2015a), respectively. The additional elements of the mass and stiffness matrix to consider lumped mass and point reinforcement attached at the (ξ_r, η_r) location of plate are calculated as follows:

Mass matrix element, m_{pq}

$$\begin{aligned} p &= (i - 1)N + j, \\ q &= (m - 1)N + nm_{pq} \\ &= \sum_{r=1}^{l_m} m_{pr} X_i(\xi_r) Y_j(\eta_r) X_m(\xi_r) Y_n(\eta_r) \end{aligned} \tag{A1}$$

$$\begin{aligned} p &= (i + M - 1)N + j, \quad q = (m + M - 1)N + nm_{pq} \\ &= \sum_{r=1}^{l_m} J_{pxr} \Theta_i(\xi_r) Y_j(\eta_r) \Theta_m(\xi_r) Y_n(\eta_r) \end{aligned} \tag{A2}$$

Table 7
Natural frequencies and mode shapes of a stiffened panel with lumped mass attachments resting on elastic springs.

Mode no.	AMM		FEM		Diff. (%)
	Natural frequency (Hz)	Mode shape	Natural frequency (Hz)	Mode shape	
1	2.06		2.05		0.49
2	2.98		2.94		1.36
3	3.57		3.52		1.42

$$\blacksquare p = (i + 2M - 1)N + j, q = (m + 2M - 1)N + n$$

$$m_{pq} = \sum_{r=1}^{l_m} J_{pyr} X_i(\xi_r) \Phi_j(\eta_r) X_m(\xi_r) \Phi_n(\eta_r) \tag{A3}$$

■ others

$$m_{pq} = 0, \tag{A4}$$

where $i, m = 1, 2, \dots, M$ and $j, n = 1, 2, \dots, N$.

Stiffness matrix element, k_{pq}

$$\blacksquare p = (i - 1)N + j, q = (m - 1)N + nk_{pq}$$

$$= \sum_{r=1}^{l_s} k_{pxr} X_i(\xi_r) Y_j(\eta_r) X_m(\xi_r) Y_n(\eta_r) \tag{A5}$$

$$\blacksquare p = (i + M - 1)N + j, q = (m + M - 1)N + nk_{pq}$$

$$= \sum_{r=1}^{l_s} k_{pxr} \Theta_i(\xi_r) Y_j(\eta_r) \Theta_m(\xi_r) Y_n(\eta_r) \tag{A6}$$

$$\blacksquare p = (i + 2M - 1)N + j, q = (m + 2M - 1)N + nk_{pq}$$

$$= \sum_{r=1}^{l_s} k_{pyr} X_i(\xi_r) \Phi_j(\eta_r) X_m(\xi_r) \Phi_n(\eta_r) \tag{A7}$$

■ others

$$k_{pq} = 0, \tag{A8}$$

where $i, m = 1, 2, \dots, M$ and $j, n = 1, 2, \dots, N$.

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