Existence and boundedness of optimal controls in infinite-horizon problems

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Optimal control problem with infinite time horizon

Consider the following problem (P):

$$J(x(\cdot),u(\cdot)) = \int_0^{\infty} f^0(t,x(t),u(t))\,dt \to \max\,,$$

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad x(0) = x_0,$$

$$u(t) \in U.$$

Here $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, $t \geq 0$, $x_0 \in G$ where G is an open convex set in $\mathbb{R}^n$, U is a nonempty closed (**not necessary bounded**) set in $\mathbb{R}^m$. The class of *admissible controls* consists of all $u(\cdot) \in L^{\infty}_{loc}([0,\infty),\mathbb{R}^m)$ such that $u(t) \in U$ for all $t \geq 0$. It is assumed that for any $u(\cdot)$ the corresponding *admissible trajectory* $x(\cdot)$ exists on $[0,\infty)$ in G and the function $t \mapsto f^0(t,x(t),u(t))$ is locally integrable on $[0,\infty)$.
An admissible pair $(x_*(\cdot),u_*(\cdot))$ is (strongly) *optimal* in problem (P) if the integral functional $J(x(\cdot),u(\cdot))$ converges and for any other admissible pair $(x(\cdot),u(\cdot))$ the following inequality holds:

$$J(x_*(\cdot),u_*(\cdot)) \geq \limsup_{T\to\infty} \int_0^{T} f^0(t,x(t),u(t))\,dt.$$

Example 1

Consider the following problem $(P1)$:

$$J(x(\cdot),u(\cdot)) = \int_0^{\infty} e^{-\rho t} x(t)u(t)\,dt \to \max\,,$$

$$\dot{x}(t) = -u(t)x(t), \qquad x(0) = x_0 > 0, \quad \rho > 0,$$

$$u(t) \in [0,\infty).$$

There is no any optimal control in $(P1)$ in the class $L^{\infty}_{loc}[0,\infty)$.

1) For any admissible pair $(x(\cdot),u(\cdot))$ we have

$$J(x(\cdot),u(\cdot)) = -\int_0^{\infty} e^{-\rho t}\dot{x}(t)\,dt = x_0 - \rho \int_0^{\infty} e^{-\rho t} x(t)\,dt < x_0.$$

2) The sequence $\{u_k(\cdot)\}_{k=1}^{\infty}$ where $u_k(t) \equiv k^2$ if $t \in [0,1/k]$ and $u_k(t) = 0$ if $t > 1/k$, $k = 1,2,\dots$, is the maximizing sequence.
3) We have

$$J(x_k(\cdot),u_k(\cdot)) = \frac{k^2 x_0}{\rho + k^2}\left(1 - e^{-\frac{k^2+\rho}{k}}\right) \to x_0 \qquad \text{as} \quad k \to \infty.$$

(A1) Regularity assumption: *For a.e.* $t \in [0, \infty)$ *partial derivatives* $f_x(t, x, u)$ *и* $f_x^0(t, x, u)$ *do exist for any* $(x, u) \in G \times U$*. Functions* $f(\cdot, \cdot, \cdot)$, $f^0(\cdot, \cdot, \cdot)$, $f_x(\cdot, \cdot, \cdot)$ *and* $f_x^0(\cdot, \cdot, \cdot)$ *are Lebesgue measurable in* t *for all* $(x, u) \in G \times U$*, continuous in* (x, u) *for a.e.* $t \in [0, \infty)$ *and locally bounded.*

(A2) Growth assumption: *For any admissible pair* $(x(\cdot), u(\cdot))$ *there exist a number* $\beta > 0$ *and a nonnegative integrable function* $\lambda : [0, \infty) \mapsto \mathbb{R}^1$ *such that for all* $\zeta \in G$*, satisfying the inequality* $\|\zeta - x_0\| < \beta$*, the Cauchy problem*

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = \zeta,$$

has a solution $x(\zeta; \cdot)$ *on* $[0, \infty)$ *in* G *and*

$$\max_{x \in [x(\zeta;t), x(t)]} \left| \langle f_x^0(t, x, u(t)), x(\zeta; t) - x(t) \rangle \right| \overset{a.e.}{\leq} \|\zeta - x_0\| \lambda(t).$$

(A3) Convexity assumption: *For any* $M > 0$ *there is a compact set* $U_M \subset U$ *such that* $\{u \in U\colon \|u\| \leq M\} \subset U_M$ *and for a.e.* $t \geq 0$ *for all* $x \in G$ *the set*

$$Q_M(t,x) = \left\{(z^0, z) \in \mathbb{R}^{n+1}\colon\ z^0 \leq f^0(t,x,u),\ z = f(t,x,u),\ u \in U_M\right\}$$

is convex.

(A4) Estimate on the "tail" of the utility functional: *There is a decreasing function* $\omega : [0,\infty) \mapsto \mathbb{R}^1$, $\omega(t) \to +0$ *as* $t \to \infty$ *such that for any* $0 \leq T \leq T'$ *for all* $(x(\cdot), u(\cdot))$ *we have*

$$\int_T^{T'} f^0(t, x(t), u(t))\, dt \leq \omega(T).$$

Along arbitrary admissible pair $(x(\cdot), u(\cdot))$ consider the following system

$$\dot{z}(t) = -\left[f_x(t, x(t), u(t))\right]^* z(t).$$

Due to $(A1)$ the normalized matrix solution $Z(\cdot)$ is well defined on $[0, \infty)$.

Lemma 1. *If admissible pair* $(x(\cdot), u(\cdot))$ *fits condition* $(A2)$ *then*

$$\left\|Z^{-1}(t) f_x^0(t, x(t), u(t))\right\| \leq \sqrt{n}\lambda(t), \qquad t \geq 0.$$

This implies that for any $T > 0$ the function $\psi_T : [0, T] \mapsto \mathbb{R}^n$ defined as

$$\psi_T(t) = Z(t) \int_t^T Z^{-1}(s) f_x^0(s, x(s), u(s))\, ds, \qquad t \in [0, T],$$

is absolutely continuous and the function $\psi \colon [0, \infty) \mapsto \mathbb{R}^n$ defined as

$$\psi(t) = Z(t) \int_t^\infty Z^{-1}(s) f_x^0(s, x(s), u(s))\, ds, \qquad t \geq 0,$$

is locally absolutely continuous.

Define $\mathcal{H} : [0, \infty) \times G \times U \times \mathbb{R}^n \to \mathbb{R}^1$ in a standard way:

$$\mathcal{H}(t, x, u, \psi) = f^0(t, x, u) + \langle \psi, f(t, x, u) \rangle,$$

$$t \in [0, \infty),\ x \in G,\ u \in U,\ \psi \in \mathbb{R}^n.$$

Main result

Theorem 1. *Assume* $(A1)$–$(A4)$ *hold and there is an admissible pair* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *such that* $J(\bar{x}(\cdot), \bar{u}(\cdot)) > -\infty$. *Assume there are a continuous positive function* $M\colon [0,\infty) \mapsto \mathbb{R}^1$, *and a positive function* $\delta\colon [0,\infty) \mapsto \mathbb{R}^1$, $\lim_{t\to\infty} \frac{\delta(t)}{t} = 0$, *such that for any admissible pair* $(x(\cdot), u(\cdot))$ *which satisfies on some set* $\mathfrak{M} \subset [0,\infty)$, $\operatorname{meas} \mathfrak{M} > 0$, *the inequality* $\|u(t)\| > M(t)$, *for a.e.* $t \in \mathfrak{M}$ *and all* $T \geq t + \delta(T)$ *we have*

$$\sup_{u\in U\colon \|u\|\leq M(t)} \mathcal{H}(t, x(t), u, \psi_T(t)) - \mathcal{H}(t, x(t), u(t), \psi_T(t)) > 0. \qquad (*)$$

Then there is an optimal control $u_*(\cdot)$ *in* (P) *and* $\|u_*(t)\| \overset{a.e.}{\leq} M(t)$.
If for a.e. $t \in \mathfrak{M}$ *inequality* $(*)$ *holds uniformly in* $T : T - \delta(T) \geq t$, *i.e.*

$$\inf_{T\colon T-\delta(T)\geq t} \left\{ \sup_{u\in U\colon \|u\|\leq M(t)} \mathcal{H}(t, x(t), u, \psi_T(t)) - \mathcal{H}(t, x(t), u(t), \psi_T(t)) \right\} > 0,$$

then any optimal control $u_*(\cdot)$ *in* (P) *satisfies* $\|u_*(t)\| \overset{a.e.}{\leq} M(t)$.

Example 2

Consider the following problem $(P2)$:

$$J(S(\cdot),u(\cdot)) = \int_0^{\infty} e^{-\rho t}\left[\ln S(t) + \ln u(t)\right]\, dt \to \max,$$

$$\dot{S}(t) = rS(t)\left(1 - \frac{S(t)}{K}\right) - u(t)S(t), \qquad S(0) = S_0,$$

$$u(t) \in (0,\infty).$$

Here $S_0 > 0$, $K > 0$, $r > 0$ and $\rho > 0$. We set $G = (0,\infty)$.
For any admissible $S(\cdot)$ we have $S(t) \leq S_{\max} = \max\{S_0, K\}$, $t \geq 0$.
Lemma 2. *There is a decreasing function* $\omega : [0,\infty) \mapsto (0,\infty)$ *such that* $\omega(t) \to +0$ *as* $t \to \infty$ *and for any* $0 \leq T < T'$ *for all admissible pairs* $(S(\cdot),u(\cdot))$ *the following inequality holds:*

$$\int_T^{T'} e^{-\rho t}\left[\ln S(t) + \ln u(t)\right]\, \mathrm{d}t < \omega(T).$$

Along any admissible pair $(S(\cdot), u(\cdot))$ we have

$$\frac{d}{dt}\left[e^{-\rho t}\ln S(t)\right] \stackrel{\text{a.e.}}{=} -\rho e^{-\rho t}\ln S(t) + re^{-\rho t} - e^{-\rho t}\left(\frac{r}{K}S(t) + u(t)\right), \quad t > 0.$$

Integrating this equality on time interval $[0, T]$, $T > 0$, we obtain

$$\int_0^T e^{-\rho t}\ln S(t)\,dt = \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \frac{r}{\rho^2}\left(1 - e^{-\rho T}\right) - \int_0^T e^{-\rho t}\left(\frac{r}{\rho K}S(t) + \frac{u(t)}{\rho}\right) dt.$$

Hence, for any $(S(\cdot), u(\cdot))$ and arbitrary $T > 0$ we have

$$\int_0^T e^{-\rho t}\left[\ln S(t) + \ln u(t)\right] dt = \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \frac{r}{\rho^2}\left(1 - e^{-\rho T}\right) - \frac{r}{\rho K}\int_0^T e^{-\rho t}S(t)\,dt + \int_0^T e^{-\rho t}\left(\ln u(t) - \frac{u(t)}{\rho}\right) dt.$$

Problem $(\tilde{P}2)$:

$$\tilde{J}(S(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\ln u(t) - \frac{u(t)}{\rho} - \frac{r}{\rho K} S(t) \right] dt \to \max,$$

$$\dot{S}(t) = rS(t) \left(1 - \frac{S(t)}{K} \right) - u(t)S(t), \qquad S(0) = S_0,$$

$$u(t) \in (0, \infty).$$

Problem $(P3)$:

$$\tilde{J}(S(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\ln u(t) - \frac{u(t)}{\rho} - \frac{r}{\rho K} S(t) \right] dt \to \max,$$

$$\dot{S}(t) = rS(t) \left(1 - \frac{S(t)}{K} \right) - u(t)S(t), \qquad S(0) = S_0,$$

$$u(t) \in [\rho, \infty).$$

Lemma 3. *Problems $(P2)$, $(\tilde{P}2)$ and $(P3)$ are equivalent.*

Let us introduce the new state variable $x(\cdot)$ in problem $(P3)$ as follows:

$$x(t) = \frac{1}{S(t)}, \qquad t \geq 0.$$

In terms of the state variable $x(\cdot)$ problem $(P3)$ can be rewritten as the following (equivalent) problem $(P4)$:

$$J(x(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\ln u(t) - \ln x(t)\right] \, dt \to \max,$$

$$\dot{x}(t) = [u(t) - r] \, x(t) + a, \qquad x(0) = x_0 = \frac{1}{S_0},$$

$$u(t) \in [\rho, \infty).$$

Here $a = r/K$. The class of admissible controls $u(\cdot)$ in problem $(P4)$ is the same as in $(P3)$. It consists of all measurable locally bounded functions $u\colon [0, \infty) \mapsto [\rho, \infty)$.

Theorem 2. *There is an optimal admissible control $u_*(\cdot)$ in problem $(P4)$. Moreover, for any optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ we have*

$$u_*(t) \overset{a.e.}{\leq} \left(1 + \frac{1}{Kx_*(t)}\right)(r + \rho), \qquad t \geq 0.$$

Proof. 1) Conditions $(A1)$–$(A4)$ of Theorem 1 are satisfied.
2) For any $T > 0$ and arbitrary $t \in [0, T]$ we get

$$-x(t)\psi_T(t) = \left[x_0 + a\int_0^t e^{-\int_0^s u(\xi)\,d\xi + rs}\,ds\right]$$
$$\times \int_t^T \frac{e^{-\rho s}}{x_0 + a\int_0^s e^{-\int_0^\tau u(\xi)\,d\xi + r\tau}\,d\tau}\,ds$$
$$\geq x_0 \int_t^T \frac{e^{-\rho s}}{x_0 + a\int_0^s e^{r\tau}\,d\tau}\,ds \geq \frac{rx_0 e^{-(r+\rho)t}}{(rx_0 + a)(r + \rho)}\left[1 - e^{-(r+\rho)(T-t)}\right].$$

For arbitrary $\delta > 0$ define the function $M_\delta \colon [0,\infty) \mapsto \mathbb{R}^1$ as follows:

$$M_\delta(t) = \frac{(rx_0 + a)(r+\rho)}{rx_0\left[1 - e^{-(r+\rho)\delta}\right]} e^{rt} + \frac{1}{\delta}, \qquad t \geq 0.$$

Then for any $T > \delta$, $t \in [0, T-\delta]$ and arbitrary admissible pair $(x(\cdot), u(\cdot))$ the function $u \mapsto \mathcal{H}(t, x(t), u, \psi_T(t))$ reaches its maximal value on $[\rho, \infty)$ at the point

$$u_T(t) = -\frac{e^{-\rho t}}{x(t)\psi_T(t)} \leq \frac{(rx_0 + a)(r+\rho)}{rx_0\left[1 - e^{-(r+\rho)(T-t)}\right]} e^{rt} \leq M_\delta(t) - \frac{1}{\delta}.$$

For a fixed $\delta > 0$ set $\delta(t) \equiv \delta$ and $M(t) \equiv M_\delta(t)$, $t \geq 0$.

Let $(x(\cdot), u(\cdot))$ be an admissible pair such that inequality $u(t) > M_\delta(t)$ holds on a set $\mathfrak{M} \subset [0,\infty)$, $\operatorname{meas} \mathfrak{M} > 0$.

Define the function $\Phi\colon [t+\delta,\infty)\mapsto\mathbb{R}^1$ as follows

$$\begin{aligned}\Phi(T) &= \sup_{u\in[\rho,M(t)]} \mathcal{H}(t,x(t),u,\psi_T(t)) - \mathcal{H}(t,x(t),u(t),\psi_T(t))\\ &= \psi_T(t)u_T(t)x(t) + e^{-\rho t}\ln u_T(t) - \left[\psi_T(t)u(t)x(t) + e^{-\rho t}\ln u(t)\right]\\ &= -e^{-\rho t} + e^{-\rho t}\left[-\rho t - \ln(-\psi_T(t)) - \ln x(t)\right]\\ &\quad - \left[\psi_T(t)u(t)x(t) + e^{-\rho t}\ln u(t)\right].\end{aligned}$$

For a.e. $T\geq t+\delta$ we get

$$\begin{aligned}\frac{d}{dT}\Phi(T) &= -\frac{e^{-\rho t}}{\psi_T(t)}\frac{d}{dT}\left[\psi_T(t)\right] - u(t)x(t)\frac{d}{dT}\left[\psi_T(t)\right]\\ &= x(t)\frac{d}{dT}\left[\psi_T(t)\right]\left[\frac{e^{-\rho t}}{-\psi_T(t)x(t)} - u(t)\right] = x(t)\frac{d}{dT}\left[\psi_T(t)\right](u_T(t)-u(t)) > 0.\end{aligned}$$

Hence,

$$\begin{aligned}\inf_{T>0\colon t\leq T-\delta}&\left\{\sup_{u\in[\rho,M(t)]} \mathcal{H}(t,x(t),u,\psi_T(t)) - \mathcal{H}(t,x(t),u(t),\psi_T(t))\right\}\\ &= \inf_{T>0\colon t\leq T-\delta}\Phi(T) = \Phi(t+\delta) > 0.\end{aligned}$$

Due to Theorem 1 there is an optimal control $u_*(\cdot)$ in $(P4)$ and

$$u_*(t) \overset{a.e.}{\leq} M_\delta(t) = \frac{(rx_0 + a)(r+\rho)}{rx_0\left[1 - e^{-(r+\rho)\delta}\right]}e^{rt} + \frac{1}{\delta}.$$

Passing to a limit in this inequality as $\delta \to \infty$ we get

$$u_*(t) \overset{a.e.}{\leq} \left(1 + \frac{1}{Kx_0}\right)(r+\rho)e^{rt}, \qquad t \geq 0.$$

3) For $\tau > 0$ the pair $(\tilde{x}_*(\cdot), \tilde{u}_*(\cdot))$: $\tilde{x}_*(t) = x_*(t+\tau)$, $\tilde{u}_*(t) = u_*(t+\tau)$, is optimal in $(P4)$ with initial condition $x(0) = x_*(\tau)$. Hence,

$$\tilde{u}_*(t) \overset{a.e.}{\leq} \left(1 + \frac{1}{K\tilde{x}_*(0)}\right)(r+\rho)e^{rt}, \qquad t \geq 0.$$

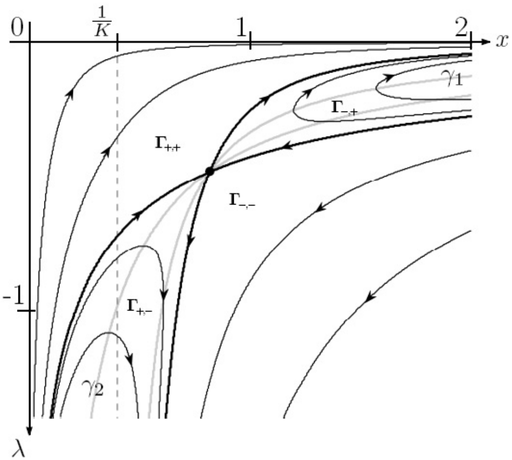
Hence, for arbitrary fixed $\tau > 0$ we have

$$u_*(t) = \tilde{u}_*(t-\tau) \overset{a.e.}{\leq} \left(1 + \frac{1}{Kx_*(\tau)}\right)(r+\rho)e^{r(t-\tau)}, \qquad t \geq \tau.$$

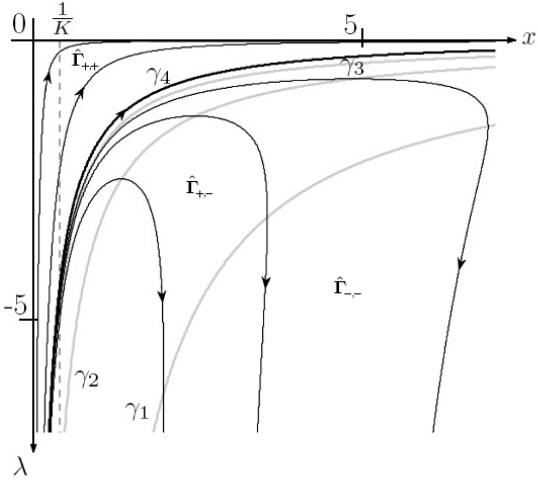
Hence.

$$u_*(t) \overset{a.e.}{\leq} \left(1 + \frac{1}{Kx_*(t)}\right)(r+\rho), \qquad t \geq 0. \qquad \square$$

Phase portrait of the current value Hamiltonian system: $r > \rho$



Phase portrait of the current value Hamiltonian system: $r \leq \rho$



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