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Relative algebraic K-theory and algebraic cyclic homology

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Abstract

Since its introduction over 40 years ago algebraic K-theory, which provides powerful invariants, still remains hard to compute. The subject of this work is the construction of an isomorphism between relative algebraic K-groups and relative algebraic cyclic homology in low dimensions, for certain nilpotent ideals. This isomorphism generalizes the Theorem of Goodwillie [Goo86] concerning rational algebras and provides a more accessible alternative to topological cyclic homology for the computation of algebraic K-groups.

Following roughly the strategy of Goodwillie, the proof is structured into several parts of varying interdependencies.

First, we construct a natural isomorphism between group homology and Lie ring homology of certain associated groups and Lie rings. This represents an integral generalization of a Theorem of Pickel [Pic78] concerning nilpotent groups and also provides a strategy for an integral version of the Theorem of Lazard [Laz65] concerning p-valued groups, which both considered homology with rational coefficients. The theory provides a bridge in form of a natural logarithm map from the homology of the multiplicative to that of the additive K-theory.

Second, we prove that the low-dimensional homotopy groups of an E_∞-space can be identified with the primitive part of its homology by using an improved version of the Hurewicz map. This represents a variant of a Theorem of Beilinson [Bei14] linking both objects up to isogeny. We apply this to the E_∞-space of relative K-theory.

Similarly by using an additive analogue we compute the primitive part of the homology of the Lie algebra homology of matrices as cyclic homology. This can be considered as an integral generalization of the Theorem of Loday, Quillen [LQ84] and Tsygan [Tsy83].

Combining the single steps we are constructing the desired isomorphism between K-theory and cyclic homology and also compare it with the negative Chern character.

Alongside the proofs we provide a comprehensive collection of required abstract tools of simplicial homotopy theory.

As an application of the main theorem we compute the lower relative K-groups of truncated polynomial rings over a subring of the rationals. This shows that our Theorem can be used to obtain new results in the computation of K-groups.

Zusammenfassung

Seit ihrer Einführung vor mehr als 40 Jahren bleibt die K-Theorie, als Lieferant mächtiger Invarianten, schwer zu berechnen. Thema der vorliegenden Arbeit ist die Konstruktion eines Isomorphismus' zwischen relativer algebraischer K-Theorie und relativer algebraischer zyklischer Homologie in niedrigen Dimensionen für gewisse nilpotente Ideale. Dieser Isomorphismus verallgemeinert den Satz von Goodwillie [Goo86] über rationale Algebren und bietet eine einfacher zugängliche Alternative zur topologischen zyklischen Homologie für die Berechnung algebraischer K-Gruppen.

Der Beweis orientiert sich grob an der Strategie von Goodwillie und kann in verschiedene Teile gegliedert werden, die mehr oder weniger unabhängig voneinander sind.

Zunächst konstruieren wir einen natürlichen Isomorphismus zwischen Gruppen- und Lie Ring-Homologie für gewisse assoziierte Gruppen und Lie Ringe. Dieser bildet eine ganzzahlige Verallgemeinerung eines Satzes von Pickel [Pic78] über nilpotente Gruppen und bietet gleichzeitig eine Beweisstrategie des Satzes von Lazard über p-bewertete Gruppen. Diese beiden Sätze treffen lediglich Aussagen für rationale Koeffizienten. Die Theorie bildet eine Brücke in Form einer natürlichen Logarithmusabbildung von multiplikativer in additive K-Theorie.

Darüber hinaus zeigen wir, dass die Homotopiegruppen eines E_∞-Raums in niedrigen Dimensionen mit dem primitiven Teil seiner Homologie identifiziert werden kann. Dies ist eine Variante eines Satzes von Beilinson [Bei14], welcher beide Objekte bis auf Isogenie in Verbindung setzt. Wir wenden dies für den E_∞-Raum der relativen K-Theorie an.

Mittels einer additiven Version dieses Satzes berechnen wir, dass der primitive Teil der Lie Algebren-Homologie von Matrizen mit der zyklischen Homologie übereinstimmt. Dies bildet eine integrale Verallgemeinerung des Satzes von Loday, Quillen [LQ84] und Tsygan [Tsy83].

Das Zusammenspiel der verschiedenen Teilschritte ermöglicht schließlich die Konstruktion des gewünschten Isomorphismus zwischen K-Theorie und zyklischer Homologie. Weiterhin vergleichen wir diesen mit dem negativen Chern Charakter.

Im Rahmen der einzelnen Beweisschritte stellen wir außerdem eine beachtliche Sammlung abstrakter Werkzeuge aus der simplizialen Homotopietheorie zusammen.

Als Anwendungsbeispiel berechnen wir die ersten relativen K-Gruppen abgeschnittener Polynomringe über einem Teilring der rationalen Zahlen. Dies zeigt, dass unser Satz neue Möglichkeiten zur Berechnung der K-Gruppen eröffnet.

Contents

1 Introduction

1.1 Algebraic K-theory

Algebraic K-theory appears in various mathematical fields, such as algebraic geometry, number theory and algebraic topology. The subject arose from the construction of the Grothendieck group K_0 of the category of projective modules, which encodes important information like the class group. While explicit algebraic constructions for lower K-groups could be achieved quite soon, it was a long search for the right definition of higher K-theory. It ended with Quillen's introduction of the plus construction for the classifying space of the general linear group, which required sophisticated topological techniques. The optimism after Quillen's computation of all the K-groups of a finite field [Qui72] did not last very long, as one realized the difficulty of calculating higher K-groups. Like in lower dimensions, there are strong links between higher K-groups and number theoretic problems. The Vandiver conjecture for example is equivalent to the knowledge of certain K-groups of the integers. Another example is the long exact localization sequence for algebraic K-theory, which is an important tool in Iwasawa theory. Starting with the sequence in dimensions 0 and 1, also the higher K-groups become more and more important.

1.2 Comparison with cyclic homology

As the K-groups itself remained mysterious, one tried to compare K-theory to simpler invariants. It began in 1965 with the *Hattori-Stallings trace map* $K_0(A) \longrightarrow A/[A, A]$, that could be extended to a map from the whole K-theory to Hochschild homology by Dennis in 1976. While it recovers some information for finite coefficients, e.g. when applied to number rings [KL02], it is a poor invariant for rational coefficients. In 1985 Connes [Con85] introduced cyclic homology as a non-commutative variant of de Rham cohomology by taking account of the canonical action of the circle on the Hochschild complex. At the same time Loday-Quillen [LQ84] and independently Tsygan [Tsy83] showed that the primitive part of rational Lie algebra homology of matrices is cyclic homology. Guided by Quillen's [Qui69a] characterization of rational homotopy groups as the primitive part of homology and the philosophy that the group homology is related to the homology of its associated Lie algebra, Goodwillie proved in [Goo86] that the relative K-theory of a nilpotent ideal in a rational algebra coincides with its cyclic homology.

While cyclic homology is built as the homotopy orbits of the circle action on the Hochschild complex, one can similarly consider the homotopy fixed points, which leads to the construction of negative cyclic homology. Karoubi [Kar87] constructed the negative Chern character as a lift of Dennis' trace map to negative cyclic homology, which canonically maps to Hochschild homology. Moreover Connes operator B provides a natural map from cyclic homology to negative cyclic homology, which can be imagined as a norm map and is an isomorphism in Goodwillie's setting. Cortiñas-Weibel [CW09] showed that Goodwillie's isomorphism composed with the negative Chern character coincides with B, in other words there is a commutative diagram

$$\begin{tikzcd} K_*(A,I) \arrow[r, equal, "\sim"] \arrow[dr, "\mathrm{ch}^-"'] & HC_{*-1}(A,I) \arrow[d, "B", "\wr"'] \\ & HC_*^-(A,I) \end{tikzcd}$$

While Goodwillie's proof worked fine for rational algebras, there was still need for a comparison theorem that worked for algebras over finite fields. This was the starting point for a completely new theory.

1.3 Topological cyclic homology

Waldhausen [HKV+02] discovered a connection of stable A-theory, which is a generalization of K-theory, and stable homotopy theory. Motivated by his "calculus of functors", Goodwillie conjectured that Dennis' trace map lifts to a map to topological Hochschild homology THH, which should be weakly equivalent to stable K-theory. Topological Hochschild homology is constructed analogous to algebraic Hochschild homology by replacing the tensor products by smash products of spectra and in the context of the symmetric monoidal category of spectra, base change along the initial map from the sphere spectrum to the Eilenberg-Maclane spectrum of the integers $\mathbb{S} \longrightarrow H\mathbb{Z}$ yields a natural map from topological Hochschild homology to algebraic Hochschild homology. Like in the construction of negative cyclic homology, one tried to involve the circle action on THH to provide a good technique to attack K-theory. This leads to the construction of topological cyclic homology and the cyclotomic trace map by Bökstedt, Hsiang and Madsen [BHM93]. Finally McCarthy [McC97] proved that it induces an isomorphism from p-completed relative K-theory for a nilpotent ideal to relative topological cyclic homology.

It must be warned that topological cyclic homology is not obtained by simply taking the homotopy fixed points of the circle action on THH, but involves a more careful treatment and the theory of equivariant stable homotopy theory. There are several computations of K-theory via the cyclotomic trace map leading also to a better understanding of topological cyclic homology, [HM03], [HM97a], [HM97b] among others, but the latter is still much more complicated than algebraic cyclic homology and to work with it requires deep understanding in (equivariant) stable homotopy theory.

1.4 Towards an integral version of Goodwillie's Theorem

The reason, why rationally algebraic cyclic homology works fine for computing K-theory, lies in the fact that the map $\mathbb{S} \longrightarrow H\mathbb{Z}$ becomes an equivalence after rational localization. In particular rationally we do not need to distinguish between topological Hochschild homology and algebraic Hochschild homology. More precisely $\mathbb{S} \longrightarrow H\mathbb{Z}$ becomes more and more connected, the more successive primes we invert, beginning with 2. The same holds for the map $THH(A) \longrightarrow HH(A)$, provided that A is flat over $\mathbb{Z}$. So it seems natural to wonder, if integrally there is still a connection between K-theory and cyclic homology in low dimensions. A first attempt into this direction was established by Brun [Bru01], who used filtrations on topological cyclic homology to prove that after p-completion

$$K_n(A, I) \cong HC_{n-1}(A, I), \qquad 0 \leq n < (p-1)/m - 2,$$

for an ideal $I \lhd A$ with A and A/I flat over $\mathbb{Z}$ and $I^m = 0$. By showing that after truncating the filtration, the cyclotomic structure becomes trivial, he could establish a natural zig-zag of weak equivalences to prove the statement. However due to this abstract approach there is no hint of a direct map that induces this isomorphism.

The present work gives a positive answer to the proposed question in a more general situation. The main goal of this work is to prove the following Theorem.

Theorem 1.4.1 (Theorem 6.3.22)
Let A be a ring with $(p-1)! \in A^\times$, for some prime number $p > 1$. Suppose A carries a finite ring filtration $A = F_0A \supset \ldots \supset F_NA = 0$, such that $\mathrm{gr}^F A$ is flat over $\mathbb{Z}$. Suppose that there is a subset $Y \subset A$, such that

(i) $\mathrm{gr}^F A = \sum_{y \in Y} \mathbb{Z} \cdot [y]$,

(ii) $y^n/n! \in F_1A$, *for all* $y \in Y \cap F_1A$ *and* $n \geq 1$.

Then for $1 \leq n < p-1$ there are isomorphisms inducing a commutative diagram

$$\begin{tikzcd}
K_n(A, F_1A) \arrow[rr, equal, "\sim"] \arrow[d, "\mathrm{ch}^-"'] & & HC_{n-1}(A, F_1A) \arrow[d, "B"] \\
HC_n^-(A, F_1A) \arrow[r, "i_A"] & HC_n^-(D_0^F A, D_1^F A) & HC_n^-(A, F_1A), \arrow[l, "i_A"']
\end{tikzcd}$$

where ch^- is the relative negative Chern character, B is Connes' operator and the two horizontal maps are induced by the canonical inclusion $A \xhookrightarrow{i_A} D_0^F A$ into the divisible closure.

Note that Connes operator B has the same connectivity as Brun's isomorphism, which gives another proof for his result. It is moreover worth noting that by following (a modification of) Goodwillie's strategy we are taking a completely different approach to the result. The main improvement in comparison with Brun's Theorem is, that we loose the dependency of the degree of nilpotency of the given ideal. Apart from that there is no need for p-completion in our setting.

Note further that every nilpotent ideal in a rational algebra satisfies the hypotheses of our Theorem and hence Goodwillie's Theorem is a direct Corollary.

1.5 Outline of the work

While proving the main result several generalizations had to be established, which partly evolved into a new theory.

Notation and categorial foundations

In chapter 2 we fix the basic notation, that we will use throughout the thesis.

Homology of associated groups and Lie rings

In chapter 3 we are generalizing a Theorem of Pickel [Pic78] linking homology of nilpotent groups and Lie rings. A lot of effort is required to set up the theory. The theory is inspired by Lazard's theory of p-valued groups [Laz65]. The main difference is, that we are using $\mathbb{N}_0$-indexed filtrations and develop an abstract approach, while he is working with valuations and his proofs have a more computational flavor. Despite that of course his axioms are slightly different, as our theory is a zero characteristic variant of his mixed characteristic theory.

The main result of this chapter can be summarized as follows.

Theorem 1.5.1 (Theorem 3.5.2)
Let $G \in \mathcal{G}rp$ and $\mathfrak{g} \in \mathcal{L}ie$ carrying filtrations F with $\operatorname{gr}^F G$ and $\operatorname{gr}^F \mathfrak{g}$ flat over $\mathbb{Z}$.

If G and $\mathfrak{g}$ are associated, then for every left/right module M over $\widehat{D}_0^F(\mathbb{Z}[G]) \overset{\lambda}{\cong} \widehat{D}_0^F(U_{\mathbb{Z}}(\mathfrak{g}))$ carrying a compatible module filtration F, there is a natural isomorphism

$$\widehat{H}_*(G, M) \cong \widehat{H}_*(\mathfrak{g}, M).$$

We introduce the notion of associated groups and Lie rings in Definition 3.5.1 and also give several examples of associated groups and Lie rings.

Proposition 1.5.2 (Proposition 3.5.5 and Remark 3.5.6)
For $X \in \mathcal{S}et$, the free (abelian) group ${}^X\mathbb{Z}$ and the free (abelian) Lie algebra $\mathcal{L}ie(\mathbb{Z}X)$ are associated via

$$\lambda : \widehat{D}_0^\Gamma \mathbb{Z}[{}^X\mathbb{Z}] \xrightarrow{\sim} \widehat{D}_0^\Gamma U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X)), \qquad x \longmapsto \exp(x).$$

Proposition 1.5.3 (Proposition 3.5.7)
Let $A \in \mathcal{A}ss$ carrying a complete filtration F, such that $\operatorname{gr}^F A$ is flat and $A = F_1 A$.

Then giving $G = 1 + A \leq (A_+)^\times$ and $\mathfrak{g} = A$ the induced filtrations, there is an isomorphism

$$\lambda_A : \widehat{D}_0^F \mathbb{Z}[1 + A] \xrightarrow{\sim} \widehat{D}_0^F U_{\mathbb{Z}}(A), \quad \underline{1 + a} \longmapsto \exp \circ s_A \circ \log \circ i_A(1 + a).$$

Moreover the following holds.

(i) *λ_A is natural in A.*

(ii) *Let $x \in A$ such that $x^n/n \in A$, for all $n \geq 1$. Then $\lambda_A(\underline{1 + x}) = 1 + x$.*

(iii) *$G = 1 + A$ and $\mathfrak{g} = A$ are associated via λ_A, if there is a subset $X \subset A$, such that*
  - *a)* $\mathrm{gr}^F A = \sum_{x \in X} \mathbb{Z} \cdot [x]$.
  - *b)* $x^n/n! \in A$, *for all $x \in X$ and $n \geq 1$.*

We want to mention, that by our approach to develop the theory from scratch, we are reproving several well-known results on the way. Worth mentioning is the Theorem of Poincaré, Birkhoff and Witt for Lie algebras over the integers 3.3.6 or the Theorem of Ado 3.3.16. Moreover there are a lot of slight generalizations of well-known results by using well-known techniques (like e.g. the Artin-Rees theory for the proof of Proposition 3.1.8). We will point out at each single step, which parts are new and which are already known.

E_∞-spaces and their homotopy groups

Following Beilinson [Bei14], the main purpose of chapter 4 is to verify the two homotopy theoretical tools below. We moreover introduce the category I of injections, that is used to construct E_∞-spaces as algebras over the I-operad of monoids that are commutative up to homotopy and also plays an important role in the subsequent chapter.

Proposition 1.5.4 (Proposition 4.3.10)
Let $k \in \mathcal{C}Ring$ with $(p-1)! \in k^\times$, for some prime number $p > 1$.
Let $X \in dg(k\text{-Mod})$ with $X_n = 0$, for all $n < c$, for some $c \geq 0$.
Then the map below is an isomorphism, for all $0 \leq n < pc$.

$$H_n X \xrightarrow{\iota_1} PH_n(\mathcal{C}om_1 X) := \ker\left(H_n(\mathcal{C}om_1(X)) \underset{(\eta\times\mathrm{id})_* + (\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(\mathcal{C}om_1(X \times X)) \right).$$

Here δ is the diagonal map and $0 \xrightarrow{\eta} X$ is the initial map of chain complexes.

Proposition 1.5.5 (Corollary 4.3.13)
Let X be a $(c-1)$-connected E_∞-space, for some $c > 0$.
Suppose $H_*(X, \mathbb{Z}) \xrightarrow{\sim} H_*(X, k)$, where $k = \mathbb{Z}[1/(p-1)!]$, for some $p > 1$.
Then the Hurewicz map induces an isomorphism, for all $0 \leq n \leq \min\{c_{c,p}, 2p + c - 4\}$ (cf. Proposition 4.3.1),

$$h = \eta_X - 1 : \pi_n X \xrightarrow{\sim} PH_n(X, k) := \ker\left(H_n(X, k) \underset{(\eta\times\mathrm{id})_* + (\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(X \times X, k) \right),$$

where $X \xrightarrow{\delta} X \times X$ *is the diagonal and* $1 \xrightarrow{\eta} X$ *is the unit map.*

Beilinson proves this in every dimension but only up to isogeny. So his and our results are generalizations in different directions of Quillen's [Qui69a] description of the rational homotopy groups as the primitive part of homology.

Cyclic homology and the Lie algebra homology of matrices

In chapter 5 we are first recalling the definition and most important facts of cyclic homology.

Roughly following Aboughazi-Ogle [AO94] the main result is to prove a generalization of the Theorem of Loday-Quillen and Tsygan.

Theorem 1.5.6 (Theorem 5.4.20)
Let $A \in k/\mathcal{R}ing$ *be flat with* $(p-1)! \in A^\times$.
Then the map ϕ *induces isomorphisms in dimensions* $0 \leq n < p-1$

$$H^\lambda_{n-1}(A) \xrightarrow{\sim} PH_n(\mathfrak{gl}_\infty A, k) = \ker\left(H_n(\mathfrak{gl}_\infty A, k) \underset{(\eta\times \mathrm{id})_*+(\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(\mathfrak{gl}_\infty A \times \mathfrak{gl}_\infty A, k)\right).$$

Here $\mathfrak{gl}_\infty A \xrightarrow{\delta} \mathfrak{gl}_\infty A \times \mathfrak{gl}_\infty A$ *is the diagonal and* $0 \xrightarrow{\eta} \mathfrak{gl}_\infty A$ *is the initial Lie algebra homomorphism.*

Moreover Connes' operator B *and the negative Chern character for Hopf algebras induce a commutative diagram*

$$\begin{tikzcd}
HC_{*-1}(A) \arrow[rrrr, "B"] \arrow[d] & & & & HC^-_*(A) \\
H^\lambda_{*-1}(A) \arrow[r, "\sim"] & PH_*(\mathfrak{gl}_\infty A, k) \arrow[r, hook] & H_*(B_*U_k(\mathfrak{gl}_\infty A)) \arrow[r, "\mathrm{ch}^-"] & HC^-_*(U_k(\mathfrak{gl}_\infty A)) \arrow[r] & HC^-_*(M_\infty A) \arrow[u, "\mathrm{trace}"']
\end{tikzcd}$$

Aboughai-Ogle give a rational proof of the Theorem of Loday-Quillen-Tsygan, which is different from the original one and adapts better to an integral generalization. However there are several steps in their proof that are trivial rationally but require deeper thoughts for the integral variant. The most important parts are the verification, that the symmetric group action is trivial on homology in low dimensions (see Corollary 5.4.5) and the following stability result, which uses the former result and explicit computations of the homology of the symmetric groups.

Proposition 1.5.7 (Proposition 5.4.9)
Let $A \in k/\mathcal{R}ing$ *be flat with* $(p-1)! \in A^\times$*, for some* $p > 1$.
Then $H_*(\mathfrak{gl}_{r-1}A, k) \longrightarrow H_*(\mathfrak{gl}_r A, k)$ *is* $(\min(r,p)-2)$*-connected, for all* $r \geq 1$.

It is also worth noting, that Theorem 5.4.20 is independent of the characteristic of the ground ring k. As far as we know, there is no result for the positive characteristic case in present literature.

The commutativity of the square in Theorem 5.4.20 is established by following Cortiñas-Weibel to construct the negative Chern character for Hopf algebras. While they were only interested in the rational case, they did not care, if the proofs are also going through integrally. In fact they do and so this section is more or less an import of their results into our notation. In particular there is no claim for originality here.

Multiplicative vs. additive K-theory

Following Quillen [Qui71a] we begin chapter 6 by defining multiplicative and additive K-theory via plus constructions for simplicial groups and for simplicial Lie algebras as well. Using a slightly different technique this was already done by Pirashvili [Pir85] a long time ago, but recently has found new interest in the context of operads (see [Liv99] and [CRS04]). Following [Lod98] we are likewise constructing the Volodin constructions parallel in the multiplicative and additive situation, which allows us to develop the two theories in perfect analogy. In both cases the plus construction of the Volodin construction can be identified with the relative K-theory and techniques of the multiplicative theory can easily be adapted to the additive one. Using the category of injections, we explicitly construct a plus-construction for $BGL(A)$, that has a natural E_∞-structure (however we found out later, that this is not new). Finally the interplay of the tools established in the chapters before allows us to prove the main theorem.

Appendix: Simplicial homotopy theory

In chapter 7 we are collecting the results of abstract simplicial homotopy theory and fixing the notation, that we need. The few given proofs are due to ourselves, but they are probably well-known in literature. Therefore let us emphasize at this point, that we do not claim the presented results to ourselves. We unfortunately could not find a reference, that takes account of everything we need.

Appendix: Simplicial homotopy theory

Same also holds for most parts of chapter 8. So let us point out, which parts are new. We do not know any reference for

Proposition 1.5.8 (Proposition 8.1.10)
Let $k \in \mathcal{C}Ring$ and $A \in \mathcal{C}\mathrm{AT}(I, k\text{-}\mathcal{A}ss)$, for some $I \in \mathcal{C}at$.
Then there is a natural map of simplicial k-modules

$$\operatorname*{hocolim}_I B(k, A_+, Y) \longrightarrow B(k, \operatorname*{hocolim}_I A_+, Y), \qquad Y \in (\operatorname*{colim}_I A_+)\text{-Mod},$$

which is a weak equivalence, if BI is contractible and A_i is a flat k-module, for all $i \in I$.

It plays an important role in the identification of the relative additive Volodin construction with the homology of the corresponding simplicial Lie algebra (cf. Proposition 6.2.8).

Other important tools are the Whitehead Theorems for simplicial groups and Lie algebras. We call it that way, because they are in perfect analogy to the well-known Theorem of Whitehead for spaces/simplicial sets. Even the proof in the group case is not new as it is adapted from a similar result of Quillen for profinite groups [Qui69b]. The innovation in the case of simplicial Lie algebras is the observation that it works in exactly the same way, when using derived Lie algebra homology, a definition of which we have not found in current literature. The technical core of the Whitehead Theorems is the Connectivity Proposition for spectral sequences 8.4.6. While it has strong similarity to Zeeman's Theorem [Zee57] and its generalizations [HR76], we do not know any reference, that states it in this form.

2 Notation and categorial foundations

2.1 Categories

In this document a **category** $\mathcal{C}$ is a class of objects $\mathrm{Obj}(\mathcal{C})$ together with (homo-)morphism sets $\mathcal{C}(X,Y)$, for each pair of objects $X, Y \in \mathcal{C}$. If $X = Y$, we also use the shorter notation $\mathcal{C}(X) = \mathcal{C}(X,X)$. A category is called **small**, if its objects form a set, otherwise it is called **large**. We will abreviate large categories by curly letters and small ones by (capital) latin letters.

Example 2.1.1
In what follows, we list the most important examples of categories, that appear here.

(i) $\mathcal{S}et$, *the category of sets with maps as homomorphisms. This may be the simplest example of a category that is not small.*

(ii) $\mathcal{G}rp$ ($\mathcal{A}b$), *the category of (abelian) groups with group homomorphisms.*

(iii) $\mathcal{R}ing$ ($\mathcal{C}Ring$), *the category of (commutative) rings.*

(iv) $R\text{-Mod}, \text{Mod-}R = R^{\mathrm{op}}\text{-Mod}, (R,S)\text{-Mod} = (R \otimes S^{\mathrm{op}})\text{-Mod}$ *with* $R, S \in \mathcal{R}ing$, *the categories of left, right and bimodules.*

(v) *The category* $\mathcal{C}at$ *of small categories with functors as morphisms.*

If the homomorphism "sets" carry more structure or more generally, if a particular category $\mathcal{C}$ is enriched over another one, we will write $\underline{\mathcal{C}}(X,Y)$ the object of morphisms from X to Y.

Example 2.1.2 (i) *The category* $\mathcal{A}b$ *is enriched over itself. For two abelian groups* $X, Y \in \mathcal{A}b$, *we call* $\mathcal{A}b(X,Y)$ *the set of homomorphisms, while* $\underline{\mathcal{A}b}(X,Y)$ *is the abelian group of homomorphisms.*

(ii) *For two small categories* $X, Y \in \mathcal{C}at$, *we call* $\underline{\mathcal{C}at}(X,Y)$ *the category of functors from* X *to* Y *with natural transformations as morphisms.*

For example $\underline{\mathcal{C}at}(\underline{1}, Y)$ *is the category of morphisms* $Y_0 \longrightarrow Y_1$, *whose morphisms are commuting squares.*

We also want to talk about functor categories with target in a large category. To keep notation simple and coherent with everything said before, we introduce $\mathcal{C}\mathrm{AT}$ as the conglomerate of all (in particular the large) categories. Of course we want the morphisms from one category to another to be functors between them. But these do not form a set, so we cannot talk about $\mathcal{C}\mathrm{AT}$ as a category [1]. Using Grothendieck's method one can come around this problem by enlarging our universe and call $\mathcal{C}\mathrm{AT}$ a category in our enlarged universe. We do not want to go more in detail, but want to use this sloppy notation to keep things simple. Further details about can be read in any book about category theory (e.g. [ML98]). With this notation we can also talk about the following category.

Example 2.1.3
For $I \in \mathcal{C}at$ and $\mathcal{C} \in \mathcal{C}\mathrm{AT}$, we also denote by $\underline{\mathcal{C}\mathrm{AT}}(I, \mathcal{C})$ the category of functors from I to $\mathcal{C}$ with natural transformations as morphisms.

If I was not small, the natural transformations would not form a set. In particular $\underline{\mathcal{C}\mathrm{AT}}(I, \mathcal{C})$ would not be a category in this case.

2.2 Limits and colimits, ends and coends

We will also introduce some general notation for limits and colimits in arbitrary categories. For $I \in \mathcal{C}at$ and $\mathcal{C} \in \mathcal{C}\mathrm{AT}$, let $X \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})$.

- We denote by $\lim X = \lim_{i\in I} X(i)$ its limit in $\mathcal{C}$, if it exists. If the category I^{op} is filtered, we will also use the notation $\varprojlim_{i\in I} X(i) = \lim X$. We use the letters $\lim X \xrightarrow{\pi_i} X(i)$ for the canonical morphisms, where $i \in I$. If $\mathcal{C}$ is **(finitely) complete**, i.e. has limits over small (finite) categories, and limits can be constructed functorially, we have an adjunction

$$\underline{\mathcal{C}\mathrm{AT}}(I, \mathcal{C})(\mathrm{const}\, X, Y) = \mathcal{C}(X, \lim Y),$$

  where $\mathrm{const}\, X$ is the constant functor, sending $i \in I$ to X and every morphism to the identity on X. Then the natural transformation $(\pi_i)_{i\in I}$ defines the counit of the adjunction. Moreover, given a natural transformation $\mathrm{const}\, X \xrightarrow{f} Y$, we call $(f_i)_{i\in I} = \lim_{i\in I} f_i : X \longrightarrow \lim Y$ the morphism induced by the universal property of the limit.

- If $I \in \mathcal{C}at$ is discrete, i.e. has no non-trivial morphisms, then $\prod_{i\in I} X(i) = \lim X$ is called a **product**. A morphism in $\mathcal{C}$ is called a **projection**, if it is one of the canonical morphisms corresponding to a product (i.e. to a limit over a discrete category). The product of a functor X indexed by a finite discrete category (i.e. set) $\{1, ..., n\}$ will be denoted by $X_1 \times ... \times X_n$ or simply $X_1 \times X_2$, if $n = 2$. For a tuple of morphisms $(C \xrightarrow{f_i} X_i)_{1\le i\le n}$, its morphism induced by the universal property is denoted by

$$(f_1, ..., f_n) : C \longrightarrow X_1 \times ... \times X_n.$$

[1] Other authors avoid this problem by also allowing the morphisms to be proper classes. What we called a category, they call a **locally presentable** category.

We will also use "$\times$" as a functor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$. So in this notation we have

$$f_1 \times f_2 = (f_1 \circ \pi_1, f_2 \circ \pi_2) \in \mathcal{C}(X_1 \times X_2, Y_1 \times Y_2), \quad f_i \in \mathcal{C}(X_i, Y_i),\ X_i, Y_i \in \mathcal{C},\ i = 1, 2.$$

The product over a constant functor $\operatorname{const} X$ indexed by a discrete category $S \in \mathcal{S}et$ is called **power** and denoted by X^S. If $\mathcal{C}$ has functorial powers, we get a natural adjunction

$$\mathcal{S}et^{\mathrm{op}}(\mathcal{C}(X, C), Y) = \mathcal{C}(X, C^Y), \qquad C \in \mathcal{C}.$$

- If $I = 1 \to 2 \leftarrow 3$ and $X \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})$, then $X_1 \times_{X_2} X_3 = \lim X$ is called a **pullback** and a square

$$\begin{tikzcd} X_4 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_3 \arrow[r] & X_2 \end{tikzcd}$$

  is called **cartesian**, if the canonical map $X_4 \xrightarrow{\sim} X_1 \times_{X_2} X_3$ is an isomorphism.

- If $I = \ 1 \rightrightarrows 2$, then $\ker X = \lim X$ is called a **kernel** of the morphism pair.

- A **terminal** object is a limit of the empty category and denoted by $*$, if it exists. For $X \in \mathcal{C}$ we let $X \xrightarrow{\varepsilon} *$ denote the unique morphism.

Dually we will also introduce analogous notation for colimits.

- We denote by $\operatorname{colim} X = \operatorname{colim}_{i \in I} X(i)$ its colimit in $\mathcal{C}$, if it exists. If the category I is filtered, we will also use the notation $\varinjlim_{i \in I} X(i) = \operatorname{colim} X$. We use the letters $X(i) \xrightarrow{\iota_i} \operatorname{colim} X$ for the canonical morphisms, where $i \in I$. If $\mathcal{C}$ is **(finitely) cocomplete**, i.e. has colimits over small (finite) categories, and colimits can be constructed functorially, we have an adjunction

$$\mathcal{C}(\operatorname{colim} X, Y) = \underline{\mathcal{C}\mathrm{AT}}(I, \mathcal{C})(X, \operatorname{const} Y).$$

  Then the natural transformation $(\iota_i)_{i \in I}$ defines the counit of the adjunction. Moreover, given a natural transformation $X \xrightarrow{f} \operatorname{const} Y$, we call $\operatorname{colim}_{i \in I} f_i : \operatorname{colim} X \longrightarrow Y$ the morphism induced by the universal property of the limit.

- If $I \in \mathcal{C}at$ is discrete, i.e. has no non-trivial morphisms, then $\coprod_{i \in I} X(i) = \operatorname{colim} X$ is called a **coproduct**. A morphism in $\mathcal{C}$ is called a **inclusion**, if it is one of the canonical morphisms corresponding to a coproduct (i.e. to a limit over a discrete category). The coproduct of a functor X indexed by a finite discrete category (i.e. set) $\{1, ..., n\}$ will be denoted by $X_1 + ... + X_n$ or simply $X_1 + X_2$, if $n = 2$. For a tuple of morphisms $(X_i \xrightarrow{f_i} C)_{1 \le i \le n}$, its morphism induced by the universal property is denoted by

$$f_1 \cup ... \cup f_n : X_1 + ... + X_n \longrightarrow C.$$

  We will also use "$+$" as a functor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$. So in this notation we have

$$f_1 + f_2 = (\iota_1 \circ f_1) \cup (\iota_2 \circ f_2) \in \mathcal{C}(X_1 \times X_2, Y_1 \times Y_2), \quad f_i \in \mathcal{C}(X_i, Y_i),\ X_i, Y_i \in \mathcal{C},\ i = 1, 2.$$

The coproduct over a constant functor const X indexed by a discrete category $S \in \mathcal{S}et$ is called a **copower** and denoted by ${}^S X$. If $\mathcal{C}$ has functorial copowers, we get a natural adjunction

$$\mathcal{C}({}^X C, Y) = \mathcal{S}et(X, \mathcal{C}(C, Y)), \qquad C \in \mathcal{C}.$$

- If $I = 1 \leftarrow 2 \rightarrow 3$ and $X \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})$, then $X_1 +_{X_2} X_3 = \operatorname{colim} X$ is called a **pushout** and a square
$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_3 \arrow[r] & X_4 \end{tikzcd}$$
is called **cocartesian**, if the canonical map $X_1 +_{X_2} X_3 \xrightarrow{\sim} X_4$ is an isomorphism.

- If $I = 1 \rightrightarrows 2$, then $\operatorname{coker} X = \operatorname{colim} X$ is called a **cokernel** of the morphism pair.

- An **initial** object is a colimit of the empty category and denoted by $\emptyset$, if it exists. For $X \in \mathcal{C}$ we let $\emptyset \xrightarrow{\eta} X$ denote the unique morphism.

If initial and terminal object coincide, it is called a zero object and denoted by 0. In the context of an abelian category, finite products and coproducts coincide and are called biproducts. These will be denoted with the symbol "$\oplus$" and the symbol "+" will only refer to the sum (i.e. the pushout over their intersection) of two subobjects of a third object in this context.

We will also use the notion of ends and coends.

Definition 2.2.1
Let $X \in \mathcal{C}\mathrm{AT}(I \times I^{\mathrm{op}}, \mathcal{C})$ with $I \in \mathcal{C}at$ and $\mathcal{C} \in \mathcal{C}\mathrm{AT}$.

(i) *The **end** over X is defined (if existing) as an object $\int_{i \in I} X(i,i) \in \mathcal{C}$ together with morphisms*
$$\int_{i \in I} X(i,i) \xrightarrow{\pi_i} X(i,i), \qquad i \in I,$$
being terminal with the property, that it induces commutative diagrams [2]
$$\begin{tikzcd} \int_{i \in I} X(i,i) \arrow[r, "\pi_i"] \arrow[d, "\pi_j"'] & X(i,i) \arrow[d, "X(\mathrm{id}_i, f)"] \\ X(j,j) \arrow[r, "X(f, \mathrm{id}_j)"] & X(i,j). \end{tikzcd} \qquad f \in I(i,j), \qquad i, j \in I.$$

[2]See also [ML98] IX.5 for more details.

(ii) *Dually the* ***coend*** *over X is defined (if existing) as an object $\int^{i\in I} X(i,i)$ together with morphisms*

$$X(i,i) \xrightarrow{\iota_i} \int_{i\in I} X(i,i), \qquad i \in I,$$

being initial with the property, that it induces commutative diagrams

$$\begin{tikzcd} X(j,i) \arrow[r, "X(f{,}\mathrm{id}_i)"] \arrow[d, "X(\mathrm{id}_j{,}f)"'] & X(i,i) \arrow[d, "\iota_i"] \\ X(j,j) \arrow[r, "\iota_j"] & \int_{i\in I} X(i,i). \end{tikzcd} \qquad f \in I(i,j), \qquad i,j \in I.$$

Remark 2.2.2
Let $X \in \mathcal{C}\mathrm{AT}(I \times I^{\mathrm{op}}, \mathcal{C})$ with $I \in \mathcal{C}at$ and $\mathcal{C} \in \mathcal{C}\mathrm{AT}$.

(i) *If $\mathcal{C}$ is complete, then the end over X is given as the kernel*

$$\int_{i\in I} X(i,i) \hookrightarrow \prod_{i\in I} X(i,i) \underset{X(\text{-},f)}{\overset{X(f,\text{-})}{\rightrightarrows}} \prod_{i\xrightarrow{f}j} X(i,j).$$

(ii) *If $\mathcal{C}$ is cocomplete, then the coend is given as the cokernel*

$$\prod_{i\xrightarrow{f}j} X(i,j) \underset{X(\text{-},f)}{\overset{X(f,\text{-})}{\rightrightarrows}} \prod_{i\in I} X(i,i) \twoheadrightarrow \int_{i\in I} X(i,i).$$

Recall the Yoneda-Lemma and its consequences for (co-)ends.

Lemma 2.2.3 (Yoneda)
Let $F \in \mathcal{C}\mathrm{AT}(I, \mathcal{S}et)$ with $i \in I \in \mathcal{C}at$. Then naturally

$$F(i) \xrightarrow{\sim} \underline{\mathcal{C}\mathrm{AT}}(I, \mathcal{S}et)(I(i,\text{-}), F), \quad f \longmapsto [I(i,j) \ni g \longmapsto F(g)(f)].$$

Corollary 2.2.4
Let $i \in I \in \mathcal{C}at$ and $\mathcal{C} \in \mathcal{C}\mathrm{AT}$. Then naturally

(i) $\underline{\mathcal{C}\mathrm{AT}}(I,\mathcal{C})(X,Y) = \displaystyle\int_i \mathcal{C}(X(i),Y(i)), \qquad X,Y \in \mathcal{C}\mathrm{AT}(I,\mathcal{C}),$

(ii) $((X(f))_{f\in I(i,j)})_{i\in I} : X(i) \xrightarrow{\sim} \displaystyle\int_j X(j)^{I(i,j)}, \qquad X \in \mathcal{C}\mathrm{AT}(I,\mathcal{C}),$

(iii) $\displaystyle\coprod_{j\in I} \coprod_{f\in I(j,i)} X(f) : \int^j {}^{I(j,i)}X(j) \xrightarrow{\sim} X(i), \qquad X \in \mathcal{C}\mathrm{AT}(I,\mathcal{C}).$

Proof.

(i) By definition of the end a tuple $f \in \prod_{i \in I} \mathcal{C}(X(i), Y(i))$ is a natural transformation, if and only if it is an element of the end.

(ii) For every $C \in \mathcal{C}$, applying $\mathcal{C}(C, \text{-})$ to the given morphism induces a bijection by (i) and the Yoneda-Lemma. Equivalenlty it is an isomorphism.

(iii) For every $C \in \mathcal{C}$, applying $\mathcal{C}(\text{-}, C)$ to the given morphism induces a bijection by (i) and the Yoneda-Lemma. Equivalenlty it is an isomorphism.

□

Corollary 2.2.5
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $f \in \mathcal{C}at(I, J)$.

(i) *Composition with f induces a functor $\underline{\mathcal{C}\mathrm{AT}}(J, \mathcal{C}) \xrightarrow{f^*} \underline{\mathcal{C}\mathrm{AT}}(I, \mathcal{C})$.*

(ii) *If $\mathcal{C}$ is complete f^* has a right adjoint*

$$\underline{\mathcal{C}\mathrm{AT}}(I, \mathcal{C})(f^* X, Y) = \mathcal{C}\mathrm{AT}(J, \mathcal{C})(X, f_* Y),$$

*where $f_* Y = \int_{i \in I} Y(i)^{I(\text{-}, f(i))}$ is called the **right Kan extension of** Y **along** f.*
If f has a left adjoint e, then $f_* = e^*$ and we do not require $\mathcal{C}$ to be complete.

(iii) *If $\mathcal{C}$ is cocomplete f^* has a left adjoint*

$$\underline{\mathcal{C}\mathrm{AT}}(J, \mathcal{C})(f_! X, Y) = \underline{\mathcal{C}\mathrm{AT}}(I, \mathcal{C})(X, f^* Y),$$

*where $f_! Y = \int^{i \in I} {}^{I(f(i), \text{-})} X(i)$ is called **left Kan extension of** X **along** f.*
If f has a right adjoint g, then $f_! = g^*$ and we do not require $\mathcal{C}$ to be cocomplete.

2.3 Symmetric monoidal categories

Definition 2.3.1
Let $(\mathcal{C}, \otimes, E)$ be a (symmetric) monoidal category.

(i) *We call $\mathcal{C}$-$\mathcal{M}ag$ the category of $\mathcal{C}$**-magmas**, whose objects are $\mathcal{C}$-objects M together with a binary operation $\mu \in \mathcal{C}(M \otimes M, M)$ and whose morphisms are $\mathcal{C}$-morphisms being compatible with the multiplication morphisms.*

(ii) *Similarly we call $\mathcal{C}$-$\mathcal{M}ag_1$ the category of **unital $\mathcal{C}$-magmas**, whose objects are magmas M together with a unit morphism $E \xrightarrow{\eta} M$ such that $\mu \circ (\mathrm{id} \otimes \eta) = \mu \circ (\eta \otimes \mathrm{id})$ is the identity on M and whose morphisms are $\mathcal{C}$-morphisms being compatible with μ and η.*

(iii) *We let $\mathcal{C}$-$\mathcal{C}om \leq \mathcal{C}$-$\mathcal{A}ss \leq \mathcal{C}$-$\mathcal{M}ag$ and $\mathcal{C}$-$\mathcal{C}om_1 \leq \mathcal{C}$-$\mathcal{A}ss_1 \leq \mathcal{C}$-$\mathcal{M}ag_1$ denote the full subcategories of **commutative magmas** and **associative magmas**.*

(iv) *Dually we define the category of (unital) (commutative, associative) $\mathcal{C}$-comagmas as $\mathcal{C}\text{-}\mathcal{M}ag^{\mathrm{op}} = ((\mathcal{C}^{\mathrm{op}})\text{-}\mathcal{M}ag)^{\mathrm{op}}$, ...*

(v) *For a magma $M \in \mathcal{C}\text{-}\mathcal{M}ag$, we call $M\text{-}\mathcal{C}$ the category of objects $X \in \mathcal{C}$ with left M-action $M \otimes X \xrightarrow{\lambda} X$ whose morphisms are $\mathcal{C}$-morphisms compatible with the action. If M is unital we moreover require that $\lambda \circ (\eta \otimes \mathrm{id})$ is the identity on X. In analogy $\mathcal{C}\text{-}M$ is the category of objects with right M-action.*

Remark 2.3.2
Let $(\mathcal{C}, \otimes, E)$ be a monoidal category. Suppose $\mathcal{C}$ has functorial countable coproducts preserved by "$\otimes$".

Then the following holds.

(i) *There is an adjunction*

$$\mathcal{C}\text{-}\mathcal{M}ag(\mathcal{M}ag(X), Y) = \mathcal{C}(X, U(Y)),$$

*where $\mathcal{M}ag(X) = \coprod_{n>0} \mathcal{M}ag^{(n)}(X)$ is the **free $\mathcal{C}$-magma**, inductively defined by*

$$\mathcal{M}ag^{(1)}(X) = X, \qquad \mathcal{M}ag^{(n)}(X) = \coprod_{0<i<n} \mathcal{M}ag^{(i)}(X) \otimes \mathcal{M}ag^{(n-i)}(X), \quad n > 1,$$

and multiplication $\mathcal{M}ag(X) \otimes \mathcal{M}ag(X) \xrightarrow{\mu} \mathcal{M}ag(X)$ given by the union of the cannical maps

$$\mathcal{M}ag^{(p)}(X) \otimes \mathcal{M}ag^{(q)}(X) \longrightarrow \mathcal{M}ag^{(p+q)}(X) \longrightarrow \mathcal{M}ag(X).$$

(ii) *Similarly we have functorial free associative $\mathcal{C}$-magmas, given by*

$$\mathcal{A}ss(X) = \coprod_{n\geq 1} \mathcal{A}ss^{(n)}(X), \qquad \mathcal{A}ss^{(n)}(X) = X^{\otimes n}, \qquad n \geq 1, \quad X \in \mathcal{C}.$$

(iii) *If $(\mathcal{C}, \otimes, E)$ is symmetric monoidal, we have a functorial free commutative, associative $\mathcal{C}$-magmas*

$$\mathcal{C}om(X) = \coprod_{n\geq 1} \mathcal{C}om^{(n)}(X), \qquad \mathcal{C}om^{(n)}(X) = X^{\otimes n}/\Sigma_n, \qquad n \geq 1, \quad X \in \mathcal{C},$$

where the symmetric group Σ_n acts by permutation of the factors using the symmetric monoidal structure of $\mathcal{C}$.

(iv) *For the unital variants, note that there is an adjunction*

$$\mathcal{C}\text{-}\mathcal{M}ag_1(\underbrace{E + X}_{X_+ :=}, Y) = \mathcal{C}\text{-}\mathcal{M}ag(X, U(Y)),$$

and similarly also for $\mathcal{A}ss$ and $\mathcal{C}om$. Combining this with the free magma adjunction, we get free unital functors by

$$\mathcal{M}ag_1(X) = E + \mathcal{M}ag(X), \qquad \mathcal{A}ss_1(X) = E + \mathcal{A}ss(X), \qquad \mathcal{C}om_1(X) = E + \mathcal{C}om(X).$$

Example 2.3.3
Consider the symmetric monoidal category $(k\text{-Mod}, \otimes_k, k)$, *for some* $k \in \mathcal{C}Ring$. *Then*

(i) $\mathcal{R}ing = (\mathbb{Z}\text{-Mod})\text{-}\mathcal{A}ss_1$ *and* $\mathcal{C}Ring = (\mathbb{Z}\text{-Mod})\text{-}\mathcal{C}om_1$.

(ii) *For* $R \in k\text{-}\mathcal{A}ss_1$ *we have* $R\text{-Mod} = R\text{-}(k\text{-Mod})$.

(iii) *For* $X \in k\text{-Mod}$ *the following holds.*

a) $\mathcal{A}ss_1(X) = \bigoplus_{n\geq 0} X^{\otimes n}$ *is the tensor algebra.*

b) $\mathcal{C}om_1(X) = \bigoplus_{n\geq 0} X^{\otimes n}/\Sigma_n$ *is the symmetric algebra.*

(iv) *We let* $k\text{-}\boldsymbol{\mathcal{L}}ie$ *denote the category of Lie algebras over* k.

(v) *For each* $N \geq 1$ *and each type of algebra we also define the* N*-nilpotent variants. For example* $k\text{-}\mathcal{A}ss^{<N} \leq k\text{-}\mathcal{A}ss$ *is the full subcategory of associative* k*-algebras* A *with* $A^N = 0$.

3 Homology of associated groups and Lie rings

Goal of this chapter is to establish an isomorphism between integral group homology and Lie ring homology (see Theorem 3.5.2). This will be the essential tool to link the multiplicative and additive Volodin constructions in chapter 6. Following ideas of Pickel [Pic78] and Lazard [Laz65] we are constructing canonical filtrations on group ring and enveloping algebra and study the structure of the associated graded algebras. Contrary to the case [Pic78] these filtrations cannot always be obtained as adic filtrations with respect to the augmentation ideal, as we will later point out. By using Artin-Rees theory, we use use these filtrations to identify group and Lie ring homology with their completed variants and then identifying the latter ones.

In this chapter and in the context of k-modules over some commutative ring $k \in \mathcal{C}Ring$, the (plain) symbol "$\otimes$" will always stand for the tensor product "$\otimes_k$" over this ring k. In the context of abelian groups, so modules over the integers, this is the tensor product over $\mathbb{Z}$. Similarly a flat k-module will always be a k-module, that is flat over the ring k, if it is not explicitly declared flat over some other ring (e.g. the integers).

3.1 Filtered modules

Definition 3.1.1
Let $k \in \mathcal{C}Ring$.

(i) *A **($\mathbb{Z}$-indexed module) filtration** F on a module $M \in k\text{-Mod}$ is a $\mathbb{Z}$-indexed sequence of submodules*

$$M \geq ... \geq F_{-1}M \geq F_0M \geq F_1M \geq ...$$

*A **filtered module** is a module together with a distinguished filtration.*

(ii) *Let $M, N \in k\text{-Mod}$ carrying filtrations F. A morphism $f \in k\text{-Mod}(M, N)$ is called m**-equicontinuous**, for some $m \in \mathbb{Z}$, if*

$$f(F_nM) \subset F_{n+m}N, \qquad n \in \mathbb{Z}.$$

*We say f **preserves** the filtration, if it is 0-equicontinuous.*

Definition 3.1.2
Let $k \in \mathcal{C}Ring$ and $M \in k$-Mod carrying a filtration F.

(i) *F is called* ***exhaustive****, if $X_\infty := \bigcup_{n \leq 0} F_n M = M$.*
It is called ***negative****, if $F_{-1}M = 0$.*

(ii) *F is called* ***Hausdorff*** *or* ***separating****, if $\bigcap_{n \geq 0} F_n M = 0$.*
It is called ***positive****, if $F_0 M = M$.*

(iii) *F is called* ***complete****, if $M \xrightarrow{\sim} \varprojlim_{n \geq 0} M/F_n M =: \widehat{M}$.*

Many other authors define filtrations as sequences of submodules getting larger and larger for growing index. As most of the filtrations appearing in this chapter are positive, we decided to define a filtration as a sequence the other way round to reduce the amount of minus signs.

Let us recall the following elementary result about module filtrations.

Proposition 3.1.3
Let $k \in \mathcal{C}Ring$ and $X, Y \in k$-Mod carrying filtrations F.
Suppose $f \in k\text{-Mod}(X, Y)$ is m-equicontinuous, for some $m \in \mathbb{Z}$.

(i) *If $\mathrm{gr}^F_* X \overset{\mathrm{gr} f}{\hookrightarrow} \mathrm{gr}^F_{*+m} Y$ is injective, then $\widehat{X}_\infty \overset{f}{\hookrightarrow} \widehat{Y}_\infty$ is injective.*

(ii) *If $\mathrm{gr}^F_* X \overset{\mathrm{gr} f}{\twoheadrightarrow} \mathrm{gr}^F_{*+m} Y$ is surjective, then $\widehat{X}_\infty \overset{f}{\twoheadrightarrow} \widehat{Y}_\infty$ is surjective.*

Proof. Let us write $X_n = F_n X$ and $Y_n = F_n Y$ for short.

(i) Let $\ell \in \mathbb{Z}$. Using the exact sequences

$$0 \longrightarrow \mathrm{gr}^F_n X \longrightarrow X_\ell / X_{n+1} \longrightarrow X_\ell / X_n \longrightarrow 0, \qquad n > \ell,$$

by induction on $n > \ell$ and using the 5-Lemma one shows that $X_\ell / X_n \hookrightarrow Y_{\ell+m}/Y_{n+\ell}$ is injective, for all $n > \ell$. Hence $\widehat{X}_\ell \overset{f}{\hookrightarrow} \widehat{Y}_{\ell+m}$ is injective, because limits are left exact, and thus $\widehat{X}_\infty \overset{f}{\hookrightarrow} \widehat{Y}_\infty$ is injective, because filtered colimits are exact.

(ii) By a similar induction as in (i) one shows that $X_\ell / X_n \hookrightarrow Y_{\ell+m}/Y_{n+\ell}$ is surjective, for all $n > \ell$. Giving the kernel $K = \ker f$ the submodule filtration induced by F, we get short exact sequences

$$0 \longrightarrow K_\ell / K_n \longrightarrow X_\ell / X_n \xrightarrow{f} Y_\ell / Y_n \longrightarrow 0, \qquad n > \ell.$$

As $(K_\ell / K_n)_{n \geq \ell}$ consists of surjections and thus $\varprojlim^1_{n > \ell} K_\ell / K_n = 0$, we see that $\widehat{X}_\ell \overset{f}{\hookrightarrow} \widehat{Y}_{\ell+m}$ is surjective, for all $\ell \in \mathbb{Z}$. Using that colimits are right exact it follows that $\widehat{X}_\infty \overset{f}{\twoheadrightarrow} \widehat{Y}_\infty$ is surjective.

□

3.1.1 Filtered algebras

Remark 3.1.4
Let $k \in \mathcal{C}Ring$.

(i) *The **discrete filtration** on a module $M \in k$-Mod is defined as*

$$F_n M = \begin{cases} M, & n \leq 0, \\ 0, & n > 0. \end{cases}$$

(ii) *If $M, N \in k$-Mod carry filtrations F, the **tensor product filtration** $F \otimes F$ on $M \otimes N$ is defined as*

$$F_n(M \otimes N) = \sum_{p+q=n} F_p M \otimes F_q N, \qquad n \in \mathbb{Z},$$

where by $F_p M \otimes F_q$ we mean its image in $M \otimes N$ under the canonical map [1].

This defines a monoidal structure on the category of filtered k-modules with 0-equicontinuous homomorphisms.

A crucial tool in the development of the theory is the following elementary observation.

Proposition 3.1.5
Let $k \in \mathcal{C}Ring$ and $M, N \in k$-Mod carry filtrations F.
Then there is a natural epimorphism $\operatorname{gr}^F M \otimes \operatorname{gr}^F N \twoheadrightarrow \operatorname{gr}^F(M \otimes N)$.
If moreover $\operatorname{gr}^F M$ or $\operatorname{gr}^F N$ is flat, then it is an isomorphism.

Proof. The natural epimorphism

$$\bigoplus_{p+q=n} M_p \otimes N_q \twoheadrightarrow \sum_{p+q=n} M_p \otimes N_q = F_n(M \otimes N), \qquad n \in \mathbb{Z},$$

maps $M_p \otimes N_{q+1} + M_{p+1} \otimes N_q \subset M_p \otimes N_q$ into $F_{n+1}(M \otimes N)$ and therefore induces the epimorphism $\operatorname{gr}^F M \otimes \operatorname{gr}^F N \twoheadrightarrow \operatorname{gr}^F(M \otimes N)$.

Suppose that $\operatorname{gr}^F M$ is flat and consider the diagram

$$\begin{tikzcd}
\bigoplus_{p+q=n} \operatorname{gr}^F_p M \otimes \operatorname{gr}^F_q N \arrow[r] \arrow[d] & \prod_{p+q=n} M/F_{p+1}M \otimes N/F_{q+1}N, \\
\operatorname{gr}^F_n(M \otimes N) \arrow[r] & \prod_{p+q=n} \operatorname{gr}^F_n(M/F_{p+1}M \otimes N/F_{q+1}N). \arrow[u]
\end{tikzcd}$$

The left vertical map is the natural epimorphism just constructed. The lower horizontal map on the factor (p, q) is induced by the quotient maps $M \twoheadrightarrow M/F_{p+1}M$ and $N \twoheadrightarrow$

[1] As this map is not injective in general, this is a bit sloppy.

$N/F_{q+1}N$ with the targets carrying the quotient filtration. Since $F_{n+1}(M/F_{p+1}M \otimes N/F_{q+1}N) = 0$, we also have maps

$$\mathrm{gr}_n^F(M/F_{p+1}M \otimes N/F_{q+1}N) \longrightarrow M/F_{p+1}M \otimes N/F_{q+1}N, \qquad p+q=n,$$

whose product is the right vertical map. The map c is defined as the composite. Let $p, q, p', q' \in \mathbb{Z}$ with $p+q = n = p'+q'$. Then by construction the composite $\pi_{(p',q')} \circ c \circ \iota_{(p,q)}$ is zero, if $(p,q) \neq (p',q')$, and is induced by the canonical injections $\mathrm{gr}_p^F M \hookrightarrow M/F_{p+1}M$ and $\mathrm{gr}_q^F N \hookrightarrow N/F_{q+1}N$, if $(p,q) = (p',q')$. It follows that c is the direct sum over all maps

$$\mathrm{gr}_p^F M \otimes \mathrm{gr}_q^F N \longrightarrow M/F_{p+1}M \otimes N/F_{q+1}N, \qquad p+q=n,$$

followed by the natural monomorphism from the direct sum into the direct product. So to prove that the left vertical map is injective, it suffices to check injectivity for all these maps. Writing it as the composite

$$\mathrm{gr}_p^F M \otimes \mathrm{gr}_q^F N \longrightarrow \mathrm{gr}_p^F M \otimes N/F_{q+1}N \longrightarrow M/F_{p+1}M \otimes N/F_{q+1}N, \qquad p+q=n,$$

the first map is injective, since $\mathrm{gr}_q^F N \hookrightarrow N/F_{q+1}N$ is injective and $\mathrm{gr}_p^F M$ is flat by assumption. Corollary 3.6.2 (i) implies that also the second map is injective, since $\mathrm{gr}_p^F M \hookrightarrow M/F_{p+1}M$ is injective and $(M/F_{p+1})/\mathrm{gr}_p^F M \cong M/F_pM$ is flat by Proposition 3.6.1 (i). □

Definition 3.1.6
Let $k \in \mathcal{C}Ring$.

*An **algebra filtration** on a k-algebra[2] A is a module filtration, that is preserved by all structure maps, where k carries the discrete and $A \otimes A$ carries the tensor product filtration.*

In other words A is an algebra (of the particular type) in the monoidal category of filtered k-modules with 0-equicontinuous homomorphisms.

3.1.2 Completed Tor-groups

Using the well-known theory of Artin-Rees we compare the Tor groups with their completions. This is needed later, when we are identifying group and Lie ring homology by comparing the saturation of group ring and enveloping algebra respecetively.

Definition 3.1.7
Let $k \in \mathcal{C}Ring$ and $A \in k\text{-}\mathcal{A}ss_1$ carrying a positive filtration F with flat $\mathrm{gr}^F A$. Let $X \in \mathrm{Mod}\text{-}A$ and $Y \in A\text{-}\mathrm{Mod}$ carrying filtrations F complatible with F on A.

*The **completed** Tor**-functor** $\widehat{\mathrm{Tor}}_*^A(X,Y)$ is defined as the homology of the completion of the bar complex*

$$B_*(X,A,Y) = \Big(\underset{0}{X \otimes Y} \xleftarrow[-1\otimes\mu]{\mu\otimes 1} \underset{1}{X \otimes A \otimes Y} \xleftarrow[\substack{-1\otimes\mu\otimes 1\\ +1\otimes 1\otimes\mu}]{\mu\otimes 1\otimes 1} \underset{2}{X \otimes A^{\otimes 2} \otimes Y} \longleftarrow \dots\Big)$$

[2]Here we allow any kind of algebra, e.g. (unital) associative algebras (with augmentation), Lie algebras, etc.

carrying the tensor product filtration.

Proposition 3.1.8
*Let $k \in \mathcal{C}Ring$ and $A \in k\text{-}\mathcal{A}ss_1$ carrying a positive filtration F, such that the associated **Rees ring** $A_F = \bigoplus_{n\geq 0} F_n A$ is left Noetherian.*
Then the following holds.

(i) $\widehat{A} = \varprojlim_{n\geq 0} A/F_n A$ *is a flat right A-module.*

So $\mathrm{Tor}_*^A(\widehat{X}, Y) \xrightarrow{\sim} \mathrm{Tor}_*^{\widehat{A}}(\widehat{X}, \widehat{Y})$, *for $X \in$ Mod-A and finitely generated $Y \in A$-Mod.*

(ii) *Suppose that $\mathrm{gr}^F A$ and $\widehat{A}$ are flat over k. Then*

$\mathrm{Tor}_*^{\widehat{A}}(\widehat{X}, \widehat{Y}) \xrightarrow{\sim} \widehat{\mathrm{Tor}_*^A}(X, Y)$, *for $X \in$ Mod-A and finitely generated $Y \in A$-Mod with $\mathrm{gr}^F Y$ and $\widehat{Y}$ flat over k.*

For right Noetherian A_F we can consider the opposite ring A^{op} to obtain the dual result.

Proof. With A_F also the quotient $A_F \twoheadrightarrow A$ is left Noetherian.

(i) This is the usual argument. Let $N \leq M \in A$-Mod and suppose $M = A{\cdot}x_1 + ... + A{\cdot}x_r$. We have to show that $\widehat{A} \otimes_A N \hookrightarrow \widehat{A} \otimes_A M$ is injective. We give M the filtration induced by F, i.e.

$$F_n M = F_n A \cdot M, \qquad n \geq 0.$$

Then $M_F = \bigoplus_{n\geq 0} F_n M = A_F \cdot x_1 + ... + A_F \cdot x_r$, and hence $N_F = \bigoplus_{n\geq 0} N \cap F_n M \leq M_F$ is finitely generated, because A_F is Noetherian. By replacing each such generator by its homogeneous compronents, we get a finite set of homogeneous generators for N_F

$$y_i \in N \cap F_{n_i} M, \qquad n_i \geq 0, \quad 1 \leq i \leq s.$$

Setting $m = \max\{n_1, ..., n_r\}$ it follows that

$$N \cap F_{n+m} M = \sum_{1\leq i\leq r} F_{n+m-n_i} A \cdot y_i \subset F_n A \cdot N, \qquad n \geq 0,$$

which proves that the submodule filtration on N is equivalent to the filtration induced by F. Hence in the commutative diagram

$$\begin{tikzcd}
\varprojlim\limits_{n\geq 0} N/(F_n A \cdot N) \arrow[r] & \varprojlim\limits_{n\geq 0} N/N \cap (F_n A \cdot M) \arrow[r, hook] & \varprojlim\limits_{n\geq 0} M/(F_n A \cdot M) \\
\widehat{A} \otimes_A N \arrow[u] \arrow[rr] & & \widehat{A} \otimes_A M, \arrow[u]
\end{tikzcd}$$

the upper left horizontal map is an isomorphism. By using finite presentations ($N \leq M$ are finitely generated and A is Noetherian), one verifies that the two vertical maps are isomorphisms. It follows that the lower horizontal map is injective and thus $\widehat{A}$ is a flat right A-module.

Let $X \in$ Mod-A and choose a projective resolution $P_*^F \xrightarrow{\simeq} Y$ for $Y \in A$-Mod. Then $\widehat{X} \in$ Mod-$\widehat{A}$ and $\widehat{A} \otimes P_*^F \xrightarrow{\simeq} \widehat{A} \otimes Y \xrightarrow{\sim} \widehat{Y}$ is a projective resolution of left $\widehat{A}$-modules, because $\widehat{A}$ is a flat right A-module. In particular

$$\mathrm{Tor}_*^A(\widehat{X}, Y) = H_*(\widehat{X} \otimes_A P_*^F) = H_*((\widehat{X} \otimes_{\widehat{A}} \widehat{A}) \otimes_A P_*^F) = H_*(\widehat{X} \otimes_{\widehat{A}} (\widehat{A} \otimes_A P_*^F)) = \mathrm{Tor}_*^{\widehat{A}}(\widehat{X}, \widehat{Y}).$$

(ii) We define the big Rees ring as $R^F A = \bigoplus_{n \in \mathbb{Z}} F_n A$. It is obtained by adjoining the central element $1 \in F_{-1}A = A$ to the usual Rees ring, which is equal to $\bigoplus_{n \geq 0} F_n A$ in this notation. In particular also $R^F A$ is Noetherian by Hilbert's basis theorem 3.6.3.

As in (i) we give Y the ($\mathbb{Z}$-indexed) filtration induced by F on A. We will inductively construct a free resolution $W_* \xrightarrow{\simeq} R^F Y = \bigoplus_{n \in \mathbb{Z}} F_n Y$ of finitely generated, graded left $R^F A$-modules, meaning that in particular the differentials preserve the grading. As Y is generated by some elements $y_{0,1}, ..., y_{0,r_0} \in Y$, we get a surjection

$$d : W_0 := (R^F A)^{r_0} \twoheadrightarrow R^F Y =: W_{-1}, \quad a \longmapsto \sum_i a_i \cdot y_{0,i}.$$

Giving W_0 the product filtration of F on each factor A, the map d preserves the grading. Now having constructed $W_m \xrightarrow{d} W_{m-1}$, for some $m \geq 0$, we know that $\ker d \leq W_m$ is finitely generated, as W_m is finitely generated and $R^F A$ is Noetherian. Replacing each generator by its homogeneous components, $\ker d$ is generated by a finite set of homogeneous elements

$$y_{m,i} \in (W_m)^{(s_{m+1,i})}, \qquad s_{m+1,i} \in \mathbb{Z}, \quad 1 \leq i \leq r_{m+1}.$$

We define $W_{m+1} = (R^F A)^{r_{m+1}}$ and give the i-th factor the grading shifted by $s_{m+1,i}$, for all $1 \leq i \leq r_{m+1}$. It follows that

$$d : W_{m+1} \longrightarrow W_m, \qquad a \longmapsto \sum_i a_i \cdot y_{m,i}$$

is a homomorphism of graded left $R^F A$-modules. So by construction $W_* \xrightarrow{\simeq} R^F Y$ is a free resolution as desired.

Now the union

$$E_* = \bigcup_{k \geq 0} (W_*)^{(k)} = \varinjlim((W_*)^{(0)} \hookrightarrow (W_*)^{(1)} \hookrightarrow ...)$$

yields a free resolution $E_* \xrightarrow{\simeq} Y$ of finitely generated left A-modules. Giving $E_m = A^{r_m}$ the product filtration, where the i-th factor A carries the shifted filtration $F_{\bullet + s_{m,i}}$, we get a complex of filtered A-modules with $R^F(E_*) = W_*$ as graded left $R^F(A)$-modules. Moreover there is an exact sequence of complexes

$$0 \longrightarrow W_* \longrightarrow W_* \longrightarrow \mathrm{gr}^F W_* \longrightarrow 0, \qquad n \in \mathbb{Z},$$

where the left map is induced by the inclusions $F_k E_* \hookrightarrow F_{k-1} E_*$, for $k \in \mathbb{Z}$. As a summary we record the following properties for $E_* \longrightarrow Y$.

a) $E_* \xrightarrow{\simeq} Y$ is a finitely generated, free resolution of left A-modules,

b) $\mathrm{gr}^F E_* \xrightarrow{\simeq} \mathrm{gr}^F Y$ is a finitely generated, free resolution of left $\mathrm{gr}^F A$-modules,

c) $\widehat{E}_* \xrightarrow{\simeq} \widehat{Y}$ is a finitely generated, free resolution of left $\widehat{A}$-modules.

Now consider the commutative diagram

$$\begin{tikzcd}
B_*(\widehat{X},\widehat{A},\widehat{Y}) \arrow[d] & \mathrm{Tot}^+ B_*(\widehat{X},\widehat{A},\widehat{E}_*) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & \widehat{X} \otimes_{\widehat{A}} \widehat{E}_* \arrow[d, "\wr"] \\
\widehat{B}_*(X,A,Y) & \mathrm{Tot}^+ \widehat{B}_*(X,A,E_*) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \widehat{X \otimes_A E_*},
\end{tikzcd}$$

As $\widehat{Y}$ and $\widehat{A}$ are flat over k, $B_*(\widehat{A},\widehat{A},\widehat{Y}) \xrightarrow{\simeq} \widehat{Y}$ is a flat resolution of left $\widehat{A}$-modules and so

$$\mathrm{Tor}_*^{\widehat{A}}(\widehat{X},\widehat{Y}) = H_*(\widehat{X} \otimes_{\widehat{A}} B_*(\widehat{A},\widehat{A},\widehat{Y})) = H_*(B_*(\widehat{X},\widehat{A},\widehat{Y})).$$

As $\widehat{E}_m$ is a free $\widehat{A}$-module by c), it follows that $B_*(\widehat{A},\widehat{A},\widehat{E}_m) \xrightarrow{\simeq} \widehat{E}_m$ is a homotopy equivalence, for evey $m \geq 0$. Hence the upper right horizontal map is a quasi-isomorphism by a spectral sequence argument. Moreover, for all $m \geq 0$, the map $B_m(\widehat{X},\widehat{A},\widehat{Y}) \xleftarrow{\simeq} \mathrm{Tot}^+ B_m(\widehat{X},\widehat{A},\widehat{E}_*)$ is a quasi-isomorphism by the Künneth formula, since $\widehat{Y}$ is flat over k. In particular also the upper left horizontal map is a quasi-isomorphism. Similarly as $\mathrm{gr}^F Y$ and $\mathrm{gr}^F A$ are flat over k, $B_*(\mathrm{gr}^F A, \mathrm{gr}^F A, \mathrm{gr}^F Y) \xrightarrow{\simeq} \mathrm{gr}^F Y$ is a flat resolution of left $\mathrm{gr}^F A$-modules and we get quasi-isomorphisms

$$B_*(\mathrm{gr}^F X, \mathrm{gr}^F A, \mathrm{gr}^F Y) \xleftarrow{\simeq} \mathrm{Tot}^+ B_*(\mathrm{gr}^F X, \mathrm{gr}^F A, \mathrm{gr}^F E_*) \xrightarrow{\simeq} \mathrm{gr}^F X \otimes_{\mathrm{gr}^F A} \mathrm{gr}^F E_*.$$

Using that $\mathrm{gr}^F A$ and $\mathrm{gr}^F Y$ are flat over k, it follows that these maps are isomorphic to the associated graded of two lower horizontal maps in the diagram above. Hence they also must be quasi-isomorphisms, as must also be the left vertical map, which finally proves (ii).

□

3.1.3 Divisible closure and saturation

Following Lazard [Laz65] we introduce the notion of the divisible closure of a filtered module. Similar to the construction of divided power algebras, this provides the necessary structure to define logarithm and exponential maps.

Proposition 3.1.9
There is a ($\mathbb{Z}$-indexed) ring filtration D on $\mathbb{Q}$, given by

$$D_0\mathbb{Q} = \mathbb{Z}, \qquad D_{-n}\mathbb{Q} = \sum_{p\ prime} p^{-\lfloor n/(p-1) \rfloor}\mathbb{Z}, \qquad D_n\mathbb{Q} = 0, \qquad n > 0.$$

It is the intersection of the (shifted) p-adic filtrations on $\mathbb{Q}$. In other words

$$D_n\mathbb{Q} = \{x \in \mathbb{Q};\ (p-1)v_p(x) \geq n,\ \textit{for all prime numbers } p\}, \qquad n \in \mathbb{Z}.$$

We call it the ***divisble*** *filtration.*

Proof. Let $x, y \in \mathbb{Z}$ be coprime. Then there are $a, b \in \mathbb{Z}$, such that $ax + by = 1$ and thus

$$(xy)^{-1} = (ax + by)(xy)^{-1} = ay^{-1} + bx^{-1} \in x^{-1}\mathbb{Z} + y^{-1}\mathbb{Z}.$$

Hence by induction on $r \geq 1$ we get

$$\begin{aligned} D_{-n}\mathbb{Q} &= \sum_{\substack{p_1,\ldots,p_r \\ \text{distinct primes}}} p_1^{-\lfloor n/(p_1-1)\rfloor} \cdots p_r^{-\lfloor n/(p_r-1)\rfloor} \cdot \mathbb{Z}, \qquad n > 0, \\ &= \{x \in \mathbb{Q};\ (p-1)v_p(x) \geq -n, \quad \text{for all primes } p\}. \end{aligned}$$

Moreover it follows that $D_m\mathbb{Q} \cdot D_n\mathbb{Q} = D_{m+n}\mathbb{Q}$, for all $m, n < 0$, proving that D is a ring filtration with the desired properties.

□

Definition 3.1.10
Let $A \in \mathcal{A}b$ carrying a positive filtration F with $\mathrm{gr}^F A$ flat over $\mathbb{Z}$. The discrete filtration F and the divisible filtration D on $\mathbb{Q}$ give rise to two positive tensor product filtrations $F := F \otimes F$ and $D^F := F \otimes D$ on $A \otimes \mathbb{Q}$:

$$F_n(A \otimes \mathbb{Q}) = F_nA \otimes \mathbb{Q}, \qquad D_n^F(A \otimes \mathbb{Q}) = \sum_{m \geq 0} F_mA \otimes D_{n-m}\mathbb{Q}, \qquad n \geq 0.$$

We also use the short notation $D_n^F(A) = D_n^F(A \otimes \mathbb{Q})$, for $n \geq 0$.

(i) *The* ***divisible closure*** *of A with respect to F is $D_0^F(A)$ and*

(ii) *the* ***saturation*** *of A with respect to F is its completion with respect to the filtration induced by F (<u>not</u> by D^F), i.e.*

$$\widehat{D}_0^F(A) = \varprojlim_{n \geq 1} D_0^F(A \otimes \mathbb{Q})/D_0^F(A \otimes \mathbb{Q}) \cap (F_nA \otimes \mathbb{Q}).$$

More generally we define $\widehat{F}_n(A \otimes \mathbb{Q})$ and $\widehat{D}_n^F(A)$ as the completions of $F_n(A \otimes \mathbb{Q})$ and $D_n^F(A)$ with respect to the filtration induced by F, for all $n \geq 0$.

Proposition 3.1.11
Let $A \in \mathcal{A}b$ carrying a positive filtration F with $\mathrm{gr}^F A$ flat over $\mathbb{Z}$. Defining $F_n(\mathrm{gr}^F A)$ as the sum of all homogeneous components of degree $\geq n$, we obtain a filtration F on $\mathrm{gr}^F A$ and for all $n \geq 0$ a natural isomorphism

$$\begin{tikzcd} D_n^F(\mathrm{gr}^F A) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{gr}^F D_n^F(A) \arrow[d, hook] \\ (\mathrm{gr}^F A) \otimes \mathbb{Q} \arrow[r, "\sim"] & \mathrm{gr}^F(A \otimes \mathbb{Q}). \end{tikzcd}$$

Proof. By definition of the filtrations D on $\mathbb{Q}$ and on $A \otimes \mathbb{Q}$, the upper horizontal map exists and is onto. The lower horizontal map is an isomorphism, since $\mathbb{Q}$ is flat over $\mathbb{Z}$. The left vertical map is injective by definition of the filtration D. Hence by commutativity the upper horizontal map must be an isomorphism, too.

□

Proposition 3.1.12
Let $A \in \mathcal{A}b$ carrying a filtration F with $\mathrm{gr}^F A$ flat over $\mathbb{Z}$ and $A = F_N A$, for some $N \geq 0$.
Then the natural map $A = F_N A \otimes D_0\mathbb{Q} \longrightarrow D_N^F A \longrightarrow \widehat{D}_N^F A$ induces isomorphisms:

(i) $D_n^F A \xrightarrow{\sim} D_n^F D_N^F A$, *for all $n \geq N$.*

(ii) $\widehat{D}_n^F A \xrightarrow{\sim} \widehat{D}_n^F \widehat{D}_N^F A$, *for all $n \geq N$.*

Proof.

(i) Since $D_0\mathbb{Q} = \mathbb{Z}$ and D is a ring filtration, it follows that

$$\sum_{\substack{p,q\geq 0,\\ p+q=n}} D_{-p}\mathbb{Q} \otimes D_{-q}\mathbb{Q} \xrightarrow{\sim} \sum_{\substack{p,q\geq 0,\\ p+q=n}} D_{-p}\mathbb{Q} \cdot D_{-q}\mathbb{Q} = D_{-n}\mathbb{Q}, \qquad n \geq 0, \tag{3.1}$$

and thus, for all $n \geq N$, the composition

$$\sum_{m\geq N} F_m A \otimes D_{n-m}\mathbb{Q} \longrightarrow \sum_{m,m'\geq N} F_m A \otimes D_{m'-m}\mathbb{Q} \otimes D_{n-m'}\mathbb{Q} \xrightarrow{\sim} \sum_{m\geq N} F_m A \otimes D_{n-m}\mathbb{Q}$$

is the identity. Hence also the first map is an isomorphism, which is equal to the map $D_n^F A \longrightarrow D_n^F D_N^F A$.

(ii) In the commutative diagram

$$\begin{tikzcd}
\mathrm{gr}^F D_n A \arrow[rr] & & \mathrm{gr}^F D_n \widehat{D}_N A \\
 & D_n \mathrm{gr}^F D_N A \arrow[r, "\sim"] & D_n \mathrm{gr}^F \widehat{D}_N A \arrow[u, "\wr"'] \\
D_n \mathrm{gr}^F A \arrow[r] \arrow[uu, "\wr"] & D_n D_N \mathrm{gr}^F A, \arrow[u, "\wr"'] &
\end{tikzcd}$$

the vertical isomorphisms are induced by Proposition 3.1.11 and the horizontal isomorphism comes from the observation that the graded of an object always coincides with the graded of its completion. As the lower horizontal map is an isomorphism by (i), it follows that the upper horizontal map is an isomorphism, which implies (ii).

□

Proposition 3.1.13
Let $A, B \in \mathcal{A}b$ carrying positive filtrations F with $\mathrm{gr}^F A, \mathrm{gr}^F B$ flat over $\mathbb{Z}$.
Then $(D_n^F A) \otimes (D_n^F B) \xrightarrow{\sim} D_n^{F\otimes F}(A \otimes B)$, for all $n \geq 0$.

Proof. Using the isomorphism (3.1) the map

$$\sum_{p+q+r+s=n} F_r A \otimes D_p\mathbb{Q} \otimes F_s B \otimes D_q\mathbb{Q} \xrightarrow{\sim} \sum_{r+s+t=n} F_r A \otimes F_s B \otimes D_t\mathbb{Q}, \qquad n \geq 0,$$

is an isomorphism. By definition this is the map $(D_n^F A) \otimes (D_n^F B) \xrightarrow{\sim} D_n^{F\otimes F}(A \otimes B)$. □

Lemma 3.1.14
Let $p > 1$ be a prime number and $m \geq 1$.
If $m = a_0 + a_1 p + ... + a_r p^r$ with $0 \leq a_i < p$ is its unique p-adic expansion, then

$$v_p(m!) = \sum_{0 \leq i \leq r} a_i (p^i - 1)/(p-1) = (m - (a_0 + ... + a_r))/(p-1).$$

In particular $(p-1)v_p(1/m!) = a_0 + ... + a_r - m \geq 1 - m$ and thus $1/m! \in D_{1-m}\mathbb{Q}$.

Proof. Any book on number theory/local fields, e.g. [Neu90] Lem. II.5.6. □

Proposition 3.1.15
Let $A \in \mathcal{A}ss$ carrying an algebra filtration F. Then logarithm and exponential series define a natural bijection

$$\log : 1 + \widehat{F}_n(A \otimes \mathbb{Q}) \overset{\sim}{\rightleftarrows} \widehat{F}_n(A \otimes \mathbb{Q}) : \exp, \qquad n \geq 1,$$

restricting to a bijection $1 + \widehat{D}^F_n(A) \overset{\sim}{\rightleftarrows} \widehat{D}^F_n(A)$, for all $n \geq 1$.

Proof. Since F is a ring filtration $F_n(A \otimes \mathbb{Q})^m \subset F_{mn}(A \otimes \mathbb{Q})$, for all $m \geq 0$, and thus the serieses

$$\log(1-x) = -\sum_{m \geq 0} \frac{x^m}{m}, \qquad \exp(x) = \sum_{m \geq 0} \frac{x^m}{m!}, \qquad x \in \widehat{F}_1(A \otimes \mathbb{Q})$$

converge. For the formal power serieses $\log(1-t), \exp(t) \in \mathbb{Q}[[t]]$ we have $\log \circ \exp(t) = t$ and $\exp \circ \log(1-t) = 1-t$. Every $x \in \widehat{F}_1(A \otimes \mathbb{Q})$ induces a ring homomorphism $\mathbb{Q}[[t]] \xrightarrow{f_x} \widehat{F}_1(A \otimes \mathbb{Q})$ sending t to x. By construction of the bijection maps, we get

- $\log \circ \exp(x) = \log \circ \exp \circ f_x(t) = f_x \circ \log \circ \exp(t) = f_x(t) = x$,
- $\exp \circ \log(1-x) = \exp \circ \log \circ f_x(1-t) = f_x \circ \exp \circ \log(1-t) = f_x(1-t) = 1-x$,

which proves the first part of the statement. By Lemma 3.1.14 we have $1/m! \in D_{1-m}\mathbb{Q}$ and thus also $1/m \in D_{1-m}\mathbb{Q}$ and we get

$$x^m/m!, x^m/m \in D^F_{mn}(A) \cdot D^F_{1-m}(A) \subset D^F_{n+(m-1)(n-1)}(A) \subset D^F_n(A), \quad x \in D^F_n(A),\ n \geq 1.$$

This proves that every summand in the serieses $1 - \exp(x)$ and $\log(1-x)$ is in $D^F_n(A)$, for $x \in D^F_n(A)$. As we gave $D^F_n(A)$ the filtration induced by F (and not by D), convergence follows from what was shown before. □

Corollary 3.1.16
Let $H \in \mathbb{Z}\text{-}\mathcal{G}rp$ carrying a positive Hopf algebra filtration F.
Then the natural bijection of Proposition 3.1.15 restricts to a bijection

$$\log : \widehat{G}^F_n(H) \overset{\sim}{\rightleftarrows} \widehat{P}^F_n(H) : \exp, \qquad n \geq 1,$$

where

- $\widehat{G}^F_n(H)$ *are the **degree-n completed group-like elements**, given by*

$$\widehat{G}^F_n(H) = (1 + \widehat{D}_n H) \cap \ker\left(\widehat{D}^F_0(H) \underset{\widehat{D}^F_0(\eta\otimes 1)\cdot \widehat{D}^F_0(1\otimes\eta)}{\overset{\widehat{D}^F_0(\delta)}{\rightrightarrows}} \widehat{D}^F_0(H \otimes H) \right), \qquad n \geq 0,$$

- $\widehat{P}^F_n(H)$ *are the **degree-n completed primitive elements**, given by*

$$\widehat{P}^F_n(H) = (\widehat{D}_n H) \cap \ker\left(\widehat{D}^F_0(H) \underset{\widehat{D}^F_0(\eta\otimes 1)+ \widehat{D}^F_0(1\otimes\eta)}{\overset{\widehat{D}^F_0(\delta)}{\rightrightarrows}} \widehat{D}^F_0(H \otimes H) \right), \qquad n \geq 0.$$

Proof. As H is a Hopf algebra, the maps $d^0 = \widehat{D}^F_0(\eta \otimes 1), d^2 = \widehat{D}^F_0(1 \otimes \eta)$ and $d^1 = \widehat{D}^F_0(\delta)$ are compatible with the multiplication $\widehat{D}^F_0(\mu)$ on $\widehat{D}^F_0(H)$. Hence for $x \in \widehat{G}^F_n(H)$ the elements $d^0(x), d^2(x) \in \widehat{D}^F_0(H \otimes H)$ commute, which implies that

$$\begin{aligned} d^1(\log(x)) &= \log(d^1(x)) = \log(d^0(x) \cdot d^2(x)) \\ &= \log(d^0(x)) + \log(d^2(x)) = d^0(\log(x)) + d^2(\log(x)), \end{aligned}$$

and thus $\log(x) \in \widehat{P}^F_n(H)$. Similarly for $x \in \widehat{P}^F_n(H)$ also the elements $d^0(x), d^2(x) \in \widehat{D}^F_0(H \otimes H)$ commute, which implies that

$$\begin{aligned} d^1(\exp(x)) &= \exp(d^1(x)) = \exp(d^0(x) + d^2(x)) \\ &= \exp(d^0(x)) \cdot \exp(d^2(x)) = d^0(\exp(x)) \cdot d^2(\exp(x)), \end{aligned}$$

and thus $\exp(x) \in \widehat{G}^F_n(H)$. □

Proposition 3.1.17
Let $A \in \mathcal{A}ss$ carrying an algebra filtration F with $A = F_1 A$.
Then there is a natural isomorphism of Lie rings

$$\ell_A : \mathrm{gr}^F(1 + \widehat{A}) \xrightarrow{\sim} \mathrm{gr}^F A, \quad [1 + a] \longmapsto [a],$$

where the Lie bracket on the left is induced by the commutators in the group $1 + \widehat{A}$ and the Lie bracket on the right is induced by the Lie bracket on A.

Proof. In degree $n \geq 1$ the map ℓ_A is given by the bijection

$$1 + F_n A / F_{n+1} A \xrightarrow{\sim} F_n A / F_{n+1} A, \quad [1 + a] \longmapsto [a].$$

It is a group homomorphism, because

$$[(1 + a) \cdot (1 + b)] = [1 + (a + b + ab)] = [1 + (a + b)], \quad a, b \in F_n A.$$

Now let $[1 + a] \in \mathrm{gr}^F_m(1 + \widehat{A})$ and $[1 + b] \in \mathrm{gr}^F_n(1 + \widehat{A})$ with $a, b \in \widehat{A}$. Then using the expansion $(1 + a)^{-1} = \sum_{i \geq 0}(-a)^i$ and $m, n \geq 1$ we see that

$$\begin{aligned} [1 + a, 1 + b] &= 1 + ((1 + a)(1 + b) - (1 + b)(1 + a)) \cdot (1 + a)^{-1}(1 + b)^{-1} \\ &= 1 + (ab - ba)(1 + a)^{-1}(1 + b)^{-1} \equiv 1 + (ab - ba) \qquad \mathrm{mod}\ F_{m+n+1}\widehat{A}, \end{aligned}$$

which proves that ℓ_A is an isomorphism of Lie rings. □

3.2 Hopf algebras

Let us recall the definition and most important basic properties of Hopf algebras.

Definition 3.2.1
A ***group object*** *in a symmetric monoidal category* $(\mathcal{C}, \otimes, E)$ *is an object* G *together with 5 structure maps*

$$G \otimes G \xrightarrow{\mu} G, \quad E \xrightarrow{\eta} G, \quad G \xrightarrow{\delta} G \otimes G, \quad G \xrightarrow{\varepsilon} E, \quad G \xrightarrow{\iota} G,$$

such that the following identities hold:

(i) $\mu(\mathrm{id} \otimes \mu) = \mu(\mu \otimes \mathrm{id})$ *and* $\mu(\mathrm{id} \otimes \eta) = \mu(\eta \otimes \mathrm{id}) = \mathrm{id}$, *i.e.* (G, μ, η) *is a monoid.*

(ii) $(\mathrm{id} \otimes \delta)\delta = (\delta \otimes \mathrm{id})\delta$ *and* $(\mathrm{id} \otimes \varepsilon)\delta = (\varepsilon \otimes \mathrm{id})\delta = \mathrm{id}$, *i.e.* (G, δ, ε) *is a comonoid.*

(iii) δ *and* ε *are monoid homomorphisms and* μ *and* η *are comonoid homomorphisms.*

(iv) $\mu(\mathrm{id} \otimes \iota)\delta = \mu(\iota \otimes \mathrm{id})\delta = \eta\varepsilon$.

A homomorphism of group objects consists of a morphism $f \in \mathcal{C}(G, H)$ *being compatible with all structure maps. We denote by* $\mathcal{C}$*-*$\mathcal{G}rp$ *the category of group objects in* $\mathcal{C}$.

Definition 3.2.2 (i) *A* ***Hopf algebra*** *over* $k \in \mathcal{C}Ring$ *is a group object in the monoidal category* $(k\text{-Mod}, \otimes, k)$.

We denote by k*-*$\mathcal{G}rp = (k\text{-Mod})$*-*$\mathcal{G}rp$ *the category of Hopf algebras over* k.

(ii) *A* ***Hopf ring*** *is a Hopf algebra over the integers, i.e. in* $(\mathcal{A}b, \otimes, \mathbb{Z})$.

3.2.1 Canonical filtrations on Hopf algebras

We are introducing the lower and colower central series. We call them this way, because they are closely linked to the particular series of groups and Lie rings, when applied to the case of a group ring resp. enveloping algebra.

Remark 3.2.3
Let $k \in \mathcal{C}Ring$ *and* $H \in k$*-*$\mathcal{G}rp$ *be a Hopf algebra.*

(i) *As* ε *is a monoid homomorphism* $\ker \varepsilon$ *carries an induced multiplication.*

(ii) *As* η *is a comonoid homomorphism* $\operatorname{coker} \eta$ *carries an induced comultiplication.*

(iii) *The two operations on* $\ker \varepsilon \cong \operatorname{coker} \eta$ *are not compatible in the sense of Definition 3.2.1 (iii) in general.*

Definition 3.2.4
Let $H \in k$*-*$\mathcal{G}rp$ *be a Hopf algebra over* $k \in \mathcal{C}Ring$.

(i) *We define* $\tilde{H}$ *as the direct summand* $\ker \varepsilon = \operatorname{coker} \eta$ *of* H.

(ii) *The* ***lower central series*** $(\Gamma_n H)_{n\geq 0}$ *on* H *is defined as the positive filtration*

$$\Gamma_0 H = H, \quad \Gamma_n H = \tilde{H}^n = \mathrm{im}\,(\tilde{H}^{\otimes n} \xrightarrow{\mu^{n-1}} H) \subset H, \qquad n \geq 1.$$

The ***indecomposables*** *of* H *are defined as* $Q(H) = \mathrm{coker}\,(H \otimes H \xrightarrow{\varepsilon\otimes 1 - \mu + 1\otimes\varepsilon} H)$.

(iii) *The* ***co-lower central series*** $(\mathrm{L}_n\, H)_{n\leq 0}$ *on* H *is defined as the negative filtration*

$$\mathrm{L}_1\, H = 0, \quad \mathrm{L}_{-n}\, H = \ker(H \xrightarrow{\delta^n} \tilde{H}^{\otimes n+1}) \subset H, \qquad n \geq 0.$$

The ***primitive elements*** *of* H *are defined as* $P(H) = \ker(H \xrightarrow{\eta\otimes 1 - \delta + 1\otimes\eta} H \otimes H)$.

Proposition 3.2.5
Let $k \in \mathcal{C}Ring$ *and* $H \in k\text{-}\mathcal{G}rp$. *Then naturally*

(i) $\mathrm{gr}_1^\Gamma H = \tilde{H}/\tilde{H}^2 \xrightarrow{\sim} Q(H)$,

(ii) $P(H) \xrightarrow{\sim} \mathrm{gr}_{-1}^{\mathrm{L}} H$.

Proof. Consider the two-sided bar construction $B_\bullet(k, H, k) \in s(k\text{-Mod})$, where k is considered as a left/right mondule over H via $H \xrightarrow{\varepsilon} k$. Then the associated complex of alternating face maps $B_*(k, H, k)$ is quasi-isomorphic to the reduced bar complex $\tilde{B}_*(k, H, k)$, which is the quotient obtained by modding out the degenerate summands, so

$$\tilde{B}_*(k, H, k) = (k \xleftarrow{\varepsilon-\varepsilon} \text{coker } \eta \xleftarrow{\varepsilon\otimes 1 - \mu + 1\otimes\varepsilon} (\text{coker } \eta)^{\otimes 2} \longleftarrow ...).$$

Moroever there is a subcomplex $B_*(\tilde{H}) \leq B_*(k, H, k)$, given by

$$B_*(\tilde{H}) = (k \xleftarrow{0} \ker\varepsilon \xleftarrow{-\mu} (\ker\varepsilon)^{\otimes 2} \xleftarrow{-\mu\otimes 1 + 1\otimes\mu} ...).$$

As the composition $B_*(\tilde{H}) \hookrightarrow B_*(k, H, k) \twoheadrightarrow \tilde{B}_*(k, H, k)$ is an isomorphism, the inclusion $\tilde{H} = \ker\varepsilon \hookrightarrow H$ induces an isomorphism

$$\mathrm{gr}_1^\Gamma H = H_1(B_*(\tilde{H})) \xrightarrow{\sim} H_1(B_*(k, H, k)) = Q(H).$$

Dually we have a two-sided cobar construction $B^\bullet(k, H, k) \in c(k\text{-Mod})$, obtained by exchanging μ for δ and ε for η. Similarly we get quasi-isomorphisms $\tilde{B}^*(k, H, k) \overset{\simeq}{\hookrightarrow} B^*(k, H, k) \overset{\simeq}{\twoheadrightarrow} B^*(\tilde{H})$, inducing the isomorphism

$$P(H) = H^1(B^*(k, H, k)) \xrightarrow{\sim} H^1(B^*(\tilde{H})) = \mathrm{gr}_{-1}^{\mathrm{L}} H.$$

∎

3.2.2 Detecting epimorphisms and monomorphisms

Using ideas of [MM65] Prop. 3.9 we are giving a criterion for a map of algebras and coalgebras resp. to be surjective and injective resp.

Definition 3.2.6
Let $k \in \mathcal{C}Ring$. Then the lower and co-lower central series also make sense for associative algebras/coalgebras.

(i) *Let $A \in k\text{-}\mathcal{A}ss$.*

   a) *The **lower central series** $(\Gamma_n A)_{n\geq 0}$ on $A \in k\text{-}\mathcal{A}ss$ is defined as the positive algebra filtration*

$$\Gamma_n A = A^n = \operatorname{im}(A^{\otimes n} \xrightarrow{\mu^{n-1}} A) \subset A, \qquad n \geq 1.$$

   b) *A is called **nilpotent**, if $\Gamma_N A = A^N = 0$, for some $N \geq 0$.*

   c) *A is called **complete**, if $A \xrightarrow{\sim} \varprojlim_{n\geq 1} A/\Gamma_n A$.*

(ii) *Dually let $C \in k\text{-}\mathcal{A}ss^{\mathrm{op}}$.*

   a) *The **co-lower central series** $(\mathrm{L}_n\, C)_{n\geq 0}$ on $C \in k\text{-}\mathcal{A}ss^{\mathrm{op}}$ is defined as negative coalgebra filtration*

$$\mathrm{L}_{-n}\, C = \ker(C \xrightarrow{\delta^n} C^{\otimes n+1}) \subset C, \qquad n \geq 0.$$

   b) *C is called **conilpotent**, if $\mathrm{L}_N\, C = C$, for some $N \geq 0$.*

   c) *A is called **cocomplete**[3], if $\bigcup_{n\leq 1} \mathrm{L}_n\, C = \varinjlim_{n\leq 1} \mathrm{L}_n\, C \xrightarrow{\sim} C$.*

We let $k\text{-}\widehat{\mathcal{A}ss}^{\mathrm{op}} \leq k\text{-}\mathcal{A}ss^{\mathrm{op}}$ and $k\text{-}\widehat{\mathcal{A}ss}_1^{\mathrm{op}} \leq k\text{-}\mathcal{A}ss_1^{\mathrm{op}}$ denote the full subcategories of cocomplete (unital) associative coalgebras over k.

Remark 3.2.7
For $k \in \mathcal{C}Ring$, there are adjunctions

$$k\text{-}\mathrm{Mod}(U(X), Y) = k\text{-}\widehat{\mathcal{A}ss}^{\mathrm{op}}(X, \widehat{\mathcal{A}ss}^{\mathrm{op}}(Y)) = k\text{-}\widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X, \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(Y)),$$

where $\widehat{\mathcal{A}ss}^{\mathrm{op}}(X) = \bigoplus_{n\geq 1} X^{\otimes n}$ is the cofree cocomplete associative coalgebra.

Proposition 3.2.8
Let $k \in \mathcal{C}Ring$ and $f \in k\text{-}\mathcal{A}ss(A, B)$ with complete A and B.

Then $A \overset{f}{\twoheadrightarrow} B$, if and only if $\mathrm{gr}_1^\Gamma A \overset{\mathrm{gr}_1^\Gamma f}{\twoheadrightarrow} \mathrm{gr}_1^\Gamma B$.

[3] However some authors call this conilpotence.

Proof. If f is surjective, so is its induced map on the quotients of indecomposables $\mathrm{gr}_1^\Gamma A \xrightarrow{\mathrm{gr}_1^\Gamma f} \mathrm{gr}_1^\Gamma B$. Vice versa suppose $\bar{f}$ is surjective. Consider the commutative square

$$\begin{tikzcd}
\mathcal{A}ss(\mathrm{gr}_1^\Gamma A) \arrow[r, "\mathcal{A}ss(\mathrm{gr}_1^\Gamma f)"] \arrow[d] & \mathcal{A}ss(\mathrm{gr}_1^\Gamma B) \arrow[d] \\
\mathrm{gr}^\Gamma A \arrow[r, "\mathrm{gr}^\Gamma f"] & \mathrm{gr}^\Gamma B,
\end{tikzcd}$$

where the graded object is taken with respect to the lower central series filtration. With $\mathrm{gr}_1^\Gamma f$ also $\mathcal{A}ss(\mathrm{gr}_1^\Gamma f)$ is epimorphic, because tensor powers and direct sums preserve epimorphisms. Moreover as $\mathrm{gr}^\Gamma A = \bigoplus_{n\geq 1} A^n/A^{n+1}$ is generated by $\mathrm{gr}_1^\Gamma A = A/A^2 = \mathrm{gr}_1^\Gamma A$ and similar for B, the vertical maps are surjective. It follows that also $\mathrm{gr}^\Gamma f$ is surjective. Hence also $A/A^n \xrightarrow{\bar{f}} B/B^n$ is surjective by induction on $n \geq 1$, using the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}_n^\Gamma A \arrow[r] \arrow[d] & A/A^{n+1} \arrow[r] \arrow[d] & A/A^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{gr}_n^\Gamma B \arrow[r] & B/B^{n+1} \arrow[r] & B/B^n \arrow[r] & 0.
\end{tikzcd}$$

By taking the inverse limit, it follows that $\hat{A} \overset{\hat{f}}{\twoheadrightarrow} \hat{B}$. As A and B are complete, the commutative square below implies that $A \overset{f}{\twoheadrightarrow} B$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\wr"'] & B \arrow[d, "\wr"'] \\
\hat{A} \arrow[r, "\hat{f}"] & \hat{B}.
\end{tikzcd}$$

$\square$

Remark 3.2.9
Let $k \in \mathcal{C}Ring$ and $f \in k\text{-}\mathcal{A}ss^{\mathrm{op}}(C, D)$ surjective.
If C is cocomplete, then so is D:

$$D = f(C) = f(\bigcup_{n\geq 1} C_n) = \bigcup_{n\geq 1} f(C_n) \subset \bigcup_{n\geq 1} D_n.$$

Similarly the following proposition is a variant of [MM65] Prop. 3.9.

Proposition 3.2.10
Let $k \in \mathcal{C}Ring$ and $f \in k\text{-}\mathcal{A}ss^{\mathrm{op}}(C, D)$ with flat D, cocomplete C and flat $\mathrm{L}_n C$, for all $n \leq 0$.
Then $C \overset{f}{\hookrightarrow} D$, if and only if $\mathrm{gr}_{-1}^{\mathrm{L}} C \overset{\mathrm{gr}_{-1}^{\mathrm{L}} f}{\hookrightarrow} \mathrm{gr}_{-1}^{\mathrm{L}} D$.

Proof. This is the dual statement to Proposition 3.2.8. If f is injective, so is its restriction to $\mathrm{gr}^{\mathrm{L}}_{-1} C = \mathrm{L}_{-1} C$. Vice versa let $n \geq 1$. By tensoring the exact sequence

$$0 \longrightarrow \mathrm{L}_{-n} C \longrightarrow C \xrightarrow{\delta^n} C^{\otimes n+1},$$

with itself and using the flatness of $\mathrm{L}_{-n} C$ and $\varinjlim_{n \geq 0} \mathrm{L}_{-n} C \xrightarrow{\sim} C$ we get an exact sequence as in the upper row of the diagram below.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{L}_{-n} C \otimes \mathrm{L}_{-n} C \arrow[r] & C \otimes C \arrow[r, "(\delta^n \otimes \mathrm{id}, \mathrm{id} \otimes \delta^n)"] & C^{\otimes n+1} \otimes C \times C \otimes C^{\otimes n+1} \\
0 \arrow[r] & \mathrm{L}_{-(n+1)} C \arrow[r] \arrow[u, dashed, "\exists! \delta|"] & C \arrow[r, "\delta^{n+1}"] \arrow[u, "\delta"] & C^{\otimes (n+2)}. \arrow[u, "(\mathrm{id}, \mathrm{id})"']
\end{tikzcd}$$

As the lower row is exact by definition of $\mathrm{L}_{-(n+1)} C$ and the right square commutes, we get a map $\delta|$ as on the left. The kernel of the right vertical map is trivial, so $\ker \delta| \xrightarrow{\sim} \ker \delta = \mathrm{L}_{-1} C = \mathrm{gr}^{\mathrm{L}}_{-1} C$. In particular we get a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker \delta| \arrow[r] \arrow[d, "\wr"'] & \mathrm{L}_{-(n+1)} C \arrow[r, "\delta|"] \arrow[d] & \mathrm{L}_{-n} C \otimes \mathrm{L}_{-n} C \arrow[d] \\
0 \arrow[r] & \mathrm{gr}^{\mathrm{L}}_{-1} C \arrow[r] \arrow[d, "\mathrm{gr}^{\mathrm{L}}_{-1} f"'] & C \arrow[r, "\delta"] \arrow[d, "f"'] & C \otimes C \arrow[d, "f \otimes f"] \\
0 \arrow[r] & \mathrm{gr}^{\mathrm{L}}_{-1} D \arrow[r] & D \arrow[r, "\delta|"] & D \otimes D.
\end{tikzcd}$$

By induction on $n \geq 1$ we will prove injectivity of

$$f|_{\mathrm{L}_{-n} C} : \mathrm{L}_{-n} C \hookrightarrow C \xrightarrow{f} D, \qquad n \geq 1.$$

This holds for $n = 1$ by assumption. If it holds for some $n \geq 1$, then also

$$f|_{\mathrm{L}_{-n} C} \otimes f|_{\mathrm{L}_{-n} C} : \mathrm{L}_{-n} C \otimes \mathrm{L}_{-n} C \longrightarrow C \otimes C \xrightarrow{f \otimes f} D \otimes D$$

is injective, as $L_{-n} C$ and D are flat by assumption. Hence $f|_{\mathrm{L}_{-(n+1)} C}$ is injective by the 5-lemma applied to the above diagram, which proves the induction step. As $f|_{\mathrm{L}_{-n} C}$ factors as $\mathrm{L}_{-n} C \xrightarrow{\mathrm{L}_{-n}(f)} \mathrm{L}_{-n} D \hookrightarrow D$, this implies that $\mathrm{L}_{-n}(f)$ is injective, for all $n \geq 1$. Since direct limits are exact, it follows that f is injective using the commutative square

$$\begin{tikzcd}
\varinjlim_{n \geq 0} \mathrm{L}_{-n} C \arrow[r, hook] \arrow[d, "\wr"'] & \varinjlim_{n \geq 0} \mathrm{L}_{-n} D \arrow[d, hook] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

□

Corollary 3.2.11
Let $k \in \mathcal{C}Ring$ be integral and $f \in k\text{-}\mathcal{A}ss^{\mathrm{op}}(C, D)$ with cocomplete and flat C.

Then $C \overset{f}{\hookrightarrow} D$, if and only if $\mathrm{gr}^{\mathrm{L}}_{-1}C \overset{\mathrm{gr}^{\mathrm{L}}_{-1}f}{\hookrightarrow} \mathrm{gr}^{\mathrm{L}}_{-1}D$.

Proof. Let $Q(k)$ denote the field of fractions of k. Then

$$0 \longrightarrow (\mathrm{gr}^{\mathrm{L}}_{-1}C) \otimes Q(k) \longrightarrow C \otimes Q(k) \xrightarrow{\delta} C \otimes C \otimes Q(k)$$

is exact, which proves that the vertical maps in the diagram below are isomorphisms

$$\begin{tikzcd}
(\mathrm{gr}^{\mathrm{L}}_{-1}C) \otimes Q(k) \arrow[r, hook, "(\mathrm{gr}^{\mathrm{L}}_{-1}f)\otimes Q(k)"] \arrow[d, "\wr"'] & (\mathrm{gr}^{\mathrm{L}}_{-1}C) \otimes Q(k) \arrow[d, "\wr"] \\
\mathrm{gr}^{\mathrm{L}}_{-1}(C \otimes Q(k)) \arrow[r, "\mathrm{gr}^{\mathrm{L}}_{-1}(f\otimes Q(k))"] & \mathrm{gr}^{\mathrm{L}}_{-1}(D \otimes Q(k)).
\end{tikzcd}$$

Moreover $(\mathrm{gr}^{\mathrm{L}}_{-1}f)\otimes Q(k)$ is injective using flatness of $Q(k)$ again. Hence by commutativity also $\mathrm{gr}^{\mathrm{L}}_{-1}(f \otimes Q(k))$ and thus $C \otimes Q(k) \xrightarrow{f\otimes Q(k)} D \otimes Q(k)$ is injective by the preceding Corollary. In the commutative square

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\
C \otimes Q(k) \arrow[r, hook, "f\otimes Q(k)"] & D \otimes Q(k),
\end{tikzcd}$$

the left vertical map is injective by flatness of C and because $k \hookrightarrow Q(k)$ as k is integral. Using commutativity f must be injective, too.

$\square$

3.3 Lie algebras and enveloping algebras

We are now introducing the Hopf algebra of an enveloping algebra of a Lie algebra and construct suitable filtrations.

3.3.1 The enveloping algebra as a Hopf algebra

Proposition 3.3.1
For every Lie algebra $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ the enveloping algebra $U_k(\mathfrak{g})$ becomes a Hopf ring via

(i) $\delta : U_k(\mathfrak{g}) \longrightarrow U_k(\mathfrak{g}) \otimes U_k(\mathfrak{g}), \qquad g \longmapsto g \otimes 1 + 1 \otimes g, \qquad g \in \mathfrak{g}.$

(ii) $\varepsilon : U_k(\mathfrak{g}) \longrightarrow k, \quad g \longmapsto 0, \qquad g \in \mathfrak{g}.$

(iii) $\iota : U_k(\mathfrak{g}) \longrightarrow U_k(\mathfrak{g})^{\mathrm{op}}, \quad g \longmapsto -g, \qquad g \in \mathfrak{g}.$

Proof. Using the universal property for enveloping algebras the given Lie ring homomorphisms on $\mathfrak{g}$ uniquely extend to the desired ring homomorphisms. Moreover it follows that

$$U_k(\mathfrak{g} \times \mathfrak{g}) \xrightarrow{\sim} U_k(\mathfrak{g}) \otimes U_k(\mathfrak{g}), \qquad (g, h) \longmapsto g \otimes 1 + 1 \otimes h, \qquad g, h \in \mathfrak{g},$$

and it suffices to check all identities for elements in $\mathfrak{g}$.

□

Remark 3.3.2
Let $k \in \mathcal{C}Ring$. Taking primitive elements induces a functor

$$k\text{-}\mathcal{G}rp \longrightarrow k\text{-}\mathcal{L}ie, \quad H \longmapsto P(H),$$

which is the right adjoint of an adjunction $k\text{-}\mathcal{G}rp(U_k(X), Y) = k\text{-}\mathcal{L}ie(X, P(Y))$, induced by the adjunction $k\text{-}\mathcal{A}ss_1(U_k(X), Y) = k\text{-}\mathcal{L}ie(X, U(Y))$.

Indeed its unit $\mathfrak{g} \xrightarrow{\eta_{\mathfrak{g}}} U_k(\mathfrak{g})$ sends $g \in \mathfrak{g}$ to the generator $g \in U_k(\mathfrak{g})$, which lies in $PU_k(\mathfrak{g})$ by construction of the coalgebra structure on $U_k(\mathfrak{g})$.

3.3.2 The co-lower central series on the enveloping algebra

The colower central series on an enveloping algebra is closely related to the filtration used in the Theorem of Poincaré, Birkhoff and Witt, as we will show in this section.

Definition 3.3.3
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$. We define an negative (associative algebra) filtration E on the enveloping algebra $U_k(\mathfrak{g})$ as the image of the the increasing algebra filtration

$$E_{-n}\mathcal{A}ss_1(\mathfrak{g}) = \mathcal{A}ss_1^{\leq n}(\mathfrak{g}) := \bigoplus_{0 \leq k \leq n} \mathfrak{g}^{\otimes k} \subset \mathcal{A}ss_1(\mathfrak{g}), \qquad n \geq 0,$$

under the quotient map $\mathcal{A}ss_1(\mathfrak{g}) \twoheadrightarrow U_k(\mathfrak{g})$ defining $U_k(\mathfrak{g})$.

The following proposition is well-known [Bou98] Chapter I, §2.6. For the convenience of the reader, we recall its simple proof.

Proposition 3.3.4
For $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ the following holds.

(i) *The filtration E on $U_k(\mathfrak{g})$ is exhaustive, i.e. $U_k(\mathfrak{g}) = \bigcup_{n \leq 0} E_n U_k(\mathfrak{g})$.*

(ii) *The natural map $\mathfrak{g} \xrightarrow{\eta_{\mathfrak{g}}} U_k(\mathfrak{g})$ extends to an epimorphism of commutative k-algebras*

$$\psi_{\mathfrak{g}} : \mathcal{C}om_1(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}^E U_k(\mathfrak{g}).$$

Proof. Let $\mathcal{A}ss_1(\mathfrak{g}) \overset{q}{\twoheadrightarrow} U_k(\mathfrak{g})$ denote the quotient map.

(i) As the filtration $\mathcal{A}ss_1^{\leq n}(\mathfrak{g})$ is exhaustive on $\mathcal{A}ss_1(\mathfrak{g})$ so is E on $U_k(\mathfrak{g})$, because

$$\bigcup_{n\geq 1} E_{-n}U_k(\mathfrak{g}) = \bigcup_{n\geq 1} q(\mathcal{A}ss_1^{\leq n}\mathfrak{g}) = q(\bigcup_{n\geq 1} \mathcal{A}ss_1^{\leq n}\mathfrak{g}) = q(\mathcal{A}ss_1\mathfrak{g}) = U_k(\mathfrak{g}).$$

(ii) We have

$$\eta_{\mathfrak{g}}^{-1}(E_0U_k(\mathfrak{g})) = \eta_{\mathfrak{g}}^{-1}(k) = 0, \qquad \eta_{\mathfrak{g}}(\mathfrak{g}) = q(\mathfrak{g}) = E_{-1}U_k(\mathfrak{g}),$$

hence $\eta_{\mathfrak{g}}$ induces a map $\mathfrak{g} \longrightarrow \mathrm{gr}^E_{-1}U_k(\mathfrak{g})$, extending to a map of k-algebras $\mathcal{A}ss_1(\mathfrak{g}) \longrightarrow \mathrm{gr}^E U_k(\mathfrak{g})$. It can be identified with $\mathrm{gr}^E q$ under the natural isomorphism $\mathcal{A}ss_1(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^E\mathcal{A}ss_1(\mathfrak{g})$. Moreover

$$[q(x), q(y)] = [\eta_{\mathfrak{g}}(x), \eta_{\mathfrak{g}}(y)] = \eta_{\mathfrak{g}}[x, y] = q[x, y] \in E_{-1}U_k(\mathfrak{g}), \qquad x, y \in \mathfrak{g},$$

which implies that $\mathcal{A}ss_1(\mathfrak{g}) \longrightarrow \mathrm{gr}^E U_k(\mathfrak{g})$ factors over $\mathcal{C}om_1(\mathfrak{g})$, as $\mathcal{A}ss_1(\mathfrak{g})$ is generated by $\mathfrak{g}$.

□

Proposition 3.3.5
For $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k$-$\mathcal{L}ie$ we have $E_nU_k(\mathfrak{g}) \subset \mathrm{L}_n\, U_k(\mathfrak{g})$, for all $n \leq 0$.
In particular $U_k(\mathfrak{g})$ is cocomplete and we get a natural map $\mathrm{gr}^E U_k(\mathfrak{g}) \longrightarrow \mathrm{gr}^{\mathrm{L}} U_k(\mathfrak{g})$.

Proof. By abuse of notation we write δ^n for the composition

$$U_k(\mathfrak{g}) \xrightarrow{\delta} U_k(\mathfrak{g})^{\otimes 2} \xrightarrow{\delta\otimes\mathrm{id}} U_k(\mathfrak{g})^{\otimes 3} \xrightarrow{\delta\otimes\mathrm{id}} \dots \xrightarrow{\delta\otimes\mathrm{id}} U_k(\mathfrak{g})^{\otimes n} \xrightarrow{\delta\otimes\mathrm{id}} U_k(\mathfrak{g})^{\otimes(n+1)},$$

which is a ring homomorphism as δ is one by definition of a Hopf algebra. For $x \in \mathfrak{g}$ we have

$$\delta^n(x) = \sum_{0\leq i\leq n} 1 \otimes \dots \otimes \underset{i}{x} \otimes \dots \otimes 1, \qquad n \geq 1,$$

and using that δ^n is a ring homomorphism it follows that the composition

$$\mathcal{A}ss_1^{\leq n}(\mathfrak{g}) \xrightarrow{q} U_k(\mathfrak{g}) \xrightarrow{\delta^n} \tilde{U}_k(\mathfrak{g})^{\otimes(n+1)},$$

is zero, for all $n \geq 1$. In particular

$$E_{-n}U_k(\mathfrak{g}) \subset \ker(U_k(\mathfrak{g}) \xrightarrow{\delta^n} \tilde{U}_k(\mathfrak{g})^{\otimes(n+1)}) = \mathrm{L}_{\text{-n}}\, U_k(\mathfrak{g}), \qquad n \geq 1.$$

□

Proposition 3.3.6
Let $k \in \mathcal{C}Ring$ be flat over $\mathbb{Z}$ and $\mathfrak{g} \in k$-$\mathcal{L}ie$ a flat Lie algebra, for which there is a monomorphism into a flat associative k-algebra $A \in k$-$\mathcal{A}ss$ [4].

[4] Such an A always exists, for every flat Lie algebra $\mathfrak{g}$, even if k is not flat over $\mathbb{Z}$. Indeed we may take $A = U_k(\mathfrak{g})$, which is flat by Corollary 3.6.2 and the general Theorem of Poincaré, Birkhoff and Witt 3.3.10. However the latter is much more difficult to prove than the special case presented here.

(i) $\psi_{\mathfrak{g}} : \mathcal{C}om_1(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^E U_k(\mathfrak{g})$.

(ii) $E_n U_k(\mathfrak{g}) = \mathrm{L}_n\, U_k(\mathfrak{g})$, *for all* $n \leq 0$.

Moreover all these are flat k*-modules. In particular also* $U_k(\mathfrak{g})$ *is flat over* k.

(iii) $\eta_{\mathfrak{g}} : \mathfrak{g} \xrightarrow{\sim} PU_k(\mathfrak{g})$.

Proof. By the universal property of the enveloping algebra the identity on A extends to a surjection $U_k(A) \twoheadrightarrow A_+ \twoheadrightarrow A_+/k = A$, where $A_+ = k \oplus A$ is the universal unital k-algebra under A of Remark 2.3.2. By the universal property of cocomplete coalgebras this map extends uniquely to a homomorphism of coalgebras $U_k(A) \longrightarrow \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(A) = \bigoplus_{n\geq 0} A^{\otimes n}$ inducing a commuting square

$$\begin{tikzcd}
\mathcal{C}om_1(A) \arrow[rr, "N"] \arrow[d, "\psi_A"'] & & \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(A) \arrow[d, equal] \\
\mathrm{gr}^E U_k(A) \arrow[r] & \mathrm{gr}^{\mathrm{L}} U_k(A) \arrow[r] & \mathrm{gr}^{\mathrm{L}} \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(A).
\end{tikzcd}$$

In homogeneous degree $n \geq 0$ the map N can be computed as

$$N(x_1 \cdots x_n) = \prod_{1\leq i\leq n} \delta^{n-1}(x_i) = \prod_{1\leq i\leq n} \Big(\sum_{1\leq j\leq n} 1 \otimes \ldots \otimes \underset{j}{x_i} \otimes \ldots \otimes 1 \Big) = \sum_{\sigma\in\Sigma_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)},$$

because all summands with at least one tensor factor equal to 1 vanish in $A_+/k = A$. Hence N is the direct sum of norm maps $A^{\otimes n}/\Sigma_n \xrightarrow{N_n} A^{\otimes n}$. The composition $(A^{\otimes n})_{\Sigma_n} \xrightarrow{N_n} A^{\otimes n} \twoheadrightarrow (A^{\otimes n})_{\Sigma_n}$ is multiplication by $n!$ and thus is injective, since A is flat over $\mathbb{Z}$. Hence N is injective and by commutativity we get that $\psi_A : \mathcal{C}om_1(A) \xrightarrow{\sim} \mathrm{gr}^E U_k(A)$ is an isomorphism and that $\mathrm{gr}^E U_k(A) \hookrightarrow \mathrm{gr}^{\mathrm{L}} U_k(A)$ is injective.

Bringing $\mathfrak{g}$ into play we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}om_1(\mathfrak{g}) \arrow[r, two heads, "\psi_{\mathfrak{g}}"] \arrow[d, hook] & \mathrm{gr}^E U_k(\mathfrak{g}) \arrow[r] \arrow[d] & \mathrm{gr}^{\mathrm{L}} U_k(\mathfrak{g}) \arrow[d] \\
\mathcal{C}om_1(A) \arrow[r, "\psi_A", "\sim"'] & \mathrm{gr}^E U_k(A) \arrow[r, hook] & \mathrm{gr}^{\mathrm{L}} U_k(A),
\end{tikzcd}$$

where the left vertical map is injective, because $\mathfrak{g} \hookrightarrow A$ and both are flat k-modules. Hence $\psi_{\mathfrak{g}}$ is an isomorphism, which proves (i). Furthermore the middle vertical map is injective. As also the lower right horizontal map is injective, so is the upper right horizontal map.

Now we use the injectivity of $\mathrm{gr}^E U_k(\mathfrak{g}) \hookrightarrow \mathrm{gr}^{\mathrm{L}} U_k(\mathfrak{g})$ to prove (ii). It is equivalent to

$$E_{n-1} U_k(\mathfrak{g}) \cap \mathrm{L}_n\, U_k(\mathfrak{g}) = E_n U_k(\mathfrak{g}), \qquad n \leq 0,$$

which implies

$$\begin{aligned}
E_m U_k(\mathfrak{g}) \cap \mathrm{L}_n\, U_k(\mathfrak{g}) &= E_m U_k(\mathfrak{g}) \cap \mathrm{L}_{m+1}\, U_k(\mathfrak{g}) \cap \mathrm{L}_n\, U_k(\mathfrak{g}) = E_{m+1} U_k(\mathfrak{g}) \cap \mathrm{L}_n\, U_k(\mathfrak{g}) \\
&= \ldots = E_{n-1} U_k(\mathfrak{g}) \cap \mathrm{L}_n\, U_k(\mathfrak{g}) = E_n U_k(\mathfrak{g}), \qquad m < n \leq 0,
\end{aligned}$$

and hence $\mathrm{L}_n U_k(\mathfrak{g}) = E_n U_k(\mathfrak{g})$, for all $n \leq 0$, as E is exhaustive. As $\mathcal{C}om_1^{\leq n}(\mathfrak{g}) \cong \mathrm{gr} E_n U_k(\mathfrak{g})$ is flat, so is also $E_n U_k(\mathfrak{g})$ by Corollary 3.6.2.

Finally consider the diagram

$$\begin{tikzcd}
\mathcal{C}om_1^{(1)}(\mathfrak{g}) \arrow[r, "\psi_{\mathfrak{g}}", "\sim"'] \arrow[d, equal] & \mathrm{gr}^E_{-1} U_k(\mathfrak{g}) \arrow[r, equal] & \mathrm{gr}^{\mathrm{L}}_{-1} U_k(\mathfrak{g}) \\
\mathfrak{g} \arrow[rr, "\eta_{\mathfrak{g}}"] & & PU_k(\mathfrak{g}). \arrow[u, "\wr"']
\end{tikzcd}$$

The upper horizontal map is an isomorphism by (i). The equality on the upper right holds by (ii). The right vertical map is an isomorphism by Proposition 3.2.5 (ii). As the diagram commutes, it follows that the lower horizontal map must be an isomorphism.

$\square$

Remark 3.3.7

Proposition 3.3.6 (ii) is false, if k is not flat over $\mathbb{Z}$.

For example let $\mathfrak{g} \in \mathbb{E}_p$-$\mathcal{L}ie$. Then in $U_{\mathbb{E}_p}(\mathfrak{g})$ we have

$$\delta(x^p) = (x \otimes 1 + 1 \otimes x)^p = x^p \otimes 1 + 1 \otimes x^p, \qquad x \in \mathfrak{g}.$$

Hence $x^p \in PU_{\mathbb{E}_p}(\mathfrak{g}) \backslash \mathfrak{g}$, for all $0 \neq x \in \mathfrak{g}$.

Corollary 3.3.8

Let $k \in \mathcal{C}Ring$ be flat over $\mathbb{Z}$ and $\mathfrak{g} \in k$-$\mathcal{L}ie$ flat over k.

Then every $\mathfrak{g} \hookrightarrow P(H)$ with flat $H \in k$-$\mathcal{G}rp$ extends to $U_k(\mathfrak{g}) \hookrightarrow H$.

If moreover k is integral, the flatness condition on H can be skipped.

Proof. As $\mathfrak{g}$ maps to the primitive part of H, the unique extension $U_k(\mathfrak{g}) \longrightarrow H$ is a homomorphism of Hopf algebras. The map $\mathfrak{g} \hookrightarrow H$ is a monomorphism into a flat associative algebra and thus by Proposition 3.3.6 (ii) the underlying coalgebra of $U_k(\mathfrak{g})$ satisfies the hypotheses of Proposition 3.2.10. Hence $U_k(\mathfrak{g}) \hookrightarrow H$ is injective, since $\mathfrak{g} = \mathrm{gr}^E_{-1} U_k(\mathfrak{g}) = \mathrm{gr}^{\mathrm{L}}_{-1} U_k(\mathfrak{g}) = PU_k(\mathfrak{g}) \hookrightarrow P(H)$ is injective by assumption.

If k is integral, then it injects into its field of fractions $Q(k)$. By flatness of $\mathfrak{g}$ and $Q(k)$ we get a monomorphism $\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes Q(k) \hookrightarrow H \otimes Q(k)$, which extends to a monomorphism $U_k(\mathfrak{g}) \longrightarrow H \longrightarrow H \otimes Q(k)$ by what we have just shown. In particular also the first map $U_k(\mathfrak{g}) \hookrightarrow H$ is injective.

$\square$

Proposition 3.3.9

Let $k \in \mathcal{C}Ring$ be integral and flat over $\mathbb{Z}$.

Then every $\mathbb{Z}_{>0}$-graded, flat Lie algebra $\mathfrak{g} \in k$-$\mathcal{L}ie$ satisfies the hypothesis of Proposition 3.3.6

Proof. There is a derivation δ on $\mathfrak{g}$, sending an element of homogeneous degree $m > 0$ to $m \cdot x$. Indeed we get

$$\delta[x, y] = (m + n) \cdot [x, y] = m \cdot [x, y] + n \cdot [x, y] = [m \cdot x, y] + [x, n \cdot y] = [\delta(x), y] + [x, \delta(y)],$$

if $x, y \in \mathfrak{g}$ have homogeneous degrees $m, n > 0$. Considering $\mathfrak{g}$ as a bimodule over $U_k(\mathfrak{g})$, acting by the adjoint action from the left and trivially from the right, this extends to a derivation

$$U_k(\mathfrak{g}) \xrightarrow{\delta} \mathfrak{g}, \quad x_1 \cdots x_n \longmapsto \sum_{1 \le i \le n} x_1 \cdots \delta(x_i) \cdots x_n = \mathrm{ad}(x_1) \circ ... \circ \mathrm{ad}(x_{n-1}) \circ \delta(x_n).$$

The composition $\mathfrak{g} \xrightarrow{\eta_\mathfrak{g}} U_k(\mathfrak{g}) \xrightarrow{\delta} \mathfrak{g}$ is again δ, which is injective as $\mathfrak{g}$ is flat over k and thus over $\mathbb{Z}$. Hence also $\eta_\mathfrak{g}$ is injective.

Now let $Q(k)$ denote the field of fractions of k. For the composition

$$\mathfrak{g} \longrightarrow \mathfrak{g} \otimes Q(k) \xrightarrow{\eta_{\mathfrak{g} \otimes Q(k)}} U_{Q(k)}(\mathfrak{g} \otimes Q(k))$$

the left map is injective, as $k \hookrightarrow Q(k)$ is injective and $\mathfrak{g}$ is flat over k, and the right map is injective by the argument above. As $U_{Q(k)}(\mathfrak{g} \otimes Q(k))$ is a vector space over $Q(k)$, it is flat over k.

□

Proposition 3.3.6 (i) is better known as the Theorem of Poincaré, Birkhoff and Witt, which also holds in a more general situation. We will state it here, although Proposition 3.3.6 will be enough for the purpose of this work.

Theorem 3.3.10 (Poincaré, Birkhoff, Witt, Higgins)
For $k \in \mathcal{C}Ring$ and flat $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ the map $\psi_\mathfrak{g} : \mathcal{C}om_1(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^E U_k(\mathfrak{g})$ is an isomorphism.

Proof. See [Hig69] Theorem 6, 7 and 8. The proof is far more involved than the proof of the special case, given in Proposition 3.3.6.

□

3.3.3 Filtrations on a Lie algebra and its enveloping algebra

We are now defining filtrations on the enveloping algebra induced by that of a Lie algebra. As a corollary we obtain the well-known Theorem of Ado.

Remark 3.3.11
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ a Lie algebra.

(i) *The* ***commutator of two submodules*** *$X_1, X_2 \le \mathfrak{g}$ is defined as the submodule $[X_1, X_2] \le \mathfrak{g}$ generated by $[x_1, x_2]$, for $x_i \in X_i$.*

(ii) *Recall from Definition 3.1.6 that a* ***Lie algebra filtration*** *on $\mathfrak{g}$ is a filtration F of submodules, such that*

$$[F_p\mathfrak{g}, F_q\mathfrak{g}] \subset F_{p+q}\mathfrak{g}, \qquad p, q \in \mathbb{Z}.$$

Remark 3.3.12
Let $k \in \mathcal{C}Ring$ and F a Lie algebra filtration on a Lie algebra $\mathfrak{g} \in k\text{-}\mathcal{L}ie$.

(i) *The Lie bracket induces a Lie algebra structure on the associated graded module* $\mathrm{gr}^F \mathfrak{g} = \bigoplus_{n\geq 1} F_n\mathfrak{g}/F_{n+1}\mathfrak{g}$.

(ii) *There is an* ***induced Hopf algebra filtration*** *on the enveloping algebra* $U_k(\mathfrak{g})$*, defined as the quotient filtration under* $\mathcal{A}ss_1(\mathfrak{g}) \twoheadrightarrow U_k(\mathfrak{g})$*, where the Hopf algebra filtration on* $\mathcal{A}ss_1(\mathfrak{g}) = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$ *is the sum of the tensor product filtrations induced by* F*.*

(iii) *If* $F_1\mathfrak{g} = \mathfrak{g}$*, then the induced filtration* F *on* $U_k(\mathfrak{g})$ *is given by*

$$F_0U_k(\mathfrak{g}) = U_k(\mathfrak{g}), \qquad F_nU_k(\mathfrak{g}) = \sum_{\substack{a_1,\ldots,a_r\geq 1,\\ a_1+\ldots+a_r\geq n}} \tilde{F}_{a_1}U_k(\mathfrak{g}) \cdot \ldots \cdot \tilde{F}_{a_r}U_k(\mathfrak{g}) \quad n \geq 1,$$

where $\tilde{F}_nU_k(\mathfrak{g}) = U_k(\mathfrak{g}) \cdot F_n\mathfrak{g}$ *is the kernel of* $U_k(\mathfrak{g}) \twoheadrightarrow U_k(\mathfrak{g}/F_n\mathfrak{g})$*, for all* $n \geq 1$*.*

(iv) *By construction the map* $\eta_\mathfrak{g} : \mathfrak{g} \longrightarrow U_k(\mathfrak{g})$ *preserves the filtrations and induces maps*

$$\mathrm{gr}^F\eta_\mathfrak{g} : \mathrm{gr}^F\mathfrak{g} \longrightarrow \mathrm{gr}^F U_k(\mathfrak{g}), \qquad \phi_\mathfrak{g} : U_k(\mathrm{gr}^F\mathfrak{g}) \longrightarrow \mathrm{gr}^F U_k(\mathfrak{g}).$$

Remark 3.3.13
Let $k \in \mathcal{C}Ring$*.*

(i) *The* ***lower central series*** $(\Gamma_n\mathfrak{g})_{n\geq 0}$ *on a Lie algebra* $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ *is given by*

$$\Gamma_0\mathfrak{g} = \Gamma_1\mathfrak{g} = \mathfrak{g}, \qquad \Gamma_{n+1}\mathfrak{g} = [\mathfrak{g}, \Gamma_n\mathfrak{g}], \qquad n \geq 0.$$

It is the initial Lie algebra filtration on $\mathfrak{g}$ *satisfying* $\Gamma_1\mathfrak{g} = \mathfrak{g}$*.*

(ii) *The lower central series defines an epimorphism preserving endofunctor*

$$\mathrm{gr}^\Gamma : k\text{-}\mathcal{L}ie \longrightarrow k\text{-}\mathcal{L}ie, \quad \mathfrak{g} \longmapsto \mathrm{gr}^\Gamma\mathfrak{g} = \bigoplus_{n\geq 1} \Gamma_n\mathfrak{g}/\Gamma_{n+1}\mathfrak{g}.$$

(iii) *By construction the induced filtration* Γ *on the enveloping algebra* $U_k(\mathfrak{g})$ *coincides with the lower central series defined for Hopf algebras in Definition 3.2.4, so there is no notational conflict occuring here.*

Proposition 3.3.14
Let $k \in \mathcal{C}Ring$ *and* F *a Lie algebra filtration on a Lie algebra* $\mathfrak{g} \in k\text{-}\mathcal{L}ie$*.*

Then $U_k(\mathrm{gr}^F\mathfrak{g}) \overset{\phi_\mathfrak{g}}{\twoheadrightarrow} \mathrm{gr}^F U_k(\mathfrak{g})$ *is always an epimorphism.*

Proof. The inclusion $\mathrm{gr}^F\mathfrak{g} \longrightarrow \mathrm{gr}^F\mathcal{A}ss_1(\mathfrak{g})$ extends to a map $\mathcal{A}ss_1(\mathrm{gr}^F\mathfrak{g}) \twoheadrightarrow \mathrm{gr}^F\mathcal{A}ss_1(\mathfrak{g})$, which is surjective being the direct sum of the natural epimorphisms

$$(\mathrm{gr}^F\mathfrak{g})^{\otimes r} \twoheadrightarrow \mathrm{gr}^F(\mathfrak{g}^{\otimes r}), \qquad r \geq 0.$$

Moreover by construction the surjection $\mathcal{A}ss_1(\mathfrak{g}) \twoheadrightarrow U_k(\mathfrak{g})$ maps $F_n\mathcal{A}ss_1(\mathfrak{g})$ onto $F_nU_k(\mathfrak{g})$, for all $n \in \mathbb{Z}$. So using the commutative square

$$\begin{tikzcd} \mathcal{A}ss_1(\mathrm{gr}^F\mathfrak{g}) \arrow[r, two heads] \arrow[d] & U_k(\mathrm{gr}^F\mathfrak{g}) \arrow[d, "\phi_{\mathfrak{g}}"] \\ \mathrm{gr}^F\mathcal{A}ss_1(\mathfrak{g}) \arrow[r, two heads] & \mathrm{gr}^F U_k(\mathfrak{g}), \end{tikzcd}$$

it follows that also $\phi_{\mathfrak{g}}$ is surjective.

□

Proposition 3.3.15
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$, carrying a Lie algebra filtration $\mathfrak{g} = F_1\mathfrak{g} \supset \ldots \supset F_n\mathfrak{g} = 0$ with $\mathrm{gr}^F\mathfrak{g} \cong k^d$, for some $n, d \geq 1$.
Then $\mathfrak{g} \hookrightarrow U_k(\mathfrak{g})/F_nU_k(\mathfrak{g})$ is injective.

Proof. This is done by induction on the rank $d \geq 1$. Let $m \geq 1$ be minimal with $\mathrm{gr}^F_m\mathfrak{g} \neq 0$. So we can assume $n > m$. If $d = 1$ then $\mathfrak{g}$ is abelian and the composition

$$\mathfrak{g} \longrightarrow U_k(\mathfrak{g}) \twoheadrightarrow U_k(\mathfrak{g})/F_{m+1}U_k(\mathfrak{g}) \twoheadrightarrow U_k(\mathfrak{g})/\tilde{U}_k(\mathfrak{g})^2$$

is injective, because by Proposition 3.2.5 we have

$$\mathrm{gr}^\Gamma_1 U_k(\mathfrak{g}) = \tilde{U}_k(\mathfrak{g})/\tilde{U}_k(\mathfrak{g})^2 \xrightarrow{\sim} Q(U_k(\mathfrak{g})) = H_1(\mathfrak{g}, k) = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}.$$

For $d > 1$ we take an element $a \in \mathfrak{g}$, such that $[a] \in \mathrm{gr}^F_m\mathfrak{g}$ can be extended to a basis of the k-module $\mathrm{gr}^F\mathfrak{g}$. We let $\mathrm{gr}^F_m\mathfrak{g} \overset{\pi_a}{\twoheadrightarrow} k$ be the corresponding projection onto the direct summand spanned by $[a]$. Since F is a Lie algebra filtration, the maps

$$q : \mathfrak{g} \twoheadrightarrow \mathrm{gr}^F_m\mathfrak{g} \xrightarrow{\pi_x} k$$

are homomorphisms of Lie algebras, the middle and the right one being abelian. The induced filtration F on the kernel $\mathfrak{n} \lhd \mathfrak{g}$ of q is a Lie algebra filtration and $\mathrm{gr}^F\mathfrak{n}$ is a direct summand of $\mathrm{gr}^F\mathfrak{g}$ and thus free of rank $< d$. Hence by the induction hypothesis we have $\mathfrak{n} \hookrightarrow U_k(\mathfrak{n})/F_nU_k(\mathfrak{n})$.

Now we have $a \in q^{-1}(1)$, the map $\mathrm{ad}(a)$ induces a derivation on $\mathfrak{n}$ and on $U_k(\mathfrak{n})$ by the universal property of the enveloping algebra. The map $1 \longmapsto a$ defines a section for q, which is a Lie algebra homomorphism. In particular $\mathfrak{g} \cong \mathfrak{n} \rtimes k$ and we get a unique action of $\mathfrak{g}$ on $U_k(\mathfrak{n})$ extending the action of $\mathfrak{n}$ and $k \cdot a \subset \mathfrak{g}$. Since $a \cdot F_q\mathfrak{n} = \mathrm{ad}(a)(F_q\mathfrak{n}) \subset F_{q+1}\mathfrak{n}$ and $\mathfrak{g} = \mathfrak{n} + k \cdot a$, we get

$$F_p\mathfrak{g} \cdot F_qU_k(\mathfrak{n}) \subset F_{p+q}U_k(\mathfrak{n}), \qquad p, q \geq 0.$$

Let A denote the k-algebra of k-linear endomorphisms on $U_k(\mathfrak{n})/F_nU_k(\mathfrak{n})$. Then A carries a ring filtration, where F_pA is the k-submodule of endomorphisms f, such that $f(F_qU_k(\mathfrak{n})) \subset F_{p+q}U_k(\mathfrak{n})$, for all $q \geq 0$. Moreover $\tilde{A} = F_1A$ is a nilpotent subalgebra and left multiplication defines a representation $\mathfrak{g} \xrightarrow{\lambda} \tilde{A}_+ = \tilde{A} \oplus k$, mapping $F_p\mathfrak{g}$ to F_pA, for all $p \geq 1$. It extends to an algebra homomorphisms $U_k(\mathfrak{g}) \longrightarrow \tilde{A}_+$, mapping $F_pU_k(\mathfrak{g})$ to F_pA, for all $p \geq 1$. In particular λ factors as $\mathfrak{g} \longrightarrow U_k(\mathfrak{g})/F_nU_k(\mathfrak{g}) \longrightarrow \tilde{A}_+$, because $\tilde{A}^n = 0$. The composition

$$k \cdot a \hookrightarrow \mathfrak{g} \longrightarrow U_k(\mathfrak{g})/F_nU_k(\mathfrak{g}) \overset{q}{\twoheadrightarrow} U_k(k)/F_nU_k(k)$$

is injective by the case $d = 1$. Using the commutative diagram

$$\begin{tikzcd}
\mathfrak{n} \arrow[r, hook] \arrow[d] & U_k(\mathfrak{n})/F_nU_k(\mathfrak{n}) \arrow[r, hook] \arrow[d] & \underline{k\text{-Mod}}(U_k(\mathfrak{n})/F_nU_k(\mathfrak{n})) \\
\mathfrak{g} \arrow[r] & U_k(\mathfrak{g})/F_nU_k(\mathfrak{g}) \arrow[r] & \tilde{A}_+, \arrow[u]
\end{tikzcd}$$

it follows that also $\mathfrak{g} \hookrightarrow U_k(\mathfrak{g})/F_nU_k(\mathfrak{g})$ is injective.

□

As a consequence we get the well-known Theorem of Ado.

Corollary 3.3.16 (Ado)
Let k be a field and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ nilpotent with $d = \dim_k \mathfrak{g} < \infty$.
Then the lower central series satisfies the hypotheses of Proposition 3.3.15.
In particular we get a finite-dimensional nilpotent representation $\mathfrak{g} \hookrightarrow U_k(\mathfrak{g})/\tilde{U}_k(\mathfrak{g})^n$, for $n \geq 1$ with $\Gamma_n\mathfrak{g} = 0$.

Proposition 3.3.17
Let $k \in \mathcal{C}Ring$ be integral and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$, carrying a Lie algebra filtration F with $F_1\mathfrak{g} = \mathfrak{g}$ and $\mathrm{gr}^F\mathfrak{g}$ flat over k.
Then $\mathrm{gr}^F\mathfrak{g} \hookrightarrow \mathrm{gr}^F U_k(\mathfrak{g})$.
If moreover k is flat over $\mathbb{Z}$, then $\mathrm{gr}^F U_k(\mathfrak{g})$ is flat and $\phi_{\mathfrak{g}} : U_k(\mathrm{gr}^F\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^F U_k(\mathfrak{g})$.

Proof. The proof will be established in several steps.

(i) First we assume that k is a field and that $\mathfrak{g}/F_n\mathfrak{g}$ is a finitely generated k-module, for all $n \geq 1$. Then $\mathrm{gr}^F(\mathfrak{g}/F_n\mathfrak{g}) = \mathrm{gr}_1^F\mathfrak{g} \oplus ... \oplus \mathrm{gr}_{n-1}^F\mathfrak{g} \cong k^d$, for some $d \geq 1$, and thus by Proposition 3.3.15 we get injections

$$\mathfrak{g}/F_n\mathfrak{g} \hookrightarrow U_k(\mathfrak{g})/F_nU_k(\mathfrak{g}) \xrightarrow{\sim} U_k(\mathfrak{g}/F_n\mathfrak{g})/F_nU_k(\mathfrak{g}/F_n\mathfrak{g}), \qquad n \geq 1,$$

where the latter is an isomorphism, because by definition $F_nU_k(\mathfrak{g})$ contains the kernel of $U_k(\mathfrak{g}) \twoheadrightarrow U_k(\mathfrak{g}/F_n\mathfrak{g})$. Equivalently $\mathrm{gr}^F\mathfrak{g} \hookrightarrow \mathrm{gr}^F U_k(\mathfrak{g})$.

(ii) Next we only assume that k is a field. For every finitely generated Lie subalgebra $\mathfrak{g}' \leq \mathfrak{g}$ we give $\mathfrak{g}'$ the subalgebra filtration. Then $\mathfrak{g}'/F_n\mathfrak{g}'$ is a finitely generated k-module, for all $n \geq 1$, and we get $\mathrm{gr}^F\mathfrak{g}' \hookrightarrow \mathrm{gr}^F U_k(\mathfrak{g}')$ by case (i). In the commutative diagram

$$\begin{tikzcd}
\varinjlim_{\mathfrak{g}' \leq \mathfrak{g} \text{ f.g.}} \mathrm{gr}^F\mathfrak{g}' \arrow[r] \arrow[d] & \mathrm{gr}^F(\varinjlim_{\mathfrak{g}' \leq \mathfrak{g} \text{ f.g.}} \mathfrak{g}') \arrow[r] \arrow[d] & \mathrm{gr}^F\mathfrak{g} \arrow[d] \\
\varinjlim_{\mathfrak{g}' \leq \mathfrak{g} \text{ f.g.}} \mathrm{gr}^F U_k(\mathfrak{g}') \arrow[r] & \mathrm{gr}^F U_k(\varinjlim_{\mathfrak{g}' \leq \mathfrak{g} \text{ f.g.}} \mathfrak{g}') \arrow[r] & \mathrm{gr}^F U_k(\mathfrak{g}),
\end{tikzcd}$$

the horizontal maps are isomorphisms, because gr^F and U_k commute with filtered colimits. Moreover the left vertical map is an isomorphism, because filtered colimits are exact. Hence also the right vertical map is injective.

(iii) Finally suppose $k \in \mathcal{C}Ring$ is an integral domain and let $Q(k)$ denote its field of fractions. We give $\mathfrak{g} \otimes Q(k)$ the filtration

$$F_n(\mathfrak{g} \otimes Q(k)) = (F_n\mathfrak{g}) \otimes Q(k), \qquad n \geq 1.$$

Then in the commutative diagram

$$\begin{tikzcd}
\mathrm{gr}^F\mathfrak{g} \arrow[r] \arrow[d] & (\mathrm{gr}^F\mathfrak{g}) \otimes Q(k) \arrow[r] \arrow[d] & \mathrm{gr}^F(\mathfrak{g} \otimes Q(k)) \arrow[d] \\
\mathrm{gr}^F U_k(\mathfrak{g}) \arrow[r] & (\mathrm{gr}^F U_k(\mathfrak{g})) \otimes Q(k) \arrow[r] & \mathrm{gr}^F(U_{Q(k)}(\mathfrak{g} \otimes Q(k))),
\end{tikzcd}$$

the upper left horizontal map is an injection, because $\mathrm{gr}^F\mathfrak{g}$ is flat and k injects into its field of fractions. Moreover the upper right horizontal map is an isomorphism by definition of the filtration on $\mathfrak{g} \otimes Q(k)$ and because $Q(k)$ is flat over k. The right vertical map is injective by (ii). Hence by commutativity $\mathrm{gr}^F\mathfrak{g} \hookrightarrow \mathrm{gr}^F U_k(\mathfrak{g})$ is injective.

Now suppose k is flat over $\mathbb{Z}$. As $\mathrm{gr}^F(U_{Q(k)}(\mathfrak{g} \otimes Q(k)))$ is a $Q(k)$-vector space, it must be flat over k. In particular

$$\mathrm{gr}^F(U_{Q(k)}(\mathfrak{g}\otimes Q(k)))\otimes \mathrm{gr}^F(U_{Q(k)}(\mathfrak{g}\otimes Q(k))) \xrightarrow{\sim} \mathrm{gr}^F(U_{Q(k)}(\mathfrak{g}\otimes Q(k))\otimes U_{Q(k)}(\mathfrak{g}\otimes Q(k)))$$

is an isomorphism and thus the coalgebra structure on $U_{Q(k)}(\mathfrak{g}\otimes Q(k))$ induces a coalgebra structure on $\mathrm{gr}^F(U_{Q(k)}(\mathfrak{g}\otimes Q(k)))$. Since the image of $\mathrm{gr}^F\mathfrak{g} \hookrightarrow \mathrm{gr}^F(U_{Q(k)}(\mathfrak{g}\otimes Q(k)))$ lies in the primitive elements, Corollary 3.3.8 yields that $U_k(\mathfrak{g}) \hookrightarrow \mathrm{gr}^F(U_{Q(k)}(\mathfrak{g}\otimes Q(k)))$ is a monomorphism. This map factors over $\mathrm{gr}^F U_k(\mathfrak{g})$ and hence also $U_k(\mathrm{gr}^F\mathfrak{g}) \hookrightarrow \mathrm{gr}^F U_k(\mathfrak{g})$ is injective and so bijective by Proposition 3.3.14. Moreover using the injection $\mathrm{gr}^F\mathfrak{g} \hookrightarrow \mathrm{gr}^F(U_{Q(k)}(\mathfrak{g} \otimes Q(k)))$ with flat target, Proposition 3.3.6 yields that $U_k(\mathrm{gr}^F\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^F U_k(\mathfrak{g})$ is flat over k.

□

3.3.4 A modified lower central series

Purpose of this section is to point out that the lower central series does not behave well enough integrally.

Proposition 3.3.18
Let $k \in \mathcal{C}Ring$ be integral with field of fractions $Q(k)$ and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$.
Then $\mathrm{gr}^F\mathfrak{g} \hookrightarrow \mathrm{gr}^F U_k(\mathfrak{g})$ and $\phi_{\mathfrak{g}} : U_k(\mathrm{gr}^F\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^F U_k(\mathfrak{g})$, for F defined by

$$F_n\mathfrak{g} = \ker(\mathfrak{g} \longrightarrow (\mathfrak{g} \otimes Q(k))/\Gamma_n(\mathfrak{g} \otimes Q(k))), \qquad n \geq 0.$$

Moreover the following holds.

(i) *If* $\mathrm{gr}^{\Gamma}\mathfrak{g}$ *is flat, then* $F = \Gamma$.

(ii) *If* $\Gamma_n\mathfrak{g} = 0$*, for some* $n \geq 1$*, then* $F_n\mathfrak{g} = \ker(\mathfrak{g} \longrightarrow \mathfrak{g} \otimes Q(k))$.

Proof. In the commutative diagrams

$$\begin{tikzcd}
\mathrm{gr}^F\mathfrak{g} \arrow[r] \arrow[d, "\mathrm{gr}^F\eta"'] & \mathrm{gr}^{\Gamma}(\mathfrak{g} \otimes Q(k)) \arrow[d, "\mathrm{gr}^{\Gamma}\eta"] \\
\mathrm{gr}^F U_k(\mathfrak{g}) \arrow[r] & \mathrm{gr}^{\Gamma}(U_{Q(k)}(\mathfrak{g} \otimes Q(k))),
\end{tikzcd}
\qquad
\begin{tikzcd}
U_k(\mathrm{gr}^F\mathfrak{g}) \arrow[r] \arrow[d] & U_{Q(k)}(\mathrm{gr}^{\Gamma}(\mathfrak{g} \otimes Q(k))) \arrow[d] \\
\mathrm{gr}^F U_k(\mathfrak{g}) \arrow[r] & \mathrm{gr}^{\Gamma}(U_{Q(k)}(\mathfrak{g} \otimes Q(k))),
\end{tikzcd}$$

the upper horizontal maps are monomorphism by construction of F and Corollary 3.3.8, respectively. The right vertical maps are injective by Proposition 3.3.17. Hence by commutativity $\mathrm{gr}^F\mathfrak{g} \hookrightarrow \mathrm{gr}^F U_k(\mathfrak{g})$ is injective and $U_k(\mathrm{gr}^F\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^F U_k(\mathfrak{g})$ is an isomorphism.

Now let $n \geq 1$. The following diagram of forgetful and inclusion functors commutes

$$\begin{tikzcd}
Q(k)\text{-}\mathcal{L}ie^{<n} \arrow[r] \arrow[d] & k\text{-}\mathcal{L}ie^{<n} \arrow[d] \\
Q(k)\text{-}\mathcal{L}ie \arrow[r] & k\text{-}\mathcal{L}ie,
\end{tikzcd}$$

where $\mathcal{L}ie^{<n} \leq \mathcal{L}ie$ is the full subcategory of n-nilpotent Lie algebras, i.e. Lie algebras $\mathfrak{g}$ with $\Gamma_n\mathfrak{g} = 0$. Hence also the corresponding diagram of their left adjoints, which are given by extension of scalars and truncation, commutes up to natural isomorphism. In other words the natural map

$$(\mathfrak{g}/\Gamma_n\mathfrak{g}) \otimes Q(k) \xrightarrow{\sim} (\mathfrak{g} \otimes Q(k))/\Gamma_n(\mathfrak{g} \otimes Q(k)), \qquad n \geq 1,$$

is an isomorphism.

(i) We have just proven that the right vertical map in the commutative square

$$\begin{tikzcd}
\mathfrak{g}/\Gamma_n\mathfrak{g} \arrow[r] \arrow[d] & (\mathfrak{g}/\Gamma_n\mathfrak{g}) \otimes Q(k) \arrow[d, "\wr"] \\
\mathfrak{g}/F_n\mathfrak{g} \arrow[r, hook] & (\mathfrak{g} \otimes Q(k))/\Gamma_n(\mathfrak{g} \otimes Q(k))
\end{tikzcd}$$

is an isomorphism. If $\mathrm{gr}^{\Gamma}\mathfrak{g}$ is flat, so is also $\mathfrak{g}/\Gamma_n\mathfrak{g}$ by Corollary 3.6.2. Hence the upper horizontal map is injective and by commuativity the left vertical map is an isomorphism. Equivalently $\Gamma_n\mathfrak{g} = F_n\mathfrak{g}$.

(ii) Using the observation above and that $\Gamma_n\mathfrak{g} = 0$, we get an isomorphism

$$\mathfrak{g} \otimes Q(k) \xrightarrow{\sim} (\mathfrak{g}/\Gamma_n\mathfrak{g}) \otimes Q(k) \xrightarrow{\sim} (\mathfrak{g} \otimes Q(k))/\Gamma_n(\mathfrak{g} \otimes Q(k)),$$

which proves that

$$F_n\mathfrak{g} = \ker(\mathfrak{g} \longrightarrow (\mathfrak{g} \otimes Q(k))/\Gamma_n(\mathfrak{g} \otimes Q(k))) = \ker(\mathfrak{g} \longrightarrow \mathfrak{g} \otimes Q(k)).$$

□

Example 3.3.19
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k$-$\mathcal{L}ie$ nilpotent.

(i) *Let $k = \mathbb{Z}$ and consider the Lie algebra of upper triangular (3×3)-matrices*

$$\mathfrak{g} = p \cdot t_3(\mathbb{Z}) = \begin{pmatrix} 0 & p\mathbb{Z} & p\mathbb{Z} \\ 0 & 0 & p\mathbb{Z} \\ 0 & 0 & 0 \end{pmatrix}, \qquad p > 1.$$

Then $\mathfrak{g}$ is nilpotent and $\mathfrak{g} \cong \mathbb{Z}^3$, but $\mathrm{gr}_1^\Gamma \mathfrak{g} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \mathbb{Z}^2 \oplus \mathbb{Z}/p$ is not free.

However as $\mathbb{Z}$ is integral, $\mathrm{gr}^F \mathfrak{g}$ is free, for the filtration F of Proposition 3.3.18.

(ii) *Let $k = \mathbb{Q}[t]/(t^2)$ and $\mathfrak{g} = \mathfrak{sl}_2(t \cdot \mathbb{Q}[t]/(t^3))$.*

Then using that $\mathfrak{sl}_2(\mathbb{Q}[t]/(t^3))$ is perfect (i.e. equals its commutator), we see that

$$\Gamma_2 \mathfrak{g} = t \cdot \mathfrak{g}, \qquad \Gamma_3 \mathfrak{g} = t^2 \cdot \mathfrak{g} = 0.$$

In particular $\mathfrak{g}$ is nilpotent and $\mathfrak{g} \cong k^3$, but $\mathrm{gr}^\Gamma \mathfrak{g}$ is k-torsion.

In fact the only Lie algebra filtration F, such that gr^F is flat, is given by $F_n \mathfrak{g} = \mathfrak{g}$, for all $n \geq 0$.

3.3.5 Free Lie algebras

Here we are showing that at least in the case of free Lie algebras, the lower central series is good enough.

Proposition 3.3.20
*Let $k \in \mathcal{C}Ring$ and $X \in k$-*Mod*.*

Then the free Lie algebra $\mathcal{L}ie(X)$ generated by X is $\mathbb{Z}_{>0}$-graded.

Moreover, if X is flat, so is also $\mathcal{L}ie(X)$ and thus Proposition 3.3.6 together with Proposition 3.3.9 imply, that $\mathcal{L}ie(X) \xrightarrow{\sim} PU_k(\mathcal{L}ie(X)) = P\mathcal{A}ss_1(X)$ is an isomorphism.

Proof. The free magma $\mathcal{M}ag(X)$ generated by X is $\mathcal{M}ag(X) = \bigoplus_{n \geq 1} \mathcal{M}ag^{(n)}(X)$ where the direct summands are inductively be constructed via

$$\mathcal{M}ag^{(1)}(X) = X, \qquad \mathcal{M}ag^{(n)}(X) = \bigoplus_{0<i<n} \mathcal{M}ag^{(i)}(X) \otimes \mathcal{M}ag^{(n-i)}(X), \quad n > 1.$$

The product map $\mathcal{M}ag(X) \otimes \mathcal{M}ag(X) \xrightarrow{\mu} \mathcal{M}ag(X)$ is given as the sum of the maps

$$\mathcal{M}ag^{(p)}(X) \otimes \mathcal{M}ag^{(q)}(X) \xrightarrow{\iota_p} \mathcal{M}ag^{(p+q)}(X).$$

In particular $\mathcal{M}ag(X)$ is a $\mathbb{Z}_{>0}$-graded k-magma. The free Lie algebra $\mathcal{L}ie(X)$ is the quotient of the free magma $\mathcal{M}ag(X)$ by the ideal $I(X)$ generated by

$$[x, x], \qquad [[x, y], z] + [[y, z], x] + [[z, x], y], \qquad x, y, z \in \mathcal{M}ag(X).$$

By replacing x, y, z by its homogeneous components we see that $I(X)$ is generated by homogeneous elements. Hence also $\mathcal{L}ie(X) = \mathcal{M}ag(X)/I(X)$ is $\mathbb{Z}_{>0}$-graded.

To prove that flatness of X implies flatness of $\mathcal{L}ie(X)$ we will use a well-known result of Hall [Hal50], who explicitly constructed a basis for the free Lie ring generated by a finite set of generators. In particular his result implies that $\mathcal{L}ie(k^n) = k \otimes \mathcal{L}ie(\mathbb{Z}^n)$ is a free k-module, for all $n \geq 1$. Since every flat module $X \in k$-Mod is a filtered colimit of finitely generated free modules and the functors $k\text{-Mod} \xrightarrow{\mathcal{L}ie} k\text{-}\mathcal{L}ie \xrightarrow{U} k\text{-Mod}$ commute with filtered colimits, the statement follows.

□

Proposition 3.3.21
Let $k \in \mathcal{C}Ring$ and let $\mathfrak{g}$ be the free Lie algebra or free abelian Lie algebra generated by a k-module $X \in k$-Mod.

Then $\phi_{\mathfrak{g}} : U_k(\mathrm{gr}^\Gamma \mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^\Gamma U_k(\mathfrak{g})$.

Proof. Let $\mathfrak{g} = \mathcal{L}ie(X)$ and consider the commutative square of ring homomorphisms under k

$$\begin{tikzcd} \mathcal{A}ss_1(X) \arrow[r] \arrow[d] & \mathrm{gr}^\Gamma \mathcal{A}ss_1(X) \arrow[d, "\varphi_X"] \\ U_k(\mathrm{gr}^\Gamma \mathfrak{g}) \arrow[r, "\phi_{\mathfrak{g}}"] & \mathrm{gr}^\Gamma U_k(\mathfrak{g}), \end{tikzcd}$$

where an $x \in X$ is mapped to $[x]$ under the upper horizontal and the left vertical map. Then the upper horizontal map is an isomorphism by construction of the lower central series. We have a factorization of forgetful functors

$$k\text{-}\mathcal{A}ss_1 \longrightarrow k\text{-}\mathcal{L}ie \longrightarrow k\text{-Mod},$$

which proves that also the composition of its left adjoints are isomorphic and hence $\mathcal{A}ss_1(X) \xrightarrow{\sim} U_k(\mathcal{L}ie(X)) = U_k(\mathfrak{g})$. In particular the right vertical map is an isomorphism and thus the left vertical map is injective. It is also surjective, because the Lie algebra $\mathrm{gr}^\Gamma \mathcal{L}ie(X)$ and hence the ring $U_k(\mathrm{gr}^\Gamma \mathcal{L}ie(X))$ is generated by $\mathrm{gr}_1^\Gamma \mathcal{L}ie(X) = X$. It follows that every map in the square and in particular $\phi_{\mathfrak{g}}$ is an isomorphism. The same arguments also apply for the abelian Lie algebra $\mathfrak{g} = X$.

□

Corollary 3.3.22
For every $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k$-$\mathcal{L}ie$, the map $U_k(\mathrm{gr}^\Gamma \mathfrak{g}) \overset{\phi_{\mathfrak{g}}}{\twoheadrightarrow} \mathrm{gr}^\Gamma U_k(\mathfrak{g})$ is surjective.

Proof. For every subset $X \subset \mathfrak{g}$ we get a homomorphism $\mathcal{L}ie(kX) \longrightarrow \mathfrak{g}$ inducing a commuting square

$$\begin{tikzcd} U_k(\mathrm{gr}^\Gamma \mathcal{L}ie(kX)) \arrow[r, "\phi_{\mathcal{L}ie(kX)}", "\sim"'] \arrow[d] & \mathrm{gr}^\Gamma U_k(\mathcal{L}ie(kX)) \arrow[d] \\ U_k(\mathrm{gr}^\Gamma \mathfrak{g}) \arrow[r, "\phi_{\mathfrak{g}}"] & \mathrm{gr}^\Gamma U_k(\mathfrak{g}). \end{tikzcd}$$

If X generates $\mathfrak{g}$, then $\mathcal{L}ie(kX) \twoheadrightarrow \mathfrak{g}$ and the right vertical map is surjective, because gr^Γ and U_k preserve epimorphisms. Hence by commutativity $\phi_{\mathfrak{g}}$ is surjective, too.

□

3.3.6 Completed Lie algebra homology

Applying the Artin-Rees theory of section 3.1.2 we are linking Lie algebra homlogy to its completion.

Definition 3.3.23
Let $k \in \mathcal{C}Ring$. Let $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ carrying a Lie algebra filtration F with $\mathrm{gr}^F U_k(\mathfrak{g})$ flat over k. Let $M \in \mathrm{Mod}\text{-}U_k(\mathfrak{g})$ carrying a filtration F, compatible with the induced filtration F on $U_k(\mathfrak{g})$.

*The **completed Lie algebra homology** is defined as $\widehat{H}_*(\mathfrak{g}, M) = \widehat{\mathrm{Tor}}_*^{U_k(\mathfrak{g})}(M, k)$, i.e. the homology of the completion $\widehat{C}_*(\mathfrak{g}, M) = \widehat{B}_*(M, U_k(\mathfrak{g}), k)$ of the **standard complex***

$$C_*(\mathfrak{g}, M) = B_*(M, U_k(\mathfrak{g}), k) = (M \otimes U_k(\mathfrak{g}) \longleftarrow M \otimes U_k(\mathfrak{g}) \otimes \longleftarrow M \otimes U_k(\mathfrak{g}) \otimes U_k(\mathfrak{g}) \longleftarrow ...).$$

Similarly we define $\widehat{H}_*(\mathfrak{g}, M) = \widehat{\mathrm{Tor}}_*^{U_k(\mathfrak{g})}(k, M)$ for $M \in U_k(\mathfrak{g})\text{-}\mathrm{Mod}$.

Proposition 3.3.24
Let $k \in \mathcal{C}Ring$ be integral, Noetherian and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ carrying a Lie algebra filtration F, such that $F_1\mathfrak{g} = \mathfrak{g}$ and $\mathrm{gr}^F\mathfrak{g} \in k\text{-}\mathrm{Mod}$ is finitely generated.

Then $U_k(\mathfrak{g})$ and the Rees ring $\bigoplus_{n\geq 0} F_n U_k(\mathfrak{g})$ are left Noetherian.

Similarly one shows that both rings are also right Noetherian.

Proof. As $\mathrm{gr}^F\mathfrak{g} \in k\text{-}\mathrm{Mod}$ is finitely generated, so is also $\mathfrak{g} \in k\text{-}\mathrm{Mod}$. Using an epimorphism $k^d \twoheadrightarrow \mathfrak{g}$ Proposition 3.3.4 yields is a surjective ring homomorphism

$$k[t_1, ..., t_d] = \mathcal{C}om_1(k^d) \twoheadrightarrow \mathcal{C}om_1(\mathfrak{g}) \overset{\psi}{\twoheadrightarrow} \mathrm{gr}^E U_k(\mathfrak{g}).$$

Since k is Noetherian, so are also $k[t_1, ..., t_d]$ and $\mathrm{gr}^E U_k(\mathfrak{g})$. Hence $U_k(\mathfrak{g})$ is left Noetherian.

Now since $\mathrm{gr}^F\mathfrak{g} \in k\text{-}\mathrm{Mod}$ is finitely generated, we have $F_N\mathfrak{g} = 0$, for some $N \geq 1$. By replacing each generator of $\mathrm{gr}^F\mathfrak{g}$ by its homogeneous components, we can assume that $\mathrm{gr}^F\mathfrak{g}$ is generated by a finite set of homogeneous elements. We choose elements $X = \{x_1, ..., x_d\} \subset \mathfrak{g}$ representing the homogeneous generators in $\mathrm{gr}^F\mathfrak{g}$. We will construct subrings $R_m \leq \bigoplus_{n\geq 0} F_n U_k(\mathfrak{g})$ by descending induction on $1 \leq m \leq N$. For $m = N$ we let $R_N = U_k(\mathfrak{g})$, which is left Noetherian as we have just proven. Having constructed R_m we adjoin the elements $X \cap F_{m-1}\mathfrak{g} \subset F_{m-1}U_k(\mathfrak{g})$ to R_m and call this ring R_{m-1}. As F is a Lie algebra filtration we have

- $R_m \cdot x + R_m = x \cdot R_m + R_m, \quad x \in X \cap F_{m-1}\mathfrak{g},$
- $yx + R_m = xy + R_m, \qquad x, y \in X \cap F_{m-1}\mathfrak{g}.$

So by the general version of Hilbert's Basis Theorem 3.6.3, we can adjoin the elements of $X \cap F_{m-1}\mathfrak{g}$ to R_m one by one, to show that also R_{m-1} is left Noetherian. By construction of the filtration F on $U_k(\mathfrak{g})$ the ideal $F_n U_k(\mathfrak{g})$ is generated by products $x_1 \cdots x_r$, where $x_i \in F_{a_i}\mathfrak{g},\ a_i \geq 0$ and $a_1 + ... + a_r \geq n$. We have proven that $\bigoplus_{n\geq 0} F_n U_k(\mathfrak{g}) = R_1$ is left Noetherian.

□

Corollary 3.3.25
If, moreover, $\mathrm{gr}^F\mathfrak{g} \in k\text{-Mod}$ *is free, then* $\mathrm{gr}^F U_k(\mathfrak{g})$ *is flat by Proposition 3.3.17 and* $\widehat{U_k(\mathfrak{g})}$ *is flat by Corollary 3.6.2 (ii). So Proposition 3.1.8 yields a natural isomorphism*

$$H_*(\mathfrak{g}, \widehat{M}) \xrightarrow{\sim} \widehat{H}_*(\mathfrak{g}, M), \qquad M \in \text{Mod-}U_k(\mathfrak{g}) \text{ or } M \in U_k(\mathfrak{g})\text{-Mod}.$$

3.3.7 Divisible closure and saturation of Lie rings

In this section we introduce the notion of divisible closure and saturation of Lie rings and show that they behave well with taking the enveloping algebra and the graded objects.

Definition 3.3.26
Let $\mathfrak{g} \in \mathcal{L}ie$ *carrying a filtration* F *with* $\mathrm{gr}^F\mathfrak{g}$ *flat over* $\mathbb{Z}$.

(i) *The* ***divisible closure*** *of* $\mathfrak{g}$ *with respect to* F *is defined as* $D_1^F(\mathfrak{g})$,

(ii) *the* ***saturation*** *of* $\mathfrak{g}$ *with resprect to* F *is defined as* $\widehat{D}_1^F(\mathfrak{g})$.

Beware the index shift compared to the abelian group version of Definition 3.1.10!

Proposition 3.3.27
Let $\mathfrak{g} \in \mathcal{L}ie$ *be a flat,* $\mathbb{Z}_{>0}$*-graded Lie ring. Defining* $F_n\mathfrak{g}$ *as the sum of all homogeneous components of degree* $\geq n$*, we obtain a Lie algebra filtration* F *on* $\mathfrak{g}$.
Then the following holds.

(i) *The map* $D_0^F\mathfrak{g} \xrightarrow{D_0^F(\eta_\mathfrak{g})} D_0^F U_\mathbb{Z}(\mathfrak{g})$ *extends to an isomorphism* $U_\mathbb{Z}(D_0^F\mathfrak{g}) \xrightarrow{\sim} D_0^F U_\mathbb{Z}(\mathfrak{g})$.

(ii) $H_n(\mathfrak{g}, D_0^F U_\mathbb{Z}(\mathfrak{g})) = 0$, *for all* $n > 0$, *considering* $D_0^F(U_\mathbb{Z}(\mathfrak{g}))$ *as a left or right module. In particular there are natural isomorphisms*
   - $H_*(\mathfrak{g}, M) \xrightarrow{\sim} \mathrm{Tor}_*^{D_0^F(U_\mathbb{Z}(\mathfrak{g}))}(M, D_0^F(U_\mathbb{Z}(\mathfrak{g})) \otimes_{U_\mathbb{Z}(\mathfrak{g})} \mathbb{Z}), \quad M \in \text{Mod-}D_0^F(U_\mathbb{Z}(\mathfrak{g}))$,
   - $H_*(\mathfrak{g}, M) \xrightarrow{\sim} \mathrm{Tor}_*^{D_0^F(U_\mathbb{Z}(\mathfrak{g}))}(\mathbb{Z} \otimes_{U_\mathbb{Z}(\mathfrak{g})} D_0^F(U_\mathbb{Z}(\mathfrak{g})), M), \quad M \in D_0^F(U_\mathbb{Z}(\mathfrak{g}))\text{-Mod}$.

Proof.

(i) In the commutative square

$$\begin{tikzcd} \mathcal{A}ss_1(D_0^F\mathfrak{g}) \arrow[r, "\sim"] \arrow[d] & D_0^F\mathcal{A}ss_1(\mathfrak{g}) \arrow[d] \\ U_\mathbb{Z}(D_0^F\mathfrak{g}) \arrow[r] & D_0^F U_\mathbb{Z}(\mathfrak{g}), \end{tikzcd}$$

the vertical maps are induced by the canonical quotient maps. The upper horizontal map is induced by the inclusion $D_0^F\mathfrak{g} \hookrightarrow D_0^F\mathcal{A}ss_1(\mathfrak{g})$ and thus an isomorphism by Proposition 3.1.13. Hence the lower map induced by map $D_0^F\mathfrak{g} \xrightarrow{D_0^F(\eta)} D_0^F U_{\mathbb{Z}}(\mathfrak{g})$ is epimorphic as the diagram commutes.

In the commutative square

$$\begin{tikzcd} U_{\mathbb{Z}}(D_0^F\mathfrak{g}) \arrow[r, two heads] \arrow[d] & D_0^F U_{\mathbb{Z}}(\mathfrak{g}) \arrow[d] \\ U_{\mathbb{Z}}(\mathfrak{g}\otimes\mathbb{Q}) \arrow[r] & U_{\mathbb{Z}}(\mathfrak{g})\otimes\mathbb{Q}, \end{tikzcd}$$

the left vertical map is injective by Corollary 3.3.8, as $D_0^F\mathfrak{g} \hookrightarrow \mathfrak{g}\otimes\mathbb{Q} \hookrightarrow U_{\mathbb{Z}}(\mathfrak{g}\otimes\mathbb{Q})$ is injective by definition of D^F and Proposition 3.3.6, which is applicable by Proposition 3.3.9, because $\mathfrak{g}$ is $\mathbb{Z}_{>0}$-graded and flat over $\mathbb{Z}$.

(ii) By (i) we have an isomorphism $U_{\mathbb{Z}}(D_0^F\mathfrak{g}) \xrightarrow{\sim} D_0^F U_{\mathbb{Z}}(\mathfrak{g})$. The homology $H_n(\mathfrak{g}, D_0^F U_{\mathbb{Z}}(\mathfrak{g})) = H_n(\mathfrak{g}, U_{\mathbb{Z}}(D_0^F\mathfrak{g}))$ is the n-th homology group of the Chevalley-Eilenberg complex $U_{\mathbb{Z}}(D_0^F\mathfrak{g})\otimes\Lambda_*\mathfrak{g}$, because $\mathfrak{g}$ is flat over $\mathbb{Z}$. Defining the usual filtration

$$F_{-n}U_{\mathbb{Z}}(D_0^F\mathfrak{g})\otimes\Lambda_*\mathfrak{g} = \sum_{0\le i\le n} \mathrm{L}_{-i}\, U_{\mathbb{Z}}(D_0^F\mathfrak{g})\otimes\Lambda_{i-n}\mathfrak{g}, \qquad n\ge 0,$$

we get a spectral sequence

$$E^1_{p,q} = H_{p+q}(\mathrm{gr}_p^F(U_{\mathbb{Z}}(D_0^F\mathfrak{g})\otimes\Lambda_*\mathfrak{g})) \quad\Rightarrow\quad H_{p+q}(\mathfrak{g}, U_{\mathbb{Z}}(D_0^F\mathfrak{g})),$$

and it suffices to show that $H_n(\mathrm{gr}^F(U_{\mathbb{Z}}(D_0^F\mathfrak{g})\otimes\Lambda_*\mathfrak{g})) = 0$, for all $n>0$. As $\mathfrak{g}$ is $\mathbb{Z}_{>0}$-graded by Proposition 3.3.9 we can apply Proposition 3.3.6 to obtain an isomorphism

$$\mathcal{C}om_1(D_0^F\mathfrak{g})\otimes\Lambda_*\mathfrak{g} \xrightarrow{\sim} \mathrm{gr}^F(U_{\mathbb{Z}}(D_0^F\mathfrak{g})\otimes\Lambda_*\mathfrak{g}).$$

The differential on the left object is given by

$$d(x_0\otimes x_1\wedge\ldots\wedge x_n) = -\sum_{1\le i\le n}(-1)^i x_0x_i\otimes x_1\wedge\ldots\wedge x_{i-1}\wedge x_{i+1}\wedge\ldots\wedge x_n,$$

and hence does not depend on the Lie algebra structure of $\mathfrak{g}$. As $\mathfrak{g}$ is flat, it is a filtered colimit of finitely generated free k-modules by Lazard-Gomorov. As filtered colimits are exact, we may therefore assume that $\mathfrak{g}$ is freely generated by some basis elements $x_1,\ldots,x_d\in\mathfrak{g}$. But then $\mathcal{C}om_1(D_0^F\mathfrak{g})\otimes\Lambda_*\mathfrak{g}$ is the Koszul complex associated to the regular sequence $x_1,\ldots,x_d\in\mathcal{C}om_1(D_0^F\mathfrak{g})$, which is exact in dimensions ≥ 1.

Explicitly, if we denote by e_i the generator 1 in dimension 1 of the Koszul complex

$$K_*(x_i) = (0\longrightarrow \underset{1}{\mathcal{C}om_1(D_0^F\mathfrak{g})} \xrightarrow{x_i} \underset{0}{\mathcal{C}om_1(D_0^F\mathfrak{g})} \longrightarrow 0),$$

then we get an isomorphism of differential graded $\mathcal{C}om_1(D_0^F\mathfrak{g})$-algebras

$$\begin{aligned} \mathcal{C}om_1(D_0^F\mathfrak{g})\otimes\Lambda_*\mathfrak{g} &\xrightarrow{\sim} K_*(x_1)\otimes_{\mathcal{C}om_1(D_0^F\mathfrak{g})}\ldots\otimes_{\mathcal{C}om_1(D_0^F\mathfrak{g})} K_*(x_d) =: K_*(x_1,\ldots,x_d), \\ 1\otimes x_i &\longmapsto 1\otimes\ldots\otimes \underset{i}{e_i}\otimes\ldots\otimes 1. \end{aligned}$$

By induction on $d \geq 1$ we get

$$H_n(K_*(x_1, ..., x_d)) = \begin{cases} \mathcal{C}om_1(D_0^F \mathfrak{g})/(x_1, ..., x_d), & n = 0, \\ 0, & n > 0. \end{cases}$$

Indeed there is a spectral sequence

$$E^2_{p,q} = H_p(K_*(x_1) \otimes_{\mathcal{C}om_1(D_0^F \mathfrak{g})} H_q(K_*(x_2, ..., x_d))) \quad \Rightarrow \quad H_{p+q}(K_*(x_1, ..., x_d)),$$

and using the induction hypothesis for $K_*(x_2, ..., x_d)$ and that

$$\mathcal{C}om_1(D_0^F \mathfrak{g})/(x_2, ..., x_d) \xhookrightarrow{x_1} \mathcal{C}om_1(D_0^F \mathfrak{g})/(x_2, ..., x_d)$$

is injective, this proves the induction step.

$\square$

Proposition 3.3.28
Let $\mathfrak{g} \in \mathcal{L}ie$ carrying a filtration F with $F_1\mathfrak{g} = \mathfrak{g}$ and $\mathrm{gr}^F \mathfrak{g}$ flat over $\mathbb{Z}$.
Then $\mathfrak{g} \longrightarrow \widehat{D}_0^F \mathfrak{g}$ induces an isomorphism $\widehat{D}_0 U_{\mathbb{Z}}(\mathfrak{g}) \xrightarrow{\sim} \widehat{D}_0 U_{\mathbb{Z}}(\widehat{D}_0^F \mathfrak{g})$.

Proof. In the diagram

$$\begin{tikzcd} U_{\mathbb{Z}}(D_0^F \mathrm{gr}^F \mathfrak{g}) \arrow[r, "\sim"] \arrow[d, "\wr"] & D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F \mathfrak{g}) \arrow[r, "D_0^F(\phi)", "\sim"'] & D_0^F \mathrm{gr}^F U_{\mathbb{Z}}(\mathfrak{g}) \arrow[d, "\wr"] \\ U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}_0^F \mathfrak{g}) \arrow[r, two heads, "\phi"] & \mathrm{gr}^F U_{\mathbb{Z}}(\widehat{D}_0^F \mathfrak{g}) \arrow[r] & \mathrm{gr}^F \widehat{D}_0^F U_{\mathbb{Z}}(\mathfrak{g}), \end{tikzcd}$$

the lower right horizontal map is induced by the extension of $\widehat{D}_0^F \mathfrak{g} \xrightarrow{\widehat{D}_0^F(\eta)} \widehat{D}_0^F U_{\mathbb{Z}}(\mathfrak{g})$ to its enveloping algebra. By Proposition 3.3.17 the upper right horizontal map is an isomorphism of flat $\mathbb{Z}$-modules and by Proposition 3.3.27 (i) the upper left horizontal map is an isomorphism. Hence by Proposition 3.1.11 also the two vertical maps are isomorphisms. The lower left horizontal map is surjective by Proposition 3.3.14 and thus an isomorphism as the diagram commutes. Hence also the right horizontal map is an isomorphism, which implies that the upper horizontal map in the diagram below is an isomorphism of flat $\mathbb{Z}$-modules.

$$\begin{tikzcd} D_0^F \mathrm{gr}^F U_{\mathbb{Z}}(\widehat{D}_0^F \mathfrak{g}) \arrow[r, "\sim"] \arrow[d, "\wr"] & D_0^F \mathrm{gr}^F \widehat{D}_0^F U_{\mathbb{Z}}(\mathfrak{g}) \arrow[d, "\wr"] \\ \mathrm{gr}^F D_0^F U_{\mathbb{Z}}(D_0^F \mathfrak{g}) \arrow[r] & \mathrm{gr}^F D_0^F \widehat{D}_0^F U_{\mathbb{Z}}(\mathfrak{g}). \end{tikzcd}$$

Again the vertical maps are isomorphisms by Proposition 3.1.11. As the diagram commutes, also the lower horizontal map is an isomorphism, which implies that the right map in the composition

$$\widehat{D}_0^F U_{\mathbb{Z}}(\mathfrak{g}) \longrightarrow \widehat{D}_0^F U_{\mathbb{Z}}(\widehat{D}_0^F \mathfrak{g}) \xrightarrow{\sim} \widehat{D}_0^F \widehat{D}_0^F U_{\mathbb{Z}}(\mathfrak{g})$$

is an isomorphism. As the composition is an isomorphism by Proposition 3.1.12, it follows that also the left map is an isomorphism.

$\square$

Proposition 3.3.29
Let $\mathfrak{g} \in \mathcal{L}ie$ carrying a Lie algebra filtration F with $F_1\mathfrak{g} = \mathfrak{g}$ and $\mathrm{gr}^F \mathfrak{g}$ flat over $\mathbb{Z}$.
Then $\widehat{D}^F_n(\mathfrak{g}) \xrightarrow{\sim} \widehat{P}^F_n U_{\mathbb{Z}}(\mathfrak{g})$ is an isomorphism[5]*, for all $n \geq 0$.*
In particular $\widehat{P}^F_1 U_{\mathbb{Z}}(\mathfrak{g})$ is the saturation of $\mathfrak{g}$.

Proof. The composition $\mathfrak{g} \xrightarrow{\eta_{\mathfrak{g}}} U_{\mathbb{Z}}(\mathfrak{g}) \xrightarrow{\eta\otimes 1-\delta+1\otimes\eta} U_{\mathbb{Z}}(\mathfrak{g}) \otimes U_{\mathbb{Z}}(\mathfrak{g})$ is zero by definition of the coalgebra structure on $U_{\mathbb{Z}}(\mathfrak{g})$. Hence it is zero after applying $\widehat{D}^F_0$ and we get a map

$$\widehat{D}^F_0\,\mathfrak{g} \longrightarrow \widehat{P}^F_0 U_{\mathbb{Z}}(\mathfrak{g}) = \ker\left(\widehat{D}^F_0 U_{\mathbb{Z}}(\mathfrak{g}) \xrightarrow{\widehat{D}^F_0(\eta\otimes 1-\delta+1\otimes\eta)} \widehat{D}^F_0\left(U_{\mathbb{Z}}(\mathfrak{g}) \otimes U_{\mathbb{Z}}(\mathfrak{g})\right)\right).$$

Taking the associated graded object we get Lie algebra homomorphisms

$$\mathrm{gr}^F \widehat{D}^F_0\,\mathfrak{g} \longrightarrow \mathrm{gr}^F \widehat{P}^F_0 U_{\mathbb{Z}}(\mathfrak{g}) \hookrightarrow P(\mathrm{gr}^F \widehat{D}^F_0 U_{\mathbb{Z}}(\mathfrak{g})) \subset \mathrm{gr}^F \widehat{D}^F_0 U_{\mathbb{Z}}(\mathfrak{g}),$$

where the right map is injective, because we give $\widehat{P}^F_0 U_{\mathbb{Z}}(\mathfrak{g}) \subset \widehat{D}^F_0 U_{\mathbb{Z}}(\mathfrak{g})$ the submodule filtration. These Lie algebra maps extend to their enveloping algebras as in the lower row of the diagram below.

$$\begin{tikzcd}
U_{\mathbb{Z}}(D^F_0 \mathrm{gr}^F \mathfrak{g}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & D^F_0 U_{\mathbb{Z}}(\mathrm{gr}^F \mathfrak{g}) \arrow[r, "D^F_0(\phi)", "\sim"'] & D^F_0 \mathrm{gr}^F U_{\mathbb{Z}}(\mathfrak{g}) \arrow[d, "\wr"'] \\
U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}^F_0 \mathfrak{g}) \arrow[r] & U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{P}^F_0 U_{\mathbb{Z}}(\mathfrak{g})) \arrow[r] & \mathrm{gr}^F \widehat{D}^F_0 U_{\mathbb{Z}}(\mathfrak{g}).
\end{tikzcd}$$

By exactly the same arguments as in the proof of Proposition 3.3.28, one shows that the vertical maps and the upper horizontal maps are isomorphisms of flat $\mathbb{Z}$-modules. In particular $\mathrm{gr}^F \widehat{D}^F_0 U_{\mathbb{Z}}(\mathfrak{g})$ and thus also the submodule $\mathrm{gr}^F \widehat{P}^F_0 U_{\mathbb{Z}}(\mathfrak{g})$ is flat, because to be flat over $\mathbb{Z}$ is equivalent be torsion-free. So $\mathrm{gr}^F \widehat{P}^F_0 U_{\mathbb{Z}}(\mathfrak{g}) \hookrightarrow P(\mathrm{gr}^F \widehat{D}^F_0 U_{\mathbb{Z}}(\mathfrak{g}))$ is an inclusion of flat $\mathbb{Z}$-modules and so the lower right horizontal map is injective by Corollary 3.3.8. As the square commutes the lower right horizontal map is also surjective and so every map in the diagram is an isomorphism.

In the commutative square

$$\begin{tikzcd}
D^F_0 \mathrm{gr}^F \mathfrak{g} \arrow[r] \arrow[d, "\wr"'] & U_{\mathbb{Z}}(D^F_0 \mathrm{gr}^F \mathfrak{g}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & D^F_0 U_{\mathbb{Z}}(\mathrm{gr}^F \mathfrak{g}) \\
\mathrm{gr}^F \widehat{D}^F_0 \mathfrak{g} \arrow[r] & U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}^F_0 \mathfrak{g}), &
\end{tikzcd}$$

the vertical maps are isomorphisms by Proposition 3.1.11 and the upper right horizontal map is an isomorphism by Proposition 3.3.27. The upper row is equal to D^F_0 applied to the map $\mathrm{gr}^F \mathfrak{g} \xrightarrow{\eta} U_{\mathbb{Z}}(\mathrm{gr}^F \mathfrak{g})$, which is injective by Proposition 3.3.9. As by construction D^F_0 is a subfunctor of tensoring by $\mathbb{Q}$, it is exact and so preserves monomorphisms. Hence the upper left and also the lower horizontal map is injective. In particular in the commutative

[5]Here $\widehat{P}^F_n$ is meant as in Corollary 3.1.16

square

$$\begin{tikzcd} \mathcal{C}om_1(\mathrm{gr}^F \widehat{D}_0^F \mathfrak{g}) \arrow[r] \arrow[d, "\psi"', "\wr"] & \mathcal{C}om_1(\mathrm{gr}^F \widehat{P}_0^F U_{\mathbb{Z}}(\mathfrak{g})) \arrow[d, "\psi"', "\wr"] \\ \mathrm{gr}^{\mathrm{L}} U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}_0^F \mathfrak{g}) \arrow[r, "\sim"] & \mathrm{gr}^{\mathrm{L}} U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{P}_0^F U_{\mathbb{Z}}(\mathfrak{g})), \end{tikzcd}$$

the vertical maps are isomorphisms by Proposition 3.3.6, because $\mathrm{gr}^F \widehat{D}_0^F \mathfrak{g} \hookrightarrow U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}_0^F \mathfrak{g})$ and $\mathrm{gr}^F \widehat{P}_0^F U_{\mathbb{Z}}(\mathfrak{g}) \longrightarrow U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{P}_0^F U_{\mathbb{Z}}(\mathfrak{g})) \xrightarrow{\sim} \mathrm{gr}^F \widehat{D}_0^F U_{\mathbb{Z}}(\mathfrak{g})$ are monomorphisms of flat $\mathbb{Z}$-modules. It follows that the upper horizontal map and so also $\mathrm{gr}^F \widehat{D}_0^F \mathfrak{g} \xrightarrow{\sim} \mathrm{gr}^F \widehat{P}_0^F U_{\mathbb{Z}}(\mathfrak{g})$ is an isomorphism, which finally implies the assertion for $n = 0$ and thus for every $n \geq 0$.

□

3.4 Groups and group rings

Parallel to the theory for Lie algebras, we are developing the analogous theory for groups. The guideline for the single-steps is exactly the same as in the Lie algebra setting.

3.4.1 The group algebra as a Hopf algebra

Proposition 3.4.1
Let $k \in \mathcal{C}Ring$ and $G \in \mathcal{G}rp$.
Then the group algebra $k[G]$ becomes a Hopf algebra via

(i) $\delta : k[G] \longrightarrow k[G] \otimes k[G], \qquad g \longmapsto g \otimes g, \qquad g \in G.$

(ii) $\varepsilon : k[G] \longrightarrow k, \quad g \longmapsto 1, \qquad g \in G.$

(iii) $\iota : k[G] \longrightarrow k[G]^{\mathrm{op}}, \quad g \longmapsto g^{-1}, \qquad g \in G.$

Proof. Using the universal property for group rings the given group homomorphisms on G uniquely extend to the desired ring homomorphisms. Moreover it follows that

$$k[G \times G] \xrightarrow{\sim} k[G] \otimes k[G], \qquad (g,h) \longmapsto g \otimes h, \qquad g,h \in G,$$

and it suffices to check all identities for elements in G.

□

Remark 3.4.2
Let $k \in \mathcal{C}Ring$. Taking grouplike elements induces a functor

$$k\text{-}\mathcal{G}rp \longrightarrow \mathcal{G}rp, \quad H \longmapsto G(H) := \{x \in H;\ \varepsilon(x) = 1,\ \delta(x) = x \otimes x\},$$

which is the right adjoint of an adjunction $k\text{-}\mathcal{G}rp(k[X], Y) = \mathcal{G}rp(X, G(Y))$, induced by the adjunction $k\text{-}\mathcal{A}ss_1(k[X], Y) = \mathcal{G}rp(X, Y^\times)$.

Indeed its unit $G \xrightarrow{\eta_{\mathfrak{g}}} k[G]$ sends $g \in G$ to $1 \cdot g$, which lies in $G(k[G])$ by construction of the coalgebra structure on $k[G]$.

3.4.2 Filtrations on a group and its group ring

Definition 3.4.3
Let $G \in \mathcal{G}rp$.

(i) *The **conjugates** of $x \in G$ under $y \in G$ are defined as ${}^y x = yxy^{-1}$ and $x^y = y^{-1}xy$.*

(ii) *The **commutator of two elements** $x, y \in G$ is defined as $[x, y] = xyx^{-1}y^{-1}$.*

(iii) *The **commutator of normal subgroups** $N_1, N_2 \lhd G$ is defined as the smallest normal subgroup $[N_1, N_2] \lhd G$ containing all elements $[n_1, n_2]$, for $n_i \in N_i$.*

(iv) *A **(positive) group filtration** on G is a positive filtration of normal subgroups $G = F_0G = F_1G \supset F_2G \supset ...$, such that*

$$[F_pG, F_qG] \subset F_{p+q}G, \qquad p, q \geq 0.$$

Lemma 3.4.4
Let $G \in \mathcal{G}rp$ and $x, y, z \in G$.

(i) $[xy, z] = [x, {}^y z] \cdot [y, z] = {}^x[y, z] \cdot [x, z]$,

(ii) $[{}^z x, {}^z y] = {}^z[x, y]$,

(iii) $[[x, y], {}^y z] \cdot [[y, z], {}^z x] \cdot [[z, x], {}^x y] = 1$

Proof. Direct computation. ∎

Remark 3.4.5
Let $k \in \mathcal{C}Ring$ and F a group filtration on a group $G \in \mathcal{G}rp$.

(i) *Using Lemma 3.4.4 the commutator bracket induces a Lie ring structure on the associated graded abelian group $\mathrm{gr}^F G = \bigoplus_{n \geq 1} F_nG/F_{n+1}G$.*

(ii) *There is an **induced positive algebra filtration** on the group ring $k[G]$, given by*

$$F_0k[G] = k[G], \qquad F_nk[G] = \sum_{\substack{a_1,\ldots,a_r \geq 1,\\ a_1+\ldots+a_r \geq n}} \tilde{F}_{a_1}k[G] \cdot \ldots \cdot \tilde{F}_{a_r}k[G] \quad n \geq 1,$$

where $\tilde{F}_nk[G] = k[G] \cdot \tilde{k}[F_nG]$ is the kernel of $k[G] \twoheadrightarrow k[G/F_nG]$, for all $n \geq 1$.

(iii) *By construction the map $\eta_G : G \longrightarrow k[G]$ preserves the filtrations and by Proposition 3.1.17 induces a (Lie) ring homomorphisms*

$$\mathrm{gr}^F \eta_G : \mathrm{gr}^F G \longrightarrow \mathrm{gr}^F k[G], \qquad \phi_G : U_k(\mathrm{gr}^F G \otimes k) \longrightarrow \mathrm{gr}^F k[G].$$

Proposition 3.4.6
Let $k \in \mathcal{C}Ring$ and F a group filtration on a group $G \in \mathcal{G}rp$.

Then $U_k(\mathrm{gr}^F G \otimes k) \overset{\phi_{\mathfrak{g}}}{\twoheadrightarrow} \mathrm{gr}^F k[G]$ is always an epimorphism.

Proof. By definition of $\tilde{F}$ we have

$$\tilde{F}_n k[G] = \ker(k[G] \twoheadrightarrow k[G/F_nG]) = k[G] \cdot \tilde{k}[F_nG] = \sum_{g \in F_nG} k[G] \cdot (g-1), \qquad n \geq 1.$$

Using this and the identity $g(h-1) = (g-1)(h-1) + (h-1)$, for $g, h \in G$, we get

$$F_n k[G] = \sum_{\substack{a_1,\dots,a_r \geq 1,\\ a_1+\dots+a_r \geq n}} \tilde{F}_{a_1} k[G] \cdot \ldots \cdot \tilde{F}_{a_r} k[G] = \sum_{\substack{a_1,\dots,a_r \geq 1,\\ a_1+\dots+a_r \geq n,\\ g_i \in F_{a_i}G}} k \cdot (g_1 - 1) \cdots (g_r - 1), \qquad n \geq 1.$$

In particular $\mathrm{gr}^F k[G]$ is generated as a k-algebra by the classes $[g-1] = \mathrm{gr}^F \eta_G[g]$, for $g \in F_nG \backslash F_{n+1}G$ and $n \geq 1$. As $U_k(\mathrm{gr}^F G \otimes k)$ is generated by $\mathrm{gr}^F G$, it follows that ϕ_G is surjective.

□

The proof for the next proposition is inspired by [Swa67]. It is the group version of Proposition 3.3.15 and modulo slight modifications the arguments are the same. For the convenience of the reader, we have written down a complete proof again.

Proposition 3.4.7
Let $k \in \mathcal{C}Ring$ with $\mathrm{char}\, k = 0$ *and $G \in \mathcal{G}rp$ carrying a group filtration F with $F_nG = 1$ and* $\mathrm{gr}^F \mathfrak{g} \cong \mathbb{Z}^d$*, for some $n, d \geq 1$.*
Then $G \hookrightarrow k[G]/F_n k[G]$ is injective.

Proof. This is done by induction on the rank $d \geq 1$. Let $m \geq 1$ be minimal with $\mathrm{gr}^F_m G \neq 0$. So we can assume $n > m$. If $d = 1$ then G is abelian and the composition

$$G \longrightarrow k[G] \twoheadrightarrow k[G]/F_{m+1}k[G] \twoheadrightarrow k[G]/\tilde{k}[G]^2,$$

is injective, because by Proposition 3.2.5 we have

$$\mathrm{gr}^\Gamma_1 k[G] = \tilde{k}[G]/\tilde{k}[G]^2 \xrightarrow{\sim} Q(k[G]) = H_1(G,k) = G/[G,G] \otimes k,$$

and the characteristic of k is zero. For $d > 1$ we take an element $a \in G$, such that $[a] \in \mathrm{gr}^F_m G$ can be extended to a basis of the $\mathbb{Z}$-module $\mathrm{gr}^F G$. We let $\mathrm{gr}^F_m G \overset{\pi_a}{\twoheadrightarrow} \mathbb{Z}$ be the corresponding projection onto the direct summand spanned by $[a]$. Since F is a group filtration, the maps

$$q : G \twoheadrightarrow \mathrm{gr}^F_m G \xrightarrow{\pi_x} \mathbb{Z}$$

are group homomorphisms. The induced filtration F on the kernel $N \lhd G$ of q is a group filtration and $\mathrm{gr}^F N$ is a direct summand of $\mathrm{gr}^F G$ and thus free of rank $< d$. Hence by the induction hypothesis we have $N \hookrightarrow k[N]/F_n k[N]$.

Now for $a \in q^{-1}(1)$, the conjugation map $\mathrm{Ad}(a) = {}^a(\text{-})$ induces an automorphism on N and on $k[N]$. The map $1 \longmapsto a$ defines a section for q, which is a group homomorphism. In particular $G \cong N \rtimes \mathbb{Z}$ and we get a unique action of G on $k[N]$ extending the action of N and $\langle a \rangle \subset G$. Since $(a-1) \cdot x = {}^a x - x = ([a,x]-1)x \in \tilde{F}_{n+1}k[N]$, for all $n \in F_nN$, we get

$$F_p k[G] \cdot F_q k[N] \subset F_{p+q} k[N], \qquad p, q \geq 0.$$

Let A denote the k-algebra of k-linear endomorphisms on $k[N]/F_nk[N]$. Then A carries a ring filtration, where F_pA is the k-submodule of endomorphisms f, such that $f(F_qk[N]) \subset F_{p+q}k[N]$, for all $q \geq 0$. Moreover $\tilde{A} = F_1A$ is a nilpotent subalgebra and left multiplication defines a representation $G \xrightarrow{\lambda} \tilde{A}_+ = \tilde{A} \oplus k$, mapping F_pG to F_pA, for all $p \geq 1$. It extends to a ring homomorphisms $k[G] \longrightarrow \tilde{A}_+$, mapping $F_pk[G]$ to F_pA, for all $p \geq 1$. In particular λ factors as $G \longrightarrow k[G]/F_nk[G] \longrightarrow \tilde{A}_+$, because $\tilde{A}^n = 0$. The composition

$$\langle a \rangle \hookrightarrow G \longrightarrow k[G]/F_nk[G] \overset{q}{\twoheadrightarrow} k[\mathbb{Z}]/F_nk[\mathbb{Z}]$$

is injective by the case $d = 1$. Using the commutative diagram

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & k[N]/F_nk[N] \arrow[r, hook] \arrow[d] & \underline{k\text{-Mod}}(k[N]/F_nk[N]) \\ G \arrow[r] & k[G]/F_nk[G] \arrow[r] & \tilde{A}_+, \arrow[u] \end{tikzcd}$$

it follows that also $G \hookrightarrow k[G]/F_nk[G]$ is injective.

□

Proposition 3.4.8
Let $G \in \mathcal{G}rp$, carrying a group filtration F with $\mathrm{gr}^F G$ flat over $\mathbb{Z}$.
Then $\mathrm{gr}^F G \otimes k \hookrightarrow \mathrm{gr}^F k[G]$ and $\phi_G : U_k(\mathrm{gr}^F G \otimes k) \xrightarrow{\sim} \mathrm{gr}^F k[G]$, for every $k \in \mathcal{C}Ring$.
Moreover $\mathrm{gr}^F k[G]$ is flat over k.

Proof. The proof is similar to that of Proposition 3.3.17 and will be established in several steps.

(i) First we assume that G/F_nG is a finitely generated, free abelian group, for all $n \geq 1$. Then $\mathrm{gr}^F(G/F_nG) = \mathrm{gr}^F_1 G \oplus ... \oplus \mathrm{gr}^F_{n-1} G \cong \mathbb{Z}^d$, for some $d \geq 1$, and thus by Proposition 3.4.7 we get injections

$$G/F_nG \hookrightarrow \mathbb{Z}[G]/F_n\mathbb{Z}[G] \xrightarrow{\sim} \mathbb{Z}[G/F_nG]/F_n\mathbb{Z}[G/F_nG], \qquad n \geq 1,$$

where the latter is an isomorphism, because by definition $F_n\mathbb{Z}[G]$ contains the kernel of $\mathbb{Z}[G] \twoheadrightarrow \mathbb{Z}[G/F_nG]$. Equivalently $\mathrm{gr}^F G \hookrightarrow \mathrm{gr}^F \mathbb{Z}[G]$.

(ii) Now we drop the finiteness condition. For every finitely generated subgroup $G' \leq G$ we give G' the subgroup filtration. Then G'/F_nG' is a finitely generated abelian group with no torsion and hence free, for all $n \geq 1$. So we get $\mathrm{gr}^F G' \hookrightarrow \mathrm{gr}^F \mathbb{Z}[G']$ by case (i). In the commutative diagram

$$\begin{tikzcd} \varinjlim_{G' \leq G \text{ f.g.}} \mathrm{gr}^F G' \arrow[r] \arrow[d] & \mathrm{gr}^F(\varinjlim_{G' \leq G \text{ f.g.}} G') \arrow[r] \arrow[d] & \mathrm{gr}^F G \arrow[d] \\ \varinjlim_{G' \leq G \text{ f.g.}} \mathrm{gr}^F \mathbb{Z}[G']) \arrow[r] & \mathrm{gr}^F \mathbb{Z}[\varinjlim_{G' \leq G \text{ f.g.}} G'] \arrow[r] & \mathrm{gr}^F \mathbb{Z}[G], \end{tikzcd}$$

the horizontal maps are isomorphisms, because gr^F and $\mathbb{Z}[\text{-}]$ commute with filtered colimits. Moreover the left vertical map is injective, because filtered colimits are exact. Hence also the right vertical map is injective. As $\mathrm{gr}^F G$ is flat, the map $\mathrm{gr}^F G \hookrightarrow \mathrm{gr}^F G \otimes \mathbb{Q} \hookrightarrow \mathrm{gr}^F \mathbb{Q}[G]$ is injective. As $\mathrm{gr}^F \mathbb{Q}[G]$ is a a vector-space over $\mathbb{Q}$, it is flat over $\mathbb{Z}$. In particular $\mathrm{gr}^F \mathbb{Q}[G] \otimes \mathrm{gr}^F \mathbb{Q}[G] \xrightarrow{\sim} \mathrm{gr}^F(\mathbb{Q}[G] \otimes \mathbb{Q}[G])$ and the coalgebra structure on $\mathbb{Q}[G]$ induces a coalgebra structure on $\mathrm{gr}^F \mathbb{Q}[G]$. As

$$\delta(g-1) = (g-1) \otimes 1 + 1 \otimes (g-1) + (g-1) \otimes (g-1), \qquad g \in G,$$

the image of $\mathrm{gr}^F G \hookrightarrow \mathrm{gr}^F \mathbb{Q}[G]$ lies in the primitive elements and so $U_{\mathbb{Z}}(\mathrm{gr}^F G) \hookrightarrow \mathrm{gr}^F \mathbb{Q}[G]$ is a monomorphism by Corollary 3.3.8. As it factors over $\mathrm{gr}^F \mathbb{Z}[G]$, it follows that also $U_{\mathbb{Z}}(\mathrm{gr}^F G) \xrightarrow{\phi_G} \mathrm{gr}^F \mathbb{Z}[G]$ is injective and thus bijective by Proposition 3.4.6.

(iii) Finally for a general commutative ring $k \in \mathcal{CRing}$, we have a commutative diagram

$$\begin{tikzcd} U_{\mathbb{Z}}(\mathrm{gr}^F G) \otimes k \arrow[r, "\phi_G \otimes \mathrm{id}"] \arrow[d] & (\mathrm{gr}^F \mathbb{Z}[G]) \otimes k \arrow[d] \\ U_k(\mathrm{gr}^F G \otimes k) \arrow[r, "\phi_G"] & \mathrm{gr}^F k[G]. \end{tikzcd} \tag{3.2}$$

As the following square of right adjoint forgetful functors commutes

$$\begin{tikzcd} k\text{-}\mathcal{A}ss_1 \arrow[r] \arrow[d] & k\text{-}\mathcal{L}ie \arrow[d] \\ \mathbb{Z}\text{-}\mathcal{A}ss_1 \arrow[r] & \mathbb{Z}\text{-}\mathcal{L}ie, \end{tikzcd}$$

so does the corresponding square of left adjoints, which are given by the enveloping algebra resp. extension of scalars. This means that the left vertical map in (3.2) is an isomorphism.

Similarly, for every $n \geq 0$, we have a commutative square of functors

$$\begin{tikzcd} k\text{-}\mathcal{A}ss_{1,F}^{<n} \arrow[r] \arrow[d] & k\text{-}\mathcal{A}ss_{1,F} \arrow[d] \\ \mathbb{Z}\text{-}\mathcal{A}ss_{1,F}^{<n} \arrow[r] & \mathbb{Z}\text{-}\mathcal{A}ss_{1,F}, \end{tikzcd}$$

where $k\text{-}\mathcal{A}ss_{1,F}$ is the category of filtered unital associative algebras with 0-equicontinuous homomorphisms and $k\text{-}\mathcal{A}ss_{1,F}^{<n} \leq k\text{-}\mathcal{A}ss_{1,F}$ is the full subcategory of those objects A with $F_n A = 0$. The functors are forgetful and inclusion functors, which commute and are right adjoints. Hence also the corresponding diagram of their left adjoints, which are given by extension of scalars and truncation, commutes up to natural isomorphism. Equivalently the natural map below is an isomorphism.

$$(\mathbb{Z}[G]/F_n\mathbb{Z}[G]) \otimes k \xrightarrow{\sim} k[G]/F_n k[G], \qquad n \geq 0.$$

Using the monomorphism $\mathrm{gr}^F G \hookrightarrow \mathrm{gr}^F \mathbb{Q}[G]$ Proposition 3.3.6 (ii) yields that $U_{\mathbb{Z}}(\mathrm{gr}^F G) \xrightarrow{\sim} \mathrm{gr}^F \mathbb{Z}[G]$ is flat. Hence by Corollary 3.6.2 (i) $\mathbb{Z}[G]/F_n\mathbb{Z}[G]$ is flat, for all $n \geq 0$. So in the diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{gr}_n^F \mathbb{Z}[G]) \otimes k \arrow[r] \arrow[d, dashed] & (\mathbb{Z}[G]/F_{n+1}\mathbb{Z}[G]) \otimes k \arrow[r] \arrow[d, "\wr"'] & (\mathbb{Z}[G]/F_n\mathbb{Z}[G]) \otimes k \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & \mathrm{gr}_n^F k[G] \arrow[r] & k[G]/F_{n+1}k[G] \arrow[r] & k[G]/F_n k[G] \arrow[r] & 0,
\end{tikzcd}$$

the upper sequence is exact by Proposition 3.6.1 (i). As the lower sequence is exact by definition, it follows that with the two right vertical maps also the left vertical map is an isomorphism. Equivalently also the right vertical map in (3.2) is an isomorphism. As the upper horizontal map is an isomorphism by (ii), we get that $U_k(\mathrm{gr}^F G \otimes k) \xrightarrow{\sim} \mathrm{gr}^F k[G]$ is an isomorphism by commutativity.

It remains to prove that $\mathrm{gr}^F G \otimes k \hookrightarrow U_k(\mathrm{gr}^F G \otimes k)$ is injective. This holds, if $\mathrm{gr}^F_{\leq n} G \otimes k \hookrightarrow U_k(\mathrm{gr}^F_{\leq n} G \otimes k)$ is injective, for all $n \geq 0$. Again by taking the filtered colimit over all finitely generated Lie subalgebras of $\mathrm{gr}^F_{\leq n} G$ we may assume that $\mathrm{gr}^F_{\leq n} G \cong \mathbb{Z}^d$, for some $d \geq 0$, and the statement follows from Proposition 3.3.15. Alternatively the injectivity of $\mathrm{gr}^F G \otimes k \hookrightarrow U_k(\mathrm{gr}^F G \otimes k)$ immediately follows from the general Theorem of Poincaré, Birkhoff and Witt 3.3.10, because $\mathrm{gr}^F G \otimes k$ is flat over k.

$\square$

3.4.3 Torsion-free nilpotent groups

For torsion-free nilpotent groups we are constructing canonical filtrations having all the good properties that we want. It is interesting to note that it is the upper central series and not the lower one, that has the better behaviour.

Remark 3.4.9 (i) *The **lower central series** $(\Gamma_n G)_{n\geq 0}$ on a group G is given by*

$$\Gamma_0 G = \Gamma_1 G = G, \qquad \Gamma_{n+1} G = [G, \Gamma_n G], \qquad n \geq 1.$$

It is the initial positive group filtration on G.

(ii) *The lower central series defines an epimorphism preserving endofunctor*

$$\mathrm{gr}^\Gamma : \mathcal{G}rp \longrightarrow \mathcal{L}ie, \quad G \longmapsto \mathrm{gr}^\Gamma G = \bigoplus_{n\geq 1} \Gamma_n G/\Gamma_{n+1} G.$$

(iii) *For any $k \in \mathcal{C}Ring$, the induced filtration Γ on the group ring $k[G]$ coincides with the lower central series defined for Hopf algebras in Definition 3.2.4, because*

$$[g,h] - 1 = (gh - hg)g^{-1}h^{-1} = ((g-1)(h-1) - (h-1)(g-1))g^{-1}h^{-1}, \quad g,h \in G.$$

So there is no notational conflict occuring here.

(iv) *The* ***upper central series*** *is defined by*

$$Z_0G = 1, \qquad Z_{n+1} = \ker(G \twoheadrightarrow G/Z_nG \twoheadrightarrow (G/Z_n)/Z(G/Z_n)), \qquad n \geq 1.$$

In particular we have $[G/Z_nG, Z_{n+1}G/Z_nG] = 1$ *and thus* $[G, Z_{n+1}G] \subset Z_nG$*, for all* $n \geq 0$.

Proposition 3.4.10
Let $G \in \mathcal{G}rp$ *be torsion-free and nilpotent. Then the map below is injective.*

$$Z_{n+1}G/Z_nG \hookrightarrow (Z_1G)^{\mathcal{A}b(Z_{n+1}G/Z_nG, Z_1G)}, \qquad x \longmapsto (f(x))_f, \qquad n \geq 1.$$

In particular $\mathrm{gr}^Z G$ *is flat over* $\mathbb{Z}$.

Proof. For the convenience of the reader we recall the proof as it is also given in [War76] Thm. 2.1. Let $x \in Z_{n+1}G$ with $x \notin Z_nG$. In particular $[x] \notin Z_nG/Z_{n-1}G = Z(G/Z_{n-1})$ and we get a $g_1 \in G$ with $1 \neq \mathrm{ad}(g_1)(x) = [g_1, x] \in Z_nG/Z_{n-1}G$. Using $[G, Z_{n+1}G] \subset Z_nG$ and Lemma 3.4.4 (i), we see that the map $Z_{n+1}G/Z_nG \xrightarrow{\mathrm{ad}(g_1)} Z_nG/Z_{n-1}G$ is a homomorphism of abelian groups. By repeating this argument, we get elements $g_1, ..., g_n \in G$ with $1 \neq \mathrm{ad}(g_n) \circ ... \circ \mathrm{ad}(g_1)(x) \in Z_1G/Z_0G = Z_1G$. So the image of x is non-trivial in coordinate $\mathrm{ad}(g_n) \circ ... \circ \mathrm{ad}(g_1)$, which proves the statement.

Now if G is torsion-free, so is also Z_1G and every power of Z_1G. Using injectivity of the maps, it follows that $\mathrm{gr}^Z G$ is torsion-free.

□

Proposition 3.4.11
For $G \in \mathcal{G}rp$ *we define a canonical group filtration* F *by setting*

$$F_0G = F_1G = G, \qquad F_{n+1}G = \ker(F_nG \longrightarrow (F_nG/[G, F_nG]) \otimes \mathbb{Q}), \qquad n \geq 1.$$

Then $\mathrm{gr}^F G$ *is flat over* $\mathbb{Z}$ *and moreover the following holds.*

(i) *If* $\mathrm{gr}^\Gamma G$ *is flat, then* $F = \Gamma$.

(ii) *If* G *is torsion-free and* $\Gamma_nG = 1$*, then also* $F_nG = 1$.

(iii) $F_nG = G \cap (1 + \tilde{\mathbb{Q}}[G]^n) \subset \mathbb{Q}[G]^\times, \qquad n \geq 1.$

In particular we have an injection $\mathrm{gr}^F G \hookrightarrow \mathrm{gr}^\Gamma \mathbb{Q}[G]$ *extending to the isomorphism* $\phi_G : U_\mathbb{Q}(\mathrm{gr}^F G \otimes \mathbb{Q}) \xrightarrow{\sim} \mathrm{gr}^\Gamma \mathbb{Q}[G]$.

Proof. By construction $\mathrm{gr}^F_n G$ is the image of F_nG in $(F_nG/[G, F_nG]) \otimes \mathbb{Q}$ and thus torsion-free, for all $n \geq 1$.

(i) By construction we have $\Gamma_nG \subset F_nG$, for all $n \geq 1$. If $\mathrm{gr}^\Gamma G$ is flat, then the map

$$\Gamma_nG/[G, \Gamma_nG] \hookrightarrow (\Gamma_nG/[G, \Gamma_nG]) \otimes \mathbb{Q}, \qquad n \geq 1,$$

is injective, which proves that $\Gamma_nG = F_nG$, by induction on $n \geq 1$.

(ii) If $\Gamma_n G = 1$, then G is nilpotent and we prove that $\Gamma_{n-i}G \subset Z_i G$, by induction on $0 \leq i \leq n$. For $i = 0$ we have $\Gamma_n G = 1 = Z_0 G$. Assuming that $\Gamma_{n-i}G \subset Z_i G$ the surjection $G/\Gamma_{n-i}G \twoheadrightarrow G/Z_i G$ maps $\mathrm{gr}^{\Gamma}_{n-i-1}G \subset Z(G/\Gamma_{n-i}G)$ to $Z(G/Z_i G) = \mathrm{gr}^{Z}_{i+1}G$ and thus we get a surjection on the cokernels

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{gr}^{\Gamma}_{n-i-1}G \arrow[r] \arrow[d] & G/\Gamma_{n-i}G \arrow[r] \arrow[d] & G/\Gamma_{n-i-1}G \arrow[r] \arrow[d, dashed, "\exists!"] & 1 \\
1 \arrow[r] & \mathrm{gr}^{Z}_{i+1} \arrow[r] & G/Z_i G \arrow[r] & G/Z_{i+1}G \arrow[r] & 1,
\end{tikzcd}$$

which proves that $\Gamma_{n-i-1}G \subset Z_{i+1}G$. In particular $Z_{n-1}G = \Gamma_1 G = G$.

Now turning everything around, we prove that $F_i G \subset Z_{n-i}G$, by induction on $1 \leq i \leq n$. Assuming $F_i G \subset Z_{n-i}G$, we get $[G, F_i G] \subset [G, Z_{n-i}G] \subset Z_{n-i-1}G$ and thus $F_i G \longrightarrow Z_{n-i}G \twoheadrightarrow Z_{n-i}G/Z_{n-i-1}G$ factors over $F_i/[G, F_i G]$. Using that $Z_{n-i}G/Z_{n-i-1}G = \mathrm{gr}^{Z}_{n-i}G$ is flat by Proposition 3.4.10 it moreover factors over $F_i G/F_{i+1}G = \mathrm{gr}^{F}_{i}G$, so we get an injection on the kernels

$$\begin{tikzcd}
1 \arrow[r] & F_{i+1}G \arrow[r] \arrow[d, dashed] & F_i G \arrow[r] \arrow[d, hook] & \mathrm{gr}^{F}_{i}G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z_{n-i-1}G \arrow[r] & Z_{n-i}G \arrow[r] & \mathrm{gr}^{Z}_{n-i}G \arrow[r] & 1,
\end{tikzcd}$$

which proves that $F_{i+1}G \subset Z_{n-i-1}G$. In particular $F_n G \subset Z_{n-n}G = Z_0 G = 1$.

(iii) We will prove that $F_n G \subset 1 + \tilde{\mathbb{Q}}[G]^n$, by induction on $n \geq 1$. The case $n = 0$ being trivial, suppose the statement holds for some $n \geq 1$. Then the composition

$$F_n G \hookrightarrow 1 + \tilde{\mathbb{Q}}[G]^n \twoheadrightarrow (1 + \tilde{\mathbb{Q}}[G]^n)/(1 + \tilde{\mathbb{Q}}[G]^{n+1}) \cong \tilde{\mathbb{Q}}[G]^n/\tilde{\mathbb{Q}}[G]^{n+1}$$

factors over $F_n G/[G, F_n G]$, because for $g \in G$ and $h \in F_n G$ we have

$$[g, h] - 1 = (gh - hg)g^{-1}h^{-1} = ((g-1)(h-1) - (h-1)(g-1))g^{-1}h^{-1} \in \tilde{\mathbb{Q}}[G]^{n+1}.$$

Since $\tilde{\mathbb{Q}}[G]^n/\tilde{\mathbb{Q}}[G]^{n+1}$ is a $\mathbb{Q}$-vector space, it also factors over $(F_n G/[G, F_n G]) \otimes \mathbb{Q}$, hence over $F_n G/F_{n+1}G$. Equivalently $F_{n+1}G \subset 1 + \tilde{\mathbb{Q}}[G]^{n+1}$, which proves the induction step.

This shows that $F = \Gamma$ on $\mathbb{Q}[G]$. Using Proposition 3.4.8 we get an injection $\mathrm{gr}^F G \hookrightarrow \mathrm{gr}^F \mathbb{Q}[G] = \mathrm{gr}^{\Gamma}\mathbb{Q}[G]$ extending to the isomorphism $\phi_G : U_{\mathbb{Q}}(\mathrm{gr}^F G \otimes \mathbb{Q}) \xrightarrow{\sim} \mathrm{gr}^{\Gamma}\mathbb{Q}[G]$. The former implies that $F_n G = G \cap (1 + \tilde{\mathbb{Q}}[G]^n)$, for all $n \geq 1$.

□

As a consequence we get a result, which was proven by several people before (for example [Jen55] Theorem 5.2 or [Swa67])

Corollary 3.4.12 (Hall, Jennings, Swan)
Let $G \in \mathcal{G}rp$ be finitely generated, torsion-free and nilpotent.

Then there is a monomorphism $G \hookrightarrow T_r(\mathbb{Z})$ into the group of upper triangular $(r \times r)$-matrices with ones on the diagonal, for some $r \geq 1$.

Proof. By Proposition 3.4.11 (ii) we get a filtration F on G, such that gr^F is flat and $F_nG = 1$, for some $n \geq 1$. Then $\mathrm{gr}^F G \hookrightarrow \mathrm{gr}^F \mathbb{Z}[G]$ and $\mathrm{gr}^F \mathbb{Z}[G]$ is flat by Proposition 3.4.8. In particular $G \hookrightarrow \mathbb{Z}[G]/F_n\mathbb{Z}[G]$ and $\mathbb{Z}[G]/F_n\mathbb{Z}[G]$ is flat. As G is finitely generated, $\tilde{\mathbb{Z}}[G]/F_n\mathbb{Z}[G]$ is a finitely generated, nilpotent (non-unital) ring. In particular it is a finitely generated $\mathbb{Z}$-module, so $\mathbb{Z}[G]/F_n\mathbb{Z}[G] \cong \mathbb{Z}^r$, for some $r \geq 1$. Left multiplcation induces a representation

$$G \hookrightarrow \mathbb{Z}[G]/F_n\mathbb{Z}[G] \hookrightarrow \underline{\mathbb{Z}\text{-Mod}}(\mathbb{Z}[G]/F_n\mathbb{Z}[G]) \cong M_r(\mathbb{Z}).$$

Hence G is isomorphic to a nilpotent subgroup of $GL_r(\mathbb{Z})$. By a base change argument, we get $G \hookrightarrow T_r(\mathbb{Z})$.

□

3.4.4 Free groups

Lemma 3.4.13
For the free group ${}^X\mathbb{Z}$ generated by a set $X \in \mathcal{S}et$, there is an isomorphism of rings

$$\begin{aligned} \varphi_X : \mathcal{A}ss_1(\mathbb{Z}X)/(X)^n &\overset{\sim}{\rightleftarrows} \mathbb{Z}[{}^X\mathbb{Z}]/\tilde{\mathbb{Z}}[{}^X\mathbb{Z}]^n : \psi_X, \qquad n \geq 0, \\ x &\longmapsto x - 1, \\ 1 + x &\longleftarrow\!\shortmid x. \end{aligned}$$

Furthermore after abelianization we get an induced isomorphism

$$\bar{\varphi}_X : \mathcal{C}om_1(\mathbb{Z}X)/(X)^n \overset{\sim}{\rightleftarrows} \mathbb{Z}[\mathbb{Z}X]/\tilde{\mathbb{Z}}[\mathbb{Z}X]^n : \bar{\psi}_X, \qquad n \geq 0.$$

Proof. The map $\varphi_X(x) = x - 1$ induces a unique $\mathbb{Z}$-linear map $\mathbb{Z}X \longrightarrow \mathbb{Z}[{}^X\mathbb{Z}]$ extending to a unique ring map $\mathcal{A}ss_1(\mathbb{Z}X) \xrightarrow{\varphi_X} \mathbb{Z}[{}^X\mathbb{Z}]$. Since $\varphi_X(X) \subset (g-1; g \in {}^X\mathbb{Z}) = \tilde{\mathbb{Z}}[{}^X\mathbb{Z}]$ we get $\varphi_X(X)^n \subset \tilde{\mathbb{Z}}[{}^X\mathbb{Z}]^n$, for all $n \geq 0$. In particular we get a ring homomorphism φ_X as desired.

In the left ring we have $(1 + (-x) + ... + (-x)^{n-1})(1 + x) = 1$ and hence $(1 + x)$ is a unit, for all $x \in X$. So $\psi_X(x) = 1 + x$ extends to a unique group homomorphism ${}^X\mathbb{Z} \longrightarrow (\mathcal{A}ss_1(\mathbb{Z}X)/(X)^n)^\times$ and a unique ring homomorphism $\mathbb{Z}[{}^X\mathbb{Z}] \xrightarrow{\psi_X} \mathcal{A}ss_1(\mathbb{Z}X)/(X)^n$. Using the formulas

$$gh - 1 = (g-1)(h-1) + (g-1) + (h-1), \quad g^{-1} - 1 = -(g-1)g^{-1}, \quad g, h \in {}^X\mathbb{Z}, \tag{3.3}$$

we see that $\tilde{\mathbb{Z}}[{}^X\mathbb{Z}] = (g-1; g \in {}^X\mathbb{Z}) = (x-1;\ x \in X)$ and thus $\psi_X(\tilde{\mathbb{Z}}[G]) \subset (X)$ and $\psi_X(\tilde{\mathbb{Z}}[G]^n) \subset (X)^n$, for all $n \geq 0$.

This proves that φ_X and ψ_X are well-defined and since $\varphi_X \circ \psi_X(x) = x$ and $\psi_X \circ \varphi_X(x) = x$, for all $x \in X$, the unversal properties show that they are inverse to each other.

□

Proposition 3.4.14
Suppose G is the free group or free abelian group generated by a set $X \in \mathcal{S}et$.
Then $\phi_G : U_{\mathbb{Z}}(\mathrm{gr}^\Gamma G) \xrightarrow{\sim} \mathrm{gr}^\Gamma \mathbb{Z}[G]$ is an isomorphism[6].

Proof. Let $G = {}^X\mathbb{Z}$ and consider the commutative square of ring homomorphisms

$$\begin{tikzcd}
\mathcal{A}ss_1(\mathbb{Z}X) \arrow[r] \arrow[d] & \mathrm{gr}^\Gamma \mathcal{A}ss_1(\mathbb{Z}X) \arrow[d, "\varphi_X"] \\
U_{\mathbb{Z}}(\mathrm{gr}^\Gamma G) \arrow[r, "\phi_G"] & \mathrm{gr}^\Gamma \mathbb{Z}[G],
\end{tikzcd}$$

where an $x \in X$ is mapped to $[x]$ under the upper horizontal and the left vertical map. Then the upper horizontal map is an isomorphism by construction and the right vertical map is an isomorphism by Lemma 3.4.13. In particular the left vertical map is injective. It is also surjective, because the Lie ring $\mathrm{gr}^\Gamma G$ and hence the ring $U_{\mathbb{Z}}(\mathrm{gr}^\Gamma G)$ is generated by $X \subset \mathbb{Z}X = \mathrm{gr}_1^\Gamma G$. It follows that every map in the square and in particular ϕ_G is an isomorphism. The same arguments also apply to the case of the free abelian group $G = \mathbb{Z}X$.

□

Corollary 3.4.15
For every $G \in \mathcal{G}rp$, we have $\phi_G : U_{\mathbb{Z}}(\mathrm{gr}^\Gamma G) \twoheadrightarrow \mathrm{gr}^\Gamma \mathbb{Z}[G]$.

Proof. For $X \subset G$ we get a map ${}^X\mathbb{Z} \longrightarrow G$ inducing a commuting square

$$\begin{tikzcd}
U_{\mathbb{Z}}(\mathrm{gr}^\Gamma({}^X\mathbb{Z})) \arrow[r, "\phi_{{}^X\mathbb{Z}}", "\sim"'] \arrow[d] & \mathrm{gr}^\Gamma \mathbb{Z}[{}^X\mathbb{Z}] \arrow[d] \\
U_{\mathbb{Z}}(\mathrm{gr}^\Gamma G) \arrow[r, "\phi_G"] & \mathrm{gr}^\Gamma \mathbb{Z}[G].
\end{tikzcd}$$

If X generates G, then ${}^X\mathbb{Z} \twoheadrightarrow G$ and the right vertical map is surjective, because $\tilde{\mathbb{Z}}[{}^X\mathbb{Z}] \twoheadrightarrow \tilde{\mathbb{Z}}[G]$ is surjective and by definition the lower central series filtration is given by powers of these ideals. By commutativity ϕ_G is surjective, too.

□

3.4.5 Completed group homology

We are proving results analogous to the Lie algebra case.

Definition 3.4.16
Let $k \in \mathcal{CR}ing$. Let $G \in \mathcal{G}rp$ carrying a positive group filtration F with $\mathrm{gr}^F G$ flat over $\mathbb{Z}$. Let $M \in \mathrm{Mod}\text{-}k[G]$ carrying a filtration F, compatible with the induced filtration F on $k[G]$.

[6] Recall that by Remark 3.4.9 the filtration Γ on $\mathbb{Z}[G]$ induced by Γ on G, as well as the lower central series filtration for Hopf algebras is given by the powers of the augmentation ideal $\tilde{\mathbb{Z}}[G]$.

The ***completed group homology*** *is defined as* $\widehat{H}_*(G,M) = \widehat{\mathrm{Tor}}_*^{k[G]}(M,k)$*, i.e. the homology of the completion* $\widehat{C}_*(G,M) = \widehat{B}_*(M,k[G],k)$ *of the* ***standard complex***

$$C_*(G,M) = B_*(M,k[G],k) = (M \otimes k[G] \longleftarrow M \otimes k[G] \otimes \longleftarrow M \otimes k[G] \otimes k[G] \longleftarrow ...).$$

Similarly we define $\widehat{H}_*(G,M) = \widehat{\mathrm{Tor}}_*^{k[G]}(k,M)$ *for* $M \in k[G]$-Mod.

Proposition 3.4.17
Let $k \in \mathcal{C}Ring$ *be Noetherian and* $G \in \mathcal{G}rp$ *carrying a positive group filtration* F*, such that* $\mathrm{gr}^F G \in \mathbb{Z}$-Mod *is finitely generated.*

Then $k[G]$ *and the Rees ring* $\bigoplus_{n\geq 0} F_n k[G]$ *are left Noetherian.*

Similarly one shows that both rings are also right Noetherian.

Proof. This is the group version of Proposition 3.3.24. We will carry it out, as there are some differences. Since $\mathrm{gr}^F G \in k$-Mod is finitely generated, we have $F_N G = 1$, for some $N \geq 1$. By replacing each generator of $\mathrm{gr}^F G$ by its homogeneous components, we can assume that $\mathrm{gr}^F G$ is generated by a finite set of homogeneous elements. We choose elements $X = \{x_1, ..., x_d\} \subset G$ representing the homogeneous generators in $\mathrm{gr}^F G$.

We will construct subrings $S_m \leq k[G]$ by descending induction on $1 \leq m \leq N$. For $m = N$ we let $S_N = k \leq k[G]$, which is Noetherian by assumption. Having constructed S_m we adjoin the elements $X \cap F_{m-1}G \subset k[G]$ and their inverses to S_m and call this ring S_{m-1}. As F is a group filtration we have

- $S_m \cdot x + S_m = x \cdot S_m + S_m, \quad x \in X \cap F_{m-1}G,$
- $yx + S_m = xy + S_m, \qquad x, y \in X \cap F_{m-1}G.$

So by the general version of Hilbert's Basis Theorem 3.6.3, we can adjoin the elements of $X \cap F_{m-1}G$ to S_m one by one, to show that also S_{m-1} is left Noetherian. We have proven that $k[G] = S_1$ is left Noetherian.

Next we will construct subrings $R_m \leq \bigoplus_{n\geq 0} F_n k[G]$ by descending induction on $1 \leq m \leq N$. For $m = N$ we let $R_N = k[G] = F_0 k[G]$, which is left Noetherian as we have just proven. Having constructed R_m we adjoin the elements $X \cap F_{m-1}G \subset F_{m-1}k[G]$ to R_m and call this ring R_{m-1}. As F is a Lie algebra filtration we have

- $R_m \cdot x + R_m = x \cdot R_m + R_m, \quad x \in X \cap F_{m-1}G,$
- $yx + R_m = xy + R_m, \qquad x, y \in X \cap F_{m-1}G.$

So by the general version of Hilbert's Basis Theorem 3.6.3, we can adjoin the elements of $X \cap F_{m-1}G$ to R_m one by one, to show that also R_{m-1} is left Noetherian. By construction of the filtration F on $k[G]$ the ideal $F_n k[G]$ is generated by products $(x_1 - 1) \cdots (x_r - 1)$, where $x_i \in F_{a_i}G,\ a_i \geq 0$ and $a_1 + ... + a_r \geq n$ (compare the proof of Proposition 3.4.6). This implies that $\bigoplus_{n\geq 0} F_n k[G] = R_1$, which is left Noetherian as we have proven.

□

Corollary 3.4.18
If moreover $\mathrm{gr}^F G \in \mathbb{Z}\text{-Mod}$ *is free, then* $\mathrm{gr}^F k[G]$ *is flat by Proposition 3.4.8 and* $\widehat{k[G]}$ *is flat by Corollary 3.6.2 (ii). So Proposition 3.1.8 yields a natural isomorphism*

$$H_*(G, \widehat{M}) \xrightarrow{\sim} \widehat{H}_*(G, M), \qquad M \in \text{Mod-}k[G] \text{ or } M \in k[G]\text{-Mod}.$$

3.5 Groups and Lie rings that are associated

The notion of associated groups and Lie rings is chosen in such a way that it allows a comparison of their particular homology. We will later give some examples, satisfying this property.

Definition 3.5.1
Let $G \in \mathcal{G}rp$ *and* $\mathfrak{g} \in \mathcal{L}ie$ *carrying filtrations* F.

Then G *and* $\mathfrak{g}$ *are called* ***associated*** *(via* λ*), if there is a commuting square of* 0-*equicontinuous isomorphisms*

$$\begin{tikzcd}
\widehat{D}_0^F \mathbb{Z}[G] \arrow[r, "\lambda", "\sim"'] & \widehat{D}_0^F U_{\mathbb{Z}}(\mathfrak{g}) \\
\widehat{I}(G) \arrow[u, hook] \arrow[r, "\lambda|", "\sim"'] & \widehat{I}(\mathfrak{g}), \arrow[u, hook]
\end{tikzcd}$$

where $\widehat{I}(G)$ *and* $\widehat{I}(\mathfrak{g})$ *are the completions of the right ideals*

$$I(G) = \tilde{\mathbb{Z}}[G] \cdot D_0^F(\mathbb{Z}[G]) \leq D_0^F(\mathbb{Z}[G]), \qquad I(\mathfrak{g}) = \tilde{U}_{\mathbb{Z}}(\mathfrak{g}) \cdot D_0^F(U_{\mathbb{Z}}(\mathfrak{g})) \leq D_0^F(U_{\mathbb{Z}}(\mathfrak{g}))$$

with respect to the submodule filtration.

3.5.1 Integral homology of associated groups and Lie rings

We are now able to prove an integral variant of Pickel's [Pic78] isomorphism between the completed group and Lie algebra homology. Note that with similar techniques it is also possible to prove an integral variant of the p-adic version due to Lazard (in fact he establishes an isomorphism on completed cohomology in [Laz65] Theorem 2.4.10). Together with the p-adic analogue of Proposition 3.5.10 this will also imply an integral variant of Lazard's isomorphism for saturated p-valued groups established in [HKN09].

Theorem 3.5.2
Let $G \in \mathcal{G}rp$ *and* $\mathfrak{g} \in \mathcal{L}ie$ *carrying filtrations* F *with* $\mathrm{gr}^F G$ *and* $\mathrm{gr}^F \mathfrak{g}$ *flat.*

If G *and* $\mathfrak{g}$ *are associated, then for every left/right module* M *over* $\widehat{D}_0^F(\mathbb{Z}[G]) \overset{\lambda}{\cong} \widehat{D}_0^F(U_{\mathbb{Z}}(\mathfrak{g}))$ *carrying a compatible module filtration* F*, there is a natural zig-zag of quasi-isomorphisms* $\widehat{C}_*(G, M) \simeq \widehat{C}_*(\mathfrak{g}, M)$.

In particular there is a natural isomorphism $\widehat{H}_*(G, M) \cong \widehat{H}_*(\mathfrak{g}, M)$.

Proof. We give a proof in case of a left module M. The case of a right module is exactly the same using the left ideal versions of $I(G)$ and $I(\mathfrak{g})$ respectively. The zig-zag of quasi-isomorphisms is given by

$$\begin{tikzcd}
C\left(\widehat{B}_*(\tilde{\mathbb{Z}}[G],\mathbb{Z}[G],M) \longrightarrow \widehat{B}_*(\mathbb{Z}[G],\mathbb{Z}[G],M)\right) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \widehat{C}_*(G,M) \\
C\left(\widehat{B}_*(I(G),D_0^F\mathbb{Z}[G],M) \longrightarrow \widehat{B}_*(D_0^F\mathbb{Z}[G],D_0^F\mathbb{Z}[G],M)\right) \arrow[d, "\lambda"', "\wr"] & \\
C\left(\widehat{B}_*(I(\mathfrak{g}),D_0^F U_{\mathbb{Z}}(\mathfrak{g}),M) \longrightarrow \widehat{B}_*(D_0^F U_{\mathbb{Z}}(\mathfrak{g}),D_0^F U_{\mathbb{Z}}(\mathfrak{g}),M)\right) & \\
C\left(\widehat{B}_*(\tilde{U}_{\mathbb{Z}}(\mathfrak{g}),U_{\mathbb{Z}}(\mathfrak{g}),M) \longrightarrow \widehat{B}_*(U_{\mathbb{Z}}(\mathfrak{g}),U_{\mathbb{Z}}(\mathfrak{g}),M)\right) \arrow[r, "\simeq"] \arrow[u, "\simeq"'] & \widehat{C}_*(\mathfrak{g},M),
\end{tikzcd}$$

which will be explained in the following.

The inclusion $\mathbb{Z}[G] \hookrightarrow D_0^F\mathbb{Z}[G]$ induces a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathbb{Z}}[G] \arrow[r] \arrow[d] & \mathbb{Z}[G] \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & I(G) \arrow[r] & D_0^F\mathbb{Z}[G] \arrow[r] & \mathbb{Z}\otimes_{\mathbb{Z}[G]}\otimes D_0^F\mathbb{Z}[G] \arrow[r] & 0.
\end{tikzcd}$$

The upper left object in the zig-zag-diagram is the cone of the map induced by the inclusion $\tilde{\mathbb{Z}}[G] \hookrightarrow \mathbb{Z}[G]$, while the object beneath is the cone of the map induced by the inclusion $I(G) \hookrightarrow D_0^F\mathbb{Z}[G]$. The map between them is induced by the left two vertical maps in the second diagram. As the upper sequence splits over $\mathbb{Z}$, we see that the upper horizontal map in the zig-zag-diagram, which is the canonical quotient map, is a quasi-isomorphism. Similarly the inclusion $U_{\mathbb{Z}}(\mathfrak{g}) \hookrightarrow D_0^F U_{\mathbb{Z}}(\mathfrak{g})$ induces a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \tilde{U}_{\mathbb{Z}}(\mathfrak{g}) \arrow[r] \arrow[d] & U_{\mathbb{Z}}(\mathfrak{g}) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & I(\mathfrak{g}) \arrow[r] & D_0^F U_{\mathbb{Z}}(\mathfrak{g}) \arrow[r] & \mathbb{Z}\otimes_{U_{\mathbb{Z}}(\mathfrak{g})}\otimes D_0^F U_{\mathbb{Z}}(\mathfrak{g}) \arrow[r] & 0,
\end{tikzcd}$$

which enables us to construct the remaining objects in the zig-zag-diagram in the same way. Now all arguments in the completed bar constructions can be replaced by their completions and it follows that by definition the isomorphism λ induces an isomorphism as depicted.

So it remains to prove that the upper and the lower vertical maps in the zig-zag-diagram are quasi-isomorphisms. To this aim we use the natural isomorphisms

$$\psi_G : U_{\mathbb{Z}}(\mathrm{gr}^F G) \xrightarrow{\sim} \mathrm{gr}^F\mathbb{Z}[G], \qquad \psi_{\mathfrak{g}} : U_{\mathbb{Z}}(\mathrm{gr}^F \mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^F U_{\mathbb{Z}}(\mathfrak{g})$$

of Proposition 3.4.8 and Proposition 3.3.17 respectively. These two propositions also imply that $\mathrm{gr}^F\tilde{\mathbb{Z}}[G] \hookrightarrow \mathrm{gr}^F\mathbb{Z}[G]$ and $\mathrm{gr}^F\tilde{U}_{\mathbb{Z}}(\mathfrak{g}) \hookrightarrow \mathrm{gr}^F U_{\mathbb{Z}}(\mathfrak{g})$ are flat over $\mathbb{Z}$. Moreover using Proposition 3.1.11 we also get natural isomorphisms

- $\varphi_G : D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G) \xrightarrow{D_0^F(\psi_G)} D_0^F \mathrm{gr}^F \mathbb{Z}[G] \xrightarrow{\sim} \mathrm{gr}^F D_0^F \mathbb{Z}[G]$,
- $\varphi_{\mathfrak{g}} : D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F \mathfrak{g}) \xrightarrow{D_0^F(\psi_{\mathfrak{g}})} D_0^F \mathrm{gr}^F U_{\mathbb{Z}}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}^F D_0^F U_{\mathbb{Z}}(\mathfrak{g})$.

In what follows we only check the group case, as the arguments for the Lie algebra case are exactly the same. By definition of $I(G)$ the map φ_G restricts to an isomorphism

$$\begin{tikzcd}
I(\mathrm{gr}^F G) = \tilde{U}_{\mathbb{Z}}(\mathrm{gr}^F G)\cdot D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G) \arrow[r, hook] \arrow[d, dashed, "\wr"] & D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G) \arrow[d, "\wr", "\varphi_G"'] \\
\mathrm{gr}^F I(G) = \mathrm{gr}^F(\tilde{\mathbb{Z}}[G]\cdot D_0^F\mathbb{Z}[G]) \arrow[r, hook] & \mathrm{gr}^F(D_0^F \mathbb{Z}[G]),
\end{tikzcd}$$

As the right objects are flat and thus torsion-free, so are also the left ones. Hence Proposition 3.1.5 implies that the associated graded of the upper vertical map in the zig-zag-diagram is isomorphic to

$$\begin{tikzcd}
C(B_*(\tilde{U}_{\mathbb{Z}}(\mathrm{gr}^F G), U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M) \arrow[r] & B_*(U_{\mathbb{Z}}(\mathrm{gr}^F G), U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M)) \arrow[d] \\
C(B_*(I(\mathrm{gr}^F G), D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M) \arrow[r] & B_*(D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G), D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M))
\end{tikzcd}$$

and it suffices to check that this map is a quasi-isomorphism. The targets of the maps, we are taking the cone of, are contractible. So by the long exact sequence for the cone, we need to show that

$$B_*(\tilde{U}_{\mathbb{Z}}(\mathrm{gr}^F G), U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M) \longrightarrow B_*(I(\mathrm{gr}^F G), D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M)$$

is a quasi-isomorphism. Taking homology yields the map

$$\mathrm{Tor}_*^{U_{\mathbb{Z}}(\mathrm{gr}^F G)}(\tilde{U}_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M) \longrightarrow \mathrm{Tor}_*^{D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G)}(I(\mathrm{gr}^F G), \mathrm{gr}^F M),$$

where we use the flatness of $\tilde{U}_{\mathbb{Z}}(\mathrm{gr}^F G), U_{\mathbb{Z}}(\mathrm{gr}^F G), I(\mathrm{gr}^F G)$ and $D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G)$ to show that homology of the particular bar complex is the stated Tor-group. Using the map of the long exact Tor-sequences associated to the map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \tilde{U}_{\mathbb{Z}}(\mathrm{gr}^F G) \arrow[r] \arrow[d] & U_{\mathbb{Z}}(\mathrm{gr}^F G) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & I(\mathrm{gr}^F G) \arrow[r] & D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G) \arrow[r] & \mathbb{Z} \otimes_{U_{\mathbb{Z}}(\mathrm{gr}^F G)} \otimes D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G) \arrow[r] & 0,
\end{tikzcd}$$

it suffices to check that

$$H_*(\mathrm{gr}^F G, M) = \mathrm{Tor}_*^{U_{\mathbb{Z}}(\mathrm{gr}^F G)}(\mathbb{Z}, \mathrm{gr}^F M) \longrightarrow \mathrm{Tor}_*^{D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G)}(\mathbb{Z} \otimes_{U_{\mathbb{Z}}(\mathrm{gr}^F G)} \otimes D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M)$$

is an isomorphism, which finally follows from Proposition 3.3.27 (ii).

$\square$

Corollary 3.5.3
Suppose that in the situation of Theorem 3.5.2 additionally $\mathrm{gr}^F G$ *and* $\mathrm{gr}^F \mathfrak{g}$ *are finitely generated and free.*

Then there are natural isomorphisms $H_*(G, M) \xrightarrow{\sim} \widehat{H}_*(G, M) \cong \widehat{H}_*(\mathfrak{g}, M) \xleftarrow{\sim} H_*(\mathfrak{g}, M)$.

Proof. This follows from Theorem 3.5.2 combined with Corollary 3.4.18 and Corollary 3.3.25 respectively.

□

Proposition 3.5.4
If in the situation of Theorem 3.5.2 also $\mathrm{gr}^F M$ *is flat, then we have natural quasi-isomorphisms*

$$\widehat{C}_*(G, M) \xrightarrow{\simeq} \widehat{C}_*(G, \mathfrak{g}, M) \xleftarrow{\simeq} \widehat{C}_*(\mathfrak{g}, M),$$

where we define

$$\begin{aligned} \widehat{C}_*(G, \mathfrak{g}, M) &:= \operatorname{coker} \left(\widehat{B}_*(I(G), D_0^F \mathbb{Z}[G], M) \longrightarrow \widehat{B}_*(D_0^F \mathbb{Z}[G], D_0^F \mathbb{Z}[G], M) \right) \\ &\overset{\lambda}{\cong} \operatorname{coker} \left(\widehat{B}_*(I(\mathfrak{g}), D_0^F U_{\mathbb{Z}}(\mathfrak{g}), M) \longrightarrow \widehat{B}_*(D_0^F U_{\mathbb{Z}}(\mathfrak{g}), D_0^F U_{\mathbb{Z}}(\mathfrak{g}), M) \right). \end{aligned}$$

Proof. It suffices to note that if $\mathrm{gr}^F M$ is flat, then

$$B_*(I(\mathrm{gr}^F G), D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M) \hookrightarrow B_*(D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G), D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F G), \mathrm{gr}^F M)$$

and hence also

$$\widehat{B}_*(I(G), D_0^F \mathbb{Z}[G], M) \longrightarrow \widehat{B}_*(D_0^F \mathbb{Z}[G], D_0^F \mathbb{Z}[G], M)$$

is injective, which implies that the natural quotient map from the cone onto the cokernel is a quasi-isomorphism. Hence by replacing cones by cokernels in the zig-zag-diagram of Theorem 3.5.2 the assertion follows.

□

3.5.2 Free (abelian) groups and free (abelian) Lie rings are associated

Proposition 3.5.5
For $X \in \mathcal{S}et$*, the free group* ${}^X\mathbb{Z}$ *and the free Lie algebra* $\mathcal{L}ie(\mathbb{Z}X)$ *are associated via*

$$\lambda : \widehat{D}_0^\Gamma \mathbb{Z}[{}^X\mathbb{Z}] \xrightarrow{\sim} \widehat{D}_0^\Gamma U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X)), \qquad x \longmapsto \exp(x).$$

Proof. We will start with the isomorphism

$$\psi_X : \widehat{\mathbb{Z}[{}^X\mathbb{Z}]} \xrightarrow{\sim} U_{\mathbb{Z}}(\widehat{\mathcal{L}ie(\mathbb{Z}}X)) = \widehat{\mathcal{A}ss_1(\mathbb{Z}}X) = \widehat{\mathbb{Z}[{}^X\mathbb{N}_0]}, \qquad x \longmapsto 1 + x,$$

which is the inverse limit of the isomorphisms ψ_X in Lemma 3.4.13. Again by Lemma 3.4.13 also its associated graded is an isomorphism $\mathrm{gr}^\Gamma \psi_X : \mathrm{gr}^\Gamma \mathbb{Z}[{}^X\mathbb{Z}] \xrightarrow{\sim} \mathrm{gr}^\Gamma U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X))$.

Hence in the commutative square

$$\begin{tikzcd} D_0\mathrm{gr}^\Gamma\mathbb{Z}[{}^X\mathbb{Z}] \arrow[r, "D_0(\mathrm{gr}^\Gamma\psi_X)", "\sim"'] \arrow[d, "\wr"'] & D_0\mathrm{gr}^\Gamma U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X)) \arrow[d, "\wr"] \\ \mathrm{gr}^\Gamma D_0(\mathbb{Z}[{}^X\mathbb{Z}]) \arrow[r, "\mathrm{gr}^\Gamma D_0(\psi_X)"] & \mathrm{gr}^\Gamma D_0(U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X))), \end{tikzcd}$$

the upper map is an isomorphism. As also the vertical maps are isomorphisms by Proposition 3.1.11, it follows that $\mathrm{gr}^\Gamma D_0(\psi_X)$ is an isomorphism, too.

Now by Proposition 3.1.15 we have $\exp(x) \in 1 + \widehat{D}_1\mathcal{A}ss_1(\mathbb{Z}X)$. As $\exp(x)$ is a unit with inverse $\exp(-x)$, the map $x \longmapsto \exp(x)$ uniquely extends to a ring homomorphism $\mathbb{Z}[{}^X\mathbb{Z}] \longrightarrow \widehat{D}_0^\Gamma U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X))$ mapping $(x^{\pm 1}-1)$ into $\widehat{D}_1 U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X))$, for all $x \in X$. Using the identity

$$gh - 1 = (g-1)(h-1) + (g-1) + (h-1), \qquad g, h \in {}^X\mathbb{Z},$$

we see that it also maps $(g-1)$ into $\widehat{D}_1 U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X))$, for all $g \in {}^X\mathbb{Z}$. It follows that the map $x \longmapsto \exp(x)$ extends to the desired map λ. Since $[(1+x)-1] = [\exp(x)-1]$ in $\mathrm{gr}^\Gamma\widehat{D}_0^\Gamma U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X))$, it follows that $\mathrm{gr}^\Gamma\lambda = \mathrm{gr}^\Gamma D_0(\psi_X)$ and thus also λ is an isomorphism.

Using the formulas (3.3) again, we see that $I({}^X\mathbb{Z}) \leq D_0^\Gamma\mathbb{Z}[{}^X\mathbb{Z}]$ is generated by $(x-1)$, for $x \in X$. By Lemma 3.1.14 we have $1/(m+1)! \in D_m\mathbb{Q}$, for all $m \geq 0$. It follows that

$$u(x) = \sum_{m\geq 0} \frac{x^m}{(m+1)!} \in \widehat{D}_0 U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X)),$$

which moreover is a unit as the series $\sum_{n\geq 0}(1-u(x))^n$ converges in $\widehat{D}_0 U_{\mathbb{Z}}(\mathcal{L}ie(\mathbb{Z}X))$ by construction of the filtration. Since we have

$$\lambda(x-1) = \exp(x) - 1 = \sum_{n\geq 0} \frac{x^{n+1}}{(n+1)!} = x \cdot u(x),$$

it follows that $\lambda(\widehat{I}({}^X\mathbb{Z})) = \widehat{I}(\mathcal{L}ie(\mathbb{Z}X))$ and hence ${}^X\mathbb{Z}$ and $\mathcal{L}ie(\mathbb{Z}X)$ are associated via λ.

□

Remark 3.5.6
Let A be a free abelian group, considered as a group G and as an abelian Lie ring $\mathfrak{g}$.
Then one can prove in a similar way, that G and $\mathfrak{g}$ are associated.
Note that the isomorphism $H_*(G,\mathbb{Z}) \cong H_*(\mathfrak{g},\mathbb{Z}) = \Lambda_* A$ is functorial in A.

3.5.3 Rings inducing associated groups and Lie rings

Proposition 3.5.7
Let $A \in \mathcal{A}ss$ carrying a complete filtration F, such that $\mathrm{gr}^F A$ is flat and $A = F_1 A$. We give $A_+ = k \oplus A$, the universal unital k-algebra of Remark 2.3.2, the algebra filtration $F_n A_+ = k \oplus F_n A$, for $n \geq 0$.

Then giving $G = 1 + A \leq (A_+)^\times$ and $\mathfrak{g} = A$ the induced filtrations, there is an isomorphism

$$\lambda_A : \widehat{D}_0^F \mathbb{Z}[1 + A] \xrightarrow{\sim} \widehat{D}_0^F U_{\mathbb{Z}}(A), \quad \underline{1 + a} \longmapsto \exp \circ s_A \circ \log \circ i_A(1 + a),$$

where the underlined expression is meant as an element in $1 + A$ and the maps

$$1 + A \xrightarrow{i_A} 1 + \widehat{D}_1^F A \xrightarrow{\log} \widehat{D}_1^F A \xrightarrow{s_A} \widehat{P}_1^F U_{\mathbb{Z}}(A) \xrightarrow{\exp} \widehat{G}_1^F U_{\mathbb{Z}}(A)$$

are those of Proposition 3.1.12, Proposition 3.1.15, Proposition 3.3.29 and Corollary 3.1.16 respectively. Moreover the following holds.

(i) *λ_A is natural in A.*

(ii) *Let $x \in A$ such that $x^n/n \in A$, for all $n \geq 1$. Then $\lambda_A(\underline{1 + x}) = 1 + x$.*

(iii) *$G = 1 + A$ and $\mathfrak{g} = A$ are associated via λ_A, if there is a subset $X \subset A$, such that*
   a) $\mathrm{gr}^F A = \sum_{x \in X} \mathbb{Z} \cdot [x]$.
   b) $x^n/n! \in A$*, for all* $x \in X$ *and* $n \geq 1$.

Proof. To check that λ_A is well-defined, consider the diagram

$$\begin{tikzcd}
1 + A \arrow[r, "i_A"] & 1 + \widehat{D}_1^F A \arrow[d, "\log"', "\wr"] & \widehat{G}_1^F U_{\mathbb{Z}}(A) \arrow[r, hook] & 1 + \widehat{D}_1^F U_{\mathbb{Z}}(A) \arrow[r, "\widehat{D}_0^F(q_A)|"] & 1 + \widehat{D}_1^F A \\
 & \widehat{D}_1^F A \arrow[r, "s_A", "\sim"'] & \widehat{P}_1^F U_{\mathbb{Z}}(A) \arrow[u, "\exp", "\wr"'] \arrow[r, hook] & \widehat{D}_1^F U_{\mathbb{Z}}(A) \arrow[u, "\exp", "\wr"'] \arrow[r, "\widehat{D}_0^F(q_A)|"] & \widehat{D}_1^F A, \arrow[u, "\exp", "\wr"']
\end{tikzcd}$$

in which the map i_A is induced by the canonical map

$$A = F_1 A \otimes D_0 \mathbb{Q} \longrightarrow D_1^F(A) \longrightarrow \widehat{D}_1^F A.$$

The map $U_{\mathbb{Z}}(A) \xrightarrow{q_A} A_+ = A \oplus \mathbb{Z}$ is the unique map extending the inclusion $A \hookrightarrow A_+$. We claim that the composition $\exp \circ s_A \circ \log \circ i_A$ is a group homomorphism.

By construction of s_A and q_A the composition $\widehat{D}_0^F(q_A) \circ s_A$ is the identity on $\widehat{D}_1^F A$, which implies that $\widehat{D}_0^F(q_A)$ restricted to $\widehat{P}_1^F U_{\mathbb{Z}}(A)$ is an isomorphism and equals $(s_A)^{-1}$. As the exponential map is natural in the ring, the two squares on the right commute, which implies that $\widehat{D}_0^F(q_A)| : \widehat{G}_1^F U_{\mathbb{Z}}(A) \xrightarrow{\sim} 1 + \widehat{D}_0^F(A)$ is a group isomorphism. Hence it suffices to check that the composition $\widehat{D}_0^F(q_A) \circ \exp \circ s_A \circ \log \circ i_A$ is a group homomorphism. But using again that the exponential map is natural, we see that

$$\widehat{D}_0^F(q_A) \circ \exp \circ s_A \circ \log \circ i_A = \exp \circ \widehat{D}_0^F(q_A) \circ s_A \circ \log \circ i_A = \exp \circ \log \circ i_A = i_A,$$

which indeed is a group homomorphism. It follows that $\exp \circ s_A \circ \log \circ i_A$ extends to ring homomorphisms $\mathbb{Z}[1 + A] \longrightarrow \widehat{D}_0^F U_{\mathbb{Z}}(A)$ and $\widehat{D}_0^F \mathbb{Z}[1 + A] \xrightarrow{\lambda_A} \widehat{D}_0^F U_{\mathbb{Z}}(A)$ by using Proposition 3.1.12.

We claim that λ_A is an isomorphism. Note that $\mathrm{gr}^F A$ and hence also $\mathrm{gr}^F(1+A)$ is flat over $\mathbb{Z}$. Consider the diagram of rings

$$\begin{tikzcd}
D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F(1+A)) \arrow[r, "D_0^F(\phi_{1+A})", "\sim"'] \arrow[d, "D_0^F U_{\mathbb{Z}}(\ell_A)"', "\wr"] & D_0^F \mathrm{gr}^F \mathbb{Z}[1+A] \arrow[r, "\sim"] & \mathrm{gr}^F \widehat{D}_0^F \mathbb{Z}[1+A] \arrow[d, "\mathrm{gr}^F \lambda_A"] \\
D_0^F U_{\mathbb{Z}}(\mathrm{gr}^F A) \arrow[r, "D_0^F(\phi_A)", "\sim"'] & D_0^F \mathrm{gr}^F U_{\mathbb{Z}}(A) \arrow[r, "\sim"] & \mathrm{gr}^F \widehat{D}_0^F U_{\mathbb{Z}}(A),
\end{tikzcd}$$

where Proposition 3.4.8 and Proposition 3.1.11 yield the isomorphisms in the upper row, and Proposition 3.3.17 and Proposition 3.1.11 yield those in the lower row. The isomorphism on the left is the one of Proposition 3.1.17. The diagram commutes, because

$$\begin{aligned}
\mathrm{gr}^F \lambda_A \circ D_0^F(\phi_{1+A})[1+a] &= \mathrm{gr}^F \lambda_A([\underline{1+a}-1]) = [\exp \circ s_A \circ \log \circ i_A(1+a) - 1] \\
&= [s_A \circ \log \circ i_A(1+a)] = [s_A \circ i_A(a)] = [a] \\
&= D_0^F(\phi_A) \circ D_0^F U_{\mathbb{Z}}(\ell_A)[1+a].
\end{aligned}$$

Hence $\mathrm{gr}^F \lambda_A$ and so λ_A is an isomorphism.

(i) By construction λ_A is natural in A.

(ii) As A is complete with respect to F, it follows that $(x^n/n)_{n\geq 1}$ converges to 0 in A and thus
$$\log(1-x) = -\sum_{n\geq 1} x^n/n \in A.$$
As the map $A \xrightarrow{i_A} \widehat{D}_1^F A$ is a 0-equicontinuous ring homomorphism, we get that
$$\begin{aligned}
\lambda_A(1+x) &= \exp \circ s_A \circ \log \circ i_A(1+x) = \exp \circ s_A \circ i_A \circ \log(1+x) \\
&= \exp \circ \log(1+x) = 1+x.
\end{aligned}$$

(iii) We have to show that λ_A maps $\widehat{I}(1+A)$ onto $\widehat{I}(A)$. Let $X' = X \cup -X$ and note that the two assumptions also hold for X replaced by X'. Let $\widehat{I}(1+X') \leq \widehat{D}_0^F \mathbb{Z}[1+A]$ and $\widehat{I}(X') \leq D_0^F U_{\mathbb{Z}}(A)$ be the closed ideals generated by $1+X'$ and X' respectively. We claim that $\widehat{I}(A) = \widehat{I}(X')$ and $\widehat{I}(1+A) = \widehat{I}(1+X')$. Therefore we let $a \in A$. We will inductively construct sequences

- $x_1, x_2, ... \in X'$, such that $(x_1 + ... + x_n)_{n\geq 1}$ converges to a, and
- $y_1, y_2, ... \in X'$, such that $((1+y_1)\cdots(1+y_n))_{n\geq 1}$ converges to $1+a$.

We may take arbitrary $x_1, y_1 \in X'$. Suppose we have constructed $x_1, ..., x_p \in X'$, such that $a \equiv x_1 + ... + x_p$ modulo $F_n A$, for some $n \geq 1$. Then by the first assumption we find $x_{p+1}, ..., x_q \in X' \cap F_{n+1} A$, such that $a \equiv x_1 + ... + x_q$ modulo $F_{n+1} A$. Similarly suppose we have constructed $y_1, ..., y_r \in X'$ with $1+a \equiv (1+y_1)\cdots(1+y_r)$ modulo $F_n A$, for some $n \geq 1$. Then we find $y_{r+1}, ..., y_s \in X'$, such that $1+a \equiv (1+y_1)\cdots(1+y_r) + y_{r+1} + ... + y_s$ modulo $F_{n+1} A$. As F is a ring filtration and $F_1 A = A$, this also implies that $1+a \equiv (1+y_1)\cdots(1+y_s)$ modulo $F_{n+1} A$.

As $\widehat{I}(X')$ is complete and contains $x_1+...+x_n$, for all $n \geq 1$, it follows that $a \in \widehat{I}(X')$. Similarly as $\widehat{I}(1+X')$ is complete and contains $\underline{1+x_1} \cdots \underline{1+x_n} - 1$ by using the formula (3.3), it also follows that $1+a \in \widehat{I}(1+X')$. As $a \in A$ was arbitrary, we get $\widehat{I}(A) = \widehat{I}(X')$ and $\widehat{I}(1+A) = \widehat{I}(1+X')$ as desired.

Now using (ii) for $x \in X'$, this implies that $\lambda_A \widehat{I}(1+A) = \lambda_A \widehat{I}(1+X') = \widehat{I}(X) = \widehat{I}(A)$, which proves that $1+A$ and A are associated via λ_A.

□

Remark 3.5.8
Proposition 3.5.7 shows that the group structure of $1+A$ *can be recovered from the Lie algebra structure of* A *and vice versa. The difficulty lies in proving that the map* λ *is infact a group homomorphism, which followed from the properties of logarithm and exponential series stated in Corollary 3.1.16.*

Historically this was proven by Lazard by using that the Hausdorff series $\log(\exp(X)\cdot\exp(Y))$, *being a rational power series in two noncommuting indeterminants* X, Y, *infact is a series of interleaved commutator brackets in* X *and* Y.

Proposition 3.5.9
Let $A = t \cdot \mathbb{Z}[t]/(t^m) \in \mathcal{A}ss$ *with* $m \geq 3$, *that we give the* (t)*-adic filtration.*

Then the following holds.

(i) *There is no* $X \subset A$ *satisfying the hypotheses of Proposition 3.5.7.*

(ii) *There is a non-canonical isomorphism* $1+A \cong A$ *and therefore non-canonically*

$$\widehat{H}_*(1+A,\mathbb{Z}) = H_*(1+A,\mathbb{Z}) = \Lambda_*(1+A) \cong \Lambda_* A = H_*(A,\mathbb{Z}) = \widehat{H}_*(A,\mathbb{Z}).$$

Proof.

(i) Assuming the opposite, by assumption (i) there must be an element $x \in X$ with $x = \pm t + at^2$, for some $a \in \mathbb{Z}[t]/(t^m)$. But then $x^2/2 = t^2/2 + bt^3$, for some $b \in \mathbb{Q}[t]/(t^m)$, contradicting assumption (ii).

(ii) There are exact sequences

$$1 \longrightarrow 1+t^{n+1} \cdot \mathbb{Z}[t]/(t^m) \longrightarrow 1+t^n \cdot \mathbb{Z}[t]/(t^m) \longrightarrow \operatorname{gr}_n(1+A) \longrightarrow 1, \qquad 1 \leq n \leq m,$$

which non-canonically split, because $\operatorname{gr}_n(1+A) \cong t^n \cdot \mathbb{Z}[t]/(t^{n+1}) \cong \mathbb{Z}$ is a free group. Hence we get non-canonical isomorphisms $1+t^n \cdot \mathbb{Z}[t]/(t^m) \cong \mathbb{Z}^{m-n} \cong t^n \cdot \mathbb{Z}[t]/(t^m)$, by induction on $1 \leq n \leq m$. The computation of the homology of an abelian finitely generated free group and Lie ring respectively are standard and can be found in any book about homological algebra. It follows from Corollary 3.4.18 and Corollary 3.3.25, that homology and completed homology are isomorphic.

□

3.5.4 Associated saturated groups and Lie rings

Proposition 3.5.10
Let $H \in \mathbb{Z}\text{-}\mathcal{G}rp$ carrying a Hopf algebra filtration F with $\mathrm{gr}^F H$ flat over $\mathbb{Z}$.
Then $\widehat{G}^F_n(H) \leq 1 + \widehat{D}^F_1(H)$ and $\widehat{P}^F_n(H) \leq \widehat{D}^F_1(H)$ with the induced filtrations are associated, for all $n \geq 1$, via

$$\lambda : \widehat{D}^F_0 \mathbb{Z}[\widehat{G}^F_n(H)] \xrightarrow{\sim} \widehat{D}^F_0 U_{\mathbb{Z}}(\widehat{P}^F_n(H)), \quad g \longmapsto \exp \circ s \circ i \circ \log(g),$$

where the the maps

$$\widehat{G}^F_n(H) \xrightarrow{\log} \widehat{P}^F_n H \xrightarrow{i} \widehat{D}^F_n \widehat{P}^F_n H \xrightarrow{s} \widehat{P}^F_n U_{\mathbb{Z}}(\widehat{P}^F_n H) \xrightarrow{\exp} \widehat{G}^F_n U_{\mathbb{Z}}(\widehat{P}^F_n H)$$

are those of Corollary 3.1.16, Proposition 3.1.12 and Proposition 3.3.29 respectively.
Note that the isomorphism λ is natural in H and there is a commutative square

$$\begin{tikzcd}
\widehat{D}^F_0 \mathbb{Z}[\widehat{G}^F_n(H)] \arrow[r, "\lambda"] \arrow[d, hook] & \widehat{D}^F_0 U_{\mathbb{Z}}(\widehat{P}^F_n(H)) \arrow[d, hook] \\
\widehat{D}^F_0 \mathbb{Z}[1 + \widehat{D}^F_1 H] \arrow[r, "\lambda_A"] & \widehat{D}^F_0 U_{\mathbb{Z}}(\widehat{D}^F_1 H),
\end{tikzcd}$$

where the lower map λ_A is that of Proposition 3.5.7 for $A = \widehat{D}^F_1(H)$ and the vertical maps are induced by the natural inclusions.

Proof. Since $\mathrm{gr}^F H$ is flat over $\mathbb{Z}$, the natural map $D^F_0 \mathrm{gr}^F H \xrightarrow{\sim} \mathrm{gr}^F \widehat{D}^F_0 H$ is an isomorphism by Proposition 3.1.11. As $\mathrm{gr}^F H \otimes \mathbb{Q}$ is flat over $\mathbb{Z}$, which means torsion-free, so is also $D^F_0 \mathrm{gr}^F H \leq \mathrm{gr}^F H \otimes \mathbb{Q}$. Giving $\widehat{P}^F_n(H) \leq \widehat{D}^F_0(H)$ the submodule filtration, we get an injection $\mathrm{gr}^F \widehat{P}^F_n(H) \hookrightarrow \mathrm{gr}^F \widehat{D}^F_0(H)$ and by the same argument as before also $\mathrm{gr}^F \widehat{P}^F_n(H)$ is flat, for all $n \geq 0$.

As $D^F_1 \mathrm{gr}^F H \xrightarrow{\sim} \mathrm{gr}^F \widehat{D}^F_1 H$ is $\mathbb{Z}_{>0}$-graded and flat, Proposition 3.3.6 combined with Proposition 3.3.9 yield that $\mathrm{gr}^F \widehat{D}^F_1 H \xhookrightarrow{\eta} PU_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}^F_1 H)$ is injective. Hence by Corollary 3.3.8 the composition $\mathrm{gr}^F \widehat{P}^F_n H \hookrightarrow \mathrm{gr}^F \widehat{D}^F_1 H \hookrightarrow PU_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}^F_1 H)$ extends to an injection $U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{P}^F_n H) \hookrightarrow U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}^F_1 H)$. As D^F_0 is a subfunctor of tensoring with $\mathbb{Q}$, it preserves monomorphisms. Hence the left vertical map in the commutative diagram below is injective.

$$\begin{tikzcd}
D^F_0 U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{P}^F_n H) \arrow[r, "D^F_0(\phi)", "\sim"'] \arrow[d, hook] & D^F_0 \mathrm{gr}^F U_{\mathbb{Z}}(\widehat{P}^F_n H) \arrow[r, "\sim"] \arrow[d] & \mathrm{gr}^F \widehat{D}^F_0 U_{\mathbb{Z}}(\widehat{P}^F_n H) \arrow[d] \\
D^F_0 U_{\mathbb{Z}}(\mathrm{gr}^F \widehat{D}^F_1 H) \arrow[r, "D^F_0(\phi)", "\sim"'] & D^F_0 \mathrm{gr}^F U_{\mathbb{Z}}(\widehat{D}^F_1 H) \arrow[r, "\sim"] & \mathrm{gr}^F \widehat{D}^F_0 U_{\mathbb{Z}}(\widehat{D}^F_1 H)
\end{tikzcd}$$

The horizontal maps are the isomorphisms of Proposition 3.3.17 and Proposition 3.1.11. It follows that the right vertical map is injective, which implies injectivity of the right vertical map in the diagram

$$\begin{tikzcd}
\widehat{G}^F_n H \arrow[r, "\log"] \arrow[d] & \widehat{P}^F_n H \arrow[r, "i"] \arrow[d, hook] & \widehat{D}^F_n \widehat{P}^F_n H \arrow[r, "s"] \arrow[d] & \widehat{P}^F_n U_{\mathbb{Z}}(\widehat{P}^F_n H) \arrow[r, "\exp"] \arrow[d] & \widehat{G}^F_n U_{\mathbb{Z}}(\widehat{P}^F_n H) \arrow[d, hook] \\
1 + A \arrow[r, "\log"] & A \arrow[r, "i"] & \widehat{D}^F_1 A \arrow[r, "s"] & \widehat{P}^F_1 U_{\mathbb{Z}}(A) \arrow[r, "\exp"] & \widehat{G}^F_1 U_{\mathbb{Z}}(A),
\end{tikzcd}$$

where the vertical maps are induced by the inclusion $\widehat{P}_n^F H \hookrightarrow \widehat{D}_1^F H = A$. As the logarithm map is natural, the lower row is

$$\exp \circ s \circ i \circ \log = \exp \circ s \circ \log \circ i, \tag{3.4}$$

which is a group homomorphism and extends to the map λ_A by Proposition 3.5.7. It follows that also the upper row is a group homomorphism, extends to the map λ and induces a commutative diagram as desired. By the same arguments as in Proposition 3.5.7 one checks that $\mathrm{gr}^F \lambda$ and hence λ is an isomorphism.

For every $n \geq 1$ we have $n^{-1} \in D_{1-n}\mathbb{Q}$ by Lemma 3.1.14 and thus

$$x^n/n \in (F_1 H)^n \otimes D_{1-n}\mathbb{Q} \subset F_n H \otimes D_{1-n}\mathbb{Q} \subset \widehat{D}_1^F H, \qquad 1 + x \in \widehat{G}_n^F H.$$

So by Proposition 3.5.7 (ii) the map $\lambda = \lambda_A|$ sends the generator $\underline{1+x} - 1 \in \widehat{I}(\widehat{G}_n^F H)$ to the generator $1 + x - 1 = x \in \widehat{P}_n^F H \subset \widehat{I}(\widehat{P}_n^F H)$. It follows that $\lambda \widehat{I}(\widehat{G}_n^F H) = \widehat{I}(\widehat{P}_n^F H)$, which proves that $\widehat{G}_n^F H$ and $\widehat{P}_n^F H$ are associated via λ.

□

Corollary 3.5.11
Let $\mathfrak{g} \in \mathcal{L}ie$ carrying a positive Lie algebra filtration F with $\mathrm{gr}^F \mathfrak{g}$ flat over $\mathbb{Z}$.
Then $\widehat{G}_1^F U_{\mathbb{Z}}(\mathfrak{g})$ and $\widehat{D}_1^F(\mathfrak{g})$ are associated.
In particular every saturated, filtered Lie ring $\mathfrak{g}$ has an associated filtered group $\widehat{G}_1^F U_{\mathbb{Z}}(\mathfrak{g})$.

Proof. The ring $\mathrm{gr}^F U_{\mathbb{Z}}(\mathfrak{g})$ is flat by Proposition 3.3.17 and by Proposition 3.3.29 we have an isomorphism $\widehat{D}_1^F \mathfrak{g} \xrightarrow{\sim} \widehat{P}_1^F U_{\mathbb{Z}}(\mathfrak{g})$. Hence $\widehat{G}_1^F U_{\mathbb{Z}}(\mathfrak{g})$ and $\widehat{D}_1^F(\mathfrak{g})$ are associated by Proposition 3.5.10.

□

Corollary 3.5.12
Let $G \in \mathcal{G}rp$ carrying a positive group filtration F with $\mathrm{gr}^F G$ flat over $\mathbb{Z}$.
Then $\widehat{G}_1^F \mathbb{Z}[G]$ and $\widehat{P}_1^F \mathbb{Z}[G]$ are associated.
In particular every saturated, filtered group G has an associated filtered Lie ring $\widehat{P}_1^F \mathbb{Z}[G]$.

3.6 Appendix

3.6.1 Flat modules

Recall some basic properties of flat modules, that can be found in any algebra book.

Proposition 3.6.1
Given $k \in \mathcal{C}Ring$ and a short exact sequence of k-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then the following holds.

(i) *If C is flat, then $A \otimes D \hookrightarrow B \otimes D$, for all $D \in k$-*Mod.

(ii) *If A and C are flat, so is B.*

(iii) *If B and C are flat, so is A.*

(iv) *If A and B are flat, then C must not be flat. Consider for example $k = \mathbb{Z}$ and*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0.$$

Proof. For $D \in k$-Mod choose a surjection $F \twoheadrightarrow D$, where $F \in k$-Mod is free. Let K denote its kernel. Applying the snake lemma to the following map of exact sequences

$$\begin{tikzcd} & A \otimes K \arrow[r] \arrow[d] & B \otimes K \arrow[r] \arrow[d] & C \otimes K \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \otimes F \arrow[r] & B \otimes F \arrow[r] & C \otimes F \arrow[r] & 0 \end{tikzcd}$$

yields a short exact sequence

$$\ker(C \otimes K \longrightarrow C \otimes F) \longrightarrow A \otimes D \longrightarrow B \otimes D.$$

As C is flat the left object is zero, which proves (i).

Now if C is flat, by (i) every monomorphism $A' \hookrightarrow B'$ induces a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \otimes A' \arrow[r] \arrow[d] & B \otimes A' \arrow[r] \arrow[d] & C \otimes A' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \otimes B' \arrow[r] & B \otimes B' \arrow[r] & C \otimes B' \arrow[r] & 0. \end{tikzcd}$$

If A and C are flat, the outer two vertical maps and thus also the middle vertical map is injective by the 5-lemma. This shows (ii). Similarly if A is flat, the middle vertical map and thus also the left vertical map is injective, which proves (iii).

□

Corollary 3.6.2
*Let $k \in \mathcal{C}Ring$ and suppose $X \in k$-*Mod *carries an exhaustive filtration F, such that* $\mathrm{gr}^F X$ *is flat over k.*

(i) *If F is bounded below, then X is flat over k.*

(ii) *If k is coherent, then $\widehat{X} = \varprojlim_{n \leq 0} X/X_n$ is flat over k.*

Proof. Let $m \in \mathbb{Z}$. Using the exact sequences

$$0 \longrightarrow \mathrm{gr}_n^F X \longrightarrow X_m/X_{n+1} \longrightarrow X_m/X_n \longrightarrow 0, \qquad n > m,$$

Proposition 3.6.1 (ii) yields that X_m/X_n is flat, by induction on $n > m$. By exactness of direct limits also $\varinjlim_{m>n} X_m/X_n \xrightarrow{\sim} X/X_n$ is flat, for all $n \in \mathbb{Z}$. If F is bounded below, there is an $n \in \mathbb{Z}$ with $X_n = 0$ and thus $X = X/X_n$ is flat.

Now suppose k is coherent and let $I \leq k$ be a finitely generated ideal. As X/X_n is flat, the map $X/X_n \otimes I \hookrightarrow X/X_n \otimes k$ is injective, for all $n \in \mathbb{Z}$. In the commutative square

$$\begin{tikzcd}
\widehat{X} \otimes I \arrow[r] & \widehat{X} \otimes k \\
\varprojlim_{n\leq 0} X/X_n \otimes I \arrow[r] \arrow[u, "\wr"] & \varprojlim_{n\leq 0} X/X_n \otimes k, \arrow[u, "\wr"']
\end{tikzcd}$$

the vertical left map is an isomorphism, because I is finitely generated and thus finitely presented as k is coherent. The lower horizontal map is injective by exactness of inverse limits. Hence also the upper horizontal map is injective, which proves that $\widehat{X}$ is flat.

□

3.6.2 Hilbert's basis theorem

Recall also the skew-commutative version of Hilbert's basis Theorem.

Theorem 3.6.3
Let $R \leq S \in \mathcal{R}ing$ and $t \in S \backslash R$. Suppose R is left Noetherian.

(i) *If $Rt + R = tR + R$, then $R[t] \leq S$ is left Noetherian.*

(ii) *If moreover $t \in S^\times$, then $R[t, t^{-1}] \leq S$ is left Noetherian.*

Proof. See for example [MR01] Theorem 1.2.10.

□

4 E_∞-spaces and their homotopy groups

4.1 The category I of injections

We introduce tha category I of injections, which plays an an important role in this chapter and will also appear in chapters 5 and 6.

Definition 4.1.1
We let $I \leq \mathcal{S}et$ denote the subcategory of sets

$$\mathbf{n} := \{1, ..., n\}, \quad n \geq 0,$$

whose morphisms are injections.

Remark 4.1.2
Considering the partial ordered set of natural numbers $\mathbb{N}_0$ as a category, we define a chain of functors

$$\mathbb{N}_0 \xrightarrow{\nu} \hat{\Delta}_{inj} \xrightarrow{\alpha} I \xrightarrow{\tau} \mathbb{N}_0,$$

where

- *ν is the inclusion functor, sending*
  - *an object $n \in \mathbb{N}_0$ to $\underline{n-1}$, for all $n \geq 0$,*
  - *a morphism $n \leq n+1$ to $d^n \in \hat{\Delta}_{inj}(\underline{n-1}, \underline{n})$, for all $n \geq 0$.*
- *α is the forgetful functor, sending $\underline{n-1}$ to $\mathbf{n}$, for all $n \geq 0$.*
- *τ is the terminal functor, sending $\mathbf{n}$ to n, for all $n \geq 0$.*

Remark 4.1.3
There is a symmetric monoidal structure on I, given by the disjoint union

$$I \times I \longrightarrow I, \quad (\mathbf{m}, \mathbf{n}) \longmapsto \mathbf{m} + \mathbf{n}.$$

Its neutral element is the empty set $\mathbf{0}$.

Moreover the forgetful functor is a strictly monoidal functor $(\hat{\Delta}_{inj}, \oplus, \underline{-1}) \xrightarrow{\alpha} (I, +, \mathbf{0})$ (see Remark 7.2.2 for the monoidal structure on $\hat{\Delta}$).

Proposition 4.1.4
The following two functors are totally final[1]

$$I \xrightarrow{\tau} \mathbb{N}_0, \qquad \hat{\Delta}_{inj} \xrightarrow{\alpha} I \xrightarrow{\tau} \mathbb{N}_0.$$

Proof. We only prove that τ is totally final. The other case can be shown in the same way. Recall that by definition τ is totally final, if and only if $B(n/\tau)$ is contractible, for all $n \in \mathbb{N}_0$. First note, that the forgetful functor

$$n/\tau \longrightarrow I, \quad (n \leq \tau(\mathbf{m})) \longmapsto \mathbf{m},$$

induces an isomorphism of n/τ with the full subcategory $I_{\geq n} \leq I$, whose objects are all $\mathbf{m} \in I$ with $m \geq n$. To show that $BI_{\geq n}$ is contractible, we define a shift functor

$$S : I_{\geq m} \longrightarrow I_{\geq m}, \quad \mathbf{n} \longmapsto \mathbf{n} + \mathbf{n}.$$

For every $f \in I(\mathbf{m}, \mathbf{m}')$ we have a commutative diagram

$$\begin{tikzcd} \mathbf{m} \arrow[r, "\iota_1"] \arrow[d, "f"'] & \mathbf{m}+\mathbf{n} \arrow[d, "f+\mathrm{id}"'] & \mathbf{n} \arrow[l, "\iota_2"'] \arrow[d, equal] \\ \mathbf{m}' \arrow[r, "\iota_1"'] & \mathbf{m}'+\mathbf{n} & \mathbf{n}, \arrow[l, "\iota_2"] \end{tikzcd}$$

so the inclusions define natural transformations $\mathrm{id}_{I_{\geq m}} \xrightarrow{\iota_1} S \xleftarrow{\iota_2} \mathrm{const}_{\mathbf{n}}$. This shows that the identity map on $BI_{\geq m}$ is homotopic to the constant map or equivalently that $BI_{\geq m}$ is contractible.

□

4.1.1 Limits over connected, non-empty categories

A detailed study of the behaviour of functors on I with limits is needed to compare their homotopy colimit with their colimit (cf. section 7.3.7). Here we are verifying the needed categorial foundations.

Lemma 4.1.5
Let $\mathcal{C}$ be a category having arbitrary pullbacks, equalizers and sequential limits.

(i) *Then $\mathcal{C}$ has limits over connected, non-empty indexing categories.*

(ii) *Suppose $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a functor, which preserves pullbacks, equalizers and sequential limits.*

Then F preserves limits over connected, non-empty indexing categories.

[1] See Definition 7.3.26.

(iii) *Let $G \hookrightarrow F$ be a natural monomorphism of functors $\mathcal{C} \longrightarrow \mathcal{D}$ and suppose F preserves pullbacks, equalizers and sequential limits. Moreover suppose that*

$$\begin{tikzcd} G(C) \arrow[r, "G(f)"] \arrow[d] & G(D) \arrow[d] \\ F(C) \arrow[r, "F(f)"] & F(D) \end{tikzcd}$$

is cartesian, for every morphism $f \in \mathcal{C}(C, D)$.

Then G preserves limits over connected, non-empty indexing categories.

Proof.

(i) Let $X \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})$ with connected, non-empty $I \in \mathcal{C}at$. Let $\mathcal{U}$ be the set of connected, non-empty subcategories $U \leq I$, such that $\lim_U X$ exists. As I is non-empty, there is an $i \in I$ and hence the discrete category with one object i lies in $\mathcal{U}$. So $\mathcal{U}$ is non-empty. Given a sequence of categories $U_1 \leq U_2 \leq ...$ in $\mathcal{U}$, we let $U = \bigcup_{n \geq 1} U_n$. Since $\mathcal{C}$ has sequential limits, the limit

$$\lim_U X = \lim(... \longrightarrow \lim_{U_2} X \longrightarrow \lim_{U_1} X),$$

exists, which proves that $U \in \mathcal{U}$. So by Zorn's Lemma there is a maximal element $M \in \mathcal{U}$.

Suppose $M \subsetneq I$. Then there is a morphism $f \in I(i, j)$ not contained in M. If $i, j \notin M$, we take an arbitrary $m \in M$. As I is connected, there is a zig-zag of morphisms

$$m \to i_1 \leftarrow i_2 \to ... \leftarrow i_n \to i.$$

As $i \notin M$, there is a i_k, such that $i_k \in M$. So by replacing f by the adjacent morphism, we may assume that $i \in M$ or $j \in M$. Let $M' \leq I$ be the subcategory generated by M and f. It is connected, as $i \in M$ or $j \in M$.

- If $i \in M$ and $j \notin M$, then $\lim_{M'} X = \lim_M X$ exists.
- If $i \notin M$ and $j \in M$, then $\lim_{M'} X = X(i) \times_{X(j)} \lim_M X$ exists, as $\mathcal{C}$ has arbitrary pullbacks.
- If $i, j \in M$, then

$$\lim_{M'} = \ker \left(\lim_M X \underset{\pi_j}{\overset{X(f)\circ\pi_i}{\rightrightarrows}} X(j) \right)$$

  exists, because $\mathcal{C}$ has arbitrary equalizers.

Hence $M' \in \mathcal{U}$, contradicting the maximality of M. It follows that $I = M$.

(ii) Replace "exists" by "is preserved by F" in the proof of (i).

(iii) By (ii) it suffices to check that G preserves pullbacks, equalizers and sequential limits.

- Given $B \longrightarrow A \longleftarrow C$ consider the commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small]
G(B \times_A C) \arrow[rr] \arrow[dd] \arrow[dr] & & G(C) \arrow[dd] \arrow[dr] & \\
& F(B \times_A C) \arrow[rr, crossing over] & & F(C) \arrow[dd] \\
G(B) \arrow[rr] \arrow[dr] & & G(A) \arrow[dr] & \\
& F(B) \arrow[rr] \arrow[from=uu, crossing over] & & F(A).
\end{tikzcd}$$

  As F preserves pullbacks, the front is cartesian. Moreover the left side is cartesian by assumption. Hence the composite of front and left side is cartesian. Equivalently the composite of the back and the right side is cartesian. As $G(A) \hookrightarrow F(A)$ is a monomorphism, this implies that the back is cartesian. In other words $G(B \times_A C) \xrightarrow{\sim} G(B) \times_{G(A)} G(C)$ and thus G preserves arbitrary pullbacks.

- Given two morphisms $f, g \in \mathcal{C}(A, B)$, there is a commutative diagram

$$\begin{tikzcd}
G(\ker(f,g)) \arrow[r] \arrow[d] & G(A) \arrow[r, shift left, "G(f)"] \arrow[r, shift right, "G(g)"'] \arrow[d] & G(B) \arrow[d] \\
F(\ker(f,g)) \arrow[r] & F(A) \arrow[r, shift left, "F(f)"] \arrow[r, shift right, "F(g)"'] & F(B).
\end{tikzcd}$$

  We have

$$\begin{aligned}
G(\ker(f,g)) &\xrightarrow{\sim} F(\ker(f,g)) \times_{F(A)} G(A) \\
&\xrightarrow{\sim} \lim\left(\begin{tikzcd} G(A) \arrow[r] & F(A) \arrow[r, shift left, "F(f)"] \arrow[r, shift right, "F(g)"'] & F(B) \end{tikzcd} \right) \\
&= \lim\left(\begin{tikzcd} G(A) \arrow[r, shift left, "G(f)"] \arrow[r, shift right, "G(g)"'] & G(B) \arrow[r, hook] & F(B) \end{tikzcd} \right) \\
&\xleftarrow{\sim} \lim\left(\begin{tikzcd} G(A) \arrow[r, shift left, "G(f)"] \arrow[r, shift right, "G(g)"'] & G(B) \end{tikzcd} \right) = \ker(G(f), G(g)),
\end{aligned}$$

  where the first map is an isomorphism by assumption on G, the second is an isomorphism, because F preserves equalizers, the third one is an equality by commutativity and the fourth map is an isomorphism, because $G(B) \hookrightarrow F(B)$ is a monomorphism. In other words $G(\ker(f,g)) \xrightarrow{\sim} \ker(G(f), G(g))$ is an isomorphism and thus G preserves arbitrary equalizers.

- Given a sequence $\ldots \to C_3 \to C_2 \to C_1$ of maps in $\mathcal{C}$, by assumption on G we have natural isomorphisms

$$G(\lim_{n\geq 1} C_n) \xrightarrow{\sim} F(\lim_{n\geq 1} C_n) \times_{F(C_m)} G(C_m), \qquad m \geq 1.$$

In particular we obtain a sequence of isomorphisms

$$\ldots \xrightarrow{\sim} F(\lim_{n\geq 1} C_n) \times_{F(C_2)} G(C_2) \xrightarrow{\sim} F(\lim_{n\geq 1} C_n) \times_{F(C_1)} G(C_1),$$

and thus the first map below is an isomorphism

$$\begin{aligned} G(\lim_{n\geq 1} C_n) &\xrightarrow{\sim} \lim_{m\geq 1} \Big(F(\lim_{n\geq 1} C_n) \times_{F(C_m)} G(C_m) \Big) \\ &\xrightarrow{\sim} \lim_{m\geq 1} \Big(\lim_{n\geq 1} F(C_n) \times_{F(C_m)} G(C_m) \Big) \\ &\xrightarrow{\sim} \lim_{n\geq m\geq 1} \big(F(C_n) \times_{F(C_m)} G(C_m) \big) \\ &\xrightarrow{\sim} \lim_{n\geq 1} \big(F(C_n) \times_{F(C_n)} G(C_n) \big) \\ &\xrightarrow{\sim} \lim_{n\geq 1} G(C_n). \end{aligned}$$

The second map is an isomorphism, as F preserves sequential limits. The third map is an isomorphism, because $\mathbb{N}_{\geq m} \subset \mathbb{N}$ is final and limits commute with pullbacks. The fourth map is an isomorphism, because the diagonal $\mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$ is final. And the last map is an isomorphism, because $F(C_n) \times_{F(C_n)} G(C_n) \xrightarrow{\pi} G(C_n)$ is an isomorphism. This proves that G preserves sequential limits.

□

4.1.2 Limits of diagrams of injections

Here we are presenting some important functors on I, that preserve limits. We are mainly interested in modified variants of these (cf. Proposition 6.3.1 and Proposition 5.4.10).

Lemma 4.1.6
Let $X \in \mathcal{C}\mathrm{AT}(C, \mathcal{S}et)$ with $C \in \mathcal{C}at$ connected and non-empty.
Suppose X maps every morphism in C to an injection.
Then $\lim_C X \xrightarrow{\pi_c} X(c)$ is injective, for all $c \in C$.

Proof. Let $c \in C$ and suppose $x_c = \pi_c x = \pi_c y = y_c$, where

$$x, y \in \lim_C X = \{z \in \prod_{c\in C}; \; X(f)(z_c) = z_d, \qquad f \in C(c,d)\}.$$

Let $d \in C$ be arbitrary. As C is connected, there is a zigzag of morphisms in C

$$c = c_1 \to c_2 \leftarrow \ldots \to c_n = d, \qquad n \geq 1.$$

We can now check that $x_{c_m} = y_{c_m}$, by induction on $1 \leq m \leq n$. Indeed by assumption the statement holds for $m = 1$. Suppose it holds for some $1 \leq m \leq n$ and consider the morphism f between c_m and c_{m+1}.

- If its target is c_{m+1}, then
$$x_{m+1} = X(f)(x_m) = X(f)(y_m) = y_{m+1}.$$
- If its target is i_m, then
$$X(f)(x_{m+1}) = x_m = y_m = X(f)(y_{m+1}),$$
and hence $x_{m+1} = y_{m+1}$, since $X(f)$ is injective.

As $c \in C$ was arbitrary, it follows that $x = y$, which proves that π_c is injective.

□

Corollary 4.1.7
The following categories have limits over connected non-empty categories.

(i) *The wide[2] subcategory category $\mathcal{S}et_{inj} \leq \mathcal{S}et$ of sets with injections as morphisms.*

(ii) *The category I of the sets $\mathbf{n} = \{1, ..., n\}$, for $n \geq 0$, with injections.*

(iii) *The category $\hat{\Delta}_{inj}$.*

The functors $\hat{\Delta}_{inj} \xrightarrow{\alpha} I \xrightarrow{\iota} \mathcal{S}et_{inj} \xrightarrow{J} \mathcal{S}et$ preserve these limits.

Proof.

(i) Let $X \in \mathcal{C}\mathrm{AT}(C, \mathcal{S}et_{inj})$ with connected and non-empty $C \in \mathcal{C}at$. Then by Lemma 4.1.6 all the projection maps $\lim_C(JX) \xrightarrow{\pi_c} JX(c)$ are injective and so they are morphisms in $\mathcal{S}et_{inj}$. Given a natural $\mathrm{const}_S \xrightarrow{s} X$ in $\mathcal{C}\mathrm{AT}(C, \mathcal{S}et_{inj})$ with $S \in \mathcal{S}et_{inj}$. Then using the universal property for limits in $\mathcal{S}et$ we get a unique map
$$U(S) \xrightarrow{\ell} \lim_C JX, \qquad x \longmapsto (s_c(x))_{c \in C},$$
such that $\pi_c \circ \ell = s_c$, for all $c \in C$. As C is non-empty, we find a $c \in C$. As $s_c = \pi_c \circ \ell$ is injective, so is also ℓ. Hence ℓ is a morphism in $\mathcal{S}et_{inj}$ and thus $\lim_C X = \lim_C JX$. In particular $\mathcal{S}et_{inj} \xrightarrow{J} \mathcal{S}et$ preserves limits over connected non-empty categories.

(ii) Let $X \in \mathcal{C}\mathrm{AT}(C, I)$ with connected and non-empty $C \in \mathcal{C}at$. Then by (i) $\lim_C(\iota X)$ exists and $\lim_C(\iota X) \xrightarrow{\pi_c} \iota X(c)$ is injective, for all $c \in C$. As C is non-empty we find a $c \in C$ and so $\lim_C(\iota X)$ is isomorphic to some subset of $\iota X(c)$. In particular there is a $\mathbf{m} \in I$ with $\iota(\mathbf{m}) \cong \lim_C(\iota X)$ and it follows that $\mathbf{m} = \lim_C X$. By construction $I \xrightarrow{\iota} \mathcal{S}et_{inj}$ preserves limits over connected, non-empty categories.

[2] A subcategory is called **wide**, if it contains every object of the larger category.

(iii) Let $X \in \mathcal{C}\mathrm{AT}(C, \hat{\Delta}_{inj})$ with connected and non-empty $C \in \mathcal{C}at$. Then by (i) $\lim_C(\iota\alpha X)$ exists. As C is non-empty we find a $c \in C$ and we give $\lim_C(\iota\alpha X)$ the initial partial order with respect to the map $\lim_C(\iota\alpha X) \xrightarrow{\pi_c} \iota\alpha X(c)$, i.e.

$$x \leq y \qquad :\Longleftrightarrow \qquad \pi_c(x) \leq \pi_c(y), \qquad x, y \in \lim_C(\iota\alpha X).$$

It is a total order, because π_c is injective and $X(c) \in \hat{\Delta}_{inj}$ is totally ordered. Hence we find an $k \geq -1$ such that $\iota\alpha(\underline{k}) \cong \lim_C(\iota\alpha X)$. We claim that $\iota\alpha(\underline{k}) \cong \lim_C(\iota\alpha X) \xrightarrow{\pi_d} \iota\alpha X(d)$ is a morphism in $\hat{\Delta}_{inj}$, for all $d \in C$. As C is connected, there is a zigzag of morphisms in C

$$c = c_1 \to c_2 \leftarrow ... \to c_n = d, \qquad n \geq 1.$$

We prove that $\iota\alpha(\underline{k}) \cong \lim_C(\iota\alpha X) \xrightarrow{\pi_{c_m}} \iota\alpha X(d)$ is a morphism in $\hat{\Delta}_{inj}$, by induction on $1 \leq m \leq k$. Indeed by assumption the statement holds for $m = 1$. Suppose it holds for some $1 \leq m \leq n$ and consider the morphism f between c_m and c_{m+1}.

- If its target is c_{m+1}, then with $X(f)$ and π_{c_m} also $\pi_{c_{m+1}} = X(f)\pi_{c_m}$ is order-preserving.
- If its target is c_m, then for every $x, y \in \underline{k}$ with $x \leq y$ we have

  $$\pi_{c_m}(x) = X(f)\pi_{c_{m+1}}(x) \leq X(f)\pi_{c_{m+1}}(y) = \pi_{c_m}(y),$$

  and hence $\pi_{c_{m+1}}(x) \leq \pi_{c_{m+1}}(y)$, because $X(c_{m+1})$ carries the initial partial order to the map $X(c_{m+1}) \xrightarrow{X(f)} X(c_m)$. Indeed there is only one possible total order on $\alpha X(c_{m+1})$, such that $X(f)$ is order preserving, because $X(f)$ is injective.

This proves the induction step and shows that $\underline{k}$ is a limit of X in $\hat{\Delta}_{inj}$. By construction $\hat{\Delta}_{inj} \xrightarrow{\alpha} I$ preserves this limit.

□

Lemma 4.1.8
Let $X \in \mathcal{C}\mathrm{AT}(C, \mathcal{S}et_{inj})$ with connected, non-empty $C \in \mathcal{C}at$. Moreover let $c \in C$ and $a \in X(c)$.

Then there is a maximal connected subcategory $c \in M \leq C$, such that a lies in the image of $\lim_M X \xrightarrow{\pi_c} X(c)$.

Proof. By Corollary 4.1.7 we have $J(\lim_C X) = \lim_C JX$, where $\mathcal{S}et_{inj} \xrightarrow{J} \mathcal{S}et$ is the canonical inclusion functor. Given a chain $c \in U_1 \leq U_2 \leq ...$ of such subcategories, then also $U = \bigcup_{n\geq 1} U_n$ has this property. Indeed if $d \in U$, then there is an U_n containing d and we let $a_d \in X(d)$ be the d-th coordinate of a lift of a under $\lim_{U_n} X \xrightarrow{\pi_c} X(c)$. Since $\lim_{U_n} X \xrightarrow{\pi_d} X(d)$ is injective, a_d is unique. Moreover using uniqueness of the lifts and the commutative diagram

$$\begin{tikzcd} \lim_{U_{n+1}} X \arrow[rr] \arrow[dr, "\pi_d"'] & & \lim_{U_n} X \arrow[dl, "\pi_d"] \\ & X(d), & \end{tikzcd}$$

it follows that a_d does not depend on $n \geq 1$. Every morphism $f \in U(d, e)$ lies in some U_n and thus $X(f)(a_d) = a_e$, which proves that $(a_d)_{d \in U} \in \lim_U X$ is a lift for a under $\lim_U X \xrightarrow{\pi_c} X(c)$. So Zorn's lemma yields a maximal connected subcategory $M \leq C$ of the desired form.

□

Proposition 4.1.9
Every pointed set $S \in \mathcal{S}et_*$ naturally induces an endofunctor $E^\bullet S \in \mathcal{C}\mathrm{AT}(\mathcal{S}et_{inj})$, given by

$$E^X(S) = S^X, \qquad E^f(S)(s)_y = \begin{cases} s_x, & y = f(x), \\ *, & y \notin f(X), \end{cases} \qquad f \in \mathcal{S}et_{inj}(X, Y).$$

It preserves limits over connected non-empty categories.

Proof. For $f \in \mathcal{S}et_{inj}(X, Y)$ we see that $X(f)(s) \in E^Y(S) = S^Y = \mathcal{S}et(Y, S)$ is the trivial extension of $s \in E^X(S) = S^X = \mathcal{S}et(X, S)$. In particular $E^f(S)$ is again injective. It follows that $E^\bullet S \in \mathcal{C}\mathrm{AT}(\mathcal{S}et_{inj})$.

Let $X \in \mathcal{C}\mathrm{AT}(C, \mathcal{S}et_{inj})$ with connected non-empty $C \in \mathcal{C}at$. We find a $c \in C$ and using the commutative diagram

$$\begin{tikzcd} E^{\lim_C X}(S) \arrow[r] \arrow[dr, "E^{\pi_c}(S)"'] & \lim_C E^X(S) \arrow[d, "\pi_c"] \\ & E^{X(c)}(S), \end{tikzcd}$$

we see that the upper horizontal map is injective, as the diagonal map is so.

To check that it is also surjective, let $s \in \lim_C E^X(S)$. By Corollary 4.1.7 we have $J(\lim_C E^X(S)) = \lim_C JE^X(S)$, where $\mathcal{S}et_{inj} \xrightarrow{J} \mathcal{S}et$ is the canonical inclusion functor. So s is a certain tuple in $\prod_{c \in C} E^{X(c)}(S) = \prod_{c \in C} S^{X(c)}$ and we have to prove that $(s_c)_a = *$, for all $c \in C$ and all $a \in X(c)$ not in the image of the map $\lim_C X \xrightarrow{\pi_c} X(c)$. By Lemma 4.1.8 there is a maximal connected subcategory $c \in M \leq C$, such that a is in the image of $\lim_C X \xrightarrow{\pi_c} X(c)$. Let $f \in C(d, e)$ be not contained in M. As C is connected, there is a zig-zag of C-morphisms connecting c with d. So by replacing f by some morphism of the zig-zag, we may assume that d or e is in M. If we had $d \in M$, then we could set $a_e := X(f)(a_d)$ to obtain a lift of a under $\lim_{M \cup \{f\}} X \xrightarrow{\pi_c} X(c)$ contradicting the maximality of M. Hence $e \in M$. Let $(a_d)_{d \in M} \in \lim_M X$ be a lift of a. Similarly if a_d had a preimage a_e under $X(d) \xrightarrow{X(f)} X(e)$, this would contradict the maximality of M. So as a_d has no preimage, we have $(s_d)_{a_d} = *$ by definition of $E^f(S)$. Using the zig-zag connecting c and d, it follows that also $(s_c)_a = *$.

□

Proposition 4.1.10
For every $n \geq 1$, the functor

$$p_n : \mathcal{S}et_{inj} \longrightarrow \mathcal{S}et_{inj}, \qquad X \longmapsto X^n,$$

preserves limits over connected, non-empty categories.

Proof. By Corollary 4.1.7 the inclusion functor $\mathcal{S}et_{inj} \xrightarrow{J} \mathcal{S}et$ preserves limits over connected, non-empty categories. Let $X \in \mathcal{C}\mathrm{AT}(I, \mathcal{S}et_{inj})$ with connected $\emptyset \neq I \in \mathcal{C}at$, we have a commutative diagram

$$\begin{tikzcd} J((\lim_I X)^n) \arrow[r] \arrow[d, "\wr"'] & J(\lim(X^n)) \arrow[d, "\wr"] \\ (\lim_I JX)^n \arrow[r, "\sim"] & \lim_I (JX)^n, \end{tikzcd}$$

where the vertical maps are bijections, because J preserves limits over connected, non-empty categories by Corollary 4.1.7. Moreover the lower horizontal map is a bijection, because limits and products always commute. Hence also the upper horizontal map is a bijection, which proves that also $p_n(\lim_I X) = (\lim_I X)^n \xrightarrow{\sim} \lim_I(X^n) = \lim_I p_n X$ is a bijection.

□

4.2 I-Operads

Here we are giving a very short introduction to I-operads with particular interest for the Barratt-Eccles operad (cf. [BE74a]), that classifies E_∞-spaces like the plus construction in K-theory.

4.2.1 Algebras over monads

Definition 4.2.1
Let $\mathcal{C}$ be a category and M a monad on $\mathcal{C}$, i.e. a monoid in $(\mathcal{C}\mathrm{AT}(\mathcal{C}), \circ, \mathrm{id}_\mathcal{C})$.

*An M-**algebra** consists of an object $X \in \mathcal{C}$ and a morphism $\mu_X \in \mathcal{C}(M(X), X)$, such that the square below commutes.*

$$\begin{tikzcd} M \circ M(X) \arrow[r, "\mu_{M(X)}"] \arrow[d, "M(\mu_X)"'] & M(X) \arrow[d, "\mu_X"] \\ M(X) \arrow[r, "\mu_X"] & X. \end{tikzcd}$$

A homomorphism of M-algebras $X \xrightarrow{f} Y$ is a map $f \in \mathcal{C}(X, Y)$, such the square below commutes.

$$\begin{tikzcd} M(X) \arrow[r, "M(f)"] \arrow[d, "\mu_X"'] & M(Y) \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

We let $\mathcal{C}$-M denote the category of M-algebras.

Remark 4.2.2 (i) *Unit and counit of an adjunction $\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y))$ define a monad GF on $\mathcal{C}$ with multiplication and unit*

$$GFGF \xrightarrow{G(\varepsilon_F)} GF, \qquad \mathrm{id}_\mathcal{C} \xrightarrow{\eta} GF.$$

Moreover F induces a functor

$$\mathcal{D} \longrightarrow \mathcal{C}\text{-}M, \quad X \longmapsto (G(X), GFG(X) \xrightarrow{G(\varepsilon_X)} G(X)).$$

(ii) *Vice versa, given a monad M on a category $\mathcal{C}$, the forgetful functor U fits in an adjunction*

$$\mathcal{C}\text{-}M(F(X), Y) = \mathcal{C}(X, U(Y)),$$

where F is the ***free*** ***M-algebra****, given by the object $M(X)$ with multiplication $MM(X) \xrightarrow{\mu_{M(X)}} M(X)$.*

4.2.2 I-operads

Remark 4.2.3
Let $(\mathcal{C}, \otimes, k)$ be a monoidal category.

(i) *There is a functor*

$$E^\bullet : k/\mathcal{C} \longrightarrow \mathcal{C}\mathrm{AT}(\hat{\Delta}_{inj}, k/\mathcal{C}), \quad (k \xrightarrow{\eta} X) \longmapsto E^\bullet(X) : [\underline{n} \longmapsto X^{\otimes \underline{n}}],$$

where $E^{d^i} X = (\mathrm{id}_X)^{\otimes i} \otimes \eta_X \otimes (\mathrm{id}_X)^{\otimes (n-i)}$, for all $0 \leq i \leq n$.

(ii) *If the monoidal structure is symmetric, (i) extends to a functor*

$$E^\bullet : k/\mathcal{C} \longrightarrow \mathcal{C}\mathrm{AT}(I, k/\mathcal{C}), \quad (k \xrightarrow{\eta} X) \longmapsto E^\bullet(X) : [\mathbf{n} \longmapsto X^{\otimes \mathbf{n}}],$$

where $E^\sigma X \in \mathcal{C}(X^{\otimes \mathbf{n}})$ is given by permutation of the tensor factors, for all $\sigma \in I(\mathbf{n})^\times = \Sigma_n$ and $n \geq 0$.

Definition 4.2.4
Let $(\mathcal{C}, \otimes, k)$ be a symmetric monoidal category.
An ***I-operad*** *is a functor $\mathcal{O} \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{C})$, such that the induced endofunctor*

$$E/\mathcal{C} \longrightarrow E/\mathcal{C}, \quad (E \xrightarrow{\eta_X} X) \longmapsto \mathcal{O} \otimes_I E^\bullet(X) = \int^{i \in I} \mathcal{O}(i) \otimes E^i(X)$$

is a monad on $E/\mathcal{C}$.

Remark 4.2.5
Let $(\mathcal{C}, \otimes, k)$ be a symmetric monoidal category, which is cocomplete.

(i) *The* ***Day convolution*** *on $\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{C})$ is defined as the bifunctor, given by the left Kan extension*

$$X * Y = +_!(X \otimes Y) = \int^{p,q} {}^{I(\text{-}, p+q)} X_p \otimes Y_q, \qquad X, Y \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{C}).$$

Using that $(I, +, \mathbf{0})$ and $(\mathcal{C}, \otimes, k)$ are symmetric monoidal, it induces a symmetric monoidal structure, whose neutral element is given by $\mathbb{O} = {}^{I(\text{-}, \mathbf{0})} k$.

(ii) *The **substitution product** on* $\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, k/\mathcal{C})$ *is defined as the bifunctor, given by*

$$X \circ Y = X \otimes_I E^\bullet Y = \int^{\mathbf{n} \in I} X_{\mathbf{n}} \otimes E^\bullet Y, \qquad X, Y \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{C}),$$

where canonically $E^\bullet Y = Y^{*\bullet} \in \mathcal{C}\mathrm{AT}(I, \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, k/\mathcal{C}))$*, using that* $(I, +, \mathbf{0})$ *and* $(\mathcal{C}, \otimes, k)$ *and hence* $(\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{C}), *, D)$ *are symmetric monoidal.*

It induces a monoidal structure, whose neutral element is given by $\mathbb{I} = {}^{I(\text{-},\mathbf{1})}k$.

(iii) *Then the functor* $(\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, k/\mathcal{C}), \circ, \mathbb{I}) \xrightarrow{M} (\mathcal{C}\mathrm{AT}(k/\mathcal{C}), \circ, \mathrm{id})$ *is monoidal.*

One could (and should!) define I-operads as monoids in $(\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, k/\mathcal{C}), \circ, \mathbb{I})$. Although this is the more elegant way to introduce I-operads, we will work with the shorter definition above. In general it differs from the definition here, but is enough for our purposes and avoids much of the abstract nonsense required to make the definitions explicit.

In the most standard references (e.g. [JLL12] or [Fre09]) only Σ-operads are considered, i.e. they use upper definitions for the subgroupoid of isomorphisms $\Sigma = I^\times \leq I$ instead using the whole category I.

See also [MMSS01] Part III for a general treatment of Day convolution on functor categories between symmetric monoidal categories.

4.2.3 Examples of I-operads

Proposition 4.2.6
Consider the category $(\mathcal{S}et, \times, *)$.

(i) *The **associative I-operad*** $\mathcal{A}ss_1 \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)$ *is given by*

$$\mathcal{A}ss_1^{(n)} := \Sigma_n \cong \operatorname*{colim}_{\hat{\Delta}_{inj}} I(\mathbf{n}, \text{-}), \qquad n \geq 0.$$

Its multiplication map $M(\mathcal{A}ss_1) \circ M(\mathcal{A}ss_1) \xrightarrow{\mu} M(\mathcal{A}ss_1)$ *is induced by*

$$\mu : \Sigma_k \times (\Sigma_{n_1} \times ... \times \Sigma_{n_k}) \hookrightarrow \Sigma_{n_1 + ... + n_k}, \quad (f, g_1, ..., g_k) \longmapsto \bar{f} \circ (g_1 + ... + g_n),$$

where $\bar{f} \in \Sigma_{n_1 + ... + n_k}$ *is given by permutation of the k blocks in* $\mathbf{n}_1 + ... + \mathbf{n}_k$*. The unit map is induced by the isomorphism* $* \xrightarrow{\eta} \mathcal{A}ss_1^{(1)}$.

Algebras over $\mathcal{A}ss_1$ *are exactly the monoids, i.e. unital associative algebras in* $(\mathcal{S}et, \times, *)$.

(ii) *The **commutative operad*** $\mathcal{C}om_1 \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)$ *is given by*

$$\mathcal{C}om_1^{(n)} = *, \quad n \geq 0,$$

together with the canonical multiplication and unit maps.

Algebras over $\mathcal{C}om_1$ *are precisely the commutative monoids.*

There are canonical morphisms of operads $\mathcal{A}ss \twoheadrightarrow \mathcal{C}om$ *and* $\mathcal{A}ss_1 \twoheadrightarrow \mathcal{C}om_1$.

Proof. See [JLL12] 9.1.3 and 13.1.3 resp. for the corresponding Σ-operads. The proof is exactly the same in our more general setting. □

Remark 4.2.7
Let $(\mathcal{C}, \otimes, E)$ be a symmetric monoidal category.

(i) *The $\mathcal{A}ss_1$-fold coproduct of E defines the associative operad in $(\mathcal{C}, \otimes, E)$, i.e.*

$$\mathcal{A}ss_1^{(n)} = {}^{\mathcal{A}ss_1^{(n)}}E = {}^{\Sigma_n}E, \qquad n \geq 0.$$

(ii) *The $\mathcal{C}om_1$-fold coproduct of E defines the commutative operad in $(\mathcal{C}, \otimes, E)$, i.e.*

$$\mathcal{C}om_1^{(n)} = {}^{\mathcal{C}om_1^{(n)}}E = E, \qquad n \geq 0.$$

Definition 4.2.8
Let $(\mathcal{C}, \otimes, E)$ be a symmetric monoidal category.
Then we get an I-operad $\mathcal{C}om_{1,\infty}$ in the symmetric monoidal category $(s\mathcal{C}, \otimes, E)$ by setting

$$\mathcal{C}om_{1,\infty}^{(n)} = {}^{E\bullet\Sigma}E \in s\mathcal{C}, \qquad n \geq 0.$$

It is usually called the ***E_∞-operad****.*
There is a canonical factorization $\mathcal{A}ss_1 \longrightarrow \mathcal{C}om_{1,\infty} \twoheadrightarrow \mathcal{C}om_1$.

Example 4.2.9 (i) *For $(\mathcal{S}et, \times, *)$ we obtain the category $s\mathcal{S}et$-$\mathcal{C}om_{1,\infty}$ of simplicial monoids, that are commutative up to homotopy.*

The operad $\mathcal{C}om_{1,\infty}$ on $s\mathcal{S}et$ is also denoted by Γ and called the ***Barratt-Eccles operad*** *after [BE74a]. The algebras $s\mathcal{S}et$-$\mathcal{C}om_{1,\infty}$ are also called* ***E_∞-spaces****.*

In [BE74b] it is shown, that E_∞-spaces are infinite loop spaces, i.e. spaces being homotopy equivalent to an n-fold loop space, for every $n \geq 0$.

(ii) *For $(\mathcal{A}b, \otimes, \mathbb{Z})$ we obtain the category $s\mathcal{A}b$-$\mathcal{C}om_{1,\infty}$ of simplicial rings, that are commutative up to homotopy.*

4.3 Homotopy groups of E_∞-spaces

4.3.1 Abelianization of E_∞-spaces

By considering infinite loop spaces as algebras over the E_∞-operad, we are giving connectivity results of the map from stable homotopy theory to the integral homology of a space. We do not claim that the obtained result are new.

Proposition 4.3.1 (Dold, Puppe)
Let $X \in s\mathcal{A}b$ be dimensionwise free.

If X is $(k-1)$-connected, the object $\mathcal{C}om^{(r)}X = X^{\otimes r}_{\Sigma_r}$ is $c_{k,r}$-connected, where

$$c_{k,r} = \begin{cases} kr-1, & 0 \le k \le 2, \\ 2r+k-3, & k \ge 2. \end{cases}$$

Proof. See [DP61] Satz 12.1.

□

Lemma 4.3.2
Let $G \in \mathcal{G}rp$ be finite with $|G| \in k^\times$.

Let $M \in dg(k[G]\text{-Mod})$, such that G acts trivially on $H_n(M)$, for all $n < m$.

Then $M \twoheadrightarrow M_G$ is $(m-1)$-connected and $H_m(M)_G = H_m(M_G)$.

Proof. There is a spectral sequence

$$E^2_{p,q} = H_p(G, H_q(M)) \quad \Rightarrow \quad H_{p+q}(G,M),$$

which by Remark 5.1.8 can be computed as

$$E^2_{p,q} = \begin{cases} H_q(M)_G = H_q(M), & p=0,\ q<m, \\ 0, & p>0,\ q<m. \end{cases}$$

It follows that $E^\infty_{p,q} = E^2_{p,q}$, for all $q<m$, and hence $H_n(M) \xrightarrow{\sim} H_n(G,M)$, for all $n<m$. Moreover

$$H_m(G,M) = E^\infty_{0,m} = E^2_{0,m} = H_0(G, H_m(M)) = H_m(M)_G,$$

and thus $H_m(M) \twoheadrightarrow H_m(G,M)$, which proves that $H_*(M) \longrightarrow H_*(G,M)$ is $(m-1)$-connected.

There is another spectral sequence

$$E^1_{p,q} = H_q(G, M_p) \quad \Rightarrow \quad H_{p+q}(G,M),$$

which by the same argument collapses at the second page, because

$$E^1_{p,q} = \begin{cases} (M_p)_G, & q=0, \\ 0, & q>0. \end{cases} \qquad E^2_{p,q} = \begin{cases} H_p(M_G), & q=0, \\ 0, & q>0. \end{cases}$$

Hence $H_*(G,M) \xrightarrow{\sim} H_*(M_G)$ and $H_*(M) \longrightarrow H_*(G,M) \xrightarrow{\sim} H_*(M_G)$ is $(m-1)$-connected.

□

Lemma 4.3.3
Let $S \in s\mathcal{S}et_*$ be $(c-1)$-connected with $c \ge 1$ and suppose $H_*(S,\mathbb{Z}) \xrightarrow{\sim} H_*(S, \mathbb{Z}[1/(p-1)!])$, for some prime number $p>1$.

Then $\pi_n \mathbb{Z}[\mathcal{C}om_{1,\infty}(S)] \xrightarrow{\sim} \pi_n \mathbb{Z}[\mathcal{C}om_1(S)]$, for all $0 \le n \le c_{c,p}$.

Proof. For a commutative ring $k \in \mathcal{C}Ring$ we define $\tilde{k}S$ as the kernel in

$$0 \longrightarrow \tilde{k}S \longrightarrow kS \longrightarrow k \longrightarrow 0,$$

where the middle term is the free k-module generated by S and the right map is induced by the terminal map $S \longrightarrow *$. Then the composition

$$\bigoplus_{n\geq 0} B_\bullet(k, k[\Sigma_n], (\tilde{k}S)^{\otimes n}) \hookrightarrow \bigoplus_{n\geq 0} B_\bullet(k, k[\Sigma_n], (kS)^{\otimes n}) \twoheadrightarrow k(\operatorname*{colim}_{n\in I} E_\bullet\Sigma_n \times S^n)$$

is an isomorphism of simplicial k-modules, so we have naturally have $k[\mathcal{C}om_{1,\infty}(S)] \cong \mathcal{C}om_{1,\infty}(\tilde{k}S)$, where $\mathcal{C}om_{1,\infty}$ on the right is considered as the Σ-operad on the symmetric monoidal category of simplicial k-modules. Now we let $k = \mathbb{Z}[1/(p-1)!]$ and consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C}om_{1,\infty}(\tilde{\mathbb{Z}}S) \arrow[r] \arrow[d] & \mathcal{C}om_{1,\infty}(\tilde{k}S) \arrow[r] \arrow[d] & \mathcal{C}om_{1,\infty}^{<p}(\tilde{k}S) \arrow[d] \\
\mathcal{C}om_{1}(\tilde{\mathbb{Z}}S) \arrow[r] & \mathcal{C}om_{1}(\tilde{k}S) \arrow[r] & \mathcal{C}om_{1}^{<p}(\tilde{k}S),
\end{tikzcd}$$

where an upper index $< p$ means that we only take the summands $0 \leq r < p$. By assumption the map $\tilde{\mathbb{Z}}S \xrightarrow{\simeq} \tilde{k}S$ is a weak equivalence. In particular it is a cofibrant resolution of simplicial abelian groups and Lemma 8.3.10 implies that the lower left horizontal map is a weak equivalence. Moreover the Künneth formula implies that $(\tilde{\mathbb{Z}}S)^{\otimes r} \xrightarrow{\simeq} (\tilde{k}S)^{\otimes r}$ is a weak equivalence, for all $r \geq 0$. Using the spectral sequence of Remark 8.2.2 for the group Σ_r, we see that also $(\tilde{\mathbb{Z}}S)^{\otimes r}_{h\Sigma_r} \xrightarrow{\simeq} (\tilde{k}S)^{\otimes r}_{h\Sigma_r}$ is a weak equivalence, for all $r \geq 0$. Summing up over all $r \geq 0$, we get that also the upper left horizontal map is a weak equivalence. As $r! \in k^\times$, for all $0 \leq r < p$, the right vertical map is a weak equivalence, because by Remark 5.1.8 we have

$$\mathcal{C}om^{(r)}_{1,\infty}(\tilde{k}S) = (\tilde{k}S)^{\otimes r}_{h\Sigma_r} = B_\bullet(k, k[\Sigma_r], (\tilde{k}S)^{\otimes r}) \overset{\simeq}{\twoheadrightarrow} k\otimes_{k[\Sigma_r]}(\tilde{k}S)^{\otimes r} = (\tilde{k}S)^{\otimes r}_{\Sigma_r} = \mathcal{C}om^{(r)}_{1}(\tilde{k}S).$$

Moreover the Künneth formula implies that $(\tilde{k}S)^{\otimes r}$ and hence $\mathcal{C}om^{(r)}_{1,\infty}(\tilde{k}S) = (\tilde{k}S)^{\otimes r}_{h\Sigma_r}$ is $(cp-1)$-connected, for all $r \geq p$. It follows that the upper right horizontal map is $(cp-1)$-connected. Similarly Proposition 4.3.1 implies that the lower right horizontal map is $c_{c,p}$-connected. It follows that π_n of the left vertical map is an isomorphism, for all $0 \leq n \leq c_{c,p} \leq cp - 1$.

□

Proposition 4.3.4
Let $S \in sSet_*$ be $(c-1)$-connected and suppose $H_*(S,\mathbb{Z}) \xrightarrow{\sim} H_*(S, \mathbb{Z}[1/(p-1)!])$, for some prime number $p > 1$.

Then the map $B\mathcal{C}om_{1,\infty}(S) \longrightarrow B\mathcal{C}om_1(S)$ is $(2p+c-3)$-connected.

If S is connected, then $\mathcal{C}om_{1,\infty}(S) \longrightarrow \mathcal{C}om_1(S)$ is $(2p+c-4)$-connected.

Proof. By using the map of I-operads $\mathcal{A}ss_1 \longrightarrow \mathcal{C}om_{1,\infty}$ induced by th inclusion of the 0-skeleton $\mathcal{A}ss_1 = (\mathcal{C}om_{1,\infty})_0 \hookrightarrow \mathcal{C}om_{1,\infty}$, we can consider $\mathcal{C}om_{1,\infty}(S)$ as a simplicial monoid, which moreover is free in every dimension. Hence by Proposition 7.3.33 the map to the group completion $\mathcal{C}om_{1,\infty}(S) \longrightarrow \mathcal{G}rp(\mathcal{C}om_{1,\infty}(S))$ induces a weak equivalence $B\mathcal{C}om_{1,\infty}(S) \xrightarrow{\simeq} B\mathcal{G}rp(\mathcal{C}om_{1,\infty}(S))$. Similarly by Proposition 7.3.33 we get a weak equivalence $B\mathcal{C}om_1(S) \xrightarrow{\simeq} B\mathcal{G}rp(\mathcal{C}om_1(S))$. Hence, once we have shown that

$$H_*(B\mathcal{C}om_{1,\infty}(S), \mathbb{Z}) \longrightarrow H_*(B\mathcal{C}om_1(S), \mathbb{Z})$$

is $(2p+c-3)$-connected, the Whitehead Theorem for simplicial groups 8.2.6 implies that $\mathcal{G}rp(\mathcal{C}om_{1,\infty}(S)) \longrightarrow \mathcal{G}rp(\mathcal{C}om_1(S))$ is $(2p+c-4)$-connected and thus $B\mathcal{C}om_{1,\infty}(S) \longrightarrow B\mathcal{C}om_1(S)$ is $(2p+c-3)$-connected. Moreover, if S is connected, so are also $\mathcal{C}om_{1,\infty}(S)$ and $\mathcal{C}om_1(S)$, and thus by Proposition 7.3.33 (ii) the vertical maps in the commutative square

$$\begin{tikzcd} \mathcal{C}om_{1,\infty}(S) \arrow[r] \arrow[d, "\simeq"'] & \mathcal{C}om_1(S) \arrow[d, "\simeq"] \\ \mathcal{G}rp(\mathcal{C}om_{1,\infty}(S)) \arrow[r] & \mathcal{G}rp(\mathcal{C}om_1(S)), \end{tikzcd}$$

are weak equivalences, which proves that also $\mathcal{C}om_{1,\infty}(S) \longrightarrow \mathcal{C}om_1(S)$ is $(2p+c-4)$-connected.

Now there is a commutative diagram

$$\begin{tikzcd} \mathcal{C}om_{1,\infty}(\Sigma S) \arrow[r, "\simeq"] \arrow[d] & B\mathcal{C}om_{1,\infty}(S) \arrow[d] \\ \mathcal{C}om_1(\Sigma S) \arrow[r, "\sim"] & B\mathcal{C}om_1(S), \end{tikzcd} \tag{4.1}$$

where the horizontal maps are induced by the terminal maps $S \xrightarrow{t} *$ via

$$(\mathcal{O}(t^{\vee(i-1)} \vee \mathrm{id} \vee t^{\vee(n-i)}))_{1\le i\le n} : \mathcal{O}(S^{\vee n}) = \mathcal{O}((S^1)_n \wedge S) \longrightarrow \mathcal{O}(X)^{\times n} = B_n\mathcal{O}(X), \qquad n \geq 0,$$

for $\mathcal{O} = \mathcal{C}om_{1,\infty}$ and $\mathcal{O} = \mathcal{C}om_1$ respectively. The lower map is an isomorphism, while the upper map is a weak equivalence by [BE74a] Lem. 4.6 or [Sch07] Lem. 3.2 combined with Prop. 4.5 loc. cit.

(i) If $c \geq 1$, then ΣS is $(c+1)$-connected and hence $\mathbb{Z}[\mathcal{C}om_{1,\infty}(\Sigma S)] \longrightarrow \mathbb{Z}[\mathcal{C}om_1(\Sigma S)]$ is $(c_{c+1,p}-1)$-connected by Lemma 4.3.3. Using (4.1) it follows that $\mathbb{Z}B\mathcal{C}om_{1,\infty}(S) \longrightarrow \mathbb{Z}B\mathcal{C}om_1(S)$ is $(c_{c+1,p}-1)$-connected, which proves the result in this case as $c_{c+1,p}-1 = 2p+c-3$.

(ii) If $c = 0$, then ΣS is connected and thus $\mathcal{C}om_{1,\infty}(\Sigma S) \longrightarrow \mathcal{C}om_1(\Sigma S)$ is $(2p+c-3)$-connected by (i). By taking the loop space we get that $\mathcal{C}om_{1,\infty}(\Sigma S) \longrightarrow \mathcal{C}om_1(\Sigma S)$ is $(2p+c-4)$-connected. Then (4.1) implies that equivalently $B\mathcal{C}om_{1,\infty}(S) \longrightarrow B\mathcal{C}om_1(S)$ is $(2p+c-4)$-connected.

□

Corollary 4.3.5
Let $k \in \mathcal{C}Ring$ with $(p-1)! \in k^\times$, for some prime number $p > 1$.
Suppose $X \in s(k\text{-Mod})$ is free in every dimension and $(c-1)$-connected with $c \geq 0$.
Then $B\mathcal{C}om_{1,\infty}(X) \longrightarrow B\mathcal{C}om_1(X)$ is $(2p+c-3)$-connected.

Proof. Let $E_\bullet X \xrightarrow{\simeq} X$ be the functorial cofibrant replacement of Corollary 7.2.32 induced by the free/forgetful functor adjunction

$$s(k\text{-Mod})(kX, Y) = s\mathcal{S}et_*(X, U(Y)).$$

Then using the Quillen spectral sequence of Theorem 7.3.18 it suffices to check that $B\mathcal{C}om_{1,\infty}(E_pX) \longrightarrow B\mathcal{C}om_1(E_pX)$ is $(2p+c-3)$-connected, for all $p \geq 0$. Since $E_pX = \tilde{k}U(E_{p-1}X)$, this map is isomorphic to

$$kB\mathcal{C}om_{1,\infty}U(E_{p-1}X) \longrightarrow kB\mathcal{C}om_1U(E_{p-1}X), \qquad p \geq 0.$$

As the functor $\tilde{k} \circ U$ preserves connectivity and X is $(c-1)$-connected, so is also $UE_{p-1}X$. Moreover $H_*(UE_{p-1}X, \mathbb{Z}) \xrightarrow{\sim} H_*(UE_{p-1}X, k)$ is an isomorphism by Lemma 4.3.6, because $E_{p-1}X \in s(k\text{-Mod})$. So we can apply Proposition 4.3.4 to conclude the proof.

□

4.3.2 Homology of simplicial abelian groups

Purpose of this section is to verify that homology behaves well with localization. Again we do not claim the results are new. We assume that all the given statements also arise in Bousfield's theory of localizations [BK72].

Lemma 4.3.6
Let $k \leq \mathbb{Q}$ be a subring and $X \in s\mathcal{A}b$.
Then $H_*(X, k) \xrightarrow{\sim} H_*(X \otimes k, k) \xleftarrow{\sim} H_*(X \otimes k, \mathbb{Z})$, considered as simplicial groups.

Proof. Let $F \xrightarrow{\simeq} X$ be a dimensionwise free replacement of X as we would get from Corollary 7.2.32 using the free/forgetful functor adjunction

$$\mathcal{A}b(kX, Y) = s\mathcal{S}et(X, U(Y)).$$

As homology is a homotopy invariant, it suffices to prove the Lemma for F.
For $M = \mathbb{Z}$ or $M = k$, Remark 8.2.2 (i) provides a spectral sequence

$$E^2_{p,q} = \pi_p \underline{H}_q(F, M) \quad \Rightarrow \quad H_{p+q}(X, M),$$

where $\underline{H}_q(F, M)_p = H_q(F_p, M)$, for all $p, q \geq 0$. Thus we can assume that F is a constant free abelian group. Since $k \leq \mathbb{Q}$, the module $F \otimes k$ is flat over $\mathbb{Z}$ and thus we have isomorphisms

$$\begin{tikzcd}
H_*(F,k) \arrow[r] \arrow[d, equal] & H_*(F\otimes k, k) \arrow[d, equal] & H_*(F \otimes k, \mathbb{Z}) \arrow[l] \arrow[d, equal] \\
(\Lambda_* F) \otimes k \arrow[r, "\sim"] & (\Lambda_*(F \otimes k)) \otimes k & \Lambda_*(F \otimes k), \arrow[l, "\sim"']
\end{tikzcd}$$

which proves the Lemma.

□

Proposition 4.3.7
Let $k \leq \mathbb{Q}$ be a subring and $X \in s\mathcal{S}et_*$ be connected with $H_*(X, \mathbb{Z}) \xrightarrow{\sim} H_*(X, k)$.
Then we have natural weak equivalences

$$\mathcal{C}om_1(X) \xrightarrow{\simeq} \mathcal{G}rp(\mathcal{C}om_1(X)) \xrightarrow{\simeq} \mathcal{G}rp(\mathcal{C}om_1(X)) \otimes k = \tilde{k}X.$$

Proof. As X is connected, so is $\mathcal{C}om_1 X$ and thus $\mathcal{C}om_1 X \xrightarrow{\simeq} \mathcal{G}rp(\mathcal{C}om_1 X)$ is a weak equivalence by Proposition 7.3.33. The proof for the second map takes a bit longer. Defining $\tilde{k}X$ as the kernel in

$$0 \longrightarrow \tilde{k}X \longrightarrow kX \longrightarrow k \longrightarrow 0,$$

we get a weak equivalence $\tilde{\mathbb{Z}}X \xrightarrow{\simeq} \tilde{k}X$ by assumption on X and thus the left map in the commutative diagram

$$\begin{tikzcd}
\mathcal{C}om_1(\tilde{k}X) \arrow[r, hook] \arrow[d, "\simeq"'] & \mathcal{C}om_1(kX) \arrow[r, two heads] \arrow[d] & k[\mathcal{C}om_1 X] \arrow[d] \\
\mathcal{C}om_1(\tilde{\mathbb{Z}}X) \arrow[r, hook] & \mathcal{C}om_1(\mathbb{Z}X) \arrow[r, two heads] & \mathbb{Z}[\mathcal{C}om_1 X]
\end{tikzcd}$$

is a weak equivalence by Lemma 8.3.10. As the horizontal compositions are isomorphisms, it follows that the right vertical map is a weak equivalence. Hence by the Künneth formula we get weak equivalences

$$\mathbb{Z}B_n\mathcal{C}om_1 X = \mathbb{Z}[\mathcal{C}om_1 X]^{\otimes n} \xrightarrow{\simeq} k[\mathcal{C}om_1 X]^{\otimes n} = kB_n\mathcal{C}om_1 X, \qquad n \geq 0,$$

and Corollary 7.3.19 implies that $\mathbb{Z}B\mathcal{C}om_1 X \xrightarrow{\simeq} kB\mathcal{C}om_1 X$ is a weak equivalence. Equivalently the left vertical map in the commutative diagram

$$\begin{tikzcd}
H_*(B\mathcal{C}om_1 X, \mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & H_*(B\mathcal{G}rp(\mathcal{C}om_1 X), \mathbb{Z}) \arrow[r] \arrow[d] & H_*(B\mathcal{G}rp(\mathcal{C}om_1 X) \otimes k, \mathbb{Z}) \arrow[d, "\wr"] \\
H_*(B\mathcal{C}om_1 X, k) \arrow[r, "\sim"] & H_*(B\mathcal{G}rp(\mathcal{C}om_1 X), k) \arrow[r, "\sim"] & H_*(B\mathcal{G}rp(\mathcal{C}om_1 X) \otimes k, k)
\end{tikzcd}$$

is an isomorphism. Using the weak equivalence $\mathcal{C}om_1 X \xrightarrow{\simeq} \mathcal{G}rp(\mathcal{C}om_1 X)$ we see that the two left horizontal maps are isomorphisms, while by Lemma 4.3.6 again also the lower right horizontal map and right vertical map are isomorphisms. Hence also the upper right horizontal map is an isomorphism and the Whitehead Theorem for simplicial groups 8.2.6 implies that $\mathcal{G}rp(\mathcal{C}om_1 X) \xrightarrow{\simeq} \mathcal{G}rp(\mathcal{C}om_1 X) \otimes k$ is a weak equivalence, because again both objects are connected. □

Corollary 4.3.8
Let $k \leq \mathbb{Q}$ be a subring and $X \in s(k\text{-Mod})$ be connected.

Considering $(X, 0)$ as a pointed simplicial set, we have natural weak equivalences

$$\mathcal{C}om_1(X,0) \xrightarrow{\simeq} \mathcal{G}rp(\mathcal{C}om_1(X,0)) \xrightarrow{\simeq} \mathcal{G}rp(\mathcal{C}om_1(X,0)) \otimes k.$$

Proof. As X is connected, the adjunction counit (see Proposition 7.2.9) is a weak equivalence

$$\varepsilon_X : S^1 \wedge \underline{s\mathcal{S}et_*}(S^1, X) \xrightarrow{\simeq} X.$$

Hence in the commutative square of simplicial group rings

$$\begin{tikzcd}
k[B\underline{s\mathcal{S}et_*}(S^1, X)] \arrow[r, equal] \arrow[d] & k[S^1 \wedge \underline{s\mathcal{S}et_*}(S^1, X)] \arrow[r, "\varepsilon", "\simeq"'] \arrow[d] & k[X] \arrow[d] \\
\mathbb{Z}[B\underline{s\mathcal{S}et_*}(S^1, X)] \arrow[r, equal] & \mathbb{Z}[S^1 \wedge \underline{s\mathcal{S}et_*}(S^1, X)] \arrow[r, "\varepsilon", "\simeq"'] & \mathbb{Z}[X]
\end{tikzcd}$$

the left vertical map is a weak equivalence by Lemma 4.3.6, because $\underline{s\mathcal{S}et_*}(S^1, X) \in s(k\text{-Mod})$. It follows that the right vertical map is a weak equivalence and we can apply Proposition 4.3.7.

$\square$

4.3.3 Homotopy groups of connected simplicial k-modules

Finally we are able to prove the advertised link between homotopy and homology using the Hurewicz map.

Proposition 4.3.9
Let $k \in \mathcal{C}Ring$ with $(p-1)! \in k^\times$, for some prime number $p > 1$.
Suppose $X \in s(k\text{-Mod})$ is $(c-1)$-connected and free in every dimension.
Then the map below is an isomorphism, for all $0 \leq n \leq c_{c,r}$ (cf. Proposition 4.3.1).

$$\pi_n X \xrightarrow{\iota_1} P\pi_n(\mathcal{C}om_1 X) := \ker\left(\pi_n(\mathcal{C}om_1(X)) \underset{(\eta\times\mathrm{id})_* + (\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} \pi_n(\mathcal{C}om_1(X \times X)) \right).$$

Here δ is the diagonal map and $0 \xrightarrow{\eta} X$ is the initial map of chain complexes.

Proof. First note that the inclusion

$$X \times X = X \oplus X \xrightarrow{\iota_1 + \iota_2} \mathcal{C}om_1(X) \otimes \mathcal{C}om_1(X),$$

extends to a natural isomorphism of simplicial commutative k-algebras $\mathcal{C}om_1(X \times X) \xrightarrow{\sim} \mathcal{C}om_1(X) \otimes \mathcal{C}om_1(X)$, because $\mathcal{C}om_1$ as a left adjoint commutes with coproducts, which are given by the tensor product in the category of commutative rings. Under this isomorphism, the map δ corresponds to the map $\mathcal{C}om_1(X) \longrightarrow \mathcal{C}om_1(X) \otimes \mathcal{C}om_1(X)$, given by $\delta = \eta \otimes \mathrm{id} + \mathrm{id} \otimes \eta$ on X. Using the commutative square

$$\begin{tikzcd}
\pi_* X \arrow[r, "\iota_1"] \arrow[d, equal] & P\pi_*(\mathcal{C}om_1 X) \arrow[d, hook] \\
\pi_* X & \mathcal{C}om_1(X), \arrow[l, "\pi_1"']
\end{tikzcd}$$

it suffices to check that $P\pi_n(\mathcal{C}om_1 X) \xrightarrow{\pi_1} \pi_n X$ is injective, for all $0 \leq n \leq c_{c,p}$. Let $a \in P\pi_n(\mathcal{C}om_1 X)$ and $0 \leq n \leq c_{c,p}$. Then there are elements $a_r \in \pi_n \mathcal{C}om_1^{(r)}(X)$, for $r \geq 0$, such that $a = \sum_{r \geq 0} a_r$. For every $r \geq p$ we have $a_r = 0$, because $0 \leq n \leq c_{c,p} \leq c_{c,r}$ and $\mathcal{C}om_1^{(r)}(X)$ is $c_{c,r}$-connected by Proposition 4.3.1. So $a = a_0 + ... + a_{p-1}$. By definition of the map δ, the following diagram commutes

$$\begin{tikzcd}
\pi_n \mathcal{C}om_1(X) \arrow[rr, "r\cdot\pi_r"] \arrow[d, "\delta"] & & \pi_n \mathcal{C}om_1^{(r)}(X) \\
\pi_n(\mathcal{C}om_1(X) \otimes \mathcal{C}om_1(X)) \arrow[d, equal] & & \\
\bigoplus_{r,s\geq 0} \pi_n(\mathcal{C}om_1^{(r)}(X) \otimes \mathcal{C}om_1^{(s)}(X)) \arrow[rr, "\pi_{1,r-1}"] & & \pi_n(\mathcal{C}om_1^{(1)}(X) \otimes \mathcal{C}om_1^{(r-1)}(X)). \arrow[uu, "\mu|"']
\end{tikzcd}$$

So we get

$$r \cdot a_r = r \cdot \pi_r(a) = \mu| \circ \pi_{1,r-1} \circ \delta(a) = \mu| \circ \pi_{1,r-1} \circ (1 \otimes a + a \otimes 1) = 0,$$

which proves that $a_2, ..., a_{p-1} = 0$, because $(p-1)! \in k^\times$. Hence $a = a_0 + a_1$ and

$$2a = \mu \circ (\eta \otimes \mathrm{id} + \mathrm{id} \otimes \eta)(a) = \mu \circ \delta(a) = a_0 + 2a_1 = 2a - a_0$$

implies $a_0 = 0$. It follows that $\pi_1 a = a_1 = 0$ implies $a = 0$, which finally proves injectivity of the maps

$$P\pi_n(\mathcal{C}om_1 X) \overset{\pi_1}{\hookrightarrow} \pi_n X, \qquad 0 \le n \le c_{c,r}.$$

□

There is a similar result in the setting of chain complexes.

Proposition 4.3.10
Let $k \in \mathcal{C}Ring$ with $(p-1)! \in k^\times$, for some prime number $p > 1$.
Let $X \in dg(k\text{-Mod})$ with $X_n = 0$, for all $n < c$, for some $c \ge 0$.
Then the map below is an isomorphism, for all $0 \le n < pc$.

$$H_n X \xrightarrow{\iota_1} PH_n(\mathcal{C}om_1 X) := \ker\left(H_n(\mathcal{C}om_1(X)) \underset{(\eta\times\mathrm{id})_* + (\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(\mathcal{C}om_1(X \times X)) \right).$$

Here δ is the diagonal map and $0 \xrightarrow{\eta} X$ is the initial map of chain complexes.

Proof. The assumption on X implies that $\mathcal{C}om^{(r)}(X) = X^{\otimes r}_{\Sigma_r}$ is not only $c_{c,r}$-connected, but even trivial in dimensions $< rc$. The rest of the proof is exactly the same as that of Proposition 4.3.9.

□

4.3.4 Homotopy groups of connected E_∞-spaces

We are giving a analogous results in the case of E_∞-spaces. Like those in the preceding section, the following results are inspired by Beilinson's first Theorem in [Bei14]. According to Beilinson, integral statements like these have not been present in literature before.

Proposition 4.3.11
Let $G \in s\mathcal{G}rp$ and $k \in \mathcal{C}Ring$.
Then the Hurewicz map induces a map

$$h = \eta_G - 1 : \pi_* G \longrightarrow P\pi_*(k[G]) := \ker\left(\pi_* k[G] \underset{(\eta\times\mathrm{id})_* + (\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} \pi_* k[G \times G] \right),$$

where $G \xrightarrow{\delta} G \times G$ is the diagonal and $1 \xrightarrow{\eta} G$ is the unit map.
The same holds for connected $G \in s\mathcal{S}et\text{-}\mathcal{A}ss_1$.

Proof. Let $[g] \in \pi_n G$, i.e. $g \in \ker(N_n G \xrightarrow{d} N_{n-1}G)$, for some $n \geq 0$. Then

$$\delta(g-1) = (g,g) - (1,1) = ((g,1)-1) + ((1,g)-1) + ((g,1)-1)\cdot((1,g)-1).$$

Since g is a cycle in $N_n G$, so are the first two summands

$$(\mathrm{id}\times\eta)(g-1) = (g,1)-1, \quad (\eta\times\mathrm{id})(g-1) = (1,g)-1$$

in $N_n k[G\times G]$. Hence by Proposition 8.1.3 (i) the third summand is zero in $\pi_n k[G\times G]$. It follows that $\delta_* \circ h = ((\eta\times\mathrm{id})_* + (\mathrm{id}\times\eta)_*)\circ h$, which proves the assertion.

When $G \in s\mathcal{S}et\text{-}\mathcal{A}ss_1$ is connected, then can take a dimensionwise free replacement $E_\bullet G \xrightarrow{\simeq} G$ like in Corollary 7.2.32 using the free/forgetful functor adjunction

$$s\mathcal{S}et\text{-}\mathcal{A}ss_1(\mathcal{A}ss_1(X), Y) = s\mathcal{S}et(X, U(Y)),$$

and Proposition 7.3.33 yields a weak equivalence to its group completion $E_\bullet G \xrightarrow{\simeq} \mathcal{G}rp(E_\bullet G)$. Hence using the commutative diagram

$$\begin{tikzcd}
\pi_* G \arrow[d, "h"'] & \pi_* E_\bullet(X) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "h"'] & \pi_* \mathcal{G}rp(E_\bullet G) \arrow[d, "h"'] \\
H_*(G,k) & H_*(E_\bullet G, k) \arrow[l, "\sim"'] \arrow[r, "\sim"] & H_*(\mathcal{G}rp(E_\bullet), k),
\end{tikzcd}$$

it follows that since the right vertical map maps into the primitive part, so do the middle and the left vertical maps.

□

Proposition 4.3.12
Let $X \in s\mathcal{S}et_*\text{-}\mathcal{C}om_1$ be $(c-1)$-connected, for some $c > 0$.
Suppose $H_*(X,\mathbb{Z}) \xrightarrow{\sim} H_*(X,k)$, where $k = \mathbb{Z}[1/(p-1)!]$, for some $p > 1$.
Then the Hurewicz map induces an isomorphism, for all $0 \leq n \leq c_{c,p}$,

$$h = \eta_X - 1 : \pi_n X \xrightarrow{\sim} PH_n(X,k) := \ker\left(H_n(X,k) \underset{(\eta\times\mathrm{id})_* + (\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(X\times X, k) \right),$$

where $X \xrightarrow{\delta} X\times X$ is the diagonal and $1 \xrightarrow{\eta} X$ is the unit map.

Proof. For $n = 0$, the maps $\delta_*, (\eta\times\mathrm{id})_*$ and $(\mathrm{id}\times\eta)_*$ coincide and are isomorphisms, which proves the statement for $n = 0$. So we can assume $n > 0$. As the statement is void, for $c = 0$, we may further assume $c > 0$. Let $1 \in X$ be the unit and considering $(X,1)$ as a pointed simplicial set, we let $\tilde{k}X$ be defined as the kernel in

$$0 \longrightarrow \tilde{k}X \longrightarrow kX \longrightarrow k \longrightarrow 0,$$

where kX in the middle means the free k-module generated by X. Then the composition

$$\mathcal{C}om_1(\tilde{k}X) = \bigoplus_{r\geq 0} (\tilde{k}X)^{\otimes r}_{\Sigma_r} \hookrightarrow \bigoplus_{r\geq 0} (kX)^{\otimes r}_{\Sigma_r} = k \coprod_{r\geq 0} X^r_{\Sigma_r} \twoheadrightarrow k \operatorname*{colim}_{\mathbf{r}\in I} E^{\mathbf{r}} X = k\mathcal{C}om_1 X$$

is an isomorphism and using Proposition 4.3.11 we get commutative diagram

$$\begin{tikzcd}
\pi_n\mathcal{C}om_1X \arrow[r, "h"] \arrow[drr, "\pi_1\circ h"'] & PH_n(\mathcal{C}om_1X,k) \arrow[r, hook] & \pi_n\mathcal{C}om_1(\tilde{k}X) \arrow[d, "\pi_1"] \\
& & \pi_n\tilde{k}X.
\end{tikzcd}$$

We claim that $\mathcal{C}om_1X \xrightarrow{\pi_1\circ h} \tilde{k}X$ is a weak equivalence and that π_1 induces isomorphisms

$$\pi_1| : PH_n(\mathcal{C}om_1X,k) \xrightarrow{\sim} \pi_n\tilde{k}X, \qquad 1\le n\le c_{c,p}.$$

This implies that also h is an isomorphism in that range and using the free/forgetful functor adjunction

$$s\mathcal{S}et_*\text{-}\mathcal{C}om_1(\mathcal{C}om_1(X),Y) = s\mathcal{S}et_*(X,U(Y)),$$

the commutative diagram

$$\begin{tikzcd}
\pi_nX \arrow[r, hook, "\eta_{U(X)}"] \arrow[d, "h"] & \pi_n\mathcal{C}om_1X \arrow[r, two heads, "U(\varepsilon_X)"] \arrow[d, "h", "\wr"'] & \pi_nX \arrow[d, "h"] \\
PH_n(X,k) \arrow[r, hook, "\eta_{U(X)}"] & PH_n(\mathcal{C}om_1X,k) \arrow[r, two heads, "U(\varepsilon_X)"] & PH_n(X,k)
\end{tikzcd}$$

then implies the assertion. Indeed since $X \overset{\eta_{U(X)}}{\hookrightarrow} \mathcal{C}om_1X \overset{U(\varepsilon_X)}{\twoheadrightarrow} X$ is the identity on X, the right vertical map is a retract of the middle one and thus also an isomorphism.

- First we prove that $\pi_1\circ h$ is a weak equivalence. For every tuple $(x_1,...,x_n)\in X^n$, we have in $\mathcal{C}om_1(\tilde{k}X)$

$$\begin{aligned}
h(x_1,...,x_n) &= x_1\cdot ...\cdot x_n - 1 = ((x_1-1)+1)\cdot ...\cdot((x_n-1)+1)-1\\
&= \sum_{a\in\{0,1\}^n}(x_1-1)^{a_1}\cdot ...\cdot(x_n-1)^{a_n}-1\\
&= \sum_{\emptyset\neq\{i_1<...<i_m\}\subset\mathbf{n}}(x_{i_1}-1)\otimes ...\otimes(x_{i_m}-1),
\end{aligned}$$

  where in the last term $1\in X$ is the unit (which coincides with the unit in $\mathcal{C}om_1(\tilde{X})$). Hence we have

$$\pi_1\circ h(x_1,...,x_n) = (x_1-1)+...+(x_n-1)\in\tilde{k}X, \qquad (x_1,...,x_n)\in X^n, \quad n\ge 0,$$

  showing that $\mathcal{C}om_1X \xrightarrow{\pi\circ h} \tilde{k}X$ is a homomorphism of simplicial monoids. Using that $\tilde{k}X\in s(k\text{-Mod})$, it factors as

$$\mathcal{C}om_1X \longrightarrow \mathcal{G}rp(\mathcal{C}om_1X) \longrightarrow \mathcal{G}rp(\mathcal{C}om_1X)\otimes k = kX/k\cdot 1_X \xrightarrow{\mathrm{id}-1_X} \tilde{k}X,$$

  where we write $1_X = 1\in X$ to avoid confusion. Since X is connected, the first two maps are weak equivalences by Proposition 4.3.7 and the third map is an isomorphism with inverse given by the canonical composition $\tilde{k}X \hookrightarrow kX \twoheadrightarrow kX/k\cdot 1_X$.

- Similarly as in the proof of Proposition 4.3.9 one shows that π_1 is an isomorphism in the range $1 \leq n \leq c_{c,p}$. However the diagonal δ here is slightly different, what requires some attention. Let

$$a \in \ker \pi_1 \subset PH_n(\mathcal{C}om_1 X, k) \subset H_n(\mathcal{C}om_1 X, k) \cong \bigoplus_{r \geq 0} \pi_n(\mathcal{C}om_1^{(r)}(\tilde{k}X)).$$

We write $a = \sum_{r \geq 0} a_r$ corresponding to the direct sum decomposition. First note that the maps $\delta, \eta \times \mathrm{id}$ and $\mathrm{id} \times \eta$ are compatible with unit and counit of the bialgebra $\mathcal{C}om_1 X$, which implies that the induced maps preserve the direct decompotisition $k \oplus \mathcal{C}om(\tilde{k}X) \longrightarrow k \oplus \mathcal{C}om(\tilde{k}X \oplus \tilde{k}X)$. As all of the three maps induce the identity on k, it follows that $a_0 = 0$. Moreover we have $a_1 = \pi_1(a) = 0$. Next we define a filtration Γ on $\mathcal{C}om_1(\tilde{k}X)$ by setting

$$\Gamma_m \mathcal{C}om_1(\tilde{k}X) = \bigoplus_{r \geq m} \mathcal{C}om_1^{(m)}(\tilde{k}X) = \mathcal{C}om(\tilde{k}X)^m, \qquad m \geq 0,$$

and similarly on $\mathcal{C}om_1(\tilde{k}X \oplus \tilde{X})$. Note that this is precisely the lower central series Γ for the associative algebra $\mathcal{C}om(\tilde{k}X)$ of Definition 3.2.6. As the three maps preserve the augmentation ideal $\mathcal{C}om(\tilde{k}X)$, they also preserve the filtration. Since we have

$$\delta(x-1) = x \otimes x - 1 \otimes 1 = 1 \otimes (x-1) + (x-1) \otimes 1 + (x-1) \otimes (x-1), \qquad x \in X,$$

it follows that the algebra homomorphism $\mathrm{gr}^\Gamma \mathcal{C}om_1(\tilde{k}X) \xrightarrow{\mathrm{gr}\delta} \mathrm{gr}^\Gamma \mathcal{C}om_1(\tilde{k}X \oplus \tilde{k}X) \cong \mathrm{gr}^\Gamma \mathcal{C}om_1(\tilde{k}X) \otimes \mathrm{gr}^\Gamma \mathcal{C}om_1(\tilde{k}X)$ is isomorphic to the δ of Proposition 4.3.9. So using the commutative diagram

$$\begin{tikzcd}
\pi_n \Gamma_m \mathcal{C}om_1(\tilde{k}X) \arrow[r, "m \cdot \pi_m"] \arrow[d, "\delta"] & \pi_n \mathcal{C}om_1^{(m)}(X) \\
\pi_n \Gamma_m \mathcal{C}om_1(\tilde{k}X \oplus \tilde{k}X) \arrow[d] & \\
\pi_n \mathrm{gr}^\Gamma_m \mathcal{C}om_1(\tilde{k}X \oplus \tilde{k}X) \arrow[d, equal] & \\
\bigoplus_{r+s=m} \pi_n(\mathcal{C}om_1^{(r)}(\tilde{k}X) \otimes \mathcal{C}om_1^{(s)}(\tilde{k}X)) \arrow[r, "\pi_{1,m-1}"] & \pi_n(\mathcal{C}om_1^{(1)}(X) \otimes \mathcal{C}om_1^{(m-1)}(X)), \arrow[uuu, "\mu|"']
\end{tikzcd}$$

we can prove that $a_m = 0$, by induction on $0 < m < p$. Indeed the statement holds for $m = 1$ and supposing it holds for some $0 < m < p$, then $\sum_{r \geq m} a_r = a \in PH_n(\mathcal{C}om_1 X, k) \cap \Gamma_m H_n(\mathcal{C}om_1 X, k)$ and the upper diagram shows

$$m \cdot a_m = m \cdot \pi_m(a) = \mu \circ \pi_{1,m-1} \circ \delta(a) = \mu \circ \pi_{1,m-1}(1 \otimes a + a \otimes 1) = 0,$$

which implies $a_m = 0$, as $m \in k^\times$. We have proven that

$$a \in \Gamma_m H_n(\mathcal{C}om_1 X, k) = \pi_n \bigoplus_{r \geq p} \mathcal{C}om_1^{(p)}(\tilde{k}X),$$

which by Propostion 4.3.1 is zero in the range $1 \le n \le c_{c,p}$, because with X also $\tilde{k}X$ is $(c-1)$-connected. This concludes the proof that $PH_n(\mathcal{C}om_1 X, k) \xrightarrow{\pi_1} \tilde{k}X$ is injective.

$\square$

Corollary 4.3.13
Let $X \in s\mathcal{S}et_*\text{-}\mathcal{C}om_{1,\infty}$ be $(c-1)$-connected, for some $c > 0$.
Suppose $H_*(X, \mathbb{Z}) \xrightarrow{\sim} H_*(X, k)$, where $k = \mathbb{Z}[1/(p-1)!]$, for some $p > 1$.
Then the Hurewicz map induces an isomorphism, for all $0 \le n \le \min\{c_{c,p}, 2p + c - 4\}$,

$$h = \eta_X - 1 : \pi_n X \xrightarrow{\sim} PH_n(X, k) := \ker\left(H_n(X, k) \underset{(\eta\times\mathrm{id})_* + (\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(X \times X, k) \right),$$

where $X \xrightarrow{\delta} X \times X$ is the diagonal and $1 \xrightarrow{\eta} X$ is the unit map.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\pi_n \mathcal{C}om_1 X \arrow[d, "h"', "\wr"] & \pi_n \mathcal{C}om_{1,\infty} X \arrow[l] \arrow[r, two heads, "U(\varepsilon_X)"] \arrow[d, "h"] & \pi_n X \arrow[d, "h"] \\
PH_n(\mathcal{C}om_1 X, k) & PH_n(\mathcal{C}om_{1,\infty} X, k) \arrow[l] \arrow[r, two heads, "U(\varepsilon_X)"] & PH_n(X, k),
\end{tikzcd}$$

where the left vertical map is an isomorphism, for $0 \le n \le c_{c,p}$ by Proposition 4.3.12. The left two horizontal maps are isomorphisms in the range $0 \le n \le 2p + c - 4$, as they are induces by the $(2p + c - 4)$-connected map $\mathcal{C}om_{1,\infty} X \twoheadrightarrow \mathcal{C}om_1 X$ of Proposition 4.3.4. Hence the middle vertical map is an isomorphism in the range $0 \le n \le \min\{c_{c,p}, 2p+c-4\}$. Using the free/forgetful functor adjunction

$$s\mathcal{S}et_*\text{-}\mathcal{C}om_{1,\infty}(\mathcal{C}om_{1,\infty} X, Y) = s\mathcal{S}et_*(X, U(Y)),$$

we see that the right two horizontal maps are induced by the retraction $\mathcal{C}om_{1,\infty} X \overset{U(\varepsilon_X)}{\twoheadrightarrow} X$ with section $X \xrightarrow{\eta_{U(X)}} \mathcal{C}om_{1,\infty}(X)$. Hence the right vertical map is a retract of the middle one and thus also an isomorphism in the desired range.

$\square$

5 Cyclic homology and the Lie algebra homology of matrices

Goal of this chapter is the integral generalization of the well-known Theorem of Loday-Quillen [LQ84] and Tsygan [Tsy83]. Alongside its proof we are giving a streamlined introduction to cyclic homology.

5.1 Cyclic homology

Nothing of this chapter is new. We are just recollecting the required definitions and elementary properties from [Lod98] that we need.

5.1.1 Variants of cyclic homology of cyclic modules

Definition 5.1.1
Let $\mathcal{C}$ be a category.

*A **cyclic $\mathcal{C}$-object** is a simplicial $\mathcal{C}$-object $X \in s\mathcal{C}$ together with endomorphisms $t_n \in \mathcal{C}(X_n)$, for all $n \geq 0$, such that*

(i) $t_n^{n+1} = \mathrm{id}_{X_n}, \qquad n \geq 0,$

(ii) $d_i t_n = \begin{cases} d_n, & i = 0, \\ t_{n-1} d_i, & 1 \leq i \leq n, \end{cases}$

(iii) $s_i t_n = \begin{cases} t_{n+1}^2 s_n, & i = 0, \\ t_{n+1} s_{i-1}, & 1 \leq i \leq n. \end{cases}$

Remark 5.1.2
Let $(\mathcal{C}, \otimes, E)$ be a symmetric monoidal category, $A \in \mathcal{C}\text{-}\mathcal{A}ss_1$.

(i) *For a A-bimodule $M \in (A \otimes A^{\mathrm{op}})\text{-}\mathcal{C}$, there is a functorial simplicial $\mathcal{C}$-object $C_\bullet(A, M)$, given by $C_n(A, M) = M \otimes A^{\otimes n}$ and*

a) $d_i = \begin{cases} \mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes(n-i-1)}, & 0 \leq i \leq n-1, \\ d_0 t_n, & i = n, \end{cases}$

b) $s_i = \mathrm{id}^{\otimes(i+1)} \otimes \eta \otimes \mathrm{id}^{\otimes(n-i)}, \qquad 0 \leq i \leq n,$

where μ denotes the multiplication map of A and M respectively, $E \xrightarrow{\eta} A$ is the unit morphism and t_n is the cyclic permutation $X \otimes A^{\otimes n} \xrightarrow{\sim} A \otimes X \otimes A^{\otimes(n-1)}$.

We call $H_*(A, M) = H_*(C_*(A, M))$ the ***homology of the associative algebra*** *A with coefficients M.*

(ii) *For $M = A$ we obtain a cyclic $\mathcal{C}$-object $C_\bullet(A) = (C_\bullet(A, A), t)$.*

We call $HH_*(A) = H_*(C_*(A))$ the ***Hochschild homology*** *of A.*

Proposition 5.1.3
Let $k \in \mathcal{C}Ring$ and X a cyclic k-module.

(i) *The maps $t' = (-1)^n t_n$ induce an action of the cyclic group $C_{\underline{n}} = \langle t' \rangle \cong \mathbb{Z}/(n+1)\mathbb{Z}$ on X_n, for each $n \geq 0$.*

(ii) *There is a bicomplex*

$$CP(X) = (... \xleftarrow{N} \underset{-2}{X_*} \xleftarrow{1-t'} \underset{-1}{X'_*} \xleftarrow{N} \underset{0}{X_*} \xleftarrow{1-t'} \underset{1}{X'_*} \xleftarrow{N} \underset{2}{X_*} \xleftarrow{1-t'} ...),$$

where

a) X_* is the complex X with differential $b = \sum_{0 \leq i \leq n} (-1)^i d_i : X_n \longrightarrow X_{n-1}$,

b) X'_* is the complex X with differential $b' = \sum_{0 \leq i < n} (-1)^i d_i : X_n \longrightarrow X_{n-1}$,

c) $N = \sum_{0 \leq i \leq n} (-1)^{in} t_n^i$ is the norm map with respect to the $C_{\underline{n}}$-action on X_n.

Proof. By computation one checks $b(1 - t') = (1 - t')b'$ and $b'N = Nb$. □

Definition 5.1.4
Let $k \in \mathcal{C}Ring$ and X a cyclic k-module.

(i) *The* ***cyclic homology*** *of X is defined as the homology $HC_*(X) = H_*(\mathrm{Tot}^\times CC(X))$, where*

$$CC(X) = (\underset{0}{X_*} \xleftarrow{1-t'} \underset{1}{X'_*} \xleftarrow{N} \underset{2}{X_*} \xleftarrow{1-t'} ...).$$

(ii) *The* ***negative cyclic homology*** *of X is defined as $HC_*^-(X) = H_*(\mathrm{Tot}^\times CC^-(X))$, where*

$$CC^-(X) = (... \xleftarrow{N} \underset{-2}{X_*} \xleftarrow{1-t'} \underset{-1}{X'_*} \xleftarrow{N} \underset{0}{X_*} \xleftarrow{1-t'} \underset{1}{X'_*}).$$

(iii) *The* ***periodic cyclic homology*** *of X is defined as $HP_*(X) = H_*(\mathrm{Tot}^\times CP(X))$, where*

$$CP(X) = (... \xleftarrow{N} \underset{-2}{X_*} \xleftarrow{1-t'} \underset{-1}{X'_*} \xleftarrow{N} \underset{0}{X_*} \xleftarrow{1-t'} \underset{1}{X'_*} \xleftarrow{N} \underset{2}{X_*} \xleftarrow{1-t'} ...).$$

(iv) *The* ***Connes homology*** *of X is defined as the homology $H_*^\lambda(X) = H_*(C^\lambda(X))$ of the* ***Connes complex*** *$C^\lambda(X)$, which is the quotient complex of X given by $C_n^\lambda(X) = X_n/(1 - (-1)^n t_n)$, for all $n \geq 0$.*

5.1.2 Connes' operator and mixed complexes

Definition 5.1.5
Let $k \in \mathcal{C}Ring$.

*A **mixed complex** is a chain complex $X \in dg(k\text{-Mod})$ together with a second differential $X_* \xrightarrow{B} \Sigma X_* = (X_{*+1}, -d)$ with $B^2 = 0$.*

Proposition 5.1.6
Let $k \in \mathcal{C}Ring$. Then every cyclic module X becomes a mixed complex via

$$B_n : X_n \xrightarrow{N} X_n \xrightarrow{s} X_{n+1} \xrightarrow{1-t'} X_{n+1}, \qquad n \geq 0,$$

*where $s = t_{n+1}s_n$. It is called **Connes' operator** after A. Connes, who introduced it first.*

Proof. Using the formula $b(1-t') = (1-t')b'$ of Proposition 5.1.3 one checks

$$bB + Bb = b(1-t')sN + (1-t')sNb = (1-t')(b's + sb')N = (1-t')N = 0.$$

□

Definition 5.1.7
For $k \in \mathcal{C}Ring$ and X a mixed complex, we define the following double complexes.

(i) $M(X) = (\underset{0}{X_*} \xleftarrow{B} \underset{1}{\Sigma X_*} \xleftarrow{\Sigma B} \underset{2}{\Sigma^2 X_*} \xleftarrow{\Sigma^2 B} ...)$,

(ii) $M^-(X) = (... \xleftarrow{\Sigma^{-2}B} \underset{-2}{\Sigma^{-2} X_*} \xleftarrow{\Sigma^{-1}B} \underset{-1}{\Sigma^{-1} X_*} \xleftarrow{B} \underset{0}{X_*})$,

(iii) $MP(X) = (... \xleftarrow{\Sigma^{-1}B} \underset{-1}{\Sigma^{-1} X_*} \xleftarrow{B} \underset{0}{X_*} \xleftarrow{\Sigma B} \underset{1}{\Sigma X_*} \xleftarrow{\Sigma^2 B} ...)$,

Remark 5.1.8
Let $k \in \mathcal{C}Ring$ and $G \in \mathcal{G}rp$ be finite with $|G| \in k^\times$.

Then the augmentation map $k[G] \xrightarrow{\varepsilon} k$ is a retraction with section

$$s = \frac{1}{|G|} \sum_{g \in G} 1 \cdot g.$$

In particular $k \in k[G]\text{-Mod}$ is projective, showing that for every $X \in k[G]\text{-Mod}$ we have

$$H_n(G, X) = \mathrm{Tor}_n^{k[G]}(k, X) = \begin{cases} X_G, & n = 0, \\ 0, & n > 0. \end{cases}$$

Proposition 5.1.9
Let $k \in \mathcal{C}Ring$ and X a cyclic k-module. Then

(i) *There is a quasi-isomorphism* $\mathrm{Tot}^\times M(X) \xrightarrow{\simeq} \mathrm{Tot}^\times CC(X)$, *given by*

$$M_{p,q}(X) \hookrightarrow CC_{p,q}(X) \times CC_{p-1,q+1}(X), \quad x \longmapsto (x, sNx).$$

Similarly we get quasi-isomorphisms

$$\mathrm{Tot}^\times M^-(X) \xrightarrow{\simeq} \mathrm{Tot}^\times CC^-(X), \qquad \mathrm{Tot}^\times MP(X) \xrightarrow{\simeq} \mathrm{Tot}^\times CP(X).$$

(ii) *If* $p > 1$ *with* $(p-1)! \in k^\times$, *then* $\mathrm{Tot}^\times CC(X) \twoheadrightarrow C^\lambda(X)$ *is* $(p-1)$*-connected.*

Proof.

(i) Using the formulas $b(1-t') = (1-t')b'$ and $b'N = Nb$ one checks that the given map is a map of chain complexes. By filtering

$$F_n M(X) = \prod_{0 \le p < n} M_{p,*}(X), \qquad F_n CC(X) = \prod_{0 \le p < 2n} CC_{p,*}(X), \qquad n \ge 0,$$

it suffices to check that the map induces a quasi-isomorphism on the associated graded complexes. But the map

$$\begin{tikzcd}
\mathrm{gr}^F_{n+1}\mathrm{Tot}^\times M(X) \arrow[r, hook] \arrow[d, equal] & \mathrm{gr}^F_{n+1}\mathrm{Tot}^\times CC(X) \arrow[d, equal] & n \ge 0 \\
\mathrm{Tot}^\times(\underset{n}{\Sigma^n X_*}) \arrow[r, hook] & \mathrm{Tot}^\times(\underset{2n}{X_*} \xleftarrow{1-t'} \underset{2n+1}{X'_*})
\end{tikzcd}$$

is a quasi-isomorphism, because the extra-degeneracy $s = t_{n+1}s_{n+1}$ induces a contraction for its cokernel X'_{*+2n+1}.

(ii) By filtering $CC(X)$ and $C^\lambda(X)$ by

$$F_r CC(X) = \prod_{0 \le n \le r} CC_{*,n}(X), \quad F_r C^\lambda(X) = \prod_{0 \le n \le r} C^\lambda_n(X), \qquad r \ge 0,$$

we obtain spectral sequences

- $E^1_{r,s} = H_s(X_r \xleftarrow{1-t'} X_r \xleftarrow{N} ...) = H_s(C_{\underline{r}}, X_r) \quad \Rightarrow \quad HC_{r+s}(X)$,
- $\bar{E}^1_{r,*} = X_r/(1-t') \quad \Rightarrow \quad H^\lambda_{r+s}(X)$,

and the map $E^1_{r,*} \longrightarrow \bar{E}^1_{r,*}$ is an isomorphism, for $0 \le r < p-1$, by Remark 5.1.8. Since also $E^1_{*,0} \xrightarrow{\sim} \bar{E}^1_{*,0}$ is an isomorphism and $E^1_{*,1} \twoheadrightarrow \bar{E}^1_{*,1} = 0$ is surjective, it follows that $E^1 \longrightarrow \bar{E}^1$ and hence $E^\infty \longrightarrow \bar{E}^\infty$ and $HC_*(X) \longrightarrow H^\lambda_*(X)$ are $(p-1)$-connected.

□

Remark 5.1.10
Let $k \in \mathcal{C}Ring$ and X a cyclic k-module.
Then there are maps of short exact sequences of bicomplexes

$$\begin{tikzcd}
0 \arrow[r] & M^-(X) \arrow[r, "S"] \arrow[d, equal] & M^-_{*-1,*+1}(X) \arrow[r, "P"] \arrow[d, hook] & X_{*-2} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M^-(X) \arrow[r, "I"] \arrow[d, "P"'] & MP(X) \arrow[r, "S"] \arrow[d] & M_{*-1,*+1}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_* \arrow[r, "I"] & M(X) \arrow[r, "S"] & M_{*-1,*+1}(X) \arrow[r] & 0,
\end{tikzcd}$$

inducing maps of long exact sequences

$$\begin{tikzcd}
\ldots \arrow[r, "S"] & H_{n-1}(X) \arrow[r, "B"] \arrow[d, "I"'] & HC^-_n(X) \arrow[r, "S"] \arrow[d, equal] & HC^-_{n-2}(X) \arrow[r, "P"] \arrow[d] & H_{n-2}(X) \arrow[r, "B"] \arrow[d, "I"'] & \ldots \\
\ldots \arrow[r, "S"] & HC_{n-1}(X) \arrow[r, "B"] \arrow[d, equal] & HC^-_n(X) \arrow[r, "I"] \arrow[d, "P"'] & HP_n(X) \arrow[r, "S"] \arrow[d] & HC_{n-2}(X) \arrow[r, "B"] \arrow[d, equal] & \ldots \\
\ldots \arrow[r, "S"] & HC_{n-1}(X) \arrow[r, "B"] & H_n(X) \arrow[r, "I"] & HC_n(X) \arrow[r, "S"] & HC_{n-2}(X) \arrow[r, "B"] & \ldots
\end{tikzcd}$$

5.2 The negative Chern character for Hopf algebras

By examination of the cyclic structure on the bar complex, we are following [CW09] to construct the negative Chern character. As we have mentioned before, the key ideas go back to them. Our essential task is a streamlined presentation of the required results we and to pay attention that everything works integrally in the same manner as rationally.

5.2.1 Cyclic modules induced by group objects

Remark 5.2.1
Let $(\mathcal{C}, \otimes, E)$ be a monoidal category and $C \in \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}$.

(i) *Then there is a simplicial $\mathcal{C}$-object $E_\bullet(C) \in s\mathcal{C}$, given by*

$$E_n(C) = C^{\otimes(n+1)}, \quad d_i = \mathrm{id}^{\otimes i} \otimes \varepsilon \otimes \mathrm{id}^{\otimes(n-i)}, \quad s_i = \mathrm{id}^{\otimes i} \otimes \delta \otimes \mathrm{id}^{\otimes(n-i)}, \qquad 0 \le i \le n.$$

(ii) *If $(\mathcal{C}, \otimes, E)$ is symmetric monoidal, then $E_\bullet(C)$ is a cyclic $\mathcal{C}$-object with $t_n \in \mathcal{C}(C^{\otimes(n+1)})^\times$ the cyclic permutation mapping the last factor to the first one.*

Proposition 5.2.2
Let $(\mathcal{C}, \otimes, k)$ be a symmetric monoidal category and $G \in \mathcal{C}\text{-}\mathcal{G}rp$ be cocommutative.
Then $E_\bullet(G)$ is a cyclic left G-module by the ***diagonal action***

$$G \otimes E_n(G) \xrightarrow{\delta^n \otimes \mathrm{id}} (G^{\otimes(n+1)}) \otimes (G^{\otimes(n+1)}) \xrightarrow{\mu} G^{\otimes(n+1)} = E_n(G), \qquad n \ge 0,$$

where by abuse of notation δ^n denotes the map $G \xrightarrow{\delta} G^{\otimes 2} \xrightarrow{\delta\otimes\mathrm{id}} \dots \xrightarrow{\delta\otimes\mathrm{id}} G^{\otimes n} \xrightarrow{\delta\otimes\mathrm{id}} G^{\otimes(n+1)}$, and $G^{\otimes(n+1)} \in \mathcal{C}\text{-}\mathcal{A}ss_1$ via factorwise G-multiplication.

Moreover there is a natural isomorphism of simplicial left G-modules $E_\bullet G \cong B_\bullet(G, G, k)$.

In particular $B_\bullet(G, G, k)$ is a cyclic left G-module.

Proof. Using that $G \xrightarrow{\delta} G \otimes G$ is a homomorphism of unital $\mathcal{C}$-magmas, it follows that the diagonal action defines a left G-module structure on $E_n(G)$, for all $n \geq 0$. By the same argument, we see that $E_\bullet(G) \in s(G\text{-}\mathcal{C})$ is a simplicial left G-module. Using that G is cocommutative, it follows that t is G-linear and hence $E_\bullet(G)$ is a cyclic left G-module. It remains to check that $E_\bullet(G) \cong B_\bullet(G, G, k)$. First note that there is an adjunction

$$G\text{-}\mathcal{C}(G \otimes X, Y) = \mathcal{C}(X, U(Y)),$$

where U is the forgetful functor. In particular we get a comonad, that is an associative comagma $B \in (\mathcal{C}\mathrm{AT}(G\text{-}\mathcal{C}), \circ, \mathrm{id})\text{-}\mathcal{A}ss_1$, given by $B(X) = G \otimes X$, for $X \in G\text{-}\mathcal{C}$. Its counit and comultiplication can be computed as

$$B(X) = G \otimes X \xrightarrow{\mu} X, \qquad B(X) = G \otimes X \xrightarrow{\mathrm{id}\otimes\eta\otimes\mathrm{id}} G \otimes G \otimes X = BB(X).$$

Associated to B there is a functorial simplicial resolution $E_\bullet(B)(X) \longrightarrow X$ and by construction $E_\bullet(B)(k) = B_\bullet(G, G, k)$, where $k \in G\text{-}\mathcal{C}$ via $G \otimes k \xrightarrow{\varepsilon\otimes\mathrm{id}} k$.

Now there is another comonad $C \in (\mathcal{C}\mathrm{AT}(G\text{-}\mathcal{C}), \circ, \mathrm{id})\text{-}\mathcal{A}ss_1$ with G acting diagonally on $C(X) := G \otimes X$, for $X \in G\text{-}\mathcal{C}$. Its counit and comultiplication are given by

$$C(X) = G \otimes X \xrightarrow{\varepsilon\otimes\mathrm{id}} X, \qquad C(X) = G \otimes X \xrightarrow{\delta\otimes\mathrm{id}} G \otimes G \otimes X = CC(X),$$

and again by construction $E_\bullet(C)(k) = E_\bullet(G)$. Using the axioms of a group object, we see that the maps

$$(\mathrm{id} \otimes \mu) \circ (\delta \otimes \mathrm{id}) : B(X) = G \otimes X \overset{\sim}{\rightleftarrows} G \otimes X = C(X) : (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \iota \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id}),$$

induce an isomorphism of comonads $B \cong C$, which proves the second assertion.

□

Proposition 5.2.3

Let $(\mathcal{C}, \otimes, k)$ be a symmetric monoidal category, $G \in \mathcal{C}\text{-}\mathcal{G}rp$ and M a G-bimodule.

Then there is an isomorphism $B_\bullet(\mathrm{Ad}(M), G, k) \cong H_\bullet(G, M)$, where $\mathrm{Ad}(M) = M$ with the right conjugation action

$$\mathrm{Ad}(M) \otimes G \xrightarrow{\mathrm{id}\otimes\delta} M \otimes G^{\otimes 2} \xrightarrow{\gamma\otimes\mathrm{id}} G \otimes M \otimes G \xrightarrow{\iota\otimes\mathrm{id}\otimes\mathrm{id}} G \otimes M \otimes G \xrightarrow{\mu} M = \mathrm{Ad}(M),$$

and γ is the isomorphism twisting the two tensor factors.

Proof. We consider $G^{\otimes n} \in \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}$ by factorwise comultiplication and define the maps in the middle of the diagram

$$\begin{tikzcd}
 & M \otimes (G^{\otimes n}) \otimes (G^{\otimes n}) \arrow[r, "\gamma\otimes\mathrm{id}"] & G^{\otimes n} \otimes M \otimes G^{\otimes n} \arrow[d, "\mu^n\otimes\mathrm{id}"] & \\
H_n(G,M) \arrow[r, equal] & M \otimes G^{\otimes n} \arrow[u, "\mathrm{id}\otimes\delta"] \arrow[r, shift left, "\sim"] & M \otimes G^{\otimes n} \arrow[l, shift left] \arrow[r, equal] \arrow[d, "\mathrm{id}\otimes\delta"] & B_n(G,M) \\
 & G \otimes M \otimes G^{\otimes n} \arrow[u, "\mu\otimes\mathrm{id}"] & M \otimes (G^{\otimes n}) \otimes (G^{\otimes n}) \arrow[d, "\gamma\otimes\mathrm{id}"] & \\
 & G \otimes M \otimes G^{\otimes n} \arrow[u, "\iota\otimes\mathrm{id}"] & G^{\otimes n} \otimes M \otimes G^{\otimes n}, \arrow[l, "\mu^{n-1}\otimes\mathrm{id}"'] &
\end{tikzcd}$$

as the composites of the upper and lower path. Using the axioms of a group object, one checks that they are inverse to each other and that the maps from left to right assemble to an isomorphism of simplicial objects.

□

Example 5.2.4
Consider the case $(\mathcal{C}, \otimes, k) = (\mathcal{S}et, \times, *)$ and $G \in \mathcal{G}rp$.

(i) *The isomorphism of Proposition 5.2.2 is given by*

$$\begin{aligned}
B_n(G,G,*) &\overset{\sim}{\rightleftarrows} E_n(G), \\
(x_0, ..., x_n) &\longmapsto (x_0, x_0x_1, ..., x_0 \cdots x_n) \\
(x_0, x_0^{-1}x_1, ..., x_{n-1}^{-1}x_n) &\longleftarrow (x_0, ..., x_n).
\end{aligned}$$

Moreover the cyclic operator t_n on $B_n(G,G,*)$ is given by

$$t_n(x_0, ..., x_n) = (x_0 \cdots x_n, (x_1 \cdots x_n)^{-1}, x_1, ..., x_{n-1}),$$

which is quite complicated compared to the simple cyclic permutation on $E_n(G)$.

(ii) *The isomorphism of Proposition 5.2.3 is given by*

$$\begin{aligned}
B_n(\mathrm{Ad}(M), G, *) &\overset{\sim}{\rightleftarrows} C_n(G,M), \\
(x_0, ..., x_n) &\longmapsto ((x_1 \cdots x_n)^{-1}x_0, x_1, ..., x_n) \\
((x_1 \cdots x_n)x_0, x_1, ..., x_n) &\longleftarrow (x_0, ..., x_n).
\end{aligned}$$

In particular, for $M = G$, this isomorphism maps the element $t_n(x_0, ..., x_n) = (x_0^{x_1 \cdots x_n}, (x_1 \cdots x_n)^{-1}, x_1, ..., x_{n-1}) \in B_n(\mathrm{Ad}(G), G, *)$ to

$$(x_n \cdot x_0^{x_1 \cdots x_n}, (x_1 \cdots x_n)^{-1}, x_1, ..., x_{n-1}),$$

which is unequal to

$$t_n((x_1 \cdots x_n)^{-1}x_0, x_1, ..., x_n) = (x_n, (x_1 \cdots x_n)^{-1}x_0, x_1, ..., x_{n-1}),$$

unless $x_0 = 1$.

In particular the upper isomorphism is not *a homomorphism of cyclic objects in general unless we restrict it to* $B_\bullet(*, G, *) = B_\bullet(\mathrm{Ad}(1), G, *) \hookrightarrow B_\bullet(\mathrm{Ad}(G), G, *)$.

Proposition 5.2.5
Let $(\mathcal{C}, \otimes, k)$ *be a symmetric monoidal category and* $G \in \mathcal{C}\text{-}\mathcal{G}rp$ *be cocommutative.*

Then the inclusion $k \overset{\eta}{\hookrightarrow} \mathrm{Ad}(G)$ *and the isomorphism of Proposition 5.2.3 induce a homomorphism of cyclic* $\mathcal{C}$*-objects*

$$c : B_\bullet(k, G, k) \hookrightarrow B_\bullet(\mathrm{Ad}(G), G, k) \xrightarrow{\sim} H_\bullet(G, G) = C_\bullet(G).$$

Proof. By explicit computation we have already seen in example 5.2.4, that this holds in the case $(\mathcal{C}, \otimes, k) = (\mathcal{S}et, \times, *)$. Of course one could also verify this in the general setting by direct computation using the axioms of group objects. However by a more conceptual approach the case $(\mathcal{C}, \otimes, k) = (\mathcal{S}et, \times, *)$ already implies the general case, as we will demonstrate in the following. We have to show that the square

$$\begin{tikzcd} B_n(k, G, k) \arrow[r] \arrow[d, "t_n"'] & C_n(G) \arrow[d, "t_n"] \\ B_n(k, G, k) \arrow[r] & C_n(G) \end{tikzcd}$$

commutes, for every $n \geq 0$. Note that by construction these are built by compositions and tensor products of the structure maps of G and the natural isomorphism twisting the tensor factors. Since G is cocommutative, these maps are homomorphisms of counital comagmas. Now by Lemma 5.2.6 below the functor $\mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}(G^{\otimes n}, \text{-})$ is strict monoidal and hence the upper square under $\mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}(G^{\otimes n}, \text{-})$ is isomorphic to the corresponding square in the category of sets

$$\begin{tikzcd} B_n(*, \mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}(G^{\otimes n}, G), *) \arrow[r] \arrow[d, "t_n"'] & C_n(\mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}(G^{\otimes n}, G)) \arrow[d, "t_n"] \\ B_n(*, \mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}(G^{\otimes n}, G), *) \arrow[r] & C_n(\mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}(G^{\otimes n}, G)), \end{tikzcd}$$

which commutes by example 5.2.4. Now on the set of homomorphisms $\mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}(G^{\otimes n}, G^{\otimes(n+1)})$ the map induced by the functor $\mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}(G^{\otimes n}, \text{-})$ is injective, because evaluation at the identity is a retraction. Hence also the square in $\mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}$ commutes.

Note that this technique also provides an alternative proof for Proposition 5.2.3.

□

Lemma 5.2.6
Let $(\mathcal{C}, \otimes, k)$ *be a symmetric monoidal category and* $C \in \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}$*. Then*

(i) *There is an adjunction*

$$\mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}({}^X C, Y) = \mathcal{S}et(X, \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, Y)),$$

where the left adjoint is given the X*-fold coproduct of* C*, for* $X \in \mathcal{S}et$*.*

(ii) *There is a natural map*

$$\prod_{1\le i\le n} \underbrace{(\varepsilon^{\otimes(i-1)} \otimes \mathrm{id} \otimes \varepsilon^{\otimes(n-i)})}_{=:\pi_i} : \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, X_1 \otimes ... \otimes Y_n) \longrightarrow \prod_{1\le i\le n} \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, X_i),$$

which is a bijection, if C is cocommutative.

In particular the functor $\mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, \text{-})$ is strict monoidal in this case via

$$\mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, X \otimes Y) \xrightarrow{\sim} \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, X) \times \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, Y), \qquad \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, k) \xrightarrow{\sim} *.$$

Proof.

(i) By the universal property of coproducts, we have a natural bijection

$$\mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}({}^X C, Y) = \prod_{x\in X} \mathcal{C}\text{-}\mathcal{A}ss_1(C, Y) = \mathcal{S}et(X, \mathcal{C}\text{-}\mathcal{A}ss_1^{\mathrm{op}}(C, Y)).$$

(ii) If C is cocommutative, then $C \xrightarrow{\delta} C \otimes C$ is a homomorphism of comagmas and hence

$$(f_i)_{1\le i\le n} \longmapsto (f_1 \otimes ... \otimes f_n) \circ \delta^{n-1}$$

is well-defined and forms an inverse:

- Using that $f_1, ..., f_n$ are homomorphisms of comagmas and by the axioms of comagmas, we have, for all $1 \le i \le n$, that

$$\begin{aligned} \pi_i \circ (f_1 \otimes ... \otimes f_n) \circ \delta^{n-1} &= ((\varepsilon f_1) \otimes ... \otimes f_i \otimes ... \otimes (\varepsilon f_n)) \circ \delta^{n-1} \qquad 1 \le i \le n \\ &= (\varepsilon \otimes ... \otimes f_i \otimes ... \otimes \varepsilon) \circ \delta^{n-1} = f_i. \end{aligned}$$

- Again by the axioms of comagmas there is a commutative diagram

$$\begin{tikzcd}
(X_1 \otimes ... \otimes X_n) \arrow[r, "\delta^{n-1}\otimes...\otimes\delta^{n-1}"] \arrow[dr, equal] & X_1^{\otimes n} \otimes ... \otimes X_n^{\otimes n} \arrow[r, equal, "\sim"] \arrow[d, "\pi_1\otimes...\otimes\pi_n"] & (X_1 \otimes ... \otimes X_n)^{\otimes n} \arrow[dl, "\pi_1\otimes...\otimes\pi_n"] \\
 & X_1 \otimes ... \otimes X_n. &
\end{tikzcd}$$

  If the upper right map is the unique isomorphism induced by the natural isomorphism permuting the tensor factors, preserving the order of $X_i^{\otimes n}$, for each $1 \le i \le n$, then the upper row is δ^{n-1} for the comagma $X_1 \otimes ... \otimes X_n$ and it follows that

$$((\pi_1 f) \otimes ... \otimes (\pi_n f)) \circ \delta^{n-1} = (\pi_1 \otimes ... \otimes \pi_n) \circ \delta^{n-1} \circ f = f.$$

□

Remark 5.2.7

Let $(\mathcal{C}, \otimes, k)$ be a symmetric monoidal category.

Then Lemma 5.2.6 (ii) implies that $X_1 \otimes ... \otimes X_n$ is a product of $X_1, ..., X_n$ in $\mathcal{C}\text{-}\mathcal{C}om_1^{\mathrm{op}}$.

In particular for $(\mathcal{C}, \otimes, k) = (\mathcal{A}b^{\mathrm{op}}, \otimes, \mathbb{Z})$ this proves that the coproduct of commutative rings is given by the tensor product.

5.2.2 Construction of the negative Chern character

The following results can also be found in [CW09]. As their notation is different from ours and they assumed the algebra to be rational, we recall their proofs, for the convenience of the reader (without the sign that has unfortunately slipped in).

Lemma 5.2.8
Let $k \in \mathcal{C}Ring$ and $A \in k\text{-}\mathcal{A}ss_1$ a unital associative k-algebra. Given a homomorphism of chain complexes of left A-modules $C \xrightarrow{f} D$, such that

(i) *C is N-connected, for some $N \in \mathbb{Z}$, i.e. $C_n = 0$, for all $n \leq N$.*

(ii) *For all $n \in \mathbb{Z}$, we have $C_n = A \otimes X_n$, for some $X_n \in k$-Mod.*

(iii) *There is a k-linear contraction s for D.*

Then f and 0 are chain homotopic via s^f, inductively defined by $s^f_N = s^f_{N-1} = ... = 0$ and

$$C_n = A \otimes X_n \xrightarrow{s^f_n} D_{n+1}, \quad a \otimes x \longmapsto a \cdot s(f_n - s^f_{n-1}d)(1 \otimes x), \qquad n > N.$$

Proof. We have to check that $f_n - ds^f_n = s^f_{n-1}d$, for all $n \in \mathbb{Z}$. For $n \leq N$, we have $C_n = 0$ and there is nothing to check. Suppose the statement is true for some $n-1 \geq N$. Then the induction hypothesis implies

$$d(f_n - s^f_{n-1}d) = (f_{n-1} - ds^f_{n-1})d = s^f_{n-2}d^2 = 0.$$

Using this, $sd + ds = 1$, the definition of s^f_n and its A-linearity, we get

$$\begin{aligned} ds^f_n(a \otimes x) &= d(a \cdot s(f_n - s^f_{n-1}d)(1 \otimes x)) = a \cdot ds(f_n - s^f_{n-1}d)(1 \otimes x) \\ &= a \cdot (1 - sd)(f_n - s^f_{n-1}d)(1 \otimes x) = a \cdot (f_n - s^f_{n-1}d)(1 \otimes x) \\ &= f_n(a \otimes x) - s^f_{n-1}(a \otimes x), \end{aligned}$$

for all $a \otimes x \in A \otimes X_n$, proving the induction step.

□

Lemma 5.2.9
Let $k \in \mathcal{C}Ring$ and $H \in k\text{-}\mathcal{G}rp$ a Hopf algebra over k.

Then there are H-linear maps $E_*(H) \xrightarrow{j^n} E_{*+2n}(H)$, for $n \geq 0$, such that

$$j^0 = \mathrm{id}, \qquad j^n b = bj^n + Bj^{n-1}, \qquad n \geq 1.$$

Proof. By setting $j^{-1} = 0$, the formula holds for j^0. Suppose we have constructed j^n, for some $n \geq 0$, satisfying $j^n b = bj^n + Bj^{n-1}$. Using this assumption and the formula $bB + Bb = 0$ of Proposition 5.1.6, we see that the map

$$f = Bj^n : E_*(H) \longrightarrow \Sigma^{2n+1} E_*(H) = (E_{*+2n+1}, -b)$$

defines a homomorphism of chain complexes:

$$fb = Bj^n b = B(bj^n + Bj^{n-1}) = Bbj^n = -bBj^n = (-b)f$$

As $E_*(H)$ is zero in negative dimensions, f maps into the kernel of $\Sigma^{2n+1}E_*(H) \twoheadrightarrow \Sigma^{2n+1}k$, which has a k-linear contraction s induced by the extra-degeneracy $s_{-1} = \eta \otimes \mathrm{id}^{\otimes n}$. So we can apply Lemma 5.2.8 to obtain an H-linear contraction $j^{n+1} := s^f$ for f. The contraction condition $s^f b + (-b)s^f = f$ readily implies

$$j^{n+1}b = s^f b = bs^f + f = bj^{n+1} + Bj^n.$$

□

Proposition 5.2.10
Let $k \in \mathcal{C}Ring$ and $H \in k\text{-}\mathcal{G}rp$ a Hopf algebra over k.

Then the map $\mathrm{Tot}^\times M^-(B_\bullet(M,H,k)) \overset{P}{\twoheadrightarrow} B_*(M,H,k)$ of Remark 5.1.10 has a natural section J, for all right H-modules $M \in \mathrm{Mod}\text{-}H$.

Proof. For $n \geq 0$, we define J^n as the composition

$$\begin{tikzcd} B_*(M,H,k) \arrow[r, "J^n"] \arrow[d, equal] & B_{*+2n}(k,H,k) \arrow[d, equal] \\ M \otimes_H B_*(H,H,k) \arrow[d, equal, "\wr"] & M \otimes_H B_{*+2n}(H,H,k) \arrow[d, equal, "\wr"] \\ M \otimes_H E_*(H) \arrow[r, "M\otimes j^n"] & M \otimes_H E_{*+2n}(H), \end{tikzcd}$$

where the vertical isomorphisms are those of Proposition 5.2.2 and j^n is the map of the preceding lemma. We define

$$J = (J^0, J^1, ...) : B_*(M,H,k) \hookrightarrow \mathrm{Tot}^\times M^-(B_\bullet(M,H,k)) = (\prod_{n\geq 0} B_{*+2n}(M,H,k), b+B).$$

Then using the relations for $(j^n)_{n\geq 0}$ we get

$$(b+B)J = (bJ^0, bJ^1 + BJ^0, bJ^2 + BJ^1, ...) = (J^0 b, J^1 b, ...) = Jb,$$

which proves that J is a homomorphism of chain complexes. By construction $PJ = \mathrm{id}$.

□

Definition 5.2.11
For $k \in \mathcal{C}Ring$ and $H \in k\text{-}\mathcal{G}rp$ the ***negative Chern character*** *is defined as the composition*

$$\begin{tikzcd} B_*(k,H,k) \arrow[rr, "\mathrm{ch}^-"] \arrow[d, "J"'] & & \mathrm{Tot}^\times M^-(H) \arrow[d, equal] \\ \mathrm{Tot}^\times M^-(B_\bullet(k,H,k)) \arrow[r] & \mathrm{Tot}^\times M^-(B_\bullet(\mathrm{Ad}(H),H,k)) \arrow[r, "\sim"] & \mathrm{Tot}^\times M^-(C_\bullet(H)). \end{tikzcd}$$

Remark 5.2.12
For $k \in \mathcal{C}Ring$ and $H \in k\text{-}\mathcal{G}rp$.
Then by construction the diagram below is commutative.

$$\begin{tikzcd}
B_\bullet(k,H,k) \arrow[r] \arrow[d, bend left, hook, "J"] \arrow[drr, "\mathrm{ch}^-"] & B_\bullet(\mathrm{Ad}(H),H,k) \arrow[r, "\sim"] & C_\bullet(H) \\
\mathrm{Tot}^\times M^-(B_\bullet(k,H,k)) \arrow[r] \arrow[u, bend left, "P"] & \mathrm{Tot}^\times M^-(B_\bullet(\mathrm{Ad}(H),H,k)) \arrow[r, "\sim"'] & \mathrm{Tot}^\times M^-(C_\bullet(H)) \arrow[u, "P"']
\end{tikzcd}$$

5.2.3 Restriction to the Chevalley-Eilenberg complex

Proposition 5.2.13
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$.
Then the antisymmetrisation map

$$e : \Lambda_*\mathfrak{g} \longrightarrow \tilde{B}_*U_k(\mathfrak{g}), \quad x_1 \wedge ... \wedge x_n \longmapsto \sum_{\sigma\in\Sigma_n} \mathrm{sgn}(\sigma)\cdot x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}$$

is a homomorphism of mixed complexes with the second differential $B = 0$ on $\Lambda_*\mathfrak{g}$.

Proof. Recall that by Proposition 5.2.5 there is a monomorphism of cyclic objects

$$c : B_\bullet(k,G,k) \hookrightarrow B_\bullet(\mathrm{Ad}(G),G,k) \xrightarrow{\sim} H_\bullet(G,\mathrm{Ad}(G)) = C_\bullet(G),$$

which in the case $G \in \mathcal{G}rp$ is given by

$$c(x_1,...,x_n) = ((x_1\cdots x_n)^{-1}, x_1,...,x_{n-1}), \qquad x_1,...,x_n \in G.$$

As j is injective, so is the map on the associated reduced complexes $\tilde{B}_*U_k(\mathfrak{g}) \xhookrightarrow{c} \tilde{C}_*U_k(\mathfrak{g})$ and it suffices to show that the composition $\Lambda_*\mathfrak{g} \xrightarrow{e} \tilde{B}_*U_k(\mathfrak{g}) \xhookrightarrow{c} \tilde{C}_*U_k(\mathfrak{g})$ is a homomorphism of mixed complexes. Recall that comultiplication and convolution on $U_k(\mathfrak{g})$ are given by

$$\delta(x) = 1\otimes x + x \otimes 1, \qquad \iota(x) = -x, \qquad x \in \mathfrak{g}.$$

So translating the formula for i to the setting $G = U_k(\mathfrak{g})$ (see also the abstract definition in Proposition 5.2.5), we get

$$\begin{aligned} c(x_1\otimes ... \otimes x_n) &= \sum_{a\in\{0,1\}^n} (-1)^{a_1+...+a_n}\cdot (x_1^{a_1}\cdots x_n^{a_n}) \otimes x_1^{1-a_1} \otimes ... \otimes x_n^{1-a_n}, \quad x_1,...,x_n \in \mathfrak{g} \\ &= 1 \otimes x_1 \otimes ... \otimes x_n, \end{aligned}$$

where the last equality holds, because all terms with values in k in any of the last n tensor factors vanish in the reduced complex $\tilde{C}_*U_k(\mathfrak{g})$. Moreover it follows that

$$\begin{aligned} Bc(x_1\otimes ... \otimes x_n) &= (1-(-1)^n t_n)sN(1\otimes x_1 \otimes ... \otimes x_n), \qquad x_1,...,x_n \in \mathfrak{g}, \\ &= (1-(-1)^n t_n)(1 \otimes N(1\otimes x_1\otimes ... \otimes x_n)) = 0, \end{aligned}$$

because at least two tensor factors of every appearing summand lie in k. This proves $Bce = 0$ and thus ce and e are homomorphisms of mixed complexes.

□

Proposition 5.2.14
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k$-$\mathcal{L}ie$. The map $M^-(\Lambda_*\mathfrak{g}) = \prod_{n \le 0} \Lambda_{*-2n}\mathfrak{g} \xrightarrow{P} \Lambda_*\mathfrak{g}$ of Remark 5.1.10 has a canonical section $J = (1, 0, 0, ...)$ and we get a diagram

$$\begin{tikzcd}
\Lambda_*\mathfrak{g} \arrow[r, "J"] \arrow[d, "e"'] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{Tot}^\times M^-(\Lambda_*\mathfrak{g}) \arrow[r, "P"] \arrow[d, "e"'] & \Lambda_*\mathfrak{g} \arrow[d, "e"'] \\
B_*U_k(\mathfrak{g}) \arrow[r, "J"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Tot}^\times M^-(B_*U_k(\mathfrak{g})) \arrow[r, "P"] & B_*U_k(\mathfrak{g}).
\end{tikzcd}$$

Then the right square commutes and the left one commutes up to homotopy.

Proof. The right square commutes, because the map $\Lambda_*\mathfrak{g} \xrightarrow{e} B_*U_k(\mathfrak{g})$ is a homomorphism of mixed complexes by Proposition 5.2.13 and P is natural in mixed complexes. To see that eJ and Je are homotopic, consider their lifts

$$\begin{tikzcd}
U_k(\mathfrak{g}) \otimes \Lambda_*\mathfrak{g} \arrow[r, "j"] \arrow[d, "\mathrm{id}\otimes e"'] & \mathrm{Tot}^\times M^-(U_k(\mathfrak{g}) \otimes \Lambda_*\mathfrak{g}) \arrow[d, "\mathrm{id}\otimes e"'] \\
B_*(U_k(\mathfrak{g}), U_k(\mathfrak{g}), k) \arrow[r, "j"] & \mathrm{Tot}^\times M^-(B_*(U_k(\mathfrak{g}), U_k(\mathfrak{g}), k)),
\end{tikzcd}$$

and the differences

$$D^n = B(\mathrm{id} \otimes e)j^n - Bj^n(\mathrm{id} \otimes e) : U_k(\mathfrak{g}) \otimes \Lambda_*\mathfrak{g} \longrightarrow B_{*+2n+1}(U_k(\mathfrak{g}), U_k(\mathfrak{g}), k), \quad n \ge 0.$$

Like in the proof of Lemma 5.2.9 we will construct $U_k(\mathfrak{g})$-linear maps $U_k(\mathfrak{g}) \otimes \Lambda_*\mathfrak{g} \xrightarrow{h^n} B_{*+2n+1}(U_k(\mathfrak{g}), U_k(\mathfrak{g}), k)$, such that

$$Bh^{n-1} + bh^n + h^n b = D^n = \begin{cases} 0, & n = 0, \\ j^n e, & n > 0. \end{cases}$$

Setting $h^0 = h^{-1} = 0$, the equality holds for $n = 0$. Suppose we have constructed h^n, for some $n \ge 0$, satisfying the equality. Then using the induction hypothesis, the relations for j^n of Lemma 5.2.9 and the formula $bB + Bb = 0$ of Proposition 5.1.6, we see that $f = j^{n+1}e - Bh^n : \Lambda_*\mathfrak{g} \longrightarrow \Sigma^{2(n+1)}B_*(U_k(\mathfrak{g}), U_k(\mathfrak{g}), k)$ is a chain map:

$$fb = j^{n+1}eb - B(h^n b) = (bj^{n+1} + Bj^n)e + B(bh^n - j^n e + Bh^{n-1}) = bj^{n+1}e - bBh^n = bf$$

As f maps into the kernel of $\Sigma^{2(n+1)}B_*(U_k(\mathfrak{g}), U_k(\mathfrak{g}), k) \cong \Sigma^{2(n+1)}E_*U_k(\mathfrak{g}) \twoheadrightarrow \Sigma^{2(n+1)}k$, which is k-linearly contractible, we get a $U_k(\mathfrak{g})$-linear contraction h^{n+1} for f. In other words

$$bh^{n+1} + h^{n+1}b = j^{n+1}e - Bh^n,$$

which proves the induction step. By construction the map

$$(k \otimes_{U_k(\mathfrak{g})} h^0, k \otimes_{U_k(\mathfrak{g})} h^1, ...) : \Lambda_*\mathfrak{g} \longrightarrow \mathrm{Tot}^\times M^-(B_*U_k(\mathfrak{g}))$$

is a contraction for D, proving that Je and eJ are chain homotopic.

□

Corollary 5.2.15
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in k\text{-}\mathcal{L}ie$.
Then the diagram below commutes up to homotopy

$$\begin{tikzcd}
\Lambda_*\mathfrak{g} \arrow[r, "\theta"] \arrow[d, "e"] & \Sigma C^\lambda_* U_k(\mathfrak{g}) \arrow[r, equal] & (C^\lambda_{*-1}U_k(\mathfrak{g}), -b) \arrow[dd, "B"] \\
B_*U_k(\mathfrak{g}) \arrow[d, "J"] & & \\
\mathrm{Tot}^\times M^-(B_*U_k(\mathfrak{g})) \arrow[r, "c"] & \mathrm{Tot}^\times M^-_*(C_*U_k(\mathfrak{g})) \arrow[r, equal] & \mathrm{Tot}^\times M^-_*(U_k(\mathfrak{g})),
\end{tikzcd}$$

where θ is the antisymmetrisation map given by

$$\theta(x_0 \wedge ... \wedge x_n) = (-1)^n \cdot \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \cdot [x_0 \otimes x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}].$$

Proof. Since in $\tilde{C}_*U_k(\mathfrak{g})$ we have

$$\begin{aligned} B\theta(x_1 \wedge ... \wedge x_n) &= (1-(-1)^n t_n)sN\theta(x_1 \wedge ... \wedge x_n) = (1-(-1)^n t_n)se(x_1 \wedge ... \wedge x_n) \\ &= 1 \otimes e(x_1 \wedge ... \wedge x_n) = ce(x_1 \wedge ... \wedge x_n), \end{aligned}$$

which shows that the square commutes up to homotopy when we replace Je by eJ. Since these two maps are homotopic by Proposition 5.2.14 the statement follows.

□

5.3 Comparing cyclic homologies

We are comparing cyclic homology and negative cyclic homology by using Connes' operator B. We show that it has poor connectivity properties integrally contrarily to the rational case. We are giving an upper bound for the connectivity and prove that essentially it is sharp by giving an example. It is remarkable that Brun [Bru01] gets exactly the same upper bounds for connectivity in his comparison of cyclic homology to topological cyclic homology. Although their construction is not comparable it seems that the "distance" of cyclic homology to both "negative cyclic homologies" is the same (called this way as they both involve taking fixed points of the circle action).

5.3.1 Homomorphisms induced by derivations

Proposition 5.3.1
Let $k \in \mathcal{C}Ring$ and $A \in k\text{-}\mathcal{A}ss_1$ together with a k-linear derivation δ on A.
Then δ induces an endomorphism $\bar{\delta}$ on the cyclic module $C_\bullet(A)$, given by

$$L_\delta = \sum_{0 \le i \le n} \mathrm{id}^{\otimes i} \otimes \delta \otimes \mathrm{id}^{\otimes(n-i)} : C_n(A) \longrightarrow C_n(A),$$

$$a_0 \otimes ... \otimes a_n \longmapsto \sum_{0 \le i \le n} a_0 \otimes ... \otimes \delta(a_i) \otimes ... \otimes a_n.$$

Moreover the induced map on periodic cyclic homology $HP_*(A) \xrightarrow{L_\delta} HP_*(A)$ *is zero.*

Proof. One checks that L_δ commutes with all the face maps d_i, degeneracies s_i and cyclic operators t_n. In [Goo85a] Thm. II.4.2 Goodwillie constructed maps $\tilde{C}_n(A) \xrightarrow{e_\delta} \tilde{C}_{n-1}(A)$ and $\tilde{C}_n(A) \xrightarrow{E_\delta} \tilde{C}_{n+1}(A)$, for all $n \geq 0$, linearly depending on δ, satisfying the forumulas

$$[e_\delta, b] = 0, \qquad [e_\delta, B] + [E, b] = \bar{\delta}, \qquad [E_\delta, B] = 0.$$

In Cor. II.4.3 he used them to define a contraction

$$h : \prod_{m \in \mathbb{Z}} \tilde{C}_{n-2m}A = \mathrm{Tot}^{\times} MP(\tilde{C}_*A)_m \longrightarrow \mathrm{Tot}^{\times} MP(\tilde{C}_*A)_{m-1} = \prod_{n \in \mathbb{Z}} \tilde{C}_{n-2m-1}A,$$
$$(x_m)_{m \in \mathbb{Z}} \longmapsto (e(x_m) + E(x_{m+1}))_{m \in \mathbb{Z}},$$

for the composition $\mathrm{Tot}^{\times} MP(\tilde{C}_*A) \xrightarrow{\bar{\delta} \circ S} \mathrm{Tot}^{\times} MP(\tilde{C}_*A)_{*-2}$. As the shift map $HP_*(A) \xrightarrow{S} HP_{*-2}(A)$ is an isomorphism on the underlying complexes, it follows that also $L_\delta = 0$ on $HP_*(A)$.

□

5.3.2 Cyclic homology and filtrations

Proposition 5.3.2
Let $k \in \mathcal{C}Ring$ and $A \in k\text{-}\mathcal{A}ss_1$ carrying a (descending) algebra filtration F (see Definition 3.1.6) with $F_0A = A$.

Then the tensor product filtration (see Remark 3.1.4) defines an induced filtration F on the cyclic module $C_\bullet(A)$. Moreover the following holds.

(i) $F_1C_\bullet(A) = C_\bullet(A, F_1A) = \ker(C_\bullet(A) \twoheadrightarrow C_\bullet(A/F_1A))$.

(ii) *If $F_{r+1}A = 0$, then $C_\bullet(A, F_1A) \twoheadrightarrow C_\bullet(A, F_1A)/F_pC_\bullet(A, F_1A)$ is $(p/r-2)$-connected.*

(iii) *If $\mathrm{gr}^F A$ is flat and $(p-1)! \in \mathrm{gr}^F A$, then $HP_*(C_\bullet(A, F_1A)/F_pC_\bullet(A, F_1A)) = 0$.*

Proof. As F is an algebra filtration, multiplication $A \otimes A \xrightarrow{\mu} A$ and $k \xrightarrow{\eta} A$ are 0-equicontinuous for the tensor product filtration on $A \otimes A$ and the discrete filtration on k (i.e. $k = F_0k \supset F_1k = 0$). Hence face maps and degeneracies are 0-equicontinuous by definition of $C_\bullet(A)$. By construction of the tensor product filtration also the cyclic operator is 0-equicontinuous, so $C_\bullet(A)$ is a filtered cyclic module.

(i) Since $F_0A = A$, by construction of the tensor product filtration we have

$$F_1C_n(A) = \sum_{0 \leq i \leq n} A^{\otimes i} \otimes F_1A \otimes A^{\otimes(n-i)} = \ker(C_n(A) \twoheadrightarrow C_n(A/F_1A)), \qquad n \geq 0.$$

(ii) It suffices to prove that the given map is an isomorphism in dimensions $< p/r - 1$ or equivalently that $F_pC_n(A, F_1A) = F_pC_n(A) = 0$, for $n < p/r - 1$. Suppose $F_pC_n(A) \neq 0$. Then there is a tuple $m_0, ..., m_n \geq 0$ with $m_0 + ... + m_n \geq p$, such that

$F_{m_1}A \otimes ... \otimes F_{m_n} \neq 0$. As $F_{r+1}A = 0$, we can assume $m_0, ..., m_n \leq r$ and it follows that

$$p \leq m_0 + ... + m_n \leq (n+1)r < p,$$

a contradiction.

(iii) As $\mathrm{gr}^F A$ is flat we have $\mathrm{gr}^F C_\bullet(A) = C_\bullet(\mathrm{gr}^F A)$ by Proposition 3.1.5 and (i) implies

$$\mathrm{gr}^F MP(C_\bullet(A, F_1A)/F_p) = \bigoplus_{1 \leq m \leq p-1} MP(C_\bullet(\mathrm{gr}^F A)^{(m)}),$$

where the upper index i refers to the i-th graded part of the graded complex. There is a k-linear derivation δ on $\mathrm{gr}^F A$, given by $\delta(x) = m \cdot x$, for homogeneous $x \in \mathrm{gr}^F_m A$ (like in the proof of Proposition 3.3.9). By Proposition 5.3.1 it induces an endomorphism on the cyclic k-module $C_\bullet(A)$. As

$$L_\delta(x_0 \otimes ... \otimes x_n) = \sum_{0 \leq i \leq n} x_0 \otimes ... \otimes m_i \cdot x_i \otimes ... \otimes x_n = (m_0 + ... + m_n) \cdot x_0 \otimes ... \otimes x_n, \quad x_i \in \mathrm{gr}^F_{m_i} A,$$

we see that L_δ is multiplication by m on $C_\bullet(\mathrm{gr}^F A)^{(m)}$. As by assumption $(p-1)! \in \mathrm{gr}^F A$, the map L_δ is an isomorphism on $\mathrm{gr}^F MP(C_\bullet(A, F_1A)/F_p)$. As L_δ is homotopic to zero by Proposition 5.3.1, it follows that $\mathrm{gr}^F MP(C_\bullet(A, F_1A)/F_p)$ is acyclic and thus also $MP(C_\bullet(A, F_1A)/F_m)$ is acyclic by induction on $1 \leq m \leq p$ using the long exact sequence. This proves (iii).

□

Corollary 5.3.3
Let $k \in \mathcal{C}Ring$ and $A \in k\text{-}\mathcal{A}ss_1$ carrying a (descending) algebra filtration F with $(p-1)! \in A = F_0A$ and flat $\mathrm{gr}^F A$.
Then the map below is $(p/r - 1)$-connected

$$HC_{*-1}(A, F_1A) \longrightarrow HC_{*-1}(C_\bullet(A, F_1A)/F_p) \xrightarrow{B} HC^-_*(C_\bullet(A, F_1A)/F_p).$$

Proof. By Proposition 5.3.2 (ii) the map $C_\bullet(A, F_1A) \twoheadrightarrow C_\bullet(A, F_1)/F_p$ is $(p/r - 2)$-connected. Using the bottom long exact sequence of Remark 5.1.10 it follows that also $HC_*(A, F_1A) \longrightarrow HC_*(C_\bullet(A, F_1A)/F_p)$ is $(p/r - 2)$-connected. Since the periodic cyclic homology $HP_*(C_\bullet(A, F_1A)/F_pC_\bullet(A, F_1A))$ vanishes by Proposition 5.3.2, the map B is an isomorphism by the middle long exact sequence of Remark 5.1.10.

□

The connectivity of the map given in the preceding corollary depends inversely proportionally on r. As this is the bottleneck for the connectivity in comparing relative K-groups and cyclic homology via the negative Chern character, it would be desirable to construct a $(p-1)$-connected map to (some modified) negative cyclic homology. However the following example demonstrates that this seems to be impossible in general.

Proposition 5.3.4
Let $k \in \mathcal{C}Ring$ and $A = \mathcal{A}ss_1^{<r}(k) = \mathbb{Z}[t]/(t^r)$, for some $r \geq 2$.

Then the (t)-adic filtration F is an algebra filtration on A and $\operatorname{gr}^F A \cong A$ is flat. For $k = \mathbb{Z}$, we have

$$HH_n(A, F_1A) = HH_n(\mathbb{Z}[t]/(t^r), (t)) = \begin{cases} \mathbb{Z}^{r-1}, & n \in 2\mathbb{N}_0, \\ \mathbb{Z}^{r-2} \oplus \mathbb{Z}/r, & n \in 1 + 2\mathbb{N}_0. \end{cases}$$

More precisely the non-zero homology groups of the homogenous parts are given by

$$\begin{array}{ll} H_{2d}(\operatorname{gr}^F_m C_\bullet(A)) = H_{2d+1}(\operatorname{gr}^F_m C_\bullet(A)) = \mathbb{Z}, & m \notin r\mathbb{N}, \\ H_{2d+1}(\operatorname{gr}^F_m C_\bullet(A)) = \mathbb{Z}/r, & m \in r\mathbb{N}, \end{array}$$

where is the integer part $d = \lfloor \frac{m-1}{r} \rfloor$. Connes operator B on $H_*(\operatorname{gr}^F_m C_\bullet(A))$ is given by

$$B : H_{2d}(\operatorname{gr}^F_m C_\bullet(A)) \xrightarrow{-m} H_{2d+1}(\operatorname{gr}^F_m C_\bullet(A)).$$

Proof. See [HM97a] section 2.1. See also [Gro94] for the original reference. □

Remark 5.3.5
Let $p > 1$ be a prime number, $k = \mathbb{Z}[(p-1)!^{-1}]$ and $A = \mathcal{A}ss_1^{<r}(k) = \mathbb{Z}[t]/(t^r)$, for some $2 \leq r \nmid p$.

Then the preceding proposition implies the following facts.

(i) *$H_*(\operatorname{gr}^F_p C_\bullet(A))$ has a non-trivial generator in dimension $1 + 2 \cdot \lfloor \frac{p-1}{r} \rfloor$. The same holds for $HC_*(\operatorname{gr}^F_p C_\bullet(A))$ by inspection of the middle long exact sequence of Remark 5.1.10.*

   In particular the map $HC_*(A, F_1A) \longrightarrow HC_*(C_\bullet(A, F_1A)/F_p)$ is not $2 \cdot (1 + \lfloor \frac{p-1}{r} \rfloor)$-connected.

(ii) *One might try to come around this problem by considering $C_*(A, F_1A) \twoheadrightarrow C_*(A, F_1A)/F_q$ with $q > p$. For $q = pr$ for example, this map is $(p-2)$-connected by Proposition 5.3.2.*

   But this ruins the connectivity of the second map

$$HC_{*-1}(C_\bullet(A, F_1A)/F_q) \xrightarrow{B} HC^-_*(C_\bullet(A, F_1A)/F_q).$$

   Indeed using the spectral sequence for the double complex $MP(\operatorname{gr}^F_p C_\bullet(A))$, we get

$$HP_n(\operatorname{gr}^F_p C_\bullet(A)) = \begin{cases} \prod_{i \in \mathbb{Z}} \mathbb{Z}/p, & n \in 1 + 2\mathbb{Z}, \\ 0, & n \in 2\mathbb{Z}, \end{cases}$$

   which implies that B is no isomorphism in any dimension as we can see from the bottom long exact sequence of Remark 5.1.10.

5.4 Homology of matrix Lie algebras

Following ideas of [AO94] we are now generalizing the Theorem of Loday-Quillen and Tsygan to the integral situation.

5.4.1 Adjoint actions on matrix Lie algebras

The main ideas for the proofs of Proposition 5.4.2 and Proposition 5.4.3 are due to Aboughazi-Ogle [AO94], who worked them out mainly in the rational setting. We will recall them to point out the technical differences in the integral setting. Moreover we need more precise statements later on.

Proposition 5.4.1
For $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ and $V \in \mathrm{Mod}\text{-}U_k(\mathfrak{g})$, the adjoint action of $\mathfrak{g}$ on $H_*(\mathfrak{g}, V)$ is trivial.

Proof. For $y \in \mathfrak{g}$, the maps

$$s_y : V \otimes \Lambda_n \mathfrak{g} \longrightarrow V \otimes \Lambda_{n+1}\mathfrak{g}, \qquad v \otimes x_1 \wedge \dots \wedge x_n \longmapsto v \otimes y \wedge x_1 \wedge \dots \wedge x_n$$

define a chain homotopy $[\text{-}, y] \simeq 0$ on the Chevalley-Eilenberg complex $V \otimes \Lambda_* \mathfrak{g}$.

If the Chevalley-Eilenberg complex does not compute Lie algebra homology, one can argue by induction on $n \geq 0$, that $[\text{-}, y]$ induces the zero map on homology. Indeed, for $n = 0$, this follows from the fact, that $H_0(\mathfrak{g}, V) = V/[V, \mathfrak{g}]$. Suppose it holds for some $n \geq 0$ and all $V \in \mathrm{Mod}\text{-}U_k(\mathfrak{g})$. Take an epimorphism $F \overset{f}{\twoheadrightarrow} V$, where F is a free $\mathfrak{g}$-module. We have a commutative diagram of long exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & H_{n+1}(\mathfrak{g}, F) \arrow[r, "f"] \arrow[d, "{[\text{-},y]}"'] & H_{n+1}(\mathfrak{g}, V) \arrow[r, "\partial"] \arrow[d, "{[\text{-},y]}"'] & H_n(\mathfrak{g}, \ker f) \arrow[r] \arrow[d, "{[\text{-},y]}"'] & \dots \\
\dots \arrow[r] & H_{n+1}(\mathfrak{g}, F) \arrow[r, "f"] & H_{n+1}(\mathfrak{g}, V) \arrow[r, "\partial"] & H_n(\mathfrak{g}, \ker f) \arrow[r] & \dots
\end{tikzcd}$$

As F is free $H_{n+1}(\mathfrak{g}, F) = 0$ and so ∂ is injective. Hence $[\text{-}, y]$ vanishes on $H_{n+1}(\mathfrak{g}, V)$ as it vanishes on $H_n(\mathfrak{g}, \ker f)$ by induction hypothesis. $\square$

Proposition 5.4.2 (Aboughazi-Ogle)
Let $A \in k/\mathcal{R}ing$ and $r \geq 1$. Then the matrix ring $M_r(A)$ is totally $\mathbb{Z}^r$-graded via

$$M_r(A)^{(v)} = \bigoplus_{\substack{1 \leq i,j \leq r, \\ v(i,j) := e_j - e_i = v}} A \cdot e_{i,j}, \qquad v \in \mathbb{Z}^r.$$

In particular:

(i) *It induces a total $\mathbb{Z}^r$-grading on the bar construction $B_* M_r(A)$.*

(ii) *With the same grading $\mathfrak{gl}_r(A)$ becomes a totally $\mathbb{Z}^r$-graded Lie algebra.*

(iii) *We get a total $\mathbb{Z}^r$-grading on the Chevalley-Eilenberg complex $\Lambda_*\mathfrak{gl}_r(A) = B_*M_r(A)^{\Sigma_*}$.*

Proof. Let $1 \leq i, j, k, \ell \leq r$ and $a, b \in A$. We have

$$(a \cdot e_{i,j}) \cdot (b \cdot e_{k,\ell}) = \delta_{j,k} \cdot (ab) \cdot e_{i,\ell},$$

where $\delta_{j,k}$ is the Kronecker delta. Moreover, if $j = k$, then

$$v(i,\ell) = e_\ell - e_i = e_\ell - e_k + e_j - e_i = v(k,\ell) + v(i,j).$$

In particular, whenever the right term does not vanish, it lies in degree $v(i,j) + v(k,\ell)$. This proves

$$M_r(A)^{(v)} \cdot M_r(A)^{(w)} \subset M_r(A)^{(v+w)}, \qquad v, w \in \mathbb{Z}^r,$$

i.e. $M_r(A)$ is a totally graded ring.

□

Proposition 5.4.3 (Aboughazi-Ogle)
Let $A \in k/\mathcal{R}ing$ be flat over k and let $r \geq 1$.

(i) *$H_*(\Lambda_*\mathfrak{gl}_r A)^{(v)}$ is* $\gcd(v_1, ..., v_r)$*-torsion, for every $v \in \mathbb{Z}^r$.*

(ii) *If $(p-1)! \in A^\times$, then $\Lambda_*\mathfrak{gl}_r(A) \twoheadrightarrow (\Lambda_*\mathfrak{gl}_r(A))^{(0)}$ is $(p-1)$-connected.*

Proof.

(i) For $1 \leq i, j, k \leq r$ and $a \in A$ we have

$$[a \cdot e_{i,j}, e_{k,k}] = (\delta_{j,k} - \delta_{i,k}) \cdot (a \cdot e_{i,j}) = v(i,j)_k \cdot (a \cdot e_{i,j}),$$

where δ is the Kronecker delta. In particular the adjoint action of $e_{k,k}$ on $(\Lambda_*\mathfrak{gl}_r A)^{(v)}$ is multiplication by v_k. As [-, $e_{k,k}$] is homotopic to 0 on homology by Proposition 5.4.1, it follows that $H_*(\Lambda_*\mathfrak{gl}_r A)^{(v)}$ is v_k-torsion and thus is $\gcd(v_1, ..., v_r)$-torsion by varying $1 \leq k \leq r$.

(ii) Consider an arbitrary generator

$$(a_1 \cdot e_{i_1,j_1}) \wedge ... \wedge (a_n \cdot e_{i_n,j_n}) \in \Lambda_n\mathfrak{gl}_r A, \qquad n \geq 0.$$

It lives in degree $v = v(i_1, j_1) + ... + v(i_n, j_n)$ and so we have $|v_k| \leq n$, for all $1 \leq k \leq r$. In particular we get $\gcd(v_1, ..., v_r) \leq n$ and $n \leq p-1$ implies $v = 0$ or $\gcd(v_1, ..., v_r) \in A^\times$. Hence in the chain of projections

$$\Lambda_*\mathfrak{gl}_r A \twoheadrightarrow \bigoplus_{\substack{v \in \mathbb{Z}^r, \\ \gcd(v_1,...,v_r) \notin A^\times}} (\Lambda_*\mathfrak{gl}_r A)^{(v)} \twoheadrightarrow (\Lambda_*\mathfrak{gl}_r A)^{(0)},$$

the first map is a quasi-isomorphism by (ii) and the the second map is an isomorphism on chains in dimensions $n \leq p-1$ and an epimorphism in dimension $n = p$.

□

Proposition 5.4.4
Let $A \in k/\mathcal{R}ing$ be flat with $(p-1)! \in A^\times$, for some $p > 1$, and $r \geq 1$.
Then conjugation by $E_r(A)$ induces the identity on $H_*(\mathfrak{gl}_r A, k)$ in dimensions $< p$.

Proof. The group $E_r(A)$ is generated by elementary matrices $1+a\cdot e_{i,j}$, for some $a \in A$ and $1 \leq i,j \leq r$ with $i \neq j$. Let α denote the automorphism of $\mathfrak{gl}_r A$ induced by conjugation with $1 + a \cdot e_{i,j}$. It extends to an automorphism $\bar{\alpha}$ of chain complexes $\Lambda_* \mathfrak{gl}_r A$. We claim that the following composition is zero

$$(\Lambda_*\mathfrak{gl}_r A)^{(0)} \xhookrightarrow{\iota} \Lambda_*\mathfrak{gl}_r A \xrightarrow{1-\bar{\alpha}} \Lambda_*\mathfrak{gl}_r A \xrightarrow{\pi} (\Lambda_*\mathfrak{gl}_r A)^{(0)}, \qquad n \geq 1,$$

where the first map is the canonical inclusion and the last map the projection. As π is $(p-1)$-connected by Proposition 5.4.3 and ι is a section for π, we get that $1 - \bar{\alpha}$ is zero on all homology groups $H_*(\mathfrak{gl}_r A, k)$ in dimension $(p-1)$, which will imply the desired statement.

(i) First suppose $\operatorname{char} A = 0$. For $b \in A$ and $1 \leq k, \ell \leq r$ we have

$$\begin{aligned}
(b \cdot e_{k,\ell})^{1+a\cdot e_{i,j}} &= (1 - a \cdot e_{i,j})(b \cdot e_{k,\ell})(1 + a \cdot e_{i,j}) \\
&= b \cdot e_{k,\ell} + \delta_{\ell,i} \cdot ba \cdot e_{k,j} - \delta_{j,k} \cdot ab \cdot e_{i,\ell} - \delta_{j,k} \cdot \delta_{\ell,i} \cdot aba \cdot e_{i,j} \\
&= b \cdot e_{k,\ell} + [b \cdot e_{k,\ell}, a \cdot e_{i,j}] + 1/2 \cdot [[b \cdot e_{k,\ell}, a \cdot e_{i,j}], a \cdot e_{i,j}],
\end{aligned}$$

where δ is the Kronecker delta. So $\alpha = \exp(\beta)$, where $\beta := [\text{-}, a \cdot e_{i,j}]$. The inner derivation β also induces an endomorphism of chain complexes $\bar{\beta}$ on $\Lambda_*\mathfrak{gl}_r A$, which is a derivation with respect to the underlying exterior algebra. Since $\beta^3 = 0$, it follows that $\bar{\beta}^{2n+1} = 0$, when restricted to $\Lambda_n \mathfrak{gl}_r A$. Hence the series $\exp(\bar{\beta})$ converges in $\mathbb{Q} \otimes \Lambda_*\mathfrak{gl}_r A$. As $\bar{\beta}$ is a derivation $\exp(\bar{\beta})$ is an algebra endomorphism of $\mathbb{Q} \otimes \Lambda_*\mathfrak{gl}_r A$. Moreover also $\bar{\alpha}$ is an algebra automorphism. As $\alpha = \exp(\beta)$ on the generators $\mathbb{Q} \otimes \mathfrak{gl}_r A \subset \mathbb{Q} \otimes \Lambda_*\mathfrak{gl}_r A$, we thus have $\bar{\alpha} = \exp(\bar{\beta})$. Now using that $i \neq j$ we see that $\bar{\beta}(\Lambda_*\mathfrak{gl}_r A)^{(0)} \subset (\Lambda_*\mathfrak{gl}_r A)^{(\neq 0)}$ and hence the composition

$$(\Lambda_*\mathfrak{gl}_r A)^{(0)} \xhookrightarrow{\iota} \Lambda_*\mathfrak{gl}_r A \xrightarrow{\bar{\beta}} \Lambda_*\mathfrak{gl}_r A \xrightarrow{\pi} (\Lambda_*\mathfrak{gl}_r A)^{(0)},$$

is zero. It follows that also $\pi(1/n!\bar{\beta}^n)\iota = 0$, for all $n \geq 1$, and so $\pi(1-\bar{\alpha})\iota = \pi(1-\exp(\bar{\beta}))\iota = 0$ on $\mathbb{Q} \otimes (\Lambda_*\mathfrak{gl}_r A)^{(0)}$. This implies $\pi(1-\bar{\alpha})\iota = 0$ on $(\Lambda_*\mathfrak{gl}_r A)^{(0)}$, because we assumed that $\operatorname{char} A = 0$ and that A is flat over k.

(ii) If $\operatorname{char} A \neq 0$ consider the ring epimorphisms from the monoid rings

$$k' := \mathbb{Z}[(k, \cdot, 1)] \twoheadrightarrow k, \qquad A' := \mathbb{Z}[(A, \cdot, 1)] \twoheadrightarrow A,$$

As $A \in k/\mathcal{R}ing$ we have $A' \in k'/\mathcal{R}ing$ by construction. For $a \in A$ we take a lift $a' \in A'$, which induces a lift $\alpha' = (\text{-})^{1+a'\cdot e_{i,j}}$ for $\alpha = (\text{-})^{1+a\cdot e_{i,j}}$, meaning that we get a commuting diagram

$$\begin{tikzcd}
(\Lambda_*\mathfrak{gl}_r A')^{(0)} \arrow[r, two heads] \arrow[d, "1-\bar{\alpha}'"'] & (\Lambda_*\mathfrak{gl}_r A)^{(0)} \arrow[d, "1-\bar{\alpha}"] \\
(\Lambda_*\mathfrak{gl}_r A')^{(0)} \arrow[r, two heads] & (\Lambda_*\mathfrak{gl}_r A)^{(0)}.
\end{tikzcd}$$

Since $\operatorname{char} A' = 0$, by (i) the left map and thus also the right map is zero.

□

Corollary 5.4.5
Let $A \in k/\mathcal{R}ing$ be flat with $(p-1)! \in A^{\times}$, for some $p > 1$, and $r \geq 1$.
Then conjugation by Σ_r induces the identity on $H_*(\mathfrak{gl}_r A, k)$ in dimensions $< p$.

Proof. Given a transposition $\tau = (k\ \ell) \in \Sigma_r$, where $1 \leq k < \ell \leq r$, we have to show that conugation by τ is the identity on homology. Let $T \in GL_r A$ which is equal to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the summand $A^2 \cong A \cdot e_k \oplus A \cdot e_\ell$ and the identity on the complement. Since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

it follows that $T \in E_r A$ and thus by Proposition 5.4.4 conjugation by T induces the identity on $H_*(\Lambda_* \mathfrak{gl}_r A)$ in dimensions $< p$. Moreover conjugation by T respects the direct sum decomposition $\Lambda_* \mathfrak{gl}_r A = (\Lambda_* \mathfrak{gl}_r A)^{(0)} \oplus (\Lambda_* \mathfrak{gl}_r A)^{(\neq 0)}$, because

$$T e_{i,j} T^{-1} = (-1)^{\delta_{k,i} + \delta_{k,j}} \cdot e_{\tau(i),\tau(j)}, \qquad 1 \leq i,j \leq r,$$

where δ is the Kronecker delta. By construction of the $\mathbb{Z}^r$-grading, $(\Lambda_n \mathfrak{gl}_r A)^{(0)}$ is spanned by

$$a_1 \cdot e_{i_1,j_1} \wedge \ldots \wedge a_n \cdot e_{i_n,j_n}, \qquad v(i_1,j_1) + \ldots + v(i_n,j_n) = 0,$$

which under conjugation by T equals

$$(-1)^{\delta_{k,i_1} + \delta_{k,j_1} + \ldots + \delta_{k,i_n} + \delta_{k,j_n}} \cdot a_1 \cdot e_{\tau(i_1),\tau(j_1)} \wedge \ldots \wedge a_n \cdot e_{\tau(i_n),\tau(j_n)}.$$

Now $0 = v(i_1,j_1) + \ldots + v(i_n,j_n) = e_{i_1} - e_{j_1} + \ldots + e_{i_n} - e_{j_n} \in \mathbb{Z}^r$ implies that

$$|\{1 \leq m \leq r;\ i_m = k\}| = |\{1 \leq m \leq r;\ j_m = k\}|,$$

and hence $\delta_{k,i_1} + \delta_{k,j_1} + \ldots + \delta_{k,i_n} + \delta_{k,j_n}$ is even. This proves that conjugation by τ equals conjugation by T on $(\Lambda_* \mathfrak{gl}_r A)^{(0)}$ and thus is trivial on homology.

□

Remark 5.4.6
Note that the conjugation action by $\Sigma_r \leq GL_r(A)$ respects the decomposition

$$\Lambda_* \mathfrak{gl}_r A = (\Lambda_* \mathfrak{gl}_r A)^{(0)} \oplus (\Lambda_* \mathfrak{gl}_r A)^{(\neq 0)},$$

and thus determines an action on $(\Lambda_* \mathfrak{gl}_r A)^{(\neq 0)}$.

5.4.2 Stability in the Lie algebra homology of matrices

The following Proposition is well-known from the computation of the homology groups of the symmetric groups (see e.g. [Nak60] or [AM04]). However there is no reference for precisely the statement that we are interested in. For the convenience of the reader we will give a short proof.

The computation is essential in the verification of the stability results. Like the integral version of the Theorem of Loday-Quillen and Tsygan, these cannot be found in present litarture.

Proposition 5.4.7
Let $k \in \mathcal{C}Ring$ with $(p-1)! \in k^\times$, for some $p > 1$.
Then $H_n(\Sigma_r, k) = 0$, for all $r \geq 1$ and $0 < n < p$.
In particular this also holds for $r = \infty$.

Proof. For $1 \leq r < p$ we have $|\Sigma_r| = r! \in k^\times$ and Remark 5.1.8 yields $H_n(\Sigma_r, k) = 0$, for all $n \geq 1$. For $r = p$, let C_p be the cyclic group with p elements, which we consider as a subgroup of Σ_p by the regular representation

$$C_p \hookrightarrow \mathcal{S}et(C_p)^\times \cong \Sigma_p.$$

The C_p-orbits Σ_p/C_p under the left C_p-action have a canonical right Σ_p-action and the quotient map $k[\Sigma_p/C_p] \twoheadrightarrow k$ has Σ_p-linear section

$$k \longrightarrow k[\Sigma_p/C_p], \qquad a \longmapsto a/(p-1)! \cdot \sum_{x \in \Sigma_p/C_p} x,$$

because by assumption $(p-1)! \in k^\times$. So we get a retraction

$$\begin{tikzcd}
H_*(C_p, k) \arrow[rr] \arrow[d, equal] & & H_*(\Sigma_p, k) \arrow[d, equal] \\
\operatorname{Tor}^{k[\Sigma_p]}(k[\Sigma_p/C_p], k) \arrow[rr, two heads] & & \operatorname{Tor}^{k[\Sigma_p]}(k, k). \arrow[ll, bend left]
\end{tikzcd}$$

Now C_p is normal in $\mathcal{G}rp(C_p)^\times \hookrightarrow \mathcal{S}et(C_p)^\times \cong \Sigma_p$, which acts trivially on $H_*(\Sigma_p, k)$. Hence the upper retraction factors as

$$H_*(C_p, k) \twoheadrightarrow H_*(C_p, k)_{\mathcal{G}rp(C_p)^\times} \twoheadrightarrow H_*(\Sigma_p, k).$$

Using the standard resolution of $k \in k[C_p]$-Mod, given by

$$k \twoheadleftarrow k[C_p] \xleftarrow{1-t} k[C_p] \xleftarrow{N} k[C_p] \xleftarrow{1-t} \ldots,$$

where $t \in C_p$ is a generator, we compute

$$H_n(C_p, k) = \begin{cases} k, & n = 0, \\ k \otimes \mathbb{F}_p, & n \in 2 \cdot \mathbb{N}_0 + 1, \\ \mathcal{A}b(\mathbb{F}_p, k), & n \in 2 \cdot \mathbb{N}_0 + 2. \end{cases}$$

with the canonical action of $\mathcal{G}rp(C_p) \cong \mathbb{F}_p^\times$. It follows that $0 = H_n(C_p, k)_{\mathcal{G}rp(C_p)^\times} \twoheadrightarrow H_n(\Sigma_p, k)$, for all $n \geq 1$, which proves the case $r = p$.

For $p < r < 2p$, we can use the same argument as before, to prove that the canonical inclusion $\Sigma_p \hookrightarrow \Sigma_r$ induces an isomorphism

$$H_*(\Sigma_p, k) \xrightarrow{\sim} H_*(\Sigma_r, k).$$

Finally we use Nakaoka's stability Theorem [Nak60] (see also [Ker05] for a shorter proof) to see that

$$H_n(\Sigma_{2p-1}, k) \xrightarrow{\sim} H_n(\Sigma_{2p}, k) \xrightarrow{\sim} H_n(\Sigma_{2p+1}, k) \xrightarrow{\sim} \dots \xrightarrow{\sim} H_*(\Sigma_\infty, k), \qquad 0 \leq n < p.$$

$\square$

Proposition 5.4.8
Let $A \in k/\mathcal{R}ing$ be flat with $(p-1)! \in k^\times$, for some $p > 1$.
Then $H_n(\Lambda_* \mathfrak{gl}_r A) \xrightarrow{\sim} H_n((\Lambda_* \mathfrak{gl}_r A)_{\Sigma_r})$, for $0 \leq n < p$ and all $r \geq 1$.
Note that the proof even works for $r = \infty$, i.e. $\mathbf{r} = \mathbb{N}$.

Proof. First we claim that

$$H_m(\Sigma_r, (\mathfrak{gl}_r A)^{\otimes n}) = 0, \qquad 0 < m < p, \qquad n \geq 0.$$

This is independent of the ring structure of A and as the functor commutes with filtered colimits in A, we can assume that $A \in k$-Mod is free, because by assumption A is flat over k. If $\bar{A} \subset A$ is a k-basis, then we obtain an isomorphism of k-modules

$$\varphi : k(\bar{A} \times \mathbf{r} \times \mathbf{r})^n \xrightarrow{\sim} (\mathfrak{gl}_r A)^{\otimes n}, \quad (a_\ell, i_\ell, j_\ell)_{1 \leq \ell \leq n} \longmapsto a_1 \cdot e_{i_1, j_1} \otimes \dots \otimes a_n \cdot e_{i_n, j_n},$$

which becomes Σ_r-linear, when we define a Σ_r-action on $(\bar{A} \times \mathbf{r} \times \mathbf{r})^n \cong \mathcal{S}et(\mathbf{n}, \bar{A} \times \mathbf{r} \times \mathbf{r})$ by setting

$$\sigma \cdot f = (\mathrm{id} \times \sigma \times \sigma) \circ f, \qquad \sigma \in \Sigma_r, \quad f \in \mathcal{S}et(\mathbf{n}, \bar{A} \times \mathbf{r} \times \mathbf{r}).$$

For $f \in \mathcal{S}et(\mathbf{n}, \bar{A} \times \mathbf{r} \times \mathbf{r})$ we define its support as

$$\operatorname{supp} f = \pi_1 \circ f(\mathbf{n}) \cup \pi_2 \circ f(\mathbf{n}),$$

where $\pi_1, \pi_2 : \bar{A} \times \mathbf{r} \times \mathbf{r} \longrightarrow \mathbf{r}$ are the projections onto the first and second factor $\mathbf{r}$. Then the isotropy group Σ_f of f with respect to the Σ_r-action is given by

$$\Sigma_f = \Sigma_{\mathbf{r} \setminus \operatorname{supp} f} = \mathcal{S}et(\mathbf{r} \setminus \operatorname{supp} f)^\times \leq \mathcal{S}et(\mathbf{r})^\times = \Sigma_r.$$

Using Shapiro's Lemma and Proposition 5.4.7 we can compute the homology of Σ_r with the free k-module generated by the orbit $\Sigma_r \cdot f$ as coefficients as

$$H_m(\Sigma_r, k(\Sigma_r \cdot f)) = H_m(\Sigma_r, k(\Sigma_r / \Sigma_f)) = H_m(\Sigma_f, k) = 0, \qquad 0 < m < p.$$

Taking the union over all orbits and using the upper isomorphism φ it follows that

$$H_m(\Sigma_r, (\mathfrak{gl}_r A)^{\otimes n}) \xleftarrow{\sim} H_m(\Sigma_r, k\mathcal{S}et(\mathbf{n}, \bar{A} \times \mathbf{r} \times \mathbf{r})) = 0, \qquad 0 < m < p,$$

which proves the claim.

Now for $n < p$ we have $n! \in k^\times$ by assumption and thus the quotient map $(\mathfrak{gl}_r A)^{\otimes n} \twoheadrightarrow \Lambda_n \mathfrak{gl}_r A$ has a Σ_r-linear section given by

$$\Lambda_n \mathfrak{gl}_r A \hookrightarrow (\mathfrak{gl}_r A)^{\otimes n}, \quad a_1 \wedge ... \wedge a_n \longmapsto 1/n! \cdot \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \cdot a_{\sigma(1)} \otimes ... \otimes a_{\sigma(n)}.$$

So our claim implies that

$$H_m(\Sigma_r, \Lambda_n \mathfrak{gl}_r A) \twoheadleftarrow H_m(\Sigma_r, (\mathfrak{gl}_r A)^{\otimes n}) = 0, \qquad 0 < m < p, \quad 0 \leq n < p.$$

In other words the spectral sequence of Remark 8.2.2 (i) for the constant group Σ_r

$$E^1_{n,m} = H_m(\Sigma_r, \Lambda_n \mathfrak{gl}_r A) \quad \Rightarrow \quad H_{n+m}(\Sigma_r, \Lambda_* \mathfrak{gl}_r A),$$

collapses in low dimensions and it follows that the edge map is an isomorphism

$$H_n(\Sigma_r, \Lambda_* \mathfrak{gl}_r A) \xrightarrow{\sim} E^2_{n,0} = H_n((\Lambda_* \mathfrak{gl}_r A)_{\Sigma_r}), \qquad 0 \leq n < p.$$

Now Corollary 5.4.5 states that Σ_r acts trivially on $H_n(\mathfrak{gl}_r A, k) = H_n(\Lambda_* \mathfrak{gl}_r A)$, for all $0 \leq n < p$, and so Proposition 5.4.7 implies that

$$H_m(\Sigma_r, H_n(\Lambda_* \mathfrak{gl}_r A)) = \begin{cases} H_n(\Lambda_* \mathfrak{gl}_r A), & m = 0, \quad 0 \leq n < p, \\ 0, & 0 < m < p, \quad 0 \leq n < p, \end{cases}$$

which is the second page of the other spectral sequence of Remark 8.2.2 (ii) converging to $H_{m+n}(\Sigma_r, \Lambda_* \mathfrak{gl}_r A)$. It follows that also the edge map

$$H_n(\Lambda_* \mathfrak{gl}_r A) = E^2_{0,n} \xrightarrow{\sim} H_n(\Sigma_r, \Lambda_* \mathfrak{gl}_r A), \qquad 0 \leq n < p$$

is an isomorphism and it remains to note that the composition

$$H_n(\Lambda_* \mathfrak{gl}_r A) \xrightarrow{\sim} H_n(\Sigma_r, \Lambda_* \mathfrak{gl}_r A) \xrightarrow{\sim} H_n((\Lambda_* \mathfrak{gl}_r A)_{\Sigma_r}), \qquad 0 \leq n < p$$

is induced by the quotient map $\Lambda_* \mathfrak{gl}_r A \twoheadrightarrow (\Lambda_* \mathfrak{gl}_r A)_{\Sigma_r}$, which finally proves the Proposition.

□

Proposition 5.4.9 (Homology stability)
Let $A \in k/\mathcal{R}ing$ be flat with $(p-1)! \in A^\times$, for some $p > 1$.
Then $H_*(\mathfrak{gl}_{r-1} A, k) \longrightarrow H_*(\mathfrak{gl}_r A, k)$ is $(\min(r,p) - 2)$-connected, for all $r \geq 1$.

Proof. The conjugation action by $\Sigma_r \leq GL_r(A)$ respects the decomposition

$$\Lambda_* \mathfrak{gl}_r A = (\Lambda_* \mathfrak{gl}_r A)^{(0)} \oplus (\Lambda_* \mathfrak{gl}_r A)^{(\neq 0)},$$

hence in the commutative diagram

$$\begin{tikzcd}
\Lambda_* \mathfrak{gl}_r A \arrow[r, two heads] \arrow[d] & (\Lambda_* \mathfrak{gl}_r A)^{(0)} \arrow[d] \arrow[l, bend right] \\
(\Lambda_* \mathfrak{gl}_r A)_{\Sigma_r} \arrow[r, two heads] & (\Lambda_* \mathfrak{gl}_r A)^{(0)}_{\Sigma_r} \arrow[l, bend left]
\end{tikzcd}$$

the horizontal maps are retractions. Moreover the upper horizontal map is $(p-1)$-connected by Proposition 5.4.3 (ii), while the left vertical map is an isomorphism in homology in dimensions $< p$ by Propposition 5.4.8. It follows that the same holds for the right vertical map and hence also for the lower horizontal map.

As this observation was independent of $r \geq 0$, we see that all the horizontal maps induce isomorphisms on homology in dimensions $< p$ in the commutative diagram

$$\begin{tikzcd}
\Lambda_* \mathfrak{gl}_{r-1} A \arrow[r] \arrow[d] & (\Lambda_* \mathfrak{gl}_{r-1} A)^{(0)} \arrow[r] \arrow[d] & (\Lambda_* \mathfrak{gl}_{r-1} A)^{(0)}_{\Sigma_r} \arrow[d] \\
\Lambda_* \mathfrak{gl}_r A \arrow[r] & (\Lambda_* \mathfrak{gl}_r A)^{(0)} \arrow[r] & (\Lambda_* \mathfrak{gl}_r A)^{(0)}_{\Sigma_r}.
\end{tikzcd}$$

Now by construction of the $\mathbb{Z}^r$-grading we have

$$(\Lambda_n \mathfrak{gl}_r A)^{(0)} = \bigoplus_{\substack{1 \leq i_1, j_1, \dots, i_n, j_n \leq r, \\ v(i_1, j_1) + \dots + v(i_n, j_n) = 0}} A \cdot e_{i_1, j_1} \wedge \dots \wedge A \cdot e_{i_n, j_n}, \qquad r, n \geq 0.$$

Moreover multiplication by an element $\sigma \in \Sigma_r$ identifies summands corresponding to tuples

$$((i_1, j_1), \dots, (i_n, j_n)) \sim ((\sigma(i_1), \sigma(j_1)), \dots, (\sigma(i_n), \sigma(j_n))).$$

Using that $0 = v(i_1, j_1) + \dots + v(i_n, j_n) = e_{i_1} - e_{j_1} + \dots + e_{i_n} - e_{j_n} \in \mathbb{Z}^r$, it follows that

$$\{i_1, j_1, \dots, i_n, j_n\} = \{i_1, \dots, i_n\},$$

whose cardinality is n at most. Thus $(\Lambda_n \mathfrak{gl}_r A)^{(0)}_{\Sigma_r}$ is spanned by the summands corresponding to tuples with $(i_m, j_m) \in \mathbf{n} \times \mathbf{n}$, for all $1 \leq m \leq n$. This proves that the right vertical map in the diagram above is an isomorphism in dimensions $< r$ and thus is $(r-2)$-connected.

□

5.4.3 The differential graded Hopf algebra of matrices

Recall from Remark 4.1.2 that there are functors

$$\mathbb{N}_0 \xrightarrow{\nu} \hat{\Delta}_{inj} \xrightarrow{\alpha} I \xrightarrow{\tau} \mathbb{N}_0,$$

where we consider the partially ordered set $(\mathbb{N}_0, \leq)$ as a category and I is the category of injections on the sets

$$\mathbf{n} = \{1, \dots, n\}, \qquad n \geq 0.$$

Proposition 5.4.10
For $A \in k/\mathcal{R}ing$, the following holds.

(i) *There is a functor*

$$M_\bullet A : I \longrightarrow k\text{-}\mathcal{A}ss, \qquad \mathbf{r} \longmapsto M_{\mathbf{r}}A = M_r A = A^{r\times r},$$

sending a morphism $f \in I(\mathbf{r}, \mathbf{s})$ to the map $M_r A = A^{\mathbf{r}\times\mathbf{r}} \xrightarrow{A^{f\times f}} A^{\mathbf{s}\times\mathbf{s}} = M_s A$.

(ii) *The block sum of matrices induces a multiplication map*

$$\mu : M_r A \times M_s A \longrightarrow M_{r+s}A, \qquad (X, Y) \longmapsto X \oplus Y := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

and the canonical morphism $\eta : 0 \xrightarrow{\sim} M_{\mathbf{0}}A$ a unit map, making $M_\bullet A$ into a symmetric monoidal functor.

(iii) *Composing with $k\text{-}\mathcal{A}ss \longrightarrow k\text{-}\mathcal{L}ie \xrightarrow{\Lambda_*} dg(k\text{-}\widehat{\mathcal{C}om_1}^{\mathrm{op}})$, we get a functor*

$$\Lambda_*\mathfrak{gl}_\bullet A : I \longrightarrow dg(k\text{-}\widehat{\mathcal{C}om_1}^{\mathrm{op}}), \qquad \mathbf{n} \longmapsto \Lambda_*\mathfrak{gl}_n A,$$

which is symmetric monoidal, when giving $dg(k\text{-}\widehat{\mathcal{C}om_1}^{\mathrm{op}})$ the monoidal structure induced by the tensor product of chain complexes.

Proof.

(i) For $f \in I(\mathbf{r}, \mathbf{s})$ the element $M_f(A)(X) \in M_s(A) = A^{s\times s}$ is obtained from $X \in M_r(A) =$ by filling up with zeroes elsewhere. Hence $M_\bullet(A)$ is a functor $I \longrightarrow \mathcal{M}on$ and maps morphisms to injections.

(ii) By construction the two maps

$$\mu \circ (\mu \times \mathrm{id}), \mu \circ (\mathrm{id} \times \mu) : M_r(A) \times M_s(A) \times M_t(A) \longrightarrow M_{r+s+t}(A)$$

are equal and similarly the diagrams commute

$$\begin{tikzcd} 1 \times M_r(A) \arrow[r, "\eta\times\mathrm{id}"] \arrow[d, "\lambda"', "\wr"] & M_0(A) \times M_r(A) \arrow[d, "\mu"] \\ M_r(A) & M_{0+r}(A), \arrow[l, "M_\lambda(A)"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} M_r(A) \times 1 \arrow[r, "\eta\times\mathrm{id}"] \arrow[d, "\rho"', "\wr"] & M_r(A) \times M_0(A) \arrow[d, "\mu"] \\ M_r(A) & M_{r+0}(A), \arrow[l, "M_\rho(A)"', "\sim"] \end{tikzcd}$$

where λ and ρ are the structure isomorphisms of the particular monoidal category. Hence $M_\bullet(A)$ is a monoidal functor. Moreover the diagram below commutes

$$\begin{tikzcd} M_r(A) \times M_s(A) \arrow[r, "\gamma", "\sim"'] \arrow[d, "\mu"'] & M_s(A) \times M_r(A) \arrow[d, "\mu"] \\ M_{r+s}(A) \arrow[r, "M_\gamma(A)", "\sim"'] & M_{s+r}(A), \end{tikzcd}$$

where γ is the braiding isomorphism of the particular monoidal category. This proves that $M_\bullet(A)$ is infact symmetric monoidal.

(iii) As the functor Λ_* is a right adjoint, it preserves products and in particular is symmetric monoidal.

$\square$

Proposition 5.4.11
Let $k \in \mathcal{C}Ring$ and $A \in k/\mathcal{R}ing$.

(i) *The subcomplexes $(\Lambda_*\mathfrak{gl}_\bullet A)^{(0)} \xhookrightarrow{i} \Lambda_*\mathfrak{gl}_\bullet A$ define a monoidal subfunctor $(I, +, \mathbf{0}) \longrightarrow (dg(k\text{-Mod}), \otimes, k)$.*

(ii) *With the monoidal structure of (i) also the quotient map is monoidal*

$$\Lambda_*\mathfrak{gl}_\bullet A \overset{q}{\twoheadrightarrow} (\Lambda_*\mathfrak{gl}_\bullet A)^{(0)},$$

which therefore is a retraction of monoidal functors.

(iii) *There is an induced coalgebra structure on the quotient $(\Lambda_*\mathfrak{gl}_\bullet A)^{(0)}$.*

Proof. By definition of the $\mathbb{Z}^r$-grading on $\Lambda_*\mathfrak{gl}_r A$, the multiplication of $\Lambda_*\mathfrak{gl}_\bullet A$ restricts to maps

$$\begin{aligned}(\Lambda_*\mathfrak{gl}_r A)^{(a)} \otimes (\Lambda_*\mathfrak{gl}_s A)^{(b)} &\longrightarrow (\Lambda_*\mathfrak{gl}_{r+s} A)^{(a,b)},\\ (a_1 \wedge ... \wedge a_n) \otimes (b_1 \wedge ... \wedge b_m) &\longmapsto a_1 \wedge ... \wedge a_n \wedge b_1 \wedge ... \wedge b_m,\end{aligned}$$

where $a \in \mathbb{Z}^r, b \in \mathbb{Z}^s$ and $(a, b) \in \mathbb{Z}^{r+s}$.

(i) In particular this proves that the multiplication of $\Lambda_*\mathfrak{gl}_\bullet A$ restricts to $(\Lambda_*\mathfrak{gl}_\bullet A)^{(0)}$, which therefore is a monoidal subfunctor.

(ii) Moreover this shows that $(\Lambda_*\mathfrak{gl}_r A)^{(\neq 0)}$ is an ideal of $\Lambda_*\mathfrak{gl}_\bullet A$, which proves (ii).

(iii) Recall that the coalgebra structure on $L := \Lambda_*\mathfrak{gl}_r A$ is induced by the norm map

$$\Lambda_*\mathfrak{gl}_r(A) = \mathcal{C}om_1(\mathfrak{gl}_r(A)[1]) \xhookrightarrow{N} \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(M_r(A)) = B_*M_r(A).$$

In particular it is given by

$$\delta(x_1 \wedge ... \wedge x_n) = \sum_{\substack{p+q=n,\\ \sigma\in \mathrm{Sh}_{p,q}}} \mathrm{sgn}(\sigma) \cdot (x_{\sigma(1)} \wedge ... \wedge x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \wedge ... \wedge x_{\sigma(n)}),$$

where $\mathrm{Sh}_{p,q} \subset \Sigma_{p+q}$ is the subset of (p, q)-shuffles, i.e. permutations $\sigma \in \Sigma_{p+q}$ with

$$\sigma(1) < ... < \sigma(p), \quad \sigma(p + 1) < ... < \sigma(p + q).$$

It follows that

$$\delta(L^{(\neq 0)}) \subset L^{(0)} \otimes L^{(\neq 0)} + L^{(\neq 0)} \otimes L^{(0)} + L^{(\neq 0)} \otimes L^{(\neq 0)},$$

which proves, that $(\Lambda_*\mathfrak{gl}_\bullet A)^{(\neq 0)}$ is a coideal of $\Lambda_*\mathfrak{gl}_\bullet A$. Explicitly $\delta(x_1 \wedge ... \wedge x_n)$ is the sum taken only over all possible decompositions lying in $L^{(0)} \otimes L^{(0)}$.

□

Remark 5.4.12
The colimit maps (symmetric) monoidal functors to (commutative) monoids in the target category. As a monoid in $dg(k\text{-}\widehat{\mathcal{C}om_1}^{\mathrm{op}})$ is a dg bialgebra, we get a commutative diagram of dg bialgebras

$$\begin{tikzcd}
d(A) := \operatorname{colim}_{\hat{\Delta}_{inj}} \alpha^*(\Lambda_*\mathfrak{gl}_\bullet A) \arrow[r, two heads, "q"] \arrow[d] & \operatorname{colim}_{\hat{\Delta}_{inj}} \alpha^*(\Lambda_*\mathfrak{gl}_\bullet A)^{(0)} = d(A)^{(0)} \arrow[d] \\
i(A) := \operatorname{colim}_{I} \Lambda_*\mathfrak{gl}_\bullet A \arrow[r, two heads, "q"] & \operatorname{colim}_{I} (\Lambda_*\mathfrak{gl}_\bullet A)^{(0)} = i(A)^{(0)},
\end{tikzcd}$$

where the horizontal maps are retractions in the category of dg algebras.

Note that the forgetful functor $dg(k\text{-}\widehat{\mathcal{C}om_1}^{\mathrm{op}}) \longrightarrow dg(k\text{-Mod})$ is a left adjoint and therefore commutes with colimits. So all the objects are also colimits in $dg(k\text{-Mod})$.

Proposition 5.4.13
For flat $A \in k/\mathcal{R}ing$ and $(p-1)! \in A^\times$, every map below is $(p-2)$-connected (cf. Remark 5.4.12)

$$\begin{tikzcd}
\Lambda_*\mathfrak{gl}_\infty A = \operatorname{colim}_{\mathbb{N}_0} (\alpha\nu)^*\Lambda_*\mathfrak{gl}_\bullet A \arrow[r, two heads] \arrow[d] & \operatorname{colim}_{\mathbb{N}_0} (\alpha\nu)^*(\Lambda_*\mathfrak{gl}_\bullet A)^{(0)} = (\Lambda_*\mathfrak{gl}_\infty A)^{(0)} \arrow[d] \\
d(A) = \operatorname{colim}_{\hat{\Delta}_{inj}} \alpha^*(\Lambda_*\mathfrak{gl}_\bullet A) \arrow[r, two heads] \arrow[d] & \operatorname{colim}_{\hat{\Delta}_{inj}} \alpha^*(\Lambda_*\mathfrak{gl}_\bullet A)^{(0)} = d(A)^{(0)} \arrow[d] \\
i(A) = \operatorname{colim}_{I} \Lambda_*\mathfrak{gl}_\bullet A \arrow[r, two heads] & \operatorname{colim}_{I} (\Lambda_*\mathfrak{gl}_\bullet A)^{(0)} = i(A)^{(0)},
\end{tikzcd}$$

Moreover the underlying $d(A) \in k\text{-}\mathcal{A}ss_1$ is a filtered colimit of free k-algebras.

Proof. The upper horizontal map is a $(p-1)$-connected by Propostion 5.4.3 (ii). Moreover by Proposition 5.4.8 the left vertical composition is $(p-2)$-connected, because $i(A) = (\Lambda_*\mathfrak{gl}_\infty A)_{\Sigma_\infty}$. As all the horizontal maps are retractions, it follows that also the right vertical composition and hence also the lower horizontal map is $(p-2)$-connected. Thus by a similar argument for the upper vertical maps, it remains to check that the upper left vertical map is $(p-2)$-connected. Let us remark here, that the proof of this fact is a product of an early stage of the work, but finally did not find an application in the later theory. So it may be interesting, but can also be skipped.

The forgetful functor $dg(k\text{-}\widehat{\mathcal{C}om_1}^{\mathrm{op}}) \xrightarrow{U} dg(k\text{-Mod})$ is left adjoint to the free functor and thus preserves colimits. Thus it suffices to check that the lower horizontal map in the commuting diagam of chain complexes

$$\begin{tikzcd}
\operatorname*{hocolim}_{\mathbb{N}_0} (\nu\alpha)^*\Lambda_*\mathfrak{gl}_\bullet A \arrow[r, two heads] \arrow[d, two heads] & \operatorname*{hocolim}_{\hat{\Delta}_{inj}} \alpha^*\Lambda_*\mathfrak{gl}_\bullet A \arrow[d, two heads] \\
\operatorname*{colim}_{\mathbb{N}_0} (\nu\alpha)^*\Lambda_*\mathfrak{gl}_\bullet A \arrow[r, two heads] & \operatorname*{colim}_{\hat{\Delta}_{inj}} \alpha^*\Lambda_*\mathfrak{gl}_\bullet A
\end{tikzcd}$$

is $(p-1)$-connected. This will be done by showing that the vertical maps are quasi-isomorphisms and the upper horizontal map is $(p-1)$-connected.

- The left vertical map is a quasi-isomorphism, because $\mathbb{N}_0$ is a filtered category and filtered colimits are exact.
- To see that the upper horizontal map is $(p-1)$-connected, we use the spectral sequence for homotopy colimits. It suffices to show that the induced map of the E^2-pages

$$\pi_r \operatorname*{hocolim}_{\mathbb{N}_0} (\alpha\nu)^* H_s(\Lambda_*\mathfrak{gl}_\bullet A) \longrightarrow \pi_r \operatorname*{hocolim}_{\hat{\Delta}_{inj}} \alpha^* H_s(\Lambda_*\mathfrak{gl}_\bullet A), \qquad r, s \geq 0, \qquad (5.1)$$

has the desired connectivity. By Corollary 5.4.5 conjugation by Σ_r induces the identity on $H_s(\mathfrak{gl}_r A, k)$, for all $0 \leq s < p$ and all $r \geq 0$. It follows that the maps

$$d^i : \Lambda_*\mathfrak{gl}_{r-1}A \longrightarrow \Lambda_*\mathfrak{gl}_r A, \qquad 0 \leq i \leq r,$$

for varying i, induce the same map on homology in dimensions $0 \leq s < p$. Hence the functors

$$\alpha^* H_s(\mathfrak{gl}_\bullet A, k) : \hat{\Delta}_{inj} \xrightarrow{\alpha} I \longrightarrow k\text{-Mod}, \qquad 0 \leq s < p,$$

factor over the functor $\hat{\Delta}_{inj} \xrightarrow{\alpha} I \xrightarrow{\tau} \mathbb{N}_0$, which by Proposition 4.1.4 is totally final. Thus by Remark 7.3.27 the maps

$$\operatorname*{hocolim}_{\mathbb{N}_0} (\alpha\nu)^* H_s(\mathfrak{gl}_\bullet A, k) \xrightarrow{\simeq} \operatorname*{hocolim}_{\hat{\Delta}_{inj}} \alpha^* H_s(\mathfrak{gl}_\bullet A, k), \qquad 0 \leq s < p,$$

are weak equivalences. In particular the map (5.1) is an isomorphism, for $r \geq 0$ and $0 \leq s < p$, and surjective, for $r = 0$ and $s = p$, which proves that the map of E^2-pages is $(p-1)$-connected.
- The hardest part is to show that the right vertical map is a quasi-isomorphism. To that aim we filter $\Lambda_*\mathfrak{gl}_\bullet A$ by dimensions, i.e.

$$F_n\Lambda_*\mathfrak{gl}_\bullet A = \bigoplus_{k \leq n} \Lambda_n \mathfrak{gl}_\bullet A, \qquad n \geq 0.$$

It follows that the differential d is (-1)-equicontinuous and therefore is zero on the associated graded object. Moreover the I-structure maps are 0-equicontinuous, so $\mathrm{gr}\Lambda_*\mathfrak{gl}_\bullet A \in \mathcal{C}\mathrm{AT}(I, dg(k\text{-Mod}))$ and we get

$$\operatorname*{hocolim}_{\mathbb{N}_0} \mathrm{gr}\Lambda_*\mathfrak{gl}_\bullet A = \mathrm{gr} \operatorname*{hocolim}_{\mathbb{N}_0} \Lambda_*\mathfrak{gl}_\bullet A \longrightarrow \mathrm{gr} \operatorname*{colim}_{\hat{\Delta}_{inj}} \Lambda_*\mathfrak{gl}_\bullet A = \operatorname*{hocolim}_{\hat{\Delta}_{inj}} \mathrm{gr}\Lambda_*\mathfrak{gl}_\bullet A.$$

By a spectral sequence argument, it suffices to check that this map is a quasi-isomorphism. As the differential of $\mathrm{gr}\Lambda_*\mathfrak{gl}_\bullet A$ is zero, we may write

$$\mathrm{gr}\Lambda_*\mathfrak{gl}_r A = \Lambda_* A^{(r\times r)} = (\Lambda_* A)^{\otimes(r\times r)} = (\Lambda_* A)^{\otimes r} \otimes (\Lambda_* A)^{\otimes r}, \qquad r \geq 0,$$

as a differential graded coalgebra and we can ignore the grading of Λ_*. As A is supposed to be flat, by a filtered colimit argument, we may further assume that A is finitely generated and free. Hence also $\Lambda_* A$ is free, say $\Lambda_* A = kX$, for some finite set X, and we can assume that $* := 1 \in k = \Lambda_0 A$ is a base point for X. Then by construction the isomorphisms

$$k(X^{r\times r}) = (kX)^{\otimes(r\times r)} \xrightarrow{\sim} (\Lambda_* A)^{\otimes(r\times r)} = \Lambda_*\mathfrak{gl}_r A, \qquad r \geq 0,$$

induce an isomorphism of functors $kq^*(E^\bullet X) \xrightarrow{\sim} \Lambda_*\mathfrak{gl}_r A$, where

$$q : I \longrightarrow I, \qquad \mathbf{r} \longmapsto \mathbf{r}^2,$$

and $E^\bullet X$ is defined as in Proposition 4.1.9, i.e. for $f \in I(\mathbf{m}, \mathbf{n})$ and $x \in E^r X = X^r$ we have

$$E^f X(x)_k = \begin{cases} x_i, & f(i) = k, \\ *, & k \notin f(\mathbf{m}). \end{cases}$$

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{gr}\,\underset{\hat{\Delta}_{inj}}{\mathrm{hocolim}}\ \Lambda_*\mathfrak{gl}_\bullet A \arrow[r] \arrow[d, equal, "\wr"'] & \mathrm{gr}\,\underset{\hat{\Delta}_{inj}}{\mathrm{colim}}\ \Lambda_*\mathfrak{gl}_\bullet A \arrow[d, equal, "\wr"] \\
\underset{\hat{\Delta}_{inj}}{\mathrm{hocolim}}\ \mathrm{gr}\Lambda_*\mathfrak{gl}_\bullet A \arrow[r] & \underset{\hat{\Delta}_{inj}}{\mathrm{colim}}\ \mathrm{gr}\Lambda_*\mathfrak{gl}_\bullet A \\
\underset{\hat{\Delta}_{inj}}{\mathrm{hocolim}}\ kq^*(E^\bullet X) \arrow[r] \arrow[u, "\wr"] \arrow[d, "\wr"'] & \underset{\hat{\Delta}_{inj}}{\mathrm{colim}}\ kq^*(E^\bullet X) \arrow[u, "\wr"'] \arrow[d, "\wr"] \\
k(\underset{\hat{\Delta}_{inj}}{\mathrm{hocolim}}\ q^*(E^\bullet X)) \arrow[r] & k(\underset{\hat{\Delta}_{inj}}{\mathrm{colim}}\ q^*(E^\bullet X)),
\end{tikzcd}$$

where the lower vertical maps are isomorphisms, as the free k-module functor is a left adjoint and therefore commutes with (homotopy) colimits. Moreover by Proposition 4.1.10 and Proposition 4.1.9 resp. the functors $I \xrightarrow{q} I \xrightarrow{E^\bullet X} \mathcal{S}et$ preserve limits over connected, non-empty categories. Hence by Corollary 7.3.32 the lower horizontal map is a quasi-isomorphism, which concludes the proof.

For every $X \in \mathcal{S}et_*$ the monoid $\mathrm{colim}_{\hat{\Delta}_{inj}} q^* E^\bullet X$ is freely generated by matrices $M \in X^{r\times r}$, that cannot be written as a block sum

$$M = \begin{pmatrix} M' & * \\ * & M'' \end{pmatrix}.$$

Hence $k[\mathrm{colim}_{\hat{\Delta}_{inj}} q^* E^\bullet X]$ is the free k-algebra generated by such X-matrices and $d(A) = \mathrm{colim}_{\hat{\Delta}_{inj}} \Lambda_*\mathfrak{gl}_\bullet A$ is a filtered colimit of these.

□

5.4.4 Primitive elements and cyclic homology

We show that the primitive elements in the quotient Chevalley-Eilenberg complex are closely related to cyclic homology and Connes homology.

Proposition 5.4.14
Let $k \in \mathcal{C}Ring$ and $A \in k/\mathcal{R}ing$ be flat.
Then there is a monomorphism of chain complexes

$$\begin{aligned} \phi : \Sigma C^\lambda_*(A, I^r_\bullet) = (C^\lambda_{*-1}(A, I^r_\bullet)) &\hookrightarrow P(\Lambda_* \mathfrak{gl}_r A)^{(0)}, \\ [a_1 \otimes ... \otimes a_n \otimes \sigma] &\longmapsto a_1 \cdot e_{\sigma(n),\sigma(1)} \wedge ... \wedge a_n \cdot e_{\sigma(n-1),\sigma(n)} \end{aligned}$$

which by definition is natural in $\mathbf{r} \in I$. Here we define $C^\lambda_*(A, I^r_\bullet)$ as the Connes complex to the semi-cyclic set[1] $C_\bullet A \otimes k[I^r_\bullet] = C_\bullet A \otimes k[I(\bullet, \mathbf{r})]$.

Proof. Let $r \geq 1$. We extend the valuation map $\mathbf{r} \times \mathbf{r} \xrightarrow{v} \mathbb{Z}^r$ to a map

$$v : \mathcal{P}(\mathbf{r} \times \mathbf{r}) = \{S \subset \mathbf{r} \times \mathbf{r}\} \longrightarrow \mathbb{Z}^r, \qquad S \longmapsto \sum_{(i,j) \in S} v(i,j) = \sum_{(i,j) \in S} e_j - e_i.$$

Then the induced comultiplication on $(\Lambda_* \mathfrak{gl}_r A)^{(0)}$ (cf. Proposition 5.4.11 and its proof) is given by

$$\delta(a_1 \cdot e_{i_1,j_1} \wedge ... \wedge a_n \cdot e_{i_n,j_n}) = \sum_{\substack{U \sqcup V = \{(i_1,j_1),...,(i_n,j_n)\}, \\ v(U)=v(V)=0}} \operatorname{sgn}(U,V) \cdot \bigwedge_{\ell \in U} a_\ell \cdot e_{i_\ell, j_\ell} \otimes \bigwedge_{\ell \in V} a_\ell \cdot e_{i_\ell, j_\ell},$$

where $\operatorname{sgn}(U,V)$ is the sign of the shuffle $\sigma \in \mathrm{Sh}_{p,q}$ with

$$U = \{\sigma(1), ..., \sigma(p)\}, \quad V = \{\sigma(p+1), ..., \sigma(p+q)\}.$$

It follows that

$$P(\Lambda_* \mathfrak{gl}_r A)^{(0)} = \sum_{\substack{\{(i_1,j_1),...,(i_n,j_n)\} \subset \mathbf{r} \times \mathbf{r} \\ \text{irreducible}}} A \cdot e_{i_1,j_1} \wedge ... \wedge A \cdot e_{i_n,j_n},$$

where we call a subset $S \subset \mathbf{r} \times \mathbf{r}$ irreducible, if

(i) $v(S) = 0$.

(ii) If $S = U \sqcup V$ with $v(U) = v(V) = 0$, then $U = \emptyset$ or $V = \emptyset$.

For injectivity, note that the definition of ϕ constitutes a lift

$$\begin{tikzcd} C_n A \otimes k[I^r_n] \arrow[r, "\bar{\phi}"] \arrow[d] & ((\mathfrak{gl}_r A)^{\otimes n})^{(0)} \arrow[d] \\ C^\lambda_n(A, I^r_\bullet) \arrow[r, "\phi"] & \Lambda_n \mathfrak{gl}_r A)^{(0)}, \end{tikzcd}$$

[1] "Semi-cyclic" means, that we do not require to have degeneracy maps.

and the lower map is obtained from the upper by taking coinvariants under canonical actions of the cyclic group C_n and the symmetric group Σ_n. The upper horizontal map is injective by construction, while two elements in the upper left object are identified under the cyclic action, if and only if they are identified under the symmetric action on the right. This proves that ϕ is injective.

Finally the maps ϕ define a homomorphism of chain complexes, because

$$\begin{aligned}
&d\phi[a_1 \otimes ... \otimes a_n \otimes \sigma] = d(a_1 \cdot e_{\sigma(n),\sigma(1)} \wedge ... \wedge a_n \cdot e_{\sigma(n-1),\sigma(n)})\\
&= \sum_{1\le i<j\le n} (-1)^{i+j} \cdot [a_i \cdot e_{\sigma(i-1),\sigma(i)}, a_j \cdot e_{\sigma(j-1),\sigma(j)}] \wedge a_1 \cdot e_{\sigma(n),\sigma(1)} \wedge ... \wedge a_n \cdot e_{\sigma(n-1),\sigma(n)}\\
&= \sum_{1\le i\le n} (-1)^{i+(i+1)} \cdot (a_i a_{i+1}) \cdot e_{\sigma(i-1),\sigma(i+1)} \wedge a_1 \cdot e_{\sigma(n),\sigma(1)} \wedge ... \wedge a_n \cdot e_{\sigma(n-1),\sigma(n)}\\
&= \sum_{1\le i\le n} (-1)^{i} \cdot a_1 \cdot e_{\sigma(n),\sigma(1)} \wedge ... \wedge (a_i a_{i+1}) \cdot e_{\sigma(i-1),\sigma(i+1)} \wedge ... \wedge a_{\sigma(n-1),\sigma(n)}\\
&= \sum_{1\le i\le n} (-1)^{i} \cdot \phi[d_i(a_1 \otimes ... \otimes a_n) \otimes d_i(\sigma)] = -\phi d[a_1 \otimes ... \otimes a_n \otimes \sigma].
\end{aligned}$$

□

Remark 5.4.15
The semi-cyclic set $\Sigma_\bullet := \alpha^* \operatorname{colim}_{\underline{r}\in\hat{\Delta}_{inj}} I^r_\bullet$ canonically becomes a cyclic set, when we define degeneracies by setting

$$s_i(\sigma)(k) = \begin{cases} d^{\sigma(i)}\sigma s^i, & k \neq i, \\ \sigma(i), & k = i, \end{cases} \qquad 0 \le i \le n, \quad \sigma \in \Sigma_n = \mathcal{S}et(\underline{n})^\times.$$

Note that $\Sigma_\bullet$ is augmented as a simplicial set and there are extra-degeneracies, given by the block sums

$$s_{-1}(\sigma) = \sigma + 1, \quad s_{n+1}(\sigma) = 1 + \sigma, \qquad \sigma \in \Sigma_n.$$

In particular $\Sigma_\bullet \xrightarrow{\simeq} *$ is a simplicial deformation retraction by Proposition 7.2.12.

Corollary 5.4.16
Let $k \in \mathcal{C}Ring$ and $A \in k/\mathcal{R}ing$ be flat.

Then the homology of the canonical map

$$\begin{array}{ccc}
C^\lambda_*(A, \Sigma_\bullet) & \longrightarrow & C^\lambda_* A \\
\| & & \| \\
\operatorname{colim}_{\underline{r}\in\hat{\Delta}_{inj}} C^\lambda_*(A, I^r_\bullet) & \longrightarrow & \operatorname{colim}_{\mathbf{r}\in I} C^\lambda_*(A, I^r_\bullet)
\end{array}$$

naturally identifies with the map $HC_*(A) \longrightarrow H^\lambda_*(A)$, which by Proposition 5.1.9 is $(p-1)$-connected, if $(p-1)! \in A^\times$.

Proof. Note that the right equality holds, because $\operatorname{colim}_{\mathbf{r}\in I} I(\text{-},\mathbf{r}) = *$ and $C^\lambda_*(A,*) = C^\lambda_* A$. Next consider the commutative square

$$\begin{tikzcd} CC_*(A,\Sigma_\bullet) \arrow[r] \arrow[d] & CC_*A \arrow[d] \\ C^\lambda_*(A,\Sigma_\bullet) \arrow[r] & C^\lambda_* A, \end{tikzcd}$$

where similarly $CC_*(A,\Sigma_\bullet)$ is defined as the cyclic complex to the cyclic module $C_\bullet A\otimes k[\Sigma_\bullet]$. Then the spectral sequence of the proof of Proposition 5.1.9

$$E^1_{r,s} = H_s(C_{\underline{r}}, C_r A\otimes k[\Sigma_r]) \quad\Rightarrow\quad HC_{r+s}(C_\bullet A\otimes k[\Sigma_\bullet])$$

collapses, because $\Sigma_{\underline{r}}$ is a free $C_{\underline{r}}$-set, for all $r\geq 0$. It follows that the left vertical map is a quasi-isomorphism.

Using the simplicial deformation retraction $\Sigma_\bullet \xrightarrow{\simeq} *$ of Remark 5.4.15, we see that also $C_\bullet A\otimes k[\Sigma_\bullet] \twoheadrightarrow C_\bullet A$ is a simplicial deformation retraction. Hence by use of the other spectral sequence for the double complex $CC_{*,*}(\text{-})$ of a cyclic module

$$E^2_{r,s} = H_r(C_{\underline{s}}, H_s(\text{-})) \quad\Rightarrow\quad HC_{r+s}(\text{-})$$

(or alternatively Connes' long exact sequence of Remark 5.1.10) we see that also the upper horizontal map is a quasi-isomorphism.

□

5.4.5 Generalizing the Theorem of Loday-Quillen-Tsygan

Having all tools in hand we are now able to prove the generalization.

Remark 5.4.17
For $A\in k/\mathcal{R}ing$ the composition of the following chain maps is the identity.

$$C^\lambda_{*-1}A \xrightarrow{\phi} \Lambda_*\mathfrak{gl}_\infty A \xrightarrow{\theta} C^\lambda_{*-1}(U_k(\mathfrak{gl}_\infty A)) \xrightarrow{\varepsilon} C^\lambda_{*-1}M_\infty(A) \xrightarrow{\text{trace}} C^\lambda_{*-1}A,$$

where

(i) *ϕ is the map induced by Proposition 5.4.14, i.e.*

$$\phi[a_0\otimes ...\otimes a_n] = a_0\cdot e_{n,0}\wedge ...\wedge a_n\cdot e_{n-1,n}.$$

(ii) *θ is the map of [Lod98] 10.2.3, given by*

$$\theta(a_0\wedge ...\wedge a_n) = \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma)\cdot[a_0\otimes a_{\sigma(1)}\otimes ...\otimes a_{\sigma(n)}].$$

(iii) *ε is the counit of the adjunction*

$$k\text{-}\mathcal{A}ss_1(U_k(X),Y) = k\text{-}\mathcal{L}ie(X,L(Y)),$$

where L maps an associative algebra Y to the Lie algebra A with Lie bracket

$$[x,y] = xy - yx, \qquad a,b\in Y.$$

(iv) trace *is the trace map on Connes homology of [Lod98] Def. 1.2.1, given by*

$$\mathrm{trace}[a^{(0)} \otimes \ldots \otimes a^{(n)}] = \sum_{f \in \mathcal{S}et(\underline{n}, \mathbb{N})} [a^{(0)}_{f(n),f(0)} \otimes a^{(1)}_{f(0),f(1)} \otimes \ldots \otimes a^{(n)}_{f(n-1),f(n)}].$$

Proposition 5.4.18
Let $A \in k/\mathcal{R}ing$ be flat with $(p-1)! \in A^\times$.

Then ϕ extends to a map of dg bialgebras, which is an isomorphism in dimensions $< p$

$$\phi : \mathcal{C}om_1(C^\lambda_{*-1}A) \longrightarrow i(A)^{(0)} = (\Lambda_* \mathfrak{gl}_\infty A^{(0)})_{\Sigma_\infty}.$$

Proof. As ϕ is the colimit of the maps ϕ of Proposition 5.4.14, it maps $C^\lambda_{*-1}(A)$ into the primitive elements of $i(A)^{(0)}$, which proves that ϕ is also a homomorphism of coalgebras. By construction θ and the trace map are Σ_∞-invariant, so $\mathrm{trace} \circ \theta$ is well-defined on $(\Lambda_* \mathfrak{gl}_\infty A^{(0)})_{\Sigma_\infty}$. Consider the following diagram

$$\begin{tikzcd}
C^\lambda_{*-1}A \arrow[r, "\phi"] \arrow[d, "\iota"'] \arrow[rr, bend left, "\mathrm{id}"] & i(A)^{(0)} \arrow[r, "\mathrm{trace}\circ\theta"] \arrow[dr, dashed, "\exists! T"'] & C^\lambda_{*-1}A \\
\mathcal{C}om_1(C^\lambda_{*-1}A) \arrow[ur, dashed, "\exists!\phi"'] & & \widehat{\mathcal{C}om}_1^{\mathrm{op}}(C^\lambda_{*-1}A). \arrow[u, "\pi"']
\end{tikzcd}$$

By the universal properties of algebras resp. coalgebras the extensions ϕ and T exist and by Proposition 7.4.11 the composition $T \circ \phi$ must be a direct sum of norm maps. As A is flat, $T \circ \phi$ is an isomorphism in dimensions $< p$ by Corollary 7.4.12, since $C^\lambda_{*-1}(A)$ is trivial in dimension 0. It follows that T is surjective and ϕ is injective in dimensions $< p$ and so it remains to check surjectivity for ϕ.

Like in the proof of Proposition 5.4.4 we consider the ring epimorphisms from the monoid rings

$$k' := \mathbb{Z}[(k, \cdot, 1)] \twoheadrightarrow k, \qquad A' := \mathbb{Z}[(A, \cdot, 1)] \twoheadrightarrow A.$$

As $A \in k/\mathcal{R}ing$ we have $A' \in k'/\mathcal{R}ing$ by construction. Writing ϕ' and T' for the particular map in the situation of A', Aboughazi-Ogle proved in [AO94] Thm. 1.1.12 that $\phi' \otimes \mathbb{Q}$ is an isomorphism. It follows that also $T' \otimes \mathbb{Q}$ is an isomorphism and using the commutative diagram

$$\begin{tikzcd}
i(A')^{(0)} \arrow[r, "T'"] \arrow[d] & \widehat{\mathcal{C}om}_1^{\mathrm{op}}(C^\lambda_{*-1}A) \arrow[d] \\
i(A')^{(0)} \otimes \mathbb{Q} \arrow[r, "T' \otimes \mathbb{Q}"] & \widehat{\mathcal{C}om}_1^{\mathrm{op}}(C^\lambda_{*-1}A) \otimes \mathbb{Q},
\end{tikzcd}$$

flatness of A' implies that the left vertical map and hence also T' is injective. As T' was already surjective in dimensions $< p$, it follows that T' and hence also ϕ' is an isomorphism in this range. Using the commutative diagram

$$\begin{tikzcd}
\mathcal{C}om_1(C^\lambda_{*-1}A') \arrow[r, "\phi'", "\sim"'] \arrow[d, two heads] & d(A')^{(0)} \arrow[d, two heads] \\
\mathcal{C}om_1(C^\lambda_{*-1}A) \arrow[r, "\phi"] & d(A)^{(0)},
\end{tikzcd}$$

we see that ϕ is surjective in dimensions $< p$, which concludes the proof.

□

Corollary 5.4.19
Let $A \in k/\mathcal{R}ing$ be flat with $(p-1)! \in A^\times$.
Then there is a zig-zag of $(p-2)$-connected homomorphisms of dg commutative coalgebras

$$\Lambda_* \mathfrak{gl}_\infty A \twoheadrightarrow (\Lambda_* \mathfrak{gl}_\infty A)^{(0)}_{\Sigma_\infty} = i(A)^{(0)} \xleftarrow{\phi} \mathcal{C}om_1(C^\lambda_{*-1} A).$$

Proof. The first map is $(p-2)$-connected by Proposition 5.4.13 and the second one by Proposition 5.4.18.

□

Theorem 5.4.20
Let $A \in k/\mathcal{R}ing$ be flat with $(p-1)! \in A^\times$.
Then the map ϕ induces isomorphisms in dimensions $0 \leq n < p-1$

$$H^\lambda_{n-1}(A) \xrightarrow{\sim} PH_n(\mathfrak{gl}_\infty A, k) = \ker \left(H_n(\mathfrak{gl}_\infty A, k) \underset{(\eta\times \mathrm{id})_* + (\mathrm{id}\times\eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(\mathfrak{gl}_\infty A \times \mathfrak{gl}_\infty A, k) \right).$$

Here $\mathfrak{gl}_\infty A \xrightarrow{\delta} \mathfrak{gl}_\infty A \times \mathfrak{gl}_\infty A$ is the diagonal and $0 \xrightarrow{\eta} \mathfrak{gl}_\infty A$ is the initial Lie algebra homomorphism.
Moreover Connes' operator B of Remark 5.1.10, the negative Chern character for the Hopf algebra $U_k(\mathfrak{gl}_\infty A)$ of Definition 5.2.11 and the antisymmetrisation map e of Proposition 5.2.13 induce a commutative diagram

$$\begin{tikzcd}
HC_{*-1}(A) \arrow[rrrr, "B"] \arrow[d] & & & & HC^-_*(A) \\
H^\lambda_{*-1}(A) \arrow[r, "\phi", "\sim"'] & PH_*(\mathfrak{gl}_\infty A, k) \arrow[r, hook, "e"] & H_*(B_* U_k(\mathfrak{gl}_\infty A)) \arrow[r, "\mathrm{ch}^-"] & HC^-_*(U_k(\mathfrak{gl}_\infty A)) \arrow[r, "\varepsilon"] & HC^-_*(M_\infty A) \arrow[u, "\mathrm{trace}"']
\end{tikzcd}$$

We call the composition $\mathrm{trace} \circ \varepsilon \circ \mathrm{ch}^-$ *the* ***(additive) negative Chern character*** *and by abuse of notation also denote it by* ch^-.

Proof. For $0 \leq n < p-1$, in the commutative diagram

$$\begin{tikzcd}
& H^\lambda_n(A) \arrow[dl] \arrow[d] \arrow[dr] & \\
PH_n(\mathfrak{gl}_\infty A, k) \arrow[r, "\sim"'] & PH_n(i(A)^{(0)}) & PH_n(\mathcal{C}om_1(C^\lambda_{*-1} A)), \arrow[l, "\phi"', "\sim"]
\end{tikzcd}$$

the horizontal maps are isomorphisms by Corollary 5.4.19, while Proposition 4.3.10 implies that the right vertical map is an isomorphism.

In the diagram

$$\begin{tikzcd}
H^\lambda_{*-1}(A) \arrow[d, "\phi"', "\wr"] \arrow[rrr, bend left, "\mathrm{id}"] & & & \\
PH_*(\mathfrak{gl}_\infty A, k) \arrow[r, "\theta"] \arrow[d, hook, "e"'] & H^\lambda_{*-1}(U_k(\mathfrak{gl}_\infty A)) \arrow[r, "\varepsilon"] \arrow[d, "B"'] & H^\lambda_{*-1}(M_\infty A) \arrow[r, "\mathrm{trace}"] \arrow[d, "B"'] & H^\lambda_{*-1}(A) \arrow[d, "B"'] \\
H_*(B_* U_k(\mathfrak{gl}_\infty A)) \arrow[r, "\mathrm{ch}^-"] & HC^-_*(U_k(\mathfrak{gl}_\infty A)) \arrow[r, "\varepsilon"] & HC^-_*(M_\infty A) \arrow[r, "\mathrm{trace}"] & HC^-_*(A)
\end{tikzcd}$$

the upper part commutes by Remark 5.4.17, while the left square commutes by Corollary 5.2.15 applied to the Lie algebra $\mathfrak{gl}_\infty A$. As Connes' operator B is natural, also the right two squares commute.

$\square$

6 Multiplicative vs. additive K-theory

6.1 Quillen's plus construction

By recalling the plus construction in the context of simplicial groups, we are following the same guideline in the setting of simplicial Lie algebras to be ably to develop the additive and multiplicative theory parallelly.

6.1.1 The plus construction for simplicial groups

Remark 6.1.1
Let $c \in s\mathcal{G}rp(G, H)$, such that in every dimension

$$c_n \cong \iota_{G_n} : G_n \longrightarrow G_n + (H/G)_n, \qquad n \geq 0,$$

where $H/G = H +_G 1 \in s\mathcal{G}rp$.

*Then by Proposition 8.1.11, every $M \in \mathbb{Z}[H/G]$-*Mod *induces a long exact sequence*

$$... \longrightarrow H_2(H/G, M) \xrightarrow{\partial} H_1(G, M) \longrightarrow H_1(H, M) \longrightarrow H_1(H/G, M) \longrightarrow 0.$$

Proposition 6.1.2
Let $G \in s\mathcal{G}rp$ and $N \lhd \pi_0 G$ be perfect, meaning that $N = [N, N]$.

Then there is cofibration $G \overset{i}{\rightarrowtail} G^+$, such that

(i) $1 \longrightarrow N \longrightarrow \pi_0 G \xrightarrow{i_*} \pi_0 G^+ \longrightarrow 1$ *is exact,*

(ii) $H_*(G, \mathbb{Z}[\pi_0 G/N]) \xrightarrow{i_*} H_*(G^+, \mathbb{Z}[\pi_0 G/N])$ *is an isomorphism.*

*In particular by Proposition 8.1.2 the map $H_*(G, M) \xrightarrow{i_*} H_*(G^+, M)$ is an isomorphism, for every $M \in \mathbb{Z}[\pi_0 G/N]$-*Mod.

Proof. The proof is established in 2 steps.

- First we assume that $N = \pi_0 G$. We let $K = \ker(G_0 \twoheadrightarrow \pi_0 G)$ and define G' as the pushout on the left

$$\begin{tikzcd} F({}^K S^0) \arrow[r] \arrow[d] & G \arrow[d] \\ F({}^K D^1) \arrow[r] & G', \end{tikzcd} \qquad \begin{tikzcd} {}^K\mathbb{Z} \arrow[r] \arrow[d] & \pi_0 G \arrow[d] \\ 1 \arrow[r] & \pi_0 G', \end{tikzcd}$$

where F is the left adjoint of the adjunction $\mathcal{G}rp(F(X), Y) = \mathcal{S}et_*(X, U(Y))$ and $S^0 \hookrightarrow D^1$ are suitable models in $s\mathcal{S}et_*$. As π_0 is a left adjoint, it preserves coproducts and pushouts. So applying π_0 to the left pushout square yields a pushout square as on the right, so $\pi_0 G' = 1$. The map $G \longrightarrow G'$ satisfies the hypothesis of Remark 6.1.1. Since $G'/G = F({}^K S^1)$, we get a long exact sequence

$$\ldots \longrightarrow H_2(F({}^K S^1), \mathbb{Z}) \xrightarrow{\partial} H_1(G, \mathbb{Z}) \longrightarrow H_1(G', \mathbb{Z}) \longrightarrow H_1(F({}^K S^1), \mathbb{Z}).$$

Looking closer at the low dimensions we get an exact sequence

$$0 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow H_2(G', \mathbb{Z}) \longrightarrow H_2(F({}^K S^1), \mathbb{Z}) \xrightarrow{\partial} H_1(G, \mathbb{Z}) \longrightarrow H_1(G', \mathbb{Z}) \longrightarrow 0.$$

The Hurewicz map induces a commuting square

$$\begin{tikzcd} (\pi_0 G)/N \arrow[r, "\sim"] \arrow[d, "h"', "\wr"] & \pi_0 G' \arrow[d, "h"', "\wr"] \\ H_1(G, \mathbb{Z}) \arrow[r] & H_1(G', \mathbb{Z}). \end{tikzcd}$$

Hence $H_1(G, \mathbb{Z}) \xrightarrow{\sim} H_1(G', \mathbb{Z})$ and

$$0 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow H_2(G', \mathbb{Z}) \longrightarrow H_2(F({}^K S^1), \mathbb{Z}) \longrightarrow 0$$

is non-canonically split exact, since the right object is the free abelian group generated by K. As $\pi_0 G' = 1$, the Hurewicz map is an isomorphism

$$h : \pi_1 G' \xrightarrow{\sim} H_2(G', \mathbb{Z}) \cong H_2(G, \mathbb{Z}) \oplus \mathbb{Z}K.$$

Let $L \subset G'_1$ be a subset of cycles, being sent to K under this map, and define G^+ as the pushout

$$\begin{tikzcd} F({}^L S^1) \arrow[r] \arrow[d] & G' \arrow[d] \\ F({}^L D^2) \arrow[r] & G^+. \end{tikzcd}$$

By the same argument as before we get a long exact sequence

$$\ldots \longrightarrow H_2(F({}^L S^2), \mathbb{Z}) \xrightarrow{\partial} H_1(G', \mathbb{Z}) \longrightarrow H_1(G^+, \mathbb{Z}) \longrightarrow H_1(F({}^L S^2), \mathbb{Z}) \longrightarrow 0,$$

whose low dimensions are given by

$$\begin{tikzcd} H_3(G', \mathbb{Z}) \arrow[r, hook] & H_3(G^+, \mathbb{Z}) \arrow[r] & H_3(F({}^L S^2), \mathbb{Z}) \arrow[r, "\partial"] \arrow[d, equal] & H_2(G', \mathbb{Z}) \arrow[r, two heads] & H_2(G^+, \mathbb{Z}) \\ & & \mathbb{Z}L \arrow[r, "\sim"] & \mathbb{Z}K. \arrow[u, hook] & \end{tikzcd}$$

In particular ∂ is injective, which proves $H_n(G,\mathbb{Z}) \xrightarrow{\sim} H_n(G',\mathbb{Z}) \xrightarrow{\sim} H_n(G^+,\mathbb{Z})$, for all $n \neq 2$. Moreover $H_2(G^+,\mathbb{Z}) = H_2(G',\mathbb{Z})/\mathbb{Z}K = H_2(G,\mathbb{Z})$, such that in all dimensions $H_*(G,\mathbb{Z}) \xrightarrow{\sim} H_*(G^+,\mathbb{Z})$. As G^+ is obtained from G by glueing cells, we see that $G \longrightarrow G^+$ is a cofibration. Since also $\pi_0 G/N = \pi_0 G' \xrightarrow{\sim} \pi_0 G^+$, this proves the statement in the first case.

- For the general case let G_N be a covering simplicial group for N, which can be defined as the kernel in

$$1 \longrightarrow G_N \longrightarrow G \longrightarrow (\pi_0 G)/N \longrightarrow 1.$$

Then G_N satisfies the hypothesis of the first case and we find a cofibration $G_N \overset{i_N}{\rightarrowtail} G_N^+$. We define the plus construction for G as the pushout on the left

$$\begin{tikzcd} G_N \arrow[r] \arrow[d, "i_N"'] & G \arrow[d, "i"] \\ G_N^+ \arrow[r] & G^+. \end{tikzcd} \qquad \begin{tikzcd} \pi_0 G_N \arrow[r] \arrow[d, "i_N"'] & \pi_0 G \arrow[d, "i"] \\ \pi_0 G_N^+ \arrow[r] & \pi_0 G^+. \end{tikzcd}$$

As π_0 preserves pushouts, also the right square is cocartesian, which proves that $(\pi_0 G)/N \xrightarrow{\sim} \pi_0 G^+$. As i_N is a cofibration, so is also i and we can apply again Remark 6.1.1 to the G^+/G-module $\mathbb{Z}[\pi_0 G^+/G] = \mathbb{Z}[\pi_0 G/N]$ and get a natural map of long exact sequences

$$\begin{tikzcd} \ldots \arrow[r, "\partial"] & H_1(G_N,\mathbb{Z}[\pi_0 G/N]) \arrow[r] \arrow[d] & H_1(G_N^+,\mathbb{Z}[\pi_0 G/N]) \arrow[r, two heads] \arrow[d] & H_1(G_N^+/G_N,\mathbb{Z}[\pi_0 G/N]) \arrow[d, equal] \\ \ldots \arrow[r, "\partial"] & H_1(G,\mathbb{Z}[\pi_0 G/N]) \arrow[r] & H_1(G^+,\mathbb{Z}[\pi_0 G/N]) \arrow[r, two heads] & H_1(G^+/G,\mathbb{Z}[\pi_0 G/N]) \end{tikzcd}$$

Since $\pi_0 G_N = N$, the simplicial groups G_N^+ and G_N act trivially on $\mathbb{Z}[\pi_0 G/N]$, which therefore can be considered as a direct sum of copies of $\mathbb{Z}$. Hence by construction $H_*(G_N,\mathbb{Z}[\pi_0 G/N]) \xrightarrow{\sim} H_*(G_N^+,\mathbb{Z}[\pi_0 G/N])$. Equivalently using the long exact sequence we get

$$H_n(G^+/G,\mathbb{Z}[\pi_0 G/N]) = H_n(G_N^+/G_N,\mathbb{Z}[\pi_0 G/N]) = 0, \qquad n \geq 1,$$

and hence $H_n(G,\mathbb{Z}[\pi_0 G/N]) \xrightarrow{\sim} H_n(G^+,\mathbb{Z}[\pi_0 G/N])$, for all $n \geq 1$. Moreover by Corollary 8.2.3 we have

$$H_0(G,\mathbb{Z}[\pi_0 G/N]) = \mathbb{Z}[\pi_0 G/N]_{\pi_0 G} = \mathbb{Z} = \mathbb{Z}[\pi_0 G/N]_{\pi_0 G^+} = H_0(G^+,\mathbb{Z}[\pi_0 G/N]),$$

which finally proves that $G \overset{i}{\rightarrowtail} G^+$ is a plus construction.

□

Remark 6.1.3
Let $G \in s\mathcal{G}rp$ and $N \lhd \pi_0 G$ be perfect.

Then $BG \xrightarrow{i} B(G^+)$ is a plus construction in the sense of Quillen.

In particular, for $A \in \mathcal{R}ing$ we have

$$K_n^{\mathcal{G}rp}(A) := \pi_{n-1} GL(A)^+ = \pi_n B(GL(A)^+) = \pi_n BGL(A)^+ = K_n(A), \qquad n \geq 1.$$

In the context of what will follow, we call $K_*^{\mathcal{G}rp}(A)$ the ***multiplicative K-theory*** *of A.*

6.1.2 The plus construction for simplicial Lie algebras

Remark 6.1.4
Let $k \in \mathcal{C}Ring$ and $c \in s(k\text{-}\mathcal{L}ie)(\mathfrak{g}, \mathfrak{h})$, such that in every dimension

$$c_n \cong \iota_{\mathfrak{g}_n} : \mathfrak{g}_n \longrightarrow \mathfrak{g}_n * (\mathfrak{h}/\mathfrak{g})_n, \qquad n \geq 0,$$

where $\mathfrak{h}/\mathfrak{g} = \mathfrak{h} *_{\mathfrak{g}} 0 \in s(k\text{-}\mathcal{L}ie)$ is a free k-module in every dimension.

Then by the Theorem of Poincaré, Birkhoff and Witt and Proposition 8.1.11, every $M \in U_k(\mathfrak{h}/\mathfrak{g})\text{-Mod}$ induces a long exact sequence

$$\ldots \longrightarrow H_2(\mathfrak{h}/\mathfrak{g}, M) \xrightarrow{\partial} H_1(\mathfrak{g}, M) \longrightarrow H_1(\mathfrak{h}, M) \longrightarrow H_1(\mathfrak{h}/\mathfrak{g}, M) \longrightarrow 0.$$

Proposition 6.1.5
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ and $\mathfrak{n} \lhd \pi_0\mathfrak{g}$ be perfect, meaning that $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}]$.

Then there is cofibration $\mathfrak{g} \overset{i}{\rightarrowtail} \mathfrak{g}^+$, such that

(i) $1 \longrightarrow \mathfrak{n} \longrightarrow \pi_0\mathfrak{g} \xrightarrow{i_*} \pi_0\mathfrak{g}^+ \longrightarrow 1$ *is exact,*

(ii) $H_*^L(\mathfrak{g}, U_k(\pi_0\mathfrak{g}/\mathfrak{n})) \xrightarrow{i_*} H_*^L(\mathfrak{g}^+, U_k(\pi_0\mathfrak{g}/\mathfrak{n}))$ *is an isomorphism.*

In particular $H_*^L(\mathfrak{g}, M) \xrightarrow{i_*} H_*^L(\mathfrak{g}^+, M)$, for $M \in U_k(\pi_0\mathfrak{g}/\mathfrak{n})\text{-Mod}$, by Proposition 8.1.2.

Proof. Replace group homology by derived Lie algebra homology in the proof of Proposition 6.1.2.

□

In [Pir85] Pirashvili constructed a plus construction for Lie algebras. His construction is (like ours) inspired by Quillen's original proof of the plus construction for spaces. He defines homology for Lie algebras by using the total left derived of the abelianization functor $s(k\text{-}\mathcal{L}ie) \longrightarrow s(k\text{-Mod})$. As this is isomorphic to our Definition 8.3.7 of derived Lie algebra homology by Proposition 8.3.16, the two constructions coincide up to weak equlivalence.

Definition 6.1.6
Let $k \in \mathcal{C}Ring$ and $A \in k\text{-}\mathcal{A}ss_1$.

*Then the **additive K-theory of** A is defined as $K_n^{\mathcal{L}ie}(A) := \pi_{n-1}\mathfrak{gl}(A)^+$, for all $n \geq 1$.*

6.2 The Volodin construction

We are introducing the well-known Volodin constructions and also provide simplicial groups and Lie algebras that up to homotopy equivalence can be identified with the two variants of the Volodin construction.

Definition 6.2.1
Let $k \in \mathcal{C}Ring$ and $A \in k/\mathcal{R}ing$ and $r \geq 1$.

(i) *For a partial order γ on $\mathbf{r}$ we define a non-unital subring*

$$t_r^\gamma A = \{a \in M_r A;\ a_{i,j} \neq 0 \ \Rightarrow\ i \overset{\gamma}{<} j\} \leq M_r A.$$

We define induced subgroups and Lie subalgebras.

a) $T_r^\gamma A := 1 + t_A^\gamma \leq GL_r A$,

b) $t_r^\gamma A \leq \mathfrak{gl}_r A$.

(ii) *We define the* ***multiplicative/additive Volodin constructions****:*

a) $X_r A = \bigcup_\gamma BT_r^\gamma A \leq BGL_r A, \qquad X(A) = \bigcup_{r\geq 1} X_r(A) \leq BGL(A)$,

b) $x_r A = \sum_\gamma \Lambda_* t_r^\gamma A \leq \Lambda_* \mathfrak{gl}_r A, \qquad x(A) = \sum_{r\geq 1} x_r(A) \leq \Lambda_* \mathfrak{gl}(A)$.

(iii) *We also define the simplicial groups/Lie algebras*

a) $Y_r A = \operatorname{hocolim}_\gamma E_\bullet(T_r^\gamma A) \in s\mathcal{G}rp, \qquad Y(A) = \varinjlim_{r\geq 0} Y_r A$,

b) $y_r A = \operatorname{hocolim}_\gamma E_\bullet(t_r^\gamma A) \in s(k\text{-}\mathcal{L}ie), \qquad y(A) = \varinjlim_{r\geq 0} y_r A$,

where E is the functorial cofibrant replacement functor in $s\mathcal{G}rp$ and $s\mathcal{L}ie$ resp. (cf. Corollary 7.2.32).

Remark 6.2.2
Let $k \in \mathcal{C}Ring$ and $A \in k/\mathcal{R}ing$ and $r \geq 1$.

(i) *If γ is the usual total order on $\mathbf{r}$, then $t_r(A) := t_r^\gamma(A)$ is the associative k-algebra of* ***upper triangular matrices*** *with zeroes on the diagonal.*

(ii) $St_r(A) = \operatorname{colim}_\gamma T_r^\gamma(A) = \pi_0 Y_r(A)$ *is the* ***unstable Steinberg group****, i.e. the free group generated by the symbols*

$$x_{i,j}(a), \qquad 1 \leq i,j \leq r, \quad i \neq j, \quad a \in A,$$

modulo the relations

a) $x_{i,j}(a) \cdot x_{i,j}(b) = x_{i,j}(a+b), \qquad 1 \leq i,j \leq r, \quad i \neq j, \quad a,b \in A$,

b) $[x_{i,j}(a), x_{k,\ell}(b)] = \begin{cases} x_{i,k}(ab), & i \neq \ell,\ j = k, \\ 1, & i \neq \ell,\ j \neq k. \end{cases}$

The ***(stable) Steinberg group*** *is $St(A) = \operatorname{colim}_{r\geq 1} St_r(A) = \pi_0 Y(A)$.*

(iii) $\mathfrak{st}_r(A) = \operatorname{colim}_\gamma t_r^\gamma(A) = \pi_0 y_r(A)$ *is the* ***Steinberg Lie algebra****, i.e. the free Lie algebra generated by the symbols*

$$x_{i,j}(a), \qquad 1 \leq i,j \leq r, \quad i \neq j, \quad a \in A,$$

modulo the relations

a) $cx_{i,j}(a) + dx_{i,j}(b) = x_{i,j}(ca+db), \qquad 1 \leq i,j \leq r, \quad i \neq j, \quad a,b \in A, \quad c,d \in k$,

b) $[x_{i,j}(a), x_{k,\ell}(b)] = \begin{cases} x_{i,k}(ab), & i \neq \ell,\ j = k, \\ 1, & i \neq \ell,\ j \neq k. \end{cases}$

*The **(stable) Steinberg Lie algebra** is* $\mathfrak{st}(A) = \operatorname{colim}_{r\geq 1} \mathfrak{st}_r(A) = \pi_0 y(A)$.

Definition 6.2.3
Let $k \in \mathcal{C}Ring$ *and* $I \lhd A \in k/\mathcal{R}ing$ *and* $r \geq 1$.

(i) *For a partial order* γ *on* $\mathbf{r}$ *we define a non-unital subring as the pullback*

$$t_r^\gamma(A, I) = t_r^\gamma(A/I) \times_{M_r(A/I)} M_r(A) \leq M_r(A).$$

We define induced subgroups and Lie subalgebras.

a) $T_r^\gamma(A, I) := 1 + t_r^\gamma(A, I) \leq GL_r A$,

b) $t_r^\gamma(A, I) := t_r^\gamma(A, I) \leq \mathfrak{gl}_r A$.

(ii) *We define the **multiplicative/additive relative Volodin constructions**:*

a) $X_r(A, I) = \bigcup_\gamma BT_r^\gamma(A, I) \leq BGL_r A, \qquad X(A, I) = \bigcup_{r\geq 1} X_r(A, I) \leq BGL(A)$,

b) $x_r(A, I) = \sum_\gamma \Lambda_* t_r^\gamma(A, I) \leq \Lambda_* \mathfrak{gl}_r A, \qquad x(A, I) = \sum_{r\geq 1} x_r(A, I) \leq \Lambda_* \mathfrak{gl}(A)$.

(iii) *We also define the homotopy pullback simplicial groups/Lie algebras*

a) $Y_r(A, I) = Y_r(A) \times^h_{GL_r(A/I)} GL_r(A), \qquad Y(A) = \varinjlim_{r\geq 0} Y_r(A, I)$,

b) $y_r(A, I) = y_r(A) \times^h_{\mathfrak{gl}_r(A/I)} \mathfrak{gl}_r(A), \qquad y(A) = \varinjlim_{r\geq 0} y_r(A, I)$.

Remark 6.2.4
Let $k \in \mathcal{C}Ring$ *and* $I \lhd A \in k/\mathcal{R}ing$ *and* $r \geq 1$.

(i) *If* γ *is the usual total order on* $\mathbf{r}$, *then* $t_r(A, I) := t_r^\gamma(A, I)$ *is the associative* k*-algebra of matrices with values in* A *above the diagonal and in* I *elsewhere.*

(ii) *We define the **relative Steinberg group***

$$St(A, I) = \operatorname*{colim}_{r\geq 0} St_r(A, I) = \pi_0 Y(A, I), \qquad St_r(A, I) = \operatorname*{colim}_{\gamma} T_r^\gamma(A, I) = \pi_0 Y(A, I).$$

*and the **relative Steinberg Lie algebra***

$$\mathfrak{st}(A, I) = \operatorname*{colim}_{r\geq 0} \mathfrak{st}_r(A, I) = \pi_0 y(A, I), \qquad \mathfrak{st}_r(A, I) = \operatorname*{colim}_{\gamma} t_r^\gamma(A, I) = \pi_0 y(A, I).$$

(iii) *Each construction relative to* $I = 0$ *of Definition 6.2.3 is exactly the particular absolute construction of Definition 6.2.1.*

6.2.1 The Volodin constructions as bar constructions

Proposition 6.2.5
Let $k \in \mathcal{C}Ring$ *and* $A \in k/\mathcal{R}ing$ *and* $r \geq 1$.
Then the canonical maps induce weak equivalences of pointed simplicial sets

$$X_r(A) = \operatorname*{colim}_{\gamma} BT_r^\gamma A \xleftarrow{\simeq} \operatorname*{hocolim}_{\gamma} BT_r^\gamma A \xrightarrow{\simeq} B \operatorname*{hocolim}_{\gamma} T_r^\gamma A = BY_r(A).$$

In particular $X(A) \simeq BY(A)$.

Proof. The functor sending a partial order γ on $\mathbf{r}$ to the pointed simplicial set $BT_r^\gamma(A)$ preserves pullbacks. Hence the first map is a weak equivalence by Corollary 7.3.32. The second map is a weak equivalence, since the nerve functor $s\mathcal{G}rp \xrightarrow{B} s\mathcal{S}et_*$ preserves homotopy colimits by Proposition 7.3.34.

$\square$

Lemma 6.2.6
For $I \lhd A \in \mathcal{R}ing$ with $A \xrightarrow{\sim} \varprojlim_{n\geq 0} A/I^n$, the map $GL_r(A) \longrightarrow GL_r(A/I)$ is surjective.

Proof. Let $a \in GL_r(A/I)$ and take lifts $x, y \in M_r(A)$ with $a = [x]$ and $a^{-1} = [y]$. It follows that $[xy] = [x][y] = aa^{-1} = 1$ and thus $xy \in 1 + M_r(I)$. Since A is I-adically complete, we see that xy is a unit in $M_r(A)$ with inverse $\sum_{n\geq 0}(xy-1)^n$. Hence $x(y(xy)^{-1}) = 1$, showing that $x \in GL_r(A)$. As $a \in GL(A/I)$ was arbitrary this proves the surjectivity of $GL_r(A) \longrightarrow GL_r(A/I)$ resp. $GL(A) \longrightarrow GL(A/I)$.

$\square$

Proposition 6.2.7
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with I-adically complete A and $r \geq 1$.
Then naturally $X_r(A, I) \simeq BY_r(A, I)$. In particular naturally $X(A, I) \simeq BY(A, I)$.

Proof. As the map $GL_r(A) \longrightarrow GL_r(A/I)$ is surjective by Lemma 6.2.6, its nerve is a Kan fibration and it follows that the natural map induces a weak equivalence

$$X_r(A, I) = X_r(A/I) \times_{BGL_r(A/I)} BGL_r(A) \xrightarrow{\simeq} X_r(A/I) \times^h_{BGL_r(A/I)} BGL_r(A).$$

Using Proposition 6.2.5 we get a natural weak equivalence

$$X_r(A/I) \times^h_{BGL_r(A/I)} BGL_r(A) \simeq BY_r(A/I) \times^h_{BGL_r(A/I)} BGL_r(A).$$

As the nerve B is a right adjoint it preserves homotopy limits and thus we get a natural weak equivalence

$$BY_r(A/I) \times^h_{BGL_r(A/I)} BGL_r(A) \xleftarrow{\simeq} B\left(Y_r(A/I) \times^h_{GL_r(A/I)} GL_r(A)\right) = BY_r(A, I).$$

$\square$

Proposition 6.2.8
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k, as well as $r \geq 1$.
Then there is a natural chain of weak equivalences

$$\Gamma x_r(A, I) \simeq B_\bullet U_k(y_r(A, I)).$$

where Γ is the Dold-Kan functor sending a chain complex to its associated simplicial module and $E_\bullet y(A, I) \xrightarrow{\simeq} y(A, I)$ is a cofibrant replacement in the sense of Corollary 7.2.32.
In particular naturally $H_*^L(y(A, I), k) = H_*(x(A, I))$.

Proof. There natural are weak equivalences

$$\Gamma x_r(A,I) = \operatorname*{colim}_{\gamma} \Gamma\Lambda_* t_r^\gamma(A,I) \xleftarrow{\simeq} \operatorname*{hocolim}_{I} \Gamma\Lambda_* t_r^\gamma(A,I) \xrightarrow{\simeq} \operatorname*{hocolim}_{\gamma} B_\bullet U_k(t_r^\gamma(A,I)).$$

Indeed the first equality holds, because Γ is part of an equivalence of categories. The left map is a weak equivalence by Proposition 7.3.29, because the functor $\gamma \longmapsto \Gamma\Lambda_* t_r^\gamma(A,I)$ preserves pullbacks. Alternatively one can argue with an induction using the Mayer-Vietoris sequence as in [Goo85b] Claim III.9. The right map is induced by the antisymmetrisation map of chain complexes

$$\Lambda_*\mathfrak{g} \longrightarrow B_*U_k(\mathfrak{g}), \quad x_1 \wedge ... \wedge x_n \longmapsto \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma)\cdot x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}, \qquad \mathfrak{g} \in k\text{-}\mathcal{L}ie,$$

which is a quasi-isomorphism, because in our case $\mathfrak{g} = t_r^\gamma(A,I)$ is flat over k. Next using the short notation

$$X \otimes_A^h Y := B_\bullet(X,A,Y), \qquad A \in s(k\text{-}\mathcal{A}ss), \quad X \in s\mathcal{A}b\text{-}A, \quad Y \in A\text{-}s\mathcal{A}b,$$

consider the diagram

$$\begin{tikzcd}
\operatorname*{hocolim}_{\gamma} k \otimes^h_{U_k(t_r^\gamma(A,I))} k & & k \otimes^h_{U_k(E_\bullet y_r(A,I))} k \\
\operatorname*{hocolim}_{\gamma} k \otimes^h_{U_k(t_r^\gamma(A/I))} U_k(t_r^\gamma(A/I)) \otimes^h_{U_k(t_r^\gamma(A,I)} k \arrow[u, "\simeq"] \arrow[d, "\simeq"'] & & k \otimes^h_{U_k(y_r(A/I))} U_k(y_r(A/I)) \otimes^h_{U_k(y_r(A,I)} k \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\operatorname*{hocolim}_{\gamma} k \otimes^h_{U_k(t_r^\gamma(A/I))} U_k(\mathfrak{st}_r(A/I)) \otimes^h_{U_k(\mathfrak{st}_r(A,I)} k \arrow[rr, "\simeq"] & & k \otimes^h_{U_k(y_r(A/I))} U_k(\mathfrak{st}_r(A/I)) \otimes^h_{U_k(\mathfrak{st}_r(A,I)} k.
\end{tikzcd}$$

Note that by assumption A/I and hence $t_r^\gamma(A/I)$ is also flat, so that by Proposition 8.3.11 we do not need cofibrant replacements. The upper vertical maps are the canonical quotient maps, which are weak equivalences, because

$$k \otimes^h_{U_k(t_r^\gamma(A/I))} U_k(t_r^\gamma(A/I)) \xrightarrow{\simeq} k, \qquad k \otimes^h_{U_k(y_r(A/I))} U_k(y_r(A/I)) \xrightarrow{\simeq} k$$

are simplicial homotopy equivalences induced by the extra-degeneracy $s_{-1} = \mathrm{id} \otimes \eta$. The lower vertical maps are induced by the natural maps

$$t_r^\gamma(A,I) \xrightarrow{\iota_\gamma} \operatorname*{colim}_{\gamma} t_r^\gamma(A,I) = \mathfrak{st}_r(A,I) = \pi_0 y(A,I) \twoheadleftarrow y(A,I).$$

The proof of Proposition 8.3.14 provides natural weak equivalences from $k \otimes^h_{U_k(\mathfrak{gl}(I))} k$ to each of the three objects

$$U_k(t_r^\gamma(A/I)) \otimes^h_{U_k(t_r^\gamma(A,I)} k, \quad U_k(\mathfrak{st}_r(A/I)) \otimes^h_{U_k(\mathfrak{st}_r(A,I)} k, \quad U_k(y_r(A/I)) \otimes^h_{U_k(y_r(A,I))} k,$$

which therefore must be weakly equivalent. Hence the two lower vertical maps are weak equivalences. Finally the lower horizontal map is the weak equivalence of Proposition 8.1.10, where the category of partial orders on $\mathbf{r}$ is contractible, because the discrete order "$\leq$" = "$=$" is an initial object.

□

6.2.2 The Steinberg action on the Volodin construction

Here we are adapting well-known results of the multiplicative setting to the additive one.

Proposition 6.2.9
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$.
Then, for all $g \in St_{r+1}(A) = \pi_0 Y_{r+1}(A)$, the two maps below are equal.

$$H_*(Y_r(A,I),\mathbb{Z}) \xhookrightarrow{\iota} H_*(Y_{r+1}(A,I),\mathbb{Z}),$$
$$H_*(Y_r(A,I),\mathbb{Z}) \xhookrightarrow{\iota} H_*(Y_{r+1}(A,I),\mathbb{Z}) \xrightarrow{(\text{-})^g} H_*(Y_{r+1}(A,I),\mathbb{Z}),$$

where $Y_r(A,I) \xhookrightarrow{\iota} Y_{r+1}(A,I)$ is the canonical inclusion and g acts via $\pi_0 Y_{r+1}(A) \longrightarrow \pi_0 Y_{r+1}(A)$.
In particular $St(A)$ acts trivially on $H_*(Y(A,I),\mathbb{Z})$.

Proof. The Steinberg group $St_{r+1}(A)$ is generated by elements $x_{i,j}(a)$ with $a \in A$ and either i or j is equal to $r+1$. Under the weak equivalence $X_r(A,I) \simeq BY_r(A,I)$ of Proposition 6.2.7 the map $(\text{-})^g \circ \iota$ with $g = x_{i,r+1}(a)$ corresponds to the map

$$\begin{aligned} X_r(A,I) = \bigcup_{\gamma \text{ on } \mathbf{r}} BT_r^{\gamma}(A,I) &\hookrightarrow \bigcup_{\gamma \text{ on } \mathbf{r}} BT_{r+1}^{\gamma'}(A,I) \subset X_{r+1}(A,I), \\ (x_1,...,x_n) &\longmapsto (x_1^g,...,x_n^g), \end{aligned}$$

where γ' is the extension of γ to $\mathbf{r+1}$ given by $i \overset{\gamma'}{<} r+1$. On the restriction to the simplicial subset an explicit homotopy is given by

$$\begin{aligned} \Delta_n^1 \times \bigcup_{\gamma \text{ on } \mathbf{r}} B_n T_{r+1}^{\gamma'}(A,I) &\longrightarrow \bigcup_{\gamma \text{ on } \mathbf{r}} B_n T_{r+1}^{\gamma'}(A,I), \\ (c_{\geq j},(x_1,...,x_n)) &\longmapsto (x_1,...,x_j g, x_{j+1}^g,...,x_n^g), \end{aligned}$$

where $c_{\geq j}$ is the map being equal to 1 precisely on the elements $j \leq k \leq n$. The other case is similar. See also [Sus81] (1.3).

□

Theorem 6.2.10 (Vaserstein, Suslin)
The simplicial abelian group $\tilde{\mathbb{Z}}X_r(A)$ is $\lfloor \frac{r-1}{2} \rfloor$-connected, for all $r \geq 2$.
In particular $X(A)$ and $Y(A)$ are acyclic.

Corollary 6.2.11
For $A \in \mathcal{R}ing$ flat over $\mathbb{Z}$, we have $x(A) \overset{\simeq}{\twoheadrightarrow} \mathbb{Z}$.

Proof. Giving A the discrete filtration $A = F_0 A \supset F_1 A = 0$, this follows from Theorem 6.3.19 and the Theorem of Vaserstein-Suslin 6.2.10.

□

Proposition 6.2.12
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k.
Then, for all $g \in \mathfrak{st}_{r+1}(A) = \pi_0 y_{r+1}(A)$, the map below is zero.

$$H_*^L(y_r(A,I),k) \overset{\iota}{\hookrightarrow} H_*^L(y_{r+1}(A,I),k) \xrightarrow{[\text{-},g]} H_*^L(y_{r+1}(A,I),k),$$

where $y_r(A,I) \overset{\iota}{\hookrightarrow} y_{r+1}(A,I)$ is the canonical inclusion and g acts via $\pi_0 y_{r+1}(A) \longrightarrow \pi_0 y_{r+1}(A,I)$.
In particular $\mathfrak{st}(A)$ acts trivially on $H_*^L(y(A,I),k) \cong H_*(x(A,I))$.

Proof. This is completely analogous. The Steinberg Lie algebra $\mathfrak{st}_{r+1}(A)$ is generated by elements $x_{i,j}(a)$ with $a \in A$ and either i or j is equal to $r+1$. Under the weak equivalence $\Gamma x_r(A,I) \simeq BU_k(y_r(A,I))$ of Proposition 6.2.8 the map $[\text{-},g] \circ \iota$ with $g = x_{i,r+1}(a)$ corresponds to the map

$$x_r(A,I) = \sum_{\gamma \text{ on } \mathbf{r}} \Lambda_* t_r^\gamma(A,I) \hookrightarrow \sum_{\gamma \text{ on } \mathbf{r}} \Lambda_* t_{r+1}^{\gamma'}(A,I) \subset x_{r+1}(A,I),$$
$$x_1 \wedge ... \wedge x_n \longmapsto \sum_{i=1}^n x_1 \wedge ... \wedge [x_i, g] \wedge ... \wedge x_n,$$

where γ' is the extension of γ to $\mathbf{r+1}$ given by $i \overset{\gamma'}{<} r+1$. On the restriction to the particular subset an explicit chain homotopy can be given by

$$\sum_{\gamma \text{ on } \mathbf{r}} \Lambda_n t_{r+1}^{\gamma'}(A,I) \longrightarrow \sum_{\gamma \text{ on } \mathbf{r}} \Lambda_{n+1} t_{r+1}^{\gamma'}(A,I), \qquad x_1 \wedge ... \wedge x_n \longmapsto (-1)^n x_1 \wedge ... \wedge x_n \wedge g.$$

The other case is similar.

□

Proposition 6.2.13
Let $k \in \mathcal{C}Ring$ and $A \in k/\mathcal{R}ing$ flat with $(p-1)! \in A^\times$.
Then $x(A) \twoheadrightarrow k$ is $(p-1)$-connected.

Proof. By Proposition 6.2.12 $\mathfrak{st}(A)$ acts trivially on $H_*(y(A),k)$. Under the weak equivalence $\Gamma x(A) \simeq B_\bullet U_k(E_\bullet y(A))$ the adjoint action of $[x_{i,j}(1), x_{j,i}(1)]$, for $i \neq j$ corresponds to the adjoint action of

$$[e_{i,j}, e_{j,i}] = e_{i,i} - e_{j,j} \in \mathfrak{gl}(A).$$

The $\mathbb{Z}^r$-grading on $\Lambda_* \mathfrak{gl}_r(A)$ of Proposition 5.4.2 induces grading of $\mathbb{Z}^\infty = \varinjlim_{r \geq 1} \mathbb{Z}^r$ on $x(A) \leq \Lambda_* \mathfrak{gl}(A)$. So by the arguments of Proposition 5.4.3 the map $\mathrm{ad}([e_{i,j}, e_{j,i}])$ is multiplication by $v_i - v_j$ on the graded summand corresponding to $v \in \mathbb{Z}^\infty$, and the proof of Proposition 5.4.3 shows that $x(A) \twoheadrightarrow x(A)^{(0)}$ is $(p-1)$-connected. For $n, r \geq 1$, the submodule $(x_r A)_n \leq \Lambda_n \mathfrak{gl}_r A$ is spanned by all summands

$$A \cdot e_{i_1,j_1} \wedge ... \wedge A \cdot e_{i_n,j_n},$$

such that for some partial order γ on $\mathbf{r}$ we have

$$i_1 \overset{\gamma}{<} j_1, \quad i_2 \overset{\gamma}{<} j_2, \quad \dots \quad i_n \overset{\gamma}{<} j_n.$$

But for any such sequence with $v(i_1, j_1) + \dots + v(i_n, j_n) = 0$, every pair (i_k, j_k) is part of a cycle

$$i_k \overset{\gamma}{<} j_k = i_{k_1} \overset{\gamma}{<} j_{k_1} = \dots . = i_{k_m} \overset{\gamma}{<} j_{k_m} = i_k,$$

contradicting the antisymmetry of γ, if $n \geq 1$. It follows that $(x_r(A)^{(0)})_n = 0$, for $n \geq 1$, and hence $x(A)^{(0)} = x_r(A)^{(0)} = k$.

□

6.2.3 Relating Volodin's and Quillen's K-Theory

Like in Suslin's comparison [Sus81] we are linking the Volodin construction to Quillen's K-theory. Again we are copying ideas from the multiplicative to the additive situation.

Lemma 6.2.14
A commutative diagram of simplicial groups

$$\begin{array}{ccc} E_1 & \xrightarrow{f_1} & B_1 \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{f_2} & B_2 \end{array}$$

with the properties below is homotopy cartesian.

(i) *The fibres $F_i = \operatorname{fib} f_i$ are simple and $\pi_0 F_1 \xrightarrow{\sim} \pi_0 F_2$.*

(ii) *The fundamental group $\pi_0 B_i$ acts trivially on $H_*(F_i, \mathbb{Z})$, for $i = 1, 2$.*

(iii) $H_*(E_1, \mathbb{Z}[\pi_0 E_2]) \xrightarrow{\sim} H_*(E_2, \mathbb{Z}[\pi_0 E_2])$ *and* $H_*(B_1, \mathbb{Z}[\pi_0 B_2]) \xrightarrow{\sim} H_*(B_2, \mathbb{Z}[\pi_0 B_2])$.

Proof. Using (ii) the edges of the second page of the Serre Spectral sequences

$$E^2_{p,q} = H_p(B_i, H_q(F_i, \mathbb{Z})) \quad \Rightarrow \quad H_{p+q}(E_i, \mathbb{Z}),$$

are given by

$$\begin{aligned} E^2_{0,q} &= H_0(B_i, H_q(F_i, \mathbb{Z})) = H_q(F_i, \mathbb{Z}), \qquad q \geq 0, \\ E^2_{p,0} &= H_p(B_i, H_0(F_i, \mathbb{Z})) = H_p(B_i, \mathbb{Z}), \qquad p \geq 0. \end{aligned}$$

Using (iii) Proposition 8.1.2 implies

$$H_*(B_1, \mathbb{Z}) \xrightarrow{\sim} H_*(B_2, \mathbb{Z}), \qquad H_*(E_1, \mathbb{Z}) \xrightarrow{\sim} H_*(E_2, \mathbb{Z}),$$

and thus the Comparison Theorem yields $H_*(F_1, \mathbb{Z}) \xrightarrow{\sim} H_*(F_2, \mathbb{Z})$. Hence by (i) and the Whitehead Theorem, we get $\pi_* F_1 \xrightarrow{\sim} \pi_* F_2$. Equivalently the square is homotopy cartesian.

□

Proposition 6.2.15
For $k \in \mathcal{C}Ring$ and $A \in k/\mathcal{R}ing$, The canonical maps induce homotopy fibration sequences

(i) $Y(A) \longrightarrow GL(A) \longrightarrow GL(A)^+$,

(ii) $Y(A) \longrightarrow E(A) \longrightarrow E(A)^+$,

(iii) $Y(A) \longrightarrow St(A) \longrightarrow St(A)^+$.

Proof. Consider the diagram

$$\begin{tikzcd} Y(A) \arrow[r] \arrow[d] & St(A) \arrow[r] \arrow[d] & E(A) \arrow[r] \arrow[d] & GL(A) \arrow[d] \\ Y(A)^+ \arrow[r] & St(A)^+ \arrow[r] & E(A)^+ \arrow[r] & GL(A)^+. \end{tikzcd}$$

We will show that each square satisfies the hypotheses of Lemma 6.2.14 and thus is homotopy cartesian. By definition of the plus construction all the vertical maps induce isomorphisms on homology with local coefficients.

- Since $\pi_0 Y(A) \xrightarrow{\sim} \pi_0 St(A) = St(A)$ the homotopy fibre of $Y(A) \longrightarrow St(A)$ is connected. Moreover $St(A) = \pi_0 Y(A)$ acts trivially on $H_*(Y(A), \mathbb{Z})$ by Proposition 6.2.9.
- The map $St(A) \twoheadrightarrow E(A)$ is a fibration with fibre $K_2(A)$.
- As source and target are constant simplicial groups the map $E(A) \hookrightarrow GL(A)$ is a fibration with trivial fibre.

So each of the three squares is homotopy cartesian and the induced map on the vertical homotopy fibres is a weak equivalence. Since

$$\pi_0 Y(A)^+ = \pi_0 Y(A)/[\pi_0 Y(A), \pi_0 Y(A)] = St(A)/[St(A), St(A)] = 1,$$

and $\tilde{H}_*(Y(A)^+, \mathbb{Z}) = \tilde{H}_*(Y(A), \mathbb{Z}) = \tilde{H}_*(X(A), \mathbb{Z}) = 0$ by Proposition 6.2.5 and Theorem 6.2.10 the Whitehead Theorem implies that $Y(A)^+$ is contractible, which finally proves the proposition.

□

Lemma 6.2.16
A commutative diagram of simplicial Lie algebras over $k \in \mathcal{C}Ring$

$$\begin{tikzcd} E_1 \arrow[r, "f_1"] \arrow[d] & B_1 \arrow[d] \\ E_2 \arrow[r, "f_2"] & B_2 \end{tikzcd}$$

with the properties below is homotopy cartesian.

(i) *The fibres $F_i = \operatorname{fib} f_i$ are simple and $\pi_0 F_1 \xrightarrow{\sim} \pi_0 F_2$.*

(ii) *The fundamental group $\pi_0 B_i$ acts trivially on $H_*(F_i, k)$, for $i = 1, 2$.*

(iii) $H_*(E_1, U_k(\pi_0 E_2)) \xrightarrow{\sim} H_*(E_2, U_k(\pi_0 E_2))$ *and* $H_*(B_1, U_k(\pi_0 B_2)) \xrightarrow{\sim} H_*(B_2, U_k(\pi_0 B_2))$.

Proof. Replace groups by Lie algebras in the proof of Lemma 6.2.14. □

Proposition 6.2.17
For $k \in \mathcal{C}Ring$ and flat $A \in k/\mathcal{R}ing$ with $(p-1)! \in A^\times$, the canonical maps induce homotopy fibration sequences up to the $(p-2)$-nd dimension[1]*.*

(i) $y(A) \longrightarrow \mathfrak{gl}(A) \longrightarrow \mathfrak{gl}(A)^+$,

(ii) $y(A) \longrightarrow \mathfrak{sl}(A) \longrightarrow \mathfrak{sl}(A)^+$,

(iii) $y(A) \longrightarrow \mathfrak{st}(A) \longrightarrow \mathfrak{st}(A)^+$.

Proof. Replace groups by Lie algebras in the proof of Proposition 6.2.15. □

6.2.4 K-theory and the relative Volodin construction

Like in [Lod98] Prop. 11.3.6 we use the results of the preceding section to link the plus construction of the relative Volodin construction to relative K-theory.

Proposition 6.2.18
Let $I \lhd A \in \mathcal{R}ing$ and A be I-adically complete.

Then the composite $Y(A, I) \longrightarrow GL(A) \longrightarrow GL(A)^+$ induces a homotopy fibration sequence

$$Y(A, I)^+ \longrightarrow GL(A)^+ \longrightarrow GL(A/I)^+.$$

Proof. Consider the diagram

$$\begin{tikzcd} Y(A,I) \arrow[r] \arrow[d] & GL(A) \arrow[d] \\ Y(A/I) \arrow[r] \arrow[d] & GL(A/I) \arrow[d] \\ * \arrow[r] & GL(A/I)^+. \end{tikzcd}$$

The upper resp. lower square is homotopy cartesian by Definition 6.2.3 and Lemma 6.2.6 resp. Proposition 6.2.17. Next consider the diagram

$$\begin{tikzcd} Y(A,I) \arrow[r] \arrow[d] & K(A,I) \arrow[r] \arrow[d] & * \arrow[d] \\ GL(A) \arrow[r] & GL(A)^+ \arrow[r] & GL(A/I)^+, \end{tikzcd}$$

[1] This means, that there is a $(p-2)$-connected map from the homotopy fibre of the right map to the left object.

where $K(A,I)$ is defined as the homotopy fibre of $GL(A)^+ \longrightarrow GL(A/I)^+$. We have just shown that the outer square is homotopy cartesian. Since by definition also the right square is homotopy cartesian, the left square must be homotopy cartesian, too. This can be seen for example by comparison of the vertical fibres and the 2-of-3 axiom for weak equivalences. So the homotopy fibre of $Y(A,I) \longrightarrow K(A,I)$ is weakly equivalent to the acylic space $Y(A)$ being the homotopy fibre of $GL(A) \longrightarrow GL(A)^+$ by Proposition 6.2.17. Thus the Serre spectral sequence for $Y(A,I) \longrightarrow K(A,I)$ collapses, proving that this map induces an isomorphism on homology. Since $K(A,I)$ is simple we get

$$Y(A,I)^+ \xrightarrow{\simeq} K(A,I)^+ \xleftarrow{\simeq} K(A,I).$$

□

Proposition 6.2.19
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k with $(p-1)! \in A^\times$.
Then the composite $y(A,I) \longrightarrow \mathfrak{gl}(A) \longrightarrow \mathfrak{gl}(A)^+$ induces a homotopy fibration sequence up to dimension $p-2$.

$$y(A,I)^+ \longrightarrow \mathfrak{gl}(A)^+ \longrightarrow \mathfrak{gl}(A/I)^+.$$

Proof. Replace group by Lie algebra in the proof of Proposition 6.2.18.

□

6.3 K-theory and group homology of matrices

6.3.1 The plus construction as an E_∞-space

Using the category of injection we explicitly construct an E_∞-structure on the plus construction of the classifying space of the general linear group. Let us point out that this idea is not new.

Proposition 6.3.1
Let $A \in \mathcal{R}ing$.

(i) *There is a functor on the category I of injections (cf. Definition 4.1.1)*

$$MM_\bullet A : I \longrightarrow \mathcal{M}on, \qquad \mathbf{s} \longmapsto MM_\mathbf{s}A = (M_sA, \cdot),$$

sending a morphism $f \in I(\mathbf{r},\mathbf{s})$ to the map MM_fA, which associates to an element $X \in M_rA$ the unique automorphism of $A^\mathbf{r}$, such that the square

$$\begin{tikzcd} A^{(\mathbf{r})} \arrow[r, "A(f)"] \arrow[d, "X"'] & A^{(\mathbf{s})} \arrow[d, "MM_f(A)(X)"] \\ A^{(\mathbf{r})} \arrow[r, "A(f)"] & A^{(\mathbf{s})} \end{tikzcd}$$

commutes and $MM_f(A)(X)$ is the identity on the complement $A^{(\mathbf{s}\setminus f(\mathbf{r}))} \leq A^{(\mathbf{s})}$.

(ii) *Moreover there is a multiplication homomorphism induced by the block sum*

$$\mu : MM_rA \times MM_sA \longrightarrow MM_{r+s}A, \qquad (X,Y) \longmapsto X \oplus Y := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix},$$

and a canonical unit morphism $\eta : 1 \xrightarrow{\sim} MM_{\mathbf{0}}A$, making $M_\bullet A$ into a symmetric monoidal functor $(I, +, \mathbf{0}) \longrightarrow (\mathcal{M}on, \times, 1)$.

(iii) *As the functors $(\text{-})^\times \longrightarrow \mathcal{G}rp \xrightarrow{B} s\mathcal{S}et$ are strictly symmetric monoidal, we also get symmetric monoidal functors*

$$GL_\bullet A : I \xrightarrow{MM_\bullet A} \mathcal{M}on \xrightarrow{(\text{-})^\times} \mathcal{G}rp, \qquad BGL_\bullet A : I \xrightarrow{GL_\bullet A} s\mathcal{S}et.$$

For an ideal $J \lhd A$, all the structure maps of $BGL_\bullet A$ restrict to a symmetric monoidal functor

$$X_\bullet(A,J) : I \longrightarrow s\mathcal{S}et, \quad \mathbf{r} \longmapsto X_r(A,J).$$

All functors preserve limits over connected non-empty categories.

Proof.

(i) For $f \in I(\mathbf{r}, \mathbf{s})$ the element $MM_f(A)(X) \in M_s(A) = A^{s\times s}$ is obtained from $X \in MM_r(A) =$ by filling up with ones on the diagonal and zeroes elsewhere. Hence $M_\bullet(A)$ is a functor $I \longrightarrow \mathcal{M}on$ and maps morphisms to injections.

(ii) See Proposition 5.4.10 (ii).

(iii) As the functors $(\text{-})^\times$ and B are right adjoints, they preserve products and in particular are symmetric monoidal. Hence the compositions $GL_\bullet A = (MM_\bullet A)^\times$ and $BGL_\bullet A$ are symmetric monoidal. Given two partial orders γ on $\mathbf{r}$ and γ' on $\mathbf{r}'$ we give $\mathbf{r} + \mathbf{r}'$ the disjoint union partial order $\gamma + \gamma'$. Then the multiplication of $M_\bullet(A)$ restricts to a map

$$\mu : T_r^\gamma(A) \times T_{r'}^{\gamma'}(A) \longrightarrow T_{r+r'}^{\gamma+\gamma'}(A),$$

which defines the monoidal structure of $X_\bullet(A,J)$.

Following the arguments in the proof of Proposition 4.1.9, one similarly uses Lemma 4.1.8 to check that $MM_\bullet(A)$ preserves limits over connected non-empty categories. Since every map $f \in I(\mathbf{r}, \mathbf{s})$ induces cartesian squares

$$\begin{tikzcd} GL_rA \arrow[r, hook, "GL_fA"] \arrow[d, hook] & GL_sA \arrow[d, hook] \\ M_rA \arrow[r, hook, "M_fA"] & M_sA, \end{tikzcd} \qquad \begin{tikzcd} X_r(A,J) \arrow[r, hook, "X_f(A,J)"] \arrow[d, hook] & X_s(A,J) \arrow[d, hook] \\ BGL_rA \arrow[r, hook, "BGL_fA"] & BGL_sA, \end{tikzcd}$$

and since B is a right adjoint and therefore preserves limits, we can apply Lemma 4.1.5 to check that also $GL_\bullet A$, $BGL_\bullet A$ and $X_\bullet(A,J)$ preserve limits over connected, non-empty indexing categories.

$\square$

Remark 6.3.2
Let $(C, \otimes, e) \xrightarrow{M} (\mathcal{C}, \otimes, E)$ *be monoidal functor.*
Then canonically $\operatorname{colim}_C M \in (\mathcal{C}, \otimes, E)\text{-}\mathcal{A}ss_1$.
Moreover $\operatorname{colim}_C M \in (\mathcal{C}, \otimes, E)\text{-}\mathcal{C}om_1$, *if* M *is symmetric monoidal.*

Proposition 6.3.3
For $A \in \mathcal{R}ing$, *the map of simplicial groups*

$$\operatorname*{hocolim}_{\mathbb{N}_0} GL_\bullet A \longrightarrow \operatorname*{hocolim}_{\hat{\Delta}_{inj}} GL_\bullet A$$

is a plus construction for $\operatorname{hocolim}_{\mathbb{N}_0} GL_\bullet A \xrightarrow{\simeq} \operatorname{colim}_{\mathbb{N}_0} GL_\bullet A = GL_\infty A$.

Proof. This was already noted in [SS12] Ex. 1.5, as the auther found out after writing down this proof. By the Whitehead Theorem for simplicial groups 8.2.6 it suffices to check that the map induces an isomorphism on homology and that

$$GL_\infty(A) = \pi_0 \operatorname*{hocolim}_{\mathbb{N}_0} GL_\bullet A \twoheadrightarrow \pi_0 \operatorname*{hocolim}_{\hat{\Delta}_{inj}} GL_\bullet A = \operatorname*{colim}_{\hat{\Delta}_{inj}} GL_\bullet(A)$$

is isomorphic to the abelianization map $GL_\infty A \twoheadrightarrow (GL_\infty A)^{ab}$. For the latter, note that in $\operatorname{colim}_{\hat{\Delta}_{inj}} GL_\bullet(A)$ we have

$$\begin{aligned}\begin{pmatrix} XY & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix} \cdot \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} YX & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}$$

for all $X, Y \in GL_r A$. This proves that the group $\operatorname{colim}_{\hat{\Delta}_{inj}} GL_\bullet(A)$ is abelian. Hence in the commutative diagram

$$\begin{tikzcd} \operatorname*{colim}_{\mathbb{N}_0} GL_\bullet A \arrow[r, two heads] \arrow[d] & \operatorname*{colim}_{\mathbb{N}_0} (GL_\bullet A)^{ab} \arrow[r, "\sim"] \arrow[d] & (\operatorname*{colim}_{\mathbb{N}_0} GL_\bullet A)^{ab} \arrow[d] \\ \operatorname*{colim}_{\hat{\Delta}_{inj}} GL_\bullet A \arrow[r, two heads] & \operatorname*{colim}_{\hat{\Delta}_{inj}} (GL_\bullet A)^{ab} \arrow[r, "\sim"] & (\operatorname*{colim}_{\hat{\Delta}_{inj}} GL_\bullet A)^{ab} \end{tikzcd}$$

the two horizontal maps on the right are isomorphisms, because abelianization is a left adjoint and therefore commutes with colimits. The middle vertical map is an isomorphism, because maps in $\hat{\Delta}_{inj}$ with same domain and codomain induce the same maps under $(GL_\bullet A)^{ab}$. It follows that also the right vertical map is an isomorphism. Moreover the lower horizontal composition is an isomorphism, because $\operatorname{colim}_{\hat{\Delta}_{inj}} GL_\bullet A$ is abelian, as we have just proven. It follows that also the lower left horizontal map is an isomorphism. This proves that $\operatorname{colim}_{\hat{\Delta}_{inj}} GL_\bullet A$ is the abelianization of $GL_\infty A$.

To see that

$$H_*(\operatorname*{hocolim}_{\mathbb{N}_0} GL_\bullet A, \mathbb{Z}) \longrightarrow H_*(\operatorname*{hocolim}_{\hat{\Delta}_{inj}} GL_\bullet A, \mathbb{Z})$$

is an isomorphism, note that the bar construction $s\mathcal{G}rp \xrightarrow{B} s\mathcal{S}et_*$ commutes with homotopy colimits by Proposition 7.3.34. As $s\mathcal{S}et_* \xrightarrow{\tilde{\mathbb{Z}}} s\mathcal{A}b$ is a left adjoint and thus also commutes with homotopy colimits, it suffices to prove that the map

$$\operatorname*{hocolim}_{\mathbb{N}_0} \tilde{\mathbb{Z}}BGL_\bullet A \longrightarrow \operatorname*{hocolim}_{\hat{\Delta}_{inj}} \tilde{\mathbb{Z}}BGL_\bullet A$$

is a weak equivalence. Using the spectral sequence for homotopy colimits of Proposition 7.3.23 it suffices to check that all the maps

$$\pi_p \operatorname*{hocolim}_{\mathbb{N}_0} H_q(GL_\bullet A, \mathbb{Z}) \xrightarrow{\sim} \pi_p \operatorname*{hocolim}_{\hat{\Delta}_{inj}} H_q(GL_\bullet A, \mathbb{Z}), \qquad p, q \geq 0$$

are isomorphisms. As the adjoint action of a group on its homology is trivial, the functor

$$H_q(GL_\bullet A, \mathbb{Z}) : \hat{\Delta}_{inj} \longrightarrow \mathcal{A}b$$

maps morphisms in $\hat{\Delta}_{inj}$ with equal domain and codomain to the same maps. In particular $H_q(GL_\bullet A, \mathbb{Z})$ factors over $\hat{\Delta}_{inj} \twoheadrightarrow \mathbb{N}_0$ and hence the maps

$$\operatorname*{hocolim}_{\mathbb{N}_0} H_q(GL_\bullet A, \mathbb{Z}) \xrightarrow{\simeq} \operatorname*{hocolim}_{\hat{\Delta}_{inj}} H_q(GL_\bullet A, \mathbb{Z}), \qquad q \geq 0,$$

are weak equivalences by Corollary 7.3.24 and Remark 7.3.25, because $\hat{\Delta}_{inj} \xrightarrow{\delta} I \xrightarrow{\tau} \mathbb{N}_0$ is totally final by Proposition 4.1.4.

□

Corollary 6.3.4
For $A \in \mathcal{R}ing$, the map

$$BGL_\infty A = \operatorname*{colim}_{\mathbb{N}_0} BGL_\bullet A \twoheadrightarrow \operatorname*{colim}_{\hat{\Delta}_{inj}} BGL_\bullet A =: D(A)$$

is a plus construction (for $BGL_\infty A$ in the sense of Quillen).
In particular the group completion $\mathcal{G}rp(D(A)) \in s\mathcal{G}rp$ is a delooping of $GL_\infty(A)^+$.

Proof. Consider the commutative diagram of pointed simplicial sets

$$\begin{tikzcd}
\operatorname*{colim}_{\mathbb{N}_0} BGL_\bullet A \arrow[d] & \operatorname*{hocolim}_{\mathbb{N}_0} BGL_\bullet A \arrow[l] \arrow[r] \arrow[d] & B \operatorname*{hocolim}_{\mathbb{N}_0} GL_\bullet A \arrow[d] \\
\operatorname*{colim}_{\hat{\Delta}_{inj}} BGL_\bullet A & \operatorname*{hocolim}_{\hat{\Delta}_{inj}} BGL_\bullet A \arrow[l] \arrow[r] & B \operatorname*{hocolim}_{\hat{\Delta}_{inj}} GL_\bullet A.
\end{tikzcd}$$

The right vertical map is a weak equivalence by Proposition 6.3.3. The right two horizontal maps are weak equivalences by Proposition 7.3.34. The upper left horizontal map is a weak equivalence, as $\mathbb{N}_0$ is a filtered category and filtered colimits are exact. The lower left horizontal map is a weak equivalence by Corollary 7.3.31, as by Corollary 4.1.7 and Proposition 6.3.1 the composite functor

$$\hat{\Delta}_{inj} \xrightarrow{\delta} I \xrightarrow{GL_\bullet A} \mathcal{G}rp \xrightarrow{B} s\mathcal{S}et$$

preserves limits over connected non-empty categories. As the diagram commutes, it follows that every map is a weak equivalence of pointed simplicial sets. As the categories $\mathbb{N}_0$ and $\hat{\Delta}_{inj}$ are connected, it follows that the colimits of pointed simplicial sets on the left are also colimits of the underlying simplicial sets. This concludes the proof for the first part.

For the second statement, note that the simplicial monoid $D(A)$ is free in every dimension. Indeed in dimension 1 it is freely generated by matrices $X \in GL_r A$, that cannot be written as a block sum $X' \oplus X''$. Similarly in dimension $n > 1$ it is freely generated by tuples $(X_1, ..., X_n) \in GL_r A^n$, that cannot be written as a block sum

$$(X_1, ..., X_n) = (X_1' \oplus X_1'', ..., X_n' \oplus X_n''), \qquad (X_1', ..., X_n') \in GL_{r'} A, \quad (X_1'', ..., X_n'') \in GL_{r''} A.$$

Hence $D(A) \xrightarrow{\simeq} \mathcal{G}rp(D(A))$ is a weak equivalence by Proposition 7.3.33, because $\pi_0 D(A) = 1$.

□

Proposition 6.3.5
For every ring $A \in \mathcal{R}ing$, the simplicial monoid $D(A) \in s\mathcal{S}et\text{-}\mathcal{A}ss_1$ is an E_∞-space, i.e. there is a multiplication map

$$\mathcal{C}om_{1,\infty} D(A) \xrightarrow{\mu} D(A),$$

making $D(A)$ into an algebra over the I-operad $\mathcal{C}om_{1,\infty} \in \mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)$ of Definition 4.2.8.

Proof. Like in the construction of the associative operad in Proposition 4.2.6 we can define multiplication maps

$$\Sigma_k \times (GL_{n_1} A \times ... \times GL_{n_k} A) \xrightarrow{\mu} GL_{n_1 + ... + n_k} A, \quad (\sigma, (X_1, ..., X_k)) \longmapsto \bar{\sigma} \circ (X_1 \oplus ... \oplus X_k),$$

where $\bar{\sigma} \in GL_{n_1 + ... + n_k} A$ is the matrix corresponding to the permutation of the k blocks in $\mathbf{n}_1 + ... + \mathbf{n}_k$. By composition with the functor

$$E : \mathcal{S}et \longrightarrow \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}, \mathcal{S}et), \quad X \longmapsto E^\bullet X = \mathcal{S}et(\text{-}, X),$$

we obtain multiplication maps

$$E\Sigma_k \times (E_\bullet GL_{n_1} A \times ... \times EGL_{n_k} A) \xrightarrow{\mu} EGL_{n_1 + ... + n_k} A,$$

which define an algebra structure on $\operatorname{colim}_{\hat{\Delta}_{inj}} EGL_\bullet A \in s\mathcal{S}et$ over the operad $\mathcal{C}om_{1,\infty} = E\Sigma \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, s\mathcal{S}et)$. By construction the maps μ respect the right action by $GL_{n_1} A \times ... \times GL_{n_k} A$ and thus induce maps

$$E\Sigma_k \times (E_\bullet GL_{n_1} A / GL_{n_1} A \times ... \times EGL_{n_k} A / GL_{n_k} A) \xrightarrow{\mu} EGL_{n_1 + ... + n_k} A / GL_{n_1 + ... + n_k} A,$$

which define an algebra structure on $D(A) = \operatorname{colim}_{\hat{\Delta}_{inj}} BGL_\bullet A \cong \operatorname{colim}_{\hat{\Delta}_{inj}} EGL_\bullet A / GL_\bullet A$.

□

Using Corollary 4.3.13 we obtain the following result. We will not need it later, so we will only sketch the proof.

Corollary 6.3.6
Let $A \in \mathcal{R}ing$ and $k \leq \mathbb{Q}$ with $(p-1)! \in k^\times$, for some prime number $p > 1$.
Then the Hurewicz map induces isomorphisms, for all $1 \leq n < p$

$$k \otimes K_n(A) \xrightarrow{\sim} PH_n(GL(A), k) = \ker \left(H_n(GL(A), k) \underset{(\eta \times \mathrm{id})_* + (\mathrm{id} \times \eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(GL(A) \times GL(A), k) \right),$$

where $GL(A) \xrightarrow{\delta} GL(A) \times GL(A)$ is the diagonal and $1 \xrightarrow{\eta} GL(A)$ is the initial group homomorphism.

Proof. (Sketch) Let $k_\infty D(A)$ be the k-completion of the space $D(A)$ in the sense of Bousfield-Kan [BK72] I. Then it can be seen that with $D(A)$ also $k_\infty D(A)$ is an infinite loop space and thus by [BE74b] Prop. 3.1 lies in $s\mathcal{S}et_*$-$\mathcal{C}om_{1,\infty}$ up to weak equivalence. As $D(A)$ is an H-space, it is simple and thus nilpotent in the sense of [BK72] I.4.3. Hence by [BK72] Prop. V.3.1 we have isomorphisms

$$k \otimes \pi_* D(A) \xrightarrow{\sim} \pi_*(k_\infty D(A)), \qquad k \otimes \tilde{H}_*(k_\infty D(A), \mathbb{Z}) \xrightarrow{\sim} \tilde{H}_*(k_\infty D(A), \mathbb{Z}),$$

which together with Corollary 4.3.13 imply the result.

□

Corollary 6.3.7
Let $J \lhd A \in \mathcal{R}ing$.
Then $X(A, J)^+ \in s\mathcal{S}et_*$-$\mathcal{C}om_{1,\infty}$ up to weak equivalence.

Proof. By Proposition 6.2.7 and by construction of the plus construction we have

$$X(A, J)^+ \simeq (BY(A, J))^+ \simeq B(Y(A, J)^+)$$

Moreover by Proposition 7.3.33 and Corollary 6.3.4 there are weak equivalences

$$D(A) \simeq BGL(A)^+ \simeq B(GL(A)^+).$$

Since $Y(A, J)^+$ is the homotopy fibre of $GL(A)^+ \twoheadrightarrow GL(A/J)^+$ by Proposition 6.2.18, we get a homotopy fibration

$$X(A, J)^+ \longrightarrow D(A) \longrightarrow D(A/J).$$

By [BE74b] Thm. A every $X \in s\mathcal{S}et_*$-$\mathcal{C}om_{1,\infty}$ is an infinite loop space (i.e. for every $n \geq 0$ there is a space $Y \in s\mathcal{S}et_*$, such that $X \simeq \Omega^n Y$). Hence $D(A)$ and $D(A/J)$ are infinite loop spaces and by the long exact sequence also $X(A, J)^+$ is an infinite loop space. Then [BE74b] Prop. 3.1. implies that $X(A, J)^+$ is weakly equivalent to an E_∞-space.

□

6.3.2 Cyclic homology and the relative Volodin construction

In the same way as in the absolute setting we can prove a relative variant of the Theorem of Loday-Quillen and Tsygan.

Proposition 6.3.8
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k and suppose $(p-1)! \in A^\times$, for some $p > 1$.

Then $\mathbb{Z}^\infty$-grading on $\Lambda_*\mathfrak{gl}_\infty A$ restricts to a grading on the subcomplex $x(A, I) \leq \Lambda_*\mathfrak{gl}_\infty A$ and the canonical quotient map onto the 0-homogeneous summand is $(p-1)$-connected

$$x(A, I) \twoheadrightarrow x(A, I)^{(0)}.$$

Proof. Using the trivial action of $\mathfrak{st}(A)$ on $H_*(x(A, I))$ in dimensions $< p$ from Proposition 6.2.12, this is proven in exactly the same way as Proposition 5.4.3. See also the proof of Proposition 6.2.13.

□

Proposition 6.3.9
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k and suppose $(p-1)! \in A^\times$, for some $p > 1$.

Then there is an action of the Steinberg group $St(A)$ on $H_*(x(A, I))$, which is trivial in dimensions $< p$.

Proof. Using Proposition 6.3.8, the proof is the same as that of Proposition 5.4.4.

□

Corollary 6.3.10
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k and suppose $(p-1)! \in A^\times$, for some $p > 1$.

Then the action of Σ_∞ on $\Lambda_*\mathfrak{gl}_\infty A$ restricts to an action on $x(A, I)$, which is trivial under H_n, for all $0 \leq n < p$.

Proof. Writing a product τ of two transpositions in Σ_∞ as composition of elementary matrices similarly as in the proof of Corollary 5.4.5, one can identify the action of τ with the conjugation action by some element of the Steinberg group, which is trivial on the particular homology groups by the preceding Proposition. So again similar arguments as in Corollary 5.4.5 imply that the infinite alternating group A_∞ acts trivially on $H_*(x(A, I))$. To see that also Σ_∞ acts trivially, let $\sigma \in \Sigma_\infty$ and take a class $x \in H_n(x(A, I))$. Then infact $x \in H_n(x_r(A, I))$, for some finite $r \geq 0$, and we can modify σ by a transposition being trivial on $\mathbf{r}$ to show that also ${}^\sigma x = x$.

□

Proposition 6.3.11
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k and suppose $(p-1)! \in A^\times$, for some $p > 1$.

Then the map $H_n(x(A, I)) \xrightarrow{\sim} H_n(x(A, I)_{\Sigma_\infty})$ is an isomorphism, for all $0 \leq n < p$.

Proof. Using Proposition 6.3.8 and Corollary 6.3.10 we again want to adapt the proof of Proposition 5.4.8. However the module structure of $x_n(A,I)$ is more complicated than that of $\Lambda_n gl_r A$. The critical point is the claim that

$$H_m(\Sigma_r, x_r(A,I)_n) = 0, \qquad 0 < m < p, \quad 0 \le n < p, \quad r \ge 1,$$

where we have to pay more attention. Note that by a cofiltered limit argument, this also implies the case $r = \infty$, which we are really interested in. Considering the colimit over γ of the retractions

$$(t_r^\gamma(A,I))^{\otimes n} \twoheadrightarrow \Lambda_n t_r^\gamma(A,I), \qquad r \ge 1,$$

it suffices to check that

$$H_m(\Sigma_r, \sum_\gamma (t_r^\gamma(A,I))^{\otimes n}) = 0, \qquad 0 < m < p, \quad 0 \le n < p, \quad r \ge 1.$$

Now since A and A/I are flat over k, we can write the morphism $A \overset{q}{\twoheadrightarrow} A/I$ as a filtered colimit of epimorphisms $A' \overset{q'}{\twoheadrightarrow} B'$ with free $A', B' \in k$-Mod. It follows that $t_r^\gamma(A,I)$ is the filtered colimit of $t_r^\gamma(A',I')$, where $I' := \ker q'$. In particular this argument shows that we may assume that A and A/I are free. In this case, note that $A \twoheadrightarrow A/I$ has a k-linear section and thus I is a direct summand of A. So we may choose a basis $\bar{I} \subset I$ and extend it to a basis $\bar{A} \subset A$. Defining

$$\bar{t}_r^\gamma(\bar{A},\bar{I}) := \{(a,i,j) \in \bar{A} \times \mathbf{r} \times \mathbf{r};\ a \in \bar{I} \quad \Rightarrow \quad i \overset{\gamma}{<} j\}, \quad \bar{x}_r(\bar{A},\bar{I}) := \bigcup_\gamma \bar{t}_r^\gamma(\bar{A},\bar{I}),$$

the Σ_r-action on $\bar{A} \times \mathbf{r} \times \mathbf{r}$ restricts to an action on $\bar{x}(\bar{A},\bar{I})$ and the isomorphism φ of Proposition 5.4.8 restricts to a $k[\Sigma_r]$-linear isomorphism $\varphi : k(\bar{x}_r(\bar{A},\bar{I}))^n \xrightarrow{\sim} x_r(A,I)^{\otimes n}$. Now by the same arguments as in Proposition 5.4.8 we can verify the claim.

□

Remark 6.3.12
For $I \lhd A \in k/\mathcal{R}ing$ the composition of the following chain maps is the identity.

$$C^\lambda_{*-1}(A,I) \xrightarrow{\phi} x(A,I) \xrightarrow{\theta} \sum_{\gamma,\ r\ge 1} C^\lambda_{*-1}(U_k(t_r^\gamma(A,I)) \xrightarrow{\varepsilon} \sum_{\gamma,\ r\ge 1} C^\lambda_{*-1} t_r^\gamma(A,I) \xrightarrow{\text{trace}} C^\lambda_{*-1}(A,I),$$

where the maps are the restrictions of those in Remark 5.4.17.

Proposition 6.3.13
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k and suppose $(p-1)! \in A^\times$, for some $p > 1$.

Then ϕ extends to the map of dg bialgebras, which is an isomorphism in dimensions $< p$

$$\phi : \mathcal{C}om_1(C^\lambda_{*-1}(A,I) \longrightarrow i(A,I)^{(0)} := x(A,I)^{(0)}_{\Sigma_\infty}.$$

Proof. By the same arguments as in Proposition 5.4.18 using Remark 6.3.12 and the proof for the rational case, given in [AO94] 1.2, implies the statement.

□

Definition 6.3.14
Let $I \lhd A \in \mathcal{R}ing$.

The ***relative (additive) negative Chern character*** *is defined as the sum over γ and r of the compositions*

$$\begin{tikzcd}
\Lambda_* t_r^\gamma(A,I) \arrow[rr] \arrow[d, "e"'] & & \mathrm{Tot}^\times M^-(A,I) \\
\tilde{B}_* U_{\mathbb{Z}}(\mathcal{T}_r^\gamma(A,I)) \arrow[r, "\mathrm{ch}^-"] & \mathrm{Tot}^\times M^-(U_{\mathbb{Z}}(T_r^\gamma(A,I))) \arrow[r, "\varepsilon"] & \mathrm{Tot}^\times M^-(t_r^\gamma A) \arrow[u, "\mathrm{trace}"']
\end{tikzcd}$$

where e is the antisymmetrization map of Proposition 5.2.13, ch^- is the negative Chern character for the Hopf algebras $U_{\mathbb{Z}}(t_r^\gamma(A,I))$ of Definition 5.2.11, $U_{\mathbb{Z}}(t_r^\gamma(A,I)) \xrightarrow{\varepsilon} t_r^\gamma(A,I)$ is the fusion map and trace *is the trace map for negative cyclic homology (see Remark 5.4.17).*

By abuse of notation we will denote it also by ch^-.

Theorem 6.3.15
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k and suppose $(p-1)! \in A^\times$, for some $p > 1$.

Then the map ϕ induces isomorphisms in dimensions $0 \le n < p-1$

$$H^\lambda_{n-1}(A,I) \xrightarrow{\sim} PH_n(x(A,I)) = \ker\left(H_n(x(A,I)) \underset{(\eta\otimes\mathrm{id})_*+(\mathrm{id}\otimes\eta)_*}{\overset{\delta_*}{\rightrightarrows}} H_n(x(A,I)\otimes x(A,I)) \right).$$

Here $x(A,I) \xrightarrow{\delta} x(A,I)\otimes x(A,I)$ is the comultiplication and $\eta : k = x(A,I)_0 \hookrightarrow x(A,I)$ is the inclusion of the zeroth dimension.

Moreover Connes' operator B of Remark 5.1.10 and the negative Chern character of Definition 6.3.14 induce a commutative diagram

$$\begin{tikzcd}
HC_{*-1}(A,I) \arrow[r, "B"] \arrow[d] & HC_*^-(A,I) \\
H^\lambda_{*-1}(A,I) \arrow[r, "\phi"] & H_*(x(A,I)) \arrow[u, "\mathrm{ch}^-"']
\end{tikzcd}$$

Proof. See the proof of Theorem 5.4.20.

□

The original plan was to link $K_n^{\mathcal{L}ie}(A)$ and $HC_{n-1}(A)$ via as stated in Conjecture 9.0.7 and then use the absolute version Theorem 5.4.20 and the long exact sequence to prove Theorem 6.3.15. However we do not know yet how to do that and so we had to reprove the Theorem in the relative setting.

6.3.3 Comparing multiplicative and additive relative K-theory

Finally we have all tools in hand to verify the main result of this work.

Proposition 6.3.16
Let $A \in \mathcal{R}ing$ carrying a ring filtration F, such that $\mathbb{Z}$ and $A = F_0A$.

Then there is an induced filtration F on $t_r(A, F_1A) \in \mathcal{A}ss$, for every $r \geq 1$, (see Definition 6.2.3 and Remark 6.2.4), given by

$$F_n t_r(A, F_1A) = \sum_{1\leq i,j\leq r} F_{\lceil (n+i-j)/r \rceil} A \cdot e_{i,j}, \qquad n \geq 1,$$

such that $F_1 t_r(A, F_1A) = t_r(A, F_1A)$ and that $\mathrm{gr}^F t_r(A, F_1A) = t_r(\mathrm{gr}^F A, \mathrm{gr}^F(F_1A))$.

Moreover suppose that $\mathrm{gr}^F A$ is flat, A is complete with respect to F and that there is a subset $Y \subset A$, such that

(i) $\mathrm{gr}^F A = \sum_{y\in Y} \mathbb{Z} \cdot [y]$,

(ii) $y^n/n! \in F_1A$, *for all $y \in Y \cap F_1A$ and $n \geq 1$.*

Then $t_r(A, I)$ satisfies the hypotheses of Proposition 3.5.7.

Proof. As

$$\left\lceil \frac{1+i-j}{r} \right\rceil = \begin{cases} 0, & 1 \leq i < j \leq r, \\ 1, & 1 \leq j \leq i \leq r, \end{cases}$$

it follows that $F_1 t_r(A, F_1A) = t_r(A, F_1A)$. Since

$$\left\lceil \frac{n+i-j}{r} \right\rceil + \left\lceil \frac{m+j-k}{r} \right\rceil \geq \left\lceil \frac{n+m+i-k}{r} \right\rceil, \qquad 1 \leq i,j,k \leq r, \quad m,n \geq 1,$$

it follows that $F_n t_r(A, F_1A) \cdot F_m t_r(A, F_1A) \subset F_{n+m} t_r(A, F_1A)$, for all $m, n \geq 1$, proving that F is a ring filtration. Moreover we have

$$\mathrm{gr}^F_n t_r(A, F_1A) = \sum_{\substack{1\leq i,j\leq r,\\ r|(n+i-j)}} \mathrm{gr}^F_{(n+i-j)/r} A \cdot e_{i,j}, \qquad n \geq 1,$$

which shows that $\mathrm{gr}^F t_r(A, F_1A) = t_r(\mathrm{gr}^F A, \mathrm{gr}^F(F_1A))$. In particular $\mathrm{gr}^F t_r(A, F_1A)$ is flat, if $\mathrm{gr}^F A$ is flat. We define

$$X = t_r(A, F_1A) \cap \bigcup_{1\leq i,j\leq r} Y \cdot e_{i,j}.$$

Then (i) implies $\mathrm{gr}^F t_r(A, F_1A) = t_r(\mathrm{gr}^F A, \mathrm{gr}^F(F_1A)) = \sum_{x\in X} \mathbb{Z} \cdot [x]$ and (ii) implies

$$(y \cdot e_{i,j})^n/n! = \begin{cases} y^n/n! \cdot e_{i,j} \in F_1A \cdot e_{i,j}, & i = j, \\ 0, & i \neq j, \end{cases}$$

for all $y \cdot e_{i,j} \in t_r(A, F_1A)$. Hence $x^n/n! \in t_r(A, F_1A)$, for all $x \in X$ and $n \geq 1$. □

Remark 6.3.17
Let $A \in \mathcal{R}ing$ carrying a finite ring filtration $A = F_0A \supset \ldots \supset F_NA = 0$ with $\mathrm{gr}^F A$ flat over $\mathbb{Z}$.

If $(N-1)! \in A^\times$, then $Y := A$ satisfies the hypothesis of Proposition 6.3.16.

Corollary 6.3.18
Let $A \in \mathcal{R}ing$ carrying a ring filtration F, such that $\mathbb{Z}$ and $A = F_0A$.

Then for every partial order γ on $\mathbf{r} = \{1, ..., r\}$, there is an induced filtration F on $t_r^\gamma(A, F_1A)$, satisfying the hypothesis of Proposition 3.5.7.

These filtrations can be chosen in a way, such that $t_r^\gamma(A, F_1A) \hookrightarrow t_r^{\gamma'}(A, F_1A)$ is $2r$-equicontinuous, for every $\gamma \subset \gamma'$.

Proof. Then γ is isomorphic to the usual total order on $\mathbf{r}$ by a permutation $\sigma_\gamma \in \Sigma_r$. So by Proposition 6.3.16 the initial filtration along the isomorphism $t_r^\gamma(A, F_1A) \xrightarrow{\sim} t_r(A, F_1A)$ induced by σ_γ satisfies the hypothesis of Proposition 3.5.7. Now by the order extension principle any partial order γ on $\mathbf{r}$ can be extended to a total order $\overline{\gamma}$ and the result follows by giving $t_r^\gamma(A, F_1A)$ the initial filtration along $t_r^\gamma(A, F_1A) \hookrightarrow t_r^{\overline{\gamma}}(A, F_1A)$. Of course the filtration depends on the chosen total extension $\overline{\gamma}$. However, if $\gamma_1 \subset \gamma_2$ and $\overline{\gamma}_1 \neq \overline{\gamma}_2$, then the filtration on $t_r^{\gamma_k}(A, F_1A)$ is given by

$$F_n t_r^{\gamma_k}(A, F_1A) = t_r^{\gamma_k}(A, F_1A) \cap \sum_{1 \le i,j \le r} F_{\lceil (n+\sigma_k(i)-\sigma_k(j))/r \rceil} A \cdot e_{i,j}, \qquad n \geq 1,$$

where $\sigma_k = \sigma_{\overline{\gamma}_k} \in \Sigma_r$, for $k = 1, 2$. Since

$$(\sigma_2(i) - \sigma_2(j)) - (\sigma_1(i) - \sigma_1(j)) \le (r-1) - (1-r) < 2r, \qquad 1 \le i, j \le r,$$

it follows that the induced map $t_r^{\gamma_1}(A, F_1A) \hookrightarrow t_r^{\gamma_2}(A, F_1A)$ is at least $2r$-equicontinous.

□

Theorem 6.3.19
Let $A \in \mathcal{R}ing$ carrying a finite ring filtration $A = F_0A \supset ... \supset F_NA = 0$, such that $\mathrm{gr}^F A$ is flat over $\mathbb{Z}$. Suppose there is a subset $Y \subset A$, such that

(i) $\mathrm{gr}^F A = \sum_{y \in Y} \mathbb{Z} \cdot [y]$,

(ii) $y^n/n! \in F_1A$, *for all $y \in Y \cap F_1A$ and $n \geq 1$.*

Then there is a natural zig-zag of weak equivalences

$$\mathbb{Z}X(A, F_1A) \simeq \Gamma x(A, F_1A), \qquad \mathbb{Z}X_r(A, F_1A) \simeq \Gamma x_r(A, F_1A), \quad r \geq 1,$$

where $X(A, F_1A)$ and $x(A, F_1A)$ are the relative Volodin constructions of Definition 6.2.3.

Proof. Suppose γ is a total order. Then by Corollary 6.3.18 and Proposition 3.5.7 the group $T_r^\gamma(A, F_1A) = 1 + t_r^\gamma(A, F_1A)$ and the Lie ring $t_r^\gamma(A, F_1A)$ are associated via some isomorphism

$$\lambda^\gamma : \widehat{D}_0^F \mathbb{Z}[T_r^\gamma(A, F_1A)] \xrightarrow{\sim} \widehat{D}_0^F U_{\mathbb{Z}}(t_r^\gamma(A, F_1A)),$$

which is natural in the underlying associative non-unital algebra $t_r^\gamma(A, F_1A)$. So by Proposition 3.5.4 λ^γ induces natural quasi-isomorphisms as in the lower row in the diagram below.

$$\begin{tikzcd}
\mathbb{Z}BT_r^\gamma(A,F_1A) \arrow[d, "\wr"'] & & \Lambda_* t_r^\gamma(A,F_1A) \arrow[d, "\simeq"'] \\
C_*(T_r^\gamma(A,F_1A),\mathbb{Z}) \arrow[d] & & C_*(t_r^\gamma(A,F_1A)) \arrow[d] \\
\widehat{C}_*(T_r^\gamma(A,F_1A),\mathbb{Z}) \arrow[r, "\simeq"] & \widehat{C}_*(T_r^\gamma(A,F_1A),t_r^\gamma(A,F_1A),\mathbb{Z}) & \widehat{C}_*(t_r^\gamma(A,F_1A)). \arrow[l, "\simeq"']
\end{tikzcd}$$

Now $t_r^\gamma(A,F_1A)$ is flat, because $\mathrm{gr}^F A$ and hence also A and F_1A are flat by Corollary 3.6.2 using $F_nA=0$. It follows that the upper right vertical map is a quasi-isomorphism. Moreover also $\mathrm{gr}^F t_r^\gamma(A,F_1A) = t_r^\gamma(\mathrm{gr}^F A, \mathrm{gr}^F(F_1A))$ is flat. Assuming that $\mathrm{gr}^F t_r^\gamma(A,I)$ is a finitely generated free abelian group for the moment, also the lower two vertical maps are quasi-isomorphisms by Proposition 3.4.17 and Proposition 3.3.24 respectively.

Note that by Corollary 6.3.18 the topology on $t_r^\gamma(A,F_1A)$ and hence on $\mathbb{Z}[T_r^\gamma(A,F_1A)]$ and $U_{\mathbb{Z}}(t_r^\gamma(A,F_1A))$ is functorial in γ in contrast to the chosen filtration F. Using this and that λ is natural in the algebra $t_r^\gamma(A,F_1A)$, all the maps in the diagram above are natural in γ and we can take the homotopy colimit over all γ we obtain a natural zig-zag

$$\begin{tikzcd}
\mathbb{Z}X_r(A,F_1A) \arrow[d, equal] & & \Gamma x_r(A,F_1A) \arrow[d, equal] \\
\operatorname*{colim}_\gamma \mathbb{Z}BT_r^\gamma(A,F_1A) & & \operatorname*{colim}_\gamma \Gamma\Lambda_* t_r^\gamma(A,F_1A) \\
\operatorname*{hocolim}_\gamma \mathbb{Z}BT_r^\gamma(A,F_1A) \arrow[u, "\simeq"'] \arrow[dr, "\simeq"] & & \operatorname*{hocolim}_\gamma \Gamma\Lambda_* t_r^\gamma(A,F_1A) \arrow[u, "\simeq"'] \arrow[dl, "\simeq"'] \\
& \operatorname*{hocolim}_\gamma \widehat{C}_*(T_r^\gamma(A,F_1A),t_r^\gamma(A,F_1A),\mathbb{Z}), &
\end{tikzcd}$$

where the upper vertical maps are the canonical quotient maps, which are quasi-isomorphisms by Proposition 6.2.7 and (the proof of) Proposition 6.2.8 respectively. This proves the assertion in the unstable situation, if $\mathrm{gr}^F A$ is a finitely generated and free abelian group. By checking that the inclusions $t_r^\gamma(A,F_1A) \longrightarrow t_{r+1}^\gamma(A,F_1)$ are continuous, one can take the colimit over $r \geq 1$ to get a quasi-isomorphism $\mathbb{Z}X(A,F_1A) \simeq x(A,F_1A)$ in this situation.

For the general case, let $S \subset A$ be a finite subset and let $t_r^{\gamma,S}(A,F_1A) \leq t_r^\gamma(A,F_1A)$ be the subring generated by $S^\gamma := S^{r\times r} \cap t_r^\gamma(A,F_1A)$. Now $F_nA=0$ implies

$$F_{(n+1)r} t_r(A,F_1A) = \sum_{1\leq i,j\leq r} F_{\lceil((n+1)r+i-j)/r\rceil} A \cdot e_{i,j} \subset \sum_{1\leq i,j\leq r} F_{n+1}A \cdot e_{i,j} = 0,$$

and by construction $F_1 t_r^\gamma(A,F_1A) = t_r^\gamma(A,F_1A)$. So as F is a ring filtration on $t_r^\gamma(A,F_1A)$, the abelian group underlying $t_r^{\gamma,S}(A,F_1A)$ is generated by the finite set of S^γ-monomials of total degree less than $(n+1)r$. In particular $\mathrm{gr}^F t_r^{\gamma,S}(A,F_1A)$ is finitely generated as an abelian group. Giving $t_r^{\gamma,S}(A,F_1A) \leq t_r^\gamma(A,F_1A)$ the subring filtration F, we see that

$\mathrm{gr}^F t_r^\gamma(A, F_1A)$ is a finitely generated abelian subgroup of the torsion-free $\mathrm{gr}^F t_r^{\gamma,S}(A, F_1A)$ and hence free. Now as A is the union of its finitely generated subrings, every of the maps in the diagram above is natural in A and S and filtered colimits are exact, we get a zig-zag of quasi-isomorphisms

$$\mathbb{Z}X_r(A, F_1A) = \operatorname*{colim}_{S\subset A \text{ finite},\gamma} \mathbb{Z}BT_r^{\gamma,S}(A, F_1A) \simeq \operatorname*{colim}_{S\subset A \text{ finite},\gamma} \Gamma\Lambda_* t_r^{\gamma,S}(A, F_1A) = \Gamma x_r(A, F_1A),$$

and similarly in the stable situation.

□

Definition 6.3.20
Let $A \in \mathcal{R}ing$.

(i) *The* ***(absolute) negative Chern character*** *is defined as the composition of*

$$K_*(A) = \pi_* GL(A)^+ \longrightarrow H_*(GL(A)^+, \mathbb{Z}) \xleftarrow{\sim} H_*(GL(A), \mathbb{Z})$$

with the negative Chern character for the Hopf algebra $\mathbb{Z}[GL(A)]$ of Definition 5.2.11, the fusion map $\mathbb{Z}[GL(A)] \xrightarrow{\varepsilon} M_\infty(A)$ and the trace map of negative cyclic homology (see Remark 5.4.17)

$$H_*(GL(A), \mathbb{Z}) = H_*(B_*\mathbb{Z}[\mathrm{GL}(A)]) \xrightarrow{\mathrm{ch}^-} HC_*^-(\mathbb{Z}[GL(A)]) \xrightarrow{\varepsilon} HC_*^-(M_\infty A) \xrightarrow{\mathrm{trace}} HC_*^-(A).$$

(ii) *Similarly, for $I \lhd A$, the* ***relative negative Chern character*** *is defined as the composition of*

$$K_*(A, I) = \pi_* X(A, I)^+ \longrightarrow H_*(X(A, I)^+, \mathbb{Z}) \xleftarrow{\sim} H_*(X(A, I), \mathbb{Z})$$

with the map induced by the sum of the compositions of the negative Chern character for the Hopf algebras $\mathbb{Z}[T_r^\gamma(A, I)]$ of Definition 5.2.11, the fusion map $\mathbb{Z}[T_r^\gamma(A, I)] \xrightarrow{\varepsilon} t_r^\gamma(A, I)$ and the trace map

$$\tilde{B}_*\mathbb{Z}[\mathcal{T}_r^\gamma(A, I)] \xrightarrow{\mathrm{ch}^-} \mathrm{Tot}^\times M^-(\mathbb{Z}[T_r^\gamma(A, I)]) \xrightarrow{\varepsilon} \mathrm{Tot}^\times M^-(t_r^\gamma A) \xrightarrow{\mathrm{trace}} \mathrm{Tot}^\times M^-(A, I).$$

See also Definition 6.3.14

By abuse of notation we will denote the two maps also by ch^-.

Lemma 6.3.21
In the situation of Theorem 6.3.19 the following holds.

(i) *After taking homotopy groups (which is homology under the Dold-Kan correspondence) the isomorphism induced by the zig-zag of Theorem 6.3.19 (ii) fits into commutative squares*

$$\begin{tikzcd}
H_*(X(A, F_1A), \mathbb{Z}) \arrow[r, equal, "\sim"] \arrow[d, "d^i"'] & H_*(x(A, I)) \arrow[d, "d^i"] \\
H_*(X(A, F_1A) \times X(A, F_1A), \mathbb{Z}) \arrow[r, equal, "\sim"] & H_*(x(A, I) \otimes x(A, I)),
\end{tikzcd}$$

where the maps d^i are given as follows

a) *On the left* d^0 *is induced by the map* $\eta \times \mathrm{id}$, *where* η *is the inclusion of the base point, and on the right by* $\mathrm{id} \otimes \eta$, *where* $k = x(A, F_1A)_0 \overset{\eta}{\hookrightarrow} x(A, F_1A)$.

b) *On the left* d^1 *is induced by the diagonal and on the right by the comultiplication on the* $x(A, I)$.

c) *On the left* d^2 *is induced by* $\mathrm{id} \times \eta$ *and on the right by* $\mathrm{id} \otimes \eta$, *where the maps* η *are like in a).*

(ii) *The isomorphism induced by the zig-zag of Theorem 6.3.19 (ii) is compatible with the relative negative Chern character, i.e. there is a commutative diagram*

$$\begin{tikzcd} H_*(X(A, F_1A), \mathbb{Z}) \arrow[rr, equal, "\sim"] \arrow[d, "\mathrm{ch}^-"'] & & H_*(x(A, I)) \arrow[d, "\mathrm{ch}^-"] \\ HC_*^-(A, F_1) \arrow[r, "i_A"] & HC_*^-(D_0^F A, D_1^F A) & HC_*^-(A, F_1A), \arrow[l, "i_A"'] \end{tikzcd}$$

where the vertical maps are given by the relative negative Chern character (see Definition 6.3.20 and Definition 6.3.14) and the two horizontal maps are induced by the canonical map into the divisible closure $A \xrightarrow{i_A} D_0^F A$ *(cf. Definition 3.1.10).*

Proof.

(i) Let γ be a total order on $\mathbf{r}$. Then the isomorphism

$$\lambda^\gamma : \widehat{D}_0^F \mathbb{Z}[T_r^\gamma(A, F_1A)] \xrightarrow{\sim} \widehat{D}_0^F U_{\mathbb{Z}}(t_r^\gamma(A, F_1A))$$

is natural in the underlying associative non-unital algebra $t_r^\gamma(A, F_1A)$. So if

$$0 \xrightarrow{\eta} t_r^\gamma(A, F_1A), \qquad t_r^\gamma(A, F_1A) \xrightarrow{\delta} t_r^\gamma(A, F_1A) \times t_r^\gamma(A, F_1A)$$

are the initial and diagonal algebra homomorphisms, the maps $d^0 = \eta \times \mathrm{id}, d^1 = \delta$ and $d^2 = \mathrm{id} \times \eta$ induce commutative diagrams

$$\begin{tikzcd} \widehat{D}_0^F \mathbb{Z}[T_r^\gamma(A, F_1A)] \arrow[r, "\sim"] \arrow[d, "d^i"'] & \widehat{D}_0^F U_{\mathbb{Z}}(t_r^\gamma(A, F_1A)) \arrow[d, "d^i"] \\ \widehat{D}_0^F \mathbb{Z}[T_r^\gamma(A, F_1A) \times T_r^\gamma(A, F_1A)] \arrow[r, "\sim"] & \widehat{D}_0^F U_{\mathbb{Z}}(t_r^\gamma(A, F_1A) \times t_r^\gamma(A, F_1A)), \end{tikzcd}$$

which are moreover natural in γ. So by taking the (homotopy) colimit over γ as in the proof of Theorem 6.3.19, we see that all the maps in the zig-zag are compatible with the d^i. As the d^i are natural in the underlying associative algebra $t_r^\gamma(A, F_1A)$, the same holds when restricted to the subalgebras $t_r^{\gamma,S}(A, F_1A)$. It remains to note that the maps d^i defined here in the proof induce the desired maps on $X(A, F_1A)$ and $x(A, F_1A)$.

(ii) There is a natural way to construct the negative Chern character for every object in the zig-zag except possibly for the middle one (cf. Proposition 3.5.4)

$$\begin{aligned}
&\widehat{C}_*(1+T,T,\mathbb{Z})\\
&= \operatorname{coker}\left(\widehat{B}_*(I(1+T),D_0^F\mathbb{Z}[1+T],\mathbb{Z}) \longrightarrow \widehat{B}_*(D_0^F\mathbb{Z}[1+T],D_0^F\mathbb{Z}[1+T],\mathbb{Z})\right)\\
&\overset{\lambda}{\cong} \operatorname{coker}\left(\widehat{B}_*(I(T),D_0^F U_{\mathbb{Z}}(T),\mathbb{Z}) \longrightarrow \widehat{B}_*(D_0^F U_{\mathbb{Z}}(T),D_0^F U_{\mathbb{Z}}(T),\mathbb{Z})\right),
\end{aligned}$$

where $T = t_r^\gamma(A, F_1A) \in \mathbb{Z}\text{-}\mathcal{A}ss$. Since the ideals $I(1+T)$ and $I(T)$ are contained in the particular augmentation ideals, we have a quotient map

$$\widehat{C}_*(1+T,T,\mathbb{Z}) \twoheadrightarrow \widehat{B}_*(D_0^F\mathbb{Z}[1+T]) \overset{\lambda}{\cong} \widehat{B}_*(D_0^F U_{\mathbb{Z}}(T)),$$

and completion of the composition with the negative Chern character map yields the desired map

$$\widehat{C}_*(1+T,T,\mathbb{Z}) \longrightarrow \operatorname{Tot}^\times \widehat{M}^-(T) \xrightarrow{\text{trace}} \operatorname{Tot}^\times \widehat{M}^-(D_0^F A, D_1^F A).$$

Since $A_n = 0$, for some $n \geq 0$, the object $M^-(D_0^F A, D_1^F A)$ is complete, if A is finitely generated over k. As we are taking the filtered colimit over finitely generated subalgebras of A, we therefore land in $M^-(D_0^F A, D_1^F A)$ as desired.

□

Theorem 6.3.22
Let $A \in \mathcal{R}ing$ with $(p-1)! \in A^\times$, for some prime $p > 1$. Suppose A carries a finite ring filtration $A = F_0A \supset \ldots \supset F_N A = 0$, such that $\operatorname{gr}^F A$ is flat over $\mathbb{Z}$. Suppose that there is a subset $Y \subset A$, such that

(i) $\operatorname{gr}^F A = \sum_{y \in Y} \mathbb{Z} \cdot [y]$,

(ii) $y^n/n! \in F_1A$, *for all* $y \in Y \cap F_1A$ *and* $n \geq 1$.

Then for $1 \leq n < p-1$ there are isomorphisms inducing a commutative diagram

$$\begin{tikzcd}
K_n(A,F_1A) \arrow[rr, equal, "\sim"] \arrow[d, "\mathrm{ch}^-"'] & & HC_{n-1}(A,F_1A) \arrow[d, "B"] \\
HC_n^-(A,F_1A) \arrow[r, "i_A"] & HC_n^-(D_0^F A, D_1^F A) & HC_n^-(A,F_1A), \arrow[l, "i_A"']
\end{tikzcd}$$

where ch^- is the relative negative Chern character of Definition 6.3.20, B is Connes' operator (see Remark 5.1.10), and the two horizontal maps are induced by the canonical inclusion $A \overset{i_A}{\hookrightarrow} D_0^F A$.

Proof. Consider the diagram

$$\begin{tikzcd}
PH_n(X(A,F_1A),\mathbb{Z}) \arrow[r, hook] \arrow[d, equal, "\wr"] & H_n(X(A,F_1A),\mathbb{Z}) \arrow[d, equal, "\wr"] \arrow[dr, "\mathrm{ch}^-"] & \\
PH_n(x(A,F_1A)) \arrow[r, hook] & H_n(x(A,F_1A)) \arrow[r, "\mathrm{ch}^-"'] & HC_n^-(D_0^F A, D_1^F A) \\
HC_{n-1}(A,F_1A) \arrow[u, "\phi", "\wr"'] \arrow[rr, "B"] & & HC_n^-(A,F_1A). \arrow[u, "i_A"']
\end{tikzcd}$$

Then the upper right triangle commutes by Lemma 6.3.21 (ii), while (i) loc. cit. implies that the upper middle vertical isomorphism restricts to the isomorphism on the left. Moreover the lower right square commutes by Theorem 6.3.15, while the left one commutes by Remark 6.3.12. Let $k = \mathbb{Z}[1/(p-1)!]$ and consider the commutative diagram

$$\begin{tikzcd}
H_*(x(A,F_1A)) \arrow[r, "\sim"] \arrow[d, equal, "\wr"'] & H_*(x(A,F_1A)) \otimes k \arrow[r, "\sim"] \arrow[d, equal, "\wr"'] & H_*(x(A,F_1A) \otimes k) \arrow[d, equal, "\wr"'] \\
H_*(X(A,F_1A),\mathbb{Z}) \arrow[r] \arrow[d, "\wr"'] & H_*(X(A,F_1A),\mathbb{Z}) \otimes k \arrow[r] \arrow[d, "\wr"'] & H_*(X(A,F_1A),k) \arrow[d, "\wr"'] \\
H_*(X(A,F_1A)^+,\mathbb{Z}) \arrow[r] & H_*(X(A,F_1A)^+,\mathbb{Z}) \otimes k \arrow[r] & H_*(X(A,F_1A)^+,k).
\end{tikzcd}$$

Since $(p-1)! \in A^\times$, the upper horizontal maps are isomorphisms, which implies that also the lower horizontal maps are isomorphisms. Since by Corollary 6.3.7 $X(A,F_1A)^+$ is a connected E_∞-space, Corollary 4.3.13 implies that

$$\pi_n X(A,F_1A)^+ \xrightarrow{\sim} PH_n(X(A,F_1A)^+,k) \xleftarrow{\sim} PH_n(X(A,F_1A),k) \xleftarrow{\sim} PH_n(X(A,F_1A),\mathbb{Z}),$$

and the result follows, because by Proposition 6.2.18 and Proposition 6.2.7 we have

$$K_*(A,F_1A) \cong \pi_{*-1} Y(A,F_1A)^+ \cong \pi_* X(A,F_1A)^+.$$

□

Corollary 6.3.23 (Goodwillie)
Let $I \lhd A \in \mathbb{Q}/\mathcal{R}ing$ be nilpotent.
Then $\mathrm{ch}^- : K_*(A,I) \xrightarrow{\sim} HC_*^-(A,I)$.

Proof. Since $\mathbb{Q} \leq A$, the I-adic filtration satisfies the hypotheses of Theorem 6.3.22. Since $(p-1)! \in A^\times$, for all $p > 1$, the result follows.

This was first proven by Goodwillie [Goo86], but it was not clear that the isomorphism is induced by the negative Chern character. This was later verified by Cortinas-Weibel in [CW09].

□

Theorem 6.3.24 (Brun)
Let $I \lhd A \in s\mathcal{R}ing$ with A and A/I flat over $\mathbb{Z}$, and $I^m = 0$, for some $m \geq 1$.

Then after p-completion in the sense of Bousfield-Kan [BK72], there is a $(p/(m-1)-2)$-connected map

$$K_*(A, I) \longrightarrow HC_*(A, I).$$

Proof. This was proven by Brun in [Bru01] using the cyclotomic trace map to topological cyclic homology. Note that he only requires A and A/I to be flat and does not make any restrictions on I^n/I^{n+1}, for $n > 0$. But our Theorem 6.3.22 is more general in some cases.

□

6.3.4 Applications of the Theorem

Proposition 6.3.25
Let $k \leq \mathbb{Q}$ with $(p-1)! \in k^\times$ and $1 \leq r \leq p$.

Then, for $0 \leq n < p-1$, we have

$$K_n(k[t]/(t^r), (t)) \cong HC_{n-1}(k[t]/(t^r), (t)) \cong \begin{cases} \bigoplus\limits_{\substack{0 \leq j \leq m, \\ 0 < a < r}} k/(a+jr), & n = 2m+1, \\ k^r, & n = 2m. \end{cases}$$

Proof. The k-algebra $k[t]/(t^r)$ with the (t)-adic-filtration satisfies the hypothesis of Theorem 6.3.22. Indeed $r \leq p$ implies $(t)^p = 0$ in $k[t]/(t^r)$, while $1/n! \cdot (t)^n \subset (t)$, for all $1 \leq n < p$, because $(p-1)! \in k^\times$ by assumption. The computation of the cyclic homology is due to [Gro94]. See also Proposition 5.3.4.

□

Remark 6.3.26 (i) *Brun [Bru01] used almost free simplicial replacements, Theorem 6.3.24 and Quillen's [Qui72] computation of $K_*(\mathbb{F}_p)$ to deduce*

$$K_n(\mathbb{Z}/p^r) = \begin{cases} 0, & n \in 2\mathbb{N}_0, \\ \mathbb{Z}/p^{m(r-1)}(p^m-1), & n = 2m-1. \end{cases}$$

We may use our Theorem to at least compute also K_{p-2}.

(ii) *Similarly one could maybe reproduce Hesselholt-Madsen's [HM97a] computation of*

$$K_*(\mathbb{F}_p[t]/(t^r), (t)) = \begin{cases} \mathbf{W}_{mr}(\mathbb{F}_p)/V_r\mathbf{W}_m(\mathbb{F}_p), & n = 2m-1, \\ 0, & n \in 2\mathbb{N}_0, \end{cases} \qquad r \geq 1,$$

here $\mathbf{W}_m(\mathbb{F}_p)$ are the big Witt vectors of length $m \geq 1$ and V_r is the Verschiebung.

7 Appendix: Simplicial homotopy theory

7.1 Model categories

7.1.1 Localizations of categories

Definition 7.1.1
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $S \subset \mathrm{Mor}(\mathcal{C})$ a subclass of morphisms.
*A category $\mathcal{C} \xrightarrow{\gamma} S^{-1}\mathcal{C}$ under $\mathcal{C}$ is called a **localization** of $\mathcal{C}$ at S, if*

(i) $\gamma(S) \subset \mathrm{Mor}(S^{-1}\mathcal{C})^{\times}$, *i.e. γ sends S to isomorphisms.*

(ii) *It is universal w.r.t. to functors satisfying (i), i.e.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \\ S^{-1}\mathcal{C}, \arrow[ur, dashed, "\exists!\tilde{F}"'] & \end{tikzcd} \qquad \forall F: \ F(S) \subset \mathrm{Mor}(\mathcal{D})^{\times}.$$

Sometimes this diagram is just required commutative up to unique natural isomorphism.

Example 7.1.2
Let $M \in \mathcal{M}on$ be a monoid, $S \subset M$.

- *Let $I(M)$ denote the category with one object, whose morphisms are the elements of M.*
- *Let $S^{-1}M$ be the monoid obtained by formally adjoining inverses for all elements in S.*

Then the canonical morphism $M \longrightarrow S^{-1}M$ induces a functor $I(M) \longrightarrow I(S^{-1}M)$, which is a localization for M.

Remark 7.1.3
If $\mathcal{C}$ is not small, then localizations are very hard to construct and do not exist in general.

Definition 7.1.4
*A **category with weak equivalences** consists of a category $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and a subclass of morphisms $w\mathcal{C} \subset \mathrm{Mor}(\mathcal{C})$, so-called **weak equivalences** (written "$\xrightarrow{\simeq}$"), such that:*

(i) $\mathrm{Mor}(\mathcal{C})^{\times} \subset w\mathcal{C}$.

(ii) *The **2-of-3 axiom** holds, i.e. for all $A, B, C \in \mathcal{C}$ and every commutative triangle*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "gf"'] & B \arrow[d, "g"] \\ & C, \end{tikzcd}$$

if 2 of the 3 morphisms are wes., so is the third.

*Its **derived category** or **homotopy category** is denoted by $D(\mathcal{C}) := \mathrm{Ho}(\mathcal{C}) := w\mathcal{C}^{-1}\mathcal{C}$, provided that it exists.*

7.1.2 Lifting properties

Definition 7.1.5
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $\ell \in \mathcal{C}(A, B), r \in \mathcal{C}(C, D)$.
If for any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\ell"'] & C \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dashed, "\exists d"] & D \end{tikzcd}$$

there exists a (not-necessarily unique) diagonal d making the diagram commutative, then one says, that

(i) *ℓ has the **left lifting property (LLP)** with respect to r and*

(ii) *r has the **right lifting property (RLP)** with respect to ℓ.*

For a subclass $S \subset \mathrm{Mor}(\mathcal{C})$ define

(i) $LLP(S) := \{f \in \mathrm{Mor}(\mathcal{C});\ f \text{ has the LLP w.r.t. all } s \in S\}$,

(ii) $RLP(S) := \{f \in \mathrm{Mor}(\mathcal{C});\ f \text{ has the RLP w.r.t. all } s \in S\}$.

Definition 7.1.6
*A **weak factorization system** on a category $\mathcal{C}$ consists of two subclasses $L, R \subset \mathrm{Mor}(\mathcal{C})$, such that*

(i) $\mathrm{Mor}(\mathcal{C}) = R \circ L$,

(ii) $L = LLP(R)$ *and* $R = RLP(L)$.

*We also say that $\mathcal{C}$ is (L, R)-**structured**.*

Proposition 7.1.7
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $L, R \subset \mathrm{Mor}(\mathcal{C})$.

Then (L, R) is a weak factorization system, if and only if the following holds.

(i) $\mathrm{Mor}(\mathcal{C}) = R \circ L$,

(ii) $L \subset LLP(R)$.

(iii) *L and R are closed under retracts, where a **retract** of a morphism $f \in \mathcal{C}(A, B)$ is a morphism $f' \in \mathcal{C}(A', B')$ inducing a commutative diagam*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"'] \arrow[rr, bend left, "\mathrm{id}_{A'}"] & A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & B \arrow[r] & B'. \end{tikzcd}$$

Proof. Suppose (L, R) is a weak factorization system. Then (i) and (ii) hold, because $L = LLP(R)$. For proving (ii), consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "s_A"'] \arrow[d, "\ell"'] \arrow[rr, bend left, "\mathrm{id}_A"] & A' \arrow[r, "r_A"'] \arrow[d, "\ell'"] & A \arrow[r, "u"] \arrow[d, "\ell"] & X \arrow[d, "r \in R"] \\ B \arrow[r, "s_B"] \arrow[rr, bend right, "\mathrm{id}_B"'] & B' \arrow[r, "r_B"] & B \arrow[r, "v"] & Y, \end{tikzcd}$$

where $\ell' \in LLP(R)$. We want to show, that there is a diagonal $d \in \mathcal{C}(B, X)$, making the right square commute. Since $\ell' \in LLP(R)$, we have a diagonal d', such that

$$\begin{tikzcd} A' \arrow[r, "r_A"] \arrow[d, "\ell'"'] & A \arrow[r, "u"] & X \arrow[d, "r \in R"] \\ B' \arrow[r, "r_B"'] \arrow[urr, dashed, "\exists d'"] & B \arrow[r, "v"'] & Y \end{tikzcd}$$

commutes. Define $d = d' s_B$ and compute

$$d\ell = d' s_B \ell = d' \ell' s_A = u r_A s_A = u, \qquad rd = r d' s_B = v r_B s_B = v,$$

hence d is a diagonal of the desired form. By duality also R is closed under retracts.

Vice versa assume the hypotheses for (L, R) given above. We have to show, that $L = LLP(R)$ and $R = RLP(L)$. Let $c \in \mathcal{C}(X, Y) \cap \in LLP(R)$ be arbitrary. Then c can be factored as $X \overset{\ell}{\rightarrowtail} Z \overset{r}{\twoheadrightarrow} Y$, where $\ell \in L$ and $r \in R$. Since $\ell \in LLP(R)$ we find a diagonal

$$\begin{tikzcd} X \arrow[r, tail, "\ell"] \arrow[d, "c"'] & Z \arrow[d, two heads, "r"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "d"] & Y, \end{tikzcd}$$

showing that c is a retract of ℓ. Hence $LLP(R) \subset L \subset LLP(R)$ and so $L = LLP(R)$. Dually one shows $R = RLP(L)$.

□

Remark 7.1.8
Given two categories $\mathcal{D}$ and $\mathcal{C}$ with an adjunction

$$\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y)).$$

Let $\ell \in \mathcal{C}(C, C')$ and $r \in \mathcal{D}(D, D')$.
Then the following lifting propblems are equivalent

$$\begin{tikzcd} F(C) \arrow[r, "a"] \arrow[d, "F(\ell)"'] & D \arrow[d, "r"] \\ F(C') \arrow[r, "b"'] \arrow[ur, dashed, "d"] & D', \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "a'"] \arrow[d, "\ell"'] & G(D) \arrow[d, "G(r)"] \\ C' \arrow[r, "b'"'] \arrow[ur, dashed, "d'"] & G(D'), \end{tikzcd}$$

where a', b' and d' corresponds to a, b and d under the adjunction bijection.
In particular the two axioms for structured adjunctions are equivalent.

Corollary 7.1.9
Given an adjunction $\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y))$, the following holds.

(i) *For every subclass $C \subset \mathrm{Mor}(\mathcal{C})$, we have $G^{-1}RLP(C) = RLP(F(C))$.*

(ii) *For every subclass $D \subset \mathrm{Mor}(\mathcal{D})$, we have $F^{-1}LLP(C) = LLP(G(D))$.*

Remark 7.1.10
Given categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ and adjunctions

$$\mathcal{A}(A, F(B, C)) = \mathcal{C}(A \otimes B, C) = \mathcal{B}(B, G(A, C)), \qquad A \in \mathcal{A},\ B \in \mathcal{B},\ C \in \mathcal{C}.$$

Let $a \in \mathcal{A}(A, A'), b \in \mathcal{B}(B, B')$ and $c \in \mathcal{C}(C, C')$.
Then the following three lifting problems are equivalent.

$$\begin{tikzcd} A \arrow[r, "u_1'"] \arrow[d, "a"'] & F(B', C) \arrow[d, "(b^*, c_*)"] \\ A' \arrow[r, "(u_2', u_3')"'] \arrow[ur, dashed, "d'"] & F(B, C) \times_{F(B, C')} F(B', C'), \end{tikzcd} \qquad \begin{tikzcd} (A \otimes B') +_{(A \otimes B)} (A' \otimes B) \arrow[r, "u_1 \cup u_2"] \arrow[d, "(a \otimes \mathrm{id}) \cup (\mathrm{id} \otimes b)"'] & C \arrow[d, "c"] \\ A' \otimes B' \arrow[r, "u_3"'] \arrow[ur, dashed, "d"] & C'. \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "u_2''"] \arrow[d, "b"'] & G(A', C) \arrow[d, "(a^*, c_*)"] \\ B' \arrow[r, "(u_1'', u_3'')"'] \arrow[ur, dashed, "d''"] & G(A, C) \times_{G(A, C')} G(A', C'). \end{tikzcd}$$

where

- u_1', u_2', u_3' *correspond to* u_1, u_2, u_3 *under the first bijection and*
- u_1'', u_2'', u_3'' *correspond to* u_1, u_2, u_3 *under the second bijection.*

In particular the three axioms for structured adjoint triples are equivalent.

Proposition 7.1.11
Given an adjunction

$$\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y)).$$

Then there is a canonical weak factorization system (L, R) *on* $\mathcal{D}$*, where*

(i) L *is the class of retracts of inclusions* $X \xhookrightarrow{\iota_X} X + F(Y)$*, where* $X \in \mathcal{D}$ *and* $Y \in \mathcal{C}$*,*

(ii) R *is the class of morphisms* r*, such that* $G(r)$ *is a retraction in* $\mathcal{C}$*.*

Proof. Recall that the adjunction bijection is given by

$$\begin{aligned} \mathcal{D}(F(X), Y) &\overset{\sim}{\rightleftarrows} \mathcal{C}(X, G(Y)), \\ f &\longmapsto G(f) \circ \eta_X, \\ \varepsilon_Y \circ F(g) &\longleftarrow g, \end{aligned}$$

where the unit $\mathrm{id}_{\mathcal{C}} \xrightarrow{\eta} GF$ and counit $FG \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{D}}$ are natural transformations. Every morphism $X \xrightarrow{f} Y$ can be factored as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota_X"'] & & Y \\ & X + FG(Y), \arrow[ur, "f \cup \varepsilon_Y"'] & \end{tikzcd}$$

and $G(f \cup \varepsilon_Y)$ is a retraction having a section

$$G(Y) \xrightarrow{\eta_{G(Y)}} GFG(Y) \xrightarrow{G(\iota_{FG(Y)})} G(X + FG(Y)).$$

Consider a general lifting problem

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "\iota_X"'] & A \arrow[d, "r"] \\ X + F(Y) \arrow[r, "(ra) \cup b"'] \arrow[ur, dashed, "d"] & B, \end{tikzcd} \qquad \begin{tikzcd} G(A) \arrow[d, "G(r)"'] \\ G(B). \arrow[u, bend right, dashed, "s"'] \end{tikzcd}$$

Let d' be the composition

$$F(Y) \xrightarrow{F(\eta_Y)} FGF(Y) \xrightarrow{FG(b)} FG(B) \xrightarrow{F(s)} FG(A) \xrightarrow{\varepsilon_A} A.$$

Then

$$\begin{aligned} rd' &= r \circ \varepsilon_A \circ F(s) \circ FG(b) \circ F(\eta_Y) = \varepsilon_B \circ FG(r) \circ F(s) \circ FG(b) \circ F(\eta_Y) \\ &= \varepsilon_B \circ \underbrace{F(G(r) \circ s)}_{=\mathrm{id}} \circ FG(b) \circ F(\eta_Y) = \varepsilon_B \circ FG(b) \circ F(\eta_Y) = b \circ \varepsilon_{F(A)} \circ F(\eta_Y) = b, \end{aligned}$$

and hence $d = a \cup d'$ is a solution to the lifting problem, because

$$d\iota_X = a, \qquad rd = (ra) \cup (rd') = (ra) \cup b.$$

Next we show, that R is closed under retracts. Suppose we are given a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "s_A"'] \arrow[d, "r'"'] \arrow[rr, bend left, "\mathrm{id}_{A'}"] & A \arrow[r, "r_A"'] \arrow[d, "r"] & A' \arrow[d, "r'"] \\ B' \arrow[r, "s_B"] \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & B \arrow[r, "r_B"] & B', \end{tikzcd} \qquad \begin{tikzcd} G(A) \arrow[d, "G(r)"'] \\ G(B). \arrow[u, bend right, "s"'] \end{tikzcd}$$

Define $s' = G(r_A)sG(s_B) \in \mathcal{C}(B', A')$. Then we have

$$r's' = r'G(r_A)sG(s_B) = G(r_B)rsG(s_B) = G(r_B)G(s_B) = \mathrm{id}_{G(B')},$$

showing that $G(r')$ is a retraction with section s'.

Now we can apply Proposition 7.1.7 to show, that (L, R) is a weak factorization system for $\mathcal{D}$.

□

Example 7.1.12
Let $R \in \mathcal{R}ing$ and consider the adjunction

$$R\text{-Mod}(RX, Y) = \mathcal{S}et(X, U(Y)).$$

Then for the induced weak factorization system (L, R) of Proposition 7.1.11 we have

(i) *L the class of monomorphisms with projective cokernel.*

(ii) *R is the class of epimorphisms.*

7.1.3 The small object argument

Definition 7.1.13 (i) *A **cardinal number** is an ordinal having greater cardinality than all its smaller ordinals.*

(ii) *We define ω as the smallest infinite cardinal.*

(iii) *A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ **preserves κ-indexed colimits**, for some cardinal number κ, if*

$$\operatorname*{colim}_{n<\kappa} F(Y_n) \xrightarrow{\sim} F(\operatorname*{colim}_{n<\kappa} Y_n),$$

for all sequences $(Y_0 \longrightarrow Y_1 \longrightarrow \ldots)$ indexed by ordinals $n < \kappa$ in $\mathcal{C}$.

(iv) *An object $X \in \mathcal{C}$ is called **κ-small**, if $\mathcal{C}(X, \text{-})$ preserves κ-indexed colimits.*

Proposition 7.1.14 (Small object argument)
Let $\mathcal{C}$ be a cocomplete category and $S \subset \mathrm{Mor}(\mathcal{C})$ a set of morphisms with κ-small domains, for some cardinal number κ.
Then $(LLP(RLP(S)), RLP(S))$ is a weak factorization system on $\mathcal{C}$.

Proof. We have to prove that $\mathrm{Mor}(\mathcal{C}) = RLP(S) \circ LLP(RLP(S))$. By [Hir03] Prop. 10.5.16 every morphism in $\mathcal{C}$ is the composition of a relative S-cell complex followed by a map in $RLP(S)$ (called the class of S-injectives there). We do not need to know what a relative S-cell complex is, it suffices to know that it is in $LLP(RLP(S))$ (called the class of S-cofibrations there) by Proposition 10.5.10 loc. cit.
□

Corollary 7.1.15
Let $\mathcal{C}$ be a category with a factorization system (L, R). Given an adjunction

$$\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y)).$$

Suppose there is an ordinal κ, such that:

(i) *There is a subset $C \subset \mathrm{Mor}(\mathcal{C})$ of morphisms with κ-small domains and $(L, R) = (LLP(RLP(C)), RLP(C))$.*

(ii) *F preserves κ-small objects.*

Then $(LLP(G^{-1}(R)), G^{-1}(R))$ is a factorization system on $\mathcal{D}$.

Proof. For all $(D \xrightarrow{c} E) \in C$ the functor $\mathcal{D}(F(D), \text{-})$ preserves κ-inductive limits by (ii), since C is κ-small. By Proposition 7.1.14 we obtain a weak factorization system $(LLP(RLP(F(C))), RLP(F(C)))$ on $\mathcal{D}$. As $RLP(F(C)) = G^{-1}RLP(C)$ by Corollary 7.1.9, this is the factorization system we wanted.
□

7.1.4 Model categories

Definition 7.1.16
*A **model category** consists of a finitely complete and cocomplete category $\mathcal{C}$ together with a **model structure**. That is three classes of morphisms $w\mathcal{C}, \mathrm{fib}\,\mathcal{C}, \mathrm{cof}\,\mathcal{C} \subset \mathrm{Mor}(\mathcal{C})$, such that:*

(i) *$(\mathcal{C}, w\mathcal{C})$ is a category with weak equivalences.*

(ii) *$(\mathrm{cof}\,\mathcal{C} \cap w\mathcal{C}, \mathrm{fib}\,\mathcal{C})$ and $(\mathrm{cof}\,\mathcal{C}, w\mathcal{C} \cap \mathrm{fib}\,\mathcal{C})$ are weak factorization systems.*

We fix the following notation.

- *The morphisms in $\mathrm{fib}\,\mathcal{C}$ are called **fibrations** and written as "$\twoheadrightarrow$".*
- *The morphisms in $\mathrm{cof}\,\mathcal{C}$ are called **cofibrations** and written as "$\rightarrowtail$".*
- *A (co-)fibration is called **trivial**, if it is also a weak equivalence.*

Proposition 7.1.17
Let $\mathcal{C}$ be a category together with three classes of morphisms $w\mathcal{C}, \operatorname{fib}\mathcal{C}, \operatorname{cof}\mathcal{C} \subset \operatorname{Mor}(\mathcal{C})$.
Then $\mathcal{C}$ is a model category, if and only if the following holds.

(M1) *$\mathcal{C}$ is closed under finite limits and colimits.*

(M2) *$w\mathcal{C}$ satisfies the 2-of-3 axiom.*

(M3) *The classes $w\mathcal{C}, \operatorname{fib}\mathcal{C}, \operatorname{cof}\mathcal{C}$ are closed under retracts.*

(M4) $\operatorname{cof}\mathcal{C} \subset LLP(\operatorname{fib}\mathcal{C} \cap w\mathcal{C})$ *and* $\operatorname{cof}\mathcal{C} \cap w\mathcal{C} \subset LLP(\operatorname{fib}\mathcal{C})$.

(M5) $\operatorname{Mor}\mathcal{C} = \operatorname{fib}\mathcal{C} \circ (\operatorname{cof}\mathcal{C} \cap w\mathcal{C}) = (\operatorname{fib}\mathcal{C} \cap w\mathcal{C}) \circ \operatorname{cof}\mathcal{C}$.

Proof. **(CM1)** is part of both descriptions. **(CM2)** is equivalent to $(\mathcal{C}, w\mathcal{C})$ being a category with weak equivalences. By Proposition 7.1.7 $(\operatorname{cof}\mathcal{C} \cap w\mathcal{C}, \operatorname{fib}\mathcal{C})$ and $(\operatorname{cof}\mathcal{C}, w\mathcal{C} \cap \operatorname{fib}\mathcal{C})$ are weak factorization systems, if and only if **(CM3)** - **(CM5)** hold. Indeed **(CM5)** and **(CM3)** imply that (trivial) (co-)fibrations are closed under retracts. Vice versa if trivial fibrations and cofibrations are closed under retracts, so are weak equivalences, since they can be factored into a trivial cofibration followed by a trivial cofibration.

□

Remark 7.1.18
Probably the four most important references for model categories are Quillen's original paper [Qui67] and the books of Goerss-Jardine [GJ09], Hirschhorn [Hir03] and Hovey [Hov99]. Unfortunately there are slight differences in their definition of a closed model category, that we will point out below.

(i) *Our definition is equivalent to Quillen's [Qui67] Definition I.5.1 of a closed model category by Remark I.5.1 loc. cit.*

(ii) *Our definition is equivalent to Goerss-Jardine's definition of a closed model category in [GJ09] II.1.1, which are precisely the axioms listed in Proposition 7.1.17.*

(iii) *Hovey [Hov99] Def. 1.1.3 and Hirschhorn [Hir03] Def. 7.1.3 require that $\mathcal{C}$ is even complete and cocomplete, i.e. has limits and colimits over all small index categories.*

7.1.5 The derived category of a model category

Definition 7.1.19
Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$.

(i) *An object X is called **fibrant**, if the unique map into the terminal object $X \longrightarrow *$ is a fibration.*

(ii) *A **fibrant replacement** of X is a weak equivalence $X \xrightarrow{\simeq} X_f$ with fibrant $X_f \in \mathcal{C}$.*
Every object has a fibrant replacement, given by a factorization of the terminal map

$$X \overset{\simeq}{\rightarrowtail} X_f \twoheadrightarrow *.$$

(iii) *A **cylinder object** for $X \in \mathcal{C}$ is a factorization of the fold map*

$$\nabla = \mathrm{id}_X \sqcup \mathrm{id}_X : X + X \overset{i_0 \sqcup i_1}{\rightarrowtail} I \cdot X \xrightarrow{\simeq} X.$$

Note that every object has a cylinder object.

(iv) *Two morphisms $f, g \in \mathcal{C}(X, Y)$ are called **(left) homotopic** via h, short $f \overset{h}{\simeq} g$, if there is a map $h \in \mathcal{C}(I \cdot X, Y)$ inducing a commutative diagram*

$$\begin{tikzcd} & X + X \arrow[dl, "\mathrm{id}_X \sqcup \mathrm{id}_X = \nabla"'] \arrow[d, tail] \arrow[dr, "f \sqcup g"] & \\ X & I \cdot X \arrow[l, "\simeq"'] \arrow[r, "h"] & Y. \end{tikzcd}$$

*Dually we also define **cofibrant** objects, **replacements** (denoted by a lower index "c"), **path objects** (denoted by $\Delta = (\mathrm{id}_X, \mathrm{id}_Y) : X \xrightarrow{\simeq} X^I \overset{(p_0,p_1)}{\twoheadrightarrow} X \times X$) and right homotopies.*

Proposition 7.1.20
Let $\mathcal{C}$ be a model category, $X \in \mathcal{C}$ cofibrant and $Y \in \mathcal{C}$ fibrant.
Then left-homotopy "$\simeq$" is an equivalence relation on $\mathcal{C}(X, Y)$ and is independent of the chosen cylinder object $I \cdot X$ for X.

Proof. See [Qui67] Lem. I.1.4 and Lem. I.1.5. Alternatively [Hir03] Prop. 7.4.5 and 7.4.7.

□

Theorem 7.1.21
Let $\mathcal{C}$ be a model category.
Then $\mathcal{C}$ has a derived category $D(\mathcal{C})$, whose objects are the same as $\mathcal{C}$ and

$$D(\mathcal{C})(X, Y) := \mathcal{C}(X_c, Y_f)/\simeq, \qquad \text{for all } X, Y \in \mathcal{C}.$$

Moreover $w\mathcal{C} = \gamma^{-1} D(\mathcal{C})^\times$, i.e. a $\mathcal{C}$-morphism is a weak equivalence, if and only if it maps to an isomorphism in the derived category.

Proof. [Qui67] Thm. I.1.1' and Cor. I.1.1. Alternatively [GJ09] II.1.

□

7.1.6 Derived functors

Definition 7.1.22
Let $\mathcal{C}$ be a category with weak equivalence having a derived category $\mathcal{C} \xrightarrow{\gamma} D(\mathcal{C})$. Suppose $\mathcal{C} \xrightarrow{G} \mathcal{D}$ is a functor into an arbitrary category $\mathcal{D}$.

(i) *The **left derived functor** LG is defined as the <u>right</u> Kan extension of G along $\mathcal{C} \xrightarrow{\gamma} D(\mathcal{C})$, it it exists.*

(ii) *The* ***right derived functor*** RG *is defined as the* <u>*left*</u> *Kan extension of* G *along* $\mathcal{C} \xrightarrow{\gamma} D(\mathcal{C})$*, if it exists.*

Lemma 7.1.23 (K. Brown)
In every model category $\mathcal{C}$ *the following holds.*

(i) *Every weak equivalence* $w : X \xrightarrow{\simeq} Y$ *between cofibrant* $X, Y \in \mathcal{C}$ *can be factored as*

$$\begin{tikzcd} X \arrow[rr, tail, "w"', "\simeq"] \arrow[dr, "\simeq"'] & & Y \arrow[dl, bend left, tail, "\simeq"] \\ & Z, \arrow[ur, -open triangle, "\simeq"'] & \end{tikzcd}$$

meaning that $Y \overset{\simeq}{\rightarrowtail} Z \xrightarrow{\simeq} Y$ *is the identity on* Y*.*

(ii) *Every weak equivalence* $w : X \xrightarrow{\simeq} Y$ *between fibrant* $X, Y \in \mathcal{C}$ *can be factored as*

$$\begin{tikzcd} X \arrow[rr, tail, "w"', "\simeq"] \arrow[dr, "\simeq"'] & & Y \\ & Z, \arrow[ur, -open triangle, "\simeq"'] \arrow[ul, bend left, -open triangle, "\simeq"] & \end{tikzcd}$$

meaning that $X \overset{\simeq}{\rightarrowtail} Z \xrightarrow{\simeq} X$ *is the identity on* X*.*

Proof. Since X and Y are cofibrant and cofibrations are stable under pushouts, we have $X \rightarrowtail X + Y \leftarrowtail Y$. Take a factorization $X + Y \rightarrowtail Z \xrightarrow{\simeq} Y$ and let $X \rightarrowtail X + Y \rightarrowtail Z$ be the composition. This is a trivial cofibration by the 2-of-3 axiom. Moreover $Y \rightarrowtail X + Y \rightarrowtail Z$ is a section for $Z \xrightarrow{\simeq} Y$ and so is also a trivial cofibration.
Statement (ii) is dual to (i).

□

Theorem 7.1.24
Let $\mathcal{C}$ *be a model category and* $\mathcal{C} \xrightarrow{F} \mathcal{D}$ *a functor into an arbitrary category* $\mathcal{D}$*.*

(i) *Suppose* F *maps trivial cofibrations between cofibrants to isomorphisms.*
Then LF *exists and is given by*

$$\eta_F : LF(X) = F(X_c) \longrightarrow F(X), \qquad \emptyset \rightarrowtail X_c \xrightarrow{\simeq} X.$$

(ii) *Suppose* F *maps trivial fibrations between fibrants to isomorphisms.*
Then RF *exists and is given by*

$$\varepsilon_F : F(X) \longrightarrow F(X_f) = RF(X), \qquad X \xrightarrow{\simeq} X_f \longrightarrow *.$$

Proof. Brown's Lemma 7.1.23 implies that F maps weak equivalences between cofibrants to isomorphisms. Hence we can apply [Qui67] Prop. I.4.1 to prove (i). Statement (ii) is dual.

□

Definition 7.1.25
Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between categories with weak equivalences. Suppose $\mathcal{C}$ and $\mathcal{D}$ have derived categories and F preserves weak equivalences.

(i) *The* ***total left derived functor*** *of F is defined as $\mathbb{L}F = L(\gamma F)$.*

(ii) *The* ***total right derived functor*** *of F is defined as $\mathbb{R}F = R(\gamma F)$.*

Theorem 7.1.26 (Quillen's adjoint functor theorem)
An adjunction between model categories $\mathcal{C}$ and $\mathcal{D}$

$$\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y)),$$

subject to the following (equivalent) hypotheses.

- *F preserves trivial cofibrations between cofibrants and cofibrations.*
- *G preserves trivial fibrations between fibrants and fibrations.*

Then the following holds.

(i) *The total derived functors of F and G exist and induce an adjunction*

$$D(\mathcal{D})(\mathbb{L}F(X), Y) = D(\mathcal{C})(X, \mathbb{R}G(Y)).$$

(ii) *$(\mathbb{L}F, \mathbb{R}G)$ form an equivalence of categories, if and only if*

a) $X \xrightarrow{\eta_X} GF(X) \longrightarrow G(F(X)_f)$ is a weak equivalence, for all cofibrant $X \in \mathcal{C}$,

b) $F(G(Y)_c) \longrightarrow FG(Y) \xrightarrow{\varepsilon_Y} Y$ is a weak equivalence, for all fibrant $Y \in \mathcal{D}$.

Proof. The existence of $\mathbb{L}F$ and $\mathbb{R}G$ follows from Theorem 7.1.24. Let $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then by construction of the derived category in Theorem 7.1.21, we have a chain of natural bijections

$$\begin{aligned} D(\mathcal{D})(\mathbb{L}F(X), \gamma Y) &= D(\mathcal{D})(\gamma F(X_c), \gamma Y) = \mathcal{D}(F(X_c)_c, Y_f)/\simeq \\ &= \mathcal{D}(F(X_c), Y_f)/\simeq = \mathcal{C}(X_c, G(Y_f))/\simeq \\ &= \mathcal{C}(X_c, C(G(Y_f)_f))/\simeq = D(\mathcal{C})(\gamma X, \gamma G(Y_f))/\simeq = D(\mathcal{C})(\gamma X, \mathbb{R}G(Y)), \end{aligned}$$

where the third and fifth equality hold by assumption on F and G. The middle bijection is induced by the adjunction and the arguments for the rest are dual to those given before.

By construction of the adjunction between the derived categories we see that its unit resp. counit is an isomorphism, if and only if the conditions a) and b) hold.

□

Definition 7.1.27
Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

(i) *A* ***Quillen adjunction*** *between* $\mathcal{C}$ *and* $\mathcal{D}$ *is an adjunction*

$$\mathcal{C}(F(X),Y)=\mathcal{D}(X,G(Y))$$

subject to the following (equivalent) hypotheses.

- *F preserves trivial cofibrations and cofibrations.*
- *G preserves trivial fibrations and fibrations.*

(ii) *A* ***Quillen equivalence*** *is a Quillen adjunction inducing an equivalence of the derived categories.*

7.1.7 Model categories induced by adjunctions

Definition 7.1.28
Let κ be a cardinal. A model category $\mathcal{C}$ is called ***κ-cofibrantly generated****, if there are subsets of $C \subset \operatorname{cof}\mathcal{C}$ and $T \subset \operatorname{cof}\mathcal{C} \cap w\mathcal{C}$, such that*

- $\operatorname{fib}\mathcal{C} \cap w\mathcal{C} = RLP(C), \qquad \operatorname{fib}\mathcal{C} = RLP(T),$
- *All domains X of morphisms in C and T are* ***κ-small*** *in $\mathcal{C}$.*

We call C (resp. T) the set of ***generating (trivial) cofibrations****.*

Theorem 7.1.29
Let $\mathcal{C}$ a κ-cofibrantly generated model category. Given an adjunction

$$\mathcal{D}(F(X),Y)=\mathcal{C}(X,G(Y)),$$

where D is an arbitrary cocomplete, finitely complete category. Suppose that:

(i) *F preserves κ-small objects.*

(ii) $LLP(G^{-1}(\operatorname{fib}\mathcal{C})) \subset G^{-1}(w\mathcal{C})$.

Then $\mathcal{D}$ is a κ-cofibrantly generated model category, where

$$w\mathcal{D}=G^{-1}(w\mathcal{C}), \qquad \operatorname{fib}\mathcal{D}=G^{-1}(\operatorname{fib}\mathcal{C}), \qquad \operatorname{cof}\mathcal{D}=LLP(\operatorname{fib}\mathcal{D}\cap w\mathcal{D}).$$

Moreover (F,G) form a Quillen adjunction, which is a Quillen equivalence, if and only if

$$\eta_X : X \xrightarrow{\simeq} GF(X), \qquad \text{for all cofibrant} \quad X \in \mathcal{C}.$$

Proof. As a functor G preserves isomorphisms. Furthermore the 2-of-3-axiom holds for $w\mathcal{D}$, hence $(\mathcal{D}, w\mathcal{D})$ is a category with weak equivalences. Let $C,T \subset \operatorname{Mor}(\mathcal{C})$ be subsets of generating cofibrations and trivial cofibrations with κ-small domains. As G preserves κ-indexed colimits, it follows that the morphisms $F(C)$ and $F(T)$ have κ-small domains. By Corollary 7.1.15 we obtain the two weak factorization systems on $\mathcal{D}$

- $(\operatorname{cof}\mathcal{D}, \operatorname{fib}\mathcal{D}\cap w\mathcal{D}) = (LLP(G^{-1}(\operatorname{fib}\mathcal{C}\cap w\mathcal{C})), G^{-1}(\operatorname{fib}\mathcal{C}\cap w\mathcal{C})),$

- $(LLP(\operatorname{fib}\mathcal{D}), \operatorname{fib}\mathcal{D}) = (LLP(G^{-1}(\operatorname{fib}\mathcal{C})), G^{-1}(\operatorname{fib}\mathcal{C}))$.

It remains to show, that $LLP(\operatorname{fib}\mathcal{D}) = \operatorname{cof}\mathcal{D} \cap w\mathcal{D}$. On the one hand we have by assumption on $LLP(\operatorname{fib}\mathcal{D})$, that

$$LLP(\operatorname{fib}\mathcal{D}) \subset LLP(\operatorname{fib}\mathcal{D}) \cap w\mathcal{D} \subset LLP(\operatorname{fib}\mathcal{D} \cap w\mathcal{D}) \cap w\mathcal{D} = \operatorname{cof}\mathcal{D} \cap w\mathcal{D}.$$

On the other hand suppose $c \in \operatorname{cof}(\mathcal{D}) \cap w\mathcal{D}$. Then c can be factored as

$$\begin{tikzcd} X \arrow[rr, "c"] \arrow[dr, "LLP(\operatorname{fib}\mathcal{D})\ni\ell"'] & & Y \\ & Z. \arrow[ur, "f\in\operatorname{fib}\mathcal{D}"'] & \end{tikzcd}$$

As c and ℓ are weak equivalences, so is f by the 2-of-3 axiom. Since by definition $\operatorname{cof}\mathcal{D} = LLP(\operatorname{fib}\mathcal{D} \cap w\mathcal{D}) \subset LLP(f)$ we find a diagonal

$$\begin{tikzcd} X \arrow[r, "\ell"] \arrow[d, "c"'] & Z \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "\exists d"] & Y, \end{tikzcd}$$

showing that c is a retraction of $\ell \in LLP(\operatorname{fib}\mathcal{D})$. By Proposition 7.1.7 the class $LLP(\operatorname{fib}\mathcal{D})$ is closed under retracts, so finally $c \in LLP(\operatorname{fib}\mathcal{D})$.

By construction G preserves (trivial) fibrations and thus (F, G) form a Quillen adjunction. Since G preserves weak equivalences the composition $X \xrightarrow{\eta_X} GF(X) \longrightarrow G(F(X)_f)$ is a weak equivalence, if and only if η_X is a weak equivalence. Thus condition (ii) a) of Quillen's adjoint functor Theorem 7.1.26 is equivalent to the given assumption. We have to prove, that it also implies condition b). Therefore let $Y \in \mathcal{D}$ be fibrant and consider the commutative diagram

$$\begin{tikzcd} & GF(G(Y)_c) \arrow[r] & GFG(Y) \arrow[r, "G(\varepsilon_Y)"] & G(Y) \\ \emptyset \arrow[r, tail] & G(Y)_c \arrow[u, "\eta_{G(Y)_c}", "\simeq"'] \arrow[r, "\simeq"] & G(Y). \arrow[u, "\eta_{G(Y)}"] \arrow[ur, "\mathrm{id}_{G(Y)}"'] & \end{tikzcd}$$

Since $G(Y)_c$ is cofibrant, the left vertical morphism is a weak equivalence by our assumption and hence the upper row is a weak equivalence by the 2-of-3 axiom. Since $w\mathcal{D} = G^{-1}w\mathcal{C}$, it follows that $F(G(Y)_c) \longrightarrow FG(Y) \xrightarrow{\varepsilon_Y} Y$ is a weak equivalence.

□

Corollary 7.1.30
Let κ be an ordinal and $\mathcal{C}$ a κ-cofibrantly generated model category. Given an adjunction

$$\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y)),$$

where D is an arbitrary cocomplete, finitely complete category. Suppose that:

(i) *G is a left adjoint functor.*

(ii) *$GF(T) \subset \operatorname{cof}\mathcal{C} \cap w\mathcal{C}$, where $T \subset \operatorname{cof}\mathcal{C} \cap w\mathcal{C}$ is a set of generating trivial cofibrations.*

Then $\mathcal{D}$ is a κ-cofibrantly generated model category with

$$w\mathcal{D} = G^{-1}(w\mathcal{C}), \qquad \operatorname{fib}\mathcal{D} = G^{-1}(\operatorname{fib}\mathcal{C}), \qquad \operatorname{cof}\mathcal{D} = LLP(\operatorname{fib}\mathcal{D} \cap w\mathcal{D}).$$

Moreover (F, G) form a Quillen adjunction, which is a Quillen equivalence, if and only if

$$\eta_X : X \xrightarrow{\simeq} GF(X), \qquad \textit{for all cofibrant} \quad X \in \mathcal{C}.$$

Proof. Assumption (i) implies that G preserves arbitrary colimits. Hence for every κ-small object $X \in \mathcal{C}$, the functor $\mathcal{C}(F(X), \text{-}) \cong \mathcal{C}(X, G(\text{-}))$ preserves κ-filtered colimits, which means that also $F(X)$ is κ-small. To check (ii), we let H denote the right adjoint of G. Applying RLP, which by definition reverses inclusions, to assumption (ii) yields

$$H^{-1}RLP(F(T)) = RLP(GF(T)) \supset RLP(\operatorname{cof}\mathcal{C} \cap w\mathcal{C}) = \operatorname{fib}\mathcal{C},$$

where we used the adjunction $\mathcal{C}(G(X), Y) = \mathcal{D}(X, H(Y))$ for the first equality. Equivalently we have

$$G^{-1}(\operatorname{fib}\mathcal{C}) = G^{-1}(RLP(T)) = RLP(F(T)) \supset H(\operatorname{fib}\mathcal{C}),$$

where we used the adjunction $\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y))$ for the second equality. Finally applying LLP yields

$$LLP(G^{-1}(\operatorname{fib}\mathcal{C})) \subset LLP(H(\operatorname{fib}\mathcal{C})) = G^{-1}LLP(\operatorname{fib}\mathcal{C}) = G^{-1}(\operatorname{cof}\mathcal{C} \cap w\mathcal{C}),$$

where again we used the adjunction $\mathcal{C}(G(X), Y) = \mathcal{D}(X, H(Y))$ for the first equality. So we can apply the preceding theorem to conclude the proof. $\square$

Theorem 7.1.31
Let κ be an ordinal and $\mathcal{C}$ a κ-cofibrantly generated model category. Given an adjunction

$$\mathcal{D}(F(X), Y) = \mathcal{C}(X, G(Y)),$$

where D is an arbitrary cocomplete, finitely complete category. Suppose that

(i) *F preserves κ-small objects.*

(ii) *There is a functorial fibrant replacement functor $q_D : D \xrightarrow{\simeq} Q(D)$, for all $D \in \mathcal{D}$.*

(iii) *There is a path object $D \xrightarrow{\simeq} D^I \xrightarrow{(p_0, p_1)} D \times D$, for all fibrant $D \in \mathcal{D}$.*

Then Theorem 7.1.29 (ii) holds and so $\mathcal{D}$ is a κ-cofibrantly generated model category via

$$w\mathcal{D} = G^{-1}(w\mathcal{C}), \qquad \operatorname{fib}\mathcal{D} = G^{-1}(\operatorname{fib}\mathcal{C}), \qquad \operatorname{cof}\mathcal{D} = LLP(\operatorname{fib}\mathcal{D} \cap w\mathcal{D}).$$

Moreover (F, G) form a Quillen adjunction, which is a Quillen equivalence, if and only if

$$\eta_X : X \xrightarrow{\simeq} GF(X), \qquad \textit{for all cofibrant} \quad X \in \mathcal{C}.$$

Proof. According to Theorem 7.1.29 we have to show that $LLP(\operatorname{fib}\mathcal{D}) \subset w\mathcal{D}$. Therefore let $\ell \in \mathcal{D}(X,Y) \cap LLP(\operatorname{fib}\mathcal{D})$. Using the fibrant replacement functor for X we get a diagonal d as on the left and using the path object for the fibrant $Q(Y)$ we get a right homotopy as on the right.

$$\begin{tikzcd} X \arrow[r, "q_X"] \arrow[d, "\ell"'] & Q(X) \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dashed, "d"] & *, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\ell"] \arrow[d, "\ell"'] & Y \arrow[r, "q_Y"] & Q(Y) \arrow[r, "s_{Q(Y)}"] & Q(Y)^I \arrow[d, two heads, "(p_0,p_1)"] \\ Y \arrow[rrr, "(q_Y, Q(\ell)d)"'] \arrow[urrr, dashed, "h"] & & & Q(Y) \times Q(Y), \end{tikzcd}$$

where the right square commutes, since $Q(\ell)d\ell = Q(\ell)q_X = q_Y\ell$ as $\mathrm{id}_{\mathcal{D}} \xrightarrow{q} Q$ is natural. Now since G is a right adjoint and preserves fibrations and weak equivalences, we see that $GQ(X), GQ(Y)$ are fibrant and that

$$GQ(Y) \xrightarrow{\simeq} GQ(Y^I) \xrightarrow{(p_0,p_1)} GQ(Y) \times GQ(Y)$$

is a path object for $GQ(Y)$. It follows that

$$G(d)G(\ell) = G(q_X), \qquad GQ(\ell)G(d) \simeq_{G(h)} G(q_Y).$$

From Theorem 7.1.21 we know that the preimage of the isomorphisms in $D(\mathcal{C})$ are precisely the weak equivalences in $\mathcal{C}$. It follows that $\gamma G(d)\gamma G(\ell) = \gamma G(q_X)$ is an isomorphism and thus $\gamma G(d)$ is a retraction. By construction of the derived category in Theorem 7.1.21 we have that $\gamma G(Q(\ell)d) = \gamma G(q_Y)$, because $GQ(Y)$ is fibrant. This implies that $\gamma GQ(\ell)\gamma G(d) = \gamma G(Q(\ell)d) = \gamma G(q_Y)$ is an isomorphism, showing that $\gamma G(d)$ is also a section and thus an isomorphism. Since $d\ell = q_X$ with isomorphisms $\gamma G(d)$ and $\gamma G(q_X)$, it follows that $\gamma G(\ell)$ is an isomorphism. i.e. $G(\ell)$ is a weak equivalence by Theorem 7.1.21, showing that $\ell \in w\mathcal{D}$ by definition.

□

7.2 Homotopy theory of simplicial objects

7.2.1 The category Δ

Definition 7.2.1
The category Δ is the category of the totally ordered sets

$$\underline{n} = \{0 < ... < n\}, \qquad n \in \mathbb{N}_0,$$

together with order-preserving maps as morphisms. For this category we also use the notation $\Delta_n^m = \Delta(\underline{n}, \underline{m})$, so that the hom-functor Δ^m is the m-th ***standard simplex*** *of simplicial sets. For $0 \leq i \leq n$ we let*

- *$d^i \in \Delta(\underline{n-1}, \underline{n})$ be the unique injection, whose image does not contain i,*
- *$s^i \in \Delta(\underline{n+1}, \underline{n})$ be the unique surjection, sending i and $i+1$ to i.*

Remark 7.2.2
The category $\hat{\Delta}$ also contains the empty set $\underline{-1} = \emptyset$ and has a (non-symmetric) monoidal structure given by the functor

$$\oplus : \hat{\Delta} \times \hat{\Delta} \longrightarrow \hat{\Delta}, \qquad (\underline{m}, \underline{n}) \longmapsto \underline{m} \oplus \underline{n} = \underline{m+n+1},$$

which maps a pair of morphisms $f_i \in \Delta(m_i, n_i)$, for $i = 1, 2$, to the morphism $f_1 \oplus f_2 \in \Delta(m_1 + m_2 + 1, n_1 + n_2 + 1)$, given by

$$(f_1 \oplus f_2)(k) = \begin{cases} f_1(k), & 0 \leq k \leq m_1, \\ f_2(k - m_1 - 1) + n_1 + 1, & m_1 + 1 \leq k \leq n_1 + n_2 + 1. \end{cases}$$

Remark 7.2.3
We also consider $\underline{n}$ as a category, for each $n \geq 0$, with morphisms "$\leq$". This means, that there is precisely one morphism from i to j, if $i \leq j$, and otherwise there is none. The category $\underline{n}$ is small.
In particular $\Delta \leq \mathcal{C}at$ is the full (2-)subcategory[1] *of objects $\underline{n}$.*

Definition 7.2.4
For $\mathcal{C} \in \mathcal{C}\mathrm{AT}$, we denote by $s\mathcal{C} = \underline{\mathcal{C}\mathrm{AT}}(\Delta^{\mathrm{op}}, \mathcal{C})$ and $c\mathcal{C} = \underline{\mathcal{C}\mathrm{AT}}(\Delta, \mathcal{C})$ the categories of simplicial and cosimplicial objects in $\mathcal{C}$. In consistence with the notation for the m-th standard simplex $\Delta^m \in s\mathcal{S}et$ we will write

- *$X_n := X(\underline{n})$, for $X \in s\mathcal{C}$.*
  Moreover we abreviate $d_i := X(d^i)$ and $s_i := X(s^i)$, for $0 \leq i \leq n$.
- *$X^n := X(\underline{n})$, for $X \in c\mathcal{C}$.*
  Moreover we abreviate $d^i := X(d^i)$ and $s^i := X(s^i)$, for $0 \leq i \leq n$.

We consider $\mathcal{C} \leq s\mathcal{C}$ as the subcategory of constant simplicial objects. So by abuse of notation $X \in \mathcal{C}$ will also stand for the constant simplicial object.

7.2.2 (Co-)skeletons

Definition 7.2.5
For any $n \geq 0$ we define the full subcategory and its inclusion functor

$$\Delta_{\leq n} := \{\underline{0}, ..., \underline{n}\} \overset{i_n}{\hookrightarrow} \Delta.$$

Definition 7.2.6
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ be finitley complete and cocomplete and $n \geq 0$.

(i) *The n-**skeleton** of a simplicial object $X \in s\mathcal{C}$ is the left Kan extension*

$$\mathrm{sk}_n X := (i_n)_!(i_n)^* X.$$

[1] We will note make precise what this means, as we will not need the notation later on.

(ii) *The n-**coskeleton** of a simplicial object $X \in s\mathcal{C}$ is the right Kan extension*

$$\mathrm{cosk}_n X := (i_n)_*(i_n)^* X.$$

Note that unit and counit of the adjunctions induce natural maps $\mathrm{sk}_n \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \mathrm{cosk}_n X$.

Definition 7.2.7
Let $n \geq 0$.

(i) *The **boundary** of the n-simplex Δ^n is defined as $\partial\Delta^n = \mathrm{sk}_{n-1}\Delta^n$.*

(ii) *For $0 \leq k \leq n$ the k-**th horn** of Δ^n is defined as*

$$\Lambda^{n,k} = \bigcup_{\substack{0 \leq i \leq n, \\ i \neq k}} d^i \Delta^{n-1} \subset \partial\Delta^n.$$

(iii) *The n-**sphere** is defined as $S^n = \Delta^n / \partial\Delta^n$.*

7.2.3 Function objects and simplicial homotopies

Proposition 7.2.8
For every complete and cocomplete category $\mathcal{C}$ there is an adjunction

$$s\mathcal{S}et(S, \underline{s\mathcal{C}}(X,Y)) = s\mathcal{C}({}^S X, Y) = s\mathcal{C}(X, \underline{s\mathcal{S}et}(S,Y)),$$

where

(i) $\underline{s\mathcal{C}}(X,Y) = \int_n \mathcal{C}(X_n, Y_n)^{\Delta^\bullet_n} = s\mathcal{C}({}^{\Delta^\bullet} X, Y)$,

(ii) ${}^S X : \underline{n} \longmapsto {}^{S_n} X_n$ *is the S-fold coproduct of X,*

(iii) $\underline{s\mathcal{S}et}(S,Y) = \int_n Y_n^{\Delta^\bullet_n \times S}$.

Note that $\underline{s\mathcal{C}}(X,Y)$ also exists for categories $\mathcal{C}$ that are neither complete nor cocomplete.

Proof. This follows from the fact that hom-functors preserve limits/colimits and the Yoneda lemmas. □

Proposition 7.2.9
For every complete and cocomplete category $\mathcal{C}$ with a zero object $*$, there is an adjunction

$$\mathcal{S}et_*(S, \underline{\mathcal{C}_*}(X,Y)) = \mathcal{C}(S \wedge X, Y) = \mathcal{C}(X, \underline{\mathcal{S}et_*}(S,Y)),$$

where

$$\underline{\mathcal{C}_*}(X,Y) = (\mathcal{C}(X,Y), (X \xrightarrow{0} Y)), \quad S \wedge X := {}^S X +_{{}^* X} * \quad \underline{\mathcal{S}et_*}(S,Y) = Y^S \times_{Y^*} *.$$

It induces an adjunction

$$s\mathcal{C}(S \wedge X, Y) = s\mathcal{S}et_*(S, \underline{s\mathcal{C}_*}(X,Y)) = s\mathcal{C}(X, \underline{s\mathcal{S}et_*}(S,Y)),$$

where

(i) $\underline{s\mathcal{C}_*}(X,Y) = (\underline{s\mathcal{C}}(X,Y),0) = \int_n \underline{\mathcal{S}et_*}((\Delta_n^\bullet)_+, \underline{\mathcal{C}_*}(X_n,Y_n))$,

(ii) $S \wedge X : \underline{n} \longmapsto S_n \wedge X_n$,

(iii) $\underline{s\mathcal{S}et_*}(S,Y) = \int_n \underline{\mathcal{S}et_*}((\Delta^\bullet)_+ \wedge S, Y)$.

Proof. Similarly this follows from the fact that hom-functors preserve limits/colimits and the Yoneda lemmas.

□

Definition 7.2.10
Let $\mathcal{C}$ be a category.

(i) *For finitely cocomplete $\mathcal{C}$, we define*

$$\pi_0 X := \operatorname{coker}\left(X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0 \right) = \operatorname{colim} X, \qquad X \in s\mathcal{C}.$$

(ii) *A **simplicial homotopy** between two maps $f_0, f_1 \in s\mathcal{C}(X,Y)$ is a map $h \in s\mathcal{C}({}^{\Delta^1}X, Y) = \underline{s\mathcal{C}}(X,Y)_1$ with $d_i(h) = f_i$.*

(iii) *Two maps $f_0, f_1 \in s\mathcal{C}(X,Y)$ are **simplicially homotopic**, if $[f_0] = [f_1]$ in $\pi_0\underline{s\mathcal{C}}(X,Y)$.*

Lemma 7.2.11
Let $s\mathcal{C}$ carrying the canonical simplicial structure and $f, g \in s\mathcal{C}(X,Y)$.
Then $f \simeq g$, if and only if there are maps $k_i \in \mathcal{C}(X_n, Y_n)$, for $0 \le i \le n+1$, satisfying in every dimension $n \ge 0$ the properties below.

(i) $d_i h_j = \begin{cases} h_{j-1} d_i, & i < j, \\ h_j d_i, & i \ge j, \end{cases} \qquad s_i h_j = \begin{cases} h_{j+1} s_i, & i < j, \\ h_j s_i, & i \ge j. \end{cases}$

(ii) $h_0 = f_n, \qquad h_{n+1} = g_n$.

Proof. Defining

$$t^i : \underline{n} \longrightarrow \underline{1}, \quad k \longmapsto \begin{cases} 0, & 0 \le k < i, \\ 1, & i \le k \le n, \end{cases}$$

there are bijections

$$T_n : \underline{n+2} \xrightarrow{\sim} \Delta_n^1, \qquad i \longmapsto t^i,$$

which induce commutative squares

$$\begin{tikzcd} \underline{n+2} \arrow[r, "T_n"] \arrow[d, "s^i"'] & \Delta_n^1 \arrow[d, "d_i"] \\ \underline{n+1} \arrow[r, "T_{n-1}"] & \Delta_{n-1}^1 \end{tikzcd} \qquad \begin{tikzcd} \underline{n+2} \arrow[r, "T_n"] \arrow[d, "d^{i+1}"'] & \Delta_n^1 \arrow[d, "s_i"] \\ \underline{n+3} \arrow[r, "T_{n+1}"] & \Delta_{n+1}^1 \end{tikzcd} \qquad 0 \le i \le n.$$

By checking the simplicial identities, one proves that the injection

$$\underline{s\mathcal{C}}(X,Y)_1 = \int_n \mathcal{C}(X_n,Y_n)^{\Delta_n^1} \hookrightarrow \prod_{n\ge 0} \mathcal{C}(X_n,Y_n)^{\Delta_n^1} \underset{\sim}{\xrightarrow{T}} \prod_{n \ge 0} \mathcal{C}(X_n,Y_n)^{\underline{n+1}}$$

surjects onto the set of tuples $((h_0, ..., h_{n+1}))_{n\geq 0}$ satisfying the relations (i). The maps

$$\underline{s\mathcal{C}}(X,Y)_0 \xleftarrow{d_0} \underline{s\mathcal{C}}(X,Y)_1 \xrightarrow{d_0} \underline{s\mathcal{C}}(X,Y)_0$$

correspond to the projections onto the factors associated to the constant maps in Δ^1_n. Hence the tuples $((h_0, ..., h_{n+1}))_{n\geq 0}$ define a homotopy from f to g, if and only if (ii) holds.

□

Proposition 7.2.12
Let $\mathcal{C}$ be a category and $X \in \hat{s}\mathcal{C}$. Suppose there are maps $s_{-1} \in \mathcal{C}(X_{n-1}, X_n)$ (resp. $s_{n+1} \in \mathcal{C}(X_{n-1}, X_n)$), for all $n \geq 0$, extending the simplicial identities[2].

Then $X \overset{d_0}{\twoheadrightarrow} X_{-1}$ is a simplicial deformation retraction with homotopy inverse s_{-1} (resp. s_{n+1}), i.e. $d_0 s_{-1} = \mathrm{id}_{X_{-1}}$ and the maps id_X and $s_{-1}d_0$ are simplicially homotopic.

Proof. One checks that the maps

$$h_i := (s_{-1})^i (d_0)^i \in \mathcal{C}(X_n), \qquad 0 \leq i \leq n+1,$$

satisfy the relations of Lemma 7.2.11 and so define a homotopy $h \in \underline{s\mathcal{C}}(X)$ from id_X to $s_{-1}d_0$. The other statement is dual. Unfortunately we could not find a reference for this elementary but crucial fact.

□

7.2.4 Reedy factorization systems

Proposition 7.2.13
Let $\mathcal{C}$ be a categroy carrying a weak factorization system (L, R).

Then there is weak factorization system (L, R) on $s\mathcal{C}$, given by

(i) $\ell \in s\mathcal{C}(X,Y) \cap L$, *if* $(X_n +_{(\mathrm{sk}_{n-1}X)_n} (\mathrm{sk}_{n-1}Y)_n \xrightarrow{\ell_n \cup \varepsilon_Y} Y_n) \in L$, *for all* $n \geq 0$.

(ii) $r \in s\mathcal{C}(X,Y) \cap R$, *if* $(X_n \xrightarrow{(r_n, \eta_X)} Y_n \times_{(\mathrm{cosk}_{n-1}Y)_n} (\mathrm{cosk}_{n-1}X)_n) \in R$, *for all* $n \geq 0$.

*It is called **Reedy factorization system** after Reedy, who constructed it first.*

Proof. There are proofs in the context of model categories given in [GJ09] 7.2., [Hir03] 15.3 and [Hov99] 5.2. The latter two also consider more generally Reedy categories and not just Δ^{op}. We present a short proof for the special case we will need later on. It suffices to check that (i) to (iii) of Proposition 7.1.7 hold. For (i), let $f \in s\mathcal{C}(X,Y)$ and recall that

$$\Delta_{\leq n} = \{\underline{0}, ..., \underline{n}\} \overset{i_n}{\hookrightarrow} \Delta, \qquad n \geq 0,$$

is the full inclusion functor. Following the same pattern we will inductively construct a factorization $(i_n)^*X \xrightarrow{\ell(n)} Z(n) \xrightarrow{r(n)} (i_n)^*Y$ in $\mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{\leq n}, \mathcal{C})$, in a way that

$$Z_n := Z(n)_n = Z(n{+}1)_n = ..., \quad \ell_n := \ell(n)_n = \ell(n{+}1)_n = ..., \quad r_n := r(n)_n = r(n{+}1)_n = ...,$$

[2] See [GJ09] I.1.3.

for all $n \geq 0$. Let $n \geq 0$ and suppose the construction has been established, for $0 \leq m < n$. Using the canonical inclusion functor $\Delta_{\leq n-1} \overset{j}{\hookrightarrow} \Delta_{\leq n}$ we get a commutative diagram

$$\begin{tikzcd}
(\mathrm{sk}_{n-1}X)_n \arrow[r, equal] \arrow[d, "\varepsilon_X"'] & (j_!(i_{n-1})^*X)_n \arrow[r, "j_!\ell(n-1)"] & j_!Z(n-1) \arrow[r, "j_!r(n-1)"] & (j_!(i_{n-1})^*Y)_n \arrow[r, equal] & (\mathrm{sk}_{n-1}Y)_n \arrow[d, "\varepsilon_Y"] \\
X_n \arrow[rrrr] \arrow[d, "\eta_Y"'] & & & & Y_n \arrow[d, "\eta_Y"] \\
(\mathrm{cosk}_{n-1}X)_n \arrow[r, equal] & (j_*(i_{n-1})^*X)_n \arrow[r, "j_*\ell(n-1)"] & j_*Z(n-1) \arrow[r, "j_!r(n-1)"] & (j_*(i_{n-1})^*Y)_n \arrow[r, equal] & (\mathrm{cosk}_{n-1}Y)_n,
\end{tikzcd}$$

giving rise to a map that we can factor as

$$X_n +_{(\mathrm{sk}_{n-1}X)_n} (j_!Z(n-1))_n \xrightarrow{\ell_n} Z_n \xrightarrow{r_n} Y_n \times_{(\mathrm{cosk}_{n-1}Y)_n} (j_*Z(n-1))_n,$$

where $\ell_n \in L$ and $r_n \in R$. Defining face maps and degeneracies by

(i) $(d_0, ..., d_n) : Z_n \xrightarrow{r_n} Y_n \times_{(\mathrm{cosk}_{n-1}Y)_n} (j_*Z(n-1))_n \xrightarrow{\pi} (j_*Z(n-1))_n \hookrightarrow (Z_{n-1})^{n+1}$,

(ii) $s_0 \cup ... \cup s_{n-1} : {}^nZ_{n-1} \twoheadrightarrow (j_!Z(n-1))_n \xrightarrow{\iota} X_n +_{(\mathrm{sk}_{n-1}X)_n} (j_!Z(n-1))_n \xrightarrow{\ell_n} Z_n$,

this constitutes an object $Z(n) = (Z_0, ..., Z_n) \in \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{\leq n}, \mathcal{C})$ with $\mathrm{sk}_{n-1}Z(n) = j_!Z(n-1)$ and $\mathrm{cosk}_{n-1}Z(n) = j_*Z(n-1)$. Moreover $\ell(n) = (\ell_0, ..., \ell_n)$ and $(r_0, ..., r_n)$ are morphisms in $\mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{\leq n}, \mathcal{C})$ by construction. In the limit we obtain a factorization $X \xrightarrow{\ell} Z \xrightarrow{r} Y$ with

$$X_n +_{(\mathrm{sk}_{n-1}Z)_n} (\mathrm{sk}_{n-1}Z)_n \xrightarrow{\ell_n \cup \varepsilon_Z} Z_n \xrightarrow{(r_n, \eta_Z)} Y_n \times_{(\mathrm{cosk}_{n-1}Z)_n} (\mathrm{cosk}_{n-1}Z)_n, \qquad n \geq 0,$$

lying in L and R of $\mathcal{C}$ respectively. This proves (i).

For (ii) consider a lifting problem

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "L \ni \ell"'] & X \arrow[d, "r \in R"] \\
B \arrow[r, "v"'] \arrow[ur, dashed, "D"] & Y.
\end{tikzcd}$$

Again we will construct a diagonal D inductively. Let $n \geq 0$, such that D_m has been constructed, for all $0 \leq m < n$, being compatible with face maps and degeneracies in that range. Then these D_m induce a map D in

$$s\mathcal{C}(\mathrm{sk}_{n-1}B, X) = s\mathcal{C}(B, \mathrm{cosk}_{n-1}X),$$

and by assumption on ℓ and r the lifting problem

$$\begin{tikzcd}
A_n +_{(\mathrm{sk}_{n-1}A)_n} (\mathrm{sk}_{n-1}B)_n \arrow[r, "u \cup D"] \arrow[d] & X_n \arrow[d, "r"] \\
B_n \arrow[r, "{(u,D)}"'] \arrow[ur, dashed, "D_n"] & Y_n \times_{(\mathrm{cosk}_{n-1}Y)_n} (\mathrm{cosk}_{n-1}X)_n
\end{tikzcd}$$

has a solution D_n, which by construction induces commutative diagrams

$$\begin{tikzcd} A_n \arrow[r, "u_n"] \arrow[d, "\ell_n"'] & X_n \arrow[d, "r_n"] \\ B_n \arrow[r, "v_n"'] \arrow[ur, "D_n"] & Y_n, \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{sk}_{n-1}B)_n \arrow[r, "D"] \arrow[d] & X_n \arrow[d] \\ B_n \arrow[r, "D"'] \arrow[ur, "D_n"] & (\mathrm{cosk}_{n-1}X)_n. \end{tikzcd}$$

Note that for $n = 0$, the left square provides a diagonal as desired. For $n > 0$ we have

- $s_0 \cup ... \cup s_{n-1} : {}^n(B_{n-1}) \twoheadrightarrow (\mathrm{sk}_{n-1}B)_n \xrightarrow{\varepsilon_B} B_n$,
- $(d_0, ..., d_n) : X_n \xrightarrow{\eta_X} (\mathrm{cosk}_{n-1}X)_n \hookrightarrow (X_n)^{n+1}$,

hence commutativity of the right square is equivalent to the conditions

$$D_n s_i = s_i D_{n-1}, \quad 0 \le i \le n-1, \qquad\qquad d_i D_n = D_{n-1} d_i, \quad 0 \le i \le n,$$

which proves that $D_0, ..., D_n$ are compatible with all face maps and degeneracies in that range. By construction $(D_0, D_1, ...)$ constitutes a morphism $s\mathcal{C}(B, X)$ solving the lifting problem from the beginning. So we have shown $L \subset LLP(R)$ or $R \subset RLP(L)$ respectively, which proves (ii).

Finally as L and R on $\mathcal{C}$ are closed under retractions by Proposition 7.1.7, so are also L and R on $s\mathcal{C}$. This proves (iii) and therefore concludes the proof.

□

7.2.5 Topological spaces and simplicial sets

Definition 7.2.14
A topological space X is called compactly generated, if $U \subset X$ is closed, if and only if $U \cap C \subset C$ is closed, for every compact subspace $C \subset X$.

Let $\mathcal{S}p \le \mathcal{T}op$ denote the category of compactly generated, Hausdorff topological spaces.

Definition 7.2.15 (i) *The **topological standard simplex** $|\Delta^\bullet| \in c\mathcal{S}p$ is defined as the fibre $|\Delta^\bullet| = (d^0)^{-1}(1)$ of the map $\mathcal{S}et(\Delta^\bullet_0, \mathbb{R}_{\ge 0}) \xrightarrow{d^0} \mathcal{S}et(\Delta^\bullet_{-1}, \mathbb{R}_{\ge 0}) = \mathbb{R}$, induced by $\underline{-1} \overset{d^0}{\hookrightarrow} \underline{0}$.*

(ii) *The **singular nerve functor** is defined as the functor*

$$S : \mathcal{S}p \longrightarrow s\mathcal{S}et, \qquad X \longmapsto \mathcal{S}p(|\Delta^\bullet|, X).$$

(iii) *The **geometric realization** is defined as the left adjoint in the adjunction*

$$\mathcal{S}p(|X|, Y) = s\mathcal{S}et(X, S(Y)),$$

which exists as $\mathcal{S}p$ is cocomplete.

Theorem 7.2.16 (Quillen)
The category $\mathcal{S}p$ of compactly generated Hausdorff spaces is a model category with the definitions below.

(i) *Weak equivalences are π_*-**isomorphisms**, i.e. maps $X \xrightarrow{f} Y$, such that*

$$\pi_0 f : \pi_0 X \xrightarrow{\sim} \pi_0 Y, \qquad \pi_n f : \pi_n(X, *) \xrightarrow{\sim} \pi_n(Y, f(*)), \qquad n > 0$$

are isomorphisms, for all $* \in X$.

(ii) *Fibrations are **Serre fibrations**, i.e. maps in $RLP\{|\Lambda^{n,k}| \hookrightarrow |\Delta^n|; 0 \le k \le n\}$.*

(iii) *Cofibrations are maps in $LLP(\mathrm{fib}\,\mathcal{S}p \cap w\mathcal{S}p)$.*

It is ω-cofibrantly generated, because $\mathrm{fib}\,\mathcal{S}p \cap w\mathcal{S}p = RLP\{|\partial\Delta^n| \hookrightarrow |\Delta^n|; n \ge 0\}$.

Proof. See [Qui67] Thm II.3.1. We have $\mathrm{fib}\,\mathcal{S}p \cap w\mathcal{S}p = RLP\{|\partial\Delta^n| \hookrightarrow |\Delta^n|; n \ge 0\}$ by [Qui67] Lem. II.3.2.

□

Theorem 7.2.17 (Quillen)
The category $s\mathcal{S}et$ of simplicial sets is a model category, where

(i) *Weak equivalences are the maps, whose topological realization is a π_*-isomorphism.*

(ii) *Fibrations are **Kan fibrations**, i.e. maps in $RLP\{\Lambda^{n,k} \hookrightarrow \Delta^n; 0 \le k \le n\}$.*

(iii) *Cofibrations are monomorphisms.*

It is ω-cofibrantly generated, because $\mathrm{fib}\,(s\mathcal{S}et) \cap w(s\mathcal{S}et) = RLP\{\partial\Delta^n \hookrightarrow \Delta^n; n \ge 0\}$.

Proof. By [Qui67] Thm II.3.3 $s\mathcal{S}et$ is a model category with weak equivalences defined as $RLP(\mathrm{cof}\,(s\mathcal{S}et)) \circ LLP(\mathrm{fib}\,(s\mathcal{S}et))$. In [Qui67] Prop. II.3.4 he shows that a map $f \in s\mathcal{S}et(X, Y)$ is in $RLP(\mathrm{cof}\,(s\mathcal{S}et)) \circ LLP(\mathrm{fib}\,(s\mathcal{S}et))$, if and only if $|f|$ is a weak equivalence. We have $\mathrm{fib}\,(s\mathcal{S}et) \cap w(s\mathcal{S}et) = RLP\{\partial\Delta^n \hookrightarrow \Delta^n; n \ge 0\}$ by [Qui67] Prop. II.2.2.

□

Theorem 7.2.18 (Quillen)
The adjunction $\mathcal{S}p(|X|, Y) = s\mathcal{S}et(X, S(Y))$ is a Quillen equivalence.

Proof. Corollary 7.1.9 implies readily that S preserves fibrations and trivial fibrations, hence the adjunction is Quillen by Theorem 7.1.26. The proof that it is a Quillen equivalence is implicit in [Qui67] II.3. For an explicit reference, see [Hov99] Thm. 3.6.7.

□

Proposition 7.2.19
The geometric realization functor $s\mathcal{S}et \longrightarrow \mathcal{S}p$ preserves finite limits.
In particular every simplicial homotopy induces a homotopy after geometric realization.

Proof. See [Hov99] Lemma 3.2.4. It follows that

$$|{}^{\Delta^1}X| = |\Delta^1 \times X| \xrightarrow{\sim} |\Delta^1| \times |X|, \qquad X \in s\mathcal{S}et,$$

and thus every simplicial homotopy induces a homotopy of spaces. □

Proposition 7.2.20
Let $X \in s\mathcal{S}et$. Then naturally

(i) $\pi_0 X = \pi_0 |X|$,

(ii) $\pi_n(X, *) := \pi_0 \underline{s\mathcal{S}et_*}(S^n, X) = \pi_n(|X|, *)$*, for all* $n > 0$ *and* $* \in X$*, if* X *is fibrant.*

Proof. See [Hov99] Lem. 3.4.3 and Prop. 3.6.3. □

7.2.6 Model categories of simplicial objects

Definition 7.2.21
A ***model category of simplicial objects*** *consists of a category* $\mathcal{C}$*, together with a model structure on* $s\mathcal{C}$*, such that the following holds.*

(SM) *For every cofibration* $x \in s\mathcal{C}(X, X')$ *and every fibration* $y \in s\mathcal{C}(Y, Y')$*, the map*

$$(x^*, y_*) : \underline{s\mathcal{C}}(X', Y) \longrightarrow \underline{s\mathcal{C}}(X, Y) \times_{\underline{s\mathcal{C}}(X, Y')} \underline{s\mathcal{C}}(X', Y')$$

is a Kan fibration of simplicial sets, which is a weak equivalence, if x *or* y *is a weak equivalence.*

In [Qui67] II.2 Quillen more generally introduced the notion of a closed simplicial model category. In II.1 loc. cit. he defined a simplicial category as a category $\mathcal{C}$, which is enriched, powered and copowered over the category of simplicial sets. In particular there are adjunctions like in Proposition 7.2.8. He then defined a closed simplicial model category as a simplicial category, which is a model category and satisfies (SM) with respect to the function objects. In particular the following holds.

Remark 7.2.22
A model category of simplicial objects is precisely a simplicial model category $s\mathcal{C}$*, whose enriched, powered and copowered structure is the canonical one of Proposition 7.2.8.*

We decided to restrict ourselves to model categories of simplicial objects, because this is the only form of closed simplicial model categories we need and so we can avoid introducing the calculus of enriched categories.

Proposition 7.2.23
if $b \in s\mathcal{S}et(B, B')$ *is a cofibration (injection) and* $c \in s\mathcal{S}et(C, C')$ *a Kan fibration, then*

$$(b^*, c_*) : \underline{s\mathcal{S}et}(B', C) \longrightarrow \underline{s\mathcal{S}et}(B, C) \times_{\underline{s\mathcal{S}et}(B, C')} \underline{s\mathcal{S}et}(B', C')$$

is a fibration. If b or c is a weak equivalence, so is also (b^*, c_*).
In particular $s\mathcal{S}et$ is a model category of simplicial objects.

Proof. This is axiom SM7 in [Qui67] Def. II.2.2. Hence the statement follows from [Qui67] Thm. II.3.3.

□

Proposition 7.2.24
Let $s\mathcal{C}$ be a model category of simplicial objects.
Let $s \in s\mathcal{S}et(S, S'), x \in s\mathcal{C}(X, X')$ be cofibrations and $y \in s\mathcal{C}(Y, Y')$ a fibration. Then

(i) $({}^{s}\mathrm{id}) \cup ({}^{\mathrm{id}}x) : ({}^{S}X') +_{({}^{S}X)} ({}^{S'}X) \longrightarrow {}^{S'}X'$ *is a cofibration, which is a weak equivalence, if a or b are weak equivalences.*

(ii) $(s^*, y_*) : \underline{s\mathcal{S}et}(S', Y) \longrightarrow \underline{s\mathcal{S}et}(S, Y) \times_{\underline{s\mathcal{S}et}(S,Y')} \underline{s\mathcal{S}et}(S', Y')$ *is a fibration, which is a weak equivalence, if a or c are weak equivalences.*

More precisely axiom **(SM)***, (ii) and (iii) are equivalent.*

Proof. Using the adjunctions of Proposition 7.2.8

$$s\mathcal{S}et(S, \underline{s\mathcal{C}}(X, Y)) = s\mathcal{C}({}^{S}X, Y) = s\mathcal{C}(X, \underline{s\mathcal{S}et}(S, Y)),$$

by Remark 7.1.10 the three lifting problems

$$\begin{tikzcd} S \arrow[r] \arrow[d, "s"'] & \underline{s\mathcal{S}et}(X', Y) \arrow[d, "(x^*, y_*)"] \\ S' \arrow[r] \arrow[ur, dashed, "d'"] & \underline{s\mathcal{S}et}(X, Y) \times_{\underline{s\mathcal{S}et}(X,Y')} \underline{s\mathcal{S}et}(X', Y'), \end{tikzcd} \qquad \begin{tikzcd} ({}^{S}X') +_{({}^{S}X)} ({}^{S'}X) \arrow[r] \arrow[d, "({}^{s}\mathrm{id})\cup({}^{\mathrm{id}}x)"'] & Y \arrow[d, "y"] \\ {}^{S'}X' \arrow[r] \arrow[ur, dashed, "d"] & Y', \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "b"'] & \underline{s\mathcal{C}}(S', Y) \arrow[d, "(s^*, y_*)"] \\ X' \arrow[r] \arrow[ur, dashed, "d''"] & \underline{s\mathcal{C}}(S, Y) \times_{\underline{s\mathcal{C}}(S,Y')} \underline{s\mathcal{C}}(S', Y') \end{tikzcd}$$

are equivalent. To prove (i), we need to check that the first problem can be solved, for every (trivial) cofibration s. So it suffices to solve the third problem, which is possible as the right vertical map is a (trivial) fibration by axiom **(SM)** of a model category of simplicial objects. Similarly one checks (ii).

□

Theorem 7.2.25
Given a model category of simplicial objects $s\mathcal{C}$ and an adjunction

$$s\mathcal{D}(F(X), Y) = s\mathcal{C}(X, U(Y)),$$

where $\mathcal{D}$ is an arbitrary cocomplete, finitely complete category. Suppose that:

(i) *F preserves κ-small objects.*

(ii) $LLP(U^{-1}(\mathrm{fib}\,(s\mathcal{C})) \subset U^{-1}(w(s\mathcal{C}))$.

Then $s\mathcal{D}$ is a κ-cofibrantly generated model category of simplicial objects with

$$w(s\mathcal{D}) = U^{-1}(w(s\mathcal{D})), \quad \mathrm{fib}\,\mathcal{C} = U^{-1}(\mathrm{fib}\,(s\mathcal{C})), \quad \mathrm{cof}\,(s\mathcal{D}) = LLP(\mathrm{fib}\,(s\mathcal{D}) \cap w(s\mathcal{D})).$$

Proof. By Theorem 7.1.29 the category $s\mathcal{D}$ is a model category of simplicial objects. Let $s \in s\mathcal{S}et(S, S')$ be a cofibration and $y \in s\mathcal{D}(Y, Y')$ a fibration. Applying U to the map

$$(s^*, y_*) : \underline{s\mathcal{S}et}(S', Y) \longrightarrow \underline{s\mathcal{S}et}(S, Y) \times_{\underline{s\mathcal{S}et}(S,Y')} \underline{s\mathcal{S}et}(S', Y')$$

we get

$$\begin{array}{ccc}
U(\underline{s\mathcal{S}et}(S', Y)) & \xrightarrow{U(s^*, y_*)} & U\left(\underline{s\mathcal{S}et}(S, Y) \times_{\underline{s\mathcal{S}et}(S,Y')} \underline{s\mathcal{S}et}(S', Y')\right) \\
\wr\downarrow & & \downarrow\wr \\
\underline{s\mathcal{S}et}(S', U(Y)) & \xrightarrow{(U(s)^*, U(y)_*)} & \underline{s\mathcal{S}et}(S, U(Y)) \times_{\underline{s\mathcal{S}et}(S,U(Y'))} \underline{s\mathcal{S}et}(S', U(Y')),
\end{array}$$

where the vertical maps are isomorphisms, because U is a right adjoint and therefore preserves arbitrary limits and ends. Since $s\mathcal{C}$ is a model category of simplicial objects, the lower map is a fibration by Proposition 7.2.24 (ii). Hence (s^*, y_*) is a fibration by definition of a fibration in $s\mathcal{D}$, which is a weak equivalence, if s or y is a weak equivalence. Equivalently axiom **(SM)** holds by Proposition 7.2.24.

□

Corollary 7.2.26
Let κ be an ordinal and $s\mathcal{C}$ a κ-cofibrantly generated model category of simplicial objects. Given an adjunction

$$s\mathcal{D}(F(X), Y) = s\mathcal{C}(X, G(Y)),$$

where $\mathcal{D}$ is an arbitrary cocomplete, finitely complete category. Suppose that:

(i) *G is a left adjoint functor.*

(ii) $GF(T) \subset \mathrm{cof}\,(s\mathcal{C}) \cap w(s\mathcal{C})$, *where* $T \subset \mathrm{cof}\,(s\mathcal{C}) \cap w(s\mathcal{C})$ *is a set of generating trivial cofibrations.*

Then $s\mathcal{D}$ is a κ-cofibrantly generated model category with

$$w(s\mathcal{D}) = G^{-1}(w(s\mathcal{C})), \quad \mathrm{fib}\,(s\mathcal{D}) = G^{-1}(\mathrm{fib}\,(s\mathcal{C})), \quad \mathrm{cof}\,(s\mathcal{D}) = LLP(\mathrm{fib}\,(s\mathcal{D}) \cap w(s\mathcal{D})).$$

Moreover (F, G) form a Quillen adjunction, which is a Quillen equivalence, if and only if

$$\eta_X : X \xrightarrow{\simeq} GF(X), \qquad \text{for all cofibrant} \quad X \in \mathcal{C}.$$

Proof. The follows readily from Corollary 7.1.30 and Theorem 7.2.25.

□

Theorem 7.2.27
Given a set $J \in \mathcal{S}et$ and an adjunction

$$s\mathcal{D}(F(X), Y) = s\mathcal{C}(X, G(Y)),$$

where $\mathcal{D}$ is an arbitrary cocomplete, finitely complete category. Suppose that

(i) *U preserves κ-small objects,*

(ii) *There is a functorial fibrant replacement $q_D : D \xrightarrow{\simeq} Q(D)$, for all $D \in s\mathcal{D}$.*

Then Theorem 7.2.25 (ii) holds and so $s\mathcal{D}$ is a κ-cofibrantly generated model category of simplicial objects via

$$w(s\mathcal{D}) = G^{-1}(w(s\mathcal{C})), \quad \mathrm{fib}\,(s\mathcal{D}) = G^{-1}(\mathrm{fib}\,(s\mathcal{C})), \quad \mathrm{cof}\,(s\mathcal{D}) = LLP(\mathrm{fib}\,(s\mathcal{D}) \cap w(s\mathcal{D})).$$

Proof. For every fibrant simplicial object $X \in s\mathcal{D}$, the construction (cf. Proposition 7.2.8)

$$\Delta = (\mathrm{id}_X, \mathrm{id}_X) : X = \underline{s\mathcal{S}et}(\Delta^0, X) \xrightarrow{s_0} \underline{s\mathcal{S}et}(\Delta^1, X) \xrightarrow{(d_0,d_1)} \underline{s\mathcal{S}et}(\Delta^0, X)^2 = X^2, \quad X \in s\mathcal{D},$$

is a path object. Indeed, the left map is a simplicial homotopy equivalence, since $\Delta^1 \xrightarrow{s^1} \Delta^0$ is a simplicial homotopy equivalence, being the left adjoint in the adjunction

$$\underline{0}(s^1(x), y) = \underline{1}(x, d^0(y)),$$

it is also weak equivalence by Proposition 7.2.19. As Δ^1 and Δ^0 are cofibrant in $s\mathcal{S}et$, Brown's lemma implies that the map

$$\underline{s\mathcal{S}et}(\Delta^0, G(X)) \xrightarrow{s^0} \underline{s\mathcal{S}et}(\Delta^1, G(X))$$

is a weak equivalence by Proposition 7.2.24 (ii). Using that G is a right adjoint and therefore commutes with limits and ends, this map is isomorphic to G applied to $\underline{s\mathcal{S}et}(\Delta^0, X)) \xrightarrow{s_0} \underline{s\mathcal{S}et}(\Delta^1, X)$, which therefore is a weak equivalence, too.

Similarly, as $\partial\Delta^1 \hookrightarrow \Delta^1$ is a cofibration, $X \in s\mathcal{D}$ and hence $G(X) \in s\mathcal{C}$ is fibrant, the map

$$(d^0, d^1)^* : \underline{s\mathcal{S}et}(\Delta^1, G(X)) = G(\underline{s\mathcal{S}et}(\Delta^1, X)) \xrightarrow{(d_0,d_1)} G(\underline{s\mathcal{S}et}(\Delta^0, X)^2) = \underline{s\mathcal{S}et}(\partial\Delta^1, G(X))$$

is a fibration, where for the equalities we again used that U is a right adjoint. So we can apply Theorem 7.1.31 to conclude the proof.

□

7.2.7 Cofibrations of simplicial objects in categories over $\mathcal{S}et$

Remark 7.2.28
Given a set $J \in \mathcal{S}et$ and for each $j \in J$ a model category $\mathcal{C}_j$.

Then the product $\mathcal{C} := \prod_{j \in J} \mathcal{C}_j$ is a model category, when we define weak equivalences/fibrations/cofibrations to be maps $f \in \mathcal{C}(X, Y)$, such that $f_j \in \mathcal{C}_j(X_j, Y_j)$ are weak equivalences/fibrations/cofibrations, for all $j \in J$.

Proposition 7.2.29
Suppose $s\mathcal{C}$ is a model category of simplicial objects by an adjunction

$$s\mathcal{C}(F(X),Y) = s\mathcal{S}et^J(X,U(Y)), \qquad J \in \mathcal{S}et.$$

Let $f \in s\mathcal{C}(X,Y)$ be ***almost free****, i.e. there are $S \in \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{surj}, \mathcal{S}et^J)$, satisfying*

(i) $Y_n = X_n + F(S_n)$, *for all $n \geq 0$,*

(ii) *Every $s \in \Delta_{surj}(\underline{n},\underline{m})$ induces a commutative square*

$$\begin{tikzcd} F(S_m) \arrow[r, "\iota_{F(S_m)}"] \arrow[d, "F(s^*)"'] & F(S_m)+X_m \arrow[r, equal] & Y_m \arrow[d, "s^*"] \\ F(S_n) \arrow[r, "\iota_{F(S_{n+1})}"] & F(S_n)+X_n \arrow[r, equal] & Y_n. \end{tikzcd}$$

(iii) $f_n : X_n \xrightarrow{\iota_{X_n}} F(S_n) + X_n = Y_n$, *for all $n \geq 0$.*

Then f is a cofibration.

Proof. By Theorem 7.2.17 the trivial fibrations in $s\mathcal{S}et$ are given by

$$\mathrm{fib}\,(s\mathcal{S}et) \cap w(s\mathcal{S}et) = RLP\{\partial\Delta^n \hookrightarrow \Delta^n; n \geq 0\}.$$

Hence by construction the trivial cofibrations in $s\mathcal{C}$ are precisely those maps $f \in s\mathcal{C}(X,Y)$ solving the lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & U_j(X) \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & U_j(Y), \end{tikzcd} \qquad n \geq 0, \quad j \in J.$$

where $U_j : s\mathcal{C} \xrightarrow{U} s\mathcal{S}et^J \xrightarrow{\pi_j} s\mathcal{S}et$, for all $j \in J$. This is equivalent to surjectivity of the maps

$$\begin{tikzcd} U_j(X_n) \arrow[r] \arrow[dd, equal] & U_j(Y_n \times_{(\mathrm{cosk}_{n-1}Y)_n} (\mathrm{cosk}_{n-1}X)_n) \arrow[d, "\wr"] \\ & U_j(Y_n) \times_{(\mathrm{cosk}_{n-1}U_j(Y))_n} (\mathrm{cosk}_{n-1}U_j(X))_n \arrow[d, equal] \\ s\mathcal{S}et(\Delta^n, U_j(X)) \arrow[r] & s\mathcal{S}et(\Delta^n, U_j(Y)) \times_{s\mathcal{S}et(\partial\Delta^n, U_j(Y))} s\mathcal{S}et(\partial\Delta^n, U_j(Y)), \end{tikzcd}$$

where the upper right vertical map is an isomorphism, since the coskeleton can be written as an end, which is preserved by U_j being a right adjoint. Since in $\mathcal{S}et$ (hence in $\mathcal{S}et^J$) the retractions are precisely the surjections, it follows that $(\mathrm{cof}\,(s\mathcal{C}), \mathrm{fib}\,(s\mathcal{C}) \cap w(s\mathcal{C}))$ is the Reedy factorization system (see Proposition 7.2.13) to the weak factorization system of Proposition 7.1.11 induced by the adjunction

$$\mathcal{C}(F(X),Y) = \mathcal{S}et^J(X,U(Y)).$$

Now let $f \in s\mathcal{C}(X, Y)$ be almost free. Then, for each $n \geq 0$, we have

$$(\mathrm{sk}_{n-1}Y)_n = (\mathrm{sk}_{n-1}X)_n + F(\mathrm{sk}_{n-1}S)_n,$$

and thus

$$\begin{tikzcd}
X_n +_{(\mathrm{sk}_{n-1}X)_n} (\mathrm{sk}_{n-1}Y)_n \arrow[rr, "\iota_{X_n} \cup \varepsilon_{Y_n}"] \arrow[d, equal] & & Y_n \arrow[d, equal] \\
X_n + F(\mathrm{sk}_{n-1}S)_n \arrow[r, "\iota"] & X_n + F(\mathrm{sk}_{n-1}S)_n + F(S_n \backslash (\mathrm{sk}_{n-1}S)_n), &
\end{tikzcd}$$

which proves that f is in L of the Reedy model structure and thus is a cofibration.

□

Corollary 7.2.30
Suppose $s\mathcal{C}$ is a model category of simplicial objects by an adjunction

$$s\mathcal{C}(F(X), Y) = s\mathcal{S}et^J(X, U(Y)), \qquad J \in \mathcal{S}et.$$

Let $Y \in s\mathcal{C}$ be ***almost free****, i.e.*

$$Y = F(S) \qquad \textit{in} \quad \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{surj}, \mathcal{C}), \qquad \textit{for some } S \in \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{surj}, \mathcal{S}et^J).$$

Then Y is cofibrant.

Remark 7.2.31
Let $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{AT}$ and given an adjunction

$$s\mathcal{C}(F(X), Y) = s\mathcal{D}(X, G(Y)), \qquad X \in \mathcal{D},\ Y \in \mathcal{C}.$$

Then the following holds.

(i) *FG is a comonad on $\mathcal{C}$, i.e. a comonoid in the monoidal category $(\mathcal{C}\mathrm{AT}(\mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$, with comultiplication and counit given by*

$$(\delta_{FG})_X : FG(X) \xrightarrow{F(\eta_{G(X)})} FG \circ FG(X), \qquad (\varepsilon_{FG})_X : FG(X) \xrightarrow{\varepsilon_X} X, \qquad X \in \mathcal{C}.$$

(ii) *The comonad structure induces a simplicial resolution $E_\bullet(FG) \xrightarrow{d_0} \mathrm{id}_{\mathcal{C}}$, for every $X \in \mathcal{C}$ a natural*

$$\begin{tikzcd}
X & FG(X) \arrow[l, "\varepsilon_X"'] \arrow[r, bend right, "F(\eta_{G(X)})"'] & (FG)^2(X) \arrow[l, shift left, "{\varepsilon_{FG(X)},\ FG(\varepsilon_X)}"'] \arrow[l, shift right] \arrow[r, bend right, shift left, "{F(\eta_{GFG(X)}),\ FGF(\eta_{G(X)})}"'] \arrow[r, bend right, shift right] & (FG)^3(X) \arrow[l, shift left=2, "{\varepsilon_{(FG)^2(X)},\ FG(\varepsilon_{FG(X)}),\ (FG)^2(\varepsilon_X)}"'] \arrow[l] \arrow[l, shift right=2] \arrow[r, bend right] \arrow[r, bend right, shift left] \arrow[r, bend right, shift right] & \cdots \arrow[l, shift left=3] \arrow[l, shift left] \arrow[l, shift right] \arrow[l, shift right=3]
\end{tikzcd}$$

(iii) *When composed with G the unit $\mathrm{id}_{\mathcal{D}} \longrightarrow GF$ induces an extra-degeneracy*

$$s_{-1} = \eta_{(GF)^{n+1}F} \in \mathcal{C}\mathrm{AT}(\mathcal{C},\mathcal{D})(G \circ (E_n(FG)), G \circ (E_{n+1}(FG))), \qquad n \geq 0.$$

In particular $G(E_\bullet(FG)) \xrightarrow{d_0} G$ is a natural simplicial deformation retraction by Proposition 7.2.12.

Corollary 7.2.32
Suppose $s\mathcal{C}$ is a model category of simplicial objects by an adjunction

$$s\mathcal{C}(F(X), Y) = s\mathcal{S}et^J(X, U(Y)), \qquad J \in \mathcal{S}et.$$

Then $E_\bullet(X) := \mathrm{diag}E_\bullet(FU)(X) \xrightarrow{d_0} X$ is a functorial cofibrant replacement for $X \in s\mathcal{C}$.

Proof. We have $j^*E_\bullet(X) = F(\mathrm{diag}S_\bullet(X))$ in $\mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{surj}, \mathcal{C})$ with

$$S_\bullet(X) = (\, U(X) \xrightarrow[\eta_{U(X)}]{} UFU(X) \underset{\substack{\eta_{UFU(X)},\\ UF(\eta_{U(X)})}}{\rightrightarrows} UFUFU(X) \,\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\, \ldots\,) \in \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{surj}, s\mathcal{S}et),$$

which proves that $E_\bullet(X)$ is almost free and hence cofibrant by Corollary 7.2.30. Moreover $U(E_\bullet(X)) \xrightarrow{d_0} U(X)$ is a simplicial deformation retraction by the preceding Remark. Hence using Proposition 7.2.19 the map $U_j(E_\bullet(X)) \xrightarrow{d_0} U_j(X)$ is a weak equivalence, for all $j \in J$. Equivalently $E_\bullet(X) \xrightarrow{d_0} X$ is a weak equivalence, which concludes the proof. □

Remark 7.2.33
Dually we can also consider the monad $UF \in (\mathcal{C}\mathrm{AT}(s\mathcal{S}et^J), \circ, \mathrm{id})$-$\mathcal{M}on$.

Then $E_\bullet(FU) = B_\bullet(F, UF, U)$ is a bar construction, where U is a left resp. F is a right UF-module by the adjunction counit resp. unit.

In particular, if $s\mathcal{C} \xrightarrow{G} \mathcal{D}$ is a functor mapping trivial cofibrations to isomorphisms, then by Theorem 7.1.24 we have

$$\mathbb{L}G(Y) = B_\bullet(GF, UF, U(Y)) := \mathrm{diag}B_\bullet(GF, UF, U)(Y), \qquad Y \in s\mathcal{C}.$$

7.2.8 Simplicial groups

Definition 7.2.34
Let $G \in s\mathcal{G}rp$.

The ***Moore complex*** *associated to G is defined as the chain complex N_*G (of possibly noncommutative groups), given by*

$$N_nG = \bigcap_{1 \leq i \leq n} \ker(G_n \xrightarrow{d_i} G_{n-1}), \qquad d = d_0| : N_nG \longrightarrow N_{n-1}G, \qquad n \geq 0.$$

Note that the subgroups $dN_nG \leq G_{n+1}$ are normal, being the image of a normal subgroup under the epimorphism d_0.

In particular $H_n(N_*G) := \ker(N_nG \xrightarrow{d} N_{n+1}G)/dN_{n-1}G$ is a group, for all $n \geq 0$.

Proposition 7.2.35
For $f \in s\mathcal{G}rp(G,H)$ the following is equivalent.

(i) $f \in s\mathcal{S}et(G,H)$ *is a Kan fibration.*

(ii) $N_nG \xrightarrow{N_nf} N_nH$ *is surjective, for all* $n > 0$.

(iii) $G_n \xrightarrow{f_n} H_n \times_{\pi_0 H} \pi_0 G$ *is surjective, for all* $n \geq 0$.

In particular every simplicial group is fibrant, when considered as a simplicial set.

Proof. See [Qui67] Prop. II.3.1. □

Remark 7.2.36
Every (dimensionwise) short exact sequence of simplicial groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

induces a fibration sequence under realization and so gives rise to a long exact sequence of homotopy groups

$$\ldots \xrightarrow{\partial} \pi_n N \longrightarrow \pi_n G \longrightarrow \pi_n H \xrightarrow{\partial} \pi_{n-1} N \longrightarrow \ldots \xrightarrow{\partial} \pi_0 N \longrightarrow \pi_0 G \longrightarrow \pi_0 H.$$

where we consider the neutral element 1 *as a base point.*

Remark 7.2.37
Recall that the natural bijection $\underline{n} + \underline{0} \longrightarrow \underline{n+1}$, *for* $n \geq 0$, *induces natural retractions*

$$X_{+1} \xrightarrow{p_X} X, \qquad d_0 : X_{+1} \xrightarrow{d_0} X_0, \qquad X \in s\mathcal{C},$$

where $\mathcal{C}$ *is an arbitrary category, and the second map is a simplicial deformation retraction by Proposition 7.2.12, since* s_0 *defines an extra degeneracy.*

Proposition 7.2.38
For every simplicial group $G \in s\mathcal{G}rp$, *there is a natural exact sequence*

$$1 \longrightarrow \Omega G \longrightarrow FG \xrightarrow{p_G} G \longrightarrow \pi_0 G \longrightarrow 1,$$

splitting up into the shorter exact sequences

$$1 \longrightarrow \Omega G \longrightarrow FG \longrightarrow \tilde{G} \longrightarrow 1, \qquad 1 \longrightarrow \tilde{G} \longrightarrow G \longrightarrow \pi_0 G \longrightarrow 1,$$

where

(i) $FG = \ker(G_{+1} \xrightarrow{d_0} G_0)$ *is the* ***cocone*** *of* G, *which is contractible by the long exact sequence as* $G_{+1} \xrightarrow{\simeq} G_0$ *is a deformation retraction.*

(ii) $\tilde{G} = \ker(G \longrightarrow \pi_0 G)$ *is the* ***universal cover*** *for* G*, since by the long exact sequence*

$$\pi_n \tilde{G} = \begin{cases} 1, & n = 0, \\ \pi_n G, & n \geq 1. \end{cases}$$

(iii) $\Omega G = \ker(FG \twoheadrightarrow \tilde{G})$ *is the* ***loop "space"*** *of* G*, since by the long exact sequence*

$$\partial : \pi_{n+1} G \cong \pi_{n+1} \tilde{G} \xrightarrow{\sim} \pi_n \Omega G, \qquad n \geq 0.$$

Proof. We claim that

$$G_0 / d_0 \ker(G_1 \xrightarrow{d_1} G_0) = \operatorname{coker}\left(G_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} G_0 \right) = \pi_0 G, \tag{7.1}$$

where the cokernel is constructed in the category of sets. Therefore let $y, y' \in G_0$.

- If $[y] = [y']$ in the cokernel, then $y' = y d_0(x)$, for some $x \in \ker d_1 \subset G_1$. Setting $z = s_0(y)x$, we see that

  $$d_0(z) = d_0 s_0(y) d_0(x) = y d_0(x), \qquad d_1(z) = y d_1(x) = y,$$

  and thus also $[y] = [y']$ in $\pi_0 G$.

- Vice versa, suppose $y = d_0(z)$ and $y' = d_1(z)$, for some $z \in G_1$. Then setting $x = z^{-1} \cdot s_0 d_1(z)$, we see that

  $$d_0(x) = d_0(z)^{-1} d_1(z), \quad d_1(x) = d_1(z)^{-1} d_1(z) = 1,$$

  and so $y' = y d_0(x)$, proving that also $[y] = [y']$ in the cokernel.

As the relation on G_0 defined by $d_0(\ker d_1)$ is transitive, this proves the assertion.

Now the (noncommutative) snake lemma applied to the map of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \ker(p_G) \arrow[r] \arrow[d, "d_0|"'] & G_{+1} \arrow[r, "p_G"] \arrow[d, "d_0"'] & G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & G_0 \arrow[r, "="] & G_0 \arrow[r] & 1 \arrow[r] & 1, \end{tikzcd}$$

induces the desired short exact sequence. ∎

Proposition 7.2.39
For every $G \in s\mathcal{G}rp$*, the following holds.*

(i) $\pi_n(|G|, 1) \cong \pi_n(G, 1) \cong H_n(N_* G)$*, for all* $n \geq 0$.

Moreover s map $f \in s\mathcal{G}rp(G, H)$ *is a weak equivalence (in* $s\mathcal{S}et$*), if and only if* $H_*(Nf)$ *is an isomorphism.*

(ii) $\pi_n G$ *is an abelian group, for all* $n \geq 1$.

(iii) *The adjoint action of G_0 on G_n induces an action of $\pi_0 G$ on $\pi_n G$, for $n \geq 1$.*

Proof. Using the simplicial identities we have

$$d_i d_0(x) = d_0 d_{i+1}(x) = d_0(1) = 1, \qquad x \in N_n G = \bigcap_{1\leq i\leq n} \ker d_i,\ 0 \leq i \leq n-1,\ n \geq 2,$$

proving that d_0 restricts to a map $N_n G \xrightarrow{d} N_{n-1}G$ and moreover that $d \circ d$ is constant 1.

(i) The first isomorphism follows from Proposition 7.2.20, because every simplicial group is fibrant by Proposition 7.2.35. By definition of the loop space, we have

$$(\Omega^n G)_0 = \ker d_0 \cap \bigcap_{1\leq i\leq n} \ker d_i = N_n G, \quad (\Omega^n G)_1 = \bigcap_{2\leq i\leq n+1} \ker d_i, \qquad n \geq 0.$$

Hence by 7.1 we have

$$\pi_n G = \pi_0 \Omega^n G = H_0(N_* \Omega^n G) = H_n(N_* G), \qquad n \geq 0,$$

where we used the long exact sequence for homotopy groups for the first equation and the long exact sequence for homology groups for the last equation.

As geometric realization preserves finite limits by Proposition 7.2.19, the geometric realization $|G|$ of a simplicial group G is a topological group and multplication by a base point $g \in |G|$ induces an isomorphism $\pi_n(|G|, 1) \xrightarrow{\sim} \pi_n(|G|, g)$. Hence $f \in s\mathcal{G}rp(G, H)$ is a weak equivalence of simplicial sets, if and only if $\pi_*(|f|, 1)$ is an isomorphism, if and only if $H_*(Nf)$ is an isomorphism.

(ii) Let $x, y \in \ker(N_n G \xrightarrow{d} N_{n-1}G) = \bigcap_{0\leq i\leq n} \ker d_i$ and define

$$z = [s_0(x), s_0(y)] \cdot [s_0(x), s_1(y)]^{-1}.$$

Then using the simplicial identities, we get

$$d_i(z) = \begin{cases} [x, y] \cdot [x, s_0 d_0(y)]^{-1} = [x, y] \cdot [x, 1]^{-1} = [x, y], & i = 0, \\ [x, y] \cdot [x, y]^{-1} = 1, & i = 1, \\ [s_0 d_1(x), s_0 d_1(y)] \cdot [s_0 d_1(x), y]^{-1} = [1, 1] \cdot [1, y]^{-1} = 1, & i = 2, \\ [s_0 d_{i-1}(x), s_0 d_{i-1}(y)] \cdot [s_0 d_{i-1}(x), s_1 d_{i-1}(y)]^{-1} = 1, & 3 \leq i \leq n. \end{cases}$$

It follows that $z \in N_{n+1}G$ and $d(z) = [x, y]$, which proves that all commutators in $\pi_n G$ are zero, or equivalently $\pi_n G$ is abelian.

(iii) Conjugation induces an adjoint action

$$\mathrm{ad} : G_0 \longrightarrow s\mathcal{G}rp(G) \xrightarrow{\pi_n} \mathcal{A}b(\pi_n G), \quad x \longmapsto \mathrm{ad}(x) = {\pi_n}^x(\text{-}).$$

Moreover, for every $z \in G_1$, there is a simplicial homotopy ${}^{d_0 z}(\text{-}) \simeq {}^{d_1 z}(\text{-})$, given by

$$\Delta^1_n \times G_n \longrightarrow G_n, \qquad (s, y) \longmapsto {}^{s^*(z)}(y).$$

In particular $G_0 \xrightarrow{\mathrm{ad}} \mathcal{A}b(\pi_n G)$ factors over $\pi_0 G$.

□

Theorem 7.2.40
Let $s\mathcal{C} \xrightarrow{G} s\mathcal{G}rp$ be a cocomplete, finitely complete category over $s\mathcal{G}rp$. Suppose that the composition with the forgetful functor $s\mathcal{C} \xrightarrow{G} s\mathcal{G}rp \xrightarrow{U} s\mathcal{S}et$ is part of an adjunction

$$s\mathcal{C}(F(X), Y) = s\mathcal{S}et(X, UG(Y)).$$

Then the category $s\mathcal{C}$ becomes a model category of simplicial objects, if we define:

(i) *Weak equivalences are maps, that are weak equivalences of underlying simplicial sets.*

(ii) *Fibrations are maps, which are Kan fibrations of simplicial sets.*

(iii) *Cofibrations are maps in $LLP(\mathrm{fib}\,(s\mathcal{C}) \cap w(s\mathcal{C}))$.*

Proposition 7.2.39 (i) implies that weak equivalences in $s\mathcal{C}$ are precisely the π_*-isomorphisms, if we define

$$\pi_* X := H_*(NG(X)) \cong \pi_*(|UG(X)|, 1), \qquad X \in s\mathcal{C}.$$

Moreover every simplicial object in $s\mathcal{C}$ is fibrant.

Proof. As every simplicial group is fibrant by Proposition 7.2.35, so is also every simplicial object in $s\mathcal{C}$, because G is a right adjoint and therefore preserves the terminal object. Hence the identity functor on $s\mathcal{C}$ is a functorial fibrant replacement and we can apply Theorem 7.2.27 to conclude the proof.

□

Remark 7.2.41
The preceding Theorem provides canonical model structures on the categories

(i) *$s\mathcal{G}rp$, $s\mathcal{A}b$, $s\mathcal{R}ing$, $s\mathcal{C}Ring$,*

(ii) *R-$s\mathcal{A}b$, for $R \in s\mathcal{R}ing$,*

(iii) *k-$s\mathcal{A}b$, $(k$-$s\mathcal{A}b)$-$\mathcal{L}ie$, $(k$-$s\mathcal{A}b)$-$\mathcal{A}ss$, $(k$-$\mathcal{A}b)$-$\mathcal{A}ss_1$, ..., for $k \in s\mathcal{C}Ring$.*

Weak equivalences/fibrations/cofibrations in these categories will always refer to this model structure, as we will not need any other model structure on these categories.

7.3 Homotopy limits and colimits

Remark 7.3.1
Let $\mathcal{C}$ be a category with weak equivalences and $I \in \mathcal{C}at$ a small category.

(i) *Then $\mathcal{C}\mathrm{AT}(I, \mathcal{C})$ is a category with weak equivalence by saying that a map $f \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})$ is a weak equivalence, if $f_i \in \mathcal{C}(X_i, Y_i)$ is a weak equivalence, for all $i \in I$.*

(ii) *Suppose $\mathcal{C}$ and $\mathcal{C}\mathrm{AT}(I,\mathcal{C})$ have derived categories, such that $w\mathcal{C} = \gamma^{-1}D(\mathcal{C})^{\times}$, i.e. weak equivalences in $\mathcal{C}$ are precisely the maps becoming isomorphisms in $D(\mathcal{C})$.*

Then by construction the constant functor $\mathcal{C} \xrightarrow{\mathrm{const}} \mathcal{C}\mathrm{AT}(I,\mathcal{C})$ preserves weak equivalences and therefore extends to a functor C inducing a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{const}"] \arrow[d, "\gamma"'] & \mathcal{C}\mathrm{AT}(I,\mathcal{C}) \arrow[d, "\gamma"] \\ D(\mathcal{C}) \arrow[r, "C"] & D(\mathcal{C}\mathrm{AT}(I,\mathcal{C})). \end{tikzcd}$$

In particular $\mathbb{R}\,\mathrm{const} = C = \mathbb{L}\,\mathrm{const}$ *is the right and left derived of* const.

Definition 7.3.2
Let $\mathcal{C}$ be a category with weak equivalences and $I \in \mathcal{C}at$ a small category.

(i) *The* ***homotopy limit*** *is defined as the total right derived functor (if it exists)*

$$\mathbb{R}\lim_I : D(\mathcal{C}\mathrm{AT}(I,\mathcal{C})) \longrightarrow D(\mathcal{C}).$$

(ii) *The* ***homotopy colimit*** *is defined as the total left derived functor (if it exists)*

$$\mathbb{L}\operatorname*{colim}_I : D(\mathcal{C}\mathrm{AT}(I,\mathcal{C})) \longrightarrow D(\mathcal{C}).$$

Remark 7.3.3
The existence of homotopy (co-)limits is quite hard to achieve in general.

(i) *In the context of model categories, the difficulty lies in constructing a suitable model structure on functor categories. While this is easy for homotopy colimits in κ-cofibrantly generated model categories (using the projective model structure[3]), it is comparably hard for homotopy limits in this situation and requires more assumptions on the category of question (to construct the analogous injective model structure).*

(ii) *However, in the context of simplicial objects, one can use the projective model structure to construct homotopy limits, too.*

7.3.1 The projective and injective model structures

Definition 7.3.4
Let $\mathcal{C}$ be a model category and $I \in \mathcal{C}at$.

(i) *A map $f \in \mathcal{C}\mathrm{AT}(I,\mathcal{C})(X,Y)$ is a* ***projective weak equivalence/fibration****, if $f_i \in \mathcal{C}(X_i, Y_i)$ is a weak equivalence/fibration, for all $i \in I$.*

Moreover f is a ***projective cofibration****, if f has the left lifting property to all trivial projective fibrations.*

If these classes of maps form a model structure on $\mathcal{C}\mathrm{AT}(I,\mathcal{C})$, we call it the ***projective model structure****.*

[3]See Definition 7.3.4.

(ii) *A map $f \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})(X, Y)$ is an* ***injective weak equivalence/cofibration****, if $f_i \in \mathcal{C}(X_i, Y_i)$ is a weak equivalence/cofibration, for all $i \in I$.*

Moreover f is an ***injective fibration****, if f has the right lifting property to all trivial projective cofibrations.*

If these classes of maps form a model structure on $\mathcal{C}\mathrm{AT}(I, \mathcal{C})$, we call it the ***injective model structure****.*

Remark 7.3.5
Let $\mathcal{C}$ be a model category and $I \in \mathcal{C}at$.

(i) *Supposing that $\mathcal{C}$ has functorial I-indexed limits, there is an adjunction*

$$\mathcal{C}\mathrm{AT}(I, \mathcal{C})(\mathrm{const}\, X, Y) = \mathcal{C}(X, \lim_I Y),$$

where $\mathrm{const}\, X$ is the constant functor, sending every map to the identity on X.

By definition the functor const *preserves weak equivalences and injective cofibrations.*

In particular by Theorem 7.1.26 the total derived functors exist and induces a Quillen adjunction

$$D(\mathcal{C}\mathrm{AT}(I, \mathcal{C}))(\mathbb{L}\,\mathrm{const}\, X, Y) = D(\mathcal{C})(X, \mathbb{R}\lim_I Y),$$

provided that the injective model structure on $\mathcal{C}\mathrm{AT}(I, \mathcal{C})$ exists.

(ii) *Dually supposing that $\mathcal{C}$ has functorial I-indexed colimits, there is an adjunction*

$$\mathcal{C}(\operatorname{colim}_I X, Y) = \mathcal{C}\mathrm{AT}(I, \mathcal{C})(X, \mathrm{const}\, Y).$$

By definition the functor const *preserves weak equivalences and projective fibrations.*

In particular by Theorem 7.1.26 the total derived functors exist and induces a Quillen adjunction

$$D(\mathcal{C})(\mathbb{L}\operatorname{colim}_I X, Y) = D(\mathcal{C}\mathrm{AT}(I, \mathcal{C}))(X, \mathbb{R}\,\mathrm{const}\, Y).$$

provided that the projective model structure on $\mathcal{C}\mathrm{AT}(I, \mathcal{C})$ exists.

Proposition 7.3.6
Let $\mathcal{C}$ be a complete model category and $I \in \mathcal{C}at$.

Then every (trivial) projective cofibration is a (trivial) injective cofibration.

Proof. Consider the functor $D\mathrm{Obj}(I) \xrightarrow{\varepsilon_I} I$. Then

$$(\varepsilon_I)^*(\varepsilon_I)_*(X)_j = \prod_{i \in I} X_i^{I(j,i)}, \qquad X \in \mathcal{C}\mathrm{AT}(D\mathrm{Obj}(I), \mathcal{C}) = \mathcal{C}^{\mathrm{Obj}(i)}.$$

Since (trivial) fibrations are stable under arbitrary products, as they are the right morphisms of a weak factorization system, the functor $(\varepsilon_I)^*(\varepsilon_I)_*$ preserves (trivial) fibrations, which we will denote by R. In other words

$$((\varepsilon_I)^*)^{-1}R \supset (\varepsilon_I)_*(R).$$

Applying LLP, which reverses inclusions, yields

$$((\varepsilon_I)^*)^{-1}R \subset LLP((\varepsilon_I)_*(R)) = ((\varepsilon_I)^*)^{-1}LLP(R),$$

where the equality follows from Corollary 7.1.9. By definition, the left is the class of projective and the right is the class of injective (trivial) cofibrations on $\mathcal{C}\mathrm{AT}(I,\mathcal{C})$.

□

Theorem 7.3.7
Let $\mathcal{C}$ be a cocomplete κ-cofibrantly generated model category.

Then the projective model structure exists, for every small category $I \in \mathcal{C}at$.

Moreoever it is λ-cofibrantly generated, where λ is a cardinal number with cardinality greater than κ and I.

If $\mathcal{C} = s\mathcal{D}$ is a model category of simplicial objects, so is also $\mathcal{C}\mathrm{AT}(I,\mathcal{C}) = \mathcal{C}\mathrm{AT}(I,s\mathcal{D}) = s\mathcal{C}\mathrm{AT}(I,\mathcal{D})$.

Proof. We will use the adjunction

$$\mathcal{C}\mathrm{AT}(I,\mathcal{C})((\varepsilon_I)_!X,Y) = \mathcal{C}\mathrm{AT}(D\mathrm{Obj}(I),\mathcal{C})(X,(\varepsilon_I)^*Y) = \mathcal{C}^{\mathrm{Obj}(I)}(X,(Y_i)_{i\in I})$$

of Remark 7.3.8 to construct the model structure. The category $\mathcal{C}\mathrm{AT}(D\mathrm{Obj}(I),\mathcal{C}) = \mathcal{C}^{\mathrm{Obj}I}$ is a λ-cofibrantly generated model category with

$$w(\mathcal{C}^{\mathrm{Obj}(I)}) = (w\mathcal{C})^{\mathrm{Obj}(I)}, \qquad \mathrm{fib}\,(\mathcal{C}^{\mathrm{Obj}(I)}) = (\mathrm{fib}\,\mathcal{C})^{\mathrm{Obj}(I)}, \qquad \mathrm{cof}\,(\mathcal{C}^{\mathrm{Obj}(I)}) = (\mathrm{cof}\,\mathcal{C})^{\mathrm{Obj}(I)}.$$

Because colimits commute, the functor $(\varepsilon_I)^*$ preserves arbitrary colimits. From the description of $(\varepsilon_I)_!$ we see that the functor $(\varepsilon_I)_!(\varepsilon_I)^*$ pointwise takes a morphism to a copower. Therefore it preserves pointwise trivial cofibrations, since trivial cofibrations in a model category are closed under coproducts. In particular we can apply Corollary 7.1.30 and see that $\mathcal{C}\mathrm{AT}(I,\mathcal{C})$ is a λ-cofibrantly generated model category. By construction it is the projective model structure.

In the context of model categories of simplicial objects, we apply Corollary 7.2.26 instead.

□

Remark 7.3.8
Let $\mathcal{C}$ be a category with arbitrary coproducts and $I \in \mathcal{C}at$. Let $D\mathrm{Obj}(I)$ denote the discrete category with objects I, i.e. the identities are the only morphisms.

Then there is a functor $D\mathrm{Obj}(I) \xrightarrow{\varepsilon_I} I$ inducing an adjunction

$$\mathcal{C}\mathrm{AT}(I,\mathcal{C})((\varepsilon_I)_!X,Y) = \mathcal{C}\mathrm{AT}(D\mathrm{Obj}(I),\mathcal{C})(X,(\varepsilon_I)^*Y) = \mathcal{C}^{\mathrm{Obj}(I)}(X,(Y_i)_{i\in I}),$$

where by Yoneda's lemma $(\varepsilon_I)_!X = \coprod_{i\in I}{}^{I(i,\text{-})}X_i$, for $X \in \mathcal{C}^{\mathrm{Obj}(I)}$.

(i) *Remark 7.2.31 provides a canonical resolution functor*

$$E_\bullet(I) : \mathcal{C}\mathrm{AT}(I,\mathcal{C}) \longrightarrow \mathcal{C}\mathrm{AT}(I,\mathcal{C}), \qquad X \longmapsto E_\bullet(I)(X),$$

such that $E_\bullet(I)(X)_i \xrightarrow{d_0} X_i$ is a simplicial homotopy equivalence, for all $i \in I$.

(ii) *For* $* \in \mathcal{C}\mathrm{AT}(I, \mathcal{S}et)$ *by Corollary 7.2.32 the map*

$$B_\bullet(I/\text{-}) = E_\bullet(I)(*) \longrightarrow *$$

is a natural projective cofibrant replacement in $\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)$*.*

Theorem 7.3.9
The injective model structure on $\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)$ *exists, for every* $I \in \mathcal{C}at$*.*

Proof. See [Lur09] A.3.3 or [GJ09] Prop. VIII.2.4.

□

7.3.2 Homotopy limits in model categories of simplicial objects

Proposition 7.3.10
For $I \in \mathcal{C}at$ *and every complete and cocomplete category* $\mathcal{C}$ *there are adjunctions*

$$\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)(S, \underline{s\mathcal{C}}(X,Y)) = \mathcal{C}\mathrm{AT}(I, s\mathcal{C})({}^{S}X, Y) = s\mathcal{C}(X, \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(S,Y)),$$

where

$$\underline{s\mathcal{C}}(X,Y) := \underline{s\mathcal{C}}(X,\text{-}) \circ Y, \quad {}^{S}X : i \longmapsto {}^{S_i}X, \quad \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(S,Y) := \int_{i\in I} \underline{s\mathcal{S}et}(S_i, Y_i).$$

Dually there are also adjunctions

$$\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, s\mathcal{S}et)(S, \underline{s\mathcal{C}}(X,Y)) = s\mathcal{C}(S \otimes_I X, Y) = \mathcal{C}\mathrm{AT}(I, s\mathcal{C})(X, \underline{s\mathcal{S}et}(S,Y)),$$

where

$$\underline{s\mathcal{C}}(X,Y) := \underline{s\mathcal{C}}(\text{-},Y) \circ X, \qquad S \otimes_I X = \int^{i\in I} {}^{S_i}X_i, \qquad \underline{s\mathcal{S}et}(S,Y) := \underline{s\mathcal{S}et}(\text{-},Y) \circ S.$$

Proof. Using Proposition 7.2.8 this follows from the Yoneda lemmas and the fact that hom-functors commute with limits/ends.

□

Proposition 7.3.11
Let $s\mathcal{C}$ *be a complete and cocomplete model category of simplicial objects and* $I \in \mathcal{C}at$*.*

(i) *Given a projective cofibration* $s \in \mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)(S, S')$*, a cofibration* $x \in s\mathcal{C}(X,Y)$ *and a projective fibration* $y \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C})(Y, Y')$*, the following holds.*

   a) $(x^*, y_*) : \underline{s\mathcal{C}}(X', Y) \longrightarrow \underline{s\mathcal{C}}(X,Y) \times_{\underline{s\mathcal{C}}(X,Y')} \underline{s\mathcal{C}}(X', Y')$ *is a projective fibration, which is a (projective) weak equivalence, if* x *or* y *are weak equivalences.*

   b) $({}^{s}\mathrm{id}) \cup ({}^{\mathrm{id}}x) : ({}^{S}X') +_{({}^{S}X)} ({}^{S'}X) \longrightarrow {}^{S'}X'$ *is a projective cofibration, which is a (projective) weak equivalence, if* s *or* x *are weak equivalences.*

c) $\underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(S', Y) \longrightarrow \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(S, Y) \times_{\underline{\mathcal{C}\mathrm{AT}(I,s\mathcal{S}et)}(S,Y')} \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(S', Y')$ *is a fibration,*

which is a weak equivalence, if s or y are weak equivalences.

(ii) *Dually given injective cofibrations $s \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, s\mathcal{S}et)(S, S')$ and $x \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C})(X, X')$ and a fibration $y \in s\mathcal{C}(Y, Y')$, the following holds.*

a) $(x^*, y_*) : \underline{s\mathcal{C}}(X', Y) \longrightarrow \underline{s\mathcal{C}}(X, Y) \times_{\underline{s\mathcal{C}}(X,Y')} \underline{s\mathcal{C}}(X', Y')$ *is a projective fibration,*

which is a (projective) weak equivalence, if x or y are weak equivalences.

b) $(s \otimes_I \mathrm{id}) \cup (\mathrm{id} \otimes_I x) : (S \otimes_I X') +_{(S\otimes_I X)} (S' \otimes_I X) \longrightarrow S' \otimes_I X'$ *is a cofibration,*

which is a weak equivalence, if s or x are weak equivalences.

c) $(s^*, y_*) : \underline{s\mathcal{S}et}(X', Y) \longrightarrow \underline{s\mathcal{S}et}(X, Y) \times_{\underline{s\mathcal{S}et}(X,Y')} \underline{s\mathcal{S}et}(X', Y')$ *is a projective fibration,*

which is a (projective) weak equivalence, if s or y are weak equivalences.

Since every projective (trivial) cofibration is also an injective (trivial) cofibration by Proposition 7.3.6, the same holds if we replace the word "injective" by "projective". Note that we do not require existence of the injective model structure here.

Proof. We can write down exactly the same proof for (i) and (ii):
By definition of $\underline{s\mathcal{C}}(\text{-}, \text{-})$ statement a) follows from Propostion 7.2.24 (ii) for the model category of simplicial objects $s\mathcal{C}$. By the same arguments as in Propostion 7.2.24 statements b) and c) are equivalent to a).

□

Proposition 7.3.12
Let $s\mathcal{C}$ be a complete and cocomplete, κ-cofibrantly generated model category of simplicial objects. Let $I \in \mathcal{C}at$ and $V \xrightarrow{\simeq} *$ be a projective cofibrant replacement in $\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)$.
Then there is a Quillen adjunction

$$\mathcal{C}\mathrm{AT}(I, s\mathcal{C})({}^V\mathrm{const}(X), Y) = s\mathcal{C}(X, \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(V, Y)).$$

As there is a natural isomorphism $\mathbb{L}({}^V\mathrm{const}) \xrightarrow{\sim} \mathbb{L}\,\mathrm{const}$, we get an isomorphism

$$\mathbb{R}\lim_I \xrightarrow{\sim} \mathbb{R}\underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(V, \text{-})$$

proving in particular the existence of the left object.

Proof. By Proposition 7.3.11 (i) b) the functor ${}^V\mathrm{const}$ preserves (trivial) cofibrations and thus the adjunction is Quillen by Theorem 7.1.26. Moreover the left derived functor $\mathbb{L}\,\mathrm{const}$ is given by

$$\mathbb{L}({}^V\mathrm{const})(X) = {}^VX_c,$$

where $X_c \xrightarrow{\simeq} X$ is a cofibrant replacement for X. Using Brown's lemma and Proposition 7.2.24, it follows that ${}^{V_i}X \longrightarrow X$ is a weak equivalence, for every cofibrant $X \in s\mathcal{C}$ and every $i \in I$. Hence in the derived category $D(s\mathcal{C})$ we have

$$\mathbb{L}({}^V\mathrm{const})(X) = {}^VX_c \xrightarrow{\sim} X_c \xleftarrow{\sim} X = \mathbb{L}\,\mathrm{const}(X), \qquad X \in s\mathcal{C},$$

proving that naturally $\mathbb{L}({}^V\mathrm{const}) \xrightarrow{\sim} \mathbb{L}\,\mathrm{const}$. Again by Quillen's adjoint functor Theorem 7.1.26 the functor $\mathbb{R}\underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(V, \text{-})$ is right adjoint to $\mathbb{L}^V \cong \mathbb{L}\,\mathrm{const}$. As also $\mathbb{R}\lim_I$ is right adjoint to $\mathbb{L}\,\mathrm{const}$, they must coincide up to natural isomorphism.

□

7.3.3 Simplicial models for homotopy (co-)limits

Proposition 7.3.13
Let $s\mathcal{C}$ be a κ-cofibrantly generated model category of simplicial objects and $X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C})$ with $I \in \mathcal{C}at$. Let $V \in \mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)$ and $W \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, s\mathcal{S}et)$ be projective cofibrant replacements of the constant functor $*$.

(i) *If $X_i \in s\mathcal{C}$ is fibrant, for all $i \in I$, then $\mathbb{R}\lim_I X = \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(V, X)$ in $D(s\mathcal{C})$.*

If $X \xrightarrow{\simeq} F(X)$ is a funtorial fibrant replacement in $s\mathcal{C}$, then

$$\mathbb{R}\lim_I X = \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(V, F(X)), \qquad X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C}).$$

(ii) *If $X_i \in s\mathcal{C}$ is cofibrant, for all $i \in I$, then $W \otimes_I X = \mathbb{L}\,\mathrm{colim}_I X$ in $D(s\mathcal{C})$.*

If $E(X) \xrightarrow{\simeq} X$ is a funtorial cofibrant replacement in $s\mathcal{C}$, then

$$W \otimes_I E(X) = \mathbb{L}\operatorname*{colim}_I X, \qquad X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C}).$$

Proof.

(i) As X is projective fibrant, using Proposition 7.3.12 we get

$$\mathbb{R}\lim_I X = \mathbb{R}\underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(V, \text{-})(X) = \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(V, X).$$

If $X \xrightarrow{\simeq} F(X)$ is a functorial fibrant replacement, for $X \in s\mathcal{C}$, it induces a functorial projective fibrant replacement on $\mathcal{C}\mathrm{AT}(I, s\mathcal{C})$, which proves (i).

(ii) As the injective model structure on $\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)$ exists by Theorem 7.3.9, the map $W \xrightarrow{\simeq} *$ can be factored as in Brown's Lemma 7.1.23 (i). So Proposition 7.3.11 (ii) b) yields a weak equivalence

$$W \otimes_I X \xrightarrow{\simeq} * \otimes_I X = \operatorname*{colim}_I X,$$

because $X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C})$ is injective cofibrant. If $E(X) \xrightarrow{\simeq} X$ is a funtorial cofibrant replacement, for $X \in s\mathcal{C}$, it induces a functorial injective cofibrant replacement on $\mathcal{C}\mathrm{AT}(I, s\mathcal{C})$, which proves (ii).

□

Using the canonical cofibrant replacements $B(I/\text{-}) \xrightarrow{\simeq} *$ and dually $B(\text{-}/I) \xrightarrow{\simeq} *$ of Remark 7.3.8, we can define canonical simplicial models.

Definition 7.3.14
Let $\mathcal{C}$ be a category with arbitrary products and coproducts and $I \in \mathcal{C}at$.

(i) *The **simplicial homotopy limit** of $Y \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C})$ is defined as*

$$\operatorname*{holim}_I X := \underline{\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)}(B(I/\text{-}), X) \in s\mathcal{C}.$$

(ii) *The **simplicial homotopy colimit** of $Y \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C})$ is defined as*

$$\operatorname*{hocolim}_I X := B(\text{-}/I) \otimes_I X \in s\mathcal{C}.$$

Corollary 7.3.15
Let $I \in \mathcal{C}at$.

(i) *Since every simplicial set is cofibrant, in the derived category $D(s\mathcal{S}et)$ we have*

$$\mathbb{L}\operatorname*{colim}_I X = \operatorname*{hocolim}_I X, \qquad X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{S}et).$$

(ii) *Let $s\mathcal{C}$ be a model category of simplicial objects, induced by an adjunction (cf. Theorem 7.2.40)*

$$s\mathcal{C}(F(X), Y) = s\mathcal{S}et(X, UG(Y)), \qquad s\mathcal{C} \xrightarrow{G} s\mathcal{G}rp \xrightarrow{U} s\mathcal{S}et.$$

Since every simplicial object in $s\mathcal{C}$ is fibrant, in the derived category $D(s\mathcal{C})$ we have

$$\mathbb{R}\lim_I X = \operatorname*{holim}_I X, \qquad X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C}).$$

7.3.4 Homotopy pullbacks and pushouts

Proposition 7.3.16
Let P be the category with three objects a, b, c and the only non-trivial morphisms $b \leftarrow a \rightarrow c$.

Then there is a projective cofibrant replacement of $* \in \mathcal{C}\mathrm{AT}(P^{\mathrm{op}}, s\mathcal{S}et)$

$$W_\bullet(P) = P(\text{-}, b) +_{P(\text{-},a)} {}^{\Delta^1}P(\text{-}, a) +_{P(\text{-},a)} P(\text{-}, c) \longrightarrow *.$$

Proof. We have

$$\begin{aligned} W_n(P) &= P(\text{-}, b) +_{P(\text{-},a)} {}^{\Delta^1_n}P(\text{-}, a) +_{P(\text{-},a)} P(\text{-}, c), \qquad n \geq 0 \\ &= P(\text{-}, b) +_{P(\text{-},a)} {}^{n+2}P(\text{-}, a) +_{P(\text{-},a)} P(\text{-}, c) \\ &= P(\text{-}, b) + {}^{n}P(\text{-}, a) + P(\text{-}, c), \end{aligned}$$

which is free in $\mathcal{C}\mathrm{AT}(P^{\mathrm{op}}, \mathcal{S}et)$, for all $n \geq 0$. Moreover the degeneracies correspond to inclusion maps under this isomorphism. Hence $W_\bullet(P)$ is almost free in $\mathcal{C}\mathrm{AT}(P^{\mathrm{op}}, s\mathcal{S}et)$ and thus projective cofibrant by Corollary 7.2.30. As the three objects

$$W_\bullet(P)(a) \cong \Delta^1, \qquad W_\bullet(P)(b) = * = W_\bullet(P)(c)$$

are contractible, $W_\bullet(P) \longrightarrow *$ is a dimensionwise homotopy equivalence. □

Definition 7.3.17
Let $\mathcal{C}$ be a category with arbitrary finite products and coproducts.

(i) *The* ***homotopy pullback*** *of an $X = (B \to A \leftarrow C) \in \mathcal{C}\mathrm{AT}(P, s\mathcal{C})$ is defined as*

$$B \times_A^h C = \underline{\mathcal{C}\mathrm{AT}(P^{\mathrm{op}}, s\mathcal{S}et)}(W_\bullet(P), X) = B \times_A \underline{s\mathcal{S}et}(\Delta^1, A) \times_A C.$$

(ii) *The* ***homotopy pushout*** *of an $X = (B \leftarrow A \to C) \in \mathcal{C}\mathrm{AT}(P^{\mathrm{op}}, s\mathcal{C})$ is defined as*

$$B +_A^h C = W_\bullet(P) \otimes_P X = B +_A {}^{\Delta^1}A +_A C.$$

7.3.5 Bisimplicial objects

Theorem 7.3.18 (Quillen)
For a bisimplicial group $G \in ss\mathcal{G}rp$ there are two converging spectral sequences

(i) $E^2_{p,q} = \pi^h_p \pi^v_q G \quad \Rightarrow \quad \pi_{p+q}\mathrm{diag}G,$

(ii) $E^2_{p,q} = \pi^v_p \pi^h_q G \quad \Rightarrow \quad \pi_{p+q}\mathrm{diag}G,$

where π^h_* resp. π^v_* means horizontal resp. vertical homotopy groups.

Proof. See [Qui66]. ∎

Corollary 7.3.19
Let $s\mathcal{C}$ be a model category of simplicial objects, induced by an adjunction (cf. Theorem 7.2.40)

$$s\mathcal{C}(F(X), Y) = s\mathcal{S}et(X, UG(Y)), \qquad s\mathcal{C} \xrightarrow{G} s\mathcal{G}rp \xrightarrow{U} s\mathcal{S}et.$$

Let $f \in ss\mathcal{C}(X, Y)$, such that $f_{p,\bullet} : \pi_* X_{p,\bullet} \longrightarrow \pi_* Y_{p,\bullet}$ is c-connected, for all $p \geq 0$.
Then $\mathrm{diag}f$ is c-connected.

Proposition 7.3.20
There is a natural weak equivalence of simplicial sets

$$B\phi^n : B\Delta/\underline{n} \xrightarrow{\simeq} B\underline{n} = \Delta^n, \qquad n \geq 0,$$

which is induced by the natural functor

$$\phi^n : \Delta/\underline{n} \xrightarrow{\simeq} \underline{n}, \quad (\underline{m} \xrightarrow{\alpha} \underline{n}) \longmapsto \alpha(m), \qquad n \geq 0.$$

Proof. Let $\underline{m} \xrightarrow{f} \underline{m}' \xrightarrow{\alpha'} \underline{n}$ be a map in $\Delta/\underline{n}$ i.e. $\alpha' f = \alpha$. Then we have $f(m) \leq m'$ and hence $\alpha(m) = \alpha' f(m) \leq \alpha'(m')$ and clearly $\alpha(m) = \alpha'(m)$, if $f = \mathrm{id}$. This shows ϕ^n is a functor. By construction it is natural in $\underline{n}$. Next consider the functors ϕ^n defined by

$$\psi^n : \underline{n} \longrightarrow \Delta/\underline{n}, \quad k \longmapsto (\underline{k} \xhookrightarrow{\iota_{k,n}} \underline{n}),$$

where $\iota_{k,n}$ is the canonical inclusion. The image of a “morphism” $k \le \ell$ in $\underline{n}$ is defined as the canonical inclusion $\underline{k} \xhookrightarrow{\iota_{k,\ell}} \underline{\ell}$.

By construction $\iota_{k,n} = \iota_{\ell,n} \circ \iota_{k,\ell}$, so $\iota_{k,\ell}$ is indeed a morphism in $\Delta/\underline{n}$. By construction $\phi^n \circ \psi^n = \mathrm{id}_{\underline{n}}$ and for any $(\underline{m} \xrightarrow{\alpha} \underline{n}) \in \Delta/\underline{n}$ the restriction $\underline{m} \xrightarrow{\tilde{\alpha}} \underline{\alpha(m)}$ of α defines a homomorphism $(\underline{m} \xrightarrow{\alpha} \underline{n}) \xrightarrow{\tilde{\alpha}} (\underline{\alpha(m)} \xhookrightarrow{\iota_{\alpha(m),\underline{n}}} \underline{n})$. This defines a natural transformation $\eta : \mathrm{id}_{\Delta/\underline{n}} \longrightarrow \psi^n \circ \phi^n$:

$$\begin{tikzcd}
\underline{m} \arrow[r, two heads, "\tilde{\alpha}"] \arrow[d, "f"'] & \underline{\alpha(m)} \arrow[d, "\iota_{\alpha(m),\alpha'(m')}"'] \arrow[dr, hook, "\iota_{\alpha(m),n}"] & \\
 & & \underline{n} \\
\underline{m'} \arrow[r, two heads, "\tilde{\alpha}'"] & \underline{\alpha'(m')} \arrow[ur, hook, "\iota_{\alpha'(m'),n}"'] &
\end{tikzcd}$$

Indeed the left square commutes, because the right triangle commutes and the outer does by assumption on f being a morphism $\alpha \xrightarrow{f} \alpha'$. In particular we get an adjunction

$$\Delta/\underline{n}(\phi^n(k), \alpha) = \underline{n}(k, \psi^n(\alpha)), \qquad n \ge 0,$$

inducing a homotopy equivalence between $B\Delta/\underline{n}$ and $\underline{n}$. Note that in contrast to ϕ^n the functor ψ^n is not(!) natural in $n \ge 0$.

□

Corollary 7.3.21
Let $s\mathcal{C}$ be a κ-cofibrantly generated model category of simplicial objects.

(i) *For injective fibrant $X \in \mathcal{C}\mathrm{AT}(\Delta, s\mathcal{C}) = cs\mathcal{C}$ the natural weak equivalence ϕ induces*

$$\phi^* : \underline{\mathcal{C}\mathrm{AT}(\Delta, s\mathcal{S}et)}(\Delta, X) \xrightarrow{\simeq} \underline{\mathcal{C}\mathrm{AT}(\Delta, s\mathcal{S}et)}(B(\Delta/\text{-}), X) = \operatorname{holim}_{\Delta} X.$$

(ii) *For injective cofibrant $X \in \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}, s\mathcal{C}) = ss\mathcal{C}$ the natural weak equivalence ϕ induces*

$$\phi_* : \operatorname{hocolim}_{\Delta^{\mathrm{op}}} X = B(\text{-}/\Delta^{\mathrm{op}}) \otimes_{\Delta^{\mathrm{op}}} X \xrightarrow{\simeq} \Delta \otimes_{\Delta^{\mathrm{op}}} X = \mathrm{diag} X.$$

Proof. By Proposition 7.3.20 the map $B(\text{-}/I) \xrightarrow{d_0} *$ is a weak equivalence of injective cofibrant functors in $\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, s\mathcal{S}et)$. As the injective model structure on $\mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)$ exists by Theorem 7.3.9, it can be factored as in Brown’s Lemma 7.1.23. So Proposition 7.2.24 yields the desired weak equivalences.

□

Corollary 7.3.22
Let $f \in ss\mathcal{S}et(X, Y)$, such that $f_{p,\bullet} : X_{p,\bullet} \xrightarrow{\simeq} Y_{p,\bullet}$ is a weak equivalence, for all $p \ge 0$. Then $\mathrm{diag} f$ *is a weak equivalence.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
\operatorname{hocolim}_{\Delta^{\mathrm{op}}} X \arrow[r, "\simeq"] \arrow[d, "\operatorname{hocolim} f"'] & \operatorname{diag} X \arrow[d, "\operatorname{diag} f"] \\
\operatorname{hocolim}_{\Delta^{\mathrm{op}}} Y \arrow[r, "\simeq"] & \operatorname{diag} Y.
\end{tikzcd}$$

As the horizontal maps are weak equivalences by Corollary 7.3.21 it suffices to check that the left vertical map is a weak equivalence. Since f is a weak equivalence of injective cofibrants in $\mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}, s\mathcal{S}et)$ this follows from Proposition 7.3.11 (ii) b) by using Brown's Lemma 7.1.23 and the existence of the injective model structure by Theorem 7.3.9.

□

Proposition 7.3.23
Let $s\mathcal{A}$ be a model category of simplicial objects, induced by an adjunction (cf. Theorem 7.2.40)

$$s\mathcal{A}(F(X), Y) = s\mathcal{S}et(X, UG(Y)), \qquad s\mathcal{A} \xrightarrow{G} s\mathcal{A}b \xrightarrow{U} s\mathcal{S}et.$$

Suppose that $\mathcal{A}$ is an abelian category and that $s\mathcal{A} \xrightarrow{G} s\mathcal{A}b$ preserves filtered colimits.

Then, for every $I \in \mathcal{C}at$ and $X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{A})$, there is a converging spectral sequence

$$E^2_{p,q} = \pi_p \operatorname*{hocolim}_I \pi_q X \quad \Rightarrow \quad \pi_{p+q} \operatorname*{hocolim}_I X.$$

Proof. By definition we have

$$\operatorname*{hocolim}_I X = B(\text{-}/I) \otimes_I X = \operatorname{diag} \int^{i \in I} {}^{B_\bullet(i/I)}(X_i)_\bullet,$$

and moreover

$$\int^{i \in I} {}^{B_p(i/I)}(X_i)_\bullet = \coprod_{i_0, \ldots, i_n \in I} {}^{I(i_0,i_1) \times \ldots \times I(i_{p-1},i_p)}(X_{i_0})_\bullet, \qquad p \geq 0.$$

By assumption the functor $s\mathcal{A} \xrightarrow{G} s\mathcal{G}rp \xrightarrow{\pi_q} \mathcal{G}rp$ preserves filtered colimits and moreover arbitrary products. Hence it preserves arbitrary coproducts, as $\mathcal{A}$ is an abelian category and thus finite products and coproducts coincide. It follows that the first of Quillen's spectral sequences (cf. Theorem 7.3.18) is precisely

$$E^2_{p,q} = \pi_p \operatorname*{hocolim}_I \pi_q G(X) \quad \Rightarrow \quad \pi_{p+q} \operatorname*{hocolim}_I G(X).$$

□

Corollary 7.3.24
In the situation of Proposition 7.3.23, we have

$$\operatorname*{holim}_I X = \mathbb{R} \lim_I X, \qquad \mathbb{L} \operatorname*{colim}_I X = \operatorname*{hocolim}_I X, \qquad X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{A}), \quad I \in \mathcal{C}at.$$

Proof. The first identity was already proven in Corollary 7.3.15. For the second identity let $E_\bullet \xrightarrow{\simeq} \mathrm{id}_{s\mathcal{A}}$ be a functorial cofibrant resolution like in Corollary 7.2.32. Then it induces a functorial injective cofibrant resolution on $\mathcal{C}\mathrm{AT}(I, s\mathcal{A})$ and we get a map

$$\mathbb{L}\operatorname*{colim}_I X = B(\text{-}/I) \otimes_I E_\bullet(X) = \operatorname*{hocolim}_I E_\bullet(X) \longrightarrow \operatorname*{hocolim}_I X.$$

where the first equality holds by Proposition 7.3.13. Now the right map induces an isomorphism on the second page of the spectral sequence for homotopy colimits of Proposition 7.3.23. In particular it is a weak equivalence of simplicial $\mathcal{A}$-objects.

□

7.3.6 Free resolutions induced by totally final functors

Remark 7.3.25
For every functor $f \in \mathcal{C}at(I, J)$, the inclusion the nerve of the comma category j/f is given by

$$B_n(j/f) = \coprod_{i_0,\dots,i_n \in I} J(j, f(i_0)) \times I(i_0, i_1) \times \dots \times I(i_{n-1}, i_n) \in \mathcal{C}\mathrm{AT}(J^{\mathrm{op}}, \mathcal{S}et), \qquad n \geq 0.$$

In particular we have $B_\bullet(\text{-}/f) \cong f_! B_\bullet(\text{-}/I)$ and the inclusion $(\text{-}/f) \longrightarrow (\text{-}/J)$ induces a natural map

$$\operatorname*{hocolim}_I f^*X = B_\bullet(\text{-}/I) \otimes_I f^*X \cong B_\bullet(\text{-}/f) \otimes_J X \longrightarrow B_\bullet(\text{-}/J) \otimes_J X = \operatorname*{hocolim}_J X,$$

where the first isomorphism is induced by the (co-)Yoneda lemma.

Definition 7.3.26
Given a functor $f \in \mathcal{C}at(I, J)$.

(i) *f is called **totally final**, if for all $j \in J$, the comma category j/f is contractible*[4].

(ii) *f is called **totally cofinal**, if for all $j \in J$, the comma category f/j is contractible.*

Remark 7.3.27
By definition, every totally final functor $f \in \mathcal{C}at(I, J)$ induces a cofibrant resolution $B_\bullet(\text{-}/f) \xrightarrow{\simeq} *$ of functors in $\mathcal{C}\mathrm{AT}(J^{\mathrm{op}}, s\mathcal{S}et)$.

In particular, for a cofibrantly generated model category of simplicial objects $s\mathcal{C}$, by Proposition 7.3.13 we get a natural weak equivalence

$$\operatorname*{hocolim}_I f^*X = B_\bullet(\text{-}/f) \otimes_J X \xrightarrow{\simeq} B_\bullet(\text{-}/J) \otimes_J X = \operatorname*{hocolim}_J X, \qquad X \in \mathcal{C}\mathrm{AT}(I, s\mathcal{C}).$$

Remark 7.3.28
Consider an adjunction $I(f(j), i) = J(j, g(i))$.

[4] It is called **final**, we one only require non-empty and connected instead of contractible.

(i) *The right adjoint g is totally final, because there is a contraction on $B_\bullet(i/g)$ induced by the natural transformation $(i \xrightarrow{\eta_i} gf(i)) \longrightarrow \mathrm{id}_{i/g}$ given by*

$$\begin{tikzcd}
i \arrow[r, equal] \arrow[d, "\eta_i"'] & i \arrow[r, equal] \arrow[d, "gf(f)\eta_i = "', "\eta_{g(j)}f"] & i \arrow[d, "f"] \\
gf(i) \arrow[r, "gf(f)"'] & gfg(j) \arrow[r, "g(\varepsilon_j)"'] & g(j).
\end{tikzcd}$$

(ii) *The left adjoint f is totally cofinal, there is a contraction on $B_\bullet(f/j)$ induced by the natural transformation $\mathrm{id}_{f/j} \longrightarrow (fg(j) \xrightarrow{\varepsilon_j} j)$ given by*

$$\begin{tikzcd}
f(i) \arrow[r, "f(\eta_i)"] \arrow[d, "f"'] & fgf(i) \arrow[r, "fg(f)"] \arrow[d, "f\varepsilon_{f(i)} = "', "\varepsilon_j fg(f)"] & fg(j) \arrow[d, "\varepsilon_j"] \\
j \arrow[r, equal] & j \arrow[r, equal] & j.
\end{tikzcd}$$

7.3.7 Homotopy colimits being colimits

Proposition 7.3.29
Let $I \in \mathcal{C}at$ and $F \in \mathcal{C}\mathrm{AT}(I, \mathcal{S}et)$, such that for every functor $G \in \mathcal{C}at(J, I)$ with connected non-empty BJ there is an object $m \in I$ and a natural transformation $\mathrm{const}_m \xrightarrow{h} G$ inducing a surjection

$$F(m) \twoheadrightarrow \lim_J (F \circ G), \qquad x \longmapsto (F(h_j)(x))_{j \in J}.$$

Then $\operatorname{hocolim}_I F \twoheadrightarrow \operatorname{colim}_I F$ *is a simplicial deformation retraction.*

Proof. Let $C(F)$ denote the **category of elements** of F, whose objects are given by the set $\coprod_{i \in I} F(i)$ and whose morphisms are given by

$$C(F)(x, y) = \{f \in I(i, j);\ y = F(f)(x)\}, \qquad x \in F(i),\ y \in F(j), \quad i, j \in I.$$

Then we have $BC(F) = B(\text{-}/I) \otimes_I F = \operatorname{hocolim}_I F$. For $\alpha \in \operatorname{colim}_I F$ we let $C(F, \alpha) \leq C(F)$ denote the full subcategory of objects $x \in \coprod_i F(i) = \mathrm{Obj}(C(F))$ with $\alpha = [x]$ under the canonical surjection $\coprod_i F(i) \twoheadrightarrow \operatorname{colim}_I F$. By construction $BC(F, \alpha) \leq BC(F)$ is the connected component corresponding to $\alpha \in \operatorname{colim}_I F = \pi_0 \operatorname{hocolim}_I F$. Now using the hypothesis for the functor

$$G_\alpha : C(F, \alpha) \longrightarrow I, \qquad F(i) \ni x \longmapsto i,$$

we get an element $m_\alpha \in I$ and a natural $\mathrm{const}_{m_\alpha} \xrightarrow{h_\alpha} G_\alpha$ inducing a surjection

$$F(m_\alpha) \twoheadrightarrow \lim_{C(F,\alpha)} (F \circ G_\alpha), \qquad x \longmapsto (F(h_{\alpha,y})(x))_{y \in C(F,\alpha)}.$$

Let $z_\alpha \in F(m_\alpha)$ be a preimage of the tuple $(y)_{y \in C(F,\alpha)}$. We define $\tilde{C}(F) \overset{S}{\hookrightarrow} C(F)$ as the inclusion of the discrete subcategory with objects z_α, for $\alpha \in \operatorname{colim}_I F$. For all $x, y \in C(F)$

and $f \in C(F)(x, y)$ we have $[x] = [y]$ in $\operatorname{colim}_I F$ and hence $z_{[x]} = z_{[y]}$. As $h_{[x]}$ is natural, we get a commuting square

$$\begin{tikzcd} z_{[x]} \arrow[r, mapsto, "F(h_{[x],x})"] \arrow[d, equal] & x \arrow[d, mapsto, "F(f)"] \\ z_{[y]} \arrow[r, mapsto, "F(h_{[y],y})"] & y. \end{tikzcd}$$

In other words the maps $(F(h_{[x],x}))_{x \in C(F)}$ define a natural transformation $SR \longrightarrow \mathrm{id}_{C(F)}$, where

$$R : C(F) \twoheadrightarrow \tilde{C}(F), \qquad x \longmapsto z_{[x]}.$$

In particular $BC(F) \overset{R}{\twoheadrightarrow} B\tilde{C}(F)$ is a deformation retraction with homotopy inverse S. As the composition

$$B\tilde{C}(F) \overset{S}{\hookrightarrow} BC(F) \cong \operatorname*{hocolim}_I F \twoheadrightarrow \operatorname*{colim}_I F$$

is an isomorphism, it follows that also $\operatorname{hocolim}_I F \twoheadrightarrow \operatorname{colim}_I F$ is a deformation retraction.

□

Remark 7.3.30
Let $I \in \mathcal{C}at$ and $F \in \mathcal{C}\mathrm{AT}(I, \mathcal{S}et)$. If I has limits over connected categories, which are preserved by F, i.e.

$$F(\lim G) \xrightarrow{\sim} \lim(F \circ G), \qquad \textit{for all} \quad G \in \mathcal{C}at(J, I), \quad J \in \mathcal{C}at, \quad \pi_0 BJ = *,$$

then the hypothesis of Proposition 7.3.29 is satisfied via

$$m := \lim_J G, \qquad h_j = \pi_j : \lim_J G \longrightarrow G(j), \quad j \in J.$$

Corollary 7.3.31
Let $I \in \mathcal{C}at$ and $F \in \mathcal{C}\mathrm{AT}(I, \mathcal{S}et)$, such that BI is contractible, $X \hookrightarrow F(i)$, for all $i \in I$, and $I \xrightarrow{F} X/\mathcal{S}et \xrightarrow{U} \mathcal{S}et$ satisfies the hypothesis of Proposition 7.3.29.

Then $\operatorname*{hocolim}_I F \twoheadrightarrow \operatorname*{colim}_I F$ is a weak equivalence of simplicial sets under X.

Proof. We can consider $(X \hookrightarrow F(i))_i$ as a natural injection $\mathrm{const}_X \hookrightarrow F$ of $\mathcal{S}et$-valued functors. Consider the diagram

$$\begin{tikzcd} \operatorname{hocolim}_I \mathrm{const}_X \arrow[r, hook] \arrow[d, "\simeq"'] & \operatorname{hocolim}_I UF \arrow[r, two heads, "\simeq"] \arrow[d, "\simeq"] & \operatorname{colim}_I UF \arrow[d, "\wr"] \\ X \arrow[r, hook] & U(\operatorname{hocolim}_I F) \arrow[r] & U(\operatorname{colim}_I F). \end{tikzcd}$$

The uppper right horizontal map is a weak equivalence by Proposition 7.3.29. A (homotopy) colimit in $X/\mathcal{S}et$ can be constructed by taking a colimit in $\mathcal{S}et$ and then identifying the images of each $x \in X$. So the left and the outer square is cocartesian. Moreover the right vertical map is an isomorphism, because BI is connected. The left vertical map can be identified with the projection $BI \times X \longrightarrow X$, which is a weak equivalence as I is contractible. As the upper left map is injective, the left square is also homotopy cartesian, which implies that the middle vertical map is a weak equivalence. Hence by commutativity the lower right horizontal map is a weak equivalence.

□

Corollary 7.3.32
Let $I \in \mathcal{C}at$ and $F \in \mathcal{C}\mathrm{AT}(I, \mathcal{S}et_*)$, such that BI is connected, $X \hookrightarrow F(i)$, for all $i \in I$, and $I \xrightarrow{F} \mathcal{S}et_* \xrightarrow{U} \mathcal{S}et$ satisfies the hypothesis of Proposition 7.3.29, e.g. if it preserves limits over connected, non-empty categories (c.f. Remark 7.3.30).

Then $\operatorname{hocolim}_I F \xrightarrow{\simeq} \operatorname{colim}_I F$ *is a weak equivalence of pointed simplicial sets.*

7.3.8 Homotopy colimits of monoids

Proposition 7.3.33
Let $M \in s\mathcal{S}et\text{-}\mathcal{A}ss_1$ be a simplicial monoid, free or free abelian in every dimension.

Then the map from M into its group completion $\mathcal{G}rp(M)$ induces weak equivalences

(i) $BM \xrightarrow{\simeq} B\mathcal{G}rp(M)$, *and*

(ii) $M \xrightarrow{\simeq} \mathcal{G}rp(M)$, *if* $\pi_0 M = 1$.

Proof. In [Qui71b] Q.5. Quillen calls simplicial monoids satisfying (i) good and proves in Prop. Q.1 that free monoids are good. It follows that also $B\mathbb{N}_0^n \xrightarrow{\simeq} B\mathbb{Z}^n$ are weak equivalences, for all $n \geq 0$. Hence free abelian monoids are good by exactness of filtered colimits. So in our situation M_n is good, for all $n \geq 0$, and thus by Prop. Q.2. loc. cit. also M is good.

For the second statement, note that by Thm. Q.4 loc. cit. the inclusion $M \hookrightarrow \mathcal{G}rp(M)$ induces an isomorphism

$$H_*(M, \mathbb{Z}) = H_*(M, \mathbb{Z})[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(\mathcal{G}rp(M), \mathbb{Z}),$$

and thus $M \xrightarrow{\simeq} \mathcal{G}rp(M)$ is a weak equivalence by the Whitehead Theorem for simplicial sets/spaces (cf. [Sri08] Cor. A.54).

□

Proposition 7.3.34
The nerve functor $s\mathcal{M}on \xrightarrow{B} s\mathcal{S}et_*$ preserves coproducts up to weak equvivalence.

In particular it also preserves homotopy colimits.

Proof. First suppose we are given a family $M_i \in \mathcal{M}on$ of free constant monoids. So $M_i = F(S_i)$, for some $S_i \in \mathcal{S}et$. Here $\mathcal{S}et \xrightarrow{F} \mathcal{M}on$ denotes the free functor being the left adjoint to the forgetful functor. The Hurewicz map induces a weak equivalence

$$S^1 \xrightarrow{\simeq} \tilde{\mathbb{Z}}(S^1) \cong B\mathbb{Z}.$$

which restricts to a weak equivalence $h : S^1 \xrightarrow{\simeq} B\mathbb{N}_0 = BF(*)$. More generally we get a natural weak equivalence of pointed simplicial sets

$$S^1 \wedge S_+ = {}^S S^1 \xrightarrow{\simeq} F({}^S *) = F(S), \qquad S \in \mathcal{S}et.$$

Hence in the commutative diagram of pointed simplicial sets

$$\begin{tikzcd}
\coprod_i S^1 \wedge (S_i)_+ \arrow[rr, "\sim"] \arrow[d, "\simeq"'] & & S^1 \wedge (\coprod_i S_i)_+ \arrow[d, "\simeq"'] \\
\coprod_i BF(S_i) \arrow[r] & B\coprod_i F(S_i) \arrow[r, "\sim"] & BF(\coprod_i S_i),
\end{tikzcd}$$

the vertical maps are weak equivalences, because weak equivalences are preserved by coproducts. Hence also the lower left horizontal map is a weak equivalence, which proves the statement in the free case.

For the case of general constant monoids $M_i \in \mathcal{M}on$, consider the commtutative square

$$\begin{tikzcd}
\coprod_i BE_\bullet(M_i) \arrow[r] \arrow[d] & B\coprod_i E_\bullet(M_i) \arrow[d] \\
\coprod_i BM_i \arrow[r] & B\coprod_i M_i,
\end{tikzcd}$$

where $E_\bullet$ is the free resolution functor obtained from the adjunction $\mathcal{M}on(F(X), Y) = \mathcal{S}et(X, U(Y))$. As B and coproducts preserve weak equivalence, the vertical maps are weak equivalences. By commutativity it suffices to prove that the upper horizontal map is a weak equivalence. This follows once we have shown that $\coprod_i BE_n(M_i) \xrightarrow{\simeq} B\coprod_i E_n(M_i)$, for all $n \geq 0$. As $E_n(M_i)$ is free, for all $n \geq 0$, we are done.

The general case follows from the constant case, because the map of bisimplicial pointed sets

$$\coprod_i B_\bullet(M_i)_\bullet \xrightarrow{\simeq} B_\bullet \coprod_i (M_i)_\bullet$$

is a weak equivalence in every dimension of M_i by the constant case.

$\square$

Corollary 7.3.35
Let $k \in s\mathcal{C}Ring$ and $A_i \in k\text{-}\mathcal{A}ss$ be flat in every dimension.

Then $\coprod_i B(A_i)_+ \xrightarrow{\simeq} B\coprod_i (A_i)_+$ is a weak equivalence of simplicial k-modules under k.

Proof. For $A_i = k[M_i]$ and $M_i \in s\mathcal{M}on$, there is a commuting diagram

$$\begin{tikzcd}
\coprod_i B^{\otimes} k[M_i] \arrow[r] & B^{\otimes} \coprod_i k[M_i] \\
k\coprod_i B^{\times} M_i \arrow[r, "\simeq"] \arrow[u, "\wr"] & kB^{\times} \coprod_i M_i, \arrow[u, "\wr"]
\end{tikzcd}$$

where $B^{\otimes}$ resp. $B^{\times}$ refers to the bar construction in algebras resp. monoids. The right verical maps are isomorphisms, because the free module functor k preserves coproducts and $kB^\times \xrightarrow{\sim} B^\otimes k$. As lower map is a weak equivalence by Proposition 7.3.34 it follows that also the lower map is a weak equivalence.

For the general case, consider the diagram

$$\begin{tikzcd} \coprod_i BE_\bullet(A_i)_+ \arrow[r] \arrow[d, "\simeq"'] & B\coprod_i E_\bullet(A_i)_+ \arrow[d, "\simeq"'] \\ \coprod_i B(A_i)_+ \arrow[r] & B\coprod_i (A_i)_+, \end{tikzcd}$$

where $E_\bullet$ is the free resolution functor induced by the adjunction $k\text{-}\mathcal{A}ss(k[X], Y) = s(\mathcal{S}et\text{-}\mathcal{A}ss)(X, U(Y))$. Again it suffices to show that in every E-dimension the upper map is a weak equivalence. But for every $n \geq 0$ there are simplicial semigroups $S_i \in s(\mathcal{S}et\text{-}\mathcal{A}ss)$ with $E_n(A_i)_+ = k[S_i]_+ = k[(S_i)_+]$. As $(S_i)_+ \in s\mathcal{M}on$ we are done by the first case.

□

7.4 Homological algebra

7.4.1 Differential graded objects aka chain complexes

Definition 7.4.1
Let $\mathcal{A}$ be an abelian category.

(i) *We call $dg\mathcal{A}$ the category of **differential graded objects** or ($\mathbb{Z}$-indexed) **chain complexes** in $\mathcal{A}$, moreover $dg_{\geq}m\mathcal{A}$ is the full subcategory of chain complexes X with $X_n = 0$, for all $n < m$.*

(ii) *By abuse of notation every object $X \in \mathcal{A}$ will also be considered as the object in $dg\mathcal{A}$, which is equal to X in dimension 0 and trivial elsewhere*

7.4.2 The induced symmetric monoidal structure

Remark 7.4.2
Let $\mathcal{A}$ be an abelian category with countable direct sums.

(i) *There is a functor $dgdg\mathcal{A} \xrightarrow{\mathrm{Tot}^+} dg\mathcal{A}$, sending a bicomplex $X \in dg\mathcal{A}$ to its **total complex** $\mathrm{Tot}^+ X$ with*

$$(\mathrm{Tot}^+ X)_n = \bigoplus_{p+q=n} X_{p,q}, \qquad X_{p,q} \xrightarrow{d^{(1)}+(-1)^p d^{(2)}} (\mathrm{Tot}^+ X)_{p+q}, \qquad p, q \in \mathbb{Z},$$

where $d^{(i)}$ is the differential in coordinate $i = 1, 2$.

(ii) *Similarly, if $\mathcal{A}$ has countable products, we define $\mathrm{Tot}^\times$ by replacing direct sums by direct products.*

Note that they coincide on bicomplexes, that are bounded below in both coordinates.

(iii) *If $\mathcal{A}$ carries a monoidal structure $(\mathcal{A}, \otimes, k)$, there is an induced tensor product on $dg\mathcal{A}$, given as the composite of* Tot^+ *with the functor*

$$dg\mathcal{A} \times dg\mathcal{A} \longrightarrow dgdg\mathcal{A}, \quad (X, Y) \longmapsto (X \otimes Y, d \otimes \mathrm{id}, \mathrm{id} \otimes d),$$

and $(dg\mathcal{A}, \otimes, k)$ is also monoidal.

(iv) *If $(\mathcal{A}, \otimes, k)$ is symmetric monoidal, so is also $(dg\mathcal{A}, \otimes, k)$ by the braiding $X \otimes Y \xrightarrow{\gamma} Y \otimes X$ given by the direct sum over all isomorphisms*

$$X_p \otimes Y_q \xrightarrow{\sim} Y_q \otimes X_p, \quad x \otimes y \longmapsto (-1)^{pq} y \otimes x, \qquad p, q \in \mathbb{Z}.$$

Remark 7.4.3
With this symmetric monoidal structure in hand we define the categories

(i) $dg(k\text{-}\mathcal{A}ss_1) = dg(k\text{-}\mathrm{Mod})\text{-}\mathcal{A}ss_1$,

(ii) $dg(k\text{-}\mathcal{C}om_1) = dg(k\text{-}\mathrm{Mod})\text{-}\mathcal{C}om_1$,

(iii) $dg(k\text{-}\mathcal{L}ie) = dg(k\text{-}\mathrm{Mod})\text{-}\mathcal{L}ie$, ...

(cf. section 2.3).

7.4.3 The model structure

Theorem 7.4.4
Let $\mathcal{A}$ be a finitely complete and cocomplete category.
Then $dg_{\geq 0}\mathcal{A}$ is a model category with the following structure.

(i) *Weak equivalences are quasi-isomorphisms, i.e. chain maps inducing isomorphisms under homology.*

(ii) *Fibrations are epimorphisms.*

(iii) *Cofibrations are monomorphisms, having projective cokernel in every dimension.*

Proof. See [Hov99] 2.3 □

Proposition 7.4.5 (i) *On the same way one can define a model structure on the category of chain complexes, that are bounded below.*

(ii) *There is also a model structure on the whole category $dg\mathcal{A}$ with weak equivalences and fibrations as before.*

However the cofibrations are more complicated.

Proof. See [Hov99] 2.3 for a detailed discussion. □

7.4.4 The Dold-Kan correspondence

Proposition 7.4.6 (Dold-Kan correspondence)
For every finitely complete abelian category $\mathcal{A}$ there is an equivalence adjunction of categories

$$s\mathcal{A}(\Gamma X, Y) = dg_{\geq 0}\mathcal{A}(X, NY),$$

where NX is the ***Moore complex*** *of X (cf. Definition 7.2.34), given by*

$$N_n X = \ker(X_n \overset{(d_1,\dots,d_n)}{\longrightarrow} (X_{n-1})^n), \qquad d = d_0|.$$

There is an isomorphism $NX \hookrightarrow X \twoheadrightarrow \tilde{X}$, where $\tilde{X}$ is the ***reduced complex****, given by*

$$\tilde{X}_n = \operatorname{coker}\left((X_{n-1})^{n-1} \overset{\sum_i s_i}{\longrightarrow} X_n\right), \qquad d = \sum_{0\leq i\leq n} (-1)^i d_i, \qquad n \geq 0.$$

Moreover there natural maps

(i) *The* ***Alexander-Whitney map***

$$\Delta : N\mathrm{diag}X \longrightarrow \mathrm{Tot}^+ N^{(1)}N^{(2)}X,$$
$$(d_0(N\mathrm{diag}X)_n \subset X_{n,n} \longrightarrow \prod_{p+q=n} N^{(1)}N^{(2)}_{p,q} = (\mathrm{Tot}^\times N^{(1)}N^{(2)})_n = (\mathrm{Tot}^+ N^{(1)}N^{(2)})_n.$$

(ii) *The* ***shuffle map****, being a section for Δ*

$$\nabla : \mathrm{Tot}^+ N^{(1)}N^{(2)}X \hookrightarrow N\mathrm{diag}X,$$
$$\sum_{\sigma\in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)\cdot s^{(1)}_{\sigma(p+q)-1} \circ \dots \circ s^{(1)}_{\sigma(p+1)-1} \circ s^{(2)}_{\sigma(p)} \circ \dots \circ s^{(2)}_{\sigma(1)-1} : N^{(1)}_p N^{(2)}_q X \longrightarrow (N\mathrm{diag}X)_{p+q},$$

where $\mathrm{Sh}_{p,q} \subset \Sigma_{p+q}$ denotes the subset of (p,q)-shuffles, which are permutations $\sigma \in \Sigma_{p+q}$ with

$$\sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q).$$

If $\mathcal{A}$ carries a monoidal structure, then using Δ and ∇ the functors N and Γ are monoidal and comonoidal.

If $(\mathcal{A}, \otimes, k)$ is symmetric monoidal, the following holds.

(i) *The functor N is symmetric monoidal and Γ is symmetric comonoidal via ∇.*

(ii) *The functors N is* <u>*not*</u> *symmetric comonoidal and Γ is not symmetric monoidal, as Δ in contrast to ∇ does not behave well with the braidings.*

Proof. See e.g. [GJ09] III.2. and/or [Wei94] 8.4 and 8.5. Note that there is a sign twist, because of the slightly different definition of the Moore complex in [Wei94] Def. 8.3.6. □

Proposition 7.4.7
Suppose $\mathcal{A} = k\text{-Mod}$, for some $k \in \mathcal{R}ing$.

Then the model structure on $dg_{\geq 0}(k\text{-Mod})$ "almost" coincides with the model structure, obtained by the adjunction

$$dg_{\geq 0}(NkX, Y) = s\mathcal{S}et(X, U\Gamma(Y)).$$

Moreover the monoidal structure maps Δ and ∇ of the functors N and Γ are weak equivalences by the Theorem of Eilenberg-Zilber (see [Wei94] 8.5).

Proof. By Proposition 7.2.39 the weak equivalences are the same, while Proposition 7.2.35 implies that also the fibrations are the same modulo the little difference in dimension 0. □

7.4.5 Differential graded algebras and coalgebras

Proposition 7.4.8
For $X \in dg(k\text{-Mod})$ let $X = X^+ \oplus X^-$, where

$$X^+ = \bigoplus_{n\in\mathbb{Z}} X_{2n}, \qquad X^- = \bigoplus_{n\in\mathbb{Z}} X_{2n+1}.$$

Then there is a ring isomorphism $S(X^+) \otimes \Lambda(X^-) \xrightarrow{\sim} \mathcal{C}om_1(X)$ induced by the ring monomorphisms

$$S(X^+) := \mathcal{C}om_1(X^+) \hookrightarrow \mathcal{C}om_1(X), \qquad \Lambda(X^-) := \mathcal{C}om_1(X^-) \hookrightarrow \mathcal{C}om_1(X),$$

where here S is the symmetric and Λ the exterior algebra functor.

Proof. By forgetting the differentials, we also get a symmetric monoidal structure on $g(k\text{-Mod}) = (k\text{-Mod})^{\mathbb{Z}}$ and we have $X = X^+ \oplus X^-$ in $g(k\text{-Mod})$. The coproduct in $g(k\text{-Mod})\text{-}\mathcal{C}om_1$ is induced by the tensor product. As $\mathcal{C}om_1$ is left adjoint to the forgetful functor, it preserves coproducts and hence

$$\mathcal{C}om_1(X^+) \otimes \mathcal{C}om_1(X^-) \xrightarrow{\sim} \mathcal{C}om_1(X^+ \oplus X^-) = \mathcal{C}om_1(X).$$

Now the following holds.

- As X^+ only has even dimensions the Σ_n-action on $(X^+)^{\otimes n}$ is just permutation of the tensor factors, hence $\mathcal{C}om_1(X^+) = S(X^+)$, the symmetric algebra.
- As X^- only has odd dimensions the Σ_n-action on $(X^-)^{\otimes n}$ is permutation tensored with the sign representation, hence $\mathcal{C}om_1(X^-) = \Lambda(X^-)$, the exterior algebra.

Similarly $\mathcal{C}om_1(X^+)$ only has even dimensions, such that the tensor product of the graded rings $\mathcal{C}om_1(X^+)$ and $\mathcal{C}om_1(X^-)$ coincides with the usual tensor product of rings. □

Proposition 7.4.9
For $X \in dg(k\text{-Mod})$ there are cocomplete, cocommutative bialgebra structures on $\mathcal{A}ss_1(X)$ and $\mathcal{C}om_1(X)$, whose comultiplication and counit are determined by

$$\delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \qquad x \in X.$$

Moreover the following holds.

(i) *The comultiplication on $\mathcal{C}om_1(X^+) = S(X^+)$ is given by*

$$\delta(x_1 \cdots x_n) = \sum_{\substack{p+q=n,\\ \sigma\in \mathrm{Sh}_{p,q}}} x_{\sigma^{-1}} \cdots x_{\sigma^{-1}(p)} \otimes x_{\sigma^{-1}(p+1)} \cdots x_{\sigma^{-1}(n)}, \qquad x_1, ..., x_n \in X^+.$$

(ii) *The comultiplication on $\mathcal{C}om_1(X^-) = \Lambda(X^-)$ is given by*

$$\delta(x_1 \wedge ... \wedge x_n) = \sum_{\substack{p+q=n,\\ \sigma\in \mathrm{Sh}_{p,q}}} \mathrm{sgn}(\sigma) \cdot x_{\sigma^{-1}} \wedge ... \wedge x_{\sigma^{-1}(p)} \otimes x_{\sigma^{-1}(p+1)} \wedge ... \wedge x_{\sigma^{-1}(n)}.$$

Here $\mathrm{Sh}_{p,q} \leq \Sigma_{p+q}$ denotes the subset of (p, q)-shuffles (c.f. Remark 7.4.6). The comultiplication is therefore also called ***shuffle (co-)product****.*

Proof. As comultiplication and counit are algebra homomorphisms, they are completely determined by their restriction to X by the universal property. Similarly the cocommutativitiy, coassociativity and counit axiom only needs to hold when restricted to X, where it holds by construction. Arguing via the universal property again it suffices to check this on $X^+ \otimes k + k \otimes X^- \subset S(X^+) \otimes \Lambda(X^-)$. The comultiplication restricts to the reduced coalgebra $\mathcal{A}ss(X) = \mathrm{coker}\,(k \xrightarrow{\eta} \mathcal{A}ss_1(X))$, such that $\mathcal{A}ss_1(X) \twoheadrightarrow \mathcal{A}ss(X)$ is a coalgebra homomorphism. For cocompleteness we have to prove that

$$\mathcal{A}ss(X) = \bigcup_{n\geq 0} \ker(\mathcal{A}ss(X) \xrightarrow{\delta^n} \mathcal{A}ss(X)^{\otimes(n+1)}).$$

The iterated comultiplication $\mathcal{A}ss_1(X) \xrightarrow{\delta^n} \mathcal{A}ss_1(X)^{\otimes(n+1)}$ is induced by its restriction to X, where we have

$$\delta^n(x) = \sum_{0\leq i\leq n} 1 \otimes ... \otimes \underset{i}{x} \otimes ... \otimes 1. \tag{7.2}$$

Hence it maps $\mathcal{A}ss^{(n)}(X) = X^{\otimes n}$ into

$$\sum_{0\leq i\leq n} \mathcal{A}ss_1(X)^{\otimes i} \otimes k \otimes \mathcal{A}ss_1(X)^{\otimes(n-i)} \subset \ker(\mathcal{A}ss_1(X)^{\otimes(n+1)} \twoheadrightarrow \mathcal{A}ss(X)^{\otimes(n+1)}),$$

which proves that $\mathcal{A}ss^{(n)}(X) \subset \ker(\mathcal{A}ss(X) \xrightarrow{\delta^n} \mathcal{A}ss(X)^{\otimes(n+1)})$. The case $\mathcal{C}om$ is similar.

Properties (i) and (ii) can be verified by induction on $n \geq 1$. Note that $\Lambda(X^-) \otimes \Lambda(X^-)$ is <u>not</u> the usual tensor product of rings here. Whenever moving two generators past each other we will get a sign twist. More precisely

$$(x \otimes y) \cdot (x' \otimes y') = -(x \wedge x') \otimes (y \wedge y'), \qquad x, y \in X^-.$$

This is the reason why we get the additional sign twist $\mathrm{sgn}(\sigma)$, for a shuffle σ. $\square$

Corollary 7.4.10
Dually for $X \in dg(k\text{-Mod})$ there is a commutative bialgebra structures on $\widehat{\mathcal{A}ss}_1(X)$, whose multiplication and unit are completely determined by the projection

$$\pi_X \circ \mu : \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X) \otimes \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X) \longrightarrow (X \otimes k) \oplus (k \otimes X) \cong X \oplus X \xrightarrow{\nabla} X, \qquad \pi_X \circ \eta : k \xrightarrow{0} X.$$

The product is also called the ***shuffle product*** *and is explicitly given by*

$$(x_1 \otimes \ldots \otimes x_m) \sqcup\!\sqcup (x_{m+1} \otimes \ldots \otimes x_{m+n}) := \sum_{\sigma \in \mathrm{Sh}_{m,n}} (x_1 \otimes \ldots \otimes x_{m+n})^{\sigma}.$$

Keep in mind that σ is permutation of the tensor factors plus a possible sign twist depending on the dimension of the permuted elements.

Proof. By the universal property of coalgebras the projection $\pi_X \circ \mu$ extends as desired and is given by

$$\mu = \varepsilon + \sum_{k \geq 0} (\pi_X \circ \mu)^{\otimes(k+1)} \circ \bar{\delta}^k,$$

where $\bar{\delta}$ denotes the comultiplication on $\widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X) \otimes \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X)$. On $\widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X)$ we have

$$\delta^k(x_1 \otimes \ldots \otimes x_m) = \sum_{0 \leq p_0 \leq \ldots \leq p_k \leq m} (x_1 \otimes \ldots \otimes x_{p_0}) \otimes (x_{p_0+1} \otimes \ldots \otimes x_{p_1}) \otimes \ldots \otimes (x_{p_{k-1}+1} \otimes \ldots \otimes x_{p_k}). \tag{7.3}$$

Moreover $\bar{\delta}^{\otimes k}$ is $\delta^{\otimes k} \otimes \delta^{\otimes k}$ composed with the braiding bringing the tensor factors in the right order. By definition under $\pi_X \circ \mu$ only those summands of

$$\delta^{\otimes k} \otimes \delta^{\otimes k}((x_1 \otimes \ldots \otimes x_m) \otimes (x_{m+1} \otimes \ldots \otimes x_{m+n}))$$

survive, which correspond to partitions $p_{-1} := 0 \leq p_0 \leq \ldots \leq p_k \leq m =: p_{k+1}$ and $q_{-1} := 0 \leq q_0 \leq \ldots \leq q_k \leq n =: q_{k+1}$ with

$$p_i - p_{i-1} \leq 1, \quad q_i - q_{i-1} \leq 1, \qquad (p_i - p_{i-1}) + (q_i - q_{i-1}) = 1, \qquad 0 \leq i \leq k+1.$$

It follows that

$$(\pi_X \circ \mu)^{\otimes(k+1)} \circ \bar{\delta}^k((x_1 \otimes \ldots \otimes x_m) \otimes (x_{m+1} \otimes \ldots \otimes x_{m+n})$$

is zero, if $m + n \neq k + 1$ and is equal to the given formula otherwise. □

Proposition 7.4.11
For $X \in dg(k\text{-Mod})$ the following holds.

(i) *The projection $\mathcal{C}om_1(X) \xrightarrow{\pi_X} X$ extends to a coalgebra map*

$$\mathcal{C}om_1(X) \xrightarrow{N} \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X),$$

which is induced by the norm maps

$$\mathcal{C}om^{(n)}(X) = (X^{\otimes n})_{\Sigma_n} \xrightarrow{N} X^{\otimes n} = \mathcal{A}ss^{(n)}(X), \qquad n \geq 0.$$

(ii) *Dually the inclusion* $X \longrightarrow \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X)$ *extends to an algebra map, which again is* N.

In particular $\mathcal{C}om_1(X) \xrightarrow{N} \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X)$ *is a bialgebra homomorphism.*

Proof.

(i) By the universal property of coalgebras the projection extends as desired and is given by

$$N = \varepsilon + \sum_{n \geq 0} (\pi_X)^{\otimes(n+1)} \circ \delta^n.$$

Using 7.2 we see that

$$(\pi_X)^{\otimes(n+1)} \circ \delta^n(x_1 \cdots x_m) = \begin{cases} \sum_{\sigma \in \Sigma_m} (x_1 \otimes \ldots \otimes x_m)^\sigma, & m = n+1, \\ 0, & m \neq n+1, \end{cases}$$

which is precisely the norm map.

(ii) The map N restricts to the inclusion of X and is an algebra homomorphism by the explicit formula of the shuffle product, because

$$\mathrm{Sh}_{p,q} \times (\Sigma_p \times \Sigma_q) \xrightarrow{\sim} \Sigma_{p+q}, \qquad (\sigma, (\alpha, \beta)) \longmapsto \sigma \circ (\alpha \oplus \beta), \qquad p, q \geq 0.$$

$\square$

Corollary 7.4.12
For $X \in dg(k\text{-Mod})$ *the norm homomorphism restricts to a map*

$$\mathcal{C}om_1(X) \xrightarrow{N} \bigoplus_{n \geq 0} (X^{\otimes n})^{\Sigma_n} =: \widehat{\mathcal{C}om}_1^{\mathrm{op}}(X) \subset \widehat{\mathcal{A}ss}_1^{\mathrm{op}}(X),$$

which is an isomorphism in dimensions $< p$*, if the following properties are satisfied.*

(i) X *is reduced, i.e.* $X_0 = 0$,

(ii) $(p-1)! \in k^\times$,

(iii) X *is flat over* k *in every dimension.*

Proof. When X is flat in every dimension, then $\widehat{\mathcal{C}om}_1^{\mathrm{op}}(X)$ is the cofree cocommutative coalgebra generated by X, because the maps

$$(X^{\otimes n})^{\Sigma_n} \hookrightarrow (X^{\otimes p} \otimes X^{\otimes q})^{\Sigma_p \times \Sigma_q} \xleftarrow{\sim} (X^{\otimes p})^{\Sigma_p} \otimes (X^{\otimes q})^{\Sigma_q}, \qquad p, q \geq 0,$$

induce a coalgebra structure on $\widehat{\mathcal{C}om}_1^{\mathrm{op}}(X)$. Hence by the dual statement of Proposition 7.4.8 we get an isomorphism

$$\widehat{\mathcal{C}om}_1^{\mathrm{op}}(X) \xrightarrow{\sim} \widehat{\mathcal{C}om}_1^{\mathrm{op}}(X^+) \otimes \widehat{\mathcal{C}om}_1^{\mathrm{op}}(X^-).$$

Consider the following diagram

$$\begin{tikzcd}
(X^+ \otimes k) \oplus (k \otimes X^-) \arrow[r, equal] \arrow[d] & (X^+ \otimes k) \oplus (k \otimes X^-) \\
\mathcal{C}om_1(X^+) \otimes \mathcal{C}om_1(X^-) \arrow[r, "N\otimes N"] \arrow[d, "\wr"'] & \widehat{\mathcal{C}om}_1^{\mathrm{op}}(X^+) \otimes \widehat{\mathcal{C}om}_1^{\mathrm{op}}(X^-) \arrow[u] \\
\mathcal{C}om_1(X) \arrow[r, "N"] & \widehat{\mathcal{C}om}_1^{\mathrm{op}}(X). \arrow[u, "\wr"']
\end{tikzcd}$$

The upper and the outer square commute. As N is a bialgebra homomorphism, the universal property of algebras resp. coalgebras implies that also the lower square commutes. Since $(p-1)! \in k^\times$, the norm map $\mathcal{C}om_1(X^+) \xrightarrow{N} \widehat{\mathcal{C}om}_1^{\mathrm{op}}(X^+)$ is an isomorphism in homogeneous degrees $0 \leq r < p$ with inverse given by the composition

$$\begin{tikzcd}[column sep=small]
(\widehat{\mathcal{C}om}_1^{\mathrm{op}})^{(r)}(X^\pm) = ((X^\pm)^{\otimes r})^{\Sigma_r} \arrow[rr] \arrow[dr] & & \mathcal{C}om_1^{(r)}(X^\pm) \xrightarrow{1/r!} \mathcal{C}om_1^{(r)}(X^\pm) \\
& (\widehat{\mathcal{A}ss}_1^{\mathrm{op}})^{(r)}(X^\pm) = ((X^\pm)^{\otimes r}) = \mathcal{A}ss_1^{(r)}(X^\pm) \arrow[ur, two heads] &
\end{tikzcd}$$

The first appearence of a p-th tensor power in $\mathcal{C}om_1(X^\pm)$ resp. $\widehat{\mathcal{C}om}_1^{\mathrm{op}}(X^\pm)$ is in dimension p, because $X_0 = 0$ and hence the lowest possibly non-zero dimension of X^- is 1 and of X^+ is 2. Hence N is an isomorphism in dimensions $< p$.

□

8 Appendix: Tor-groups and homology for simplicial objects

8.1 Simplicial rings and modules

Throughout this chapter we give $s\mathcal{R}ing = s(\mathbb{Z}\text{-}\mathcal{A}ss_1)$ the model structure induced by Theorem 7.2.40 using the adjunction

$$\mathcal{A}ss_1(\mathcal{A}ss_1(\mathbb{Z}X), Y) = \mathcal{S}et(X, UG(Y)),$$

where $\mathbb{Z}\text{-}\mathcal{A}ss_1 \xrightarrow{G} \mathcal{G}rp$ sends a ring to underlying additive group. Similarly, for a simplicial ring $R \in s\mathcal{R}ing$, we give $R\text{-}s\mathcal{A}b$ and $s\mathcal{A}b\text{-}R$ resp. the model structure induced by the adjunctions

$$R\text{-}s\mathcal{A}b(R \otimes \mathbb{Z}X, Y) = s\mathcal{G}rp(X, U(Y)), \qquad s\mathcal{A}b\text{-}R(\mathbb{Z}X \otimes R, Y) = s\mathcal{G}rp(X, U(Y))$$

where the forgetful functor $R\text{-}s\mathcal{A}b \xrightarrow{G} s\mathcal{S}et$ resp. $s\mathcal{A}b\text{-}R \xrightarrow{G} s\mathcal{S}et$ sends a module to its underlying additive simplicial group.

For a simplicial object X we will denote by $E_\bullet(X) \longrightarrow X$ the canonical functorial almost free replacement of Corollary 7.2.32.

Definition 8.1.1
Let $R \in s\mathcal{R}ing$ and $X \in s\mathcal{A}b\text{-}R$ as well as $Y, Z \in R\text{-}s\mathcal{A}b$.

(i) $\mathrm{Tor}_*^R(X,Y) = \pi_*\mathbb{L}(X \otimes_R \text{-})(Y) = \pi_*\mathbb{L}(\text{-} \otimes_R Y)(X) = \pi_*(X \otimes_R^L Y) \in \mathcal{A}b,$

(ii) $\mathrm{Ext}_R^*(Y,Z) = \pi_{-*}\mathbb{R}\underline{R\text{-}s\mathcal{A}b}(Y, \text{-})(Z) = \pi_{-*}\mathbb{L}\underline{R\text{-}s\mathcal{A}b}(\text{-}, Z)(Y) \in \mathcal{A}b.$

Proposition 8.1.2
Let $k \in s\mathcal{C}Ring$ and $f \in k\text{-}\mathcal{A}ss(A,B)$, such that

$$f_* : \mathrm{Tor}_*^{A_+}(k, \pi_0 B) \xrightarrow{\sim} \mathrm{Tor}_*^{B_+}(k, \pi_0 B).$$

*Then $f_* : \mathrm{Tor}_*^{A_+}(k, M) \xrightarrow{\sim} \mathrm{Tor}_*^{B_+}(k, M)$ is an isomorphism, for all $M \in \pi_0 B$-*Mod.

Proof. We will show by induction on $n \geq 0$ that $\mathrm{Tor}_*^{A_+}(k, M) \longrightarrow \mathrm{Tor}_*^{B_+}(k, M)$ is $(n-1)$-connected, for all $M \in \pi_0 B$-Mod. Let K denote the kernel of $F := \pi_0 B^{(M)} \twoheadrightarrow M$ and consider the associated long exact sequences

$$\begin{tikzcd}
\ldots \arrow[r, "\partial"] & \operatorname{Tor}_0^{A+}(k,K) \arrow[r] \arrow[d] & \operatorname{Tor}_0^{A+}(k,F) \arrow[r, two heads] \arrow[d, "\wr"'] & \operatorname{Tor}_0^{A+}(k,M) \arrow[d] \\
\ldots \arrow[r, "\partial"] & \operatorname{Tor}_0^{B+}(k,K) \arrow[r] & \operatorname{Tor}_0^{B+}(k,F) \arrow[r, two heads] & \operatorname{Tor}_0^{B+}(k,M).
\end{tikzcd}$$

It follows that $\operatorname{Tor}_0^{A+}(k,M) \twoheadrightarrow \operatorname{Tor}_0^{B+}(k,M)$, which proves the case $n=0$. Suppose the statement holds for some $n \geq 0$ and all M. Then using the induction hypothesis, for K and M, the 5-lemma for the sequence

$$\begin{tikzcd}
\operatorname{Tor}_n^{A+}(k,K) \arrow[r] \arrow[d] & \operatorname{Tor}_n^{A+}(k,F) \arrow[r] \arrow[d, "\wr"'] & \operatorname{Tor}_n^{A+}(k,M) \arrow[r, "\partial"] \arrow[d] & \operatorname{Tor}_{n-1}^{A+}(k,K) \arrow[r] \arrow[d, "\wr"'] & \operatorname{Tor}_{n-1}^{A+}(k,F) \arrow[d, "\wr"'] \\
\operatorname{Tor}_n^{B+}(k,K) \arrow[r] & \operatorname{Tor}_n^{B+}(k,F) \arrow[r] & \operatorname{Tor}_n^{B+}(k,M) \arrow[r, "\partial"] & \operatorname{Tor}_{n-1}^{B+}(k,K) \arrow[r] & \operatorname{Tor}_{n-1}^{B+}(k,F)
\end{tikzcd}$$

yields $\operatorname{Tor}_n^{A+}(k,M) \xrightarrow{\sim} \operatorname{Tor}_n^{B+}(k,M)$. By applying the same argument for $M=K$, we also get $\operatorname{Tor}_n^{A+}(k,K) \xrightarrow{\sim} \operatorname{Tor}_n^{B+}(k,K)$. Applying the 5-lemma again for the sequence

$$\begin{tikzcd}
\operatorname{Tor}_{n+1}^{A+}(k,K) \arrow[r] \arrow[d] & \operatorname{Tor}_{n+1}^{A+}(k,F) \arrow[r] \arrow[d, "\wr"'] & \operatorname{Tor}_{n+1}^{A+}(k,M) \arrow[r, "\partial"] \arrow[d] & \operatorname{Tor}_n^{A+}(k,K) \arrow[r] \arrow[d, "\wr"'] & \operatorname{Tor}_n^{A+}(k,F) \arrow[d, "\wr"'] \\
\operatorname{Tor}_{n+1}^{B+}(k,K) \arrow[r] & \operatorname{Tor}_{n+1}^{B+}(k,F) \arrow[r] & \operatorname{Tor}_{n+1}^{B+}(k,M) \arrow[r, "\partial"] & \operatorname{Tor}_n^{B+}(k,K) \arrow[r] & \operatorname{Tor}_n^{B+}(k,F)
\end{tikzcd}$$

we also get $\operatorname{Tor}_{n+1}^{A+}(k,M) \twoheadrightarrow \operatorname{Tor}_{n+1}^{B+}(k,M)$, proving that $\operatorname{Tor}_*^{A+}(k,M) \longrightarrow \operatorname{Tor}_*^{B+}(k,M)$ is n-connected.

□

8.1.1 Tor-spectral sequences

Proposition 8.1.3
Let $R \in s\mathcal{R}ing$ and $Y \in R\text{-}s\mathcal{A}b$.

(i) *The action $R \otimes Y \longrightarrow Y$ defines an action $\pi_n R \otimes \pi_n Y \longrightarrow \pi_n(R \otimes Y) \longrightarrow \pi_n Y$, $n \geq 0$. We have $[ry] = 0$ in $\pi_n Y$, for all $[r] \in \pi_n R$ and $[y] \in \pi_n Y$, when $n > 0$.*

(ii) *The action $R_0 \otimes Y \longrightarrow R \otimes Y \longrightarrow Y$ defines an action $\pi_0 R \otimes \pi_n Y \longrightarrow \pi_n Y$, $n \geq 0$.*

Proof. The proof is exactly the same as in Proposition 7.2.39.

(i) Let $r \in \ker(N_n R \xrightarrow{d} N_{n-1} R) = \bigcap_{0 \leq i \leq n} \ker d_i$ and $y \in \ker(N_n Y \xrightarrow{d} N_{n-1} Y)$. As $n > 0$, we can define

$$z = s_0(r) \cdot s_0(y) - s_0(r) \cdot s_1(y) \in Y_{n+1}.$$

Then using the simplicial identities, we get

$$d_i(z) = \begin{cases} r \cdot y - r \cdot s_0 d_0(y) = r \cdot y - r \cdot 0 = r \cdot y, & i = 0, \\ r \cdot y - r \cdot y = 0, & i = 1, \\ s_0 d_1(r) \cdot s_0 d_1(y) - s_0 d_1(r) \cdot y = 0 \cdot 0 - 0 \cdot y = 0, & i = 2, \\ s_0 d_{i-1}(r) \cdot s_0 d_{i-1}(y) - s_0 d_{i-1}(r) \cdot s_1 d_{i-1}(y) = 0, & 3 \leq i \leq n. \end{cases}$$

It follows that $z \in N_{n+1}Y$ and hence $[ry] = [d(z)] = 0$ in $\pi_n Y$.

(ii) The constant simplicial ring of zero vertices $R_0 \leq R$ acts on $\pi_n Y$ via

$$R_0 \longrightarrow s\mathcal{A}b(Y) \xrightarrow{\pi_n} \mathcal{A}b(\pi_n Y), \quad r \longmapsto \pi_n(r \cdot \text{-}).$$

Moreover, for every $z \in R_1$, there is a simplicial homotopy $(d_0 z \cdot \text{-}) \simeq (d_1 z \cdot \text{-})$ of homomorphisms of simplicial abelian groups, given by

$$^{\Delta^1_n} Y_n \longrightarrow Y_n, \qquad \iota_s(y) \longmapsto s^*(r) \cdot y.$$

In particular the action $R_0 \longrightarrow k\text{-Mod}(\pi_n Y)$ factors over $\pi_0 R$.

□

Proposition 8.1.4 (Quillen)
For $R \in s\mathcal{R}ing$ and $X \in s\mathcal{A}b\text{-}R$ as well as $Y \in R\text{-}s\mathcal{A}b$, there are natural spectral sequences

(i) $E^2_{p,q} = \pi_p \operatorname{Tor}^{R_\bullet}_q(X_\bullet, Y_\bullet) \quad \Rightarrow \quad \operatorname{Tor}^R_{p+q}(X, Y)$,

where $\operatorname{Tor}^{R_\bullet}_q(X_\bullet, Y_\bullet)$ means taking the Tor-group dimensionwise.

(ii) $E^2_{p,q} = \operatorname{Tor}^{\pi_* R}_p(\pi_* X, \pi_* Y)_q \quad \Rightarrow \quad \operatorname{Tor}^R_{p+q}(X, Y)$,

the left object being the q-th homogeneous component of the graded $\operatorname{Tor}^{\pi_* R}_p(\pi_* X, \pi_* Y)$.

(iii) $E^2_{p,q} = \operatorname{Tor}^R_p(X, \pi_q Y) \quad \Rightarrow \quad \operatorname{Tor}^R_{p+q}(X, Y)$,

where $\pi_q Y \in R\text{-}s\mathcal{A}b$ via $R \otimes \pi_q Y \twoheadrightarrow \pi_0 R \otimes \pi_q Y \xrightarrow{\mu} \pi_q Y$ using Proposition 8.1.3.

(iv) $E^2_{p,q} = \operatorname{Tor}^R_p(\pi_q X, Y) \quad \Rightarrow \quad \operatorname{Tor}^R_{p+q}(X, Y)$.

Proof. See [Qui67] II, §6: Theorem 6.

□

The next corollary demonstrates that a dimensionwise flat replacement will be sufficient to compute Tor-groups.

Corollary 8.1.5
Let $k \in \mathcal{C}Ring$ and $R \in s(k\text{-}\mathcal{A}b)$, as well as $X \in s\mathcal{A}b\text{-}R$ and $Y \in R\text{-}s\mathcal{A}b$. Suppose $X_n \in \mathcal{A}b\text{-}R_n$ is flat, for all $n \geq 0$.

Then the natural map $X \otimes^L_R Y \xrightarrow{\simeq} X \otimes_R Y$ is a weak equivalence.

In particular, if $Y_n, R_n \in k\text{-Mod}$ are flat, for all $n \geq 0$, then $B_\bullet(X, R, Y) \simeq X \otimes^L_R Y$.

By symmetry the same also holds, if we interchange the roles of X and Y.

Proof. Let $E \longrightarrow \mathrm{id}$ be the functorial almost free replacement in R-$s\mathcal{A}b$ of Corollary 7.2.32. Then $X \otimes_R E(Y)$ is the diagonal of the bisimplicial k-module

$$X_p \otimes_{R_p} E_q(Y_p), \qquad p, q \geq 0.$$

where $E \longrightarrow \mathrm{id}$ is the almost free replacement in R_p-$\mathcal{A}b$. The associated spectral sequence (cf. Theorem 7.3.18) is the first one of Proposition 8.1.4

$$E^2_{p,q} = \pi_p \pi_q (X \otimes_R E(Y)) = \pi_p \operatorname{Tor}_q^{R_\bullet}(X_\bullet, Y_\bullet) \quad \Rightarrow \quad \operatorname{Tor}^R_{p+q}(X, Y).$$

Hence by assumption on X we get

$$E^2_{p,q} = \begin{cases} \pi_p(X \otimes_R Y), & q = 0, \\ 0, & q > 0, \end{cases}$$

which proves that the spectral sequence collapses, so that the natural map $X \otimes^L_R Y = X \otimes_R E(Y) \longrightarrow X \otimes_R Y$ is a weak equivalence.

Now if $Y_n, R_n \in k$-Mod are flat, for all $n \geq 0$, then the free/forgetful functor adjunction

$$R\text{-}\mathcal{A}b(R \otimes M, N) = s(k\text{-Mod})(M, U(N))$$

induces a weak equivalence $F(Y) \xrightarrow{\simeq} Y$ with $F(Y)_n = (R_n)^{\otimes n+1} \otimes Y_n$ being flat over R_n, for all $n \geq 0$. Hence the maps $X \otimes^L_R Y \xleftarrow{\simeq} X \otimes^L_R F(Y) \xrightarrow{\simeq} X \otimes_R F(Y)$ are weak equivalences. As $X \otimes_R F(Y) = B_\bullet(X, R, Y)$, this finishes the proof.

□

The following Corollary will be crucial to construct the generalized Hochschild-Serre spectral sequences in the situation of simplicial groups and Lie algebras.

Corollary 8.1.6
For $f \in s\mathcal{R}ing(R, S), X \in s\mathcal{A}b\text{-}S$ and $Y \in R\text{-}s\mathcal{A}b$ there is a converging spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^S_p(X, \operatorname{Tor}^R_q(S, Y)) \quad \Rightarrow \quad \operatorname{Tor}^R_{p+q}(X, Y).$$

Proof. Let $EY \xrightarrow{\simeq} Y$ be an almost free replacement in R-$s\mathcal{A}b$. Then $S \otimes_R EY$ is almost free in S-$s\mathcal{A}b$, implying that

$$\operatorname{Tor}^S_*(X, S \otimes_R EY) = \pi_*(X \otimes_S S \otimes_R EY) = \pi_*(X \otimes_R EY) = \operatorname{Tor}^R_*(X, Y).$$

Thus the Tor spectral sequence of Proposition 8.1.4 (iii) is

$$\begin{array}{ccc} \operatorname{Tor}^S_p(X, \operatorname{Tor}^R_q(S, Y)) & \Rightarrow & \operatorname{Tor}^R_{p+q}(X, Y) \\ \| & & \| \\ \operatorname{Tor}^S_p(X, \pi_q(S \otimes_R EY)) & \Rightarrow & \operatorname{Tor}^S_{p+q}(X, S \otimes_R EY). \end{array}$$

□

8.1.2 Tor-groups and homotopy colimits

We need the following generalization of Shapiro's lemma.

Proposition 8.1.7
Let $k \in \mathcal{C}Ring$ and $c \in k\text{-}\mathcal{A}ss_1(A, B)$, such that B is flat over A.
Then, for all $X \in$ Mod-A and $Y \in B$-Mod, there is a natural weak equivalence

$$X \otimes_A E_\bullet c^*(Y) \xrightarrow{\simeq} X \otimes_A c^*(E_\bullet Y) = (X \otimes_A B) \otimes_B E_\bullet Y.$$

In particular $\mathrm{Tor}^A_*(X, Y) = \mathrm{Tor}^B_*(X \otimes_A B, Y)$.

Proof. For $S \in k\text{-}\mathcal{A}ss_1$ consider the free/forgetful functor adjunction

$$S\text{-}\mathcal{A}b(F_S(X), Y) = \mathcal{S}et(X, U_S(Y)).$$

Moreover we let $(c_!, c^*)$ denote the adjunction

$$B\text{-}s\mathcal{A}b(c_! X, Y) = A\text{-}s\mathcal{A}b(X, c^* Y),$$

i.e. $c_! = (B \otimes_A \text{-})$ and c^* is the forgetful functor. Then c induces a map of monads $(U_A F_A) \longrightarrow (U_B F_B)$ and a map of right $(U_A F_A)$-modules $F_A \xrightarrow{\eta} c^* c_! F_A = c^* F_B$ (cf. Remark 7.2.33). In particular we obtain a weak equivalence

$$E_\bullet c^* Y = B_\bullet(F_A, U_A F_A, U_B(Y)) \xrightarrow{\simeq} B_\bullet(c^* F_B, U_B F_B, U_B(Y)) = c^* E_\bullet Y,$$

inducing the desired map. By Corollary 8.1.5 it is also a weak equivalence, because B is flat over A and thus $c^* E_\bullet Y \xrightarrow{\simeq} c^* Y$ is a dimensionwise flat replacement.

□

Proposition 8.1.8
Let $k \in \mathcal{C}Ring$ and $A_i \in k\text{-}\mathcal{A}ss$ and $A = \coprod_i A_i$.
Then the right $(A_i)_+$-module inclusions $A_i \hookrightarrow A$ extend to a right A_+-module isomorphism

$$\bigoplus_{i \in I} A_i \otimes_{(A_i)_+} A_+ \xrightarrow{\sim} A.$$

In particular, if $A_i \in k$-Mod is flat, for all i, then Proposition 8.1.7 yields a natural weak equivalence

$$\bigoplus_i B_\bullet(A_i, (A_i)_+, (\iota_i)^* Y) \xrightarrow{\simeq} B_\bullet(A, A_+, Y), \qquad Y \in A_+\text{-Mod}.$$

On homotopy groups, we thus have $\bigoplus_i \mathrm{Tor}^{(A_i)_+}_*(A_i, Y) = \mathrm{Tor}^{A_+}_*(A, Y)$ by Corollary 8.1.5.

Proof. The algebra A is the free algebra generated by the subalgebras A_i. Hence explicitly we have

$$A_+ = \bigoplus_{\substack{i_1 \neq \ldots \neq i_n, \\ n \geq 0}} A_{i_1} \otimes \ldots \otimes A_{i_n} = (A_i)_+ \otimes \bigoplus_{\substack{i \neq i_1 \neq \ldots \neq i_n, \\ n \geq 0}} A_{i_1} \otimes \ldots \otimes A_{i_n}, \qquad i \in I.$$

It follows that

$$\bigoplus_i A_i \otimes_{(A_i)_+} A_+ = \bigoplus_{\substack{i\neq i_1\neq \ldots \neq i_n,\\ n\geq 0}} A_i \otimes A_{i_1} \otimes \ldots \otimes A_{i_n} = A,$$

which is A_+-linear from the right and induced by the inclusions $A_i \hookrightarrow A$. Now suppose A_i is free over k, for all i. Then A_+ is free over $(A_i)_+$, for all i. Using Proposition 8.1.7 this proves

$$\mathrm{Tor}_*^{A_+}(A, M) = \bigoplus_i \mathrm{Tor}_*^{A_+}(A_i \otimes_{(A_i)_+} A_+, M) = \bigoplus_i \mathrm{Tor}_*^{(A_i)_+}(A_i, M).$$

□

Proposition 8.1.9
Let $k \in \mathcal{C}Ring$ and $A \in \mathcal{C}\mathrm{AT}(I, k\text{-}\mathcal{A}ss_1)$, for some $I \in \mathcal{C}at$.
Then there is a natural map of bisimplicial k-modules

$$\operatorname*{hocolim}_I C_\bullet(A, X) \longrightarrow C_\bullet(\operatorname*{hocolim}_I A, X), \qquad X \in (\operatorname*{colim}_I A, \operatorname*{colim}_I A)\text{-Mod},$$

where the homotopy colimits are taken dimensionwise.

Proof. Let k-$\mathcal{A}ct$ be the category, whose objects are pairs (A, X), where $A \in k\text{-}\mathcal{A}ss_1$ and $X \in (A, A)$-Mod. Its homomorphisms are given by

$$k\text{-}\mathcal{A}ct((A, X), (B, Y)) = \coprod_{f \in k\text{-}\mathcal{A}ss_1(A,B)} (A, A)\text{-Mod}(X, f^*Y) \subset k\text{-}\mathcal{A}ss_1(A, B) \times k\text{-Mod}(X, Y),$$

with composition $(f, g) \circ (f', g') = (ff', gg')$. Given a set of objects $(A_i, X_i) \in k\text{-}\mathcal{A}ct$, define $A = \coprod_i A_i \in k\text{-}\mathcal{A}ss_1$ and $X = \bigoplus_i A \otimes_{A_i} X_i \otimes_{A_i} A \in (A, A)$-Mod. Then $(A, X) = \coprod_i (A_i, X_i)$ in k-$\mathcal{A}ct$, because

$$\begin{aligned}
&k\text{-}\mathcal{A}ct((A, X), (B, Y))\\
=&\{(f, g) \in k\text{-}\mathcal{A}ss_1(A, B) \times k\text{-Mod}(X, Y);\ g(axa') = f(a)g(x)f(a')\}\\
=&\{(f, g) \in \prod_i k\text{-}\mathcal{A}ss_1(A_i, B) \times k\text{-Mod}(X_i, Y);\ g_i(axa') = f_i(a)g_i(x)f_i(a')\}\\
=&\prod_i k\text{-}\mathcal{A}ct((A_i, X_i), (B, Y)).
\end{aligned}$$

In particular we can define homotopy colimits in $\mathcal{A}$-$\mathcal{A}ct$ and in our given situation we have $\operatorname{hocolim}_I(A, X) = (\operatorname{hocolim}_I A, X')$, for some bimodule X' over $\operatorname{hocolim}_I A$. As the inclusion maps $A_i \xrightarrow{\iota_i} \operatorname{colim}_I A$ define a natural transformation $(A, X) \longrightarrow \mathrm{const}(\operatorname{colim}_I A, X)$ in $\mathcal{C}\mathrm{AT}(I, k\text{-}\mathcal{A}ct)$, we get a map

$$(\operatorname*{hocolim}_I A, X') = \operatorname*{hocolim}_I (A, X) \longrightarrow (\operatorname*{colim}_I A, X).$$

It factors over $(\operatorname{hocolim}_I A, X)$, because the action of $\operatorname{hocolim}_I A$ on X is induced by the action of $\operatorname{colim}_I A$. Now using that the construction $C_\bullet$ defines a functor

$$k\text{-}\mathcal{A}ct \longrightarrow s(k\text{-Mod}), \quad (A, X) \longmapsto C_\bullet(A, X),$$

we get the desired map as the composition

$$\operatorname*{hocolim}_I C_\bullet(A, X) \longrightarrow C_\bullet \operatorname*{hocolim}_I (A, X) \longrightarrow C_\bullet(\operatorname*{hocolim}_I A, X).$$

□

Proposition 8.1.10
Let $k \in \mathcal{C}Ring$ and $A \in \mathcal{C}\mathrm{AT}(I, k\text{-}\mathcal{A}ss)$, for some $I \in \mathcal{C}at$.
Then there is a natural map of simplicial k-modules

$$\operatorname*{hocolim}_I B(k, A_+, Y) \longrightarrow B(k, \operatorname*{hocolim}_I A_+, Y), \qquad Y \in (\operatorname*{colim}_I A_+)\text{-Mod},$$

which is a weak equivalence, if BI is contractible and A_i is a flat k-module, for all $i \in I$.

Proof. Setting $X = Y \otimes k \in (\operatorname{colim}_I A_+, \operatorname{colim}_I A_+)$-Mod, the map is the diagonal of the map of Proposition 8.1.9. We claim that the diagonal of the following commutative square of bisimplicial k-modules is homotopy cocartesian

$$\begin{tikzcd}
{}^{BI}Y \arrow[rr] \arrow[d, equal] & & Y \arrow[d, equal] \\
\operatorname{hocolim}_I B_0(k, A_+, Y) \arrow[rr] \arrow[d] & & B_0(k, \operatorname{hocolim}_I A_+, Y) \arrow[d] \\
\operatorname{hocolim}_I B_\bullet(k, A_+, Y) \arrow[rr] & & B_\bullet(k, \operatorname{hocolim}_I A_+, Y).
\end{tikzcd}$$

This will imply the statement, because if BI is contractible, then the upper horizontal map and thus also the lower horizontal map is a weak equivalence. As the diagonal perserves weak equivalences by Corollary 7.3.19 or Corollary 7.3.22, it suffices to check that the square is homotopy cocartesian in every dimension $n \geq 0$ of the homotopy colimit. By setting

$$S(i, n) := \coprod_{i_1, \dots, i_n \in I} I(i, i_1) \times \dots \times I(i_{n-1}, i_n), \qquad i \in I, \quad n \geq 0,$$

we see that the in an arbitrary category cocomplete category $\mathcal{C}$, the simplicial homotopy colimit is given by

$$(\operatorname*{hocolim}_I C)_n = \coprod_{i_0, \dots, i_n \in I} {}^{I(i_0, i_1) \times \dots \times I(i_{n-1}, i_n)} C_{i_0} = \coprod_{i \in I} {}^{S(i,n)} C_i, \qquad C \in \mathcal{C}\mathrm{AT}(I, \mathcal{C}), \quad n \geq 0.$$

So in dimension $n \geq 0$ (in homotopy colimit direction), the diagram from above can be identified with the outer square of the diagram

$$\begin{tikzcd}
\coprod_i{}^{S(i,n)} Y \arrow[rr] \arrow[d, "\simeq"'] & & Y \arrow[d, "\simeq"] \\
\coprod_i{}^{S(i,n)} B_\bullet((A_i)_+, (A_i)_+, Y) \arrow[r] \arrow[d] & B_\bullet(\coprod_i{}^{S(i,n)}(A_i)_+, \coprod_i{}^{S(i,n)}(A_i)_+, Y) \arrow[d] \\
\coprod_i{}^{S(i,n)} B_\bullet(k, (A_i)_+, Y) \arrow[r] & B_\bullet(k, \coprod_i{}^{S(i,n)}(A_i)_+, Y).
\end{tikzcd}$$

Here the lower vertical maps are induced by the augmentation maps and the upper vertical maps are given by the extra-degeneracies $s_{-1} = \eta \otimes \mathrm{id}$ for the bar constructions in the middle. As the latter are weak equivalences, it suffices to check that the lower square is homotopy cocartesian. As the two vertical maps are surjective, we can equivalently prove that it is homotopy cartesian. But the induced map on their kernel is the map

$$\coprod_i{}^{S(i,n)} B_\bullet(A_i, (A_i)_+, Y) \xrightarrow{\simeq} B_\bullet(\coprod_i{}^{S(i,n)} A_i, \coprod_i{}^{S(i,n)}(A_i)_+, Y),$$

which by Proposition 8.1.8 is a weak equivalence, as by assumption A_i is flat over k, for all $i \in I$. This concludes the proof.

□

Proposition 8.1.11
Let $k \in s\mathcal{C}Ring$ and $c \in k\text{-}\mathcal{A}ss(A, B)$, such that in every dimension

$$c_n \cong \iota_{A_n} : A_n \longrightarrow A_n * (B/A)_n, \qquad n \geq 0,$$

and $(B/A)_n = B_n *_{A_n} 0 \in k\text{-}\mathcal{A}ss$ is a flat k-module, for all $n \geq 0$.
Then, for all $M \in (B/A)_+\text{-Mod}$, there is a natural long exact sequence

$$\ldots \longrightarrow \mathrm{Tor}_2^{(B/A)_+}(k, M) \xrightarrow{\partial} \mathrm{Tor}_1^{A_+}(k, M) \longrightarrow \mathrm{Tor}_1^{B_+}(k, M) \longrightarrow \mathrm{Tor}_1^{(B/A)_+}(k, M).$$

Proof. There are two ways to prove this. One can either use the properties of the map $c_+ \in k\text{-}\mathcal{A}ss_1(A_+, B_+)$ to show that the quotient map

$$\mathrm{hocolim}\,(k \leftarrow A_+ \xrightarrow{c} B_+) \xrightarrow{\simeq} (B/A)_+$$

is a weak equivalence. Then applying Proposition 8.1.10 we obtain a homotopy cocartesian square

$$\begin{tikzcd}
B_\bullet(k, A_+, M) \arrow[r] \arrow[d] & B_\bullet(k, B_+, M) \arrow[d] \\
B_\bullet(k, k, M) \arrow[r] & B_\bullet(k, (B/A)_+, M).
\end{tikzcd}$$

As $B_\bullet(k, k, M) \xrightarrow{\simeq} M$, this yields the long exact sequence.

Alternatively one can use Proposition 8.1.8 to prove the result more directly, which will be done in more detail in the following. First note that for any associative algebra $R \in k\text{-}\mathcal{A}ss$ and $M \in R\text{-Mod}$, the long exact Tor-sequence to

$$0 \longrightarrow R \longrightarrow R_+ \longrightarrow k \longrightarrow 0$$

yields natural maps $\partial : \mathrm{Tor}^{R_+}_{n+1}(k, M) \longrightarrow \mathrm{Tor}^{R_+}_n(R, M)$, being isomorphisms for $n \geq 1$ and injective for $n = 0$. Now by Proposition 8.1.8 the inclusion $A \longrightarrow B$ induces an exact sequence of right B_+-modules

$$0 \longrightarrow A \otimes_{A_+} B_+ \longrightarrow B \longrightarrow C \longrightarrow 0,$$

where the cokernel C is isomorphic to $(B/A)_n \otimes_{((B/A)_n)_+} (B_n)_+$, in every dimension $n \geq 0$. We get an induced long exact sequence by applying $\mathrm{Tor}^{B_+}_*(\text{-}, M)$ and the statement follows by identifying

$$\mathrm{Tor}^{B_+}_*(A \otimes_{A_+} B_+, M) = \mathrm{Tor}^{A_+}_*(A, M), \qquad \mathrm{Tor}^{B_+}_*(C, M) = \mathrm{Tor}^{(B/A)_+}_*(B/A, M).$$

By assumption $(B/A)_n$ is flat and B_n is the coproduct of A_n and $(B/A)_n$ in every dimension $n \geq 0$. So the first equality follows from Proposition 8.1.7. We denote by $E^S_\bullet(X) \xrightarrow{\simeq} X$ an almost free replacement of $X \in S_+\text{-Mod}$, for $S \in k\text{-}\mathcal{A}ss$. We have

$$C \otimes_{B_+} E^B_\bullet(M) \xleftarrow{\simeq} C \otimes_{B_+} E^B_\bullet E^{B/A}_\bullet(M) \xrightarrow{\simeq} C \otimes_{B_+} E^{B/A}_\bullet(M).$$

Indeed, since $B_n \cong A_n * (B/A)_n$ and $C_n \cong (B/A)_n \otimes_{((B/A)_n)_+} (B_n)_+$ is free relative to $((B/A)_n)_+$, in every dimension $n \geq 0$, the right map is a weak equivalence. This proves the second identity.

□

8.2 Simplicial groups

8.2.1 Homology and cohomology

Definition 8.2.1
*The **homology** and **cohomology** of a simplicial group $G \in s\mathcal{G}rp$ with coefficients $M \in k[G]\text{-}s\mathcal{A}b$, where $k \in s\mathcal{R}ing$, are defined as*

(i) $H_*(G, M) = \mathrm{Tor}^{k[G]}_*(k, M) = \pi_*(k \otimes^L_{k[G]} M) = \mathbb{L}_*(\text{-} \otimes_{k[G]} \text{-})(k, M)$,

(ii) $H^*(G, M) = \mathrm{Ext}^*_{k[G]}(k, M) = \mathbb{R}^{-*}\underline{k[G]\text{-}s\mathcal{A}b}(\text{-}, \text{-})(M, k)$.

Remark 8.2.2
Let $k \in s\mathcal{R}ing$, $G \in s\mathcal{G}rp$ and $M \in k[G]\text{-}s\mathcal{A}b$.
Then by Proposition 8.1.4 (i) and (iii) we have spectral sequences

(i) $E^2_{p,q} = \pi_p \underline{H}_q(G, M) \quad \Rightarrow \quad H_{p+q}(G, M)$,
where $\underline{H}_q(G, M)_p = H_q(G_p, M_p)$, for all $p, q \geq 0$.

(ii) $E^2_{p,q} = H_p(G, H_q M) \quad \Rightarrow \quad H_{p+q}(G, M)$,

where $H_q M \in k[G]\text{-}s\mathcal{A}b$ *via* $k[G] \otimes H_q M \twoheadrightarrow \pi_0 k[G] \otimes H_q M \xrightarrow{\mu} H_q M$.

Corollary 8.2.3
Let $k \in s\mathcal{R}ing$, $G \in s\mathcal{G}rp$ *and* $M \in k[G]\text{-}s\mathcal{A}b$.

(i) $H_0(G, M) = (\pi_0 M)_{\pi_0 G} = (\pi_0 k) \otimes_{(\pi_0 k)[\pi_0 G]} (\pi_0 M)$,

(ii) $H_1(G, M) = (\pi_0 G)/[\pi_0 G, \pi_0 G] \otimes M$, *if* G *acts trivially on a constant* $M \in \mathcal{A}b$.
If G *is* $(r-1)$*-reduced, for some* $r \geq 1$*, we also have* $H_{r+1}(G, M) = (\pi_r G) \otimes M$.

Proof. In the notation of (i) of the preceding remark, we have natural isomorphisms

$$\underline{H}_0(G, M)_p = H_0(G_p, M_p) = \mathrm{Tor}_0^{k_p[G_p]}(k_p, M_p) = k_p \otimes_{k_p[G_p]} M_p = (M_p)_{G_p}, \qquad p \geq 0.$$

(i) Using the spectral sequence loc. cit., we get an isomorphism

$$H_0(G, M) = E^2_{0,0} = \pi_0 \underline{H}_0(G, M) = \pi_0(k \otimes_{k[G]} M) = (\pi_0 k) \otimes_{(\pi_0 k)[\pi_0 G]} (\pi_0 M) = (\pi_0 M)_{\pi_0 G},$$

where the fourth equality follows from the fact that tensor product are right exact and for the sequence

$$\mu \otimes \mathrm{id} - \mathrm{id} \otimes \mu : k \otimes k[G] \otimes M \longrightarrow k \otimes M,$$

taking the cokernel, which is $k \otimes_{k[G]} M$, commutes with taking the colimit over Δ^{op}, which is application of π_0.

(ii) If $M \in \mathcal{A}b$ is constant and G acts trivially, then we have

$$\underline{H}_1(G, M)_p = H_1(G_p, M) = G_p/[G_p, G_p] \otimes M, \qquad p \geq 0,$$

proving that $\pi_0 \underline{H}_1(G, M) = (\pi_0 G)/[\pi_0 G, \pi_0 G] \otimes M$, as tensoring with M and the abelianization functor of groups are left adjoints and thus commute with the colimit over Δ^{op}, which is application of π_0. So the result follows from the low term exact sequence for the spectral sequence

$$H_2(G, M) \longrightarrow E^2_{2,0} \xrightarrow{d} E^2_{0,1} \longrightarrow H_1(G, M) \longrightarrow E^2_{1,0} \longrightarrow 0,$$

since $\underline{H}_0(G, M)_p = M_{G_p} = M$ is constant and thus

$$E^2_{p,0} = \pi_p \underline{H}_0(G, M) = 0, \qquad p \geq 1.$$

Note that in the second case, the complex $\underline{H}_q(G_\bullet, M)$ is $(r-1)$-reduced, for $q \geq 1$, and hence

$$E^2_{p,q} = \pi_p \underline{H}_q(G, M) = 0, \qquad 0 < p < r,\ q \geq 1.$$

So we can deduce an exact sequence

$$H_{r+2}(G, M) \longrightarrow E^2_{r+2,0} \xrightarrow{d} E^2_{r,1} \longrightarrow H_{r+1}(G, M) \longrightarrow E^2_{r+1,0} \longrightarrow 0,$$

and the statement follows by using the isomorphism $\pi_r \underline{H}_1(G, M) = (\pi_r G) \otimes M$.

$\square$

8.2.2 The Homology spectral sequence for a fibration

We may deduce the following Proposition, which is probably well-known.

Proposition 8.2.4
Let $k \in s\mathcal{R}ing$, $N \lhd G \in s\mathcal{G}rp$ and $M \in k[G]$-$s\mathcal{A}b$. Then there is a spectral sequence

$$E^2_{p,q} = H_p(G/N, H_q(N, M)) \quad \Rightarrow \quad H_{p+q}(G, M),$$

where $H_q(N, M) \in k[G/N]$-$s\mathcal{A}b$ via

$$k[G/N] \otimes H_q(N, M) \twoheadrightarrow \pi_0 k[G/N] \otimes H_q(N, M) \xrightarrow{\mu} H_q(N, M).$$

Proof. By Corollary 8.1.6 we have a spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{k[G/N]}(k, \mathrm{Tor}_q^{k[G]}(k[G/N], M)) \quad \Rightarrow \quad \mathrm{Tor}_{p+q}^{k[G]}(k, M) = H_{p+q}(G, M).$$

Since $k[G]$ is free over $k[N]$ every almost free replacement of M in $k[G]$-$s\mathcal{A}b$ is also almost free over $k[N]$. This implies

$$\begin{aligned}\mathrm{Tor}_*^{k[G]}(k[G/N], M) &= \pi_*(k[G/N] \otimes_{k[G]} EM) \cong \pi_*(k \otimes_{k[N]} k[G] \otimes_{k[G]} EM) \\ &\cong \pi_*(k \otimes_{k[N]} EM) = \mathrm{Tor}_*^{k[N]}(k, M) = H_*(N, M),\end{aligned}$$

and hence $E^2_{p,q} = \mathrm{Tor}_p^{k[G/N]}(k, \mathrm{Tor}_q^{k[G]}(k[G/N], M)) = H_p(G/N, H_q(N, M))$.

□

8.2.3 Homology of free groups

Proposition 8.2.5
For every $G \in s\mathcal{G}rp$ there is a natural epimorphism of simplicial abelian groups

$$\tilde{\mathbb{Z}}BG \twoheadrightarrow \tilde{\mathbb{Z}}B(G/[G,G]) \twoheadrightarrow B(G/[G,G]).$$

It is a weak equivalence, if G is almost free.
In particular $H_{*+1}(G, \mathbb{Z}) \xrightarrow{\sim} \pi_* \mathbb{L}(\text{-}/[\text{-},\text{-}])(G)$, for all $G \in s\mathcal{G}rp$.

Proof. The first map is induced by the natural epimorphism $G \twoheadrightarrow G/[G,G]$. Using the $B(G/[G,G]) \in s\mathcal{A}b$, the second map is the counit of the free/forgetful functor adjunction

$$s\mathcal{A}b(\tilde{\mathbb{Z}}X, Y) = s\mathcal{S}et_*(X, Y).$$

Now suppose G is almost free and consider both objects as the diagonal of a bisimplicial object, where the second dimension is induced by the functor B. We compute the first pages of the associated spectral sequences (cf. Theorem 7.3.18) as

$$E^1_{p,q} = \tilde{H}_p(BG_q \wedge \mathbb{Z}) = \tilde{H}_p(G_q, \mathbb{Z}) \xrightarrow{\sim} \pi_p(B(G_q/[G_q, G_q]) = \begin{cases} G_q/[G_q, G_q], & p = 1, \\ 0, & p \neq 1. \end{cases}$$

Moreover a generator $x \in G_q$ induces elements $[x-1] = [x] \in H_1(BG_q \wedge \mathbb{Z})$, which is mapped to $[x] \otimes a \in \pi_1(G_q/[G_q, G_q]) = G_q/[G_q, G_q] \otimes A$, for all $a \in A$. This proves that the given map induces an isomorphism.

We give $s\mathcal{A}b$ the model structure of Theorem 7.2.40, induced by the adjunction

$$s\mathcal{A}b(X/[X,X], Y) = s\mathcal{G}rp(X, I(Y)),$$

where I is the inclusion functor. Since I preserves (trivial) fibrations, its left adjoint preserves (trivial) cofibrations and so it has a total left derived functor by Theorem 7.1.26. Hence we have

$$BG \wedge \mathbb{Z} \xleftarrow{\simeq} BE(G) \wedge \mathbb{Z} \xrightarrow{\simeq} B(EG/[EG, EG]) = B\mathbb{L}(\text{-}/[\text{-},\text{-}])(G), \qquad G \in s\mathcal{G}rp,$$

where $EG \xrightarrow{\simeq} G$ is an almost free replacement. As B shifts the homotopy groups by 1 dimension, we get

$$H_{*+1}(G, \mathbb{Z}) = \pi_{*+1} BG \wedge \mathbb{Z} = \pi_{n+1} B\mathbb{L}(\text{-}/[\text{-},\text{-}])(G) = \pi_* \mathbb{L}(\text{-}/[\text{-},\text{-}])(G).$$

□

8.2.4 The Whitehead Theorem

The following proposition is completely analogous to the Whitehead Theorem for simplicial profinite groups, presented in [Qui69b].

Proposition 8.2.6 (Whitehead-Quillen)
Let $f \in s\mathcal{G}rp(G, H)$ and $c \geq 0$.

(i) *Suppose $\pi_0 G = 1$.*

Then G is c-connected, if and only if $\tilde{H}_*(G, \mathbb{Z})$ is $(c+1)$-connected.

(ii) *Suppose $\pi_0 f$ is an isomorphism.*

Then f is c-connected, if and only if $H_*(f, \mathbb{Z}[\pi_0 H])$ is $(c+1)$-connected.

Proof.

(i) By Corollary 8.2.3 (ii) we have

$$H_1(G, \mathbb{Z}) = \pi_0 G/[\pi_0 G, \pi_0 G] = 0,$$

so there is nothing to show for the case $c = 0$.

Suppose we have proven the equivalence for all simplicial groups, for some $c \geq 0$. Let $G \in s\mathcal{G}rp$ be connected and suppose $\tilde{H}_*(G, \mathbb{Z})$ is $(c+2)$-connected. Since G is connected, by Proposition 7.2.38 we have a short exact sequence of simplicial groups

$$1 \longrightarrow \Omega G \longrightarrow FG \xrightarrow{d_{n+1}} G \longrightarrow 1.$$

Since G is connected, the action of G on $H_*(\Omega G, \mathbb{Z})$ is trivial, implying that

$$H_*(G, H_0(\Omega G, \mathbb{Z})) = H_*(G, \mathbb{Z}), \qquad H_0(G, H_*(\Omega G, \mathbb{Z})) = H_*(\Omega G, \mathbb{Z}),$$

by Corollary 8.2.3 (i). In particular the homology spectral sequence of Proposition 8.2.4

$$E^2_{p,q} = H_p(G, H_q(\Omega G, \mathbb{Z})) \quad \Rightarrow \quad H_{p+q}(FG, \mathbb{Z})$$

satisfies assumption (F) of Proposition 8.4.6 (i). As FG is contractible, $\tilde{H}_*(FG, \mathbb{Z})$ is acyclic. Hence the map $H_*(FG, \mathbb{Z}) \longrightarrow H_*(G, \mathbb{Z})$ is $(c+1)$-connected, because by assumption $\tilde{H}_*(G, \mathbb{Z})$ is $(c+2)$-connected. So by Proposition 8.4.6 (i) we see that $\tilde{H}_*(\Omega G, \mathbb{Z})$ is $(c+1)$-connected. Moreover $\pi_0 \Omega G \cong \pi_1 G$ is abelian by Proposition 7.2.39 (iii), implying that

$$\pi_0 \Omega G = \pi_0 \Omega G / [\pi_0 \Omega G, \pi_0 \Omega G] = H_1(\Omega G, \mathbb{Z}),$$

by Corollary 8.2.3 (ii). But the first homology group is zero, because $1 \leq (c+1)$. So by induction hypothesis ΩG is c-connected or equivalently G is $(c+1)$-connected.

Vice versa, if G is $(c+1)$-connected, then ΩG is c-connected and by induction hypothesis $\tilde{H}_*(\Omega G, \mathbb{Z})$ is $(c+1)$-connected. Equivalently $H_*(FG, \mathbb{Z}) \longrightarrow H_*(G, \mathbb{Z})$ is $(c+1)$-connected by Proposition 8.4.6 (i) again, proving that $\tilde{H}_*(G, \mathbb{Z})$ is $(c+2)$-connected, as FG is contractible. Be aware that we still have a surjection in dimension $(c+2)$.

(ii) In the model category of simplicial groups we can factor f into a trivial cofibration followed by a fibration $G \overset{\simeq}{\rightarrowtail} G' \xrightarrow{f'} H$. By replacing f by f' we can therefore assume that f is a fibration. Then Proposition 7.2.35 implies that f is surjective, because $\pi_0 f$ is an isomorphism. Suppose $H_*(f, \mathbb{Z}[\pi_0 H])$ is $(c+1)$-connected. By Corollary 8.2.3 we have

$$\begin{aligned} H_0(H, H_*(N, \mathbb{Z}[\pi_0 H])) &= (H_*(N, \mathbb{Z}) \otimes \mathbb{Z}[\pi_0 H])_{\pi_0 H} = H_*(N, \mathbb{Z}), \\ H_*(H, H_0(N, \mathbb{Z}[\pi_0 H])) &= H_*(H, \mathbb{Z}[\pi_0 H]), \end{aligned}$$

so again hypothesis (F) of Proposition 8.4.6 (i) holds for the homology spectral sequence

$$E^2_{p,q} = H_p(H, H_q(N, \mathbb{Z}[\pi_0 H])) \quad \Rightarrow \quad H_{p+q}(G, \mathbb{Z}[\pi_0 H]),$$

and it follows that $H_*(N, \mathbb{Z})$ is $(c+1)$-connected. Since $\pi_0 f$ is an isomorphism, the long exact sequence yields an exact sequence

$$\pi_1 G \xrightarrow{\pi_1 f} \pi_1 H \xrightarrow{\partial} \pi_0 N \longrightarrow 1,$$

proving that $\pi_0 N$ is abelian. As $1 \leq (c+1)$, we get

$$\pi_0 N = (\pi_0 N)/[\pi_0 N, \pi_0 N] = H_1(N, \mathbb{Z}) = 0,$$

by Corollary 8.2.3 (ii). So N satisfies the hypothesis of (i) and therefore must be c-connected. Equivalently f is c-connected.

Vice versa if f is c-connected, so is N and thus $H_*(N, \mathbb{Z})$ is $(c+1)$-connected by (i). Equivalently $H_*(f, \mathbb{Z}[\pi_0 H])$ is $(c+1)$-connected by the argument using the spectral sequence.

□

8.3 Simplicial Lie algebras

Proposition 8.3.1
Let $k \in \mathcal{C}Ring$. For every $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$, the following holds.

(i) *$\pi_n\mathfrak{g}$ is a subquotient Lie algebra of $\mathfrak{g}_n$, for all $n \geq 0$.*

(ii) *$\pi_n\mathfrak{g}$ is abelian, for all $n \geq 1$.*

(iii) *The adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_n$ induces a Lie algebra action of $\pi_0\mathfrak{g}$ on $\pi_n\mathfrak{g}$, for $n \geq 1$.*

Proof. The proof is exactly the same as its multiplicative analogue Proposition 7.2.39.

(i) The submodule $N_n\mathfrak{g} \leq \mathfrak{g}_n$ is normal, being the intersection of kernels. Since

$$[x, d_0(y)] = [d_0s_0(x), d_0(y)] = d_0[s_0(x), y] \in d_0N_{n+1}\mathfrak{g}, \qquad x \in \mathfrak{g}_n,\ y \in N_{n+1}\mathfrak{g},$$

also $d_0N_{n+1}\mathfrak{g}$ is normal in $\mathfrak{g}_n$. It follows that $d_0N_{n+1}\mathfrak{g}$ also is normal $\ker(N_n\mathfrak{g} \xrightarrow{d} N_{n-1}(\mathfrak{g}))$, proving that $\pi_n\mathfrak{g}$ is a subquotient Lie algebra of $\mathfrak{g}_n$.

(ii) Taking the adjoint action in every dimension we get $\mathrm{Ad}(\mathfrak{g}) \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$. By Proposition 8.1.3 (i) we get an induced action

$$\pi_n\mathfrak{g} \otimes \pi_n\mathfrak{g} \longrightarrow \pi_nU_k(\mathfrak{g}) \otimes \pi_n\mathfrak{g} \longrightarrow \pi_n\mathfrak{g}, \qquad x \otimes y \longmapsto [x, y],$$

which is trivial in dimension $n > 0$. Equivalently $\pi_n\mathfrak{g}$ is abelian, for all $n > 0$.

(iii) By Proposition 8.1.3 (ii) we get an induced action of $\pi_0\mathfrak{g} \longrightarrow \pi_0U_k(\mathfrak{g})$ on $\pi_n\mathfrak{g}$, for every $n \geq 0$.

□

8.3.1 Homology and cohomology

Definition 8.3.2
For $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ a simplicial Lie algebra, $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.
The ***(dimensionwise) homology*** *and* ***cohomology*** *of $\mathfrak{g}$ with coefficients M is*

(i) $H_*(\mathfrak{g}, M) = \mathrm{Tor}_*^{U_k(\mathfrak{g})}(k, M) = \mathbb{L}_*(k \otimes_{U_k(\mathfrak{g})} \text{-})(M) = \pi_*(k \otimes^L_{U_k(\mathfrak{g})} M)$,

(ii) $H^*(\mathfrak{g}, M) = \mathrm{Ext}^*_{U_k(\mathfrak{g})}(k, M) = \mathbb{R}^{-*}\underline{U_k(\mathfrak{g})\text{-}s\mathcal{A}b}(k, \text{-})(M) = \mathbb{R}^{-*}\underline{U_k(\mathfrak{g})\text{-}s\mathcal{A}b}(\text{-}, \text{-})(k, M)$.

Remark 8.3.3
For $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ a simplicial Lie algebra, $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.

(i) *For constant $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ its homology agrees with the usual Lie algebra homology.*

(ii) *If $\mathfrak{h} \longrightarrow \mathfrak{g}$ is a homotopy equivalence, then so is $U_k(\mathfrak{h}) \longrightarrow U_k(\mathfrak{g})$ and hence $H_*(\mathfrak{h}, M) \xrightarrow{\sim} H_*(\mathfrak{g}, M)$ by Proposition 8.1.4 (ii).*

It is not clear right from the definition, wheather U_k preserves general weak equivalences between Lie algebras. Therefore it is also not clear, if this dimensionwise homology is a homotopy invariant for simplicial Lie algebas.

Remark 8.3.4
Let $k \in \mathcal{C}Ring$, $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ and $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.
Then by Proposition 8.1.4 (i) and (iii) we have spectral sequences

(i) $E^2_{p,q} = \pi_p \underline{H}_q(\mathfrak{g}, M) \quad \Rightarrow \quad H_{p+q}(\mathfrak{g}, M)$,
where $\underline{H}_q(\mathfrak{g}, M)_p = H_q(\mathfrak{g}_p, M_p)$, for all $p, q \geq 0$.

(ii) $E^2_{p,q} = H_p(\mathfrak{g}, H_q M) \quad \Rightarrow \quad H_{p+q}(\mathfrak{g}, M)$,
where $H_q M \in \mathfrak{g}\text{-}s\mathcal{A}b$ via $U_k(\mathfrak{g}) \otimes H_q M \twoheadrightarrow \pi_0 U_k(\mathfrak{g}) \otimes H_q M \xrightarrow{\mu} H_q M$.

Corollary 8.3.5
Let $k \in \mathcal{C}Ring$, $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ and $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.

(i) $H_0(\mathfrak{g}, M) = H_0(\mathfrak{g}, M) = (\pi_0 M)_{\pi_0 \mathfrak{g}} = (\pi_0 k) \otimes_{U_k(\pi_0 \mathfrak{g})} (\pi_0 M)$,

(ii) $H_1(\mathfrak{g}, M) = (\pi_0 \mathfrak{g})/[\pi_0 \mathfrak{g}, \pi_0 \mathfrak{g}] \otimes M$, *if $\mathfrak{g}$ acts trivially on a constant $M \in k\text{-}s\mathcal{A}b$.*

Proof. In the notation of Remark 8.3.4 (i) we have natural isomorphisms

$$\underline{H}_0(\mathfrak{g}, M)_p = H_0(\mathfrak{g}_p, M_p) = \mathrm{Tor}_0^{U_k(E_\bullet \mathfrak{g}_p)}(k, M_p) = k \otimes_{U_k(\mathfrak{g}_p)} M_p = M_p/[\mathfrak{g}_p, M_p], \qquad p \geq 0.$$

(i) Using the spectral sequence loc. cit., we get an isomorphism

$$H_0(G, M) = E^2_{0,0} = \pi_0 \underline{H}_0(\mathfrak{g}, M) = (\pi_0 M)/[\pi_0 \mathfrak{g}_p, \pi_0 M] = k \otimes_{U_k(\pi_0 \mathfrak{g})} (\pi_0 M).$$

(ii) If $M \in k\text{-}s\mathcal{A}b$ is constant and $\mathfrak{g}$ acts trivially, then we have

$$\underline{H}_1(\mathfrak{g}, M)_p = H_1(\mathfrak{g}_p, M) = \mathfrak{g}_p/[\mathfrak{g}_p, \mathfrak{g}_p] \otimes M, \qquad p \geq 0,$$

proving that $\pi_0 \underline{H}_1(\mathfrak{g}, M) = (\pi_0 \mathfrak{g})/[\pi_0 \mathfrak{g}, \pi_0 \mathfrak{g}] \otimes M$. So the result follows from the low term exact sequence for the spectral sequence

$$H_2(\mathfrak{g}, M) \longrightarrow E^2_{2,0} \xrightarrow{d} E^2_{0,1} \longrightarrow H_1(\mathfrak{g}, M) \longrightarrow E^2_{1,0} \longrightarrow 0,$$

since $\underline{H}_0(\mathfrak{g}, M)_p = M/[\mathfrak{g}_p, M] = M$ is constant and thus

$$E^2_{p,0} = \pi_p \underline{H}_0(\mathfrak{g}, M) = 0, \qquad p \geq 1.$$

□

Remark 8.3.6
Let $k \in \mathcal{C}Ring$, $\mathfrak{n} \lhd \mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ and $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.
Then by Corollary 8.1.6 there is a spectral sequence

$$E^2_{p,q} = H_p(\mathfrak{g}/\mathfrak{n}, \mathrm{Tor}_q^{U_k(\mathfrak{g})}(U_k(\mathfrak{g}/\mathfrak{n}), M)) \quad \Rightarrow \quad H_{p+q}(\mathfrak{g}, M).$$

However it is unclear (at least to the author), wheather $H_*(\mathfrak{n}, M) \longrightarrow \mathrm{Tor}_*^{U_k(\mathfrak{g})}(U_k(\mathfrak{g}/\mathfrak{n}), M)$ is an isomorphism in general. We will later see by using the Theorem of Poincaré, Birkhoff and Witt 3.3.10, that this is true, if $\mathfrak{g}$ is flat over k in every dimension. For the non-simplicial case this is better known as the Hochschild-Serre spectral sequence [GH53]. In [Wei94] 7.5.2, the map of question is claimed to be an isomorphism without any hint of a proof.

8.3.2 Derived homology and cohomology

We already pointed out two possible flaws of the (dimensionwise) homology of simplicial Lie algebras, as it was introduced in the preceding section:

(i) It is unclear, wheather dimensionwise homology is a homotopy invariant.

(ii) The homology spectral sequence 8.3.6 may have a dissatisfactory form for later applications.

In this section we therefore introduce derived homology and cohomology of simplicial Lie algebras to fix these problems. It is also shown that it agrees with dimensionwise homology in the most important cases.

Definition 8.3.7
For $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ a simplicial Lie algebra, $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.
*The **derived homology** and **derived cohomology** of $\mathfrak{g}$ with coefficients in M are defined as*

(i) $H_*^L(\mathfrak{g}, M) = H_*(E_\bullet\mathfrak{g}, M) = \mathrm{Tor}_*^{U_k(E_\bullet\mathfrak{g})}(k, M)$,

(ii) $H_L^*(\mathfrak{g}, M) = H^*(E_\bullet\mathfrak{g}, M) = \mathrm{Ext}^*_{U_k(E_\bullet\mathfrak{g})}(k, M)$.

where $E_\bullet\mathfrak{g} \xrightarrow{\simeq} \mathfrak{g}$ is an almost free replacement of $\mathfrak{g}$.

Remark 8.3.8
Let $k \in \mathcal{C}Ring$, $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ and $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.
Then by Proposition 8.1.4 (i) and (iii) we have spectral sequences

(i) $E^2_{p,q} = H_p\underline{H}_q^L(\mathfrak{g}, M) \quad \Rightarrow \quad H^L_{p+q}(\mathfrak{g}, M)$,
where $\underline{H}_q^L(\mathfrak{g}, M)_p = H_q^L(\mathfrak{g}_p, M_p)$, for all $p, q \geq 0$.

(ii) $E^2_{p,q} = H_p^L(\mathfrak{g}, H_qM) \quad \Rightarrow \quad H^L_{p+q}(\mathfrak{g}, M)$,
where $H_qM \in \mathfrak{g}\text{-}s\mathcal{A}b$ via $U_k(E_\bullet\mathfrak{g}) \otimes H_qM \twoheadrightarrow \pi_0U_k(\mathfrak{g}) \otimes H_qM \xrightarrow{\mu} H_qM$.

Remark 8.3.9
Let $k \in \mathcal{C}Ring$, $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ and $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.
Then from Corollary 8.3.5 we get isomorphisms

(i) $H_0^L(\mathfrak{g}, M) = H_0(E_\bullet\mathfrak{g}, M) = H_0(\mathfrak{g}, M) = (\pi_0M)_{\pi_0\mathfrak{g}} = (\pi_0k) \otimes_{U_k(\pi_0\mathfrak{g})} (\pi_0M)$,

(ii) $H_1^L(\mathfrak{g}, M) = H_1(E_\bullet \mathfrak{g}, M) = H_1(\mathfrak{g}, M) = (\pi_0\mathfrak{g})/[\pi_0\mathfrak{g}, \pi_0\mathfrak{g}] \otimes M$, *if* $\mathfrak{g}$ *acts trivially on a constant* $M \in k\text{-Mod}$.

Lemma 8.3.10
Let $k \in \mathcal{C}Ring$ *and* $X \in s(k\text{-Mod})$ *flat over* k.
Then every almost free replacement $E_\bullet(X) \longrightarrow X$ *in* $s(k\text{-Mod})$ *induces a weak equivalence* $\mathcal{C}om_1(E_\bullet(X)) \longrightarrow \mathcal{C}om_1(X)$.

Proof. Considering $E_\bullet(X)$ as a bisimplicial object and using Quillen's spectral sequence (cf. Theorem 7.3.18), it suffices to prove that $\mathcal{C}om_1(E_\bullet X_n) \xrightarrow{\simeq} \mathcal{C}om_1(X_n)$ is a weak equivalence, for all $n \geq 0$. In other words, we can assume that X is a constant simplicial k-module. As X is flat over k, it is a filtered colimit of free k-modules. As $E_\bullet$, π_* and $\mathcal{C}om_1$ commute with filtered colimits, we therefore can assume that X is a free k-module. But then $E_\bullet(X) \longrightarrow X$ and thus also $\mathcal{C}om_1(E_\bullet X) \xrightarrow{\simeq} \mathcal{C}om_1(X)$ is a simplicial homotopy equivalence. In particular it is a weak equivalence.

□

Proposition 8.3.11
Let $k \in \mathcal{C}Ring$ *and* $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ *with* $\mathfrak{g}_n$ *flat over* k, *for all* $0 \leq n < c$.
Then $H_*^L(\mathfrak{g}, M) \longrightarrow H_*(\mathfrak{g}, M)$ *is* c*-connected, for all* $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$.

Proof. Suppose that $\mathfrak{g} \in k\text{-}\mathcal{L}ie$ is flat over k and take an almost free replacement of k-modules $E_\bullet(\mathfrak{g}) \longrightarrow \mathfrak{g}$. Then by Lemma 8.3.10 the upper horizontal map is a weak equivalence in the commutative diagram

$$\begin{tikzcd}
\mathcal{C}om_1(E_\bullet \mathfrak{g}) \arrow[r, "\simeq"'] \arrow[d, "\wr"'] & \mathcal{C}om_1(\mathfrak{g}) \arrow[d, "\wr"] \\
\mathrm{gr}^{\mathrm{L}} U_k(E_\bullet \mathfrak{g}) \arrow[r, "\simeq"'] & \mathrm{gr}^{\mathrm{L}} U_k(\mathfrak{g}),
\end{tikzcd}$$

where L is the colower central series of Definition 3.2.4. Again by flatness the Theorem of Poincaré, Birkhoff and Witt yields that the two vertical maps are isomorphisms. Hence by commutativity the lower horizontal map is a weak equivalence. By induction on $n \geq 0$ using the long exact sequence, it follows that $\mathrm{L}_n\, U_k(E_\bullet \mathfrak{g}) \xrightarrow{\simeq} \mathrm{L}_n\, U_k(\mathfrak{g})$ is a weak equivalence, for all $n \geq 0$, and hence $U_k(E_\bullet \mathfrak{g}) \xrightarrow{\simeq} U_k(\mathfrak{g})$ is a weak equivalence, since filtered colimits are exact. If $\mathfrak{g}$ is not flat, we still have

$$\pi_0 U_k(E_\bullet \mathfrak{g}) \xrightarrow{\sim} \pi_0 U_k(\mathfrak{g}) = U_k(\mathfrak{g}), \qquad \pi_1 U_k(E_\bullet \mathfrak{g}) \twoheadrightarrow \pi_1 U_k(\mathfrak{g}) = 0,$$

meaning that $U_k(E_\bullet \mathfrak{g}) \longrightarrow U_k(\mathfrak{g})$ is 0-connected. Now let $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ be flat in dimensions $< c$. By considering $U_k(E_\bullet \mathfrak{g}_\bullet) \longrightarrow U_k(\mathfrak{g}_\bullet)$ as a map of bisimplicial k-modules, the right one constant in the first coordinate, the map on the first page of the associated spectral sequences

$$\pi_* U_k(E_\bullet \mathfrak{g}_p) \longrightarrow \pi_* U_k(\mathfrak{g}_p), \qquad p \geq 0,$$

is an isomorphism for $0 \leq p < c$ and 0-connected, for $p \geq c$. In particular it is c-connected, implying that also $U_k(E_\bullet \mathfrak{g}) \longrightarrow U_k(\mathfrak{g})$ is c-connected. Using Tor-spectral sequence of

Proposition 8.1.4 (ii), it follows that also

$$H_*^L(\mathfrak{g}, M) = \mathrm{Tor}_*^{U_k(E_\bullet)}(k, M) \longrightarrow \mathrm{Tor}_*^{U_k(\mathfrak{g})}(k, M) = H_*(\mathfrak{g}, M)$$

is c-connected.

□

Remark 8.3.12
We do not know, if the map $H_*^L(\mathfrak{g}, M) \longrightarrow H_*(\mathfrak{g}, M)$ is an isomorphism in general.
Since we definitely know that H_*^L is a homotopy invariant (and H_* maybe not), we will work with derived Lie algebra homology.

8.3.3 The Homology spectral sequence for a fibration

Lemma 8.3.13
For $k \in \mathcal{C}Ring$ and $\mathfrak{n} \lhd \mathfrak{g} \in k\text{-}\mathcal{L}ie$ the following holds.

(i) $U_k(\mathfrak{g})_\mathfrak{n} \xrightarrow{\sim} U_k(\mathfrak{g}/\mathfrak{n})$ *as $U_k(\mathfrak{g}/\mathfrak{n})$-modules.*

(ii) *If $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{n}$ is a retraction of Lie algebras, every section induces an isomorphism of $U_k(\mathfrak{n})$-modules $U_k(\mathfrak{n}) \otimes U_k(\mathfrak{g}/\mathfrak{n}) \xrightarrow{\sim} U_k(\mathfrak{g})$.*

Proof.

(i) Let $\mathfrak{g} \overset{q}{\twoheadrightarrow} \mathfrak{g}/\mathfrak{n}$ and $U_k(\mathfrak{g}) \overset{r}{\twoheadrightarrow} U_k(\mathfrak{g})_\mathfrak{n}$ be the canonical quotient maps. Then q induces a bijection natural in $M \in U_k(\mathfrak{g}/\mathfrak{n})\text{-}s\mathcal{A}b$

$$U_k(q)^* = (\text{-} \circ U_k(q)) : U_k(\mathfrak{g}/\mathfrak{n})\text{-}s\mathcal{A}b(U_k(\mathfrak{g}/\mathfrak{n}), M) \xrightarrow{\sim} U_k(\mathfrak{g})\text{-}s\mathcal{A}b(U_k(\mathfrak{g}), M).$$

Similarly r induces a bijection natural in $M \in U_k(\mathfrak{g}/\mathfrak{n})\text{-}s\mathcal{A}b$

$$r^* = (\text{-} \circ r) : U_k(\mathfrak{g}/\mathfrak{n})\text{-}s\mathcal{A}b(U_k(\mathfrak{g})_\mathfrak{n}, M) \xrightarrow{\sim} U_k(\mathfrak{g})\text{-}s\mathcal{A}b(U_k(\mathfrak{g}), M),$$

because r is epimorphic and for all $f \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b(U_k(\mathfrak{g}), M)$ we have

$$f([\mathfrak{n}, U_k(\mathfrak{g})]) = [q(\mathfrak{n}), f(U_k(\mathfrak{g}))] = [0, f(U_k(\mathfrak{g}))] = 0.$$

As $[\mathfrak{n}, U_k(\mathfrak{g}/\mathfrak{n})] = 0$, the map $U_k(q)$ factors as $U_k(\mathfrak{g})_\mathfrak{n} \overset{s}{\twoheadrightarrow} U_k(\mathfrak{g}/\mathfrak{n})$, which therefore induces a bijection natural in $M \in U_k(\mathfrak{g}/\mathfrak{n})\text{-}s\mathcal{A}b$

$$s^* = (\text{-} \circ s) : U_k(\mathfrak{g}/\mathfrak{n})\text{-}s\mathcal{A}b(U_k(\mathfrak{g}/\mathfrak{n}), M) \xrightarrow{\sim} U_k(\mathfrak{g}/\mathfrak{n})\text{-}s\mathcal{A}b(U_k(\mathfrak{g})_\mathfrak{n}, M).$$

Equivalently s is an isomorphism of $U_k(\mathfrak{g}/\mathfrak{n})$-modules.

(ii) If $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{n}$ is a retraction, then $\mathfrak{g} \cong \mathfrak{n} \rtimes \mathfrak{g}/\mathfrak{n}$ and we have $U_k(\mathfrak{n}) \otimes_\theta U_k(\mathfrak{g}/\mathfrak{n})$ for some θ by Proposition 8.5.2. In particular $U_k(\mathfrak{g}) \cong U_k(\mathfrak{n}) \otimes_\theta U_k(\mathfrak{g}/\mathfrak{n}) = U_k(\mathfrak{n}) \otimes U_k(\mathfrak{g}/\mathfrak{n})$ as $U_k(\mathfrak{n})$-modules.

□

Proposition 8.3.14
Let $k \in \mathcal{C}Ring$, $\mathfrak{n} \lhd \mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$ and $M \in U_k(\mathfrak{g})\text{-}s\mathcal{A}b$. Then there is a spectral sequence

$$E^2_{p,q} = H^L_p(\mathfrak{g}/\mathfrak{n}, H^L_q(\mathfrak{n}, M)) \quad \Rightarrow \quad H^L_{p+q}(\mathfrak{g}, M),$$

where $H^L_q(N, M) \in U_k(\mathfrak{g}/\mathfrak{n})\text{-}s\mathcal{A}b$ via

$$U_k(\mathfrak{g}/\mathfrak{n}) \otimes H^L_q(\mathfrak{n}, M) \twoheadrightarrow \pi_0 U_k(\mathfrak{g}/\mathfrak{n}) \otimes H^L_q(\mathfrak{n}, M) \xrightarrow{\mu} H^L_q(\mathfrak{n}, M).$$

Proof. By Corollary 7.2.32 there is a functorial almost free replacement $E_\bullet$ induced by the adjunction

$$k\text{-}\mathcal{L}ie(\mathcal{L}ie(kX), Y) = \mathcal{S}et(X, U(Y)).$$

It preserves epimorphisms and thus $E_\bullet(\mathfrak{g}) \twoheadrightarrow E_\bullet(\mathfrak{g}/\mathfrak{n})$. Let $\mathfrak{n}'$ denote its kernel and let us write $\mathfrak{g}' = E_\bullet(\mathfrak{g})$ for short. Then $\mathfrak{g}'/\mathfrak{n}' = E_\bullet(\mathfrak{g}/\mathfrak{n})$ and the long exact sequence of homotopy groups applied to the diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{n}' \arrow[r] \arrow[d] & \mathfrak{g}' \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{g}'/\mathfrak{n}' \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \mathfrak{n} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{g}/\mathfrak{n} \arrow[r] & 0,
\end{tikzcd}$$

implies that also $\mathfrak{n}' \xrightarrow{\simeq} \mathfrak{n}$. For every $n \geq 0$ the map $E_n(\mathfrak{g}) \twoheadrightarrow E_n(\mathfrak{g}/\mathfrak{n})$ is of the form $\mathcal{L}ie(kX) \twoheadrightarrow \mathcal{L}ie(kY)$, induced by some surjection $r \in \mathcal{S}et(X, Y)$. For every section s of r there is an isomorphism of k-modules

$$kX \xrightarrow{\sim} kY \oplus k\tilde{X}, \quad x \longmapsto r(x) + (x - sr(x)),$$

where $\tilde{X} = X \backslash sY$. Under this isomorphism the map $kX \overset{kr}{\twoheadrightarrow} kY$ corresponds to the projection $kY \oplus k\tilde{X} \overset{\mathrm{id}+0}{\twoheadrightarrow} kY$. So $E_n(\mathfrak{g}) \twoheadrightarrow E_n(\mathfrak{g}/\mathfrak{n})$ is isomorphic to

$$\mathcal{L}ie(k\tilde{X}) + \mathcal{L}ie(kY) \overset{\mathrm{id}\cup 0}{\twoheadrightarrow} \mathcal{L}ie(kY),$$

whose kernel by Proposition 8.5.5 is the free Lie algebra

$$\mathcal{L}ie(U_k\mathcal{L}ie(kY) \otimes k\tilde{X}) = \mathcal{L}ie(\mathcal{A}ss_1(kY) \otimes \tilde{X}) = \mathcal{L}ie(k(\mathcal{A}ss_1(Y) \times \tilde{X})).$$

It follows that $\mathfrak{n}'$ is dimensionwise free.

Now by Corollary 8.1.6 we have a converging spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{U_k(\mathfrak{g}'/\mathfrak{n}')}(k, \mathrm{Tor}_q^{U_k(\mathfrak{g}')}(U_k(\mathfrak{g}'/\mathfrak{n}'), M)) \quad \Rightarrow \quad \mathrm{Tor}_{p+q}^{U_k(\mathfrak{g}')}(k, M).$$

As $\mathfrak{g}'_n/\mathfrak{n}'_n \in k\text{-}\mathcal{L}ie$ is free, the map $\mathfrak{g}'_n \twoheadrightarrow \mathfrak{g}'_n/\mathfrak{n}'_n$ has a section homomorphism, for all $n \geq 0$. So using the $U_k(\mathfrak{n}'_n)$-module isomorphism $U_k(\mathfrak{g}'_n) \cong U_k(\mathfrak{n}'_n) \otimes U_k(\mathfrak{g}'_n/\mathfrak{n}'_n)$ of Lemma 8.3.13 (ii) we see that $U_k(\mathfrak{g}'_n)$ is a free $U_k(\mathfrak{n}'_n)$-module in every dimension $n \geq 0$. In particular every cofibrant replacement of M in $U_k(\mathfrak{g}')\text{-}s\mathcal{A}b$ is also cofibrant over $U_k(\mathfrak{n}')$. This implies

$$\begin{aligned}
\mathrm{Tor}_*^{U_k(\mathfrak{g}')}(U_k(\mathfrak{g}'/\mathfrak{n}'), M) &= \pi_*(U_k(\mathfrak{g}'/\mathfrak{n}') \otimes_{U_k(\mathfrak{g}')} EM) \cong \pi_*(k \otimes_{U_k(\mathfrak{n}')} U_k(\mathfrak{g}') \otimes_{U_k(\mathfrak{g}')} EM) \\
&\cong \pi_*(k \otimes_{U_k(\mathfrak{n}')} EM) = \mathrm{Tor}_*^{U_k(\mathfrak{n}')}(k, M) = H_*(\mathfrak{n}', M),
\end{aligned}$$

and hence the spectral sequence looks like

$$E^2_{p,q} = H_p(\mathfrak{g}'/\mathfrak{n}', H_q(\mathfrak{n}', M)) = H^L_p(\mathfrak{g}/\mathfrak{n}, H^L_q(\mathfrak{n}, M)) \quad \Rightarrow \quad H_{p+q}(\mathfrak{g}', M) = H^L_{p+q}(\mathfrak{g}, M).$$

□

8.3.4 The Whitehead Theorem

Proposition 8.3.15
Let $k \in \mathcal{C}Ring, c \geq 0$ and $f \in s(k\text{-}\mathcal{L}ie)(\mathfrak{g}, \mathfrak{h})$.

(i) *Suppose $\pi_0\mathfrak{g} = 1$.*
Then $\mathfrak{g}$ is c-connected, if and only if $H^L_*(\mathfrak{g}, k)$ is $(c+1)$-connected.

(ii) *Suppose $\pi_0 f$ is an isomorphism.*
Then f is c-connected, if and only if $H^L_*(f, U_k(\pi_0\mathfrak{h}))$ is $(c+1)$-connected.

Proof. Using the Lie algebra analogues Proposition 8.3.1, Corollary 8.3.5 and the Hochschild-Serre spectral sequence instead of the group versions, the proof is exactly the same as in Proposition 8.2.6. Slightly more difficult is only the proof of the equivalence

$$H_q(\mathfrak{n}, k) = 0 \quad \forall\, 0 \leq q \leq c \qquad \Longleftrightarrow \qquad H_0(\mathfrak{h}, H_q(\mathfrak{n}, U_k(\pi_0\mathfrak{n}))) = 0 \quad \forall\, 0 \leq q \leq c.$$

Applying $H^L_*(\mathfrak{h}, \text{-})$ to the short exact Künneth sequence yields a long exact sequence

$$\begin{aligned} \ldots \longrightarrow H^L_1(\mathfrak{h}, \mathrm{Tor}^k_1(H^L_{q-1}(\mathfrak{n}, k), U_k(\pi_0\mathfrak{h}))) &\xrightarrow{\partial} H^L_0(\mathfrak{h}, H^L_q(\mathfrak{n}, k) \otimes U_k(\pi_0\mathfrak{h})) \\ \longrightarrow H^L_0(\mathfrak{h}, H^L_q(\mathfrak{n}, U_k(\pi_0\mathfrak{h}))) &\longrightarrow H^L_0(\mathfrak{h}, \mathrm{Tor}^k_1(H^L_{q-1}(\mathfrak{n}, k), U_k(\pi_0\mathfrak{h}))) \longrightarrow 0. \end{aligned}$$

Hence if $H_q(\mathfrak{n}, k)$ vanishes, for $0 \leq q \leq c$, also $H_0(\mathfrak{h}, H_q(\mathfrak{n}, U_k(\pi_0\mathfrak{n})))$ vanishes in that range. Vice versa, we will show by induction on $0 \leq q \leq c$ that $H_q(\mathfrak{n}, k) = 0$. Suppose $H_i(\mathfrak{n}, k)$ is zero for all $0 \leq i < q$ and some $0 \leq q \leq c$. Then in particular the first term of the extract of the long exact sequence is zero. Hence with the third term also the second term is zero, which by Remark 8.3.9 (i) can be computed as

$$H^L_0(\mathfrak{h}, H^L_q(\mathfrak{n}, k) \otimes U_k(\pi_0\mathfrak{h})) = (H^L_q(\mathfrak{n}, k) \otimes U_k(\pi_0\mathfrak{h}))_{\pi_0\mathfrak{h}} = H^L_q(\mathfrak{n}, k).$$

This proves the induction step.

□

8.3.5 Homology of free Lie algebras

Proposition 8.3.16
Let $k \in \mathcal{C}Ring$ and $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$.
Then there is a natural epimorphism of simplicial k-modules

$$B^{\otimes}U_k(\mathfrak{g})/k \longrightarrow B^{\otimes}U_k(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])/k = \mathcal{C}om(B^{\times}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])) \twoheadrightarrow B^{\times}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]).$$

It is a weak equivalence, if $\mathfrak{g}$ is an almost free simplicial Lie algebra.
In particular $H^L_{*+1}(\mathfrak{g}, k) \xrightarrow{\sim} \pi_*\mathbb{L}(\text{-}/[\text{-},\text{-}])(\mathfrak{g})$, for all $\mathfrak{g} \in s(k\text{-}\mathcal{L}ie)$.

Proof. The first map is induced by the quotient map $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, the equality middle is induced by the equality $U_k(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) = \mathcal{C}om_1(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$, using that $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian, and the identifications

$$B_n^{\otimes}\mathcal{C}om_1(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) = \mathcal{C}om_1(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^{\otimes n} = \mathcal{C}om_1(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]^n) = \mathcal{C}om_1(B_n^{\times}\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]), \qquad n \geq 0.$$

The second map is the projection onto the particular summand. That the given map is a weak equivalence, for almost free $\mathfrak{g}$ is exactly the same as in the case of simplicial groups (see Proposition 8.2.5).

□

8.4 Comparison theorems for homology spectral sequences

8.4.1 Connectivity of graded objects

Definition 8.4.1
Let $\mathcal{A}$ be an abelian category and $k \in \mathbb{Z}$.

(i) *A $\mathbb{Z}$-graded object $X \in \mathcal{A}^{\mathbb{Z}}$ is called k-**connected**, if:* $\quad X_n = 0, \quad$ *for all $n \leq k$.*

(ii) *A morphism $f \in \mathcal{A}^{\mathbb{Z}}(X,Y)$ is called k-**connected**, if*

$$X_n \xrightarrow{f_n} Y_n \qquad \text{is an} \begin{cases} \text{isomorphism}, & n \leq k, \\ \text{epimorphism}, & n = k+1. \end{cases}$$

Remark 8.4.2
Let $\mathcal{A}$ be an abelian category and $k \in \mathbb{Z}$.

(i) *Every object $X \in dg\mathcal{A}$ is k-connected, if and only if $H_*X \in \mathcal{A}^{\mathbb{Z}}$ is k-connected.*

(ii) *For a morphism $f \in dg\mathcal{A}(X,Y)$, the following is equivalent.*
   a) f is k-connected.
   b) $\mathrm{hofib} f$ is k-connected.
   c) $H_*\mathrm{hofib} f$ is k-connected.
   d) $H_*f \in \mathcal{A}^{\mathbb{Z}}(H_*X, H_*Y)$ is k-connected.

8.4.2 Basic comparison theorem

Proposition 8.4.3
Let $\mathcal{A}$ be an abelian category, $k \geq 0$ and $f \in dg\mathcal{A}(X,Y)$.
Suppose the underlying map of graded objects $f \in \mathcal{A}^{\mathbb{Z}}(X,Y)$ is k-connected. Then

(i) *The induced map on the cycles $Z_*(f) \in \mathcal{A}^{\mathbb{Z}}(Z_*X, Z_*Y)$ is k-connected.*

(ii) *The induced map on the boundaries $B_*(f) \in \mathcal{A}^{\mathbb{Z}}(B_*X, B_*Y)$ is $(k-1)$-connected.*

(iii) *The induced map on homology $H_*(f) \in \mathcal{A}^{\mathbb{Z}}(H_*X, H_*Y)$ are k-connected.*

Proof. This is well-known. As the proof is short, we will carry it out for the convenience of the reader. Since $U(f)$ is k-connected, the 5-lemma applied to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z_nX \arrow[r] \arrow[d, "Z_nf"'] & X_n \arrow[r, "d"] \arrow[d, "f_n"'] & X_{n-1} \arrow[d, "f_{n-1}"'] \\ 0 \arrow[r] & Z_nY \arrow[r] & Y_n \arrow[r, "d"] & Y_{n-1} \end{tikzcd}$$

yields that Z_nf is an isomorphism, if $n \leq k$ and an epimorphism, if $n = k+1$, meaning that Z_*f is k-connected. Similarly the 5-lemma applied to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z_nX \arrow[r] \arrow[d, "Z_nf"'] & X_n \arrow[r, "d"] \arrow[d, "f_n"'] & B_{n-1}X \arrow[r] \arrow[d, "f_{n-1}"'] & 1 \\ 0 \arrow[r] & Z_nY \arrow[r] & Y_n \arrow[r, "d"] & B_{n-1}Y \arrow[r] & 1 \end{tikzcd}$$

yields that B_*f is $(k-1)$-connected. Finally the 5-lemma applied to the commutative diagram

$$\begin{tikzcd} X_{n+1} \arrow[r, "d"] \arrow[d, "f_{n+1}"'] & Z_nX \arrow[r] \arrow[d, "Z_nf"'] & H_nX \arrow[r] \arrow[d, "H_nf"'] & 0 \\ Y_{n+1} \arrow[r, "d"] & Z_nY \arrow[r] & H_nY \arrow[r] & 0 \end{tikzcd}$$

yields that H_*f is k-connected.

□

Corollary 8.4.4
Let $E \xrightarrow{f} \bar{E}$ be a homomorphism of spectral sequences in an abelian category $\mathcal{A}$.
If $f^r \in \mathcal{A}^{\mathbb{Z}}(E^r, \bar{E}^r)$ is k-connected, for some r, so is f^s, for all $s \geq r$.

Proof. Using Proposition 8.4.3 we see that also

$$f^{r+1} : E^{r+1} = H_*(E^r) \xrightarrow{H_*(f^r)} H_*(\bar{E}^r) = \bar{E}^{r+1}$$

is k-connected. Hence by induction f^s is k-connected, for all $s \geq r$.

□

8.4.3 Connectivity of homology spectral sequences

Proposition 8.4.5
Let E^r be a first quadrant spectral sequence with $r \geq 2$, weakly converging to H.
Then there is a sequence of $\mathbb{N}_0$-graded objects

$$E^2_{0,*} \twoheadrightarrow E^\infty_{0,*} \hookrightarrow H_* \twoheadrightarrow E^\infty_{*,0} \hookrightarrow E^2_{*,0}.$$

Moreover all these maps are isomorphisms in dimension 0.

Proof. Let $n \geq 0$ and $r \geq 2$. Since $E^r_{0,n} \xrightarrow{d} E^r_{-r,n+r-1} = 0$, for all $r \geq 1$, we have exact sequences

$$E^r_{r,n-r+1} \xrightarrow{d} E^r_{0,n} \longrightarrow E^{r+1}_{0,n} \longrightarrow 0, \qquad r \geq 1,$$

where the left term is zero, if $n - r + 1 < 0$. Hence the first morphism can be defined as the composite

$$E^2_{0,n} \twoheadrightarrow E^3_{0,n} \twoheadrightarrow \ldots \twoheadrightarrow E^{n+2}_{0,n} = E^\infty_{0,n} \hookrightarrow H_n, \qquad n \geq 0.$$

Similarly, since $0 = E^r_{n+r,1-r} \xrightarrow{d} E^r_{n,0}$, for all $r \geq 2$, we have exact sequences

$$0 \longrightarrow E^{r+1}_{n,0} \longrightarrow E^r_{n,0} \xrightarrow{d} E^r_{n-r,r-1}, \qquad r \geq 2,$$

where the right term is zero, if $n - r < 0$. Hence the second morphism can be defined as the composite

$$H_n \twoheadrightarrow E^\infty_{n,0} = E^{n+1}_{n,0} \hookrightarrow \ldots \hookrightarrow E^2_{n,0}, \qquad n \geq 0.$$

Since E^r lies in the first quadrant, we have $E^2_{0,0} = E^\infty_{0,0} = (\mathrm{Tot} E^\infty_{*,*})_0$, showing that the maps are isomorphisms in dimension 0.

□

Ideas for the proof of one direction in (i) of the proposition below are based on computations in spectral sequence as they were given in [Sri08] Prop. 2.5 a). We do not know, wheather a proof in this generality is available in present literature. The statement is also similar to Zeeman's Comparison Theorem [Zee57], the proof of which uses similar techniques but is slightly more complicated. Infact our proposition implies and even tightens Zeemans comparison in some cases, as we will demonstrate in the subsequent section. On the other hand it is probably implied by the generalized comparison theorem, whose hardly readible proof is given in [HR76].

Proposition 8.4.6
Let E^r be a first quadrant spectral sequence with $r \geq 2$, weakly converging to H.

(i) *Suppose, that for every $q \geq 0$:*

$$\textbf{(F)} \quad E^2_{0,*}/E^2_{0,0} \;\; q\text{-connected} \quad \Longrightarrow \quad E^2_{p,*}/E^2_{p,0} \;\; q\text{-connected}, \qquad \text{for all } p > 0.$$

Then the following is equivalent.

a) $H_* \longrightarrow E^2_{*,0}$ *is $(k-1)$-connected.*

b) $E^2_{0,*}/E^2_{0,0}$ *is $(k-1)$-connected.*

(ii) *Suppose, that for every $p \geq 0$:*

$$\textbf{(B)} \quad E^2_{*,0}/E^2_{0,0} \;\; p\text{-connected} \quad \Longrightarrow \quad E^2_{*,q}/E^2_{0,q} \;\; p\text{-connected}, \qquad \text{for all } q > 0.$$

Then the following is equivalent.

a) $E^2_{0,*} \longrightarrow H_*$ *is $(k-1)$-connected.*

b) $E^2_{*,0}/E^2_{0,0}$ *is* k*-connected.*

Proof.

(i) To see that a) implies b), we will show by induction on $0 < n \leq k$, that $E^2_{0,*}/E^2_{0,0}$ is $(n-1)$-connected. There is nothing to check, for $n = 1$. Suppose the statement holds for some $0 < n < k$. Since $E^r_{0,n} \xrightarrow{d} E^r_{-r,n+r-1} = 0$, for all $r \geq 1$, we have exact sequences

$$E^r_{r,n-r+1} \xrightarrow{d} E^r_{0,n} \longrightarrow E^{r+1}_{0,n} \longrightarrow 0, \qquad r \geq 1.$$

- For $2 \leq r < n+1$ we have $0 < n-r+1 < n$ and thus $E^2_{0,n-r+1} = 0$ by induction hypothesis, which with **(F)** implies $E^2_{r,n-r+1} = 0$ and so $E^r_{r,n-r+1} = 0$.
- For $r = n+1$ we have an even longer exact sequence

$$0 \longrightarrow E^{r+1}_{n+1,0} \longrightarrow E^r_{n+1,0} \xrightarrow{d} E^r_{0,n} \longrightarrow E^{r+1}_{0,n} \longrightarrow 0, \qquad r \geq 1.$$

  Since $r = n+1 \leq k$ and thus $E^\infty_{r,0} \xrightarrow{\sim} E^2_{r,0}$ by a), the left map is an isomorphism and so $d = 0$.

It follows that $E^2_{0,n} \xrightarrow{\sim} ... \xrightarrow{\sim} E^{n+2}_{0,n} = E^\infty_{0,n} = \mathrm{gr}_0 H_n = 0$, because $n > 0$ and by a)

$$H_n \xrightarrow{\sim} \mathrm{gr}_n H_n = E^\infty_{n,0} \xrightarrow{\sim} E^2_{n,0}.$$

Vice versa, we assume b) and let $0 < n \leq k$. Since $0 = E^r_{n+r,1-r} \xrightarrow{d} E^r_{n,0}$, for all $r \geq 2$, we have exact sequences

$$0 \longrightarrow E^{r+1}_{n,0} \longrightarrow E^r_{n,0} \xrightarrow{d} E^r_{n-r,r-1}, \qquad r \geq 2.$$

- For $2 \leq r \leq k$, we have $0 < r-1 < k$ and thus $E^2_{0,r-1} = 0$ by b), which with **(F)** implies $E^2_{n-r,r-1} = 0$ and so $E^r_{n-r,r-1} = 0$.
- For $r > k$, we have $n - r \leq k - r < 0$ and thus again $E^r_{n-r,r-1} = 0$.

It follows that $H_n \twoheadrightarrow \mathrm{gr}_n H_n = E^\infty_{n,0} = E^{n+1}_{n,0} \xrightarrow{\sim} ... \xrightarrow{\sim} E^2_{n,0}$.

Now suppose $0 < n < k$. Since $E^2_{0,*}/E^2_{0,0}$ is $(k-1)$-connected, so is $E^r_{p,*}/E^2_{p,0}$ by **(F)**, for all $p > 0$ and $r \geq 2$. Then $E^\infty_{n-q,q} = 0$, for all $0 < q \leq n$. Equivalently $H_n \xrightarrow{\sim} \mathrm{gr}_n H_n = E^\infty_{n,0}$ and thus

$$H_n \xrightarrow{\sim} \mathrm{gr}_n H_n = E^\infty_{n,0} = E^{n+1}_{n,0} \xrightarrow{\sim} ... \xrightarrow{\sim} E^2_{n,0}.$$

(ii) To see that a) implies b), we will show by induction on $0 < n \leq k+1$, that $E^2_{*,0}/E^2_{0,0}$ is $(n-1)$-connected. There is nothing to check, for $n = 1$. Suppose the statement holds for some $0 < n < k+1$. Since $0 = E^r_{n+r,1-r} \xrightarrow{d} E^r_{n,0}$, for all $r \geq 2$, we have exact sequences

$$0 \longrightarrow E^{r+1}_{n,0} \longrightarrow E^r_{n,0} \xrightarrow{d} E^r_{n-r,r-1}, \qquad r \geq 2.$$

- For $2 \le r < n$ we have $0 < n - r < n$ and thus $E^2_{n-r,0} = 0$ by induction hypothesis, which with **(B)** implies $E^2_{n-r,r-1} = 0$ and so $E^r_{n-r,r-1} = 0$.
- For $r = n$ we have an even longer exact sequence

$$0 \longrightarrow E^{r+1}_{n,0} \longrightarrow E^r_{n,0} \xrightarrow{d} E^r_{0,n-1} \longrightarrow E^{r+1}_{0,n-1} \longrightarrow 0.$$

Since $r - 1 = n - 1 < k$ and thus $E^2_{0,r-1} \xrightarrow{\sim} E^\infty_{0,r-1}$ by a), it follows that the right map is an isomorphism and so $d = 0$.

It follows that $0 = \mathrm{gr}_n H_n = E^\infty_{n,0} = E^{n+1}_{n,0} \xrightarrow{\sim} ... \xrightarrow{\sim} E^2_{n,0}$, because $n > 0$ and by a)

$$E^2_{0,n} \twoheadrightarrow E^\infty_{0,n} = \mathrm{gr}_0 H_n \xrightarrow{\sim} H_n.$$

Vice versa, we assume b) and let $0 < n \le k$. Since $E^2_{*,0}/E^2_{0,0}$ is k-connected, so is $E^r_{*,q}/E^2_{0,q}$ by **(B)**, for all $q > 0$ and $r \ge 2$. Then $E^\infty_{p,n-p} = 0$, for all $0 < p \le n$. Equivalently $E^\infty_{0,n} = \mathrm{gr}_0 H_n \xrightarrow{\sim} H_n$ and thus

$$E^2_{0,n} \twoheadrightarrow ... \twoheadrightarrow E^{n+2}_{0,n} = E^\infty_{0,n} = \mathrm{gr}_0 H_n \xrightarrow{\sim} H_n.$$

Now suppose $0 < n < k$. Since $E^r_{0,n} \xrightarrow{d} E^r_{-r,n+r-1} = 0$, for all $r \ge 1$, we have exact sequences

$$E^r_{r,n-r+1} \xrightarrow{d} E^r_{0,n} \longrightarrow E^{r+1}_{0,n} \longrightarrow 0, \qquad r \ge 1.$$

- For $2 \le r \le k$, we have $E^2_{r,0} = 0$ by b), which with **(B)** implies $E^2_{r,n-r+1} = 0$ and so $E^r_{r,n-r+1} = 0$.
- For $r > k$, we have $n - r + 1 < k - r + 1 \le 0$ and thus $E^r_{r,n-r+1} = 0$.

It follows that $E^2_{0,n} \xrightarrow{\sim} ... \xrightarrow{\sim} E^{n+2}_{0,n} = E^\infty_{0,n} = \mathrm{gr}_0 H_n \xrightarrow{\sim} H_n$.

□

8.4.4 Comparison with Zeeman's Theorem

Theorem 8.4.7 (Zeeman)
Let $E^r \xrightarrow{f^r} \bar{E}^r$ be a map of first quadrant spectral sequences in abelian category, $r \ge 2$, each of which gives rise to an exact sequence

$$0 \longrightarrow E^2_{p,0} \otimes E^2_{0,q} \longrightarrow E^2_{p,q} \longrightarrow \mathrm{Tor}_1(E^2_{p-1,0}, E^2_{0,q}) \longrightarrow 0, \qquad p, q \ge 0.$$

Then the following holds.

(i) *If $f^2_{*,0}$ is $(p-1)$- and $f^2_{0,*}$ is $(q-1)$-connected, then $f^\infty_{*,*}$ is $\min(p-2, q-1)$-connected.*

(ii) *If $f^\infty_{*,*}$ is $(n-1)$- and $f^2_{*,0}$ is $(p-1)$-connected, then $f^2_{0,*}$ is $\min(n-2, p-3)$-connected.*

(iii) *If $f^\infty_{*,*}$ is $(n-1)$- and $f^2_{0,*}$ is $(q-1)$-connected, then $f^2_{*,0}$ is $\min(n-1, q-1)$-connected.*

Proof. See [Zee57]. □

We want to apply Zeeman's Theorem to the Serre spectral sequence of a fibration to be able to compare it with Proposition 8.4.6.

Corollary 8.4.8
Given a map of fibre sequences with 1*-connected base spaces* B *and* $\bar{B}$

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"'] & E \arrow[r, -{Triangle[open]}] \arrow[d, "e"'] & B \arrow[d, "b"'] \\ \bar{F} \arrow[r] & \bar{E} \arrow[r, -{Triangle[open]}] & \bar{B}, \end{tikzcd}$$

Then the following holds.

(i) *If* $\tilde{\mathbb{Z}}b$ *is* $(p-1)$*- and* $\tilde{\mathbb{Z}}f$ *is* $(q-1)$*-connected, then* $\tilde{\mathbb{Z}}e$ *is* $\min(p-2, q-1)$*-connected.*

(ii) *If* $\tilde{\mathbb{Z}}e$ *is* $(n-1)$*- and* $\tilde{\mathbb{Z}}b$ *is* $(p-1)$*-connected, then* $\tilde{\mathbb{Z}}f$ *is* $\min(n-2, p-3)$*-connected.*

(iii) *If* $\tilde{\mathbb{Z}}e$ *is* $(n-1)$*- and* $\tilde{\mathbb{Z}}f$ *is* $(q-1)$*-connected, then* $\tilde{\mathbb{Z}}b$ *is* $\min(n-1, q-1)$*-connected.*

Note that we get similar implications on homotopy groups by applying the 5*-Lemma to the long exact sequence.*

Proof. This follows immediately from Zeeman's comparison Theorem 8.4.7 using the induced maps between their Serre spectral sequences

$$H_p(b, H_q(f, \mathbb{Z})) : H_p(B, H_q(F, \mathbb{Z})) \longrightarrow H_p(\bar{B}, H_q(\bar{F}, \mathbb{Z})), \qquad p, q \geq 0.$$

□

Now we will use Proposition 8.4.6.

Proposition 8.4.9
For a fibre sequence $F \longrightarrow E \longrightarrow B$ *with* 1*-connected base* B*, the following holds.*

(i) $\tilde{\mathbb{Z}}E \longrightarrow \tilde{\mathbb{Z}}B$ *is* $(k-1)$*-connected, if and only if* $\tilde{\mathbb{Z}}F$ *is* $(k-1)$*-connected.*

(ii) $\tilde{\mathbb{Z}}F \longrightarrow \tilde{\mathbb{Z}}E$ *is* $(k-1)$*-connected, if and only if* $\tilde{\mathbb{Z}}B$ *is* k*-connected.*

Note that we get similar implications on homotopy groups by applying the 5*-Lemma to the long exact sequence.*

Proof. This follows by applying the Proposition 8.4.6 (i) to the Serre spectral sequence

$$E^2_{p,q} = H_p(B, H_q(F, \mathbb{Z})) \quad \Longrightarrow \quad H_{p+q}(E, \mathbb{Z}).$$

□

Proposition 8.4.10
Given a fibration between fibre sequences with 1*-connected base spaces* B *and* $\bar{B}$

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"'] & E \arrow[r, -open triangle] \arrow[d, "e"'] & B \arrow[d, "b"'] \\ \bar{F} \arrow[r] & \bar{E} \arrow[r, -open triangle] & \bar{B}. \end{tikzcd}$$

Then the following holds.

(i) *If* $\tilde{\mathbb{Z}}f$ *is* $(q-1)$*- and* $\tilde{\mathbb{Z}}b$ *is* $(p-1)$*-connected, then* $\tilde{\mathbb{Z}}e$ *is* $\min(p-1, q-1)$*-connected.*

(ii) *If* $\tilde{\mathbb{Z}}e$ *is* $(n-1)$*- and* $\tilde{\mathbb{Z}}b$ *is* $(p-1)$*-connected, then* $\tilde{\mathbb{Z}}f$ *is* $\min(n-1, p-2)$*-connected.*

(iii) *If* $\tilde{\mathbb{Z}}e$ *is* $(n-1)$*- and* $\tilde{\mathbb{Z}}f$ *is* $(q-1)$*-connected, then* $\tilde{\mathbb{Z}}b$ *is* $\min(n-1, q)$*-connected.*

Note that we get similar implications on homotopy groups by applying the 5*-Lemma to the long exact sequence.*

Proof. Statement (i) follows from Corollary 8.4.4. For (ii) and (iii), consider the commutative diagram

$$\begin{tikzcd} \operatorname{fib} f \arrow[r] \arrow[d] & \operatorname{fib} e \arrow[r, -open triangle] \arrow[d] & \operatorname{fib} b \arrow[d] \\ F \arrow[r] \arrow[d, "f"'] & E \arrow[r, -open triangle] \arrow[d, "e"'] & B \arrow[d, "b"'] \\ \bar{F} \arrow[r] & \bar{E} \arrow[r, -open triangle] & \bar{B}. \end{tikzcd}$$

Since limits preserve fibrations, the upper right horizontal map is a fibration. Moreover $\operatorname{fib} f$ is its fibre.

Using Proposition 8.4.9 (i) we get the following implications.

$$\begin{array}{ll} & \tilde{\mathbb{Z}}f \text{ is } (q-1)\text{- and } \tilde{\mathbb{Z}}b \text{ is } (p-1)\text{-connected,} \\ \Longleftrightarrow & \tilde{\mathbb{Z}}\operatorname{fib} f \text{ is } (q-1)\text{-connected and } \tilde{\mathbb{Z}}\operatorname{fib} b \text{ is } (p-1)\text{-connected.} \\ \Longrightarrow & \tilde{\mathbb{Z}}\operatorname{fib} f \longrightarrow \tilde{\mathbb{Z}}\operatorname{fib} e \text{ is } (p-2)\text{-connected (or } \tilde{\mathbb{Z}}\operatorname{fib} e \longrightarrow \tilde{\mathbb{Z}}\operatorname{fib} b \text{ is } (q-1)\text{-connected).} \\ \Longleftrightarrow & \tilde{\mathbb{Z}}\operatorname{fib} e \text{ is } \min(p-1, q-1)\text{-connected.} \\ \Longleftrightarrow & \tilde{\mathbb{Z}}e \text{ is } \min(p-1, q-1)\text{-connected.} \end{array}$$

Similarly using Proposition 8.4.9 (i) we get the following implications.

$$\begin{array}{ll} & \tilde{\mathbb{Z}}e \text{ is } (n-1)\text{- and } \tilde{\mathbb{Z}}b \text{ is } (p-1)\text{-connected,} \\ \Longleftrightarrow & \tilde{\mathbb{Z}}\operatorname{fib} e \text{ is } (n-1)\text{-connected and } \tilde{\mathbb{Z}}\operatorname{fib} b \text{ is } (p-1)\text{-connected.} \\ \Longrightarrow & \tilde{\mathbb{Z}}\operatorname{fib} e \longrightarrow \tilde{\mathbb{Z}}\operatorname{fib} b \text{ is } \min(n-1, p-2)\text{-connected.} \\ \Longleftrightarrow & \tilde{\mathbb{Z}}\operatorname{fib} f \text{ is } \min(n-1, p-2)\text{-connected.} \\ \Longleftrightarrow & \tilde{\mathbb{Z}}f \text{ is } \min(n-1, p-2)\text{-connected.} \end{array}$$

And in the same way for (iii)

$$
\begin{aligned}
& \tilde{\mathbb{Z}}e \text{ is } (n-1)\text{- and } \tilde{\mathbb{Z}}f \text{ is } (q-1)\text{-connected,} \\
\Longleftrightarrow\quad & \tilde{\mathbb{Z}}\mathrm{fib}\, e \text{ is } (n-1)\text{-connected and } \tilde{\mathbb{Z}}\mathrm{fib}\, f \text{ is } (q-1)\text{-connected.} \\
\Longrightarrow\quad & \tilde{\mathbb{Z}}\mathrm{fib}\, f \longrightarrow \tilde{\mathbb{Z}}\mathrm{fib}\, e \text{ is } \min(n-2, q-1)\text{-connected.} \\
\Longleftrightarrow\quad & \tilde{\mathbb{Z}}\mathrm{fib}\, b \text{ is } \min(n-1, q)\text{-connected.} \\
\Longleftrightarrow\quad & \tilde{\mathbb{Z}}b \text{ is } \min(n-1, q)\text{-connected.}
\end{aligned}
$$

□

8.5 Appendix: Semi-direct tensor products of rings

Proposition 8.5.1
Let $k \in \mathcal{C}Ring$ and $R, S \in k\text{-}\mathcal{A}ss$. Suppose S is generated as a k-algebra by a k-submodule $X \leq S$. For $\theta \in k\text{-}\mathrm{Mod}(S \otimes R, R \otimes S)$ we let $R \otimes_\theta S$ be the k-module $R \otimes S$ together with the multiplication map

$$\mu_{R\otimes_\theta S} : (R \otimes_\theta S) \otimes (R \otimes_\theta S) \longrightarrow (R \otimes_\theta S), \qquad (r \otimes s) \otimes (r' \otimes s') \longmapsto r \cdot \theta(s \otimes r') \cdot s',$$

where $R \otimes S$ carries the canonical (R, S)-bimodule structure.
Then $R \otimes_\theta S$ is associative, if the following two squares commute

$$
\begin{tikzcd}
X \otimes R \otimes R \arrow[r, "\theta\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & R \otimes S \otimes R \arrow[r, "\mathrm{id}\otimes\theta"] & R \otimes R \otimes S \arrow[d, "\mu\otimes\mathrm{id}"] \\
X \otimes R \arrow[rr, "\theta"] & & R \otimes S,
\end{tikzcd}
\qquad
\begin{tikzcd}
S \otimes R \otimes R \arrow[r, "\theta\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & R \otimes S \otimes R \arrow[r, "\mathrm{id}\otimes\theta"] & R \otimes R \otimes S \arrow[d, "\mu\otimes\mathrm{id}"] \\
S \otimes R \arrow[rr, "\theta"] & & R \otimes S.
\end{tikzcd}
$$

Moreover if R and S are unital and

$$\theta(1 \otimes r) = r \otimes 1, \qquad \theta(x \otimes 1) = 1 \otimes x, \qquad r \in R,\ x \in X,$$

then $1 \otimes 1 \in R \otimes_\theta S$ is a unit and the canonical maps $R \longrightarrow R \otimes_\theta S \longleftarrow S$ are universal algebra homomorphisms, meaning that there is a unique algebra homomorphism in

$$
\begin{tikzcd}
R \arrow[r] \arrow[dr, "f"'] & R \otimes_\theta S \arrow[d, dashed, "\exists!"] & S \arrow[l] \arrow[dl, "g"] \\
& T, &
\end{tikzcd}
$$

for every pair of morphisms $(f, g) \in k\text{-}\mathcal{A}ss_1(R, T) \times k\text{-}\mathcal{A}ss_1(S, T)$ with

$$\mu_T(f \otimes g)\theta(x \otimes r) = g(x) \cdot f(r), \qquad x \in X,\ r \in R.$$

Proof. In a canonical way $M := R \otimes_\theta S = R \otimes S \in (R, S)\text{-Mod}$. Moreover if we define

1) $(r \otimes s) \cdot r' = r \cdot \theta(s \otimes r'), \quad s' \cdot (r \otimes s) = \theta(s' \otimes r) \cdot s, \qquad$ for all $r, r' \in R,\ s, s' \in S.$

then right from the definition we get

2) $m \cdot (r \otimes s) = (m \cdot r) \cdot s, \quad (r \otimes s) \cdot m = r \cdot (s \cdot m), \qquad$ for all $r \in R,\ s \in S,\ m \in M,$

3) $r \cdot (m \cdot r') = (r \cdot m) \cdot r', \quad s \cdot (m \cdot s') = (s \cdot m) \cdot s', \qquad$ for all $r, r' \in R,\ m \in M,\ s, s' \in S,$

and the two squares commute, if and only if

4) $\theta(s\otimes rr') = \theta(s\otimes r)\cdot r', \quad \theta(ss'\otimes r) = s\cdot\theta(s'\otimes r), \qquad$ for all $r, r' \in R,\ s, s' \in S.$

It suffices to check that M is associative under the weakened condition, that the left equation of 4) only holds for $x \in X$. First of all, the right equation of 4) implies

$$(ss')\cdot(r''\otimes s'') \overset{1}{=} \theta(ss'\otimes r'')\cdot s'' \overset{4}{=} (s\cdot\theta(s'\otimes r''))\cdot s'' \overset{3}{=} s\cdot(\theta(s'\otimes r'')\cdot s'') \overset{1}{=} s\cdot(s'\cdot(r''\otimes s'')),$$

for $s, s', s'' \in S$ and $r'' \in R$. Hence

5) $(ss')\cdot m = s\cdot(s'\cdot m), \quad$ for all $s, s' \in S,\ m \in M.$

We will prove by induction on $n \geq 1$, that

6) $(s\cdot m)\cdot m' = s\cdot(m\cdot m'), \quad$ for all $s \in S_n = \sum_{1\leq i\leq n} X^i, \quad m, m' \in M.$

Let $x \in X, r \in R, s \in S$ and $m \in M$. We have $s\cdot m = r_1\otimes s_1 + ... + r_n\otimes s_n$, for some $r_i \in R,\ s_i \in S$ and $1 \leq i \leq n$. Hence

$$\begin{aligned} x\cdot((r\otimes s)\cdot m) &\overset{2}{=} x\cdot(r\cdot(s\cdot m)) = x\cdot(r\cdot(\sum_i r_i\otimes s_i)) = \sum_i x\cdot(rr_i\otimes s_i) \\ &\overset{1}{=} \sum_i \theta(x\otimes rr_i)\cdot s_i \overset{4}{=} \sum_i(\theta(x\otimes r)\cdot r_i)\cdot s_i) \\ &\overset{2}{=} \sum_i \theta(x\otimes r)\cdot(r_i\otimes s_i) = \theta(x\otimes r)\cdot(s\cdot m). \end{aligned}$$

Similarly we have $\theta(x\otimes r) = r_1\otimes s_1 + ... + r_n\otimes s_n$, for some other $r_i \in R,\ s_i \in S$, so

$$\begin{aligned} (x\cdot(r\otimes s))\cdot m &\overset{1}{=} (\theta(x\otimes r)\cdot s)\cdot m = ((\sum_i r_i\otimes s_i)\cdot s)\cdot m = \sum_i(r_i\otimes s_i s)\cdot m \\ &\overset{2}{=} \sum_i r_i\cdot((s_i s)\cdot m) \overset{5}{=} \sum_i r_i\cdot(s_i\cdot(s\cdot m)) \\ &\overset{2}{=} \sum_i(r_i\otimes s_i)\cdot(s\otimes m) = \theta(x\otimes r)\cdot(s\cdot m). \end{aligned}$$

Putting both together this proves $x\cdot((r\otimes s)\cdot m) = (x\cdot(r\otimes s))\cdot m$, which implies the case $n = 1$. Suppose 6) holds for some $n \geq 1$. Let $s \in S_n,\ x \in X,\ m', m'' \in M$ with $m' = r'\otimes s'$ and use the induction hypothesis (I) two times to obtain

$$\begin{aligned} ((sx)\cdot m')\cdot m'' &= ((sx)\cdot(r'\otimes s'))\cdot m'' \overset{1}{=} (\theta(sx\otimes r')\cdot s')\cdot m'' \\ &\overset{4}{=} ((s\cdot\theta(x\otimes r'))\cdot s')\cdot m'' \overset{3}{=} (s\cdot(\theta(x\otimes r')\cdot s'))\cdot m'' \\ &\overset{1}{=} (s\cdot(x\cdot(r'\otimes s')))\cdot m'' = (s\cdot(x\cdot m'))\cdot m'' \\ &\overset{I}{=} s\cdot((x\cdot m')\cdot m'') \overset{I}{=} s\cdot(x\cdot(m'\cdot m'')) \overset{5}{=} (sx)\cdot(m'\cdot m''). \end{aligned}$$

Since $S_{n+1} = S_n + S_n \cdot X$, this proves the induction step. As $S = \sum_{n\geq 1} S_n$ statement 6) holds for all $s \in S$. Now for $r, r'' \in R,\ s, s'' \in S$ and $m' \in M'$ we get

$$\begin{aligned}((r \otimes s) \cdot m') \cdot (r'' \otimes s'') &\stackrel{2}{=} (r \cdot (s \cdot m')) \cdot (r'' \otimes s'') \stackrel{2}{=} ((r \cdot (s \cdot m')) \cdot r'') \cdot s'' \\ &\stackrel{3}{=} (r \cdot ((s \cdot m') \cdot r'')) \cdot s'' \stackrel{5}{=} (r \cdot (s \cdot (m' \cdot r''))) \cdot s'' \\ &= r \cdot ((s \cdot (m' \cdot r'')) \cdot s'') \stackrel{3}{=} r \cdot (s \cdot ((m' \cdot r'') \cdot s'')) \\ &\stackrel{2}{=} r \cdot (s \cdot (m' \cdot (r'' \otimes s''))) \stackrel{2}{=} (r \otimes s) \cdot (m' \cdot (r'' \otimes s'')),\end{aligned}$$

which finally proves that M is associative.

If R and S are unital, we will prove that $\theta(s \otimes 1) = 1 \otimes s$, for all $s \in S_n$, by induction on $n \geq 1$. By hypothesis this is true for $n = 1$. Suppose it holds for some $n \geq 1$ and let $s \in S_n$ and $x \in X$. Then applying the induction hypothesis (I) two times we get

$$\theta(sx \otimes 1) \stackrel{4}{=} s \cdot \theta(x \otimes 1) \stackrel{I}{=} s \cdot (1 \otimes x) \stackrel{1}{=} \theta(s \otimes 1) \cdot x \stackrel{I}{=} (1 \otimes s) \cdot x = 1 \otimes sx,$$

which proves the induction step, as $S_{n+1} = S_n + S_n \cdot X$. Now we can verify

7) $1_S \cdot (r \otimes s) = \theta(1_S \otimes r) \cdot s = (r \otimes 1_S) \cdot s = r \otimes s$, for all $r \in R,\ s \in S$.

8) $(r \otimes s) \cdot 1_R = r \cdot \theta(s \otimes 1_R) = r \cdot (1_R \otimes s) = r \otimes s$, for all $r \in R,\ s \in S$.

This implies that for all $m \in M$ we have

$$(1_R \otimes 1_S) \cdot m \stackrel{2}{=} 1_R \cdot (1_S \cdot m) \stackrel{7}{=} 1_R \cdot m = m = m \cdot 1_S \stackrel{8}{=} (m \cdot 1_R) \cdot 1_S \stackrel{2}{=} m \cdot (1_R \otimes 1_S).$$

The canonical maps $R \longrightarrow R \otimes_\theta S \longleftarrow S$ are algebra homomorphisms, because

- $(r \otimes 1) \cdot (r' \otimes 1) = r \cdot \theta(1 \otimes r') \cdot 1 = r \cdot (r' \otimes 1) \cdot 1 = rr' \otimes 1$, for all $r, r' \in R$.
- $(1 \otimes s) \cdot (1 \otimes s') = 1 \cdot \theta(s \otimes 1) \cdot s' = 1 \cdot (1 \otimes s) \cdot s' = 1 \otimes ss'$, for all $s, s' \in S$.

Now let $(f, g) \in k\text{-}\mathcal{A}ss_1(R, T) \times k\text{-}\mathcal{A}ss_1(S, T)$ with the given properties. Define $f \otimes_\theta g = \mu_T(f \otimes g) \in k\text{-Mod}(R \otimes S, T)$. We will prove by induction on $n \geq 1$ that

9) $(f \otimes_\theta g)(s \cdot m) = g(s) \cdot ((f \otimes_\theta g))(m)$, for all $s \in S_n,\ m \in M$.

For $x \in X,\ r \in R,\ s \in S$ we have by assumption

$$\begin{aligned}(f \otimes_\theta g)(x \cdot (r \otimes s)) &\stackrel{1}{=} (f \otimes_\theta g)(\theta(x \otimes r) \cdot s) = (f \otimes_\theta g)\theta(x \otimes r) \cdot g(s) \\ &= g(x) \cdot f(r) \cdot g(s) = g(x) \cdot (f \otimes_\theta g)(r \otimes s),\end{aligned}$$

which proves the case $n = 1$. Suppose it holds for some $n \geq 1$. Let $s \in S_n,\ x \in X$ and $r \in R$. Then using the induction hypothesis (I) we get

$$\begin{aligned}(f \otimes_\theta g)((sx) \cdot m) &= (f \otimes_\theta g)(s \cdot (x \cdot m)) \stackrel{I}{=} g(s) \cdot (f \otimes_\theta g)(x \cdot m) \\ &\stackrel{I}{=} g(s) \cdot (g(x) \cdot (f \otimes_\theta g)(m)) = g(sx) \cdot (f \otimes_\theta g)(m),\end{aligned}$$

which proves the induction step, because $S_{n+1} = S_n + S_n \cdot X$. As $S = \sum_{n\geq 1} S_n$ statement 9) holds for all $s \in S$. Now let $r \in R,\ s \in S$ and $m \in M$. Then

$$\begin{aligned}(f \otimes_\theta g)((r \otimes s) \cdot m) &\overset{2}{=} (f \otimes_\theta g)(r \cdot (s \cdot m)) = f(r) \cdot (f \otimes_\theta g)(s \cdot m) \\ &\overset{9}{=} f(r) \cdot (g(s) \cdot (f \otimes_\theta g)(m)) = (f \otimes_\theta g)(r \otimes s) \cdot (f \otimes_\theta g)(m),\end{aligned}$$

which proves that $(f \otimes_\theta g)$ is an algeba homomorphism, since also $(f \otimes_\theta g)(1 \otimes 1) = f(1) \cdot g(1) = 1$. It is the only one being compatible with f and g, because

$$(f \otimes_\theta g)(r \otimes s) = (f \otimes_\theta g)(r \otimes 1) \cdot (f \otimes_\theta g)(1 \otimes s) = f(r) \cdot g(s), \qquad r \in R,\ s \in S.$$

□

Proposition 8.5.2
Let $k \in \mathcal{C}Ring$ and $\mathfrak{h}, \mathfrak{n} \in k\text{-}\mathcal{L}ie$. Suppose $\delta \in k\text{-}\mathcal{L}ie(\mathfrak{h}, \underline{k\text{-Der}}(\mathfrak{n}))$, where $\underline{k\text{-Der}}(\mathfrak{n})$ denotes the Lie algebra of k-linear derivations on $\mathfrak{n}$.
Then $\mathfrak{h} \hookrightarrow \mathfrak{n} \rtimes_\delta \mathfrak{h}$ extends to an isomorphism of $U_k(\mathfrak{n})$-modules

$$U_k(\mathfrak{n}) \otimes U_k(\mathfrak{h}) \xrightarrow{\sim} U_k(\mathfrak{n} \rtimes_\delta \mathfrak{h}),$$

which becomes an algebra isomorphism $U_k(\mathfrak{n}) \otimes_\theta U_k(\mathfrak{h}) \xrightarrow{\sim} U_k(\mathfrak{n} \rtimes_\delta \mathfrak{h})$ for some θ.

Proof. For every derivation d on $\mathfrak{n}$ the composition $\mathfrak{n} \xrightarrow{d} \mathfrak{n} \xrightarrow{\eta_\mathfrak{n}} U_k(\mathfrak{n})$ extends uniquely to a k-linear derivation on $U_k(\mathfrak{n})$. So δ induces a homomorphism of Lie algebras

$$\mathfrak{h} \xrightarrow{\delta} \underline{k\text{-Der}}(\mathfrak{n}) \longrightarrow \underline{k\text{-Der}}(U_k(\mathfrak{n})) \hookrightarrow \underline{k\text{-Mod}}(U_k(\mathfrak{n})),$$

where $\underline{k\text{-Mod}}(U_k(\mathfrak{n}))$ is the associative algebra of k-linear endomorphisms on $U_k(\mathfrak{n})$. Using this we get a homomorphism of Lie algebras

$$\theta' : \mathfrak{h} \longrightarrow \underline{k\text{-Mod}}(U_k(\mathfrak{n})) \otimes U_k(\mathfrak{h}), \qquad h \longmapsto \delta(h) \otimes 1 + 1 \otimes h,$$

where the tensor product on the right carries the canonical factorwise associative multiplication. By the universal property θ' extends uniquely to an algebra homomorphism $U_k(\mathfrak{h}) \longrightarrow \underline{k\text{-Mod}}(U_k(\mathfrak{n})) \otimes U_k(\mathfrak{h})$, which we will denote by the same letter. Let θ be the adjoint to θ'. In the notation of the proof of Proposition 8.5.1 this definition immediately implies

- $\theta(ss' \otimes r) = \theta'(ss')(r) = \theta'(s)\theta(s')(r) = s \cdot \theta(s' \otimes r), \quad$ for all $s, s' \in U_k(\mathfrak{h}),\ r \in U_k(\mathfrak{n})$,
- $\theta(1 \otimes r) = \theta'(1)(r) = (\mathrm{id} \otimes 1)(r) = r \otimes 1, \quad$ for all $r \in U_k(\mathfrak{n})$.
- $\theta(h \otimes 1) = \delta(h)(1) \otimes 1 + 1 \otimes h = 1 \otimes h, \quad$ for all $h \in \mathfrak{h}$,

Moreover, for $h \in \mathfrak{h}$ and $r, r' \in U_k(\mathfrak{n})$ we have

$$\begin{aligned}\theta(h \otimes r) \cdot r' &= (\delta(h)(r) \otimes 1 + r \otimes h) \cdot r' = \delta(h)(r) \cdot \theta(1 \otimes r') + r \cdot \theta(h \otimes r') \\ &= \delta(h)(r) \cdot (r' \otimes 1) + r \cdot (\delta(h)(r') \otimes 1 + r' \otimes h) \\ &= \delta(h)(r)r' \otimes 1 + r\delta(h)(r') \otimes 1 + rr' \otimes h \\ &= \delta(h)(rr') \otimes 1 + 1 \otimes rr' = \theta(h \otimes rr').\end{aligned}$$

Hence $U_k(\mathfrak{h}) \otimes_\theta U_k(\mathfrak{n})$ is a unital, associative k-algebra by Proposition 8.5.1. Moreover we have natural Lie algebra homomorphisms

$$\mathfrak{n} \longrightarrow U_k(\mathfrak{n}) \longrightarrow U_k(\mathfrak{n}) \otimes_\theta U_k(\mathfrak{h}) \longleftarrow U_k(\mathfrak{h}) \longleftarrow \mathfrak{h},$$

and since for all $h \in \mathfrak{h}$ and $n \in \mathfrak{n}$ we have

$$\begin{aligned}[1 \otimes h, n \otimes 1] &= (1 \otimes h) \cdot (n \otimes 1) - (n \otimes 1) \cdot (1 \otimes h) = \theta(h \otimes n) - n \otimes h \\ &= \delta(h)(n) \otimes 1 + n \otimes h - n \otimes h = \delta(h)(n),\end{aligned}$$

these extend uniquely to a Lie algebra homomorphism $\mathfrak{g} \longrightarrow U_k(\mathfrak{n}) \otimes_\theta U_k(\mathfrak{h})$ and thus to an algebra homomorphism $U_k(\mathfrak{g}) \xrightarrow{a} U_k(\mathfrak{n}) \otimes_\theta U_k(\mathfrak{h})$. The inclusion maps $\mathfrak{n} \longrightarrow \mathfrak{g} \longleftarrow \mathfrak{h}$ induce a k-linear map

$$b : U_k(\mathfrak{n}) \otimes_\theta U_k(\mathfrak{h}) \longrightarrow U_k(\mathfrak{g}) \otimes U_k(\mathfrak{g}) \xrightarrow{\mu} U_k(\mathfrak{g}),$$

which satisfies

$$b\theta(h \otimes r) = \delta(h)(r) \cdot 1 + r \cdot h = h \cdot r, \qquad h \in \mathfrak{h},\ r \in U_k(\mathfrak{n}).$$

Hence by Proposition 8.5.1 b is an algebra homomorphism being compatible with i and j. We have $ab(1 \otimes s) = a(s) = 1 \otimes s$, for all $s \in \mathfrak{h}$ and thus also for all $s \in U_k(\mathfrak{h})$ by the universal property of the enveloping algebra using that ab is an algebra homomorphism. As ab is left $U_k(\mathfrak{n})$-linear, it follows that ab is the identity. Similarly we have

$$ba(n + h) = b(n \otimes 1 + 1 \otimes h) = n + h, \qquad n + h \in \mathfrak{n} + \mathfrak{h} = \mathfrak{g},$$

which proves that also ba is the identity by the universal property of the enveloping algebra. By construction b is precisely the $U_k(\mathfrak{n})$-linear extension of $U_k(\mathfrak{h}) \longrightarrow U_k(\mathfrak{g})$.

□

8.5.1 Free products as semi-direct products

Proposition 8.5.3
For $G, H \in \mathcal{G}rp$ there is a short exact sequence[1]

$$1 \longrightarrow {}^G H \xrightarrow{i} G * H \xrightarrow{\mathrm{id} \cup 0} G \longrightarrow 1,$$

where $i = \coprod_{g \in G} i_g$ and $H \xrightarrow{i_g} G + H$ sends h to ghg^{-1}, for all $g \in G$.

[1]Note that the coproduct of groups is the free product, which usually is denoted by '$*$'.

Proof. Let G be acting from the left on the G-fold copower ${}^G H$ by permuting the coproduct factors, i.e.

$$\sigma : G \longrightarrow \mathcal{G}rp({}^G H), \qquad x \longmapsto \coprod_{y \in G} \iota_{xy}.$$

The group homomorphisms $H \xrightarrow{\iota_1} {}^G H \longrightarrow {}^G H \rtimes_\sigma G \longleftarrow G$ glue to a group homomorphism $G * H \xrightarrow{a} {}^G H \rtimes_\sigma G$. By construction we have

$$i\sigma(x)(\iota_y h) = i(\iota_{xy} h) = xyh(xy)^{-1} = x(yhy^{-1})x^{-1} = xi(\iota_y h)x^{-1}, \qquad x, y \in G,\ h \in H,$$

hence i extends to a homomorphism of groups ${}^G H \rtimes_\sigma G \xrightarrow{b} G * H$. For all $g \in G$ and $h \in H$ we have

- $ab(1, g) = a(g) = (1, g)$,
- $ab(\iota_g h, 1) = a(ghg^{-1}) = a(g)a(h)a(g)^{-1} = a(g)(\iota_1 h, 1)a(g)^{-1} = \iota_g h$,

which proves that $ab = \mathrm{id}$ by the universal property of the semi-direct product. Similarly we have

$$ba(g) = b(1, g) = g, \qquad ba(h) = b(\iota_1 h, 1) = 1 \cdot h = h, \qquad g \in G,\ h \in H,$$

and hence $ba = \mathrm{id}$ by the universal property of coproducts.

$\square$

Remark 8.5.4
In particular for $G \in \mathcal{G}rp$ and a free group $H = {}^X\mathbb{Z}$ generated by a set $X \in \mathcal{S}et$, we get an exact sequence

$$1 \longrightarrow {}^{G \times X}\mathbb{Z} \longrightarrow G * {}^X\mathbb{Z} \xrightarrow{\mathrm{id} \cup 0} G \longrightarrow 1.$$

Proposition 8.5.5
*Let $k \in \mathcal{C}Ring$. For $\mathfrak{g} \in k$-$\mathcal{L}ie$ and $X \in k$-*Mod *there is a natural exact sequence of Lie algebras*[2]

$$0 \longrightarrow \mathcal{L}ie(U_k(\mathfrak{g}) \otimes X) \xrightarrow{i} \mathfrak{g} * \mathcal{L}ie(X) \xrightarrow{\mathrm{id} \cup 0} \mathfrak{g} \longrightarrow 0.$$

Proof. Using the adjoint action $\mathfrak{g} * \mathcal{L}ie(X)$ becomes a $U_k(\mathfrak{g})$-module and the inclusion map $X \longrightarrow \mathcal{L}ie(X) \longrightarrow \mathfrak{g} * \mathcal{L}ie(X)$ extends uniquely to a $U_k(\mathfrak{g})$-linear map $U_k(\mathfrak{g}) \otimes X \longrightarrow \mathfrak{g} * \mathcal{L}ie(X)$ and to the Lie algebra homomorphism i. Moreover left multiplication by an element $g \in \mathfrak{g}$ induces an endomorphism of $U_k(\mathfrak{g}) \otimes X$ extending uniquely to a k-linear derivation on $\mathcal{L}ie(U_k(\mathfrak{g}) \otimes X)$. This defines a k-linear map $\mathfrak{g} \longrightarrow \underline{k\text{-Der}}(\mathcal{L}ie(U_k(\mathfrak{g}) \otimes X)$. Since

$$\delta([g, h])(1 \otimes x) = [g, h] \otimes x = g \cdot (h \otimes x) - h \cdot (g \otimes x) = [\delta(g), \delta(h)](x), \qquad g, h \in \mathfrak{g},\ x \in X,$$

[2]To avoid confusion and to emphasize the similarity, we also denote the coproduct in the category of Lie algebras by '$*$' here.

it follows that δ is a homomorphism of Lie algebras and we can form the semi-direct product $\mathcal{L}ie(U_k(\mathfrak{g}) \otimes X) \rtimes_\delta \mathfrak{g}$. There are canonical Lie algebra homomorphisms

$$\mathcal{L}ie(X) \longrightarrow \mathcal{L}ie(U_k(\mathfrak{g}) \otimes X) \longrightarrow \mathcal{L}ie(U_k(\mathfrak{g}) \otimes X) \rtimes_\delta \mathfrak{g} \longleftarrow \mathfrak{g},$$

which glue to a homomorphism of Lie algebras $\mathfrak{g} + \mathcal{L}ie(X) \xrightarrow{a} \mathcal{L}ie(U_k(\mathfrak{g}) \otimes X) \rtimes_\delta \mathfrak{g}$. For all $g_0, ..., g_n \in \mathfrak{g}$ and $x \in X$ we have by construction

$$i\delta(g_0)((g_1 \cdots g_n) \otimes x) = i((g_0 \cdots g_n) \otimes x) = \mathrm{ad}(g_0) \circ ... \circ \mathrm{ad}(g_n)(x) = [g_0, i((g_1 \cdots g_n) \otimes x)],$$

hence i extends to a homomorphism $\mathcal{L}ie(U_k(\mathfrak{g}) \otimes X) \rtimes_\delta \mathfrak{g} \xrightarrow{b} \mathfrak{g} + \mathcal{L}ie(X)$. For all $g_0, ..., g_n \in \mathfrak{g}$ and $x \in X$ we have

- $ab(0, g_0) = a(g_0) = (0, g_0)$,
- $ab((g_1 \cdots g_n) \otimes x, 0) = a(\mathrm{ad}(g_1) \circ ... \circ \mathrm{ad}(g_n)(x)) = \mathrm{ad}(g_1) \circ ... \circ \mathrm{ad}(g_n)(1 \otimes x) = (g_1 \cdots g_n) \otimes x$,

which proves that $ab = \mathrm{id}$ by the universal property of semi-direct products. Similarly

$$ba(g) = b(0, g) = g, \qquad ba(x) = b(1 \otimes x) = i(1 \otimes x) = x, \qquad g \in \mathfrak{g},\ x \in X,$$

which proves that also $ba = \mathrm{id}$ by the universal property of coproducts.

$\square$

9 Conclusion and Outlook

First of all we want to mention that our strategy presented in chapter 3 can also be used to analogously develop a theory for p-valued groups. Then the proof of Theorem 3.5.2 also provides a plan to prove a mixed characteristic version, which generalizes Lazard's rational isomorphism $\widehat{H}_*(G, M) \cong \widehat{H}_*(\mathfrak{g}, M)$ to integral coefficients. A first step in this direction was made by [HKN09], but using our techniques could improve this result even further. Working this out in detail could be a future project in close range.

A direct application could be a mixed characteristic version of the main Theorem 6.3.22, which would be a variant of Beilinson's Theorem [Bei14].

Conjecture 9.0.6
Let $p > 1$ be a prime number, $p \in I \lhd A \in \mathcal{R}ing$, such that $A \xrightarrow{\sim} \varprojlim_{n\geq 0} A/I^n$ and $\mathrm{gr}A = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is finitely generated and free over $\mathrm{gr}\mathbb{Z}_p = \mathbb{F}_p[t]$.

If r is the maximum of 2 and the stable range of A, then there are natural isomorphisms

$$K_n(A, I; \mathbb{Z}_p) \cong HC_{n-1}(A, I; \mathbb{Z}_p), \qquad 1 \leq n \leq p - r.$$

The stable range condition on A is needed, because we cannot hope to be able to identify the p-completed homology of $X_r(A, I)$ with that of $x_r(A, I)$, for arbitrary large $r \geq 1$. Moreover the finite generation property is needed so that we can apply the Theorems of Suslin [Sus84] and Suslin-Yufryakov [SY86] to verify that K-theory commutes with the projective limits over all quotients A/I^n.

Although the main problem could be solved, we want to suggest some further possible improvements. As we already mentioned in section 6.3.2 we originally wanted to identify $\pi_{*-1}\mathfrak{gl}(A)^+$ with $PH_*(\mathfrak{gl}A, k) \cong HC_{*-1}(A)$ in low dimensions like in the multiplicative setting, and then use the long exact sequence to prove the relative statement. This naturally leads to the following

Conjecture 9.0.7
For $k \in \mathcal{C}Ring$ and flat $A \in k/\mathcal{R}ing$ with $p > 1$, the following seems reasonable.

The Hurewicz map and the maps of Remark 5.4.17 below are isomorphisms

$$K_n^{\mathcal{L}ie}(A) = \pi_{n-1}\mathfrak{gl}(A)^+ \xrightarrow{\sim} PH_n(\mathfrak{gl}_\infty A, k) \xrightarrow{\sim} HC_{n-1}(A), \qquad 0 \leq n < p.$$

Now the map $\mathbb{S} \longrightarrow H\mathbb{Z}$ is $(2p-3)$-connected after inverting $(p-1)!$ (cf. Proposition 4.3.4), the natural question arises, if we could also extend our isomorphism up to this range. As the number p (up to a constant) appears as an upper bound several times, we will discuss each of its appearances.

First, dimension p is the limit in the identification of the homotopy groups with the primitive part of homology (cf. Corollary 4.3.13), which only works up to dimension $p-1$, if the space of question is connected. Of course, if the first relative cyclic homology group vanishes, so does the first relative algebraic K-group by the main Theorem 6.3.22. In this case the map automatically gets $(2p-3)$-connected by Corollary 4.3.13. Another idea could be to deloop both constructions before taking homology. This could be done using the bar construction to the monoid structure of

$$D(A,J) = \operatorname*{colim}_{\hat{\Delta}_{inj}} X_\bullet(A,J), \qquad d(A,J) = \operatorname*{colim}_{\hat{\Delta}_{inj}} x_\bullet(A,J).$$

However one should also be able to construct a zig-zag of space level maps between relative multiplicative K-theory and relative additive K-theory.

Conjecture 9.0.8
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k and $(p-1)! \in A^\times$, for some $p > 1$.

Then there is a zig-zag of simplicial sets linking $Y(A,I)^+$ and $y(A,I)^+$ and inducing isomorphisms on homotopy groups in dimensions $< 2p-2$.

Second, the upper bound p appears, when going down to the quotient

$$(\Lambda_* \mathfrak{gl}_r A)^{(0)} \twoheadrightarrow (\Lambda_* \mathfrak{gl}_r A)^{(0)}_{\Sigma_r}.$$

From Proposition 5.4.14 and Corollary 5.4.16 we know that on the primitive part this map corresponds to the map

$$CC_*(A) \simeq C^\lambda_*(A, \Sigma_\bullet) \twoheadrightarrow C^\lambda_*(A),$$

which also is only $(p-1)$-connected by Proposition 5.1.9. So one should maybe try to prove the following conjecture to get better connectivity.

Conjecture 9.0.9
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$ with A and A/I flat over k and $(p-1)! \in A^\times$, for some $p > 1$.

Then the map $\mathcal{C}om_1(C^\lambda_{*-1}(A, \Sigma_\bullet)) \longrightarrow d(A)^{(0)} = \operatorname{colim}_{\hat{\Delta}_{inj}} (\Lambda_* \mathfrak{gl}_\bullet A)^{(0)}$ is $(2p-2)$-connected or even a weak equivalence.

Third, we only have connectivity $< (p-1)$, when going down to the quotient

$$\Lambda_* \mathfrak{gl}_r A \twoheadrightarrow \bigoplus_{a \in \mathbb{Z}^r} (\Lambda_* \mathfrak{gl}_r A)^{(a)} \otimes k/\gcd(a_1, ..., a_r)k \twoheadrightarrow (\Lambda_* \mathfrak{gl}_r A)^{(0)}.$$

Using Proposition 5.4.3 (i) one can see that the first map is a quasi-isomorphism, while the second one is only $(p-1)$-connected. If the right object corresponds to $\mathcal{C}om_1(C^\lambda_{*-1}(A, \Sigma_\bullet))$, then the left object may correspond to a variant of cyclic homology, maybe topological cyclic homology. So it also seems natural to ask

Question 9.0.10
Let $k \in \mathcal{C}Ring$ and $I \lhd A \in k/\mathcal{R}ing$.

Are the groups $K_*^{\mathcal{L}ie}(A, I)$ and ${}_{Hk}TC_*(A, I)$ isomorphic in positive dimensions?

By the latter we mean the relative topological cyclic homology over the base ring spectrum Hk (i.e. the Eilenberg-Maclane ring spectrum to the ring k) in the sense of [ABG$^+$14].

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Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Ort, Datum, Unterschrift

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