Strategic interaction under uncertainty

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für Sonja, Heinz-Jürgen, Jenny und Nina

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Chapter 1

Introduction

The theory of strategic interaction or, game theory, for short, plays an important role in economics. It can offer insights into situations in which two or more interacting individuals choose actions that jointly affect the payoff of each party. Game-theoretic applications cover a wide range of economic, political and social situations such as auctions, contract formation, bargaining situations, political competition, and public good provision, to only name a few. This broad scope of application makes it a powerful concept. Most games involve some kind of uncertainty. For instance, players may be uncertain about the strategy choice of other players or they may lack information about the strategic environment.

Game theory is closely tied to decision theory. In fact, the former can be viewed as the natural extension of the latter. In the words of Myerson (1991, p. 5): "The logical roots of game theory are in Bayesian decision theory. Indeed, game theory can be viewed as an extension of decision theory [...]. Thus, to understand the fundamental ideas of game theory, one should begin by studying decision theory." Bayesian decision theory assumes that decision makers' subjective beliefs can be represented by unique probability measures and that they update their prior beliefs in accordance with Bayes' rule when receiving new information. Furthermore, Bayesian decision-makers usually are subjective expected utility maximizers. Savage (1954) provided an axiomatic foundation for the Bayesian approach. His subjective expected utility theory has become the leading model of choice under uncertainty.

However, Ellsberg (1961) questioned the descriptive adequacy of subjective expected utility theory. He exemplified that the choice behavior of many subjects is not consistent with Savage's theory when facing "ambiguous uncertainty", or "ambiguity", that is, a situation in which some events have known probabilities, whereas for other ones the probabilities are unknown. Ellsberg's observation has received powerful empirical support in the last decades (see Camerer and Weber, 1992). In this thesis, the term "uncertainty" will be used as a generic term to cover both ambiguity and non-ambiguous uncertainty ("risk"). To represent behavior as observed by Ellsberg, several alternatives to subjective expected utility theory have been suggested in recent years. Two prominent alternatives are Choquet expected utility theory of Schmeidler (1989) and the multiple prior approach of Gilboa and Schmeidler (1989). More recent examples are the smooth ambiguity model of Klibanoff et al. (2005) and the variational model of Maccheroni et al. (2006).

The main goal of this thesis is to shed some light on the impact of ambiguity-sensitive behavior on strategic decision-making in interactive situations. As Crawford (1990, p. 152) appropriately expressed it: "In recent years, non-expected utility decision models have given us significantly better explanations of observed behavior in nonstrategic environments. These successes, and the weight of the experimental evidence against the expected utility hypothesis, suggest that much might be learned about strategic behavior by basing applications of game theory on more general models of individual decisions under uncertainty." In this spirit, the present thesis investigates non-cooperative game models that are based on alternative models of individual decision-making under uncertainty. The main body of this dissertation consists of three chapters (Chapters 4, 5 and 6), each of which studies strategic interaction under uncertainty. Chapter 4 and 5 explore formal models in which uncertainty arises from exogenous chance moves and incomplete information, respectively. While the game studied in Chapter 4 does not involve private information, the model in Chapter 5 allows for private information. Chapter 6 experimentally examines the extent to which a lack of information about others' preferences affects subject behavior. It is shown that a strategic ambiguity model as well as a quasi Bayesian model of incomplete information explain the findings better than standard Nash equilibrium. The results of chapters 4 and 6 are based on collaborative work with Boris Wiesenfarth (Chapter 4), and Christoph Brunner and Hannes Rau (Chapter 6).

This thesis is organized as follows. Chapter 2 outlines the decision-theoretic foundations of the interactive models studied in this work. First, the historical development of modern decision theory is briefly reviewed. I recall in some detail the fundamentals of subjective expected utility theory as well as the experiments by Ellsberg (1961). Finally, alternative models of choice under uncertainty are considered, especially, the Choquet expected utility model and the multiple prior model. These models will be used in subsequent chapters. Chapter 3 discusses some conceptual foundations of non-cooperative game theory. It starts with sketching the historical roots of modern game theory. Basic concepts such as the concept of a game and the Nash equilibrium concept are recalled. The last part of this chapter deals with different sources of uncertainty in games. In the context of strategic uncertainty, I describe generalized equilibrium concepts that allow for players whose preferences are not represented by expected utility functionals. Furthermore, I review the class of Bayesian games introduced by Harsanyi (1967-68) to analyze games of incomplete information.

In Chapter 4, a Hotelling duopoly game that incorporates ambiguous uncertainty about the market demand is examined. The key assumption of this model is that firms' beliefs are represented by neo-additive capacities introduced by Chateauneuf et al. (2007). The related literature is reviewed and the model is specified. Moreover, this chapter discusses implications for possible applications of the Capacity model and limitations of the existing models. Chapter 5 investigates the extent to which we can distinguish expected and uncertainty-averse non-expected utility players on the basis of their behavior. A model of incomplete information games is used in which players can choose mixed strategies. First, this model is illustrated by two examples and described in detail. The following part of the chapter provides the results. Subsequently, I discuss the underlying model and introduce a generalized equilibrium concept. Chapter 6 reports on the results of the aforementioned experimental study testing whether revealing players' preferences to each other leads to more equilibrium play. Chapter 7 concludes with an overall summary.

In terms of the style of this thesis, definitions, examples, figures, tables, etc. are numbered per chapter (e.g., Example 2.1), except axioms, which are consecutively numbered as well as footnotes. In the text, italics indicate definitions. The first time an abbreviation is used, both the spelled-out version and short form are presented, where one of which is written in parentheses. Notations are defined when they are used for the first time in the text. A unified notation is used across all of the chapters, with only a few exceptions. For instance, in Chapter 4, "σ^2" denotes the variance of a random variable, while, in the other chapters, a mixed strategy profile is denoted by "σ". For the sake of simplicity, generic female pronouns are used in this thesis, i.e., "she" stands for "he or she" et cetera.

Chapter 2

Decision-theoretic foundations

2.1 Historical background

Theories of decision-making under uncertainty have a long tradition, reaching back at least to the 18th century. During this time, Daniel Bernoulli (1738/1956) explicitly formulated the idea of expected utility maximization in order to solve the so-called St. Petersburg paradox. This paradox is based on a gamble with infinite expected value. Decision-makers are only willing to pay a finite (and rather small) price to enter such a gamble, although its expected value is infinite. At Bernoulli's time, this was paradoxical since it contradicted the prevailing opinion that the expected monetary value is an adequate decision criterion in uncertain situations.

After Bernoulli, no further seminal contributions to decision theory appeared until the early 20th century. At this time, in particular, Ramsey (1926) and de Finetti (1937) renewed the idea of "subjective probabilities" in the context of decision problems under uncertainty. According to this idea, probabilities reflect subjective degrees of belief, i.e., personal assessments of relative likelihoods. In contrast, for "objectivists" or "frequentists" such as John Venn: "[...] all which Probability discusses is the statistical frequency of events, or, if we prefer so to put it, the quantity of belief with which any one of these events should be individually regarded [...]" (Venn, 1888, p. 29). Following this interpretation, probabilities explain stable relative frequencies that remain comparatively constant across large numbers of trials. Hence, probabilities reflect objective evidence, or, more precisely, probabilities measure the physical tendency of an event to occur.

In the view of Ramsey and de Finetti, individuals act as if they attach subjective probabilities to the states of the world, even if there is no objective probabilistic information available. To put it differently, individuals maximize expected utility with respect to their subjective beliefs. Independently, Ramsey and de Finetti suggested to infer subjective probabilities from choices between bets. In their words: "The old-established way of measuring a person's belief is to propose a bet, and see what are the lowest odds which he will accept. This method I regard as fundamentally sound" (Ramsey, 1926, p. 73) and "It is a question simply of making mathematically precise the trivial and obvious idea that the degree of probability attributed by an individual to a given event is revealed by the conditions under which he would be disposed to bet on that event" (de Finetti, 1937, p. 101). Ramsey and de Finetti used axiomatic approaches, which take the existence of utilities and monetary payoffs, respectively, as given, to derive subjective probabilities from preferences over bets. Both papers can be seen as precursors of modern decision theory, which mainly studies axioms of rational decision-making.

The first axiomatic foundation for the concept of utility and expected utility maximization was published by von Neumann and Morgenstern (1944). In their theory, the objects of choice are lotteries, i.e., probability distributions over outcomes. Von Neumann and Morgenstern identified a parsimonious set of seemingly reasonable axioms on preferences over lotteries which are necessary and sufficient for the existence of a utility function on the set of outcomes and for the expected utility criterion. Compared to Ramsey and de Finetti, the approach from von Neumann and Morgenstern operates conversely: it takes the existence of the probabilities as given and shows that numerical utilities of outcomes can be derived from preferences over lotteries.

The seminal paper of Leonard Savage (1954) synthesized the ideas of Ramsey, de Finetti and von Neumann and Morgenstern. Following Kreps (1988, p. 120), Savage's theory is "[...] the crowning glory of choice theory [...]." Savage showed how to obtain utilities, subjective probabilities and the expected utility decision criterion without taking probabilities and utilities as primitives. His approach will be discussed in more detail in Section 2.2.

After the papers by von Neumann and Morgenstern and Savage, "paradoxes" were exposed that questioned the descriptive adequacy of their approaches. Allais (1953) challenged von Neumann and Morgenstern's theory by showing that in situations where probabilities are given, a decision-maker's utilities and probabilities may not combine linearly. Subsequently, Ellsberg (1961) questioned Savage's subjective expected utility theory. He exemplified that individuals frequently display preferences which are not consistent with a subjective probability measure when they face ambiguous uncertainty. Section 2.3 elaborates further on the so-called Ellsberg paradox since it gave rise to the non-probabilistic generalizations of Savage's theory described in Section 2.4.

In the same time period, apart from the literature on expected utility theory, a notable contribution to decision theory was made by Wald (1950). Inspired by von Neumann and Morgenstern's analysis of games, Wald suggested the maxmin decision criterion, sometimes also called "Wald criterion". This criterion is another ancestor to the literature on non-probabilistic decision theory reviewed in Section 2.4. It is intuitive and easy to apply: in an uncertain situation, a maxmin decision-maker looks at the worst potential consequence of each alternative and then chooses the alternative with the best worst-case outcome. The maxmin principle was generalized by Arrow and Hurwicz (1972). According to their generalized criterion, a decision-maker evaluates an alternative by a convex combination of its worst and its best consequence.

2.2 Subjective expected utility theory

This section describes subjective expected utility theory in the sense of Savage (1954). The focus lies on the elements challenged by Ellsberg (1961). At first, we consider Savage's original framework and then the framework proposed by Anscombe and Aumann (1963).

Savage's framework consists of four elements: a set Ω (*the states of the world*), a set X (*the outcomes or consequences*), a set $\mathcal{F}$ (*the acts*), and a binary relation $\succsim$ on the set $\mathcal{F}$ (*the decision-maker's preferences*). According to Savage (1954, p. 9), a state of the world $\omega \in \Omega$ is "a description of the world, leaving no relevant aspect undescribed," and the true state is "the state that does in fact obtain [...]." Ex-ante, the decision-maker does not know the true state of the world. Hence, we can think of the set Ω as an exhaustive list of all scenarios that may be encountered. Any subset $E \subset \Omega$ is called an *event*. The set X comprises "[...] anything at all about which the person could possibly be concerned" (Savage, 1954, p. 14). An object of choice is an act, which is defined as "[...] a function attaching a consequence to each state of the world" (Savage, 1954, p. 14). In other words, acts are functions from states to consequences. Hence, the set of all acts is $\mathcal{F} = \{f \mid f : \Omega \to X\}$. Since the decision-maker does not know the true state of the world, she is uncertain about which consequence $f(\omega) \in X$ will result from an act f. However, she knows all possible consequences of an act. Savage's setting assumes a binary relation $\succsim \subseteq \mathcal{F} \times \mathcal{F}$, which represents the decision-maker's preferences.

Savage postulated seven restrictions or axioms on the preference relation $\succsim$. He showed that a decision-maker, whose preferences satisfy his axioms, will behave as if she possesses a utility function over the outcomes and a unique subjective prior distribution over the states. Moreover, when choosing among acts, the decision-maker will choose the one with the highest expected utility according to her utility function and her subjective belief. One of Savage's main axioms is the so-called Sure-Thing Principle. Roughly, it requires that the preference between two acts should not depend on the states of the world where both acts have identical consequences. Formally, the axiom is formulated as follows:

Axiom 1 (Sure-Thing Principle). *For all acts $f, g, f', g' \in \mathcal{F}$ and every event $E \subset \Omega$, if*

$f(\omega) = f'(\omega)$ and $g(\omega) = g'(\omega)$ for all $\omega \in E$ and

$f(\omega) = g(\omega)$ and $f'(\omega) = g'(\omega)$ for all $\omega \notin E$,

then $f \succsim g \Leftrightarrow f' \succsim g'$.

The Sure-Thing Principle is a separability axiom. It says that when comparing two acts, it suffices to consider the states of the world in which these acts yield different outcomes. The following example illustrates the rationale behind this axiom.

Example 2.1. *There are three states of the world, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and three consequences, $X = \{a, b, c\}$. Consider the event $E = \{\omega_1, \omega_2\}$ and a decision-maker who has to choose between the acts given in Table 2.1.*

Table 2.1: Example for the Sure-Thing Principle

	ω_1	ω_2	ω_3
f	a	b	c
g	c	a	c
f'	a	b	b
g'	c	a	b

Suppose that the decision-maker in Example 2.1 prefers, for some reason, act f to act g. Then, the Sure-Thing Principle requires that she has to prefer f' over g'. The reason behind this is the following. On the one hand, f and g as well as f' and g' differ only if event E occurs. In this case, f corresponds to f' and g corresponds to g'. Therefore, it seems reasonable that the preference ranking between f and g should be the same as that of f' and g'. To put it differently, when comparing two acts, we may "eliminate" any state of the world in which both acts yield identical consequences. The axiom implies a characteristic feature of subjective expected utility theory, namely event-separability. That is, preferences over acts are separable across mutually exclusive events. The Sure-Thing Principle seems intuitive and reasonable. In the words of Savage (1954, p. 21): "[...] I know of no other extralogical principle governing decisions that finds such ready acceptance." Nonetheless, a considerable share of individuals violate this axiom in Ellsberg's experiments, which is further discussed in Section 2.3.

Savage's representation theorem can be expressed as follows: a preference relation $\succsim$ over $\mathcal{F}$ satisfies his seven axioms if and only if there exists a unique probability measure π on Ω and a function $u : X \to \mathbb{R}$, which is unique up to positive linear transformations, such that for every $f, g \in \mathcal{F}$,

$$f \succsim g \;\; \Leftrightarrow \;\; \int_\Omega u(f(\omega))\, d\pi(\omega) \geq \int_\Omega u(g(\omega))\, d\pi(\omega). \tag{2.1}$$

In this equation, the probability measure π represents the decision-maker's subjective belief, and the utility function u on the outcomes represents her taste.

Savage does not put any restrictions on the set of outcomes X, e.g., on its topology. However, this generality comes at the cost of having an infinite set of states Ω. The main motivation of Anscombe and Aumann (1963) is to develop a simplified subjective expected utility framework: "The novelty of our presentation, if any, lies in the double use of utility theory, permitting the very simple and plausible assumptions and the simple construction and proof" (Anscombe and Aumann, 1963, p. 203). The key assumption of Anscombe and Aumann's approach is that there are two sources of uncertainty: a *roulette lottery*, which refers to uncertainty generated by an objective randomization device such as a roulette wheel and a *horse lottery*, which refers to a source of subjective uncertainty like a horse race. By using this setup, Anscombe and Aumann showed that it is possible to derive subjective expected utility for a finite set of states and with a smaller set of axioms. In their framework, the set of roulette lotteries, $\Delta(X)$, is the set of all probability distributions with finite support on the set of consequences X, formally,

$$\Delta(X) = \left\{ \mu : X \to [0,1] \;\middle|\; \begin{array}{l} \#\{x \in X \mid \mu(x) > 0\} < \infty, \\ \sum_{x \in X} \mu(x) = 1 \end{array} \right\}.$$

The objects of choice are "horse-then-roulette lotteries", i.e., functions from states to $\Delta(X)$. The set of all "Anscombe-Aumann acts" is denoted by $\mathcal{H}$. These acts can be viewed as two-stage lotteries. At first, "horse races" take place to determine the state of the world, i.e., there is a lottery with unknown probabilities where the outcome is a state $\omega \in \Omega$, which determines the roulette lottery $h(\omega)$. Afterwards, the roulette lottery $h(\omega)$ is played out, in which outcome $x \in X$ obtains with a known probability. Henceforth, $h(\omega)(x)$ denotes the probability that lottery $h(\omega)$ assigns to outcome $x \in X$. The set $\mathcal{H}$ is endowed with a mixture operation: mixtures of acts are performed statewise, that is, for all $h, g \in \mathcal{H}$ and $\alpha \in [0,1]$, $(\alpha h + (1-\alpha)g)(\omega) = \alpha h(\omega) + (1-\alpha)g(\omega)$.

A central axiom of Anscombe and Aumann's approach is the independence axiom, which is adopted from von Neumann and Morgenstern (1944):

Axiom 2 (Independence)**.** *For all $h, g, h' \in \mathcal{H}$ and all $\alpha \in (0,1)$,*

$h \succsim g \Leftrightarrow \alpha h + (1-\alpha)h' \succsim \alpha g + (1-\alpha)h'$.

The approaches, which are discussed in detail in Section 2.4, use an Anscombe-Aumann setup. A core feature of both approaches is that they weaken Axiom 2.

Anscombe and Aumann showed that a preference relation $\succsim$ over $\mathcal{H}$ satisfies their axioms if and only if there exists a unique subjective probability measure π on Ω and a utility function u, which is unique up to positive linear transformations, such that for every $h, g \in \mathcal{H}$,

$$h \succsim g \ \Leftrightarrow \ \int_{\Omega} \left[\sum_{x \in X} h(\omega)(x)u(x) \right] d\pi(\omega) \geq \int_{\Omega} \left[\sum_{x \in X} g(\omega)(x)u(x) \right] d\pi(\omega). \tag{2.2}$$

This representation involves double integration: given an act $h \in \mathcal{H}$, one first determines the decision-maker's expected utility of the roulette lottery $h(\omega)$ for every state $\omega \in \Omega$. Then, the expectation of these expected utility values is taken under the subjective measure π.

In the literature, there are several papers dealing with variations of the two frameworks described in this section. For instance, Gul (1992) derives the subjective expected utility theorem in a finite state variant of Savage's setting with topological restrictions on the set of consequences X. Sarin and Wakker (1997) simplify Anscombe and Aumann's setup by reducing their two-stage approach to a single-stage approach.

2.3 Ellsberg's experiments

Daniel Ellsberg (1961) questioned the descriptive adequacy of the subjective expected utility theory. He illustrated with the help of thought experiments that, when faced with a special type of uncertainty, called "ambiguity", the behavior of most subjects is not consistent with a unique subjective prior distribution. Ellsberg viewed *ambiguity* as a situation in which the probabilities of some events are known, while for other events they are unknown: "What is at issue might be called the ambiguity of this information, a quality depending on the amount, type, reliability and 'unanimity' of information, and giving rise to one's degree of 'confidence' in an estimate of relative likelihoods" (Ellsberg, 1961, p. 657). In the recent literature, ambiguity is mostly associated with situations where probabilities are imperfectly known. For instance, according to Camerer and Weber (1992, p. 330): "Ambiguity is uncertainty about probability, created by missing information that is relevant and could be known." That means, ambiguity also refers to situations without known probabilities. The term "Knightian uncertainty" is often used as a synonym for ambiguity. The reason is that Knight (1921) has already distinguished between measurable uncertainty ("risk"), which can be represented by probabilities, and unmeasurable uncertainty ("uncertainty"), which cannot.

Ellsberg proposed two thought experiments. The first experiment stems from Knight (1921) and is as follows.

Two-urn experiment. *There are two urns, I and II, each containing 100 balls, which are either red or black. In urn I, there are 50 black balls and 50 red balls. The composition of urn II is unknown. A subject can choose one of the two urns and bet on the color of a ball drawn from this urn. There are four possible bets: "choose to draw from urn I, bet on red" (f); "choose to draw from urn I, bet on black" (g); "choose to draw from urn II, bet on red" (f′) and "choose to draw from urn II, bet on black" (g′). If the subject wins the bet, she gets $ 100, otherwise nothing. Table 2.2 summarizes the bets in the experiments.*

Table 2.2: Ellsberg's two-urn experiment

Urn I

	Red	Black
f	100	0
g	0	100

Urn II

	Red	Black
f'	100	0
g'	0	100

Ellsberg reports that a majority of the people he asked "under nonexperimental conditions" are indifferent between betting on red and on black, given that the ball is drawn from the same urn. That is, $f \sim g$ and $f' \sim g'$. Given a bet on a particular color, they strictly prefer the urn with known proportions, i.e., $f \succ f'$ and $g \succ g'$. Furthermore, Ellsberg observed a small minority of subjects with exactly opposite preferences, i.e., these subjects strictly prefer to bet on Urn II. Both patterns of choices are not compatible with the idea that the decision-maker has probabilistic beliefs. To see why, consider a subject with preferences $f \succ f'$ and $g \succ g'$. From the observation $f \succ f'$, we may infer that the subject considers it as more probable that a red ball is drawn from Urn I than from Urn II. Similarly, from $g \succ g'$, we may infer that she considers it to be more probable that a black ball is drawn from Urn I than from Urn II. Apparently, these judgments are not consistent with a well-defined probability distribution. Hence, the choices of the subject cannot be explained by subjective expected utility maximization. We can conclude that the subject must violate some of the Savage axioms.

Ellsberg's second experiment is a direct test of Savage's Sure-Thing Principle.

One-urn experiment. *There is an urn containing 30 red balls and 60 others that are black and yellow in unknown proportion. One ball is to be drawn random from the urn. There are two choice situations. In the first choice situation, a subject is asked to choose between the following bets: "bet on red" (f) and "bet on black" (g). In the second choice situation, the subject can choose between the options: "bet on red or yellow" (f') and "bet on black or yellow" (g'). The subject gets \$ 100 if she wins the bet and nothing otherwise. Table 2.3 summarizes the bets in the experiments.*

Table 2.3: Ellsberg's one-urn experiment

	Red	Black	Yellow
f	100	0	0
g	0	100	0
f'	100	0	100
g'	0	100	100

In the second experiment, Ellsberg frequently observed the following choice pattern: $f \succ g$ and $g' \succ f'$. This pattern is a direct violation of Axiom 1. Again, it is not consistent with a subjective prior distribution. To see why, let R,B and Y denote the event that the ball drawn is red, black or yellow and denote by $\pi(E)$ the subjective probability of event E. From $f \succ g$, we may infer that $\pi(R) > \pi(B)$, and from $g' \succ f'$ that $\pi(R \cup Y) < \pi(B \cup Y)$. Obviously, this is not consistent with a probability distribution.

To sum up, Ellsberg observed that a majority of his subjects strictly prefer the urn with known proportions in the two-urn experiment and bets with known probabilities in the one-urn experiment. These subjects can be termed “ambiguity averse” since they apparently try to avoid ambiguous uncertainty. By now, there is ample empirical evidence supporting Ellsberg’s observations, for a survey see Camerer and Weber (1992).

In an early reply to Ellsberg, Howard Raiffa (1961) suggested that ambiguous uncertainty can be eliminated by randomization over acts. He proposed the following example:

Example 2.2 (cf. Raiffa, 1961, p. 693-694)**.** *Consider Ellsberg’s one-urn experiment. Suppose a fair coin is tossed and the subject is asked to choose between the following options: “Act f is taken if heads appears and act g' is taken if tails appears” (A) and “Act g is taken if heads appears and act f' is taken if tails appears” (B).*

The options in Raiffa’s example can be viewed as objective 50-50 mixtures over Ellsberg’s one-urn bets, i.e., Option A corresponds to 1/2 f+1/2 g’ and Option B to 1/2 g+ 1/2 f’. The consequences of either option depend on the coin toss and the selection of a ball. Following Raiffa, either option leads to an “objective” 50-50 chance of getting $ 100 and 0. He claimed that “[..] options A and B are objectively identical!” (Raiffa, 1961, p. 694). If we accept Raiffa’s view, we can conclude that ambiguity averse subjects must prefer mixtures over acts. The uncertainty aversion axiom of Schmeidler (1989) is in line with this conclusion (see Axiom 3 in Section 2.4). However, Raiffa’s view is controversial. To my knowledge, there exists no strong evidence for his view.

2.4 Non-expected utility theory

Subjective expected utility theory has two characteristic features (see equation (2.1) and (2.2)): (1) The decision-maker’s beliefs are probabilistic, i.e., her beliefs can be represented by a single probability distribution defined over the states of the world and (2) the decision-maker applies the subjective expected utility decision criterion. Consequently, by taking expectations, beliefs are used in a linear manner.

The features (1) and (2) are not inseparable. Machina and Schmeidler (1992) characterize decision-makers that meet (1), but fail (2). In the words of Machina and Schmeidler

(1992, p. 747): “[...] 'What does it take for choice behavior that does not necessarily conform to the expected utility hypothesis to nonetheless be based on probabilistic beliefs?' We will call such an agent a *probabilistically sophisticated* ***non****-expected utility maximizer*.” Probabilistic sophistication is a weaker criterion than expected utility. Any subjective expected utility maximizer is probabilistically sophisticated, but the converse is not true. Apparently, the Ellsberg paradox, described in the previous section, challenges not only subjective expected utility theory but also probabilistic sophistication.

Non-expected utility models can be roughly divided into two groups. In the first group, we find non-expected utility models in which decision-makers are probabilistically sophisticated. The second group consists of those which allow for non-probabilistic beliefs. In the last decades, various non-expected utility models were developed. It is beyond the scope of the present work to discuss all these models. Therefore, this section describes two prominent approaches, which are relevant for this work: the non-additive prior or Choquet model and the multiple prior model. Both approaches have been suggested in the wake of the overwhelming empirical evidence confirming Ellsberg's observations. Hence, they belong to the second group of non-expected utility models where decision-makers do not necessarily have probabilistic beliefs. In the Choquet approach, subjective beliefs are represented by a non-additive measure called capacity, in the multiple priors model, by a set of priors.

Non-additive prior

Following Etner et al. (2012), the Choquet expected utility model, developed by David Schmeidler (1989), is the first axiomatically sound model of decision-making under ambiguity. In this model, beliefs are characterized by capacities.

Definition 2.1 (cf. Schmeidler, 1986, p. 255)**.** Let Ω be a set of states of nature and $\mathcal{A}$ a nonempty sigma-algebra of subsets of Ω. A *capacity* is a real-valued set function $\nu : \mathcal{A} \to \mathbb{R}$ that satisfies the following properties:

(i) $\nu(\varnothing) = 0$ and $\nu(\Omega) = 1$ (normalization),

(ii) for any $E, F \in \mathcal{A}$, $E \subseteq F$ implies $\nu(E) \leq \nu(F)$ (monotonicity).

A capacity is a generalization of a probability measure that does not necessarily satisfy the property of sigma-additivity. Hence, a capacity can be seen as a non-additive measure or prior. Choquet (1954) introduced an integration operation with respect to capacities:

Definition 2.2 (cf. Schmeidler, 1986, p. 255-256)**.** The *Choquet integral* of a bounded and $\mathcal{A}-$measurable function $\phi : \Omega \to \mathbb{R}$ with respect to a capacity ν is

$$\int_\Omega \phi \, d\nu = \int_{-\infty}^{0} \big[\nu(\{\omega \mid \phi(\omega) \geq t\}) - 1\big] \, dt + \int_{0}^{\infty} \big[\nu(\{\omega \mid \phi(\omega) \geq t\})\big] \, dt,$$

where the integrals on the right-hand side of this equation are Riemann integrals.

Remark 2.1 (cf. Schmeidler, 1986, p. 257)**.** Suppose ϕ has a finite range and takes the values $\phi_1 > \phi_2 > \cdots > \phi_k > 0$. That is, ϕ_i, $i = 1, \ldots, k$, denotes the ith highest value of ϕ. Let $E_i \subseteq \Omega$ be the event in which outcome ϕ_i occurs, i.e., E_i is the preimage of ϕ_i under ϕ. Note that the collection of sets $\{E_i\}_{i=1}^{n}$ is a partition of Ω. Let $\phi_{k+1} := 0$, then, the Choquet integral of ϕ with respect to a capacity ν can be expressed as :

$$\int_\Omega \phi \, d\nu = \sum_{i=1}^{k} (\phi_i - \phi_{i+1}) \, \nu\Big(\bigcup_{j=1}^{i} E_j\Big).$$

In economic applications, we often consider situations in which the function ϕ has a finite range, e.g., if the state space is assumed to be finite. The expression of the Choquet integral in Remark 2.1 is helpful for gaining intuition about it. We may view the capacity ν as a decision-maker's belief, the function ϕ as an act and the values ϕ_i as the consequences of ϕ. Interpreted in this way, the decision-maker evaluates an act according to the Choquet integral as follows. She considers first the lowest outcome of the act and then she adds subsequent potential gains, weighted by her subjective assessments of the occurrence of these gains.

Schmeidler (1989) uses the Anscombe-Aumann setup described in Section 2.2. His axiomatization of Choquet expected utility theory is based on a weaker independence axiom. In contrast to Axiom 2, Schmeidler's comonotonic independence axiom postulates independence only for comonotonic acts. One important property of the Choquet Integral is that it is additive for comonotonic functions. Comonotonicity stands for common monotonicity and is defined as follows:

Definition 2.3 (cf. Schmeidler, 1989, p. 575)**.** Two acts $h, g \in \mathcal{H}$ are said to be *comonotonic* if there are no $\omega, \omega' \in \Omega$ such that $h(\omega) \succ h(\omega')$ and $g(\omega') \succ g(\omega)$.

In the Anscombe-Aumann setting, the first-stage outcomes are roulette lotteries. For this reason, in Definition 2.3, monotonicity refers to preference orderings over lotteries. The definition says that two acts are comonotonic if they induce the same ranking of states in terms of the desirability of their outcomes. In other words, the same state yields the most preferred lottery under both acts, the same state yields the second most preferred lottery, and so on down the line.

Schmeidler (1989) proved that a preference relation $\succsim$ over $\mathcal{H}$ satisfies his axioms if and only if there exists a unique capacity ν on Ω and a utility function u, which is unique up to positive linear transformations, such that for every $h, g \in \mathcal{H}$,

$$h \succsim g \;\Leftrightarrow\; \int_\Omega \left[\sum_{x \in X} h(\omega)(x) u(x) \right] d\nu \geq \int_\Omega \left[\sum_{x \in X} g(\omega)(x) u(x) \right] d\nu. \tag{2.3}$$

This representation differs from the Anscombe-Aumann representation (2.2) solely in that the outer integral is a Choquet integral taken with respect to a capacity ν.

The following example illustrates that the Ellsberg paradox can be resolved by using the Choquet approach.

Example 2.3. *Consider Ellsberg's one-urn experiment and a decision-maker with Choquet expected utility preferences. Denote by ν the capacity which represents the decision-maker's belief. Let $\Omega = \{R, B, Y\}$ be the state space where R,B and Y denote the event that the ball drawn is red, black or yellow.*

If we assume that $u(100) > u(0)$ and take the normalization $u(0) := 0$, the decision-maker in Example 2.3 evaluates the four one-urn bets as follows:

$\int_\Omega u(f)\, d\nu = u(100)\nu(R)$ and $\int_\Omega u(g)\, d\nu = u(100)\nu(B)$,

$\int_\Omega u(f')\, d\nu = u(100)\nu(R \cup Y)$ and $\int_\Omega u(g')\, d\nu = u(100)\nu(B \cup Y)$.

For some capacities, e.g., for the capacity

$$\nu(E) = \begin{cases} 1/3 \text{ if } R \subseteq E, \\ 2/3 \text{ if } B \cup Y \subseteq E \\ 0 \text{ otherwise} \end{cases} \quad \text{for } E \subset \Omega, \text{ and } \nu(\Omega) = 1,$$

the decision-maker prefers f over g and g' over f'. This is the pattern that Ellsberg (1961) frequently observed in his one-urn experiment.

Schmeidler (1989) also introduced a definition of uncertainty aversion, which is one of the most commonly used definitions in the literature. However, there are alternative definitions of uncertainty aversion, e.g., those provided by Epstein (1999) and Ghirardato and Marinacci (2002). According to Schmeidler's definition, a preference relation reveals uncertainty aversion if a mixture over any two acts is weakly preferred to whichever of the two acts that is not strictly preferred over the other.

Axiom 3 (Uncertainty aversion). *A preference relation $\succsim$ on $\mathcal{H}$ reveals uncertainty aversion if for all $h, g \in \mathcal{H}$ and any $\alpha \in [0,1]$, $h \succsim g$ implies $\alpha h + (1-\alpha)g \succsim g$.*

This definition is in line with the argument of Raiffa (1961) in that it requires a preference for mixtures. Preferences that satisfy Axiom 3 are represented by a quasiconcave function.[1] In the context of the Choquet expected utility functional, Schmeidler (1989) shows that quasiconcavity is equivalent to a convex capacity:

Definition 2.4. A capacity ν is said to be *convex*, if for all events $E, F \in \mathcal{A}$, it holds that $\nu(E) + \nu(F) \leq \nu(E \cup F) + \nu(E \cap F)$.

A decision-maker whose beliefs are represented by a proper convex capacity puts more weight on bad outcomes than an expected utility maximizer would. If the inequality in Definition 2.4 is reversed, the capacity ν is called *concave*. When integrating with respect to a concave capacity, more weight is placed on good outcomes. If a capacity is both concave and convex, then it is additive, that is, a probability measure.

[1] A function $V : M \to \mathbb{R}$ is called *quasiconcave* on a convex subset M of a real vector space if for all $x, y \in M$ and all $\alpha \in [0,1]$, it holds that $V(\alpha x + (1-\alpha)y) \geq \min\{V(x), V(y)\}$.

The Choquet expected utility model is a very general approach. For instance, Wakker (1990) showed that the anticipated utility model by Quiggin (1982) and Yaari (1987) can be considered as a special case of the Choquet approach.

Multiple priors

Another prominent model of decision-making under ambiguity is the multiple prior model introduced by Gilboa and Schmeidler (1989). The key idea of this approach is that, in case of ambiguous uncertainty, an individual has too little information to form a unique prior probability distribution. Therefore, this model assumes that the individual considers a set of priors as possible.

The maxmin expected utility model of Gilboa and Schmeidler (1989) is closely related to the criterion suggested by Wald (1950) (cf. Section 2.1). In the words of Gilboa and Schmeidler (1989, p. 143): "Hence our main result can be considered as an axiomatic foundation of Wald's criterion."

Gilboa and Schmeidler (1989) also use the Anscombe-Aumann framework. The crucial axioms in their setting are uncertainty aversion and certainty-independence. The uncertainty aversion axiom is similar to Axiom 3. Certainty-independence is weaker than Axiom 2 in that it only requires independence with respect to constant acts, i.e., acts that yield the same roulette lottery in every state of the world.

The beliefs of a decision-maker whose preferences satisfy the axioms of Gilboa and Schmeidler (1989) can be represented by a closed and convex set of probability measures on the states of the world, C. Furthermore, the decision-maker evaluates an act by the minimal expected utility over all priors in her prior set. Consequently, she maximizes her minimal expected utility when choosing among acts, formally, for every $h, g \in \mathcal{H}$,

$$h \succsim g \;\Leftrightarrow\; \min_{\pi \in C} \int_{\Omega} \left[\sum_{x \in X} h(\omega)(x) u(x) \right] d\pi(\omega) \geq \min_{\pi \in C} \int_{\Omega} \left[\sum_{x \in X} g(\omega)(x) u(x) \right] d\pi(\omega). \quad (2.4)$$

It can be easily shown that ambiguity-averse behavior in Ellsberg's experiments can be explained by the maxmin expected utility model. For instance, consider Example 2.3 and suppose that the decision-maker is a maxmin expected utility maximizer and her beliefs are represented by the prior set

$$C = \{\pi \mid \pi \text{ is a probability distribution over } \{R, B, Y\} \text{ and } \pi(R) = 1/3\}.$$

Ghirardato et al. (2004) provide an axiomatization of the so-called $\alpha-$maxmin expected utility model. This model is akin to the decision rule proposed by Arrow and Hurwicz (1972). An $\alpha-$maxmin expected utility decision-maker considers not only the minimal expected utility over all priors in her prior set as in the representation (2.4) but also the maximal expected utility. She evaluates an act by a convex combination of minimal and maximal expected utility, where the parameter of the convex combination, $\alpha \in [0,1]$, can be interpreted as a measure of the decision-maker's ambiguity attitude. However, as shown by Eichberger et al. (2011), Ghirardato et al.'s axiomatization has a flaw: if we restrict attention to finite state spaces, $\alpha-$MEU preferences satisfy Ghirardato et al.'s axioms if and only if $\alpha = 0$ or $\alpha = 1$. That is, the preferences are either maxmin or maxmax expected utility preferences.

The maxmin and Choquet expected utility model have a nonempty intersection. As shown by Schmeidler (1986), the models coincide if the prior set, which represents the decision-maker's beliefs, is the core of a convex capacity. The *core* of a capacity is the set of all probability measures that assign a higher probability to each event than the capacity.

Definition 2.5. Let ν be a capacity defined on a nonempty sigma-algebra of subsets $\mathcal{A}$ of a state space Ω. The *core* of the capacity ν is the set

$$\{\pi : \mathcal{A} \to \mathbb{R} \mid \pi \text{ is additive}, \pi(\Omega) = \nu(\Omega), \text{ and } \pi(E) \geq \nu(E) \text{ for all } E \in \mathcal{A}\}.$$

The Choquet expected utility of an act f with respect to a convex capacity ν coincides with maxmin expected utility of f when the prior set C equals the set of probabilities in the core of ν:

$$\int_\Omega u(f)\, d\nu = \min_{\pi \in core(\nu)} \int_\Omega u(f)\, d\pi(\omega).$$

Chapter 3

Game-theoretic foundations

3.1 Historical background

The birth of formal game theory is often attributed to Zermelo (1913).[2] Some textbooks state comparatively general propositions under the heading of Zermelo such as Fudenberg and Tirole (1991, p. 91): "A finite game of perfect information has a pure-strategy Nash equilibrium." Others stick closer to Zermelo's original work, for example, Eichberger (1993, p. 9): "In Chess, either white can force a win, or black can force a win, or both sides can force a draw." In fact, Zermelo presented his analysis in the context of the game of chess. However, in his introductory words, Zermelo remarked that his considerations also apply, in principle, to all two-person games without chance moves.

Subsequently, in the 1920s, several papers on game theory were published. For instance, Kalmár (1928-1929) offered a generalization of Zermelo's work. Particularly important for the further development of game theory were a series of notes by Emile Borel that appeared between 1921 and 1927,[3] and, especially, the paper "Zur Theorie der Gesellschaftsspiele" of von Neumann (1928).

Borel introduced the idea of pure and randomized strategies. Given a game, he wanted to know whether it is possible to find an optimal strategy, that is, a strategy "[...] that gives the player who adopts it a superiority over every player who does not adopt it"

[2]However, applications of game theory to economics were published considerably earlier, e.g., Cournot (1838/1897).

[3]See, for instance, Borel (1921/1953 , 1924/1953 , 1927/1953).

(Borel, 1921/1953, p. 97). However, Borel's investigations were restricted either to symmetric two-person zero-sum games or to specific examples of games.

Von Neumann's paper can be viewed as the starting point of modern game theory. He introduced a general description of the concept of a game based on its "rules" and proved the well-known minimax theorem.[4] By using this theorem, von Neumann showed that, for every finite two-person zero-sum game, there exists a unique numerical value m (the maxmin value) and a mixed strategy for each player (the maxmin strategy) such that, given the other player's maxmin strategy, player 1's highest possible payoff is m and that of player 2 is $-m$. Moreover, player 1's maxmin strategy is a best response to player 2's maxmin strategy and vice versa. In the last part of his paper, von Neumann aimed at determining values analogous to the maxmin value for the players of three-person zero-sum games. In this context, he developed the basic ideas of cooperative game theory. He argued that two of the players can form a coalition against the third one and suggested a solution based on the basic values for the players together with gains and losses, which depend on which of the players actually cooperate with each other.

The ideas of von Neumann (1928) were further developed and extended by von Neumann and Morgenstern (1944). In their pathbreaking treatise, they provided a comprehensive conceptual framework for the theory of games, which includes a general set-theoretical definition of a game. Without such a systematic framework, game theory would be a collection of individual examples and special families of games. The first part of von Neumann and Morgenstern's book (approximately the first 200 pages) is devoted to the conceptual framework and to two-person zero-sum games, while the remaining part is mainly concerned with cooperative game theory. Their book did not provide a solution concept for finite non-cooperative n-player games. This gap was closed by the celebrated equilibrium concept of Nash (1950, 1951). The treatise of von Neumann and Morgenstern and Nash's equilibrium concept are the cornerstones of modern game theory. The following subsection further discusses some of these basic concepts.

[4]The minimax theorem can be expressed as follows (see von Neumann, 1928, p. 307-311). Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be compact and convex sets. If $f : X \times Y \to \mathbb{R}$ is a continuous function that is quasiconcave in x for each fixed $y \in Y$ and quasiconvex in y for each fixed $x \in X$, then $\max_x \min_y f(x,y) = \min_y \max_x f(x,y)$.

3.2 Basic concepts of non-cooperative game theory

The forms that are usually used for representing games are the extensive form and the normal (or strategic) form. An *extensive-form game* is a detailed description of a game that captures its dynamic structure. It specifies, amongst others, the game tree (i.e., who moves when), what the players can do when they move, and what players know when it is their turn to move. A formal definition of this representation is omitted here because it is not directly relevant for the present thesis, and it would require additional notation which would be of little use in the remainder of this work. The reader is referred to textbooks such as Eichberger (1993) and Mas-Colell et al. (1995). A simpler way to describe a game is to use the normal form, which can be viewed as a reduced version of the extensive form.

Definition 3.1. A *normal-form game* is a set $G_N = \left\langle I, \{S_i\}_{i \in I}, X, \gamma, \{v_i\}_{i \in I} \right\rangle$, where

(1) I is a *finite set of players*;

(2) S_i is the *finite set of pure strategies of player i*. Let $S = \bigtimes_{i \in I} S_i$;

(3) X is a *set of outcomes*;

(4) $\gamma : S \to X$ is *an outcome function*, which maps from strategy profiles onto outcomes;

(5) $v_i : X \to \mathbb{R}$ is the *utility function of player i*, which assigns a number to each outcome.

Note that an element of the set X captures the consequences for all players that are induced by a strategy combination. For instance, an outcome of a strategy profile can be a vector of real numbers which represent monetary gains and losses of the players.

Remark 3.1. The set which consists of the elements (1)-(4) of the normal-form description above is called *normal game-form*.

Remark 3.2. Definition 3.1 is often simplified by leaving out the elements (3)-(5) and replacing these by *the payoff function of each player i*, $u_i : S \to \mathbb{R}$, which corresponds to the composition $u_i := v_i \circ \gamma$.

Henceforth, unless noted otherwise, I will use the simplification described in Remark 3.2.

The key elements of any game are players' strategies. Several games can be analyzed by using strategic dominance, which occurs when a player's strategy is "better" than another strategy, no matter what the other players do. More precisely, a player's strategy *strictly dominates* another of her strategies if it leads to a strictly higher payoff, regardless of the strategies played by the opponents of that player. A strategy *weakly dominates* another strategy if it never yields a lower payoff and, for some of the opponents' strategy profiles, it yields a higher payoff. We say that a strategy is *strictly (weakly) dominant* if it strictly (weakly) dominates all other strategies. These notions of dominance also apply to mixed strategies.

The concept of mixed strategies captures the idea that a player may randomize over her pure strategies by using a random device for selecting a pure strategy. Another possible interpretation is that mixed strategies represent a player's uncertainty about the pure strategy choices of her opponents. We will come back to this point in Section 3.3. A mixed strategy is a probability distribution over pure strategies. Formally, a *mixed strategy* of player i is a function $\sigma_i : S_i \rightarrow [0,1]$, which assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$, where $\sum\limits_{s_i \in S_i} \sigma_i(s_i) = 1$. Denote by Σ_i the set of all mixed strategies of player i (i.e., the set of all probability distributions over S_i), and let $\Sigma = \bigtimes\limits_{i \in I} \Sigma_i$. Nash (1950, 1951) assumed that players are expected utility maximizers, and that they have rational expectations, i.e., their beliefs are consistent with the mixed strategies that are actually played.

Assumption 3.1. Player i's payoff, $EU_i(\sigma)$, from a mixed strategy combination $\sigma \in \Sigma$ is the expected value of the payoffs from the corresponding pure strategy profiles:

$$EU_i(\sigma) = \sum_{s \in S} \left(\prod_{j \in I} \sigma_j(s_j) \right) u_i(s).$$

Since, according to Assumption 3.1, any combination of degenerate mixed strategies is payoff equivalent to a pure strategy profile, one can view a pure strategy as the special case of a mixed strategy. Consequently, mixed strategies can be seen as a natural generalization of pure strategies.

We may now turn to Nash's solution concept for non-cooperative games. A Nash equilibrium of a game is a strategy profile in which each player plays a best response to the strategies of the other players. From now on, denote "player i's opponents" by "$-i$".

Definition 3.2. A strategy combination $(\sigma_i^*, \sigma_{-i}^*) \in \Sigma$ is an *equilibrium point* (or, a *Nash equilibrium*) in game G_N if

$$\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i} EU_i\left(\sigma_i, \sigma_{-i}^*\right) \text{ for each player } i.$$

As shown by Nash (1950, 1951), there exists an equilibrium point in every finite normal-form game. Furthermore, for the special case of two-person zero-sum games, Nash's concept coincides with the solution concept suggested by von Neumann (1928). In these games, Nash equilibrium strategies are maxmin strategies, which are defined as follows.

Definition 3.3. A strategy of player i is a *maxmin strategy* if it solves

$$\max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} EU_i\left(\sigma_i, \sigma_{-i}\right).$$

Several refinements of Nash equilibrium have been proposed. A comprehensive survey can be found in van Damme (1983). The remaining part of this subsection describes the concept of subgame perfection since this refinement will be used in Chapter 4. Subgame perfection was introduced by Selten (1965) and captures the requirement of sequential rationality in dynamic games. Firstly, we need to specify what a subgame is. Many extensive-form games contain parts that could be viewed as games in themselves. Such smaller games that are embedded in a larger one and the game as a whole are called subgames.

Definition 3.4. A *subgame* is a subset of a game that meets the following properties:

(i) It starts with a single node, i.e., the initial node is a singleton information set.

(ii) It contains all the nodes that are successors of the initial node and only these nodes.

(iii) If a node in a particular information set is contained in the subgame, then so are all of the nodes that are elements of this information set.

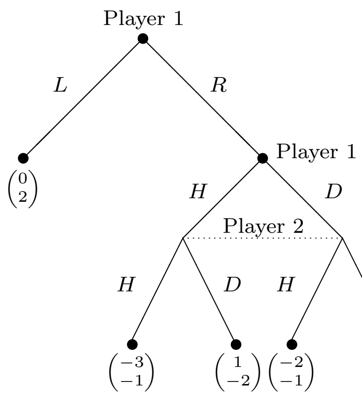
The notion of a subgame shall be briefly illustrated with the help of Example 3.1 below. The game in the example has two subgames: the game itself and a proper subgame that corresponds to the part of the game beginning with player 1's decision between H and D. Furthermore, the game has three pure strategy Nash equilibria:

(Play L and H if R, Play H),

(Play L and D if R, Play H),

(Play R and D if R, Play D).

Example 3.1 (cf. Mas-Colell et al., 1995, p. 274). *Consider the following game.*



However, (Play D, Play D) is the sole Nash equilibrium in the proper subgame when we consider it separately. Therefore, player 2's intention to play H should the game reach the subgame can be considered as a "non-credible threat". Since player 1 can expect that both players will play D in the subgame, she should play R. Hence, only the last of the three Nash equilibria above is sequentially rational. The concept of subgame perfection eliminates non-credible threats by requiring that a solution to a dynamic game must induce a Nash equilibrium in every subgame.

Definition 3.5. A strategy combination $\sigma^* \in \Sigma$ in a game is a *subgame perfect Nash equilibrium (SPNE)* if it induces a Nash equilibrium in every subgame of the game.

3.3 Uncertainty in games

In general, one can distinguish between two sources of uncertainty in games. The first source can be termed *strategic* or *endogenous uncertainty* and refers to a player's uncertainty about the strategy choice of other players. This source is inherent in the strategic situation. The second source, *environmental* or *exogenous uncertainty*, arises from chance moves or from *incomplete information*.[5] The latter refers to a situation in which some or

[5]The distinction between chance moves and incomplete information is fuzzy. For instance, a game with payoff uncertainty can also be viewed as a very special case of an incomplete information game.

all players may lack information about the "rules" of a game or, equivalently, about its normal (or extensive) form. For instance, a player can be uncertain about other players' or her own payoffs, strategy spaces, et cetera. In the model of Harsanyi (1967-68), which will be described in this section, exogenous uncertainty can be associated with nature's move at the beginning of the game. Furthermore, this section describes several approaches that have been suggested to capture more generally strategic uncertainty.

Strategic uncertainty

In Section 3.2, it was pointed out that mixed strategies can be interpreted as a player's uncertainty about the strategy choice of other players. According to this interpretation, Assumption 3.1 can be questioned since it implicitly requires that a player must bear as much uncertainty about her own strategy choice as her opponents do. This section discusses generalizations of Nash equilibrium for complete information games that do not require Assumption 3.1. That is, these equilibrium concepts allow for players with non-expected utility preferences over the lotteries induced by mixed strategy combinations.

Crawford (1990) shows that Nash equilibrium may fail to exist when players preferences cannot be represented by functions that are quasiconcave in the probabilities. To accommodate more general preferences, he introduces an "equilibrium in beliefs", which exists in any finite complete information normal-form game where players have continuous preference functions. For simplicity, we will introduce this concept for two-person games, as Crawford does. In Chapter 5, a n-player version of equilibrium in beliefs will be described and used. Suppose that instead of Assumption 3.1, player i's preferences over mixed strategy profiles are represented by an arbitrary continuous function $V_i : \Sigma_1 \times \Sigma_2 \to \mathbb{R}$. For any set $T \subseteq \Sigma_{-i}$ of mixed strategies, $\text{conv}[T] \subseteq \Sigma_{-i}$ denotes the convex hull of T. An element $\beta_i \in \text{conv}[T]$ can be viewed as a representation of player i's beliefs about the other player's strategy choice from T since any expectation taken with respect to a second-order probability distribution over T will lie in its convex hull. Player i's best responses to a belief β_i about her opponent's strategy choice are

$$R_i(\beta_i) = \{\sigma_i \mid \sigma_i \in \underset{\sigma_i \in \Sigma_i}{\arg\max}\ V_i\left(\sigma_i, \beta_i\right)\}.$$

An equilibrium in beliefs is a belief system in which player 1's belief β_1 about player 2's mixed strategy lies in the convex hull of 2's best replies, given her belief β_2 about player 1's mixed strategy, and vice versa.

Definition 3.6 (cf. Crawford, 1990, p. 139)**.** An *equilibrium in beliefs* in a two-person game G_N is a belief system (β_1^*, β_2^*) such that

$$\beta_2^* \in \text{conv}[R_1(\beta_1^*)] \text{ and } \beta_1^* \in \text{conv}[R_2(\beta_2^*)].$$

The notion of equilibrium in beliefs coincides with Nash equilibrium if players' preferences are quasiconcave. In this case, the concept can be considered as a formal foundation for the interpretation of mixed strategies as players' beliefs: "[...] what I have called equilibrium in beliefs is therefore often viewed simply as an alternative interpretation of Nash equilibrium, with the equilibrating variables viewed not as players' strategy choices, but as their beliefs" (Crawford, 1990, p. 140).

The equilibrium in beliefs approach maintains the assumption that players have probabilistic beliefs. In the literature, there are several equilibrium concepts that allow for players with non-probabilistic beliefs. These papers on strategic ambiguity can be roughly divided into two groups. The first group consists of Klibanoff (1996), Lo (1996), and Lehrer (2012), who assume that players explicitly randomize. They provide equilibrium concepts with weaker requirements regarding the consistency between beliefs and strategies than Nash equilibrium. In contrast, the approach of the second group, which includes Dow and Werlang (1994), Eichberger and Kelsey (2000, 2014), and Marinacci (2000), is based on the interpretation of a mixed strategy as a player's belief about the pure strategy choices of his opponents. The equilibrium definitions of these papers require consistency conditions between the beliefs that players hold. In Chapter 6, the concept of Eichberger and Kelsey (2014) will be applied and described in some detail.

The approach of the first group has the drawback that, typically, players' beliefs will not coincide with the strategies that are actually played. A criticism of the approach of the second group is that it has limited abilities to predict behavior since it usually does not specify the strategies that are played.

Incomplete information

In analyzing a game with incomplete information, we need to deal with infinite hierarchies of beliefs. For example, consider a game in which player i does not know some parameter of the game. In such a game, player i's strategy choice will depend on her beliefs about this parameter, her beliefs about the beliefs of the other players about the parameter, her beliefs about the other players' beliefs about her own beliefs about the parameter, and so on ad infinitum. A game model that explicitly captures these processes would be very complicated and difficult to analyze.

To overcome this difficulty, Harsanyi (1967-68) suggested another approach in which each player's hierarchy of beliefs is summarized in a single entity, called the player's type. At the beginning of a game, a "nature move" determines each player's type. Players' uncertainty about the rules of the game is fully represented by the uncertainty about the types. Harsanyi assumed that players are Bayesian expected utility maximizers and that they have a common prior distribution over the type space.[6] He showed that, under this assumption, his approach can be used to transform incomplete information games into game-theoretically equivalent games with complete, but imperfect, information, commonly known as *Bayesian games*.[7] Formally, a Bayesian game is defined as follows.

Definition 3.7. A *Bayesian game* is an ordered set $G_B = \langle I, \{A_i, \Theta_i, u_i, \pi_i\}_{i\in I}\rangle$, where

(1) I is a *finite set of players.*

(2) A_i is the *finite set of actions of player i.* Let $A = \bigtimes_{i\in I} A_i$.

(3) Θ_i is the *finite set of potential types of player i.* Let $\Theta = \bigtimes_{i\in I} \Theta_i$.

(4) $u_i : A \times \Theta \to \mathbb{R}$ is *player i's payoff function.*

(5) π_i is *player i's belief about the other players' types.*

In a Bayesian game, a strategy $\sigma_i \in \Sigma_i$ of player i prescribes a mixed action for each possible type of player i, formally, a strategy is a mapping $\sigma_i : \Theta_i \to \Delta(A_i)$, where $\Delta(A_i)$ denotes the set of all probability measures over A_i.

[6]Decision-makers are said to be *Bayesian expected utility maximizers* if they have expected utility preferences and update their beliefs according to Bayes' rule in light of new information.

[7]A game has *imperfect information* if all or some players, when making any decision, are not perfectly informed about other players' and/or own previous moves and/or about previous chance moves.

A Bayesian game can be solved by using either players' interim or their ex-ante expected utility. These notions of expected utility can be thought of as issues of timing.

At the "interim stage", each player has learned her type but not the types of the other players. The *interim expected utility* of player i with type θ_i of a strategy profile σ is

$$EU_i(\sigma \mid \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \pi(\theta_{-i}|\theta_i) \left(\sum_{a \in A} \left(\prod_{j \in I} \sigma_j(a_j \mid \theta_j) \right) u_i(a, \theta_i, \theta_{-i}) \right), \tag{3.1}$$

where $\pi(\theta_{-i}|\theta_i)$ denotes the probability of θ_{-i} under the condition that i knows she is of type θ_i, and $\sigma_j(a_j \mid \theta_j)$ is the probability of action a_j that strategy σ_j prescribes for θ_j.

At the "ex-ante stage", players know nothing about anyone's actual type. The *ex-ante expected utility* of player i from a mixed strategy profile σ is

$$EU_i(\sigma) = \sum_{\theta_i \in \Theta_i} \pi(\theta_i) \left(EU_i(\sigma \mid \theta_i) \right). \tag{3.2}$$

A solution to a Bayesian game can be defined by using either (3.1) or (3.2).

Definition 3.8. A strategy combination $\sigma^* \in \Sigma$ is

(i) an *interim Bayesian Nash equilibrium* in game G_B if

$$\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i} EU_i\left(\sigma_i, \sigma_{-i}^* \mid \theta_i\right) \text{ for each player } i \text{ and all } \theta_i \in \Theta_i,$$

(ii) an *ex-ante Bayesian Nash equilibrium* in game G_B if

$$\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i} EU_i\left(\sigma_i, \sigma_{-i}^*\right) \text{ for each player } i.$$

The two equilibrium notions in Definition 3.8 are equivalent if $\pi(\theta_i) > 0$ for all $\theta_i \in \Theta_i$ (see Harsanyi, 1967-68, p. 181, 321). In Harsanyi's framework, this equivalence results from the fact that Bayesian expected utility maximizers show dynamically consistent behavior. In models with non-expected utility players, interim equilibrium concepts may differ from ex-ante equilibrium concepts. In games without private information, players do not know their types. Consequently, interim approaches cannot be used to analyze these games. In Chapter 4 and 5, an ex-ante equilibrium concept will be used since we mainly consider games with payoff uncertainty but without private information. In Chapter 6, an interim concept will be used in the context of the quasi Bayesian model.

Chapter 4

Spatial competition under uncertainty

This chapter, based on Kauffeldt and Wiesenfarth (2014), aims at analyzing the impact of demand uncertainty on firms' product design decisions under duopolistic competition. A well-known and widely studied model of product differentiation is the location-then-price duopoly game by Hotelling (1929).[8] While there is a vast literature on extensions of Hotelling's model, there are almost no models that incorporate demand ambiguity (see Section 4.1.1). To our knowledge, the only exception is Król (2012). In his model, firms face *complete ignorance*, i.e., they do not have any probabilistic information.

In this chapter, we develop a general Hotelling model that incorporates partial ambiguous uncertainty about the market demand which can also be interpreted as the degree of firms' confidence in their prior beliefs. Our model is based on the Choquet expected utility approach of Schmeidler (1989) described in Section 2.4. More specifically, the model assumes that firms' beliefs are represented by so-called neo-additive capacities introduced by Chateauneuf et al. (2007).[9] This type of capacity is particularly useful in the context of firms since it explicitly incorporates the classical decision criteria as special cases, i.e., it incorporates the expected utility criterion and the Wald and Arrow/Hurwicz criterion

[8]The "location" in Hotelling's game is typically interpreted as a position in a geographical or product type space. In this chapter, we focus on the latter interpretation.

[9]Neo stands for "non-extreme-outcome".

addressed in Section 2.1.[10]

This study makes a valuable contribution to the literature by providing additional analytical tools for understanding product differentiation under demand uncertainty.[11] In our view, the capacity model offers more plausible explanations for some real-life phenomena. Furthermore, it provides a unifying framework for the model of Król (2012) and the probabilistic model of Meagher and Zauner (2004). In fact, the models of Meagher and Zauner[12] (or MZ, for short) and Król are special cases of the capacity model. We believe that our model adds to these models by filling their gaps. This will be explained in more detail in Section 4.3. In particular, Król's analysis is based on variations of the size of the support of the uncertainty. However, according to the commonly used updating rules, new information will decrease the size of the support but it will not increase the support.

This chapter is organized as follows. In the next section, we review the related literature and describe our model in detail. Section 4.2 provides our results. First, we derive firms' pure strategy subgame-perfect product design choices for the Hotelling game with ambiguity. Then, we carry out a comparative static analysis with respect to all model parameters and discuss their implications for equilibrium product characteristics and Choquet expected profits. Section 4.3 is concerned with implications for possible applications of the Hotelling model under demand location uncertainty. Finally, Section 4.4 concludes with a summary of the main results and a discussion of our findings. Unless noted otherwise, the proofs of the propositions are given in Section 4.5.

4.1 Preliminaries

4.1.1 Related literature

Hotelling's original model consists of two firms and uniformly distributed consumers along a compact interval facing linear transportation costs. At the first stage of the game,

[10] This aspect will be further elaborated in Section 4.1.3 and Remark 4.1 and 4.4.

[11] Besides firms' ambiguity attitude, we distinguish four different sources of ambiguous uncertainty and determine their influence on firms' product design choices: the variance of firms' prior beliefs, the degree of ambiguity, the size of the support of the uncertainty and the magnitude of the parameter of consumers' quadratic cost functions.

[12] With a technical restriction. For more details, see Section 4.2, especially Remark 4.1 and 4.3.

firms choose simultaneously their locations on this interval. At the second stage, firms face price competition. Several papers study extensions or variants of Hotelling 's model, see, e.g., Gabszewicz and Thisse (1992) for a survey. In an early paper, D'Asprémont et al. (1979) show that a subgame-perfect Nash equilibrium is not guaranteed under linear cost functions. As a resort to this complication, D'Asprémont et al. replaced Hotelling's original assumption of linear transportation costs by quadratic ones. In the literature, Hotelling models with quadratic cost functions are frequently referred to as "AGT-models", where AGT stands for "D'Asprémont, Gabszewicz and Thisse".

There are several papers that examine Hotelling models with demand uncertainty. For instance, Balvers and Szerb (1996) consider a Hotelling model that incorporates random shocks on the quality of each firm's product under the assumption that there is no price competition. Harter (1996) studies a model with demand location uncertainty where firms enter the market sequentially. Similar to Harter, Casado-Izaga (2000),[13] and MZ (2004, 2005) discuss extensions of Hotelling's model in which demand uncertainty is introduced by enabling the midpoint of the consumer interval to be probabilistic. MZ (2005) generalize Casado-Izaga (2000) by parametrizing the length of the support. They find that equilibrium differentiation increases in the size of the support. MZ (2004) restrict this support to compact subsets of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ but allow for a broad class of density functions. Again, MZ show that uncertainty constitutes a differentiation force.

All the contributions above assume that firms' beliefs are represented by a unique and common prior. As mentioned above, an exception is the model of Król (2012). He introduces complete ignorance into the framework of MZ and examines, amongst others, the influence of firms' ambiguity attitudes on their decisions given firms use the Arrow/Hurwicz decision criterion. Król finds that uncertainty can be an agglomeration force if firms are sufficiently pessimistic.

[13] Casado-Izaga (2000) assume that consumers are uniformly distributed on the interval $[\Theta, \Theta + 1]$ where Θ is drawn from a uniform distribution $[0, 1]$. Consequently, the midpoint of the consumer interval follows implicitly a uniform distribution on $[\frac{1}{2}, \frac{3}{2}]$.

4.1.2 The basic model

Our framework is inspired by the modified AGT-model of MZ (2004). There are two firms, $i = 1, 2$, interacting in a two-stage Hotelling duopoly game. Each firm produces a homogeneous commodity at constant marginal production costs which are normalized to zero. At the first stage of the game, firms select simultaneously their product characteristics x_i from the real line under the assumption that $x_1 \leq x_2$. At the second stage, firms face price competition setting prices $p_i \in \mathbb{R}_+$ simultaneously as well.
Furthermore, there is a unit mass of consumers, each consumer being uniquely characterized by a specific taste, $s \in \mathbb{R}$, representing her ideal commodity. Consumer tastes are assumed to be uniformly distributed on an interval of the form $[M - \frac{1}{2}, M + \frac{1}{2}]$ where $M \in \mathbb{R}$. A customer whose taste is located at s and consumes product i, faces a disutility from not consuming her ideal product. Consumers' utility losses depend on the squared distance between s and the selected product design x_i, formally $t(s-x_i)^2$ where $t \in \mathbb{R}_{++}$.[14] In addition, customers need to pay the price p_i of product i. As a consequence, total costs are given by $p_i + t(s - x_i)^2$. Moreover, we assume that customers purchase one unit of the homogeneous good from the firm that brings about the lowest total costs. Implicitly, this guarantees that consumers' outside option is non-binding. In other words, there is no reservation price.

In the certainty model M and t are fixed and exogenously given parameters known to both firms throughout the game. In the risk model of MZ (2004), M is unknown to both firms, whereas the scaling parameter t is normalized to 1. In the model of Król (2012), firms face ambiguity with respect to (t, M) resolving ambiguity with the Arrow/Hurwicz α-maxmin criterion. Similar to these models, we presume that the realization $(\hat{t}, \hat{M})$ of (t, M) is revealed to both firms before the price competition.

Assumption 4.1. Uncertainty is resolved at the second stage of the game before the price competition.

The rationale behind this assumption lies in the fact that most firms are able to adjust prices more easily than product designs (see MZ, 2004). For instance, if actual sales

[14] The parameter t allows for an up- or downward distortion of this quadratic disutility.

volumes differ from estimated sales volumes, firm managers usually are in the position to readjust retail prices accordingly.

In addition, we assume that firms dispose of some probabilistic information condensed in a common prior π. We refer to π as "reference probability distribution" or "reference prior". Henceforth, let E_π denote the expectation taken with respect to π. Similar to the risk case, we need several assumptions concerning the reference probability π which are summarized in Assumption 4.2.

Assumption 4.2. The reference prior π on (t, M) satisfies the subsequent requirements:

(R1) The variance of M exist: $E_\pi[M^2] < \infty$.

(R2) The expectation of M is normalized to zero: $E_\pi[M] = 0$.

(R3) The distribution of M has no atoms.

(R4) The support of M is given by the symmetric interval $[-L, L] \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$.

(R5) The support of t is given by the interval $[\underline{t}, \overline{t}]$ where $\underline{t} \in (0, 1]$ and $\overline{t} \geq 1$.

(R6) The expectation of t is normalized to 1: $E_\pi[t] = 1$.

(R7) The random variables t and M are uncorrelated.

At the first stage of the game, the random variable M enters quadratically into each firm's objective function.[15] This observation provides a justification why firms' product design choices solely depend on the first and second moment of M. On these grounds, Assumption (R1) guarantees the existence of best-response functions. Moreover, taking (R1) and (R4) together, we can formulate the following lemma which proves to be very useful for the mathematical considerations in the comparative statics section.

Lemma 4.1. *The Requirements* (R_1) *and* (R_4) *imply*

$$E_\pi[M] \in [-L, L] \text{ and } \mathrm{Var}_\pi(M) \in [0, L^2].$$

[15]See equation (4.1) and Lemma 4.5.

The Requirements (R2) and (R6) are introduced for reasons of symmetry and tractability. Requirement (R3) is purely technical in nature and can be replaced in order to allow for discrete distributions or mixtures of continuous and discrete distributions. (R4) makes sure that the support of M is a compact subset of the interval $[-\frac{1}{2}, \frac{1}{2}]$ restricting the size of uncertainty to be relatively small. Furthermore, it assures that the extreme intervals for possible realizations of the consumer distribution $[-L-\frac{1}{2}, -L+\frac{1}{2}]$ and $[L-\frac{1}{2}, L+\frac{1}{2}]$ always have a non-empty intersection. Without this assumption, we would have to consider three cases:

(1) The firm located left becomes a monopolist.

(2) Both firms share the market.

(3) The firm located right becomes a monopolist.

Cases (1) and (3) only occur if the size of uncertainty is large enough.[16] In this study, we intend to restrict the analysis to the duopoly case (2). Therefore, we restrict the size of uncertainty so that only the second case applies. Furthermore, (R4) and (R5) imply that the support of π is a subset of $[\underline{t}, \overline{t}] \times [-L, L]$. Lastly, (R7) ensures that we can disentangle the effects of t and M.

4.1.3 Demand ambiguity

We introduce ambiguity by assuming that firms' beliefs are represented by non-additive probabilities or capacities. Our analysis relies on a distinct class of capacities, called neo-additive capacities, axiomatized by Chateauneuf et al. (2007).

Definition 4.1 (cf. Eichberger et al., 2009, p. 359)**.** Let π be a probability distribution on $\Omega = [\underline{t}, \overline{t}] \times [-L, L]$ satisfying Assumption 4.2 and let $\mathcal{A}$ be an algebra of events defined on Ω. Then, for $\alpha, \delta \in [0, 1]$, a neo-additive capacity ν is defined by $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$, and $\nu(E) = \delta\alpha + (1-\delta)\pi(E)$ where $E \in \mathcal{A}$ is a nonempty and strict subset of Ω.

[16] See MZ (2005) for a detailed investigation of these cases for the risk model.

From our point of view, neo-additive capacities display several nice features. The parameter δ can be interpreted as a measure of ambiguity or of firms' confidence in the common reference prior π. Thus, one can contemplate our model as a setting where firms exhibit uncertainty with respect to their prior beliefs due to imprecise or unreliable information. Moreover, the parameter α describes firms' attitude towards ambiguity. The higher α, the more pessimistic firm managers are. As a result, neo-additive capacities allow for a clear separation between the degree of ambiguity and firms' ambiguity attitude which is, as we want to argue in this chapter, essential for many economic applications. Consequently, we assume that the neo-additive capacity represents firms' ex-ante uncertainty.

Assumption 4.3. Each firm's belief on (t, M) is represented by a neo-additive capacity.

The rationale speaking for the introduction of neo-additive capacities lies in the fact that firms might not completely trust the information available at the time of making their product choice. There are multiple reasons why this might be the case, e.g., firms introducing newly innovated products into the market might dispose of data on similar products that are already established in the market but have no data on the new good. It seems plausible that firms use this data to predict the market outcome, still firms cannot account for short-term trends in consumer tastes. Furthermore, data reliability is closely tied to the comparability of the reference product with the newly innovated product. The more heterogeneous both products are, the less plausible it seems to rely on available data on the reference product. Neo-additive capacities allow for a model of partial information in which firms have a certain stock of data available whose reliability might be questionable up to a certain degree. Interpreted in this way, the model by Król (2012) represents a situation where firms have ex-ante no information about the distribution of consumer tastes or completely distrust the information available at the time of making their product design choices. Neo-additive capacities allow for an additional interpretative component with respect to a multitude of possible real-world applications of Hotelling models under uncertainty by adding an additional explanatory source for increasing or decreasing product differentiation under ambiguity.

4.2 Results

4.2.1 Subgame-perfect Nash equilibrium with ambiguity

In this section, we determine equilibrium product differentiation under ambiguity by backward induction. In a first step, we solve the price subgame at the second stage where the midpoint M of the consumer distribution and the cost parameter t are fixed and known to both firms.

Price subgame

According to Assumption 4.1, the realization $(\hat{t}, \hat{M})$ is known to both firms at the second stage. Equilibrium prices are zero if firms do not differentiate their products. Otherwise, firms' equilibrium prices depend on the distance between firms' averaged product design $\bar{x} := \frac{x_1+x_2}{2}$ and the realized midpoint $\hat{M}$. There is an interior equilibrium where each firm charges a positive price:

Lemma 4.2 (Interior price equilibrium)**.** *If $x_1 \leq x_2$ and $(\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}]$, firms charge the subsequent equilibrium prices:*

$$p_1^* = \frac{2}{3}\hat{t}\Delta_x\left(\bar{x} - \hat{M} + \frac{3}{2}\right) \quad \textit{and} \quad p_2^* = \frac{2}{3}\hat{t}\Delta_x\left(-\bar{x} + \hat{M} + \frac{3}{2}\right)$$

Proof. See Anderson et al. (1997, p. 107) and Meagher and Zauner (2004, p. 203).

Apart from the interior equilibrium, there are two more boundary equilibria where one of the two firms becomes a monopolist:

Lemma 4.3 (Boundary price equilibria)**.** *If $x_1 \leq x_2$ and $(\hat{M} - \bar{x}) \notin [-\frac{3}{2}, \frac{3}{2}]$, firms charge the subsequent equilibrium prices:*

$$p_1^* = 2\hat{t}\Delta_x\left(\bar{x} - \hat{M} - \tfrac{1}{2}\right) \quad \textit{and} \quad p_2^* = 0 \quad \textit{if} \quad (\hat{M} - \bar{x}) < -\tfrac{3}{2}$$

or

$$p_2^* = 2\hat{t}\Delta_x\left(\hat{M} - \bar{x} - \tfrac{1}{2}\right) \quad \textit{and} \quad p_1^* = 0 \quad \textit{if} \quad (\hat{M} - \bar{x}) > \tfrac{3}{2}.$$

Proof. See Anderson et al. (1997, p. 107) and Meagher and Zauner (2004, p. 203).

Product design competition

As shown in the previous subsection, one obtains for a fixed pair (x_1, x_2) of product characteristics a unique equilibrium for the price subgame. By making use of the equilibrium prices from Lemma 4.2 and 4.3, we obtain firms' second stage profits for the realization $(\hat{t}, \hat{M})$ depending on firms' product characteristics:

$$\Pi_i(x_i, x_j, \hat{t}, \hat{M}) = \begin{cases} \hat{t}\,\Delta_x \left[1 + 2\,(-1)^i(\hat{M} - \bar{x})\right] & \text{for} \quad (-1)^i\,(\hat{M} - \bar{x}) > \frac{3}{2} \\ \hat{t}\,\Delta_x \left[3(-1)^i + 2(\hat{M} - \bar{x})\right]^2 /18 & \text{for} \quad (\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}] \\ 0 & \text{otherwise} \end{cases}, \tag{4.1}$$

where $\bar{x} := \frac{x_1+x_2}{2}$, $\Delta_x := x_2 - x_1$ and $j := 3 - i$.

In the following, we elaborate on each firm's objective function at the first stage of the game. In order to do so, we rely on the fact that the second piece of (4.1) is monotonic in $(\hat{t}, \hat{M})$ as specified in Lemma 4.4 below.

Lemma 4.4. *If the condition $(\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}]$ is met, firm i's profit function $\Pi_i(x_i, x_j, \hat{t}, \hat{M})$ is strictly increasing in $\hat{t}$, strictly decreasing in $\hat{M}$ for firm 1, and strictly increasing for firm 2, provided that $x_1 < x_2$.*

At the first stage of the game, the distribution of (t, M) is unknown. In accordance with Assumption 4.3 and Definition 4.1, firms consider the Choquet expected value of their first stage profits, which we denote by $\mathrm{CEU}_i[x_i, x_j, t, M]$ for firm $i \in \{1, 2\}$. Note that the Choquet-expectation is taken with respect to a neo-additive capacity ν. Following Lemma 3.1 in Chateauneuf et al. (2007), we obtain the representation (4.2) of firm i's Choquet expected profit at the first stage of the game.

$$\begin{aligned} \mathrm{CEU}_i[x_i, x_j, t, M] = \int \Pi_i(x_i, x_j, t, M)d\nu &= (1-\delta)\,\mathrm{E}_\pi[\Pi_i(x_i, x_j, t, M)] \\ &+ \delta[(1-\alpha)\max\{\Pi_i(x_i, x_j, \hat{t}, \hat{M}) \mid (\hat{t}, \hat{M}) \in \mathrm{supp}(t, M)\}\cdot \\ &+ \alpha \min\{\Pi_i(x_i, x_j, \hat{t}, \hat{M}) \mid (\hat{t}, \hat{M}) \in \mathrm{supp}(t, M)\}] \end{aligned} \tag{4.2}$$

Remark 4.1. These Choquet expected profits allow for a nice interpretation, namely that they generalize Hotelling models treated in the literature before. For $\delta = 0$ and a constant scaling factor $t = 1$, we obtain the model of MZ (2004) with a normalized mean of M. In case of $\delta = 1$ and $\bar{t} = 1$, the framework boils down to the model of Król (2012). Thus, we can consider these specifications as extreme cases of the capacity model.

At first, consider the second part of equation (4.2). Making use of Lemma 4.4, we obtain for $(\hat{M} - \bar{x}) \in [-\frac{3}{2}, \frac{3}{2}]$ the following explicit functional relationships:

$$\begin{aligned} \max\{\Pi_1(x_i, x_j, \hat{t}, \hat{M}) \mid (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\} &= \Pi_1(x_1, x_2, \bar{t}, -L) \\ \min\{\Pi_1(x_i, x_j, \hat{t}, \hat{M}) \mid (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\}] &= \Pi_1(x_1, x_2, \underline{t}, L) \\ \max\{\Pi_2(x_i, x_j, \hat{t}, \hat{M}) \mid (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\} &= \Pi_2(x_1, x_2, \bar{t}, L) \\ \min\{\Pi_2(x_i, x_j, \hat{t}, \hat{M}) \mid (\hat{t}, \hat{M}) \in \operatorname{supp}(t, M)\}] &= \Pi_2(x_1, x_2, \underline{t}, -L) \end{aligned} \quad . \tag{4.3}$$

Remark 4.2. One can interpret these results as follows. Firm 1's best-case scenario occurs when the realized midpoint $\hat{M}$ of the consumer interval equals the lower support boundary $-L$. This is true since we assume, w.l.o.g. (without loss of generality), that firm 1 is the firm whose product characteristic is located left of firm 2's product characteristic. Therefore, it is more convenient for firm 1 if the consumer distribution is located closer to its own product design. Similarly, firm 1's worst-case scenario occurs when the midpoint of the consumer interval takes as realization the upper support boundary L. For firm 2 the opposite is true.

The first term of firm i's Choquet expected profit equals the "usual" expectation of its profit function with respect to the reference prior $\mathrm{E}_\pi[\Pi_i(x_1, x_1, t, M)]$. In order to elaborate on this part, we need the following Lemma which can be considered as an analogue to the global competition lemma in Meagher and Zauner (2004).[17]

Lemma 4.5 (Global competition)**.** *Under Assumptions 1,2, and 3, one has at any pure strategy SPNE for the Hotelling game with ambiguous demand location uncertainty that the support* $[-L, L]$ *of* M *is contained in* $[\bar{x} - \frac{3}{2}, \bar{x} + \frac{3}{2}]$*, formally* $[-L, L] \subset \left[\bar{x} - \frac{3}{2}, \bar{x} + \frac{3}{2}\right]$.

[17] For the Hotelling model under certainty, Anderson et al. (1997) point out a similar property in footnote 8.

Lemma 4.5 proves very useful when it comes to determining firms' subgame-perfect product design choices. In fact, due to Lemma 4.3, one could expect that there are equilibria where, for some realizations of uncertainty, one or the other firm can monopolize the market. However, according to Lemma 4.5, firm i's objective function at the first stage of the game is given by the Choquet expected value of the second piece of (4.1). The global competition lemma implies that $\mathrm{E}_\pi[\Pi_i(x_i, x_j, t, M)]$ depends solely on the the mean vector $\mathrm{E}_\pi[(t, M)] = (\mu_t, \mu_M)$ and the variance-covariance matrix

$$\mathrm{Cov}_\pi(t, M) = \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_M^2 \end{pmatrix}.$$

The following lemma provides an explicit mathematical form for $\mathrm{E}_\pi[\Pi_i(x_i, x_j, t, M)]$.

Lemma 4.6. *If $x_1 \leq x_2$ w.l.o.g., then, under Assumptions 1,2, and 3, at any pure strategy SPNE for the Hotelling game under uncertainty, firms choose product characteristics, (x_1^*, x_2^*), such that firm i's expected profit w.r.t. the reference prior π is*

$$\begin{aligned} \mathrm{E}_\pi[\Pi_i(x_i^*, x_j^*, t, M)] &= \mu_t \int_{-L}^{L} (-1)^j \, \frac{2}{9} \, (x_j^* - x_i^*) \, \left(\bar{x}^* - \left(M + \frac{3}{2}(-1)^i\right)\right)^2 f_\pi(M) dM \\ &= \frac{(-1)^j}{18} \, \mu_t \, (x_j^* - x_i^*) \Big\{ (2\bar{x}^* - 3(-1)^i)^2 \\ &\quad - 4\mu_M(2\bar{x}^* - 3(-1)^i) + 4(\mu_M + \sigma_M^2) \Big\}, \end{aligned} \tag{4.4}$$

where $\bar{x}^* = x_i^* + x_j^*$.

Next, after specifying firms' first-stage objective functions, we derive subgame-perfect product designs. Firm i's best reply, $R_i^*(\hat{x}_j)$, given the product characteristic choice of firm j, $\hat{x}_j$, is

$$R_i^*(\hat{x}_j) := \underset{x_i \in \mathbb{R}}{\arg\max} \Big\{ (1-\delta) \, \mathrm{E}_\pi[\Pi_i(x_i, \hat{x}_j, t, M)] + \delta \big[(1-\alpha)\Pi_i(x_i, \hat{x}_j, \bar{t}, -L) + \alpha \Pi_i(x_i, \hat{x}_j, \underline{t}, L) \big] \Big\}.$$

Solving for firms' mutual best replies, one obtain firms' subgame-perfect equilibrium differentiation as stated in the following proposition.

Proposition 4.1. *Under Assumptions 1,2, and 3, there is a unique pure strategy SPNE for the Hotelling game under ambiguity. Firms' equilibrium locations are given by*

$$x_1^* = \frac{\delta\left(-(\alpha-1)(2L+3)^2\overline{t}+\alpha(3-2L)^2\underline{t}-4\sigma^2-9\right)+4\sigma^2+9}{4(\delta((\alpha-1)(2L+3)\overline{t}+\alpha(2L-3)\underline{t}+3)-3)}$$

$$x_2^* = \frac{\delta\left((\alpha-1)(2L+3)^2\overline{t}-\alpha(3-2L)^2\underline{t}+4\sigma^2+9\right)-4\sigma^2-9}{4(\delta((\alpha-1)(2L+3)\overline{t}+\alpha(2L-3)\underline{t}+3)-3)}.$$

The equilibrium differentiation, $\Delta_x^* := x_2^* - x_1^*$, *is*

$$\Delta_x^* = \frac{\delta\left((\alpha-1)(2L+3)^2\overline{t}-\alpha(3-2L)^2\underline{t}+4\sigma^2+9\right)-4\sigma^2-9}{2(\delta((\alpha-1)(2L+3)\overline{t}+\alpha(2L-3)\underline{t}+3)-3)},$$

and firms' Choquet expected equilibrium profits are given by

$$\mathrm{CEU}_i[x_1^*, x_2^*, t, M] = -\frac{\left(\delta\left(-(\alpha-1)(2L+3)^2\overline{t}+\alpha(3-2L)^2\underline{t}-4\sigma^2-9\right)+4\sigma^2+9\right)^2}{36(\delta((\alpha-1)(2L+3)\overline{t}+\alpha(2L-3)\underline{t}+3)-3)}.$$

Remark 4.3. It is worthwhile to highlight and discuss some special cases of this equilibrium. Setting $\delta = 1$ and $\overline{t} = 1$, which corresponds to a situation under complete ignorance, one obtains the equilibrium of Król (2012) in its full generality. Setting $\delta = 0$ and $\underline{t} = \overline{t} = 1$, we obtain the equilibrium in MZ (2004) with the slight difference that we impose a probability with zero mean. The normalization $\mathrm{E}_\pi[M] = 0$ ensures symmetry and is, in our view, not a strong restriction. We can interpret this assumption in the following way: both firms determine the expected midpoint of the consumer interval and align all possible product designs symmetrically around this mean. If the mean is nonzero, firms can transform the set of all product characteristics to be centered around zero. After determining their product characteristic choices in the normalized setting, firms may retransform their product characteristic decision into the non-normalized product space and obtain the optimal product design. For consumer distributions with nonzero mean there are no solutions in closed-form for firms' subgame-perfect product characteristic choices. Nevertheless, it is plausible to argue that both firms shift their subgame-perfect locations into the direction of this mean.

4.2.2 Comparative statics

The capacity model yields interesting comparative static results. In this section, we discuss basic properties of firms' product design choices with respect to changes in the underlying model parameters. Similar to Król (2012), the following proposition examines c.p. (ceteris paribus) variations in the global ambiguity attitude α.

Proposition 4.2 (Variation in firms' ambiguity attitude α)**.** *Under the Assumptions 1,2, and 3, one can observe at any SPNE of the Hotelling game under ambiguity the subsequent effects on optimal product designs:*

$$\frac{\partial x_1^*}{\partial \alpha} \geq 0 \qquad \textit{and} \qquad \frac{\partial x_2^*}{\partial \alpha} \leq 0.$$

The results of Proposition 4.2 are related to the findings in Król (2012) stating that a higher degree of pessimism leads to lower product differentiation. This finding extends to our model, with the difference that the magnitude of the effect is weakened the more confidence firms have in the reference prior π. In case of full confidence, or absence of ambiguity, firms' attitude towards ambiguity becomes irrelevant for their product differentiation choices. To give some intuition: for a high degree of pessimism α, each firm puts a higher weight on the maxmin criterion than on the maxmax criterion. Therefore, the worst-case scenario becomes increasingly important. The worst-case of firm 1 is that the expectation of M equals L. As the expectation moves to the right and firm 1 considers this expectation as relevant, firm 1 has an incentive to select a product characteristic located on the right hand side of its initial characteristics. Similarly, for firm 2, the worst-case scenario corresponds to left boundary of the support $-L$. Since firm 2 places increasingly more weight to this worst-case, there is an incentive for the latter to relocate to the left. All in all, equilibrium differentiation decreases. To sum up these findings, we conclude that, contrary to risk in the models of MZ, ambiguous uncertainty is not per se a differentiation force. What matters is ambiguity attitude of both firms. We call this attitude the degree of global optimism or pessimism since we consider a market where both firms exhibit the same ambiguity attitude. Hence, attitude towards ambiguity becomes a global characteristic of the market and could be interpreted as "market sentiment".

As a next step, we examine c.p. variations in the variance of the reference prior σ^2.

Proposition 4.3 (Variation in the variance σ^2)**.** *If* $0 \leq \delta < 1$ *and the Assumptions 1, 2, and 3 hold, one has at any SPNE for the Hotelling game under ambiguity that optimal product designs react in the following way to an increase in* σ^2*:*

$$\frac{\partial x_1^*}{\partial \sigma^2} < 0 \qquad \text{and} \qquad \frac{\partial x_2^*}{\partial \sigma^2} > 0.$$

Uncertainty, as measured by the variance of the underlying distribution, constitutes a differentiation force. As argued by MZ (2004), the intuition here is that, in the Hotelling game, firms are confronted with two countervailing effects. If a firm selects, at given prices, a product characteristic that is more far away from the realized midpoint $\hat{M}$ than the characteristic selected by its competitor, it loses market share (*demand effect*). At the same time, however, one can observe that increasing product differentiation weakens price competition and leads to higher equilibrium prices (*price effect*). Due to the assumption of quadratic cost functions, the price effect dominates the demand effect. If a firm faces demand location uncertainty, the negative effect of loosing market shares in some realizations of uncertainty is not so dramatic as in the certainty case since there are other realizations of M where the latter's product design is better located than before. Consequently, an increasing variance of the underlying probability distribution strengthens the dominance of the price effect. Therefore, equilibrium differentiation is even more excessive than under certainty. Of course, the same interpretation applies for the capacity model as long as $0 \leq \delta < 1$ with the sole difference that the effect of a c.p. increase in σ^2 is weaker the less confident firms are in the reference prior π.

The following proposition examines c.p. variations in the lower and upper support boundary of the transportation cost parameter.

Proposition 4.4 (Variations in the magnitude of the support boundaries of t)**.** *If* $0 < \alpha \leq 1$ *and* $0 < \delta \leq 1$*, then, under the Assumptions 1, 2, and 3, one has at any SPNE for the Hotelling game under ambiguity that*

$$\frac{\partial x_1^*}{\partial \underline{t}} > 0 \qquad \textit{and} \qquad \frac{\partial x_2^*}{\partial \underline{t}} < 0.$$

Similarly, for $0 \leq \alpha < 1$ *and* $0 < \delta \leq 1$*, one obtains*

$$\frac{\partial x_1^*}{\partial \bar{t}} < 0 \qquad \textit{and} \qquad \frac{\partial x_2^*}{\partial \bar{t}} > 0.$$

The first part of Proposition 4.4 is quite similar to the respective statement in Król (2012). Variations in the support of the transportation cost parameter can be interpreted as fluctuations in the magnitude of uncertainty around t. As $\underline{t}$ approaches one, the overall size of uncertainty with respect to t decreases. A c.p. increase in $\underline{t}$ solely affects the pessimistic part of firms' first-stage profit functions. This deceases firms' equilibrium product differentiation. The following considerations explain why this is the case. Comparing the Hotelling model with a standard symmetric Bertrand competition, we observe the following important difference. In the standard Bertrand scenario, firms offer homogeneous products. The only Nash equilibrium in pure strategies is that firms set prices equal to marginal costs, implying zero profits for both firms. In a Bertrand world with heterogeneous products this finding is no longer true. By introducing transportation costs, the Hotelling framework adds an additional distinctive feature to a homogeneous and symmetric Bertrand competition rendering products per se more heterogeneous. It is therefore intuitive that a higher transportation cost weakens competition between firms.

In the Hotelling model there are two countervailing incentives at work that determine firms' product design choices. One is that firms want to locate in the center of the Hotelling interval in order to obtain a higher market share. This is because firms' market share depends on the so-called indifferent consumer ξ.[18] All consumers located left of ξ strictly prefer to purchase the good from the firm located left. On the other hand,

[18]The indifferent consumer ξ can be obtained by equating total costs $p_1 + t(\xi - x_1)^2 = p_2 + t(\xi - x_2)^2$ and solving this expression for ξ.

consumers located right of ξ strictly prefer to purchase the good from the other firm. If the firm located left c.p. relocates to the right, then the indifferent consumer also shifts to the right. In this case, the market share of this firm increases and, as a consequence, also its profits. A similar argument holds for the rival firm. If the firm located at the right c.p. relocates to the left, then its market share increases, and hence also its profit. To sum up, the firm located left has an incentive to relocate to the right and the firm located to the right has an incentive to relocate to the left.

The second incentive is that firms want to differentiate their products more in order to weaken price competition. If product differentiation gets lower, price competition gets stronger since both products become increasingly homogeneous. Therefore, in the limit, the only distinguishing feature of a product boils down to its price. Now, if price competition is weakened by a higher transportation cost, it is plausible that firms have an incentive to reduce product differentiation in order to obtain a higher market share. To summarize the results. Increasing uncertainty with respect to the transportation cost parameter t entail a higher degree of product differentiation.

Next, we explore a c.p. increase in firms' confidence level δ. Our findings can be summarized in the following way: if firms' attitude towards ambiguity exhibits sufficiently strong optimism, one can conclude that a lower confidence into the reference prior increases equilibrium differentiation. Opposite results hold for sufficiently pessimistic firms. Furthermore, there is an intermediate value of global pessimism α^* such that firms' equilibrium differentiation remains unchanged no matter which global confidence level firms might assign to the reference probability distribution of the midpoint M. The following proposition makes this precise.

Proposition 4.5 (Variation in the confidence level δ). *Under the Assumptions 1, 2, and 3, one has at any SPNE for the Hotelling game under ambiguity that*

$$\frac{\partial x_1^*}{\partial \delta} = -\frac{\partial x_2^*}{\partial \delta} = \begin{cases} < 0 & \text{for} \quad 0 \leq \alpha < \alpha^* \\ = 0 & \text{for} \quad \alpha = \alpha^* \\ > 0 & \text{for} \quad 1 \geq \alpha > \alpha^* \end{cases}$$

where $\alpha^* = \alpha^*(\underline{t}, \overline{t}, \sigma^2, L)$ *is a cutoff-value defined by*

$$\alpha^* = \frac{(2L+3)(3L-2\sigma^2)}{(2L+3)\overline{t}(3L-2\sigma^2)-(2L-3)\underline{t}(3L+2\sigma^2)}.$$

Taking these results together we obtain for Δ^*

$$\frac{\partial \Delta^*}{\partial \delta} = \begin{cases} > 0 & \textit{for} \quad 0 \leq \alpha < \alpha^* \\ = 0 & \textit{for} \quad \alpha = \alpha^* \\ < 0 & \textit{for} \quad 1 \geq \alpha > \alpha^*. \end{cases}$$

Finally, we consider variations in the support size L. As Proposition 4.6 shows, our model replicates similar comparative static results as in Król (2012) by varying the length L of the support of the midpoint M: an increase in the support fosters decreasing product differentiation if firms are sufficiently pessimistic. For an intermediate value of pessimism firms do not relocate. If firms are sufficiently optimistic, an increase in L yields higher equilibrium differentiation. Apparently, similar product differentiation choices might be generated by variations in the size of the support L, as compared to variations in the confidence level δ. It indispensable to notice meaningful differences between the two sources of ambiguity since the degree of ambiguity or of firms' confidence in the reference prior plays a central role in this chapter (see Remark 4.4 and the subsequent discussion).

Proposition 4.6 (Variation in the size of the support L)**.** *If* $0 < \delta \leq 1$ *and Assumptions 1, 2, and 3 hold, on has at any SPNE for the Hotelling game under ambiguity that*

$$\frac{\partial x_1^*}{\partial L} = -\frac{\partial x_2^*}{\partial L} = \begin{cases} < 0 & \textit{for} \quad 0 \leq \alpha < \hat{\alpha} \\ = 0 & \textit{for} \quad \alpha = \hat{\alpha} \\ > 0 & \textit{for} \quad 1 \geq \alpha > \hat{\alpha} \end{cases}$$

where $\hat{\alpha} \in [0,1]$ *is a cutoff-value with* $\hat{\alpha} = \hat{\alpha}(\delta, \underline{t}, \overline{t}, \sigma^2)$. *Taking these results together we* *obtain for* Δ^*

$$\frac{\partial \Delta^*}{\partial L} = \begin{cases} > 0 & \text{for} \quad 0 \leq \alpha < \hat{\alpha} \\ = 0 & \text{for} \quad \alpha = \hat{\alpha} \\ < 0 & \text{for} \quad 1 \geq \alpha > \hat{\alpha}. \end{cases}$$

Remark 4.4. The effects of c.p. variations of L and δ might go in similar directions, but the magnitude of both effects is different. In fact, both effects are interrelated. An increase in the support has a stronger impact on equilibrium differentiation if firms' confidence in the reference prior is low. In case of full confidence, changes in the support do not affect firms' product design decisions.

Besides the magnitude of the effects, there is a clear difference between both sources of uncertainty concerning economic applications. The support $[-L, L]$ of M consists of all midpoint realizations of the consumer interval which firms view as possible before they perform their design choices. What would it actually mean if L were an endogenous variable? It would mean that firms adjust their views on possible demand realizations in light of new information by including or excluding certain market demand scenarios. However, this is problematic since it does not take into account how firms update their beliefs. None of the commonly used updating rules lead to an increase in the support size.

In the context of the multiple prior approach on which Król's model is based, the most commonly used updating rule is the generalized Bayesian updating rule. According to this rule, a decision-maker updates each prior from her prior set in accordance with *Bayes' rule*[19] when receiving new information (see Jaffray, 1992). It is well-known that Bayesian updating reduces the support of the prior distribution by shifting the whole probability mass to the "true" event.

Eichberger et al. (2010) analyze three updating rules for neo-additive capacities on which our model is based.[20] They show that under all three rules the updated capacity is still neo-additive with new optimism and confidence parameters, and a reference

[19]Bayes' rule follows from the law of conditional probability. Let Ω be a finite state space. For any two events $E, F \subseteq \Omega$ with prior probabilities $\pi(E)$ and $\pi(F)$, the posterior probability of E conditional on F is $\pi(E \mid F) = \frac{\pi(E \cap F)}{\pi(F)} = \frac{\pi(F|E) \cdot \pi(E)}{\pi(F)}$.

[20]These updating rules are the optimistic updating rule, the pessimistic updating rule, also called Dempster-Shafer updating rule, and the generalized Bayesian updating rule.

probability distribution that is updated in accordance with Bayes' rule. Consequently, the support of the posterior reference distribution is a subset of the support of the prior reference distribution.

The c.p. effect of variations in the support size shows how firms would behave in a different environment where they face less or more uncertainty, as measured by the support size. However, the previous considerations suggest that c.p. increases in the support size should not be used to explain changes in firms' behavior in the same environment when they receive new information.

In contrast, c.p. variations in the confidence level δ retain the assumption of an exogenously fixed support length. Firms know possible upper and lower bounds of demand and consider demand uncertainty defined on a fixed support. The reference prior π might reflect firms' ex-ante information about the market environment, e.g., firms might have observable data or can pursue market research to estimate an underlying probability distribution for market demand. Under the assumption that firm managers are sufficiently pessimistic, increasing product differentiation might have different reasons. One explanation could be that firms become more optimistic, that is, due to a change in the market environment firms adjust their ambiguity attitudes to account for the new situation. On the other hand, it is possible that firms obtain more reliable data on market outcomes which increases their confidence in the data available. However, they do not readjust their attitude towards ambiguity. In such a scenario, a higher confidence in the reference prior weakens the impact of pessimism on product differentiation choices.

4.3 Applications

The purpose of this section is to provide the reader with additional insight into the mechanics of the capacity approach. First, we describe the limitations of the models of MZ (2004) and Król (2012). Subsequently, we reconsider the real-life example proposed by Król. In particular, we discuss implications of confidence and pessimism in the context of this example.

Limitations of the existing models

Meagher and Zauner's probabilistic model predicts that uncertainty, as measured by the variance of the underlying distribution, constitutes a differentiation force. Their model presupposes that firms have a common prior about the distribution of the market demand. This is plausible when firms can rely on sufficient past data and consumer preferences are comparatively stable. If these assumptions are not met, the model may fail to accurately predict firms' behavior. For instance, the model predicts that there will be low product differentiation in markets that have little demand fluctuations. However, it is possible that firms highly differentiate their products even if the market demand does not fluctuate much. For example, think of firms which introduce products with a new design into an existing market.

In the model of Król (2012), uncertainty, as measured by the size of the support of the midpoint, can either be a differentiation or an agglomeration force depending on firms' attitudes towards uncertainty. The model predicts that pessimistic firms tend to differentiate their products less which seems not implausible. Król's model has the advantage that firms may end up acting as if they have different priors. On the other hand, it has the drawback that it does not incorporate probabilistic information. In other words, firms' choices are governed by degenerate probability measures. Instead of probabilistic information, Król examines variations in the size of the support of the midpoint. In our view, in many economic applications, it is more plausible that the size of the support is an exogenously fixed parameter, see, e.g., the discussion of Example 4.1. Furthermore, as argued in Section 4.2.2, an increasing support size cannot be justified on the basis of commonly used updating rules.

An example

In the following, we reconsider Król's example of the mutual funds market. One reason why this market is so apt to be discussed in a Hotelling framework is that there exists a relatively clear measure of firms' product differentiation. We will discuss this measure in the sequel. Moreover, demand fluctuates due to partially unobservable factors, e.g., subjective evaluations. For this reason, it is plausible that ambiguity is prevalent in this market.

Example 4.1 (Król, 2012, p. 599)**.** *Consider the managed mutual funds market. We may interpret a portfolio's position ranging from safe investments to risky portfolios as a location in the product space. Data about the daily returns of the fifteen most popular actively managed US mutual funds indicate that, after the financial crisis 2008, fund managers tend to differentiate their products less.*

What is the reason for fund managers' behavior in Example 4.1? Król's explanation is based on two arguments. Firstly, he claims that, before the crisis, financial firms' did not consider the post-crisis range of investor behavior as possible.[21] Therefore, the crisis forced firms to revise their beliefs. Secondly, Król interprets conservative stress test simulations following the crisis as a signal sent out to the competitors that a firm uses a worst-case-based approach for decision-making.

As already argued earlier in this chapter, the weakness of the first argument is that it does not take into account how firms update their beliefs. The second argument refers to government-imposed stress tests after the crisis. If we interpret these stress tests as signals, then the strategy of a firm is independent of its type. Since each type sends the same signal, no new information is revealed to the other firms. In our view, it is debatable whether stress test simulations induced a shift in fund managers' ambiguity attitude towards a more pessimistic preference approach, or whether exactly those fund managers knew more clearly that investors would prefer more secure assets after the crisis.

In our view, a more plausible explanation for lower post-crisis product differentiation is that firms were more confident about investors' demand. In the context of this explanation, we may keep the standard assumption of stable preferences. That is, we assume that fund managers' ambiguity attitudes remained relatively stable even though government stress tests were imposed. It is not implausible to assume that fund managers are rather optimistic individuals. Moreover, it is likely that fund managers know the whole range of

[21] In particular, the shift of consumer preferences toward safe investments due to decreasing stock prices during the crisis.

possible investor behavior due to market research and historical data.[22] This means that variations in the support of the midpoint of the consumer distribution cannot account for the observation of decreasing product differentiation. The demand of investors is governed by subjective evaluations that depend on numerous observable and unobservable factors such as recent stock market developments or individual future expectations. Therefore, it is plausible to assume that firms faced highly fluctuating demand before the financial crisis. At the end and shortly after the crisis, one may assume that firms were more confident about investors' demand since it was obvious that the majority of investors would go for rather safe assets. To sum up, given that fund managers are sufficiently optimistic, an increase of their confidence, $\delta < 0$, leads to lower equilibrium differentiation. This is in line with the capacity model, see Proposition 4.5.

4.4 Summary

In this chapter, we develop a general Hotelling model incorporating demand ambiguity that provides a unifying framework for the model under risk of Meagher and Zauner (2004) and the model under complete ignorance of Król (2012). Ambiguity is introduced by assuming that firms' beliefs are represented by neo-additive capacities. We analyze firms' optimal product characteristic choices and find that there exists a unique subgame-perfect Nash equilibrium in pure strategies for the Hotelling game under ambiguity.

Our model incorporates a variety of different sources of uncertainty. First of all, as in MZ's model, there is the variance σ^2 of the reference probability π. Secondly, as in Król's model, we have the length L of the support interval of the midpoint of the consumer interval M. A third measure of uncertainty is given by the confidence, or degree of ambiguity, parameter δ. The last source of uncertainty lies in the support $[\underline{t}, \overline{t}]$ of the transportation cost parameter t.

While a ceteris paribus increase in σ^2 leads to a higher equilibrium product differentiation, the direction of the effect of an increase of L is not clear. As our results show,

[22]Financial firms' can rely on past data of various historical economic crises including stock market crashes (e.g., the Great Depression in the 1930s), bubbles (e.g. dot-com bubble in 2000), and financial crises (e.g., Asian financial crisis in 1997).

this effect strongly depends on firms' ambiguity attitude α and the degree of ambiguity δ. Similarly, an increase of δ can trigger off opposing effects. When firms are pessimistic enough, equilibrium differentiation decreases, when firms are rather optimistic product differentiation is going to increase. One can also argue the other way round. For a given confidence level, increasing pessimism yields lower equilibrium differentiation, whereas an increase in optimism increases equilibrium differentiation.

These considerations show that one should be very cautious when it comes to drawing conclusions from real-world applications of Hotelling models under uncertainty. In our view, it is indispensable to clearly identify the driving factors of an observed increase or decrease in product differentiation since the conclusions from observed firm behavior might change in light of different sources of uncertainty. In particular, it might be very important for official regulatory procedures whether observed product differentiation choices are to be attributed to perceived changes in data-reliability or whether firms feature more or less optimistic behavioral patterns. Hence, it seems worthwhile for policymakers to disentangle the effect of confidence and ambiguity attitude on product differentiation.

4.5 Proofs

Proof of Lemma 4.1. The support of M is restricted to the interval $[-L, L] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$. The mean and the variance of M exists. For the mean we can perform the following line of estimates:

$$\mathrm{E}_\pi[M] = \int\limits_{\mathbb{R}} M d\mathbb{P} \leq \int\limits_{\mathbb{R}} L d\mathbb{P} = L \int\limits_{\mathbb{R}} 1 d\mathbb{P} = L$$

and

$$\mathrm{E}_\pi[M] = \int\limits_{\mathbb{R}} M d\mathbb{P} \geq \int\limits_{\mathbb{R}} -L d\mathbb{P} = -L \int\limits_{\mathbb{R}} 1 d\mathbb{P} = -L$$

Similarly, for the second moment of M we obtain

$$\mathrm{E}_\pi[M^2] = \int\limits_{\mathbb{R}} M^2 d\mathbb{P} \leq \int\limits_{\mathbb{R}} L^2 d\mathbb{P} = L^2 \quad \text{and} \quad \mathrm{E}_\pi[M^2] = \int\limits_{\mathbb{R}} M^2 d\mathbb{P} \geq 0$$

and for the variance σ^2 we conclude

$$\sigma_M^2 = \mathrm{E}_\pi[M^2] - \mathrm{E}_\pi[M]^2 \le \mathrm{E}_\pi[M]^2 \le L^2 \quad \text{and} \quad \sigma_M^2 \ge 0.$$

□

Proof of Lemma 4.4. Lemma 4.5 implies that firms' second-stage profits at the realization $(\hat{t}, \hat{M})$ equal the second piece of (4.1):

$$\Pi_1 = \tfrac{1}{18}\hat{t}(x_2 - x_1)[-3 + 2(\hat{M} - \bar{x})]^2 \text{ and } \Pi_2 = \tfrac{1}{18}\hat{t}(x_2 - x_1)[3 + 2(\hat{M} - \bar{x})]^2.$$

Both profit functions are continuously differentiable with respect to $\hat{t}$ and $\hat{M}$. Differentiation with respect to $\hat{t}$ yields

$$\frac{\partial \Pi_1}{\partial \hat{t}} = \frac{2}{9}(x_2 - x_1)\left[\frac{x_1 + x_2}{2} - \hat{M} + \frac{3}{2}\right]^2 > 0$$
$$\frac{\partial \Pi_2}{\partial \hat{t}} = -\frac{2}{9}(x_1 - x_2)\left[\frac{x_1 + x_2}{2} - \hat{M} - \frac{3}{2}\right]^2 > 0.$$

Differentiation with respect to $\hat{M}$ yields

$$\frac{\partial \Pi_1^*}{\partial \hat{M}} = \underbrace{-\frac{4}{9}\hat{t}(x_2 - x_1)}_{<0}\underbrace{\left[\frac{x_1 + x_2}{2} - \hat{M} + \frac{3}{2}\right]}_{>0} < 0$$
$$\frac{\partial \Pi_2^*}{\partial \hat{M}} = \underbrace{\frac{4}{9}\hat{t}(x_1 - x_2)}_{<0}\underbrace{\left[\frac{x_1 + x_2}{2} - \hat{M} - \frac{3}{2}\right]}_{<0} > 0.$$

□

Proof of Lemma 4.5. The proof of the lemma follows exactly the same line of arguments as in the proof of Lemma 3.1 in Król (2012, p. 602) with a slight modification in case 3. There are three different cases to be considered.

1. Case 1 refers to a situation where either firm 1 or firm 2 can monopolize the market for certain realizations of the midpoint M. If firm 1 can monopolize the market for certain realizations of M, we can conclude that firm 1 will monopolize the market if $\hat{M} = -L$, since w.lo.g. firm 1 is the firm left of firm 2. Similarly, we can conclude

that firm 2 can monopolize the market for $\hat{M} = L$. This is finding is impossible. If firm 1 monopolizes the market for the realization $\hat{M} = -L$, we have by Lemma 3.1, equation (4.3) that $\frac{x_1+x_2}{2} - \frac{3}{2} > -L$. If firm 2 monopolizes the market, we have by (4.3) that $\frac{x_1+x_2}{2} + \frac{3}{2} < L$. Thus, we must have that $L + \frac{x_1+x_2}{2} > \frac{3}{2}$ and $L - \frac{x_1+x_2}{2} > \frac{3}{2}$ holds at the same time implying $\left|\frac{x_1+x_2}{2}\right| < L - \frac{3}{2}$. This is a contradiction since L is assumed to be smaller than $\frac{1}{2}$.

2. Case 2 describes a scenario where one of the two firms can monopolize the market for each realization $\hat{M}$ of uncertainty. If firm j is a monopolist, the other firm can deviate from its original location in order to obtain a positive market share and therefore make strictly positive profits. Król suggests the location $x_{-j} = -x_j$.

3. Case 3 refers to a situation where, w.l.o.g., firm 1 can monopolize the market for some realizations of uncertainty, in particular the realization $\hat{M} = -L$ and for the remaining realizations, in particular the realization $\hat{M} = L$, there exists a competitive equilibrium. Consider now the profit function of firm 2 in case of a competitive equilibrium[23] :

$$\frac{\partial \Pi_2}{\partial x_2}(x_1, x_2, L, t) = \frac{t\,(2\,L - 3\,x_2 + x_1 + 3)\,(2\,L - x_2 - x_1 + 3)}{18}$$

We want to show that

$$\frac{\partial \Pi_2}{\partial x_2}(x_1, x_2, L, t) < 0.$$

We determine the sign of both brackets. Consider the expression in within the second bracket first. We have

$$2L + 3 - x_1 - x_2 > 0 \quad \Leftrightarrow \quad 2L + 3 > x_1 + x_2 \quad \Leftrightarrow \quad L + \frac{3}{2} > \bar{x}$$

The last condition corresponds to the requirement for a competitive solution in cases where the midpoint $M = L$ realizes. Therefore it must be, by assumption, positive. The second bracket is negative. The monopolistic outcome for the midpoint

[23]We consider the profit function of firm 2, Król considers the profit function of firm 1.

realization $M = -L$ requires $L + \bar{x} > \frac{3}{2}$. Solving this inequality for x_2, we obtain $x_2 > 3 - 2L - x_1$. By using this inequality, we can conduct an estimation for the expression in the first bracket:

$$3 + 2L + x_1 - 3x_2 < 8L - 6 + 4x_1 < 8L - 8 < 0$$

The last inequality follows from the fact that $L < \frac{1}{2}$ and $x_1 < 0$. Thus, we proved that

$$\frac{\partial \Pi_2}{\partial x_2}(x_1, x_2, L, t) < 0.$$

This finding shows that firm 2 has an incentive to move leftwards in order to reduce both firms' product differentiation and that a strict competitive solution does not exist under the above stated parameter restrictions. Since we consider a symmetric scenario, a similar argument holds for a scenario where firm 2 becomes a monopolist. For the remaining cases $\hat{M} = L$ and $\hat{M} = -L$ there is a competitive solution.

□

Proof of Lemma 4.6. The first part of firms' Choquet expected profit is

$$\mathrm{E}_\pi[\Pi_i(x_1, x_2, t, M)] = \int_{-L}^{L} (-1)^j \frac{2}{9} t (x_j - x_i) \left(\frac{x_i + x_j}{2} - \left(M + \frac{3}{2}(-1)^i \right) \right)^2 f(M) dM.$$

This expectation is of the form

$$\mathrm{E}_\pi[g_i(t)\, h_i(M)]$$

with real-valued Borel-measurable functions g_i and h_i for $i = 1, 2$. We define

$$g_i(t) = t \qquad \text{and} \qquad h_i(M) = (-1)^j \frac{2}{9} t (x_j - x_i) \left(\frac{x_i + x_j}{2} - \left(M + \frac{3}{2}(-1)^i \right) \right)^2.$$

By $(R7)$, t and M are uncorrelated. By Lemma 5.20 in Meintrup and Schäffler (2005, p. 131), we obtain that $g_i(t)$ and $h_i(M)$ are uncorrelated as well. Thus, we can conclude

$$\mathrm{E}_\pi[\Pi_i^*(x_1, x_2, t, M)] = E_\pi[g_i(t)\, h_i(M)] = \mathrm{E}_\pi[g_i(t)]\ \mathrm{E}_\pi[h_i(M)] = \mu_t\ \mathrm{E}_\pi[h_i(M)].$$

In the following, we can rely on the results in Meagher and Zauner (2004, p. 205), since $\mathrm{E}_\pi[h_i(M)]$ is equal to firm i's expected profit function in the risk case. Thus,

$$\mathrm{E}_\pi[\Pi_i(x_1, x_2, t, M)] = t_\mu \int_{-L}^{L} (-1)^j \frac{2}{9}(x_j - x_i)\left(\frac{x_i + x_j}{2} - \left(M + \frac{3}{2}(-1)^i\right)\right)^2 f(M)dM$$

$$= \frac{(-1)^j}{18} t_\mu (x_j - x_i)\{(x_i + x_j - 3(-1)^i)^2$$

$$- 4\mu_M(x_i + x_j - 3(-1)^i) + 4(\mu_M + \sigma_M^2)\}.$$

□

Proof of Proposition 4.1. We derive expected CEU profits at the first stage of the game. We obtain for firm 1:

$$\begin{aligned}&\mathrm{CEU}_1[x_1, x_2, \alpha, \delta, \underline{t}, \overline{t}, \sigma^2, L]\\ &:= \delta\left(\frac{2\,(1-\alpha)\,\overline{t}\,\,(x_2 - x_1)\,\left(L + \frac{x_2+x_1}{2} + \frac{3}{2}\right)^2}{9} + \frac{2\,\alpha\,\underline{t}\,\,(x_2 - x_1)\,\left(-L + \frac{x_2+x_1}{2} + \frac{3}{2}\right)^2}{9}\right)\\ &+ \frac{(1-\delta)\,\,(x_2 - x_1)\,\left((x_2 + x_1 + 3)^2 + 4\,\sigma^2\right)}{18}.\end{aligned} \tag{4.5}$$

Similarly, we obtain for firm 2

$$\begin{aligned}&\mathrm{CEU}_2[x_1, x_2, \alpha, \delta, \underline{t}, \overline{t}, \sigma^2, L]\\ &:= \delta\left(\frac{2\,\alpha\,\underline{t}\,\,(x_2 - x_1)\,\left(L + \frac{x_2+x_1}{2} - \frac{3}{2}\right)^2}{9} + \frac{2\,(1-\alpha)\,\overline{t}\,\,(x_2 - x_1)\,\left(-L + \frac{x_2+x_1}{2} - \frac{3}{2}\right)^2}{9}\right)\\ &+ \frac{(1-\delta)\,\,(x_2 - x_1)\,\left((x_2 + x_1 - 3)^2 + 4\,\sigma\right)}{18}.\end{aligned} \tag{4.6}$$

Taking the derivative of (4.5) with respect to x_1 yields

$$\begin{aligned}\frac{\partial\,\mathrm{CEU}_1}{\partial x_1} := &-\frac{2\delta\,(1-\alpha)\,\bar{t}\left(L+\frac{x_2+x_1}{2}+\frac{3}{2}\right)^2}{9}+\frac{2\delta\,(1-\alpha)\,\bar{t}\,(x_2-x_1)\,\left(L+\frac{x_2+x_1}{2}+\frac{3}{2}\right)}{9}\\ &+\frac{2\delta\,\alpha\,\underline{t}\,(x_2-x_1)\,\left(-L+\frac{x_2+x_1}{2}+\frac{3}{2}\right)}{9}-\frac{2\delta\,\alpha\,\underline{t}\left(-L+\frac{x_2+x_1}{2}+\frac{3}{2}\right)^2}{9}\\ &-\frac{(1-\delta)\,\left((x_2+x_1+3)^2+4\,\sigma\right)}{18}+\frac{(1-\delta)\,(x_2-x_1)\,(x_2+x_1+3)}{9}.\end{aligned} \tag{4.7}$$

Similarly, we take the derivative of (4.6) with respect to x_2

$$\begin{aligned}\frac{\partial\,\mathrm{CEU}_2}{\partial x_2} := &\frac{2\delta\,\alpha\,\underline{t}\left(L+\frac{x_2+x_1}{2}-\frac{3}{2}\right)^2}{9}+\frac{2\delta\,\alpha\,\underline{t}\,(x_2-x_1)\,\left(L+\frac{x_2+x_1}{2}-\frac{3}{2}\right)}{9}\\ &+\frac{2\delta\,(1-\alpha)\,\bar{t}\,(x_2-x_1)\,\left(-L+\frac{x_2+x_1}{2}-\frac{3}{2}\right)}{9}+\frac{2\delta\,(1-\alpha)\,\bar{t}\left(-L+\frac{x_2+x_1}{2}-\frac{3}{2}\right)^2}{9}\\ &+\frac{(1-\delta)\,\left((x_2+x_1-3)^2+4\,\sigma\right)}{18}+\frac{(1-\delta)\,(x_2-x_1)\,(x_2+x_1-3)}{9}.\end{aligned} \tag{4.8}$$

Now, we solve the following system of equations:

$$\frac{\partial\,\mathrm{CEU}_1}{\partial x_1}=0 \quad \text{and} \quad \frac{\partial\,\mathrm{CEU}_2}{\partial x_2}=0. \tag{4.9}$$

and obtain three solution pairs. The first solution pair (x_1^*, x_2^*) is given by:

$$x_1^* = \frac{\delta\left(-(\alpha-1)(2L+3)^2\bar{t}+\alpha(3-2L)^2\underline{t}-4\sigma^2-9\right)+4\sigma^2+9}{4(\delta((\alpha-1)(2L+3)\bar{t}+\alpha(2L-3)\underline{t}+3)-3)}$$

$$x_2^* = \frac{\delta\left((\alpha-1)(2L+3)^2\bar{t}-\alpha(3-2L)^2\underline{t}+4\sigma^2+9\right)-4\sigma^2-9}{4(\delta((\alpha-1)(2L+3)\bar{t}+\alpha(2L-3)\underline{t}+3)-3)}$$

The second pair of solutions (x_1^{**}, x_2^{**}) is given by:

$$x_1^{**} = \Bigg(- \Big(\delta(\delta((\alpha-1)^2(2L+3)^2\bar{t}^2 + 2(\alpha-1)\bar{t}(2L(6\alpha L\underline{t} - L + 3) - 9\alpha\underline{t} + 9)$$

$$+ \alpha\underline{t}(4L(\alpha(L-3)\underline{t} + L + 3) + 9(\alpha\underline{t} - 2)) + 4\sigma^2(-\alpha\bar{t} + \alpha\underline{t} + \bar{t} - 1) + 9)$$

$$+ 4(\alpha-1)\bar{t}((L-3)L + \sigma^2) - 2\alpha\underline{t}(2L(L+3)$$

$$+ 2\sigma^2 - 9) + 2(-9(\alpha-1)\bar{t} + 4\sigma^2 - 9)) - 4\sigma^2 + 9\Big)^{\frac{1}{2}}$$

$$+ \delta(-(\alpha-1)(2L+3)\bar{t} + \alpha(3-2L)\underline{t} - 3) + 3\Bigg)$$

$$\cdot \Big(2(\delta((\alpha-1)\bar{t} - \alpha\underline{t} + 1) - 1)\Big)^{-1}$$

and

$$x_2^{**} = -\Bigg(\Big(\delta(\delta((\alpha-1)^2(2L+3)^2\bar{t}^2 + 2(\alpha-1)\bar{t}(2L(6\alpha L\underline{t} - L + 3) - 9\alpha\underline{t} + 9)$$

$$+ \alpha\underline{t}(4L(\alpha(L-3)\underline{t} + L + 3) + 9(\alpha\underline{t} - 2)) + 4\sigma^2(-\alpha\bar{t} + \alpha\underline{t} + \bar{t} - 1) + 9)$$

$$+ 4(\alpha-1)\bar{t}((L-3)L + \sigma^2) - 2\alpha\underline{t}(2L(L+3) + 2\sigma^2 - 9)$$

$$+ 2(-9(\alpha-1)\bar{t} + 4\sigma^2 - 9)) - 4\sigma^2 + 9\Big)^{\frac{1}{2}}$$

$$- \delta((\alpha-1)(2L+3)\bar{t} + \alpha(2L-3)\underline{t} + 3) + 3\Bigg)$$

$$\cdot \Big(2(\delta((\alpha-1)\bar{t} - \alpha\underline{t} + 1) - 1)\Big)^{-1}$$

Finally, the last pair of solutions (x_1^{***}, x_2^{***}) is given by:

$$\begin{aligned} x_1^{***} =& \Bigg(\Big(\delta(\delta((\alpha-1)^2(2L+3)^2\bar{t}^2 + 2(\alpha-1)\bar{t}(2L(6\alpha L\underline{t} - L + 3) - 9\alpha\underline{t} + 9) \\ &+ \alpha\underline{t}(4L(\alpha(L-3)\underline{t} + L + 3) + 9(\alpha\underline{t} - 2)) + 4\sigma^2(-\alpha\bar{t} + \alpha\underline{t} \\ &+ \bar{t} - 1) + 9) + 4(\alpha-1)\bar{t}((L-3)L + \sigma^2) - 2\alpha\underline{t}(2L(L+3) + 2\sigma^2 - 9) \\ &+ 2(-9(\alpha-1)\bar{t} + 4\sigma^2 - 9)) - 4\sigma^2 + 9\Big)^{\frac{1}{2}} \\ &+ \delta(-(\alpha-1)(2L+3)\bar{t} + \alpha(3-2L)\underline{t} - 3) + 3 \Bigg) \\ &\cdot \Big(2(\delta((\alpha-1)\bar{t} - \alpha\underline{t} + 1) - 1) \Big)^{-1} \end{aligned}$$

and

$$\begin{aligned} x_2^{***} = & \Bigg(\Big(\delta(\delta((\alpha-1)^2(2L+3)^2\bar{t}^2 + 2(\alpha-1)\bar{t}(2L(6\alpha L\underline{t} - L + 3) \\ &- 9\alpha\underline{t} + 9) + \alpha\underline{t}(4L(\alpha(L-3)\underline{t} + L + 3) + 9(\alpha\underline{t} - 2)) + 4\sigma^2(-\alpha\bar{t} + \alpha\underline{t} \\ &+ \bar{t} - 1) + 9) + 4(\alpha-1)\bar{t}((L-3)L + \sigma^2) - 2\alpha\underline{t}(2L(L+3) + 2\sigma^2 - 9) \\ &+ 2(-9(\alpha-1)\bar{t} + 4\sigma^2 - 9)) - 4\sigma^2 + 9\Big)^{\frac{1}{2}} \\ &+ \delta((\alpha-1)(2L+3)\bar{t} + \alpha(2L-3)\underline{t} + 3) - 3 \Bigg) \\ &\cdot \Big(2(\delta((\alpha-1)\bar{t} - \alpha\underline{t} + 1) - 1) \Big)^{-1} \end{aligned}$$

The first pair of solutions (x_1^*, x_2^*) satisfies the global competition condition according to Lemma 4.5. We demonstrate this by using Wolfram Mathematica version 10.0.0.0. You can find the code at the end of the proof section. The problem is analyzed in sections 5 to 7 of the code. Mathematica returns the value "true" for the first pair of solutions. The solution pairs (x_1^{**}, x_2^{**}) and (x_1^{***}, x_2^{***}) do not fulfill the global competition condition

$$L - \frac{3}{2} < \bar{x} < -L + \frac{3}{2}.$$

This is examined in sections 8 and 9 of our code. Therefore, we define, in a first step, the means

$$\overline{x}_2 = \frac{x_1^{**} + x_2^{**}}{2} \qquad \text{and} \qquad \overline{x}_3 = \frac{x_1^{***} + x_2^{***}}{2}.$$

Using numerical optimization techniques, we obtain that the range of $\overline{x}_2$ is given by $[1, 2]$. Similarly, the range of $\overline{x}_3$ is given by $[-2, -1]$. Moreover, $\overline{x}_2$ attains its minimum value 1 for $L = \frac{1}{2}$. This implies that $\overline{x}_2 \geq 1$. However, the global competition condition would require that $\overline{x}_2 < -\frac{1}{2} + \frac{3}{2} = 1$. This is a contradiction. Similarly, $\overline{x}_3$ attains its maximum value -1 for $L = \frac{1}{2}$. As a consequence, we can infer that $\overline{x}_3 \leq -1$. In order to meet the requirements of Lemma 4.5, $\overline{x}_3$ also needs to satisfy $\overline{x}_3 > \frac{1}{2} - \frac{3}{2} = -1$. This excludes (x_1^{***}, x_2^{***}) as a feasible solution.

As a next step, we show that the first pair of solutions is indeed a maximizer for both firms. The second order derivative of equation (4.5) and (4.6) evaluated at (x_1^*, x_2^*) yields

$$\begin{aligned}
\frac{\partial^2 \operatorname{CEU}_i}{\partial x_i^2} := \Big(&\delta(\delta(3(\alpha-1)^2(2L+3)^2\overline{t}^2 \\
&+ 2(\alpha-1)\overline{t}(2L(10\alpha L\underline{t} - L + 9) - 27\alpha\underline{t} + 27) \\
&+ \alpha\underline{t}(3\alpha(3-2L)^2\underline{t} + 4L(L+9) - 54) \\
&+ 4\sigma^2(-\alpha\overline{t} + \alpha\underline{t} + \overline{t} - 1) + 27) + 4(\alpha-1)\overline{t}((L-9)L \\
&+ \sigma^2) - 2\alpha\underline{t}(2L(L+9) + 2\sigma^2 - 27) - 54((\alpha-1)\overline{t} + 1) \\
&+ 8\sigma^2) - 4\sigma^2 + 27\Big) \\
&\cdot \Big(18(\delta((\alpha-1)(2L+3)\overline{t} + \alpha(2L-3)\underline{t} + 3) - 3)\Big)^{-1}
\end{aligned}$$

for both firms. First, we examine the sign of the denominator. It is

$$\begin{aligned}
&18(\delta((\alpha-1)(2L+3)\overline{t} + \alpha(2L-3)\underline{t} + 3) - 3) \\
&= 36\alpha\delta L\overline{t} + 36\alpha\delta L\underline{t} + 54\alpha\delta\overline{t} - 54\alpha\delta\underline{t} - 36\delta L\overline{t} - 54\delta\overline{t} + 54\delta - 54 \\
&\leq 36\delta L\overline{t} + 18\alpha\delta\underline{t} + 54\delta\overline{t} - 54\alpha\delta\underline{t} - 36\delta L\overline{t} - 54\delta\overline{t} + 54 - 54 \\
&= -36\alpha\delta\underline{t} \\
&\leq 0.
\end{aligned}$$

Hence, the denominator is negative. Subsequently, we show that the numerator is nonnegative. Taking the derivative of the numerator with respect to $\bar{t}$ yields

$$\begin{aligned}&-2(1-\alpha)\delta(\delta(3(\alpha-1)(2L+3)^2\bar{t}+2L(L(10\alpha\underline{t}-1)+9)\\&-27\alpha\underline{t}-2\sigma^2+27)+2((L-9)L+\sigma^2)-27)\end{aligned}$$

Given the parameter restrictions of the model, this expression is non-negative. We verify this in sections 13 and 14 of the Mathematica code. Hence, the numerator becomes smaller as we insert the minimum value 1 for $\bar{t}$. Doing so, we obtain after several steps of algebra

$$\begin{aligned}h=&\Big(\alpha^2\delta^2\Big(4L^2(\underline{t}+3)(3\underline{t}+1)-36L\Big(\underline{t}^2-1\Big)+27(\underline{t}-1)^2\Big)\\&-2\alpha\delta\Big(2L^2(9\delta\underline{t}+7\delta+\underline{t}-1)\\&+18L(\delta(-\underline{t})+\delta+\underline{t}+1)-(\underline{t}-1)(2(\delta-1)\sigma^2+27)\Big)\\&+4\delta(L((4\delta-1)L+9)+\sigma^2)-4\sigma^2+27\Big)\end{aligned}$$

It remains to be demonstrated that this expression is non-negative. Using Mathematica, we check whether h can be negative under the restrictions $0\leq\alpha\leq 1$, $0\leq\delta\leq 1$, $0\leq\underline{t}\leq 1$, $0\leq L\leq\frac{1}{2}$ and $0\leq\sigma^2\leq\frac{1}{4}$, see sections 15 and 16 of the code. Mathematica returns the value "false".

We obtain the equilibrium profits by inserting the equilibrium locations x_i^* for $i=1,2$ into (4.5) and (4.6). After several steps of algebra, we obtain

$$\mathrm{CEU}_i=-\frac{\left(\delta\left(-(\alpha-1)(2L+3)^2\bar{t}+\alpha(3-2L)^2\underline{t}-4\sigma^2-9\right)+4\sigma^2+9\right)^2}{36(\delta((\alpha-1)(2L+3)\bar{t}+\alpha(2L-3)\underline{t}+3)-3)}.$$

The competitive differentiation is given by

$$\begin{aligned}\Delta_x^*&=x_2^*-x_1^*=2x_2^*\\&=\frac{\delta\left((\alpha-1)(2L+3)^2\bar{t}-\alpha(3-2L)^2\underline{t}+4\sigma^2+9\right)-4\sigma^2-9}{2(\delta((\alpha-1)(2L+3)\bar{t}+\alpha(2L-3)\underline{t}+3)-3)}.\end{aligned}$$

What remains to show is that non of the firms has an incentive to "jump over" its opponent. That is, firm 1 has no incentive to choose a product design to the right of that from firm 2. Similarly, firm 2 does not want to be on the left of firm 1. Our proof follows the proof of Anderson et al. (1997, p. 113-114). Given any location x_2, if firm 1 chooses its best reply, $R_1^*(x_2)$, under the restriction $x_1 \leq x_2$, its profit equals $\widehat{\mathrm{CEU}}_1 = \mathrm{CEU}_1[R_1^*(x_2), x_2, \alpha, \delta, \underline{t}, \bar{t}), \sigma^2, L]$. At first, we show that under global competition, cf. Lemma 4.5, firm 1's optimal profit is increasing in x_2. By using the envelope theorem on firm 1's profit function, we obtain:

$$
\begin{aligned}
\frac{\widehat{\mathrm{CEU}}_1}{\partial x_2} =& \frac{2\bar{t}\;(1-\alpha)\;\delta\left(\left(L+\frac{x_2+R_1^*(x_2)}{2}+\frac{3}{2}\right)^2+(x_2-R_1^*(x_2))\left(L+\frac{x_2+R_1^*(x_2)}{2}+\frac{3}{2}\right)\right)}{9} \\
&+\frac{2\,\alpha\,\underline{t}\,\delta\left((x_2-R_1^*(x_2))\left(-L+\frac{x_2+R_1^*(x_2)}{2}+\frac{3}{2}\right)+\left(-L+\frac{x_2+R_1^*(x_2)}{2}+\frac{3}{2}\right)^2\right)}{9} \\
&+(1-\delta)\left(\frac{\left(4\,\sigma^2+(x_2+R_1^*(x_2)+3)^2\right)}{18}+\frac{(x_2-R_1^*(x_2))\;(x_2+R_1^*(x_2)+3)}{9}\right) \\
&> 0.
\end{aligned}
$$

To see that the sign of this derivative is positive, note that the quadratic terms are positive and the distance $(x_2 - R_1^*(x_2))$ is nonnegative. Furthermore, by assumption, the parameters $\underline{t}, \bar{t}, \alpha$ and δ are all nonnegative. Let us consider the other terms. The term $\left(L + \frac{x_2+R_1^*(x_2)}{2} + \frac{3}{2}\right)$ in the first part of the derivative is positive, since, by the global competition lemma, $\frac{x_2+R_1^*(x_2)}{2} + \frac{3}{2} > -L$. Since it holds also that $\frac{x_2+R_1^*(x_2)}{2} + \frac{3}{2} > L$, the term $\left(-L + \frac{x_2+R_1^*(x_2)}{2} + \frac{3}{2}\right)$ in the second part of the derivative is positive. Finally, the term $(x_2 + R_1^*(x_2) + 3)$ is positive, since by the global competition lemma: $\frac{x_2+R_1^*(x_2)}{2} \geq L - \frac{3}{2}$ and due to the restriction of L, see Assumption 4.2 (R4): $L - \frac{3}{2} \geq -2$. Taken together, all terms are positive or nonnegative, which proves that $\frac{\partial \widehat{\mathrm{CEU}}_1}{\partial x_2} > 0$. Now, suppose firm 1 locates to the other side of its rival. Then, given any location x_2, if firm 1 chooses its best reply, $R_1^*(x_2)$, now, restricting its location to $x_1 \geq x_2$, its profit is decreasing in x_2.

Applying the envelope theorem yields:

$$
\begin{aligned}
\frac{\partial \widehat{\mathrm{CEU}}_1}{\partial x_2} =& \frac{2\,(1-\alpha)\,\delta \bar{t}\left((R_1^*(x_2)-x_2)\left(-L+\frac{R_1^*(x_2)+x_2}{2}-\frac{3}{2}\right)-\left(-L+\frac{R_1^*(x_2)+x_2}{2}-\frac{3}{2}\right)^2\right)}{9}\\
&+\frac{2\,\alpha\,\delta\,\underline{t}\left((R_1^*(x_2)-x_2)\left(L+\frac{R_1^*(x_2)+x_2}{2}-\frac{3}{2}\right)-\left(L+\frac{R_1^*(x_2)+x_2}{2}-\frac{3}{2}\right)^2\right)}{9}\\
&+(1-\delta)\left(\frac{-4\,\sigma^2-(R_1^*(x_2)+x_2-3)^2}{18}+\frac{(R_1^*(x_2)-x_2)\,(R_1^*(x_2)+x_2-3)}{9}\right)\\
&< 0.
\end{aligned}
$$

By using the global competition lemma, the sign of the derivative can be proved analogously as before.

To sum up, we have that $\frac{\partial \widehat{\mathrm{CEU}}_1}{\partial x_2} > 0$ for $x_1 \leq x_2$ and $\frac{\partial \widehat{\mathrm{CEU}}_1}{\partial x_2} < 0$ for $x_1 \geq x_2$. This means that there is a unique "straddle" point, $\tilde{x}$, such that if firm 2 chooses a characteristic $x_2 > \tilde{x}$, firm 1 will optimally locate to the left of firm 2. Otherwise, if $x_2 < \tilde{x}$, firm 1 chooses an optimal characteristic to the right of firm 2. Due to the symmetry of the model, the same line of arguments apply to firm 2. Hence, given firms' equilibrium product characteristics from the first part of the proof, it holds that $x_1^* < \tilde{x} < x_2^*$. That is, neither firm has an incentive to jump over its opponent, which completes our proof. □

Before starting with the proofs of the comparative static analysis, we want to point out that for many of the estimations performed in the subsequent five proofs, we make use of the following intrinsic parameter restrictions:

- upper and lower support boundaries for M: $0 < L \leq \frac{1}{2}$
- upper and lower bound of the confidence parameter: $0 \leq \delta \leq 1$
- upper and lower bound of ambiguity attitude: $0 \leq \alpha \leq 1$
- upper and lower bound of the variance of M: $0 \leq \sigma^2 \leq L^2 \leq \frac{1}{4}$
- upper and lower bound of the transportation cost parameter: $0 < \underline{t} \leq 1 \leq \bar{t}$

Proof of Proposition 4.2. The derivative of x_1^* with respect to α is given by

$$\frac{\partial x_1^*}{\partial \alpha} = -\frac{\delta(2L-3)\underline{t}(2\delta L(2L+3)\overline{t}-(\delta-1)(3L+2\sigma^2))+(\delta-1)\delta(2L+3)\overline{t}(3L-2\sigma^2)}{2(\delta((\alpha-1)(2L+3)\overline{t}+\alpha(2L-3)\underline{t}+3)-3)^2}$$

The denominator is positive. Therefore, the sign of the derivative is determined by its numerator. We analyze the sign of this expression in two steps. The first part of the numerator is

$$g_1 := -\delta(2L-3)\underline{t}(2\delta L(2L+3)\overline{t}+(1-\delta)(3L+2\sigma^2))$$

Due to the fact that $L < \frac{1}{2}$, one can infer that $2L-3 < 0$. Hence, one obtains $g_1 > 0$. The second part of the numerator is

$$g_2 := (1-\delta)\delta(2L+3)\overline{t}(3L-2\sigma^2)$$

Since $\sigma^2 < L^2 < L$, one can infer that

$$3L-2\sigma^2 > 3L-2L = L > 0.$$

Therefore, one has $g_2 > 0$ as well. This proves that $\frac{\partial x_1^*}{\partial \alpha} > 0$ and $\frac{\partial x_2^*}{\partial \alpha} = -\frac{\partial x_1^*}{\partial \alpha} < 0$. □

Proof of Proposition 4.3. The derivative of x_1^* with respect to σ^2 is given by

$$\frac{\partial x_1^*}{\partial \sigma^2} = \frac{1-\delta}{\delta((\alpha-1)(2L+3)\overline{t}+\alpha(2L-3)\underline{t}+3)-3}$$

The numerator is non-negative since $1-\delta \geq 0$ for $0 \leq \delta \geq 1$. It is strictly positive for $0 \leq \delta < 1$. For the denominator, observe that $\delta(\alpha-1)(2L+3)\overline{t} \leq 0$ and $\delta(\alpha(2L-3)\underline{t}+3) \leq 0$, since $L < \frac{1}{2}$. Hence, the denominator is smaller or equal -3 and therefore negative. Thus, $\frac{\partial x_1^*}{\partial \sigma^2} \leq 0$ and $\frac{\partial x_2^*}{\partial \sigma^2} = -\frac{\partial x_1^*}{\partial \sigma^2} \geq 0$. For $\delta = 1$ both x_1^* and x_2^* are independent of σ^2. Therefore $\frac{\partial x_2^*}{\partial \sigma^2} = \frac{\partial x_1^*}{\partial \sigma^2} = 0$. □

Proof of Proposition 4.4. We have

$$\frac{\partial x_1^*}{\partial \underline{t}} = \frac{\alpha\delta\left(2(\alpha-1)\delta L\left(4L^2-9\right)\overline{t}+(\delta-1)(2L-3)(3L+2\sigma^2)\right)}{2(\delta((\alpha-1)(2L+3)\overline{t}+\alpha(2L-3)\underline{t}+3)-3)^2}$$

It is obvious that the denominator is positive. Turning to the numerator, we can see that $4L^2 - 9 \leq 0$ since $L < \frac{1}{2}$. Therefore, we can conclude that

$$2\alpha\delta(\alpha-1)\delta L\left(4L^2-9\right)\bar{t} \geq 0.$$

Similarly, since $2L - 3 < 0$ and $\alpha - 1 \leq 0$, one can infer that

$$\alpha\delta(\delta-1)(2L-3)(3L+2\sigma^2) \geq 0.$$

Consequently, the numerator is positive and $\frac{\partial x_1^*}{\partial \underline{t}} > 0$. Since $x_2^* = -x_1^*$, it follows that $\frac{\partial x_2^*}{\partial \underline{t}} = -\frac{\partial x_1^*}{\partial \underline{t}} < 0$. The derivative of x_1^* with respect to $\bar{t}$ is given by

$$\frac{\partial x_1^*}{\partial \bar{t}} = -\frac{(\alpha-1)\delta(2L+3)(2\alpha\delta L(2L-3)\underline{t}+(\delta-1)(3L-2\sigma^2))}{2(\delta((\alpha-1)(2L+3)\bar{t}+\alpha(2L-3)\underline{t}+3)-3)^2}$$

Clearly, the denominator is positive. Turning to the numerator, observe that the factor $-(\alpha-1)\delta(2L+3)$ is positive. Moreover, since $L < \frac{1}{2}$, one can infer

$$2\alpha\delta L(2L-3)\underline{t} \leq 0.$$

As a next step, we can show that

$$3L - 2\sigma^2 \geq 3L - 2L^2 > 3L - 2L = L > 0.$$

This implies $(\delta-1)(3L-2\sigma^2)) \leq 0$. In total, the numerator is negative. Therefore, one has $\frac{\partial x_1^*}{\partial \underline{t}} < 0$ and $\frac{\partial x_2^*}{\partial \underline{t}} > 0$. □

Proof of Proposition 4.5. The derivative of x_1^* with respect to δ is given by

$$\frac{\partial x_1^*}{\partial \delta} = \frac{(\alpha-1)(2L+3)\bar{t}(3L-2\sigma^2)-\alpha(2L-3)\underline{t}(3L+2\sigma^2)}{2(\delta((\alpha-1)(2L+3)\bar{t}+\alpha(2L-3)\underline{t}+3)-3)^2}$$

It is straightforward to see that the denominator is positive. The first part of the numerator is given by

$$g_3 := (\alpha - 1)(2L + 3)\bar{t}(3L - 2\sigma^2).$$

Since

$$3L - 2\sigma^2 \geq 3L - 2L = L > 0,$$

one can infer that $g_3 \leq 0$. Defining

$$g_4 := -\alpha(2L - 3)\underline{t}(3L + 2\sigma^2),$$

one obtains by $2L - 3 < 0$ that $g_4 \geq 0$. As a consequence, one can infer that $\frac{\partial x_1^*}{\partial \delta} > 0$ if $g_3 + g_4 > 0$ and $\frac{\partial x_1^*}{\partial \delta} < 0$ if $g_3 + g_4 < 0$. Moreover, one has $\frac{\partial x_1^*}{\partial \delta} = 0$ if $g_3 + g_4 = 0$. Solving the equation $-g_3 = g_4$ for α, one obtains the unique solution

$$\alpha^* := \frac{(2L + 3)(3L - 2\sigma^2)}{(2L + 3)\bar{t}(3L - 2\sigma^2) - (2L - 3)\underline{t}(3L + 2\sigma^2)}$$

Besides, one can see that $g_3 + g_4 > 0$ whenever $\alpha > \alpha^*$ and $g_3 + g_4 < 0$ whenever $\alpha < \alpha^*$. This establishes that the numerator has, for every parameter constellation, a unique zero α^* where $\frac{\partial x_1^*}{\partial \delta} < 0$ for all $0 \leq \alpha < \alpha^*$, $\frac{\partial x_1^*}{\partial \delta} = 0$ for $\alpha = \alpha^*$ and $\frac{\partial x_1^*}{\partial \delta} > 0$ for all $1 \geq \alpha > \alpha^*$. Since $x_2^* = -x_1^*$, we obtain the postulated result for x_2^* without reexamining the respective derivative. □

Proof of Proposition 4.6. The derivative of ${x_1}^*$ with respect to L is given by

$$\begin{aligned}\frac{\partial x_1^*}{\partial L} = &- \delta((\alpha - 1)\bar{t}((\delta - 1)(12L - 4\sigma^2 + 9) - 24\alpha\delta L\underline{t})\\ &+ \alpha\underline{t}(-\alpha\delta(3 - 2L)^2\underline{t} - (\delta - 1)(12L + 4\sigma^2 - 9)) + (\alpha - 1)^2\delta(2L + 3)^2\bar{t}^2)\\ &\cdot (2(\delta((\alpha - 1)(2L + 3)\bar{t} + \alpha(2L - 3)\underline{t} + 3) - 3)^2)^{-1}\end{aligned}$$

As we can see, the denominator is positive. Therefore the sign of the derivative solely depends on the numerator. Since $\delta \geq 0$ it is sufficient to consider the sign of numerator divided by δ. We denote this expression with $(*)$. Inserting $\alpha = 0$ into expression $(*)$

yields

$$\begin{aligned}
&\bar{t}\left((\delta-1)(12L-4\sigma^2+9)-\delta(2L+3)^2\bar{t}\right)\\
&\leq \delta[-12L+4\sigma^2-9]\\
&\leq \delta[-12L-8]\\
&=-4\delta(3L+2)\\
&<0
\end{aligned}$$

This shows that the derivative is strictly negative for $\alpha = 0$. Similarly, inserting $\alpha = 1$ into $(*)$, we obtain

$$\underline{t}\left((\delta-1)(12L+4\sigma^2-9)+\delta(3-2L)^2\underline{t}\right) \tag{4.10}$$

We establish that expression (4.10) is strictly positive. It is

$$12L+4\sigma^2-9 \leq 6+1-9=-2<0.$$

As a consequence, we obtain

$$(\delta-1)(12L+4\sigma^2-9) \geq 0.$$

This shows that the numerator is positive. Now, we demonstrated that $\frac{\partial x_1^*}{\partial L} < 0$ for $\alpha = 0$, and $\frac{\partial x_1^*}{\partial L} > 0$ for $\alpha = 1$. The derivative is continuous. By the intermediate value theorem for continuous functions, we obtain that there is $\hat{\alpha} \in (0, 1)$ such that $\frac{\partial x_1^*}{\partial L} = 0$ for $\alpha = \hat{\alpha}$. What remains to be shown is that $\hat{\alpha}$ is unique. In this case, we know that x_1^* is strictly decreasing in L for values of α smaller that $\hat{\alpha}$, constant for $\alpha = \hat{\alpha}$, and increasing for $1 \geq \alpha > \hat{\alpha}$. Solving expression $(*)$ for α, we know that we find at least one zero, the zero $\hat{\alpha}$ in the interval $[0, 1]$. Since $(*)$ is a quadratic function in α, we can conclude that it has one more root $\hat{\hat{\alpha}}$. This root cannot be located in the interval $[0, 1]$ as well. This we show by making use of a proof by contradiction. Assume, w.l.o.g., that $\hat{\hat{\alpha}}$ was in the interval $[0, 1]$ as well and that $\hat{\alpha} < \hat{\hat{\alpha}}$. We can distinguish two cases. Case 1 is that the quadratic function has a global maximum, and case 2 is that the quadratic function has a global minimum. Since we can find both roots in the interval $[0, 1]$, the global maximum, or alternatively the global minimum, are also located in this interval. Assume now that we have a quadratic function with a global maximum. In this case, we have that $(*)$ is smaller zero for $\alpha < \hat{\alpha}$, equal to zero for $\alpha \in \{\hat{\alpha}, \hat{\hat{\alpha}}\}$, and smaller zero for $\alpha \in (\hat{\hat{\alpha}}, 1]$. The last statement contradicts that $(*)$ is larger zero for $\alpha = 1$ what we already showed above. For a global minimum a similar line of arguments holds. Since both roots are located in the interval $[0, 1]$, we can deduce that the minimum is located in this interval as well. In this case we can conclude that $(*)$ is larger than zero for $\alpha < \hat{\alpha}$, equal to zero for $\alpha \in \{\hat{\alpha}, \hat{\hat{\alpha}}\}$, and again larger zero for $\alpha \in (\hat{\hat{\alpha}}, 1]$. The first statement contradicts that $(*)$ is smaller zero for $\alpha = 0$. To sum up, we have only one root in $[0, 1]$. $\square$

"1. Define Objectives for Firm 1 and Firm 2";

```
f1[x1_, x2_, alpha_, delta_, tlow_, thigh_, sigma_, L_] :=
  delta * ((2 * thigh * (1 - alpha) * (x2 - x1) * (L + (x2 + x1) / 2 + 3 / 2)^2) / 9 +
      (2 * alpha * tlow * (x2 - x1) * (-L + (x2 + x1) / 2 + 3 / 2)^2) / 9) +
   ((1 - delta) * (x2 - x1) * ((x2 + x1 + 3)^2 + 4 * sigma)) / 18;

f2[x1_, x2_, alpha_, delta_, tlow_, thigh_, sigma_, L_] :=
  delta * ((2 * alpha * tlow * (x2 - x1) * (L + (x2 + x1) / 2 - 3 / 2)^2) / 9 +
      (2 * (1 - alpha) * thigh * (x2 - x1) * (-L + (x2 + x1) / 2 - 3 / 2)^2) / 9) +
   ((1 - delta) * (x2 - x1) * ((x2 + x1 - 3)^2 + 4 * sigma)) / 18;
```

"2. Introduce Parameter Restrictions";

```
assumptions = And[L == 1 / 2, 0 ≤ alpha ≤ 1,
   0 < tlow <= 1, 0 <= delta ≤ 1, 0 <= sigma ≤ L^2, thigh ≥ 1];
```

"3. Define the Midpoint Between Both Firms";

```
mean = (x1 + x2) / 2;
```

"4. Solving for Mutual Best Responses";

```
solutions =
  FullSimplify[Solve[{D[f1[x1, x2, alpha, delta, tlow, thigh, sigma, L], x1] == 0,
     D[f2[x1, x2, alpha, delta, tlow, thigh, sigma, L], x2] == 0}, {x1, x2}]];
```

"5. Store Solutions in a Table";

```
TableForm[Table[{solutions[[i, 1, 2]], solutions[[i, 2, 2]]},
   {i, Length[solutions]}], TableHeadings → {{"1", "2", "3"}, {"x1", "x2"}},
  TableAlignments → Center, TableSpacing → {3, 4}];
```

"6. Verify Whether Solution Satisfies the Global Competition Condition";

```
Table[{i, FullSimplify[(And[-3 / 2 < -L - mean, L - mean < 3 / 2] /. solutions[[i]]),
   assumptions]}, {i, Length[solutions]}]
```

```
{{1, True}, {2, 2 + 2 delta (-1 + thigh - alpha thigh + alpha tlow) >
    √(9 - 4 sigma + delta (-18 - 23 (-1 + alpha) thigh + 11 alpha tlow +
          4 sigma (2 + (-1 + alpha) thigh - alpha tlow) + delta (9 + 16 (-1 + alpha)^2
              thigh^2 + 4 sigma (-1 + thigh - alpha thigh + alpha tlow) + alpha tlow
              (-11 + 4 alpha tlow) - (-1 + alpha) thigh (-23 + 12 alpha tlow)))) &&
   2 delta (1 + (-1 + alpha) thigh - alpha tlow) <
    2 +
     √(9 - 4 sigma + delta (-18 - 23 (-1 + alpha) thigh + 11 alpha tlow +
           4 sigma (2 + (-1 + alpha) thigh - alpha tlow) + delta (9 + 16 (-1 + alpha)^2
               thigh^2 + 4 sigma (-1 + thigh - alpha thigh + alpha tlow) + alpha tlow
               (-11 + 4 alpha tlow) - (-1 + alpha) thigh (-23 + 12 alpha tlow))))},
 {3, 2 delta (1 + (-1 + alpha) thigh - alpha tlow) <
    2 +
     √(9 - 4 sigma + delta (-18 - 23 (-1 + alpha) thigh + 11 alpha tlow +
           4 sigma (2 + (-1 + alpha) thigh - alpha tlow) + delta (9 + 16 (-1 + alpha)^2
               thigh^2 + 4 sigma (-1 + thigh - alpha thigh + alpha tlow) + alpha tlow
               (-11 + 4 alpha tlow) - (-1 + alpha) thigh (-23 + 12 alpha tlow)))) &&
   2 delta (1 + (-1 + alpha) thigh - alpha tlow) + √(9 - 4 sigma +
        delta (-18 - 23 (-1 + alpha) thigh + 11 alpha tlow + 4 sigma
            (2 + (-1 + alpha) thigh - alpha tlow) + delta (9 + 16 (-1 + alpha)^2 thigh^2 +
              4 sigma (-1 + thigh - alpha thigh + alpha tlow) + alpha tlow (-11 +
                 4 alpha tlow) - (-1 + alpha) thigh (-23 + 12 alpha tlow)))) < 2}}
```

"7. Verify That the First Pair of Solutions Satisfies the Global Competition Condition";

```
x1st = (9 + 4 sigma +
     delta (-9 - 4 sigma - (-1 + alpha) (3 + 2 L)^2 thigh + alpha (3 - 2 L)^2 tlow)) /
   (4 (-3 + delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow)));
```

```
x2st = -x1st;
```

```
Simplify[x2st > 3 / 2 - L, assumptions]
```

```
3 delta + 4 sigma > 3 + 4 delta (sigma + alpha tlow)
```

```
Reduce[{3 delta + 4 sigma > 3 + 4 delta (sigma + alpha tlow), assumptions},
 {alpha, delta, sigma, tlow}]
```

```
False
```

"8. Define the Mean for the Second and Third Pair of Solutions";

```
mean2 =
  ((3 + delta (-3 - (-1 + alpha) (3 + 2 L) thigh + alpha (3 - 2 L) tlow) - √(9 - 4 sigma +
            delta (4 (-1 + alpha) ((-3 + L) L + sigma) thigh + 2 (-9 + 4 sigma -
                    9 (-1 + alpha) thigh) - 2 alpha (-9 + 2 L (3 + L) + 2 sigma) tlow +
                delta (9 + (-1 + alpha)^2 (3 + 2 L)^2 thigh^2 + 4 sigma
                    (-1 + thigh - alpha thigh + alpha tlow) + alpha tlow
                    (9 (-2 + alpha tlow) + 4 L (3 + L + alpha (-3 + L) tlow)) + 2 (-1 +
                      alpha) thigh (9 - 9 alpha tlow + 2 L (3 - L + 6 alpha L tlow)))))) /
       (2 (-1 + delta (1 + (-1 + alpha) thigh - alpha tlow))) -
      ((3 - delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow) +
          √(9 - 4 sigma + delta (4 (-1 + alpha) ((-3 + L) L + sigma)
                  thigh + 2 (-9 + 4 sigma - 9 (-1 + alpha) thigh) - 2 alpha
                  (-9 + 2 L (3 + L) + 2 sigma) tlow + delta (9 + (-1 + alpha)^2 (3 + 2 L)^2
                      thigh^2 + 4 sigma (-1 + thigh - alpha thigh + alpha tlow) + alpha
                      tlow (9 (-2 + alpha tlow) + 4 L (3 + L + alpha (-3 + L) tlow)) +
                    2 (-1 + alpha) thigh (9 - 9 alpha tlow + 2 L
                        (3 - L + 6 alpha L tlow)))))) /
        (2 (-1 + delta (1 + (-1 + alpha) thigh - alpha tlow))))) / 2;

mean3 =
  ((3 + delta (-3 - (-1 + alpha) (3 + 2 L) thigh + alpha (3 - 2 L) tlow) + √(9 - 4 sigma +
            delta (4 (-1 + alpha) ((-3 + L) L + sigma) thigh + 2 (-9 + 4 sigma -
                    9 (-1 + alpha) thigh) - 2 alpha (-9 + 2 L (3 + L) + 2 sigma) tlow +
                delta (9 + (-1 + alpha)^2 (3 + 2 L)^2 thigh^2 + 4 sigma
                    (-1 + thigh - alpha thigh + alpha tlow) + alpha tlow
                    (9 (-2 + alpha tlow) + 4 L (3 + L + alpha (-3 + L) tlow)) + 2 (-1 +
                      alpha) thigh (9 - 9 alpha tlow + 2 L (3 - L + 6 alpha L tlow)))))) /
       (2 (-1 + delta (1 + (-1 + alpha) thigh - alpha tlow))) +
      (-3 + delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow) +
          √(9 - 4 sigma + delta (4 (-1 + alpha) ((-3 + L) L + sigma) thigh +
                2 (-9 + 4 sigma - 9 (-1 + alpha) thigh) - 2 alpha (-9 + 2 L (3 + L) +
                    2 sigma) tlow + delta (9 + (-1 + alpha)^2 (3 + 2 L)^2 thigh^2 +
                    4 sigma (-1 + thigh - alpha thigh + alpha tlow) + alpha tlow
                    (9 (-2 + alpha tlow) + 4 L (3 + L + alpha (-3 + L) tlow)) + 2 (-1 +
                      alpha) thigh (9 - 9 alpha tlow + 2 L (3 - L + 6 alpha L tlow)))))) /
       (2 (-1 + delta (1 + (-1 + alpha) thigh - alpha tlow)))) / 2;
```

"9. Determine the Mean's Range for the Second and Third Pair of Solutions";

```
NMinimize[{mean2, 0 <= L ≤ 1 / 2, 0 ≤ alpha ≤ 1, 0 <= tlow <= 1, 0 <= delta ≤ 1,
  0 <= sigma ≤ L^2, thigh ≥ 1}, {alpha, delta, tlow, thigh, sigma, L}]
```

{1., {alpha → 1., delta → 1., tlow → 0.878054, thigh → 1., sigma → 0.139562, L → 0.5}}

```
NMaximize[{mean2, 0 <= L ≤ 1 / 2, 0 ≤ alpha ≤ 1, 0 <= tlow <= 1, 0 <= delta ≤ 1,
  0 <= sigma ≤ L^2, thigh ≥ 1}, {alpha, delta, tlow, thigh, sigma, L}]
```

{2., {alpha → 2.6438×10^{-9}, delta → 1.,
  tlow → 0.0272573, thigh → 2.23364, sigma → 0.234253, L → 0.5}}

```
NMinimize[{mean3, 0 <= L ≤ 1 / 2, 0 ≤ alpha ≤ 1, 0 <= tlow <= 1, 0 <= delta ≤ 1,
  0 <= sigma ≤ L^2, thigh ≥ 1}, {alpha, delta, tlow, thigh, sigma, L}]
```

$\{-2., \{\text{alpha} \to 2.6438 \times 10^{-9}, \text{delta} \to 1.,$
$\text{tlow} \to 0.0272573, \text{thigh} \to 2.23364, \text{sigma} \to 0.234253, \text{L} \to 0.5\}\}$

```
NMaximize[{mean3, 0 <= L ≤ 1 / 2, 0 ≤ alpha ≤ 1, 0 <= tlow <= 1, 0 <= delta ≤ 1,
  0 <= sigma ≤ L^2, thigh ≥ 1}, {alpha, delta, tlow, thigh, sigma, L}]
```

$\{-1., \{\text{alpha} \to 1., \text{delta} \to 1., \text{tlow} \to 0.365717,$
$\text{thigh} \to 1.79008, \text{sigma} \to 1.80912 \times 10^{-22}, \text{L} \to 0.5\}\}$

"10. Second-Order Derivative Firm 1";

```
FullSimplify[ D[f1[x1, x2, alpha, delta, tlow, thigh, sigma, L], {x1, 2}]]
```

$\frac{1}{9}(-6-3\,x1-x2+\text{delta}\,(6+3\,x1+\text{alpha tlow}\,(-6+4\,L-3\,x1-x2)+x2+(-1+\text{alpha})\,\text{thigh}\,(6+4\,L+3\,x1+x2)))$

```
Secondorderderivative1[x1_, x2_, alpha_, delta_, tlow_, thigh_, sigma_, L_] :=
  1/9 (-6 - 3 x1 - x2 + delta (6 + 3 x1 + alpha tlow (-6 + 4 L - 3 x1 - x2) +
        x2 + (-1 + alpha) thigh (6 + 4 L + 3 x1 + x2)));
```

"11. Second-Order Derivative Firm 2";

```
FullSimplify[D[f2[x1, x2, alpha, delta, tlow, thigh, sigma, L], {x2, 2}]]
```

$\frac{1}{9}(-6+x1+3\,x2+\text{delta}\,(6-x1+(-1+\text{alpha})\,\text{thigh}\,(6+4\,L-x1-3\,x2)-3\,x2+\text{alpha tlow}\,(-6+4\,L+x1+3\,x2)))$

```
Secondorderderivative2[x1_, x2_, alpha_, delta_, tlow_, thigh_, sigma_, L_] :=
 1/9 (-6 + x1 + 3 x2 + delta (6 - x1 + (-1 + alpha) thigh (6 + 4 L - x1 - 3 x2) -
       3 x2 + alpha tlow (-6 + 4 L + x1 + 3 x2)))
```

"12. Second-Order Derivatives Evaluated at Equilibrium Candidate Positions";

```
FullSimplify[Secondorderderivative1[(9 + 4 sigma +
      delta (-9 - 4 sigma - (-1 + alpha) (3 + 2 L)^2 thigh + alpha (3 - 2 L)^2 tlow)) /
    (4 (-3 + delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow))), (-9 -
      4 sigma + delta (9 + 4 sigma + (-1 + alpha) (3 + 2 L)^2 thigh - alpha (3 - 2 L)^2 tlow)) /
    (4 (-3 + delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow))),
  alpha, delta, tlow, thigh, sigma, L]]
```

```
(27 - 4 sigma + delta
     (8 sigma + 4 (-1 + alpha) ((-9 + L) L + sigma) thigh - 54 (1 + (-1 + alpha) thigh) -
       2 alpha (-27 + 2 L (9 + L) + 2 sigma) tlow + delta (27 + 3 (-1 + alpha)^2
            (3 + 2 L)^2 thigh^2 + 4 sigma (-1 + thigh - alpha thigh + alpha tlow) +
          alpha tlow (-54 + 4 L (9 + L) + 3 alpha (3 - 2 L)^2 tlow) +
          2 (-1 + alpha) thigh (27 - 27 alpha tlow + 2 L (9 - L + 10 alpha L tlow))))) /
 (18 (-3 + delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow)))
```

```
FullSimplify[Secondorderderivative2[(9 + 4 sigma +
      delta (-9 - 4 sigma - (-1 + alpha) (3 + 2 L)^2 thigh + alpha (3 - 2 L)^2 tlow)) /
    (4 (-3 + delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow))), (-9 -
      4 sigma + delta (9 + 4 sigma + (-1 + alpha) (3 + 2 L)^2 thigh - alpha (3 - 2 L)^2 tlow)) /
    (4 (-3 + delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow))),
  alpha, delta, tlow, thigh, sigma, L]]
```

```
(27 - 4 sigma + delta
     (8 sigma + 4 (-1 + alpha) ((-9 + L) L + sigma) thigh - 54 (1 + (-1 + alpha) thigh) -
       2 alpha (-27 + 2 L (9 + L) + 2 sigma) tlow + delta (27 + 3 (-1 + alpha)^2
            (3 + 2 L)^2 thigh^2 + 4 sigma (-1 + thigh - alpha thigh + alpha tlow) +
          alpha tlow (-54 + 4 L (9 + L) + 3 alpha (3 - 2 L)^2 tlow) +
          2 (-1 + alpha) thigh (27 - 27 alpha tlow + 2 L (9 - L + 10 alpha L tlow))))) /
 (18 (-3 + delta (3 + (-1 + alpha) (3 + 2 L) thigh + alpha (-3 + 2 L) tlow)))
```

"13. Take the Derivative of the Numerator With Respect to Thigh";

```
FullSimplify[
 D[(27 - 4 sigma + delta (8 sigma + 4 (-1 + alpha) ((-9 + L) L + sigma) thigh - 54
          (1 + (-1 + alpha) thigh) - 2 alpha (-27 + 2 L (9 + L) + 2 sigma) tlow +
        delta (27 + 3 (-1 + alpha)^2 (3 + 2 L)^2 thigh^2 +
            4 sigma (-1 + thigh - alpha thigh + alpha tlow) +
            alpha tlow (-54 + 4 L (9 + L) + 3 alpha (3 - 2 L)^2 tlow) + 2 (-1 + alpha)
             thigh (27 - 27 alpha tlow + 2 L (9 - L + 10 alpha L tlow))))), thigh]]
```

```
2 (-1 + alpha) delta
 (-27 + 2 ((-9 + L) L + sigma) + delta (27 - 2 sigma + 3 (-1 + alpha) (3 + 2 L)^2 thigh -
       27 alpha tlow + 2 L (9 + L (-1 + 10 alpha tlow))))
```

"14. Check Whether the Derivative Can be Negative";

```
assumptions = And[0 <= L ≤ 1 / 2, 0 ≤ alpha ≤ 1,
   0 < tlow ≤ 1, 0 <= delta ≤ 1, 0 <= sigma ≤ L^2, thigh >= 1];
```

```
Reduce[
 {2 (-1 + alpha) delta (-27 + 2 ((-9 + L) L + sigma) + delta (27 - 2 sigma + 3 (-1 + alpha)
        (3 + 2 L)^2 thigh - 27 alpha tlow + 2 L (9 + L (-1 + 10 alpha tlow)))) < 0,
  assumptions}, {alpha, delta, sigma, L, tlow, thigh}]
False
```

"15. Evaluate the Numerator of the Second-Order Derivative at Thigh=1";

```
Num[alpha_, delta_, tlow_, thigh_, sigma_, L_] :=
  (27 - 4 sigma + delta (8 sigma + 4 (-1 + alpha) ((-9 + L) L + sigma) thigh -
      54 (1 + (-1 + alpha) thigh) - 2 alpha (-27 + 2 L (9 + L) + 2 sigma) tlow +
      delta (27 + 3 (-1 + alpha)^2 (3 + 2 L)^2 thigh^2 + 4 sigma (-1 + thigh - alpha thigh +
            alpha tlow) + alpha tlow (-54 + 4 L (9 + L) + 3 alpha (3 - 2 L)^2 tlow) +
          2 (-1 + alpha) thigh (27 - 27 alpha tlow + 2 L (9 - L + 10 alpha L tlow)))));

FullSimplify[Num[alpha, delta, tlow, 1, sigma, L]]

27 - 4 sigma + delta (4 (L (9 + (-1 + 4 delta) L) + sigma) +
    alpha^2 delta (27 (-1 + tlow)^2 + 4 L^2 (3 + tlow) (1 + 3 tlow) - 36 L (-1 + tlow^2)) -
    2 alpha (-(27 + 2 (-1 + delta) sigma) (-1 + tlow) +
       18 L (1 + delta + tlow - delta tlow) + 2 L^2 (-1 + tlow + delta (7 + 9 tlow))))

27 - 4 sigma + delta (4 (L (9 + (-1 + 4 delta) L) + sigma) +
    alpha^2 delta (27 (-1 + tlow)^2 + 4 L^2 (3 + tlow) (1 + 3 tlow) - 36 L (-1 + tlow^2)) -
    2 alpha (-(27 + 2 (-1 + delta) sigma) (-1 + tlow) +
       18 L (1 + delta + tlow - delta tlow) + 2 L^2 (-1 + tlow + delta (7 + 9 tlow))))

27 - 4 sigma + delta (4 (L (9 + (-1 + 4 delta) L) + sigma) +
    alpha^2 delta (27 (-1 + tlow)^2 + 4 L^2 (3 + tlow) (1 + 3 tlow) - 36 L (-1 + tlow^2)) -
    2 alpha ((-27 - 2 (-1 + delta) sigma) (-1 + tlow) +
       18 L (1 + delta + tlow - delta tlow) + 2 L^2 (-1 + tlow + delta (7 + 9 tlow))))
```

```
"16. Can The Numerator
   Evaluated at Thigh=1 Become Negative?";

Reduce[{27 - 4 sigma + delta (4 (L (9 + (-1 + 4 delta) L) + sigma) +
        alpha^2 delta (27 (-1 + tlow)^2 + 4 L^2 (3 + tlow) (1 + 3 tlow) - 36 L (-1 + tlow^2)) -
        2 alpha (-(27 + 2 (-1 + delta) sigma) (-1 + tlow) +
           18 L (1 + delta + tlow - delta tlow) + 2 L^2 (-1 + tlow + delta (7 + 9 tlow)))) <
    0, 0 <= L ≤ 1 / 2, 0 ≤ alpha ≤ 1, 0 <= tlow ≤ 1, 0 <= delta ≤ 1,
   0 <= sigma ≤ L^2}, {alpha, delta,
   tlow,
   sigma,
   L}]

False
```

Chapter 5

Strategic behavior in games with payoff uncertainty

This Chapter, which is based on Kauffeldt (2016), investigates the extent to which we can distinguish between expected and non-expected utility players on the basis of their behavior. A model of incomplete information games is used in which players can choose mixed strategies.

Most real-life strategic situations involve incomplete information. To analyze games of incomplete information, Harsanyi (1967-68) introduced the concept of Bayesian games. One key assumption of Harsanyi's approach is that players are Bayesian expected utility maximizers and share a prior distribution over the state space (see Chapter 3). However, as exemplified by Ellsberg (1961), individuals frequently violate the expected utility (or, briefly, EU) hypothesis when they face ambiguity (see Chapter 2).

A number of models of incomplete information games with non-EU players have been proposed (see the literature below). However, to my knowledge, it has not been systematically investigated whether, and if so, under which conditions, these models predict behavior that differs from the behavior predicted by models with EU players. Such an investigation is useful not only for gaining theoretical insights, but also as a guide to design experiments testing one model against another.

In this chapter, I offer a first attempt at systematically examining the question of when one can distinguish EU from non-EU players in the context of an increasingly used model. The key assumption of this model is that players behave as expected utility maximizers with correct beliefs concerning mixed strategy combinations but face ambiguity[24] about the environment. Comparable models were first introduced by Bade (2011a) and Azrieli and Teper (2011). Therefore, I shall occasionally refer to this model as the "Bade-Azrieli-Teper-model" (or, briefly, BAT-model). For instance, the applications described later on in this section are based on BAT-type models.

Unfortunately, ambiguity-neutral and ambiguity-averse players sometimes behave observationally equivalent, which means that it is impossible to identify non-EU players by observing equilibrium actions. I show that EU and non-EU players can be distinguished from each other by looking at their best-response correspondences. More precisely, the strategic behavior of uncertainty-averse non-EU players can differ substantially from the behavior of EU players in both: the use (*hedging behavior*) and the response (*reversal behavior*) to mixed strategies. From a decision theory perspective, both characteristics are due to the same cause, namely a preference for mixtures. However, from a game theory perspective, it does matter whether a player prefers to randomize over her strategies or whether she exhibits a preference for mixed strategy combinations of the opponents.

In the formal analysis, attention is restricted to games with payoff uncertainty but without private information. However, the results can also be applied to incomplete information games with private information (see Remark 5.2). The first main theorem shows that EU and non-EU players behave observationally equivalent whenever the non-EU players do not exhibit hedging behavior. In other words, the absence of hedging behavior is sufficient for observational equivalence. That is, we need hedging behavior if we want to distinguish EU from non-EU players by observing equilibrium actions. The second main theorem shows that non-EU players behave as if they were EU players if and only if they do not exhibit hedging and reversal behavior. To put it differently, hedging and/or reversal behavior are necessary and sufficient for the existence of behavioral differences between EU and non-EU players, which means that there are no other behavioral

[24]The main theorems also apply to non-EU players who are probabilistically sophisticated in the sense of Machina and Schmeidler (1992), but this chapter focuses on players with non-probabilistic beliefs.

differences. The propositions provide necessary and sufficient conditions for the existence of hedging and reversal behavior in terms of the payoff structure of a game. However, for tractability reasons, we consider here only two-person two-strategies games and players with maxmin expected utility preferences (see Chapter 2). In laboratory experiments, both restrictions are frequently satisfied.

The majority of the literature on games played by non-EU players has focused on games with complete information in which players face only strategic uncertainty (see Chapter 3). There is a relatively small, but growing, literature on incomplete information games played by non-EU players. Epstein and Wang (1996) offer a general framework that provides a foundation for a "type-space" approach à la Harsanyi with non-EU players. Eichberger and Kelsey (2004) generalize perfect Bayesian equilibrium for the case of two-person games where players have non-additive beliefs. In their Dempster-Shafer equilibrium, players maximize Choquet expected utility. Kajii and Ui (2005) investigate a model in which all players have maxmin expected utility preferences. Their model differs from Bayesian games in that it does not assume a common prior over the states. Instead, there is a set of priors for each player, which may vary among players. Bade (2011a) and Azrieli and Teper (2011) consider more general preferences. Their models assume that players choose mixtures as their strategies, and there is no ambiguity about the probabilities of mixed strategies, i.e., no strategic ambiguity. However, players face ambiguity about the environment. The papers differ in that Bade (2011a) requires payoffs to be state-independent, and in that Azrieli and Teper (2011) do not rule out correlation devices and diverging beliefs.

There is an increasing number of papers on applications of incomplete information games that use BAT-type models. These papers examine games with payoff ambiguity but without private information. For instance, Bade (2011b) studies electoral competition between two parties in a two-stage game by assuming that parties are uncertain about voters' marginal rates of substitution between issues. Król (2012) investigates ambiguous demand in the context of a two-stage product-type-then-price competition game. Aflaki (2013) examines the tragedy of the commons in which players face ambiguity concerning the size of the resource endowment. Bade (2011a) and Król (2012) use maxmin expected utility preferences. Aflaki (2013) additionally considers Choquet expected utility, and smooth ambiguity preferences introduced by Klibanoff et al. (2005). Another example is the Hotelling model under demand ambiguity developed in the previous chapter.

This chapter is organized as follows. The following section gives two examples to illustrate the BAT-model, and the notions of hedging and reversal behavior. Furthermore, the BAT-model is described in detail. Section 5.2 provides the results. The subsequent section discusses the underlying model, and introduces a generalized equilibrium concept. Finally, Section 5.4 concludes with a summary of the main results. Unless noted otherwise, the proofs of the results are given in Section 5.5.

5.1 Preliminaries

5.1.1 Two examples

In the following, we consider two examples.[25] These examples illustrate the BAT-model and our notions of hedging and reversal behavior. In addition, they show potential economic applications of the BAT-model.

Example 5.1 (Discrete Cournot duopoly with uncertain demand)**.** *There are two firms, $i \in \{1, 2\}$, which produce a homogeneous product. The firms compete in quantities, and decide simultaneously whether to produce a low quantity normalized to one, $q_l = 1$, or a high quantity, $q_h = 2$. Marginal costs of production are constant and normalized to one. The market price, p, depends on the total quantity in the industry, Q, and on an uncertain state of the world, $\omega \in \{\omega_1, \omega_2\}$: $p = A(\omega) - b(\omega) \cdot Q$, where $(A, b)(\omega_1) = (6, \frac{3}{2})$ and $(A, b)(\omega_2) = (2, 0)$. When choosing whether to produce q_l or q_h, neither firm knows the state of the world. Firms' state-dependent profits are:*

[25]The equilibria for the games in the examples and the formal derivation of players' best-response correspondences are given in Section 5.5.

ω_1:

	q_l	q_h
q_l	$2, 2$	$\frac{1}{2}, 1$
q_h	$1, \frac{1}{2}$	$-2, -2$

ω_2:

	q_l	q_h
q_l	$1, 1$	$1, 2$
q_h	$2, 1$	$2, 2$

Since firms' profits depend on a state of nature, every pure strategy profile induces a state-contingent vector of profits for both firms. For instance, the strategy profile (q_l, q_l) induces the vector $f_i(q_l, q_l) = (f_i^{\omega_1}(q_l, q_l), f_i^{\omega_2}(q_l, q_l)) = (2, 1)$ for each firm i. Every mixed strategy profile generates a probability distribution over pure strategy profiles. In a given state ω, each firm's payoff from a mixed profile corresponds to its expected profit with respect to this distribution. Hence, every mixed profile induces state-contingent vectors of expected profits. For example, suppose firm 2 (column) plays q_l with 1/4 probability (and q_h with 3/4 probability). Denote firm 2's strategy by $\sigma_2(q_l) = \frac{1}{4}$.[26] If firm 1 (row) plays q_l with certainty (i.e., the strategy $\sigma_1(q_l) = 1$), then its vector induced by the mixed profile $(\sigma_1(q_l), \sigma_2(q_l)) = (1, \frac{1}{4})$ equals $f_1(1, \frac{1}{4}) = \frac{1}{4} f_1(q_l, q_l) + \frac{3}{4} f_1(q_l, q_h) = (\frac{7}{8}, 1)$.

Now, assume that each firm $i \in \{1, 2\}$ has the following non-EU preferences over state-contingent (expected) profits: $V_i(f_i) = \min\{f_i^{\omega_1}, f_i^{\omega_2}\}$. Then, firm i's best-response correspondence, R_i, takes the form illustrated in Figure 5.1.

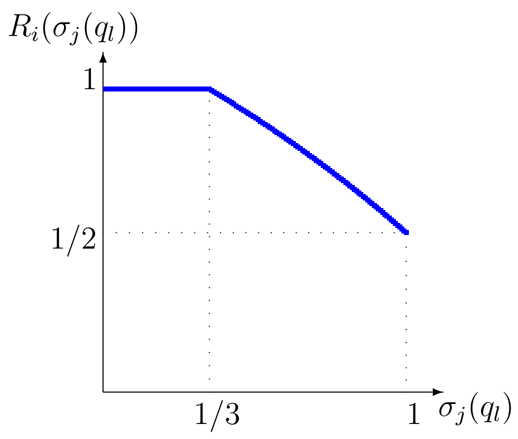


Figure 5.1: Best-response correspondence of a firm in Example 5.1

[26] In this section, mixed strategies are denoted by their first component, $\sigma_j(q_l) = (\sigma_j(q_l), \sigma_j(q_h))$, since $\sigma_j(q_l) = 1 - \sigma_j(q_h)$.

As Figure 5.1 shows, firm i has a unique best response to all strategies of firm j. Furthermore, its unique best response is a mixed strategy if j plays q_l with more than 1/3 probability. EU players would never show this type of strategic behavior. They use mixed strategies to make the other players indifferent between playing their pure strategies, for instance, like in matching pennies-type games, to avoid exploitation by their opponents. However, for an EU player, mixed strategies are always weakly optimal: if a mixed strategy is a best response to some strategy profile of the other players, then, at the same time, all pure strategies to which it assigns positive probability are best responses. Consequently, mixed strategies are never unique best responses.

Why are non-EU players able to behave differently? The reason is that they randomize over their pure strategies not only for strategic purposes, but also as a kind of "hedging" against environmental uncertainty. In Example 5.1, q_l is a strictly dominant strategy in ω_1 and q_h in ω_2 for both firms. If firm j chooses a strategy $\sigma_j(q_l) \leq 1/3$, then firm i's expected profit in state ω_1 is lower than in ω_2, regardless of its strategy choice. Therefore, firm i will play its strictly dominant strategy in ω_1, $\sigma_i(q_l) = 1$. Otherwise, if $\sigma_j(q_l) > 1/3$, firm i seeks to smooth its expected profits across states by playing a mixed strategy. For instance, given $\sigma_j(q_l) = 1$, firm i will play q_l (and q_h) with 1/2 probability, which induces the vector $f_i\big(\frac{1}{2}, 1\big) = \big(\frac{3}{2}, \frac{3}{2}\big)$.

This is not new from a decision theory perspective. In an early reply to Ellsberg (1961), Raiffa (1961) claimed that ambiguous uncertainty can be eliminated by randomizing (see Example 2.2 and the subsequent discussion). Furthermore, Schmeidler (1989) defines uncertainty-aversion as a weak preference for randomization (see Axiom 3). More recently, Battigalli et al. (2013) study a framework of mixed extensions of decision problems under uncertainty that involves preference for randomization as an expression of uncertainty-aversion. They and other authors, e.g., Gilboa and Schmeidler (1989) and Saito (2013), use the term "hedging" to refer to situations in which decision-makers prefer randomized choices. Although some confusion may arise from the connotations of this traditional term,[27] I will follow this terminology and refer to a preference for playing mixed strategies

[27]For instance, it could be associated with "hedging" in finance, which refers to activities that reduce portfolio risk.

as "hedging behavior".[28]

In the game theory literature, only a few authors, e.g., Klibanoff (1996) and Lo (1996), explicitly discuss a preference for randomized strategies in the context of their models, which involve strategic ambiguity but no environmental uncertainty.

Example 5.2 (Uncertain investment). *There is an investor, I, with initial wealth 1 and a fund manager, M. The investor decides whether to invest her money in the fund, In, or keep it at the bank, Bk, with a guaranteed payoff of 1. The fund manager chooses an investment strategy: he can either speculate on falling or on rising share prices. For simplicity, suppose he can either buy one stock, S, or a put option on the stock, P. Initially, stock and put are worth 1. The future stock value $q^s(\omega)$ depends on an uncertain state of the world, $\omega \in \{\omega_1, \omega_2\}$, where $q^s(\omega_1) = 6$ and $q^s(\omega_2) = 0$. The strike price of the put is 6, hence, its future value is $q^p(\omega) = 6 - q^s(\omega)$. The fee for the fund manager is performance-based: he gets 1 if the investment is successful, otherwise 0. Players' state-dependent payoffs are:*

	S	P
Bk	$2, 0$	$2, 0$
In	$5, 1$	$0, 0$

ω_1

	S	P
Bk	$2, 0$	$2, 0$
In	$0, 0$	$5, 1$

ω_2

Again, suppose that each player $i \in \{I, M\}$ has the following non-EU preferences over state-contingent (expected) payoffs: $V_i(f_i) = \min\{f_i^{\omega_1}, f_i^{\omega_2}\}$. Then, the players' best-response correspondences, R_i, are given in Figure 5.2. The fund manager (left graph) has a weakly dominant mixed strategy: buying the stock and the put with 1/2 probability. The investor (right graph) has no preference for mixed strategies. However, she shows the second type of strategic behavior which differs from behavior of EU players: she prefers to keep her money at the bank if the investor buys the stock or the put with high probability. Otherwise, if the fund manager's action is sufficiently uncertain

[28] Klibanoff (2001) suggests the term "objectifying behavior". In my opinion, another suitable alternative is "Raiffa behavior" since he was was the first who pointed to this effect.

for her, she will invest in the fund. In other words, her preference for strategy Bk over In, given S or P, reverses for some mixtures of S and P. Therefore, I will refer to this type of behavior as "reversal behavior". In contrast, if an EU player prefers to play a particular strategy in response to two strategies of her opponent, she will still prefer to play this strategy in response to any mixture of the two strategies. More formally, the preimage of each of her best responses is convex under her best-response correspondence.[29]

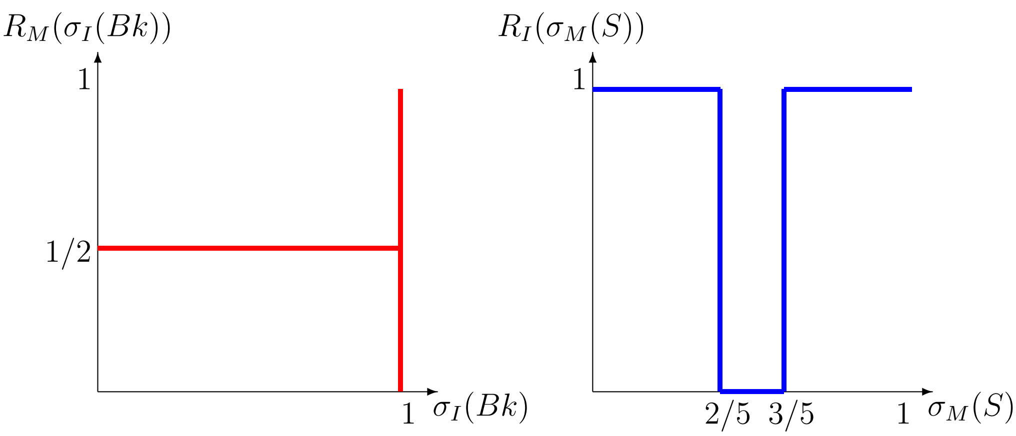


Figure 5.2: Players' best-response correspondences in Example 5.2

The reason for reversal behavior in Example 5.2 is that, no matter what the investor chooses, her expected profit in ω_1 is lower than in ω_2 if $\sigma_M(S) > 1/2$ and higher if $\sigma_M(S) < 1/2$. According to her objective function $V_I = \min\{f_I^{\omega_1}, f_I^{\omega_2}\}$, she will maximize $f_I^{\omega_1}$ if $\sigma_M(S) > 1/2$, and, otherwise, $f_I^{\omega_2}$. Hence, given $\sigma_M(S) > 1/2$, the investor's best-response correspondence equals her best responses in ω_1, and, otherwise, her best responses in ω_2.

To summarize, non-EU players can behave differently than EU players in that they may prefer randomized strategies and/or change their preferences for strategies due to mixture operations of one of their opponents. As an aside, note that in both cases, the matrix-form is an unsatisfactory representation of the game.

[29]Note that this holds only for two-person games. For the general case, see Definition 5.4.

5.1.2 The basic model

This section describes the class of games to be studied. First, we introduce the notion of a canonical game. This is a more detailed description of a game than a game-form but it is less detailed than a complete description of a game.

Definition 5.1. *A canonical normal-form game with incomplete information* (or canonical game, for short) is an ordered set $G = \langle I, \Omega, \{A_i, u_i\}_{i \in I} \rangle$, where

(1) $I = \{1, \dots, n\}$ is a *finite set of players*;

(2) $\Omega = \{\omega_1, \dots, \omega_m\}$ is a *finite set of states of nature*;

(3) A_i is the *finite set of pure actions of player* i. Let $A = \bigtimes_{i \in I} A_i$;

(4) $u_i : A \times \Omega \to \mathbb{R}$ is *the payoff function of player* i.

Players' payoffs (4) depend not only on an action profile, $a \in A$, but also on an uncertain state of the world (2). Note that (4) is a simplification (see Remark 3.2): let $\gamma : A \times \Omega \to X$ be an outcome function that maps from action profiles and states onto a set of outcomes X, and let $v_i : X \to \mathbb{R}$ be player i's utility function on the outcomes. Player i's payoff function corresponds to the composition $u_i := v_i \circ \gamma : A \times \Omega \to \mathbb{R}$.

Remark 5.1. A canonical game is a game-form together with fixed state-dependent payoffs.

According to (4), an action profile $a \in A$ induces payoff $u_i(a, \omega)$ in state $\omega \in \Omega$ for each $i \in I$. Hence, every action profile induces a payoff vector or, an act, $f_i(a) = \big(u_i(a, \omega_1), \dots, u_i(a, \omega_m)\big) \in \mathbb{R}^m$ for each $i \in I$. Definition 5.1 does not include private information. I shall restrict attention to this case to avoid cumbersome notation. As the following remark shows, private information can be easily introduced into the game.

Remark 5.2 (cf. Bade, 2011a and Azrieli and Teper, 2011)**.** Private information can be introduced into the game by defining an information partition P_i of Ω for each $i \in I$ which specifies players' strategy sets. A *pure strategy* of player i is then a P_i-measurable function $s_i : \Omega \to A_i$. If player i has no private information, the partition P_i is trivial. In this case, player i's strategies correspond to her actions.

Remark 5.3. The results in this paper hold also for canonical games with private information.

Of particular interest in this paper are mixed actions. The *mixed extension of a canonical game* involves, in addition to the elements of Definition 5.1, players' mixed action sets. Recall that a *mixed action* of player i is a function $\sigma_i : A_i \to [0,1]$ where $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$. As in Chapter 3, the set of all mixed actions of player i is denoted by Σ_i, and $\sigma_i(a_i)$ denotes the probability which σ_i assigns to action a_i. The BAT-model does not allow for strategic ambiguity, which means that players' have correct beliefs about mixed action combinations. To put it differently, the probabilities $\sigma_i(a_i)$ can be viewed as "objective" or known probabilities. It is assumed that, in any given state $\omega \in \Omega$, players' preferences w.r.t. a mixed profile $\sigma \in \Sigma$ have an EU representation.

Assumption 5.1. Fix a state $\omega \in \Omega$, then player i's payoff from a mixed profile $\sigma \in \Sigma$ is

$$f_i^{\omega}(\sigma) = \sum_{a \in A} \left(\prod_{j \in I} \sigma_j(a_j) \right) u_i(a, \omega) \text{ for each } i \in I.$$

According to Assumption 5.1, every mixed action profile $\sigma \in \Sigma$ induces a vector of expected payoffs $f_i(\sigma) = \left(f_i^{\omega_1}, \ldots, f_i^{\omega_m}\right) \in \mathbb{R}^m$, which is a convex combination of player i's payoff vectors induced by pure strategy profiles $f_i(a)$, formally,

$$f_i(\sigma) = \sum_{a \in A} \left(\prod_{j \in I} \sigma_j(a_j) f_i(a) \right).$$

In other words, mixed strategies induce statewise mixtures of acts.

Given the actions of the other players, any degenerate mixed action is payoff equivalent to a pure action. Therefore, we may associate the set of player i's pure actions, A_i, with the subset of Σ_i that contains i's degenerate mixed actions. Henceforth, depending on the context, the symbols a_i and A_i may also stand for (the set of) i's degenerate mixed actions. Furthermore, we denote the set of all canonical games by Γ.

5.1.3 Preferences over acts and equilibrium points

Definition 5.1 is not sufficient to characterize the solution of a game. In order to obtain a solvable game from a canonical game $G \in \Gamma$, we need to specify each player i's preferences, $\succsim_i$, over m-dimensional payoff vectors, as in the examples in Section 5.1.1. That is, for each $i \in I$, there exists a function $V_i : \mathbb{R}^m \to \mathbb{R}$ such that, for all $f, g \in \mathbb{R}^m$,

$$f \succsim_i g \;\Leftrightarrow\; V_i(f) \;\geq\; V_i(g).$$

The preference ordering $\succsim_i$ of each player i induces, through the associated payoff vectors, a preference ordering on action profiles and hence on actions for any given action combination of the other players. Let $\succsim = \{\succsim_i\}_{i \in I}$ denote the collection of players' preferences over acts. I shall refer to the set $\langle G, \succsim \rangle$ as G played by, or, with $\succsim$-players, or simply as game. The analysis in this chapter focuses on the representation function $V_i(\cdot)$ of player i's preferences. Throughout Section 5.2, we will impose the following restrictions on $V_i(\cdot)$.[30]

Assumption 5.2. For each $i \in I$, function V_i is continuous and quasiconcave on $\mathbb{R}^m$, and monotonic, i.e., for all $f, g \in \mathbb{R}^m$, $f(\omega) \geq (>) g(\omega)$ for all $\omega \in \Omega$ implies $V_i(f) \geq (>) V_i(g)$.

According to Assumption 5.2, the underlying preference relation $\succsim_i$ of each player i is complete, transitive, and monotonic. Furthermore, it satisfies uncertainty-aversion in the sense of Schmeidler (1989), which translates into quasiconcavity of the representation function. There is a huge variety of preferences that are consistent with Assumption 5.2. For instance, maxmin expected utility preferences, Choquet expected utility preferences if the capacity is convex, and smooth ambiguity-averse preferences. For more details, compare Cerreira-Vioglio et al. (2011), who identify the representation of preferences that satisfy the properties mentioned above.

The following examples illustrate two possible representation functions. Let $\Delta(\Omega)$ be the set of all probability measures over Ω, and $\mathscr{C}$ be the collection of all nonempty, closed and convex subsets of $\Delta(\Omega)$. An element of $\Delta(\Omega)$ (i.e., a probability vector or prior) is denoted by $\pi = (\pi(\omega_1), \ldots, \pi(\omega_m))$ where $\pi(\omega)$ is the probability of state $\omega \in \Omega$.

Example 5.3 (Expected utility). *The belief of an EU player i is represented by a unique prior $\pi_i \in \Delta(\Omega)$. She evaluates a state-contingent vector $f \in \mathbb{R}^m$ by the expected utility with respect to her prior:*

[30] In Section 5.3, we discuss an equilibrium concept that allows for preferences, which are not represented by quasiconcave functions.

$$EU_{\pi_i}(f) = f \cdot \pi_i^\top \text{ where } f \text{ is a row vector and } \pi_i^\top \text{ a column vector.}$$

Hence, for all $f, g \in \mathbb{R}^m$*, it holds that,* $f \succsim_i^{EU} g \;\Leftrightarrow\; f \cdot \pi_i^\top \;\geq\; g \cdot \pi_i^\top$.

Consequently, a game played by EU players is a game $\langle G, \succsim^{EU} \rangle$ where each player i has EU preferences, $\succsim_i^{EU}$, i.e., i's preferences are represented by an EU function, $V_i = EU_{\pi_i}$.

Example 5.4 (Maxmin expected utility)**.** *The key idea of the maxmin expected utility approach is that, in case of ambiguous uncertainty, an individual i has too little information to form a unique prior probability distribution* $\pi_i \in \Delta(\Omega)$*. For this reason, she considers a set of priors* $C_i \in \mathscr{C}$ *as possible. A maxmin expected utility player i evaluates an act* $f \in \mathbb{R}^m$ *by the minimal expected utility over all priors in her prior set:*

$$MEU_{C_i}(f) = \min_{\pi \in C_i} \{EU_\pi(f)\}.$$

Hence, for all $f, g \in \mathbb{R}^m$*, it holds that,* $f \succsim_i^{MEU} g \;\Leftrightarrow\; \min_{\pi \in C_i} \{EU_\pi(f)\} \;\geq\; \min_{\pi \in C_i} \{EU_\pi(g)\}$.

Finally, we turn to the solution of a game $\langle G, \succsim \rangle$. From now on, occasionally, I abuse notation and write $V_i(\sigma)$ instead of $V_i(f_i(\sigma))$. An (ex-ante) equilibrium for (the mixed extension of) a normal-form game with incomplete information is defined as follows:

Definition 5.2. An *equilibrium point in a game* $\langle G, \succsim \rangle$ is a profile $(\sigma_i^*, \sigma_{-i}^*) \in \Sigma$ such that

$$\sigma_i^* \;\in\; \arg\max_{\sigma_i \in \Sigma_i} \; V_i\left(\sigma_i, \sigma_{-i}^*\right) \text{ for each player } i.$$

Azrieli and Teper (2011) show that, under Assumption 5.1, uncertainty-aversion, i.e., quasiconcavity of players' objective functions, is necessary and sufficient for equilibrium existence, provided that players' preferences are continuous and monotonic. Their theorem is essentially similar to the existence theorem of Debreu (1952), which shows that in a finite game Nash equilibrium is guaranteed to exist if players preferences are nonlinear but quasiconcave in their own strategies.

5.1.4 Hedging behavior and reversal behavior

The *best-response correspondence* of player i is a multivalued mapping $R_i : \Sigma_{-i} \rightrightarrows \Sigma_i$ defined by $R_i(\sigma_{-i}) = \{\sigma_i \mid \sigma_i \in \arg\max_{\sigma_i \in \Sigma_i} \; V_i\left(\sigma_i, \sigma_{-i}\right)\}$. Furthermore, a pure action $a_i \in A_i$

is said to be contained in the *support of a mixed action* $\sigma_i \in \Sigma_i$ if σ_i assigns a strictly positive probability to a_i, formally $\text{supp}(\sigma_i) = \{a_i \in A_i \mid \sigma_i(a_i) > 0\}$.

Definition 5.3. Player i with preferences $\succsim_i$ represented by function V_i exhibits *hedging behavior* in $G \in \Gamma$ if there exists a mixed action $\sigma'_i \in \Sigma_i$ which satisfies

(i) $\sigma'_i \in R_i(\sigma'_{-i})$ for some $\sigma'_{-i} \in \Sigma_{-i}$ and

(ii) $V_i(\sigma'_i, \sigma'_{-i}) > V_i(a_i, \sigma'_{-i})$ for some $a_i \in \text{supp}(\sigma'_i)$.

Property (i) in Definition 5.3 restricts the notion of hedging behavior to actions that are contained in a player's best-response correspondence. This is necessary since if a mixed action is not a best response, an EU player may also prefer it over particular pure actions from its support. However, this is not possible if the mixed action is a best response due to the linearity of the EU functional.[31] Furthermore, non-EU players may strictly prefer mixed actions. This is the case when property (ii) holds for all $a_i \in \text{supp}(\sigma'_i)$. As a consequence, mixed actions can be unique best responses.

The second type of strategic behavior refers to players' behavior concerning randomizing operations of the other players.

Definition 5.4. Let $(\sigma'_j, \overline{\sigma}_{-j}), (\sigma''_j, \overline{\sigma}_{-j}) \in \Sigma_{-i}$, where $\overline{\sigma}_{-j}$ denotes a fixed strategy profile of all players except player i and player j. Player i with preferences $\succsim_i$ exhibits *reversal behavior* in $G \in \Gamma$ if there exist actions $a'_i, a''_i \in A_i$ such that

(i) $a'_i \in R_i(\sigma'_j, \overline{\sigma}_{-j})$, $a'_i \in R_i(\sigma''_j, \overline{\sigma}_{-j})$, and $a'_i \notin R_i(\alpha\sigma'_j + (1-\alpha)\sigma''_j, \sigma'_{-j})$ for some $\alpha \in (0,1)$ and/or

(ii) $a'_i \in R_i(\sigma'_j, \overline{\sigma}_{-j})$, $a'_i \in R_i(\sigma''_j, \overline{\sigma}_{-j})$, and $a''_i \notin R_i(\sigma'_j, \overline{\sigma}_{-j})$ and/or $a''_i \notin R_i(\sigma''_j, \overline{\sigma}_{-j})$, and $a''_i \in R_i(\alpha\sigma'_j + (1-\alpha)\sigma''_j, \overline{\sigma}_{-j})$ for some $\alpha \in (0,1)$.

Definition 5.4 is more technical in nature. Condition (i) refers to a situation like in Example 5.2 where an action is a best response to some action profiles but not to all convex combinations of the profiles. Condition (ii) describes a situation in which a pure action is a best response to a convex combination of two action profiles but not to both profiles, and, at the same time, there exists another action which is a best response to

[31] This property can be easily shown, see, for instance, Dekel et al. (1991, p. 236).

both profiles. Due to the linearity of the EU function, (ii) is also not possible if player i is an EU player.

5.2 Results

5.2.1 Main theorems

This section analyzes the extent to which the strategic behavior of EU players can be distinguished from the behavior of non-EU players. It turns out that if non-EU players exhibit neither hedging nor reversal behavior in a game, they are *quasi-expected utility players*. That is, they behave as if if they were EU players. In this case, it is impossible to distinguish between the players on the basis of their strategic behavior.

In general, it is difficult to infer players' preferences from their equilibrium actions. Bade (2011a) shows that the sets of equilibria of a two-player game and its ambiguous act extension are "observationally equivalent" in the sense that their supports coincide. My first main theorem is similar in nature but holds also for n-player games with state-dependent payoffs. More precisely, it shows that one cannot identify non-EU players by observing equilibrium actions whenever the players do not show hedging behavior.

Theorem 5.1. *Fix a canonical game $\bar{G} \in \Gamma$. Consider players with preferences $\succsim$. If none of the players shows hedging behavior in $\bar{G}$, then, under Assumption 5.1 and 5.2, for any equilibrium point $\sigma^* \in \Sigma$ in $\langle \bar{G}, \succsim \rangle$, there exist priors $\{\pi_i\}_{i \in I}$ such that σ^* is an equilibrium point in game $\langle \bar{G}, \succsim^{EU} \rangle$ in which each player i is an expected utility maximizer and her beliefs about nature are represented by π_i.*

Theorem 5.1 shows that if players do not show hedging behavior, we need to consider their beliefs about nature or their best-response correspondences to distinguish EU from non-EU players. However, on the basis of beliefs, we are only able to identify non-EU players who are not probabilistically sophisticated in the sense of Machina and Schmeidler (1992). Moreover, from an experimental point of view, it might be difficult to measure players' beliefs. In complete information games, eliciting players' ex-ante beliefs about their opponents' strategy choice may affect their decisions in the game. In addition, there is evidence that players' ex-post beliefs are biased (see Rubinstein and Salant, 2016). These difficulties could also limit the ability to measure players' beliefs about nature.

To state the second main theorem, we need to introduce the notion of best response equivalence. Roughly, two games are said to be best response equivalent if player i's best responses coincide in both games for each $i \in I$. The precise definition is as follows.

Definition 5.5. Let R_i^G be the best-response correspondence of player i in $G \in \Gamma$. Two games $\langle G, \succsim \rangle$ and $\langle G', \succsim' \rangle$ are said to be *best response equivalent* if they consist of the same number of players and the same set of pure actions for each player, and if

$$R_i^G = R_i^{G'} \text{ for each player } i \in I.$$

The second theorem says that, in any given two-person game, players exhibit neither hedging nor reversal behavior if and only if there exists a game with EU players that is best response equivalent to that game. In other words, non-EU players behave strategically as if they were EU players if and only if they do not exhibit hedging and reversal behavior. Consequently, hedging and reversal behavior are the sole behavioral differences between EU and non-EU players.

Theorem 5.2. *Consider a two-person game $\langle G, \succsim \rangle$, then, under Assumption 5.1 and 5.2, the following statements are equivalent:*

(i) Each player $i \in I$ exhibits neither hedging behavior nor reversal behavior in G.

(ii) There exists a game $\langle G', \succsim^{EU} \rangle$ which is best response equivalent to $\langle G, \succsim \rangle$.

Taken together, non-EU players who do not exhibit hedging and reversal behavior in a two-person canonical game G cannot be distinguished from EU players by observing their equilibrium actions due to Theorem 5.1, and behave structurally as if they were EU players by Theorem 5.2. Therefore, these players can be termed quasi-expected utility players.

According to Definition 5.5, two games, which are best response equivalent, have the same number of players, and, for each player, the same set of pure actions. However, the games may still have different state spaces and state-dependent payoffs. One may ask under which conditions the game with EU players in statement (ii) of Theorem 5.2 has the same canonical game structure as the game with non-EU players, i.e., under which conditions it holds that $G = G'$. Proposition 5.1 gives sufficient conditions for this to be true in the context of two-person two-action games, which will be treated in Section 5.2.2. The proposition says that, given that players exhibit neither hedging nor reversal behavior, it suffices that each player has a strictly dominant strategy and/or her state-dependent payoffs satisfy the following condition: there exists two states of the world such that in one of the two states, player i's first pure action is strictly worse than the second if the opponent plays her first pure action and strictly better than the second if the opponent plays her second pure strategy, and vice versa in the other state of the world. This condition can also be viewed as the requirement that player i's state-dependent utility function has strictly decreasing differences in one state of the world and strictly increasing differences in another state.[32] However, note that the requirement (ii) in Proposition 5.1 below is slightly stronger than the notion of increasing (decreasing) differences since it requires increasing (decreasing) differences with respect to the reference point 0.

Proposition 5.1. *Fix a two-person two-actions game $\bar{G} \in \Gamma$. Let $A_i = \{a_i', a_i''\}$ be player i's action set. Consider players with preferences $\succsim$ who do not exhibit hedging and reversal behavior in $\bar{G}$. If, for each player i, at least one of the following conditions is met*

(i) player i has a strictly dominant strategy (strategic dominance),

(ii) there exist $\omega', \omega'' \in \Omega$ such that (strictly increasing and decreasing differences)

$$u_i(a_i', a_{-i}', \omega') - u_i(a_i'', a_{-i}', \omega') < 0 < u_i(a_i', a_{-i}'', \omega') - u_i(a_i'', a_{-i}'', \omega') \text{ and}$$

$$u_i(a_i', a_{-i}', \omega'') - u_i(a_i'', a_{-i}', \omega'') > 0 > u_i(a_i', a_{-i}'', \omega'') - u_i(a_i'', a_{-i}'', \omega''),$$

then, there exist priors $\{\pi_i\}_{i \in I}$ such that $\langle \bar{G}, \succsim^{EU} \rangle$ is best response equivalent to $\langle \bar{G}, \succsim \rangle$.

The results of this section show that we are only able to behaviorally distinguish non-EU from EU players if the non-EU players exhibit hedging and/or reversal behavior. In the next section, we will provide conditions under which players show hedging or reversal behavior, especially in terms of the payoff structure of a game.

[32] A function $f : X \times Y \to \mathbb{R}$ has *strictly increasing differences* if for $x', x'' \in X$, $x'' > x'$, and for $y', y'' \in Y$, $y'' > y'$, it holds that $f(x'', y'') - f(x', y'') > f(x'', y') - f(x', y')$.

5.2.2 Existence of hedging and reversal behavior

General preferences

In this section, we consider games where players are strictly uncertainty-averse. A player i is said to be *strictly uncertainty-averse in* $G \in \Gamma$ if her objective function V_i is strictly quasiconcave on the feasible payoffs in the game. The feasible payoffs correspond to the convex hull of player i's payoff vectors induced by pure action profiles, formally $conv\{f_i(a) \mid a \in A\}$. Furthermore, the games to be studied in this section have a special property, called property (P).

Definition 5.6. A canonical game $G \in \Gamma$ has property (P) if

$$f_i(a_i', \sigma_{-i}) \neq f_i(a_i'', \sigma_{-i}),$$

for each $i \in I$ and all $a_i', a_i'' \in A_i$, $a_i' \neq a_i''$ and any given $\sigma_{-i} \in \Sigma_{-i}$.

In these games, the existence of hedging behavior is closely tied to the existence of strictly dominant strategies, as the following proposition demonstrates.

Proposition 5.2. *In any canonical game* $G \in \Gamma$ *that meets property (P), and that is played by strictly uncertainty-averse players, the following statements are equivalent:*

(i) Some players have no strictly dominant pure strategies.

(ii) Some players exhibit hedging behavior in G*.*

Proof. The proof of the proposition is straightforward. □

Although Proposition 5.2 is simple from a mathematical point of view, it has two interesting implications. Firstly, if we observe a mixed equilibrium in a canonical game like the one in the proposition and we know that the players are strictly uncertainty-averse, then we can conclude that some players show hedging behavior.

Corollary 5.1. *If we observe a mixed equilibrium in a game* $\left\langle G, \succsim^{UA} \right\rangle$ *where* G *meets property (P), and the players are strictly uncertainty-averse, then some players exhibit hedging behavior.*

Secondly, suppose that it is known that a player is strictly uncertainty-averse but his particular objective function V_i is unknown. Then, we can exclude that the player exhibits hedging behavior if and only if she has a pure action that is strictly dominant in each state of the world.

Corollary 5.2. *A strictly uncertainty-averse player i shows no hedging behavior in a canonical game G which satisfies property (P) if and only if she has a pure action $a_i' \in A_i$ such that $u_i(a_i', a_{-i}, \omega) > u_i(a_i, a_{-i}, \omega)$ for all $\omega \in \Omega$ and all $a_i \in A_i$, $a_i \neq a_i'$ and any $a_{-i} \in A_{-i}$.*

Maxmin expected utility

This section studies two-person two-strategies games played by players whose preferences are represented by maxmin expected utility (or MEU players, for short). Experiments on game theory are often based on two-player two-strategies games.

It is worth noting that the results of this section hold also for uncertainty-averse players with Choquet expected utility (or, briefly, CEU) preferences. This follows from the fact that uncertainty-averse CEU preferences (i.e., CEU with convex capacities) correspond to MEU preferences where the prior set equals the set of probabilities in the core of the capacity (see Section 2.4 in Chapter 2). Hence, preferences that can be represented by CEU can also be represented by MEU.[33]

The focus of this section lies on hedging behavior due to Theorem 5.1. All results, except Proposition 5.4, provide conditions under which we can exclude hedging and reversal behavior, respectively, for all possible prior sets. The negations of the results give existence conditions. One may think of a situation similar to that discussed in the context of Corollary 5.2: suppose that we know that player i has MEU preferences but his particular prior set C_i is unknown.

Ghirardato et al. (1998) and Klibanoff (2001) provide useful results concerning hedging behavior. They examine additivity and preference for mixtures, respectively, in the context of single-person decision problems and MEU preferences. Comonotonicity is a

[33] Under uncertainty-aversion, MEU is even a strict generalization of CEU (see Klibanoff, 2001).

natural starting point for the question of additivity of the MEU functional.[34] However, comonotonicity does not ensure additivity as an example in Klibanoff (1996) illustrates. Ghirardato et al. (1998) show that we need a stronger condition for additivity of the MEU functional, called affine-relatedness.

Definition 5.7. Two vectors $f, g \in \mathbb{R}^m$ are *affinely related* if there exist $a \geq 0$ and $b \in \mathbb{R}$ such that $f^\omega = ag^\omega + b$ and/or $g^\omega = af^\omega + b$ for all $\omega \in \Omega$.

Definition 5.7 says that f and g are affinely related if either f or g is constant or there exist $a > 0$ and $b \in \mathbb{R}$ such that $f^\omega = ag^\omega + b$. We say that two vectors $f, g \in \mathbb{R}^m$ are *negatively affinely related* if f is affinely related to $-g$. Affine-relatedness implies comonotonicity, but the converse is not true. For the special case of two states of nature, affine-relatedness is equivalent to comonotonicity.

Proposition 5.3 (Ghirardato et al., 1998, p. 409)**.** *For $f, g \in \mathbb{R}^m$, the following statements are equivalent:*

(i) f and g are affinely related.

(ii) $MEU_{C_i}(f + g) = MEU_{C_i}(f) + MEU_{C_i}(g)$ for all $C_i \in \mathscr{C}$.

Another relation, which we will need later, is dominance-relatedness.

Definition 5.8. Two vectors $f, g \in \mathbb{R}^m$ are *dominance related* if $f^\omega \geq g^\omega$ and/or $g^\omega \geq f^\omega$ for all $\omega \in \Omega$.

Two vectors $f, g \in \mathbb{R}^m$ are said to be *strictly dominance related* if $f^\omega > g^\omega$ or $g^\omega > f^\omega$ for all $\omega \in \Omega$. Furthermore, a vector f is *constant* if $f^\omega = f^{\omega'}$ for all $\omega, \omega' \in \Omega$.

In the sequel, as before, we denote player i's action set by $A_i = \{a_i', a_i''\}$. By using negative affine-relatedness, we obtain a strong existence result for hedging behavior. Fix an action of the other player, if player i's pure actions induce negatively affinely related payoff vectors, then player i will show hedging behavior for all prior sets contained in a particular subset of $\mathscr{C}$. The following proposition makes this precise.

Proposition 5.4. *Fix a $\overline{\sigma}_{-i} \in \Sigma_{-i}$. Let $\mathscr{C}^*$ be the collection of all closed, convex and nonempty subsets of $\Delta(\Omega)$ which contain some π', π'' such that*

[34] In our context, two vectors $f, g \in \mathbb{R}^m$ are comonotonic if $(f^\omega - f^{\omega'})(g^\omega - g^{\omega'}) \geq 0$ for all $\omega, \omega' \in \Omega$.

(i) $f_i(a_i', \overline{\sigma}_{-i})\pi' > f_i(a_i'', \overline{\sigma}_{-i})\pi'$ *and* $f_i(a_i', \overline{\sigma}_{-i})\pi'' < f_i(a_i'', \overline{\sigma}_{-i})\pi''$ *and*

(ii) $f_i(a_i', \overline{\sigma}_{-i})\pi' \neq f_i(a_i', \overline{\sigma}_{-i})\pi''$ *and* $f_i(a_i'', \overline{\sigma}_{-i})\pi' \neq f_i(a_i'', \overline{\sigma}_{-i})\pi''$.

If $f_i(a_i', \overline{\sigma}_{-i})$ *and* $f_i(a_i'', \overline{\sigma}_{-i})$ *are negatively affinely related, then, a* MEU_{C_i} *player shows hedging behavior for all* $C_i \in \mathscr{C}^*$.

In general the set $\mathscr{C}^*$ can vary strongly across different payoff vectors. Apparently, the set does not contain singletons, but it can be empty. For instance, $\mathscr{C}^*$ is empty when $f_i(a_i', \overline{\sigma}_{-i})$ and $f_i(a_i'', \overline{\sigma}_{-i})$ are dominance related.

The first lemma illustrates that, given an action of the opponent, player i does not show hedging behavior for all prior sets if and only if i's pure actions induce payoff vectors that are strictly dominance related and/or affinely related.

Lemma 5.1. *Fix a* $\overline{\sigma}_{-i} \in \Sigma_{-i}$. *The following statements are equivalent:*

(i) $f_i(a_i', \overline{\sigma}_{-i})$ *and* $f_i(a_i'', \overline{\sigma}_{-i})$ *are (a) strictly dominance related or (b) affinely related.*

(ii) Given $\overline{\sigma}_{-i}$, *a* MEU_{C_i} *player* i *shows no hedging behavior for all* $C_i \in \mathscr{C}$.

Proof. We omit the proof since Lemma 5.1 is a variant of Theorem 2 in Klibanoff (2001).

The next proposition gives the most important result in this section. It states that, in many games, player i shows neither hedging nor reversal behavior for all prior sets if and only if player i's payoff vectors, which are induced by pure action profiles, are pairwise affinely related.

Proposition 5.5. *Consider a two-player two-strategies canonical game* $G \in \Gamma$ *in which* $f_i(a_i', \sigma_{-i})$ *and* $f_i(a_i'', \sigma_{-i})$ *are not strictly dominance related for any* $\sigma_{-i} \in \Sigma_{-i}$ *and* $f_i(a_i', a_{-i}) \neq f_i(a_i'', a_{-i})$ *for any* $a_{-i} \in A_{-i}$. *If at most one of the vectors from the set* $\{f_i(a) \mid a \in A\}$ *is constant, the following statements are equivalent:*

(i) Player i*'s payoff vectors induced by pure action profiles are pairwise affinely related.*

(ii) A MEU_{C_i} *player* i *shows neither hedging nor reversal behavior in* G *for all* $C_i \in \mathscr{C}$.

If statement (i) of Proposition 5.5 is true, the function MEU_{C_i} is additive on the convex hull of player i's payoff vectors for any prior set $C_i \in \mathscr{C}$. Hence, for every prior set C_i, there exist a prior $\pi_i' \in \Delta(\Omega)$ such that a MEU_{C_i} player and a $EU_{\pi_i'}$ player behave identically.

The statement in Proposition 5.5 is restricted to canonical games where, given any action of the other player, the induced vectors of player i's pure actions are not strictly dominance related. However, this is not a strong restriction. To see why, note that, in most games, there exists a closed and convex subset, $\tilde{\Sigma}_{-i} \subseteq \Sigma_{-i}$, that satisfies the property that the induced vectors of player i's pure actions are not strictly dominance related.[35] Proposition 5.5 can be analogously applied to these games: player i shows no hedging and reversal behavior for all prior sets only if MEU_{C_i} is additive on the set of all vectors which are induced by the profiles which involve elements of $\tilde{\Sigma}_{-i}$.

Furthermore, Proposition 5.5 is restricted to canonical games where at most one of player i's payoff vectors induced by pure action profiles is constant, and where any two pure action profiles induce different payoff vectors. The last two propositions discuss the existence of hedging behavior for canonical games which do not satisfy these properties.

At first, we consider the case where more than one of player i's payoff vectors induced by pure action profiles is constant. In this case, a MEU_{C_i} player i exhibits no hedging behavior if and only if the MEU_{C_i} functional is additive for all induced vectors of the game and/or the vectors of one of player i's actions are constant, given any action of the other player.

Proposition 5.6. *Consider a two-player two-strategies canonical game $G \in \Gamma$ in which there exists a $\sigma_{-i} \in \Sigma_{-i}$ such that $f_i(a_i', \sigma_{-i})$ and $f_i(a_i'', \sigma_{-i})$ are not strictly dominance related. If at least two of the vectors from $\{f_i(a) \mid a \in A\}$ are constant, the following statements are equivalent:*

(i) (a) Player i's payoff vectors induced by pure action profiles are pairwise affinely related and/or (b) the vectors $f_i(a_i', a_{-i}')$ and $f_i(a_i', a_{-i}'')$ are constant.

(ii) A MEU_{C_i} player i shows no hedging behavior in G for all $C_i \in \mathscr{C}$.

Finally, we turn to games in which player i's pure actions can induce equal payoff vectors, given an action of the other player. In these games, a MEU_{C_i} player i shows no hedging behavior if and only if the MEU_{C_i} functional is additive and/or player i's pure

[35] Exceptions are games where, for any given action of the opponent, one action induces a payoff vector which strictly dominates the vector induced by the other action.

actions induce equal vectors, given any pure action of the opponent.[36]

Proposition 5.7. *Consider a two-player two-strategies canonical game $G \in \Gamma$ where $f_i(a_i', \sigma_{-i})$ and $f_i(a_i'', \sigma_{-i})$ are not strictly dominance related for any $a_{-i} \in \Sigma_{-i}$ and some non-degenerate $\sigma_{-i} \in \Sigma_{-i}$. Furthermore, it holds that $f_i(a_i', a_{-i}) = f_i(a_i'', a_{-i})$ for some $a_{-i} \in A_{-i}$. If at most one of the vectors from $\{f_i(a) \mid a \in A\}$ is constant, the following statements are equivalent:*

(i) (a) Player i's payoff vectors induced by pure action profiles are pairwise affinely related and/or (b) $f_i(a_i', a_{-i}) = f_i(a_i'', a_{-i})$ for any given $a_{-i} \in A_{-i}$.

(ii) A MEU_{C_i} player i shows no hedging behavior in G for all $C_i \in \mathscr{C}$.

5.3 Discussion

This section first discusses a preference for randomization, which is a key aspect of this chapter. Subsequently, an essential assumption of the BAT-model is questioned. Finally, I introduce a generalized equilibrium concept that allows for players who are not uncertainty-averse.

5.3.1 Preference for randomization

A preference for randomization plays a central role in this chapter. One may ask whether there exists evidence for such a preference. There is little experimental literature on this topic. One study by Dominiak and Schnedler (2011) finds no evidence for a preference for mixtures. However, the study is about single-person decisions and does not explicitly test for ex-ante and ex-post randomization attitudes, which I will elaborate on in the next subsection.

A further question is whether a preference for randomization leads to an infinite sequence of randomization operations: suppose a player strictly prefers a 1/2-mixture of two pure actions a_1 and a_2 over either alone, say, he prefers to flip a coin to determine his strategy choice. After flipping the coin, it turns out to be a_1. Due to his preferences

[36] Note that the latter is not necessarily equivalent to the case where all payoff vectors induced by pure action profiles are equal since it is still possible that $f_i(a_i', a_{-i}') \neq f_i(a_i', a_{-i}'')$.

before the coin flip, one may think that he would strictly prefer to flip the coin again and again...ad infinitum. An argument against this view is dynamic consistency, as Machina (1989) eloquently argues. In addition, an infinite sequence of randomization operations is impossible when mixed actions are generated by some kind of exogenous random device and players accept binding commitments to play a pure action based on the outcome of this device.

5.3.2 The model

Assumption 5.1 of the model is crucial for the existence of hedging and reversal behavior. Recall that Assumption 5.1 states that a mixed action profile induces an expected utility value in each state of the world. There is no compelling reason for this assumption. Alternatively, we could have assumed that players' payoffs from a mixed profile equals the expectation of the representation function values of pure action profiles taken with respect to the distribution given by the mixed profile:

Assumption 5.1′. Player i's payoff from a mixed profile $\sigma \in \Sigma$ is

$$U_i(\sigma) = \sum_{a \in A} \left(\prod_{j \in I} \sigma_j(a_j) V_i\big(f_i(a)\big) \right).$$

To see the difference between Assumption 5.1 and 5.1′, consider the state-dependent payoff matrices of Example 5.1:

	q_l	q_h
q_l	$2, 2$	$\frac{1}{2}, 1$
q_h	$1, \frac{1}{2}$	$-2, -2$

ω_1

	q_l	q_h
q_l	$1, 1$	$1, 2$
q_h	$2, 1$	$2, 2$

ω_2

Again, suppose that each firm $i \in \{1, 2\}$ has the following non-EU preferences over state-contingent (expected) profits: $V_i(f_i) = \min\{f_i^{\omega_1}, f_i^{\omega_2}\}$. Recall that, under Assumption 5.1, the mixed profile $(\sigma_1(q_l), \sigma_2(q_l)) = (1, \frac{1}{4})$ induces the vector $f_1(1, \frac{1}{4}) = (\frac{7}{8}, 1)$ for firm 1. Hence, firm 1's payoff from this mixed profile equals $V_1(1, \frac{1}{4}) = \frac{7}{8}$.

In contrast, under Assumption 5.1′, firm 1's payoff from $(\sigma_1(q_l), \sigma_2(q_l)) = (1, \frac{1}{4})$ is $U_1(1, \frac{1}{4}) = \frac{1}{4}V_1(q_l, q_l) + \frac{3}{4}V_1(q_l, q_h) = \frac{5}{8}$. Assumption 5.1′ implies that a player's objective function is linear in both her strategies and the strategies of the other players. Consequently, under Assumption 5.1′, players exhibit neither hedging nor reversal behavior. That is, they are quasi-expected utility players.

Which assumption is the "correct" one? In the class of games we consider in this chapter there are two sources of uncertainty: strategic risk and ambiguous uncertainty about the environment. From a decision theory perspective, this situation can be viewed as a two-stage lottery that involves

1. An ambiguous lottery which represents Nature's move.
2. A risky lottery which is given by players' mixed strategies.

In my view, the underlying assumption of the model depends on how players evaluate the two-stage lottery above. This is closely tied to the distinction between *ex-ante* and *ex-post randomization*. These notions can be thought of as how players perceive the sequence of lottery 1. and 2., i.e., whether Nature's move takes place before or after the randomization by mixed strategies. In a recent paper, Eichberger et al. (2014) show that dynamically consistent individuals will be indifferent to ex-ante randomizations, but may exhibit a strict preference for ex-post randomizations. Following this result, we can associate Assumption 5.1 with ex-post randomization and Assumption 5.1′ with ex-ante randomization.

Finally, it is worth mentioning that the model studied in this chapter does not allow for strategic ambiguity. As a consequence, it avoids the drawbacks of strategic ambiguity models described in Section 3.3. From my point of view, it would be desirable to get an appropriate generalization of the model with a richer state space that incorporates strategic ambiguity. For instance, we can think of a BAT-model based on the Choquet approach. Independence between strategies and the environment could be introduced by a Fubini Theorem for non-additive measures (see, e.g., Ghirardato, 1997 and Chateauneuf and Lefort, 2008). Moreover, such a model could build on an generalized equilibrium concept, e.g., on the concept of Eichberger and Kelsey (2014) discussed in Chapter 6.

5.3.3 Equilibrium without uncertainty-aversion

So far, we have assumed that players are uncertainty-averse. Azrieli and Teper (2011) show that the equilibrium according to Definition 5.2 may fail to exist when players are not ambiguity-averse (or ambiguity-neutral). In the following, I define an equilibrium for the BAT-model that allows for more general preferences. This equilibrium concept is based on the notion of equilibrium in beliefs introduced by Crawford (1990), discussed in Section 3.3, and a n-player version of equilibrium in beliefs defined in Zimper (2007).

Let $\Delta(\Sigma_i)$ be the set of all probability distributions with finite support on player i's mixed strategy set. An element $\beta_j^i \in \Delta(\Sigma_i)$ is a *belief of player j about player i's mixed strategy choice*. Consequently, an element of the product space $\beta_i \in \bigtimes_{j\in I\setminus\{i\}} \Delta(\Sigma_j)$ is *a belief of player i about the mixed strategy choices of the other players*. Let $\beta_i(\sigma_{-i})$ be the probability with which player i believes that her opponents will play the strategy combination σ_{-i}. In analogy to Assumption 5.1, we assume that, in any given state $\omega \in \Omega$, each player's payoff from a mixed strategy $\sigma_i \in \Sigma_i$ and a belief $\beta_i \in \bigtimes_{j\in I\setminus\{i\}} \Delta(\Sigma_j)$ corresponds to her expected utility. The probability which is assigned to a particular strategy combination of the other players equals the expectation taken with respect to the belief β_i.

Assumption 5.3. Fix a state $\omega \in \Omega$, then player i's payoff from a mixed strategy $\sigma_i \in \Sigma_i$ and a belief $\beta_i \in \bigtimes_{j\in I\setminus\{i\}} \Delta(\Sigma_j)$ is

$$f_i^\omega(\sigma_i, \beta_i) = \sum_{a\in A} \Big(\sigma_i(a_i) \cdot \big[\sum_{\sigma_{-i}\in \text{supp}(\beta_i)} \beta_i(\sigma_{-i}) \cdot \sigma_{-i}(a_{-i})\big]\Big) u_i(a, \omega) \text{ for each } i \in I.$$

According to Assumption 5.3, every mixed action σ_i together with a belief β_i induces a vector of expected payoffs:

$$f_i(\sigma_i, \beta_i) = \big(f_i^{\omega_1}(\sigma_i, \beta_i), \dots, f_i^{\omega_m}(\sigma_i, \beta_i)\big).$$

For each player i, let $V_i : \mathbb{R}^m \to \mathbb{R}$ be a continuous function that represents her preferences $\succsim_i$ over over m-dimensional payoff vectors. Then, players best responses to their beliefs can be defined in the usual way.

Definition 5.9. Consider a player i with preferences $\succsim_i$ represented by function V_i. *Player i's best responses to her belief β_i* are

$$R_i(\beta_i) = \{\sigma_i \mid \sigma_i \in \arg\max_{\sigma_i \in \Sigma_i} V_i\left(f_i(\sigma_i, \beta_i)\right)\}.$$

Now, we can introduce the solution concept for a game played by players whose preferences are not necessarily represented by quasiconcave functions.

Definition 5.10. An *equilibrium in beliefs* in a game $\langle G, \succsim \rangle$ is a beliefs system $(\beta_i^*, \beta_{-i}^*)$ such that, for each player $i \in I$,

(i) $\beta_j^{i*} \in \Delta(\Sigma_i)$ is the identical belief of all players $j \neq i$ about player i's mixed strategy choice and

(ii) $\sigma_i \in R_i(\beta_i^*)$ for all $\sigma_i \in \text{supp}(\beta_j^*)$, i.e., player i's mixed strategies in the support of β_j^{i*} are best responses to her belief β_i^*.

An equilibrium in beliefs exists in any game $\langle G, \succsim \rangle$ in which the preferences of each player i are represented by a continuous functions V_i. This follows from the existence result in Zimper (2007, p. 69). Finally, observe that the equilibrium according to Definition 5.10 coincides with the equilibrium in the sense of Definition 5.2 whenever players' preferences are represented by quasiconcave functions.

5.4 Summary

This chapter examines the extent to which we can distinguish expected from non-expected utility players on the basis of their behavior. Both types of players sometimes behave observationally equivalent, which means that we cannot infer players' preferences by observing their equilibrium actions. It is shown that expected and uncertainty-averse non-expected utility players can be distinguished from each other on the basis of their best responses. Non-expected utility players may use mixed strategies differently, called hedging behavior, and may respond differently to mixed strategy combinations, called reversal behavior.

The first main theorem shows that if non-expected utility players do not exhibit hedging behavior, then they behave observationally equivalent to expected utility players. The second main theorem states that hedging and/or reversal behavior are necessary and sufficient for distinguishing expected from non-expected utility players by looking at their best responses, i.e., these are the sole behavioral differences between the players. Furthermore, this chapter provides necessary and sufficient conditions for the existence of hedging and reversal behavior in terms of the payoff structure of two-person two-strategies games. In the last part of this chapter, I discuss the underlying model and introduce an equilibrium concept that allows for players who are not uncertainty-averse.

The analysis of this chapter provides insights into the BAT-model. It is useful for economic applications of this model and can serve as a guide to design experiments testing the model. Furthermore, this study can be a starting point for further experimental and theoretical research. Besides the point raised in Section 5.3, one interesting question is, for instance, whether hedging or reversal behavior can be strategically exploited.

5.5 Proofs

Example 5.1 and 5.2 (Best-response correspondences and equilibria).

Example 5.1. Given a strategy profile $(\sigma_1, \sigma_2) = \big(\sigma_1(q_l), \sigma_2(q_l)\big)$, firm 1's state-dependent expected profits are

$$f_1^{\omega_1}(\sigma_1, \sigma_2) = \big(\tfrac{1}{2}\sigma_1[5 - 3\sigma_2] + 3\sigma_2 - 2\big) \text{ and } f_1^{\omega_2}(\sigma_1, \sigma_2) = \big(2 - \sigma_1\big).$$

If $\sigma_2 \leq 1/3$, then $f_1^{\omega_1}(\sigma_1, \sigma_2) \leq f_1^{\omega_2}(\sigma_1, \sigma_2)$ for all $\sigma_1 \in [0, 1]$. In this case, firm 1 will maximize $f_1^{\omega_1}(\sigma_1, \sigma_2)$ by playing $\sigma_1 = 1$. Otherwise, for any given $\sigma_2 > 1/3$, there exists a mixed strategy σ_1' such that $f_1^{\omega_1}(\sigma_1', \sigma_2) = f_1^{\omega_2}(\sigma_1', \sigma_2)$, which maximizes $V_1\big(f(\sigma_1, \sigma_2)\big) = \min\big\{f_1^{\omega_1}(\sigma_1, \sigma_2),\ f_1^{\omega_2}(\sigma_1, \sigma_2)\big\}$. By setting $f_1^{\omega_1}(\sigma_1, \sigma_2) = f_1^{\omega_2}(\sigma_1, \sigma_2)$, we obtain $\sigma_1' = \big(8 - 6\sigma_2\big)/\big(7 - 3\sigma_2\big)$. Due to the symmetry of the game, the same argumentation applies to firm 2. Consequently, firm i's best-response correspondence is:

$$R_i(\sigma_j(q_l)) = \begin{cases} 1, & \text{if } \sigma_j(q_l) \leq 1/3 \\ \big(8 - 6\sigma_j(q_l)\big)/\big(7 - 3\sigma_j(q_l)\big), & \text{if } \sigma_j(q_l) > 1/3 \end{cases}.$$

The game has only one equilibrium: $\big(\sigma_1^*(q_l), \sigma_2^*(q_l)\big) \approx (0.74, 0.74)$.

Example 5.2. Given a strategy profile $(\sigma_M, \sigma_I) = \big(\sigma_M(S), \sigma_I(Bk)\big)$, players' state-dependent expected profits are

$$f_M^{\omega_1}(\sigma_M, \sigma_I) = \big(\sigma_M[1-\sigma_I]\big) \text{ and } f_M^{\omega_2}(\sigma_M, \sigma_I) = \big(\sigma_M[\sigma_I - 1] + 1 - \sigma_I\big), \text{ and}$$

$$f_I^{\omega_1}(\sigma_M, \sigma_I) = \big(\sigma_I[2-5\sigma_M] + 5\sigma_M\big) \text{ and } f_I^{\omega_2}(\sigma_M, \sigma_I) = \big(\sigma_I[5\sigma_M - 3] + 5 - 5\sigma_M\big).$$

If $\sigma_I = 1$, M is indifferent between all of his strategies since $f_1^{\omega_1}(\sigma_M, 1) = 0 = f_1^{\omega_2}(\sigma_M, 1)$ for all $\sigma_M \in [0,1]$. Otherwise, M's unique best response is $\sigma_M = 1/2$, where $f_1^{\omega_1}(1/2, \sigma_I) = f_1^{\omega_2}(1/2, \sigma_I)$ for all $\sigma_I \in [0,1]$. Hence,

$$R_M(\sigma_I(Bk)) = \begin{cases} 1/2, & \text{if } \sigma_I(Bk) \in [0,1) \\ [0,1], & \text{if } \sigma_I(BK) = 1 \end{cases}$$

Let $R_I(\sigma_M(S) \mid \omega)$ be the investor's best-response correspondence in state $\omega \in \{\omega_1, \omega_2\}$. Since $f_I^{\omega_1}(\sigma_M, \sigma_I) \leq (\geq) f_I^{\omega_2}(\sigma_M, \sigma_I)$ for $\sigma_M \leq (\geq) 1/2$ and all $\sigma_I \in [0,1]$, I's best response correspondence is

$$R_I(\sigma_M(S)) = \begin{cases} R_I(\sigma_M(S) \mid \omega_1), & \text{if } \sigma_M(S) \leq 1/2 \\ R_I(\sigma_M(S) \mid \omega_2), & \text{if } \sigma_M(S) \geq 1/2 \end{cases}$$

The game has one equilibrium where the investor buys the stock: $\big(\sigma_M^*(S), \sigma_I^*(Bk)\big) = (0.5, 0)$, and infinitely many equilibria where she keeps her money: $\{\big(\sigma_M^*(S), \sigma_I^*(Bk)\big) \mid \sigma_M^*(S) \in [0, \frac{2}{5}] \cup [\frac{3}{5}, 1] \text{ and } \sigma_I^*(Bk) = 1\}$.

Notation 5.1. From now on, $f, g, h, k \in \mathbb{R}^m$ denote row payoff vectors and $\pi \in \Delta(\Omega)$ column probability vectors. A zero vector of proper dimension is denoted by $\mathbf{0}$. The following convention for ordering relations will be used. For real numbers, the relations $=, >, \geq$ are defined as usual. If $x, y \in \mathbb{R}^n$, $n > 1$, then

$$x = y \Leftrightarrow x_i = y_i \text{ for } i = 1, \ldots, n;$$

$$x \geqq y \Leftrightarrow x_i \geqq y_i \text{ for } i = 1, \ldots, n;$$

$$x \geq y \Leftrightarrow x \geqq y \text{ and } x \neq y;$$

$$x > y \Leftrightarrow x_i > y_i \text{ for } i = 1, \ldots, n.$$

Furthermore, for any set S, ∂S denotes the boundary of S, $int(S)$ the interior of S, and $cl(S)$ the closure of S. Matrix operations, e.g. matrix multiplication, inner product, and scalar multiplication, et cetera, are defined as usual. The same holds true for set operations such as intersection, union, set difference, et cetera.

Proof of Theorem 5.1. Essentially, the proof of this theorem is based on a separating hyperplane argument. We will use a theorem of the alternatives to establish this argument.

Fix a canonical game $\bar{G} \in \Gamma$ and consider players with preferences $\succsim$. Suppose $(\sigma_i^*, \sigma_{-i}^*) \in \Sigma$ is an equilibrium for the game $\langle \bar{G}, \succsim \rangle$. Consider an arbitrary player i. Let V_i be a function which represents i's preferences $\succsim_i$ and satisfies Assumption 5.2. We prove the theorem by showing that if $\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i} V_i(\sigma_i, \sigma_{-i}^*)$, then there exists a $\pi_i \in \Delta(\Omega)$ such that $\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i} EU_{\pi_i}(\sigma_i, \sigma_{-i}^*)$, whenever player i with preferences $\succsim_i$ shows no hedging behavior in $\bar{G}$. In other words, if player i's best response to σ_{-i}^* is σ_i^*, given that her preferences are $\succsim_i$, then there exists a prior such that σ_i^* is also a best response to σ_{-i}^* if i's preferences are $\succsim_i^{EU}$. This proves the theorem since we considered an arbitrary player i. The proof for general finite strategy spaces is a bit tedious and confusing. For this reason, the proof is given for four actions, $A_i = \{a_1, a_2, a_3, a_4\}$, the generalization is straightforward. Given σ_{-i}^*, let $f, g, h, k \in \mathbb{R}^m$ be the payoff vectors induced by i's pure actions, i.e., $f = f_i(a_1, \sigma_{-i}^*)$, $g = f_i(a_2, \sigma_{-i}^*)$, et cetera. Hence, i's payoffs are

	σ_{-i}^*
a_1	f
a_2	g
a_3	h
a_4	k

We distinguish two cases: player i's equilibrium strategy σ_i^* in $\langle \bar{G}, \succsim \rangle$ is 1. a degenerate mixed action (resp. a pure action) or 2. a proper mixed action.

Case 1. W.l.o.g., we may assume that $\sigma_i^* = a_1$ is i's equilibrium action in $\langle \bar{G}, \succsim \rangle$. Given that i exhibits no hedging behavior, we need to show that there exists a prior $\pi_i \in \Delta(\Omega)$ such that $EU_{\pi_i}(a_1, \sigma_{-i}^*) \geq EU_{\pi_i}(a_i, \sigma_{-i}^*)$ for $a_i \in \{a_1, a_2, a_3, a_4\}$. Note that this is equivalent to

$$\exists \pi_i \in \Delta(\Omega): \;\; (f-g)\pi_i \geq 0, (f-h)\pi_i \geq 0, \text{ and } (f-k)\pi_i \geq 0 \tag{5.1}$$

Let I be a $m \times m$ identity matrix and define

$$x = \begin{pmatrix} \pi_i \\ \\ \gamma \end{pmatrix} \in \mathbb{R}^{(m+1)},\; B = \begin{bmatrix} & 0 \\ I & \vdots \\ & 0 \end{bmatrix} \in \mathbb{R}^{m \times (m+1)},\; C = \begin{bmatrix} f-g & 0 \\ f-h & 0 \\ f-k & 0 \end{bmatrix} \in \mathbb{R}^{3 \times (m+1)}, \text{ and}$$
$$D = \begin{pmatrix} 1 & \dots & 1 & -1 \end{pmatrix} \in \mathbb{R}^{1 \times (m+1)}.$$

Then, condition (5.1) is equivalent to the system:

$$Bx \geq \mathbf{0},\; Cx \geqq \mathbf{0}, \text{ and } Dx = 0 \tag{5.2}$$

$Bx \geq \mathbf{0}$ ensures nonnegativity of the probabilities and $Dx = 0$ translates into $\sum_{\omega \in \Omega} \pi_i(\omega) = \gamma$, which can be normalized to $\sum_{\omega \in \Omega} \pi_i(\omega) = 1$. $Cx \geqq \mathbf{0}$ is the condition that a_1 is a best response to σ_{-i}^*.

Claim. System (5.2) *has a solution* $x \in \mathbb{R}^{(m+1)}$.

Proof. By Tucker's theorem of the alternative, cf. Mangasarian (1969, p. 29), either (5.2) has a solution $x \in \mathbb{R}^{(m+1)}$ or the equation $B^\top y^2 + C^\top y^3 + D^\top y^4 = \mathbf{0}$ has a solution $(y^2, y^3, y^4) \in \mathbb{R}^m \times \mathbb{R}^3 \times \mathbb{R}$ with $y^2 > \mathbf{0}$ and $y^3 \geqq \mathbf{0}$, which equals

$$\begin{bmatrix} \begin{pmatrix} y_1^2 \\ \vdots \\ y_m^2 \end{pmatrix} + (f-g)y_1^3 + (f-h)y_2^3 + (f-k)y_3^3 + \begin{pmatrix} y^4 \\ \vdots \\ y^4 \end{pmatrix} \\ -y^4 \end{bmatrix} = \mathbf{0} \tag{5.3}$$

Since $y^4 = 0$ and $y^2 > \mathbf{0}$, (5.3) has a solution iff (if and only if) there exists $y_1^3, y_2^3, y_3^3 \geq 0$ such that $(f-g)y_1^3 + (f-h)y_2^3 + (f-k)y_3^3 < \mathbf{0}$. This condition is equivalent to the existence of $\alpha, \beta \in [0,1]$ such that $f < \alpha g + \beta h + (1-\alpha-\beta)k$. Given σ_{-i}^*, the right-hand side of this inequality corresponds to the induced payoff vector of the following mixed action of player i: $\sigma_i' = \big(\sigma_i'(a_1), \sigma_i'(a_2), \sigma_i'(a_3), \sigma_i'(a_4)\big) = \big(0, \alpha, \beta, 1-\alpha-\beta\big)$. Hence, $f^\omega(a_1, \sigma_{-i}^*) < f^\omega(\sigma_i', \sigma_{-i}^*)$ for all $\omega \in \Omega$. Then, by Assumption 5.2 (monotonicity), $V_i(a_1, \sigma_{-i}^*) < V_i(\sigma_i', \sigma_{-i}^*)$ - a contradiction to the initial assumption that a_1 is the equilibrium strategy σ_i^* of player i in $\langle \bar{G}, \succsim \rangle$. Consequently, (5.3) has no solution, which proves that (5.2) has a solution. □

Case 2. The proof of the second case follows the same line as the proof of the first case. W.l.o.g. assume that player i's equilibrium strategy, σ_i^*, is a proper mixed action with $\text{supp}(\sigma_i^*) = \{a_1, a_2\}$. We need to show that there exists a prior $\pi_i \in \Delta(\Omega)$ such that such that $EU_{\pi_i}(\sigma_i^*, \sigma_{-i}^*) \geq EU_{\pi_i}(a_i, \sigma_{-i}^*)$ for $a_i \in \{a_1, a_2, a_3, a_4\}$. This is equivalent to the condition

$$\exists \pi_i \in \Delta(\Omega): \ (f-g)\pi_i = 0, (f-h)\pi_i \geq 0, (f-k)\pi_i \geq 0, (g-h)\pi_i \geq 0, (g-k)\pi_i \geq 0$$

which can be expressed as

$$Bx \geq \mathbf{0},\ Cx \geqq \mathbf{0}, \text{ and } Dx = \mathbf{0}, \tag{5.4}$$

where $x = \begin{pmatrix} \pi_i \\ \gamma \end{pmatrix} \in \mathbb{R}^{(m+1)}$, $B = \begin{bmatrix} & 0 \\ I & \vdots \\ & 0 \end{bmatrix} \in \mathbb{R}^{m \times (m+1)}$, $C = \begin{bmatrix} f-h & 0 \\ f-k & 0 \end{bmatrix} \in \mathbb{R}^{2 \times (m+1)}$,

and

$$D = \begin{bmatrix} 1 & \dots & 1 & -1 \\ & f-g & & 0 \end{bmatrix} \in \mathbb{R}^{2\times(m+1)}.$$

Claim. System (5.4) *has a solution* $x \in \mathbb{R}^{(m+1)}$.

Proof. According to Tucker's theorem, the alternative to the claim is that the system

$$\begin{bmatrix} \begin{pmatrix} y_1^2 \\ \vdots \\ y_m^2 \end{pmatrix} + (f-h)y_1^3 + (f-k)y_2^3 + (f-g)y_2^4 + \begin{pmatrix} y_1^4 \\ \vdots \\ y_1^4 \end{pmatrix} \\ -y_1^4 \end{bmatrix} = \mathbf{0} \tag{5.5}$$

has a solution $(y^2, y^3, y^4) \in \mathbb{R}^m \times \mathbb{R}^2 \times \mathbb{R}^2$ with $y^2 > \mathbf{0}$ and $y^3 \geqq \mathbf{0}$.

Equation (5.5) has a solution iff $(f-h)y_1^3+(f-k)y_2^3+(f-g)y_2^4 < 0$ for some $y_1^3, y_2^3 \geq 0$ and $y_2^4 \in \mathbb{R}$. For $y_2^4 \geq 0$, one obtains the same contradiction as before. If $y_2^4 < 0$, then there are $\alpha, \beta \in [0,1]$ such that $(\alpha+\beta)f+(1-\alpha-\beta)g < \alpha h+\beta k+(1-\alpha-\beta)f$. Given σ^*_{-i}, let σ'_i be the mixed action of i that induces the vector on the left-hand side of the inequality and σ''_i the action that induces the vector on the right-hand side. By Assumption 5.2 (monotonicity), $V_i(\sigma'_i, \sigma^*_{-i}) < V_i(\sigma''_i, \sigma^*_{-i})$. Furthermore, since i shows no hedging behavior, it holds that $V_i(\sigma'_i, \sigma^*_{-i}) = (\alpha+\beta)V_i(a_1, \sigma^*_{-i}) + (1-\alpha-\beta)(a_2, \sigma^*_{-i}) = V_i(\sigma^*_i, \sigma^*_{-i})$. Consequently, $V_i(\sigma^*_i, \sigma^*_{-i}) < V_i(\sigma''_i, \sigma^*_{-i})$, a contradiction to the initial assumption that σ^*_i is i's equilibrium strategy. This proves that (5.5) has no solution which implies that (5.4) has a solution. □

Since player i was chosen arbitrarily, for any given equilibrium point σ^* of $\langle \bar{G}, \succsim \rangle$, there exists a prior π_i for each $i \in I$ such that σ^* is an equilibrium point of $\langle \bar{G}, \succsim^{EU} \rangle$, which proves the theorem. □

In order to prove Theorem 5.2, we need the following lemma:

Lemma 5.2. *Let* Δ^d *be the d-dimensional unit simplex,* $d < \infty$, *and let* $\mathscr{B}$ *be a finite collection of closed, convex and nonempty sets . If*

(i) $\bigcup_{B \in \mathscr{B}} B = \Delta^d$ *and*

(ii) $int(B') \cap int(B'') = \varnothing$ *for all* $B', B'' \in \mathscr{B}$,

then each B *in* $\mathscr{B}$ *is a polyhedron.*

Proof. If $\mathscr{B}$ is a singleton, then the statement is trivial by (i). Assume that $\mathscr{B}$ is not a singleton. Since each B in $\mathscr{B}$ is closed (i.e., $\partial B \subseteq B$), (ii) implies that $B' \cap B'' = \partial B' \cap \partial B''$ for all $B', B'' \in \mathscr{B}$. Furthermore, by (i), if $x \in \partial B'$, then $x \in \partial B''$ for some $B'' \in \mathscr{B}$ and/or $x \in \partial\Delta^d$, formally $\partial B' = \left[\bigcup_{B'' \in \mathscr{B} \setminus B'} (B' \cap B'')\right] \cup (\partial B' \cap \Delta^d)$.

It holds that $\partial B' = cl(\partial B')$ because $\partial B'$ is closed. Hence, $\partial B' = int(\partial B') \,\dot{\cup}\, \partial\partial B'$. Due to $\partial\partial B' = \partial B'$, it follows that $int(\partial B') = \varnothing$. Therefore, $int(B' \cap B'') = \varnothing$. Furthermore, $(\partial B' \cap \partial B'')$ is closed and convex since it is an intersection of closed and convex sets (recall that $B' \cap B'' = \partial B' \cap \partial B''$). Taken together, $(\partial B' \cap \partial B'')$ is a closed and convex set with empty interior, which implies that $(\partial B' \cap \partial B'')$ is contained in a hyperplane. In addition, $(\partial B' \cap \partial\Delta^d)$ is contained in a hyperplane since $\partial\Delta^d$ is contained in a hyperplane. Thus, $(\partial B' \cap \partial B'')$ is contained in a hyperplane for all $B'' \in \mathscr{B} \setminus B'$ and $(\partial B' \cap \partial\Delta^d)$ is contained in a hyperplane. Therefore, $\partial B'$ is contained in the union of finitely many hyperplanes, formally $\partial B' \subseteq \bigcup_{n \in N} H_n$, where H_n is a hyperplane and N an index set. Let $\mathscr{H}_n$ be a half-space, which is associated with hyperplane n. Then, there exists n half-spaces such that $B' \subseteq \bigcap_{n \in N} \mathscr{H}_n$ since B' is a convex set. Furthermore, it holds that $B' \supseteq \bigcap_{n \in N} \mathscr{H}_n$ since the boundary of B' is contained in the hyperplanes associated with the half-spaces. Consequently, B' equals the intersection of finitely many half-spaces. That is, B' is a polyhedron, which proves the claim, since B' was chosen arbitrarily. □

Proof of Theorem 5.2. "(i) $\implies$ (ii)". The proof is based on the following fact: consider a finite two-player normal-form game with complete information or with incomplete information and EU players. Let $i \in \{1, 2\}$ and $j = 3 - i$ denote the players. Then, for each player i, it holds that the preimages of i's pure actions under her best-response correspondence are either empty or polyhedral subsets of the set of j's mixed strategies, Σ_j, which corresponds to the $|A_j|$-dimensional unit simplex. For example, consider the preimages of player i's pure strategies in the well-known Rock-paper-scissors-game given in Figure 5.3.[37]

[37]Figure 5.3 shows the two-dimensional projection of Σ_j.

Consequently, if, for each player i, the preimages of i's pure actions under her best response correspondence in a two-player game $\langle G, \succsim \rangle$ satsify

(*) the union of all preimages equal Σ_j and

(**) the preimage of every pure action is either empty or a polyhedron,

then there exists a two-player game $\langle G', \succsim^{EU} \rangle$ which is best response equivalent to $\langle G, \succsim \rangle$. In other words, (*) and (**) imply statement (ii) of the theorem. Therefore, if statement (i) implies (*) and (**), then (i) implies (ii).

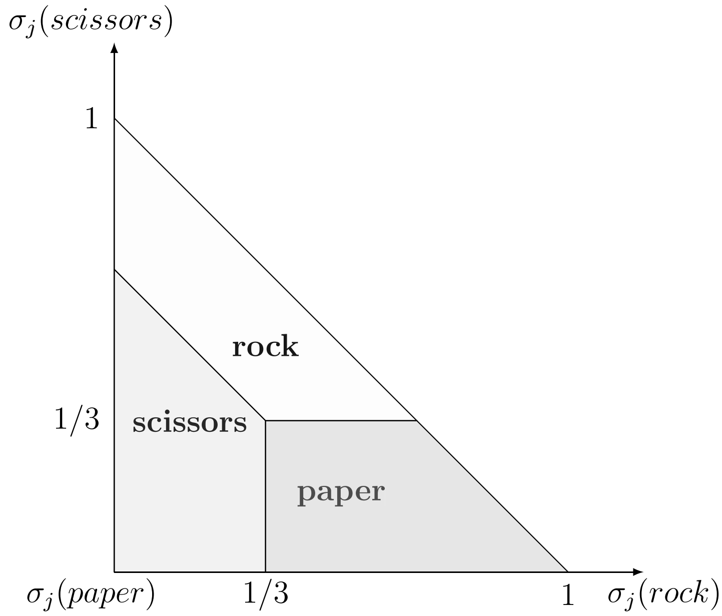


Figure 5.3: Preimages of pure strategies under a player's best-response correspondence in Rock-paper-scissors

Consider player i and suppose she has K pure actions: $A_i = \{a_i^1, \dots, a_i^K\}$. Let B_k be the preimage of action $a_i^k \in A_i$ under i's best-response correspondence, formally $B_k = \{\sigma_j \in \Sigma_j \mid \sigma_j \in R_i(a_i^k)\}$.

Claim. (i) implies (*).

Proof. By statement (i) of the theorem, i exhibits no hedging behavior in G. This implies that for every $\sigma_j \in \Sigma_j$, there exists a pure action $a_i \in A_i$ which is a best response to σ_j, i.e., $\bigcup_{i=1}^{K} B_k \supseteq \Sigma_j$. Furthermore, by the definition of a best-response correspondence,

$\bigcup_{i=1}^{K} B_k \subseteq \Sigma_j$. Hence,

$$\bigcup_{i=1}^{K} B_k = \Sigma_j, \tag{5.6}$$

which means that (i) implies (*). □

Claim. (i) implies (**).

Proof. Due to (i), i shows no reversal behavior in G. The negation of condition (ii) in Definition 5.4 implies:

$$int(B_k) \cap int(B'_k) = \varnothing \tag{5.7}$$

for all $k, k' \in \{1, \ldots, K\}$, $k \neq k'$.

W.l.o.g., we may assume that $B_k \neq \varnothing$ and $B_k \neq B_{k'}$ for all $k, k' \in \{1, \ldots, K\}$, $k \neq k'$. According to Assumption 5.2, the function $V_i(\cdot)$, which represents i's preferences, is continuous. Therefore, each B_k is closed. Furthermore, the negation of condition (i) in Definition 5.4 implies that each B_k is convex. Considering these properties together with equation (5.6) and (5.7) and using Lemma 5.2, we see that each B_k is a polyhedron. □

To sum up, $(i) \Rightarrow$ (*) and (**) $\Rightarrow (ii)$.

"(i) $\Longleftarrow$ (ii)". The examples in Section 5.1.1 illustrate that $\neg(i) \Rightarrow \neg(ii)$, which is logically equivalent to (i) $\Leftarrow$ (ii). □

Notation 5.2. From now on, $f, g, h, k \in \mathbb{R}^m$ denote player i's payoff vectors which are induced by pure actions profiles in a given two-person two-strategies canonical game, i.e., i's payoff matrix is:

	a'_{-i}	a''_{-i}
a'_i	f	g
a''_i	h	k

Proof of Proposition 5.1. In some parts of the proof, the argumentation is based on theorems of the alternative like in the proof of Theorem 5.1. These parts of the proof will be only sketched.

(i) Consider player i and suppose she has a strictly dominant strategy in $\langle \bar{G}, \succsim \rangle$. W.l.o.g. assume that a_i' is strictly dominant. If

$$\exists \pi_i \in \Delta(\Omega) : (f - h)\pi_i > 0 \text{ and } (g - k)\pi_i > 0, \tag{5.8}$$

then a_i' is also a strictly dominant strategy for i in case she has EU preferences and the subjective prior π_i. By applying Motzkin's theorem, cf. Mangasarian (1969, p. 28-29), we obtain an alternative to condition (5.8). This alternative has a solution iff $\alpha f + (1 - \alpha)g \leqq \alpha h + (1 - \alpha)k$ for some $\alpha \in [0, 1]$. Then, by Assumption 5.2 (monotonicity), there exists a $\sigma_{-i} \in \Sigma_{-i}$ such that $V_i(a_i', \sigma_{-i}) \leq V_i(a_i'', \sigma_{-i})$. This contradicts the assumption that a_i' is a strictly dominant strategy. Consequently, (5.8) has a solution.

(ii) Suppose i has no strictly dominant strategy. By Theorem 5.2, $\langle \bar{G}, \succsim \rangle$ is best response equivalent to some $\langle G', \succsim^{EU} \rangle$. Let $f', g', h', k' \in \mathbb{R}^m$ be i's payoff vectors induced by pure action profiles in G'. Note that $\langle G', \succsim^{EU} \rangle$ is best response equivalent to a two-person complete information game with identical action sets, where player i's payoffs equal the expected utility values: $U_{f'} = EU_{\pi_i}(f')$, $U_{g'} = EU_{\pi_i}(g')$, et cetera (see matrix (a) below). Furthermore, it is well-known that player i's best response sets are unaffected if we transform her payoff matrix (a) into matrix (b) where $z > 0$ and $\varepsilon, \delta \in \mathbb{R}$ (see, e.g., Weibull, 1995).

(a)

	a_{-i}'	a_{-i}''
a_i'	$U_{f'}$	$U_{g'}$
a_i''	$U_{h'}$	$U_{k'}$

(b)

	a_{-i}'	a_{-i}''
a_i'	$zU_{f'} + \varepsilon$	$zU_{g'} + \delta$
a_i''	$zU_{h'} + \varepsilon$	$zU_{k'} + \delta$

To sum up, $\langle \bar{G}, \succsim \rangle$ is best response equivalent to $\langle G', \succsim^{EU} \rangle$ and $\langle G', \succsim^{EU} \rangle$ is best response equivalent to a complete information game where player i's payoff matrix is matrix (b) above. Consequently, the second part of the proposition is proven if

$$\begin{gathered} \exists \pi_i \in \Delta(\Omega), z > 0, \varepsilon, \delta \in \mathbb{R} : \\ f\pi_i = zU_{f'} + \varepsilon, h\pi_i = zU_{h'} + \varepsilon, g\pi_i = zU_{g'} + \delta \text{ and } k\pi_i = zU_{k'} + \delta. \end{gathered} \tag{5.9}$$

By using Motzkin's theorem again, we obtain an alternative to (5.9) which has a solution iff $(f-h)y_1^4 + (g-k)y_3^4 \leqq \mathbf{0}$ and $(U_{f'} - U_{h'})y_1^4 + (U_{g'} - U_{k'})y_3^4 > 0$ for some $y_1^4, y_3^4 \in \mathbb{R}$. For $y_1^4 = 0$, $y_3^4 = 0$, $y_1^4, y_3^4 > 0$ and $y_1^4, y_3^4 < 0$, we get a similar contradiction as in case of a strictly dominant strategy. If $y_1^4 > 0$ and $y_3^4 < 0$, the first part of the alternative condition equals $(f-h)a \leqq (g-k)$ for some $a > 0$. However, by restriction (ii) of the proposition, there exists a $\omega'' \in \Omega$ such that $(f^{\omega''} - h^{\omega''}) > 0$ and $(g^{\omega''} - k^{\omega''}) < 0$ which contradicts this condition. Similarly, restriction (ii) contradicts the first part of the alternative if $y_1^4 < 0$ and $y_3^4 > 0$. Therefore, (5.9) has a solution, which completes the proof.

□

Proof of Proposition 5.4. Consider a two-players two-strategies game and fix a $\bar{\sigma}_{-i} \in \Sigma_{-i}$. Let $f(a_i', \bar{\sigma}_{-i}) = f$ and $f(a_i'', \bar{\sigma}_{-i}) = g$ denote player i's payoff vectors induced by her pure actions. Note that every vector induces, through expectation, an ordering on probabilities. The proof is based on the fact that affine-relatedness implies that the induced orderings of two vectors are identical (see Ghirardato et al., 1998). That is, if f and $-g$ are affinely related, then f and g induce opposite orderings on the probabilities. Assume that the set $\mathscr{C}^*$ is nonempty, which implies that f and g are not dominance related and non-constant. Take an arbitrary $C_i \in \mathscr{C}^*$. Since there are $\pi', \pi'' \in C_i$ such that $f\pi' \neq f\pi''$ and $g\pi' \neq g\pi''$, it holds that $\arg\min_{\pi \in C_i} \{EU_\pi(f)\} \cap \arg\max_{\pi \in C_i} \{EU_\pi(f)\} = \varnothing$ and $\arg\min_{\pi \in C_i} \{EU_\pi(g)\} \cap \arg\max_{\pi \in C_i} \{EU_\pi(g)\} = \varnothing$. Furthermore, if f and g are negatively affinely related, it holds that $\arg\min_{\pi \in C_i} \{EU_\pi(f)\} \cap \arg\min_{\pi \in C_i} \{EU_\pi(g)\} = \varnothing$. Then, by Lemma 1 in Ghirardato et al. (1998), $MEU_{C_i}(f+g) \neq MEU_{C_i}(f) + MEU_{C_i}(g)$. Hence, we are done if $MEU_{C_i}(f) = MEU_{C_i}(g)$. W.l.o.g. assume that $MEU_{C_i}(f) > MEU_{C_i}(g)$. Let $MEU_{C_i}(f) = f\tilde{\pi}$. Since there is a $\pi'' \in C_i$ such that $f\pi'' < g\pi''$, we have that $f\tilde{\pi} \leq f\pi'' < g\pi''$. Moreover, it holds that $g\pi'' \leq g\tilde{\pi}$ because f is negatively affinely related to g. Hence, $f\tilde{\pi} < g\tilde{\pi}$. Then, for sufficiently high $\alpha \in [0,1] : MEU_{C_i}(\alpha f + (1-\alpha)g) = \alpha f\tilde{\pi} + (1-\alpha)g\tilde{\pi} > f\tilde{\pi} = MEU_{C_i}(f)$. This means that there exists a mixed action, $(\sigma_i(a_i'), \sigma_i(a_i'')) = (\alpha, 1-\alpha)$, which is a strictly better response to $\bar{\sigma}_{-i}$ than a_i'. Since we assumed that a_i' is a strictly better response to $\bar{\sigma}_{-i}$ than a_i'', player i exhibits hedging behavior, which proves the proposition. □

Before proving Proposition 5.5, we need a couple of lemmas.

Lemma 5.3. *Fix a two-player two-strategies canonical game* $\bar{G} \in \Gamma$ *where at most one of* $f, g, h, k \in \mathbb{R}^m$ *is constant and* f, h *and* g, k *are not strictly dominance related. If a* MEU_{C_i} *player* i *exhibits no hedging behavior in* $\bar{G}$ *for all* $C_i \in \mathscr{C}$*, then one of the following statements is true:*

(i) f, h *and* g, k *are affinely related, there is no* $\omega' \in \Omega$ *such that* $f^{\omega'} = h^{\omega'}$ *and* $g^{\omega'} = k^{\omega'}$*, and* h *is weakly dominated by* f *and* k *by* g *or vice versa.*

(ii) f, h *and* g, k *are affinely related and* $f = h$ *and/or* $g = k$*.*

(iii) f, g, h, k *are pairwise affinely related.*

(iv) $f, -g, h, -k$ *are pairwise affinely related.*

Proof. Since f, h and g, k are not strictly dominance related, by Lemma 5.1, if a MEU_{C_i} player i shows no hedging behavior in $\bar{G}$ for all $C_i \in \mathscr{C}$, then f, h and g, k are affinely related. Hence, for all $\omega \in \Omega$, it holds that,

(*) $h^\omega = a' f^\omega + b'$ for some $a' \geq 0$, $b' \in \mathbb{R}$ and

(**) $k^\omega = a'' g^\omega + b''$ for some $a'' \geq 0$, $b'' \in \mathbb{R}$.

Furthermore, either

(***) $\alpha f + (1-\alpha) g$ and $\alpha h + (1-\alpha) k$ are strictly dominance related for all $\alpha \in (0, 1)$

or

(****) $\alpha' f^\omega + (1-\alpha') g^\omega = a'''[\alpha' h^\omega + (1-\alpha') k^\omega] + b'''$ for some $a''' \geq 0$, $b''' \in \mathbb{R}$.

(****) follows whenever (***) is false. To see why, observe that if (***) is false, then i shows no hedging behavior in $\bar{G}$ for all $C_i \in \mathscr{C}$ only if $\alpha' f + (1-\alpha')$ is affinely related to $\alpha' h + (1-\alpha') k$ whenever $\alpha' f + (1-\alpha')$ and $\alpha' h + (1-\alpha') k$ are not strictly dominance related. Hence, there exist $\alpha' \in (0, 1)$ such that, for all $\omega \in \Omega$ (****) is true.

Claim. (***) implies (i).

Proof. Suppose (***) is true. Since f, h and g, k are not strictly dominance related (***) is true iff there is no $\omega' \in \Omega$ such that $f^{\omega'} = h^{\omega'}$ and $g^{\omega'} = k^{\omega'}$, and h is weakly dominated by f and k by g or vice versa. □

Claim. (*), (**), and (****) imply (ii) or (iii) or (iv).

Proof. Suppose (***) is not true. Then, there exist $\alpha' \in (0,1)$ such that $\alpha' f + (1-\alpha')g$ and $\alpha' h + (1-\alpha')k$ are not dominance related and non-constant.

(ii) If $f = h$, then (**) implies (****). Similarly, (*) implies (****) whenever $g = k$.

W.l.o.g. assume that f and g are non-constant and let $f \neq h$ and $g \neq k$.

(iii) If f is affinely related to g, then (*) and (**) imply that h and k are affinely related, either because one of the vectors is constant or by transitivity. Hence, all vectors are pairwise affinely related.

(iv) If f and g are not affinely related, then (*),(**), (****) imply that $f^\omega = \tilde{b} g^\omega + \hat{b}$ for all $\omega \in \Omega$ and some $\tilde{b}, \hat{b} \in \mathbb{R}$ where $\tilde{b} \neq 0$, otherwise f is constant. Since f and g are not affinely related, it holds that $\tilde{b} < 0$, which means that f is affinely related to $-g$. Then, h and $-k$ are affinely related by transitivity or because one of the vectors is constant.

□

□

Lemma 5.4. *Let at most one of the payoff vectors f, g, h, k be constant and $f, -g, h, -k$ be pairwise affinely related. If there exist $\pi', \pi'' \in \Delta(\Omega)$ such that $\frac{f\pi' - f\pi''}{g\pi'' - g\pi'} \neq \frac{h\pi' - h\pi''}{k\pi'' - k\pi'}$, then $\alpha f + (1-\alpha)g$ is not affinely related to $\alpha h + (1-\alpha)k$ for some $\alpha \in (0,1)$.*

Proof. W.l.o.g. assume that $\frac{f\pi' - f\pi''}{g\pi'' - g\pi'} > \frac{h\pi' - h\pi''}{k\pi'' - k\pi'}$ and $f\pi' > f\pi''$. The latter implies that $h\pi' > h\pi''$, $g\pi' < g\pi''$, and $k\pi' < k\pi''$, since $f, -g, h, -k$ are pairwise affinely related. Therefore, it holds that $\alpha f\pi' + (1-\alpha)g\pi' \geq \alpha f\pi'' + (1-\alpha)g\pi''$ for all $\alpha \geq \frac{g\pi'' - g\pi'}{g\pi'' - g\pi' + f\pi' - f\pi''}$ and $\alpha h\pi' + (1-\alpha)k\pi' \geq \alpha h\pi'' + (1-\alpha)k\pi''$ for all $\alpha \geq \frac{k\pi'' - k\pi'}{k\pi'' - k\pi' + h\pi' - h\pi''}$. Furthermore, $\frac{f\pi' - f\pi''}{g\pi'' - g\pi'} > \frac{h\pi' - h\pi''}{k\pi'' - k\pi'}$ implies that $\frac{g\pi'' - g\pi'}{g\pi'' - g\pi' + f\pi' - f\pi''} < \frac{k\pi'' - k\pi'}{k\pi'' - k\pi' + h\pi' - h\pi''}$. Consequently, $\alpha f\pi' + (1-\alpha)g\pi' > \alpha f\pi'' + (1-\alpha)g\pi''$ and $\alpha h\pi' + (1-\alpha)k\pi' < \alpha h\pi'' + (1-\alpha)k\pi''$ for all $\alpha \in \left(\frac{g\pi'' - g\pi'}{g\pi'' - g\pi' + f\pi' - f\pi''}, \frac{k\pi'' - k\pi'}{k\pi'' - k\pi' + h\pi' - h\pi''}\right)$. That is, there exist $\alpha \in (0,1)$ such that $\alpha f + (1-\alpha)g$ and $\alpha h + (1-\alpha)k$ induce different orderings on probabilities, which means that these payoff vectors are not affinely related. □

Lemma 5.5. *Let* $e(\omega) = \frac{(k^\omega - g^\omega)}{(k^\omega - g^\omega + f^\omega - h^\omega)}$ *for* $\omega \in \Omega$ *and define the sets:*
$E_- = \{e(\omega) \mid (k^\omega - g^\omega + f^\omega - h^\omega) < 0\}$ *and* $E_+ = \{e(\omega) \mid (k^\omega - g^\omega + f^\omega - h^\omega) > 0\}$.
The following statements are equivalent.

(i) $\alpha f + (1-\alpha)g$ *strictly dominates* $\alpha h + (1-\alpha)k$ *for some* $\alpha \in [0,1]$.

(ii) (a) For each $\omega \in \Omega$*:* $f^\omega > h^\omega$ *and/or* $g^\omega > k^\omega$ *and (b)* $\max\{E_+\} < \min\{E_-\}$.

Proof. Statement (i) says that there exist $\alpha \in [0,1]$ which solves the following system of linear inequalities:

$$\begin{gathered} \alpha f^{\omega_1} + (1-\alpha)g^{\omega_1} > \alpha h^{\omega_1} + (1-\alpha)k^{\omega_1} \\ \vdots \\ \alpha f^{\omega_m} + (1-\alpha)g^{\omega_m} > \alpha h^{\omega_m} + (1-\alpha)k^{\omega_m} \end{gathered}$$

This system is solvable iff each inequality has a nonempty solution set, which corresponds to condition (ii)(a), and the solutions sets of all inequalities have a nonempty intersection, which is equivalent to condition (ii)(b). □

Proof of Proposition 5.5. The proof of "(i) $\Longrightarrow$ (ii)" is trivial. "(i) $\Longleftarrow$ (ii)". Under the assumptions of the proposition, Lemma 5.3 shows that statement (ii) implies either (i) or $f, -g, h, -k$ are pairwise affinely related. Suppose that (ii) implies the latter. By the assumptions of the proposition, it holds that $\nexists\, \alpha \in [0,1] : \alpha f + (1-\alpha)g > \alpha h + (1-\alpha)k$ or vice versa. The negation of Lemma 5.5 implies that

$$f^{\omega'} \leq h^{\omega'} \text{ and } g^{\omega'} \leq k^{\omega'} \text{ for some } \omega' \in \Omega \text{ and/or } \max\{E_+\} \geq \min\{E_-\} \text{ and} \tag{5.10}$$

$$h^{\omega''} \leq f^{\omega''} \text{ and } k^{\omega''} \leq g^{\omega''} \text{ for some } \omega'' \in \Omega \text{ and/or } \max\{E_-\} \geq \min\{E_+\}. \tag{5.11}$$

At first, consider the case where the first condition of (5.10) and/or (5.11) is violated. W.l.o.g. assume that the first condition of (5.10) is violated. That is, for each $\omega \in \Omega$: $f^\omega > h^\omega$ and/or $g^\omega > k^\omega$. Furthermore, $\max\{E_+\} \geq \min\{E_-\}$, otherwise $\alpha f + (1-\alpha)g > \alpha h + (1-\alpha)k$ for some $\alpha \in [0,1]$. If $f^\omega > h^\omega$ for all $\omega \in \Omega$ and/or $g^\omega > k^\omega$ for all $\omega \in \Omega$, then f strictly dominates h and/or g strictly dominates k, which contradicts the assumptions of the proposition. Therefore, suppose that there are $\omega', \omega'' \in \Omega$ such that

$f^{\omega'} \leq h^{\omega'}$ and $g^{\omega''} \leq k^{\omega''}$. Let $e(\omega_+) \in \max\{E_+\}$ and $e(\omega_-) \in \min\{E_-\}$. Due to $f^{\omega_-} > h^{\omega_-}$ and/or $g^{\omega_-} > k^{\omega_-}$, it holds that $e(\omega_-) > 0$. If $g^{\omega_+} \geq k^{\omega_+}$, then $e(\omega_+) \leq 0 < e(\omega_-)$ - a contradiction. Therefore, $g^{\omega_+} < k^{\omega_+}$ and $f^{\omega_+} > h^{\omega_+}$, which implies that $e(\omega_+) < 1$. If $f^{\omega_-} \geq h^{\omega_-}$, then $e(\omega_-) \geq 1 > e(\omega_+)$ - a contradiction. Therefore, $f^{\omega_-} < h^{\omega_-}$ and $g^{\omega_-} > k^{\omega_-}$. Taken together, we have that,

(*) $g^{\omega_+} < k^{\omega_+}$ and $f^{\omega_+} > h^{\omega_+}$; $f^{\omega_-} < h^{\omega_-}$ and $g^{\omega_-} > k^{\omega_-}$.

W.l.o.g. we may assume that $f^{\omega_+} \leq f^{\omega_-}$. Then, since f is affinely related to h and negatively affinely related to g and k,

(**) $h^{\omega_+} < h^{\omega_-}$, $g^{\omega_+} \geq g^{\omega_-}$ and $k^{\omega_+} > k^{\omega_-}$.

Now, consider the prior set $\bar{C}_i = \{\beta\delta_{\omega_+} + (1-\beta)\delta_{\omega_-} \mid \beta \in [0,1]\}$ where δ_ω denotes the measure concentrated on $\omega \in \Omega$. Then, by (*) and (**), $MEU_{\bar{C}_i}(f) = f^{\omega_+} > h^{\omega_+} = MEU_{\bar{C}_i}(h)$ and $MEU_{\bar{C}_i}(g) = g^{\omega_-} > k^{\omega_-} = MEU_{\bar{C}_i}(k)$. This means that action a_i' is the unique best response of a $MEU_{\bar{C}_i}$ player i to a_{-i}' and a_{-i}''. Consequently, it needs to hold that a_{-i}' is the unique best response to $\alpha a_{-i}' + (1-\alpha)a_{-i}''$ for all $\alpha \in [0,1]$. Otherwise, player i exhibits reversal behavior, which contradicts statement (ii). Let $\underline{\alpha} = \frac{k^{\omega_+} - g^{\omega_+}}{k^{\omega_+} - g^{\omega_+} + f^{\omega_+} - h^{\omega_+}} \in (0,1)$ and $\overline{\alpha} = \frac{g^{\omega_-} - k^{\omega_-}}{g^{\omega_-} - k^{\omega_-} + h^{\omega_-} - f^{\omega_-}} \in (0,1)$. Then, $\alpha f^{\omega_+} + (1-\alpha) g^{\omega_+} \leq \alpha h^{\omega_+} + (1-\alpha) k^{\omega_+}$ for all $\alpha \in [0, \underline{\alpha}]$ and $\alpha f^{\omega_-} + (1-\alpha) g^{\omega_-} \leq \alpha h^{\omega_-} + (1-\alpha) k^{\omega_-}$ for all $\alpha \in [\overline{\alpha}, 1]$. Player i exhibits no reversal behavior only if $\underline{\alpha} < \frac{g^{\omega_+} - g^{\omega_-}}{g^{\omega_+} - g^{\omega_-} + f^{\omega_-} - f^{\omega_+}} < \overline{\alpha}$, which is equivalent to

(***) $\frac{k^{\omega_+} - g^{\omega_+}}{f^{\omega_+} - h^{\omega_+}} < \frac{g^{\omega_+} - g^{\omega_-}}{f^{\omega_-} - f^{\omega_+}}$ and $\frac{k^{\omega_-} - g^{\omega_-}}{f^{\omega_-} - h^{\omega_-}} > \frac{g^{\omega_+} - g^{\omega_-}}{f^{\omega_-} - f^{\omega_+}}$.

However, (*), (**), (***), and the affine-relatedness condition from Lemma 5.4, $\frac{f^{\omega_-} - f^{\omega_+}}{g^{\omega_+} - g^{\omega_-}} = \frac{h^{\omega_-} - h^{\omega_+}}{k^{\omega_+} - k^{\omega_-}}$, lead to a contradiction (see the Mathematica code at the end of this proof). That is, either a $MEU_{\bar{C}_i}$ player exhibits reversal behavior or there exists a $C_i \in \mathscr{C}$ such that a MEU_{C_i} player exhibits hedging behavior. Consequently, the first condition of (5.10) and (5.11) need to be both fulfilled. This implies that there are $\omega', \omega'' \in \Omega$ such that

(****) $f^{\omega''} - f^{\omega'} \geq h^{\omega''} - h^{\omega'}$ and $g^{\omega'} - g^{\omega''} \leq k^{\omega'} - k^{\omega''}$.

Define the prior set $\tilde{C}_i = \{\beta\delta_{\omega'} + (1-\beta)\delta_{\omega''} \mid \beta \in [0,1]\}$. If the inequalities in (****) are strict, it holds that $\frac{f^{\omega''} - f^{\omega'}}{g^{\omega'} - g^{\omega''}} > \frac{h^{\omega''} - h^{\omega'}}{k^{\omega'} - k^{\omega''}}$, which means that a $MEU_{\tilde{C}_i}$ player i shows hedging behavior due to Lemma 5.4. At least one of the inequalities in (****) is not strict iff

(*****) ($f^{\omega'} = h^{\omega'}$ and $f^{\omega''} = h^{\omega''}$) and/or ($g^{\omega'} = k^{\omega'}$ and $g^{\omega''} = k^{\omega''}$).

Consider (*****) with "and". Then, $f^{\omega'} = h^{\omega'}$ and $g^{\omega'} = k^{\omega'}$. By the proposition, at most one of the acts is constant. Suppose that f is constant, which implies that g, h, k are not constant. Since $f \neq h$ and $g \neq k$, there exists a $\pi' \in \Delta(\Omega)$ such that $h^{\omega'} \neq h\pi'$, which implies that $f\pi' \neq h\pi'$, and $g^{\omega'} \neq g\pi'$, $k^{\omega'} \neq k\pi'$ and $g\pi' \neq h\pi'$. Define the prior set $\hat{C}_i = \{\beta\delta_{\omega'} + (1-\beta)\pi' \mid \beta \in [0,1]\}$. Since $f, -g, h, -k$ are pairwise affinely related either $MEU_{\hat{C}_i}(f) = f^{\omega'}$ and $MEU_{\hat{C}_i}(h) = h^{\omega'}$ or $MEU_{\hat{C}_i}(g) = g^{\omega'}$ and $MEU_{\hat{C}_i}(k) = k^{\omega'}$, but not both. W.l.o.g. assume that $MEU_{\hat{C}_i}(f) = f^{\omega'}$ and $MEU_{\hat{C}_i}(h) = h^{\omega'}$. Given $\alpha a'_{-i} + (1-\alpha)a''_{-i}$, a $MEU_{\hat{C}_i}$ player is indifferent between her actions for all $\alpha \in [0, \alpha']$ and strictly prefer one of her pure actions for $\alpha \in [\alpha'', 1]$, where α' is sufficiently low and α'' is sufficiently large. That is, a $MEU_{\hat{C}_i}$ player shows reversal behavior - a contradiction. Similarly, one can show that (*****) with "or" yields a contradiction.

To sum up, if (ii) implies that $f, -g, h, -k$ are pairwise affinely related, we obtain a contradiction to the assumptions of the proposition, which proves that (ii) implies (i). □

```
Define
f⁺ = f', f⁻ = f'' etc.
f, h and g, k are affinely related :
h = a * f + b
k = a' * g + b'
Affine - relatedness condition :
(f'' - f') / (g' - g'') = (h'' - h') / (k' - k'') = (a * (f'' - f')) / (a' * (g⁺ - g⁻))
implies that a = a'.
Condition (*)
g' < a * g' + b', f' > a * f' + b, f'' < a * f'' + b, g'' > a * g'' + b'
Condition (**)
a * f' + b < a * f'' + b, g' ≥ g'', f' ≤ f'', a * g' + b' > a * g'' + b'
Condition (***)
((a - 1) * g' + b') / ((1 - a) * f' - b) < (g' - g'') / (f'' - f') and
((1 - a) * g'' - b') / ((a - 1) * f'' + b) > (g' - g'') / (f'' - f')
implies that
((a - 1) * g' + b') / ((1 - a) * f' - b) < ((1 - a) * g'' - b') / ((a - 1) * f'' + b).
These conditions together yield a contradiction :
In[38]:= Reduce[((1 - a) * g'' - b') / ((a - 1) * f'' + b) > ((a - 1) * g' + b') / ((1 - a) * f' - b) &&
   f' < f'' && g' > g'' && ((1 - a) * g'' - b') > 0 &&
   ((a - 1) * f'' + b) > 0 && ((a - 1) * g' + b') > 0 && ((1 - a) * f' - b) > 0 &&
   (g' - g'') / (f'' - f') > ((a - 1) * g' + b') / ((1 - a) * f' + b) && f' < f'' &&
   g' > g'' && ((1 - a) * f' + b) > 0 && ((a - 1) * g' + b') > 0 && a > 0 &&
   (g' - g'') / (f'' - f') < ((a - 1) * g'' + b') / ((1 - a) * f'' + b) && f' < f'' &&
   g' > g'' && ((1 - a) * f'' + b) > 0 && ((a - 1) * g'' + b') > 0 && a > 0]
Out[38]= False
```

Proof of Proposition 5.6. The proof of "(i)(a) $\implies$ (ii)" and of "(i)(b) $\implies$ (ii)" is straightforward. We prove "(i) $\Longleftarrow$ (ii)" by its contrapositive "$\neg(i) \Rightarrow \neg(ii)$". Suppose that $\neg(i)(a)$ and $\neg(i)(b)$ is true. Then, it holds that f, g, h, k are not pairwise affinely related and if f (resp. h) is constant, then g (resp. k) is not constant and vice versa. There are two cases to consider:

Case 1. Let f and h be constant and g and k be non-constant. Since f, g, h, k are not pairwise affinely related, it needs to hold that g is not affinely related to k. By the proposition, there exists $\alpha' \in [0,1]$ such that $\alpha' f + (1-\alpha')g$ and $\alpha' h + (1-\alpha')k$ are not strictly dominance related. By Lemma 5.1, a MEU_{C_i} player i exhibits no hedging behavior for all $C_i \in \mathscr{C}$ only if $\alpha' f + (1-\alpha')g$ is affinely related to $\alpha' h + (1-\alpha')k$, i.e., (*) $\alpha' f^\omega + (1-\alpha')g^\omega = a[\alpha' h^\omega + (1-\alpha')k^\omega] + b$ for all $\omega \in \Omega$ and some $a > 0, b \in \mathbb{R}$.

Since f and h are constant, (*) is equivalent to $g^\omega = ak^\omega + \tilde{b}$ for all $\omega \in \Omega$ and some $a > 0, \tilde{b} \in \mathbb{R}$, which means that g is affinely related to k - a contradiction. Therefore, a MEU_{C_i} player i shows hedging behavior, whenever $\neg(i)(a)$ is true.

Case 2. Let f and k be constant and g and h be non-constant. This case can be proven similarly to the previous one.

Therefore, "$\neg(i) \Rightarrow \neg(ii)$" $\Leftrightarrow$ "(i) $\Leftarrow$ (ii)". □

Proof of Proposition 5.7. The proof of "(i)(a) $\Longrightarrow$ (ii)" and of "(i)(b) $\Longrightarrow$ (ii)" is straightforward. As in the previous proof, we prove "(i) $\Longleftarrow$ (ii)" by its contrapositive. Let $\neg(i)$ be true. Then, f, g, h, k are not pairwise affinely related and if $f = h$ (resp. $g = k$), then $g \neq k$ (resp. $f \neq h$). W.l.o.g. assume that $f = h$ and $g \neq k$. Since g and k are not strictly dominance related, $\alpha f + (1-\alpha)g$ and $\alpha h + (1-\alpha)k$ are not strictly dominance related for all $\alpha \in [0,1]$. Due to Lemma 5.1, if there exists a $\alpha' \in [0,1]$ such that $\alpha' f + (1-\alpha')g$ is not affinely related $\alpha' h + (1-\alpha')k$, then a MEU_{C_i} player i shows hedging behavior for some $C_i \in \mathscr{C}$, i.e., $\neg(ii)$ is true. If g is affinely related to k, then (*) $g^\omega = a' k^\omega + b'$ for all $\omega \in \Omega$ and some $a' > 0, b' \in \mathbb{R}$. Let $\alpha' \in (0,1)$. If $\alpha' f + (1-\alpha')g$ is affinely related to $\alpha' h + (1-\alpha')k$, then (**) $\alpha' f^\omega + (1-\alpha')g^\omega = a[\alpha' h^\omega + (1-\alpha')k^\omega] + b$ for all $\omega \in \Omega$ and some $a > 0, b \in \mathbb{R}$. If f, g, h, k are not pairwise affinely related, (*) and (**) cannot be true at the same time. Hence, "$\neg(i) \Rightarrow \neg(ii)$" $\Leftrightarrow$ "(i) $\Leftarrow$ (ii)". □

Chapter 6

Nash equilibrium behavior and uncertainty about others' preferences

In applications of game theory, it is frequently assumed that agents' preferences are commonly or at least mutually known. In recent years, this assumption has been increasingly questioned and relaxed. For example, Healy (2011) finds that subjects fail to accurately predict other subjects' preferences over possible outcomes in two-person two-strategies games (or, briefly, 2×2 games). In this chapter, which is based on Brunner et al. (2015), we test whether knowledge about other player's preferences has a significant effect on the frequency of equilibrium play. We find that subjects are indeed significantly more likely to play a Nash equilibrium strategy when they are informed about their opponent's preferences over the possible outcomes of the game.

Whenever it is unlikely that players know each others preferences, it might be advisable to use a more general equilibrium concept such as the strategic ambiguity models described in Section 3.3, or Bayesian Nash equilibrium rather than the standard Nash equilibrium. We will elaborate further on this point but will first discuss our experiment in more detail.

The experiment consists of two treatments, called "baseline" and "info". Both treatments have two stages. In stage 1, we let subjects rank eight monetary payoff pairs (they will be referred to as "payment-pairs"). These are then used to construct four different 2×2 games. In stage 2, each subject in both treatments plays each of these games exactly once. The two treatments differ in that the preferences elicited in stage 1 are only revealed in stage 2 of treatment info.

This design allows us to avoid the assumption that subjects only care about their *own* monetary payments. Instead, we can use the preferences elicited in stage 1 to describe the game that our subjects are playing. This is illustrated in Example 6.1 below, which corresponds to one of the games our subjects play in stage 2.

Example 6.1. *Consider the prisoner's-dilemma-type game-form in Figure 6.1. The numbers in the matrix correspond to the amount of money paid to the players, where the first number is the row player's payment and the second number is the column player's payment. Suppose that the two players, $i \in \{r, c\}$, where r stands for row and c for column,*

(a) are selfish payment maximizers and only care about their own payments. That is, each player's preferences over payment-pairs $(x_r, x_c) \in \mathbb{R}^2$ are represented by a strictly monotone increasing utility function $v_i(x_i)$ that depends only on his own payment or

(b) have other-regarding preferences represented by a function $\tilde{v}_i : \mathbb{R}^2 \to \mathbb{R}$,

then the games that result in cases (a) and (b) are depicted in Figure 6.2.

	L	R
U	$4,4$	$8,3$
D	$3,8$	$7,7$

Figure 6.1: Prisoner's-dilemma-type game-form

In Example 6.1, the game that results if players are selfish (a) is a prisoner's-dilemma-type game. In this case, the game has only one Nash equilibrium (U, L), i.e., everyone defects. That is not necessarily true for the induced game (b), where players have social preferences. For example, if $\tilde{v}_1(7,7) > \tilde{v}_1(8,3)$ and $\tilde{v}_2(7,7) > \tilde{v}_2(3,8)$, then mutual cooperation, (D, R), is a Nash equilibrium in (b).

	L	R
U	$v_r(4), v_c(4)$	$v_r(8), v_c(3)$
D	$v_r(3), v_c(8)$	$v_r(7), v_c(7)$

(a) Players with selfish preferences

	L	R
U	$\tilde{v}_r(4,4), \tilde{v}_c(4,4)$	$\tilde{v}_r(8,3), \tilde{v}_c(8,3)$
D	$\tilde{v}_r(3,8), \tilde{v}_c(3,8)$	$\tilde{v}_r(7,7), \tilde{v}_c(7,7)$

(b) Players with social preferences

Figure 6.2: Induced games in Example 6.1

While we can accommodate preferences that depend on both players' monetary payments, we maintain the assumption that the specific game form, other subjects' preferences, or any other factors have no effect on subjects' utility. We will discuss evidence suggesting that such considerations do not play an important role in the games used in this study.

Our main result is that subjects are much more likely to play a Nash equilibrium strategy in treatment info compared to treatment baseline. Therefore, subjects not only fail to accurately predict other players' preferences, the lack of such information also significantly affects their behavior.

If players do not know each other, concepts that are more general than Nash equilibrium might provide a more reliable prediction. In our experiment, we find that a strategy is more likely to be played when it cannot lead to the lowest ranked payment-pair (maxmin) or when it can result in the realization of the highest ranked one (maxmax). Intuitively, if a subject is uncertain about the strategy choice of her opponent, then, depending on her attitude towards uncertainty, she will try to avoid the lowest ranked payment-pair, or, to reach the highest ranked one. We show that the strategic ambiguity model of Eichberger and Kelsey (2014) can rationalize such strategy choices (see Section 6.3). This model allows for optimistic responses to strategic ambiguity. The other models mentioned in Section 3.3, except Marinacci (2000), assume ambiguity-averse behavior. While these models can explain maxmin strategy choices, they cannot rationalize maxmax behavior.

Another possibility is to take a Bayesian approach by modeling a situation where preferences are not mutually known as a game of incomplete information, and using the approach of Harsanyi (1967-68) to transform that game into a Bayesian game. Players with different preferences can be thought of as different types and it is then assumed that the prior distribution of types is commonly known. Such more general models have increasingly been developed in various fields.[38] We show that the behavior observed in our baseline treatment is consistent with a noisy version of Bayesian Nash Equilibrium,

[38]In auction theory, for example, the assumption that all bidders are risk neutral and that this is commonly known has been relaxed. Instead, the prior distribution of risk preferences rather than other bidders' actual risk preferences are assumed to be commonly known (see, e.g., Hu and Zou, 2015).

which we call *Quasi-Bayesian Nash Equilibrium* (QBNE).

The papers closest to this study are Healy (2011) and a recent working paper by Wolff (2014), who considers three-player public good games. In contrast to our experiment, Wolff does not reveal the primitives of a game, i.e., subjects' preferences over the material outcomes, to other subjects. Instead, he discloses subjects' best-response correspondences in one of his treatments and finds a smaller effect than we do. In his experiment, cognitive limitations might weaken the effect of such a disclosure due to his more complex setting.

This chapter is organized as follows. The next section describes the experimental design. Then, we discuss our results. In Section 6.3, we describe the aforementioned strategic ambiguity model in detail, and show that the observed maxmin and maxmax strategy choices can be explained by this model. Furthermore, we specify our notion of Quasi-Bayesian Nash Equilibrium, and show that the observed behavior in our baseline treatment is consistent with a QBNE. Section 6.4 concludes with a summary.

6.1 Experimental design, lab and payment details

Experimental design

Our experiment consists of two treatment with two stages each. In the first stage of both treatments, we elicit subjects' preferences over eight different payment-pairs. These payment-pairs are then used to construct four different 2×2 games. In stage 2 of each treatment, subjects play each one of these games exactly once. In treatment "info", subjects can see how their opponent ranked the four payment-pairs of the current game, whereas in treatment "baseline", this information is not disclosed.

We will now describe stage 1 in more detail, which is identical in both treatments. Subjects are asked to create an ordinal ranking over the following set X of eight payment-pairs:

$$X = \{(8,3),\ (7,7),\ (8,5),\ (4,4),\ (2,6),\ (3,8),\ (3,3),\ (2,2)\} \subset \mathbb{R}^2 \qquad (6.1)$$

The first number, x_1, corresponds to the amount of money (in Euros) paid to the decision-maker. The second number, x_2, is paid to some other subject (the "recipient"). Subjects are informed that they will not interact with the recipient in any other way in either stage of the experiment.

The order in which the payment-pairs appear on the screen was randomly determined beforehand and remains constant in all sessions. Subjects rank the payment-pairs by assigning a number between one and eight to each pair, where lower numbers indicate a higher preference. The same number can be assigned to multiple payment-pairs, thus allowing for indifference. In treatment info, subjects are told that their rankings would be disclosed to other participants at a later stage of the experiment. In treatment baseline, we made it clear that this information would not be revealed. After subjects confirm their ranking, they proceed to stage 2, in which they play the following four one-shot 2×2 games (all numbers are payments in Euro):

Game 1

	L	R
U	$4,4$	$8,3$
D	$3,8$	$7,7$

Game 3

	L	R
U	$4,4$	$8,3$
D	$3,3$	$7,7$

Game 2

	L	R
U	$5,8$	$7,7$
D	$6,2$	$3,3$

Game 4

	L	R
U	$8,3$	$2,2$
D	$7,7$	$3,8$

Figure 6.3: Games in the experiment

These games were selected because we conjectured that social preferences might play some role here. Moreover, they could be constructed using only 8 payment-pairs and exhibit some diversity with respect to the number of pure strategy Nash equilibria under the assumption that subjects are selfish payment maximizers.

All subjects play each game exactly once, each time against a different anonymous opponent. Games are played one after another and feedback about the outcome is only provided at the end of the experiment when subjects are paid, but not while subjects still make decisions.

In both treatments, subjects can see how they ranked the four payment-pairs of the current game in stage 1. This information is displayed by assigning 1-4 stars to each outcome, where more stars indicate a better outcome. In treatment info, subjects are shown their own *and* their opponent's ranking in matrix form (see Figure 6.4). Just like in the payment matrix, the first entry corresponds to the subject's own ranking while the second entry reveals the opponent's ranking. In treatment baseline, subjects were shown the same rankings matrix but this matrix only contained their own rankings.

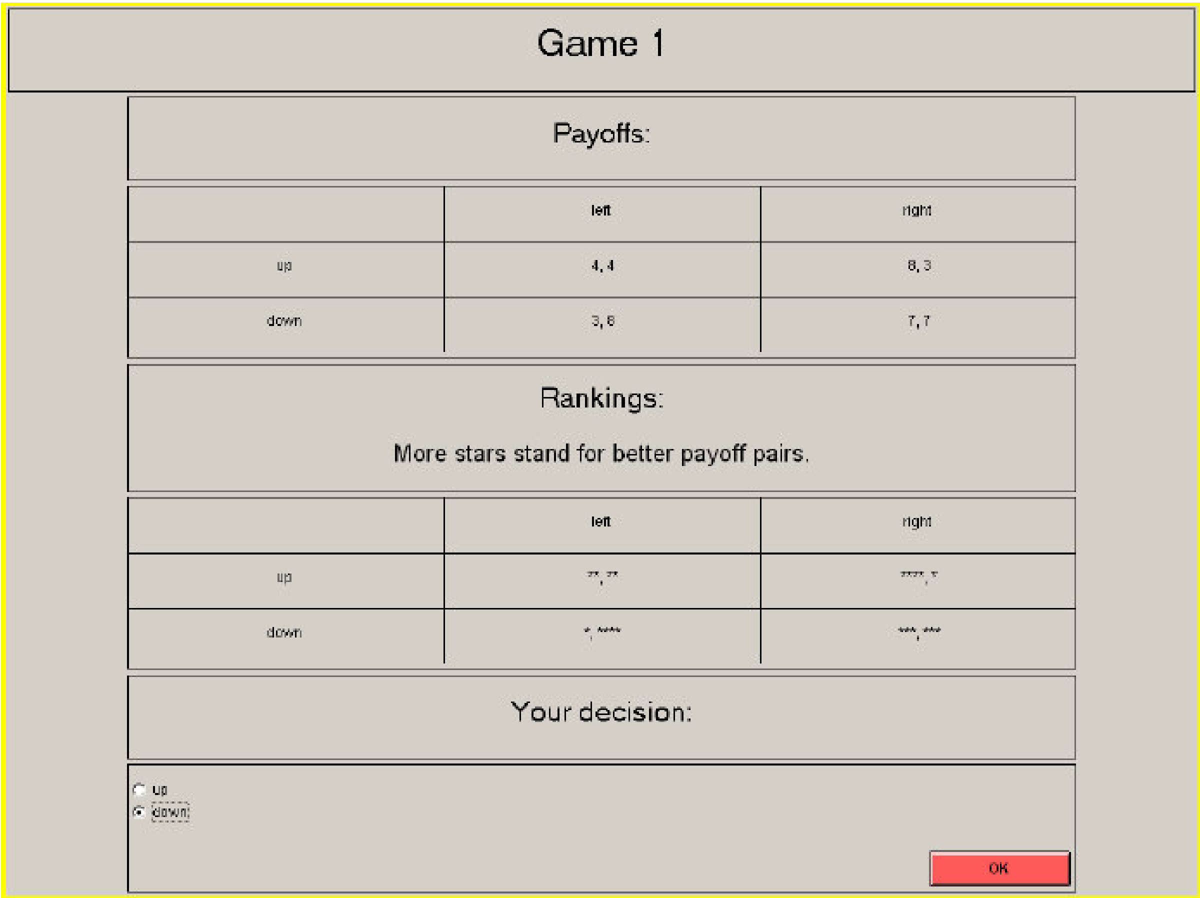


Figure 6.4: Information screen

Lab and payment details

In both treatments, subjects' payments are determined by a *random incentive system (RIS)*, which is a frequently used mechanism in experimental economics. Each subject is paid for exactly one of her decisions, which is randomly selected at the end of the experiment. If a decision from stage 1 is chosen, two of the eight payment-pairs from (6.1) are randomly selected. The subject is then paid the first number, x_1, of the payment pair that she ranked more highly in stage 1. The second number, x_2, is paid to some other subject. The probability that stage 1 is paid is $\frac{7}{8}$ while stage 2 is paid with a probability of $\frac{1}{8}$. These probabilities are consistent with selecting each of the $\binom{8}{2}$ possible pairs of payment pairs and each of the four decisions made in stage 2 with equal probability. Paying stage 1 with a substantially higher probability also reduces the odds that subjects might misrepresent their preferences. This issue will be discussed in more detail in Section 6.2.2.

Subjects were given printed instructions and could only participate after successfully answering several test questions. Test questions as well as the rest of the experiment were programmed using Z-Tree (Fischbacher, 2007). All sessions were conducted between August and October 2014 at the AWI-Lab of the University of Heidelberg. Subjects from all fields of study were recruited using Orsee (Greiner, 2004). Fewer than half of the subjects were economics students. Sessions lasted about 40-50 minutes on average. The following table summarizes the number of participants per session as well as average payments:

Table 6.1: Summary of treatment information

Treatment	Sessions	Subjects	Average payments
baseline	8	84	€ 12.36
info	7	80	€ 12.59

6.2 Results

In this section, we first characterize subjects' preferences as measured in stage 1 of the experiment. Then, we discuss the possibility that subjects might misrepresent their true preferences and that preferences might change when subjects are shown their opponents' preferences. We find no evidence for either of these effects and thus proceed to present the main treatment effect: subjects are more likely to play an equilibrium strategy in treatment info than in treatment baseline. This effect can be observed in each of the four games, though it is not significant when we only use the data from one single game. We also find that a strategy is played more often when it can lead to the highest ranked payment-pair and less often when it can lead to the lowest ranked one.

6.2.1 Characterization of measured preferences

In stage 1 of the experiment, we elicit subjects' preferences over the set of payment-pairs $X \subset \mathbb{R}^2$ defined in equation (6.1). To characterize subjects' preferences, we introduce four properties: pareto-efficiency, strict pareto efficiency, maximization of own payoff, and maximization of total payoff. These properties are defined as follows:

Definition 6.1 (Pareto efficiency)**.** A subject's preferences $\succsim$ on X are said to satisfy *pareto-efficiency* if, for all $x, y \in X$, $x \succ y$ whenever $x_1 \geq y_1$ and $x_2 \geq y_2$ with at least one inequality strict.

Definition 6.2 (Strict pareto efficiency)**.** A subject's preferences $\succsim$ on X are said to satisfy *strict pareto-efficiency* if, for all $x, y \in X$, $x \succ y$ whenever $x_1 > y_1$ and $x_2 > y_2$.

Definition 6.3 (Maximization of own payoff)**.** A subject is said to *maximize his own payoff* if, for all $x, y \in X$, $x \succ y$ whenever $x_1 > y_1$.

Definition 6.4 (Maximization of total payoff)**.** A subject is said to *maximize total payoff* if, for all $x, y \in X$, $x \succ y$ whenever $x_1 + x_2 > y_1 + y_2$.

Table 6.2 shows the fraction of subjects whose preferences are consistent with the properties defined above.

Table 6.2: Measured preferences

	Pareto efficiency	Strict pareto efficiency	Maximization of own payoff	Maximization of total payoff	n
Percentage consistent	69.5%	92.7%	54.9%	1.8%	164

6.2.2 Did we manage to elicit subjects' true preferences?

When preferences are elicited in stage 1 of the experiment, subjects in treatment info are aware that these preferences will be revealed to other subjects. However, they are not informed about the specific games that are played in stage 2. Hence, subjects did not have the information necessary to figure out what kind of misrepresentation might be most advantageous: in some 2×2 games, it could be beneficial to be perceived as having social preferences whereas in other games, the contrary is more likely (e.g., in the chicken game). Moreover, recall that a decision made in stage 2 affects a subject's payment with a probability of only 1/8. Therefore, it does not seem plausible that a rational subject would misrepresent her preferences in stage 1.

We test the claim that subjects truthfully state their preferences in stage 1 of treatment info by using the frequency with which subjects play strictly dominated strategies in stage 2 of the experiment. To identify strategies that are strictly dominated, we use the preferences elicited in stage 1. If these reflect a subject's true preferences, a rational subject should never play such a strictly dominated strategy. In contrast, if subjects strategically misrepresent their preferences in stage 1, a strategy that we classify as strictly dominated may in fact not be dominated according to the subjects' true preferences. Since preferences in treatment baseline are not revealed to other subjects, it is clear that subjects in treatment baseline have no reason to misrepresent their preferences. Therefore, we can compare the frequency with which subjects play a strictly dominated strategy in the two treatments to test the claim that preferences are truthfully revealed in stage 1 of treatment info. If that claim is true, no difference should be observed. Otherwise, subjects should be more likely to play a strictly dominated strategy in treatment info than in baseline.

Table 6.3: Violations of strict dominance

Treatment	Subjects	Games played	Games with dominated strategy	Dominated strategy played	Subjects who played dominated strategy at least once
Baseline	84	336	136	23.53%	32.14%
Info	80	320	140	25.71%	33.75%

Table 6.3 shows how often subjects play a strictly dominated strategy using the preferences stated in stage 1 to define the according games. Each subject played 4 games, thus resulting in 336 games played in treatment baseline and 320 in info. In 136 of these games in treatment baseline and 140 in info, one of the strategies was strictly dominated.

In roughly a quarter of these cases, the strictly dominated strategy was played.

In order to check the assumption that subjects do not misrepresent their preferences in both treatments, we run a regression using the 136 games in treatment baseline as well as the 140 games in treatment info as observations. The dependent variable "dominated" is a dummy variable that assumes a value of 1 if the strictly dominated strategy was played. The only explanatory variable other than the intercept is a treatment dummy ("info"). We run a probit regression and compute robust standard errors clustered by subject (see Table 6.4). The coefficient estimate for the treatment dummy is not significantly different from 0. Hence, the null hypothesis cannot be rejected:

Result 6.1. *Subjects are equally likely to play a strictly dominated strategy in both treatments.*

We therefore maintain the assumption that subjects truthfully state their preferences in stage 1 of the experiment in both treatments.

Table 6.4: Probit regression "dominated", robust standard errors clustered by subject

Dependent variable: Dominated	Coefficient	SE
Info	0.07	0.18
Constant	-0.72^{***}	0,13
n		276
Clusters		160
Pseudo R^2		0.0006

*** significant at 1% level

In psychological game theory, Rabin (1993) and Dufwenberg and Kirchsteiger (2004) introduced models of reciprocity in which players reward kind actions and punish unkind ones. Reciprocity could lead to a problem equivalent to the misrepresentation of preferences discussed in this section. For instance, consider Game 1 in stage 2 of treatment info. Suppose an own-payoff maximizer (row) is matched with a total-payoff maximizer (column). The row player might then believe that column will cooperate (play R), even though column expects row to defect (play U). This expected kindness on the part of column might then induce row to also cooperate, thus violating our assumption that only outcomes matter. In other words, subjects' preferences might change once they are shown their opponents' payment rankings in stage 2 of treatment info. If so, the preferences we use in our analysis would no longer correspond to subjects' true preferences. Since such preference adjustments are only possible in treatment info but not in treatment baseline, we can use Result 6.1 to argue that such effects probably do not matter much in our experiment. If they did, one would expect to observe that subjects play strictly dominated strategies more often in treatment info compared to treatment baseline.

6.2.3 Main results

The evidence discussed in the previous section suggests that subjects indeed state their true preferences in stage 1 of the experiment in both treatments. We can therefore proceed to discuss our main hypothesis that subject behavior is more consistent with the Nash equilibrium when preferences are mutually known. Aumann and Brandenburger (1995) showed that the Nash equilibrium does not necessarily require *common knowledge* (i.e., all know, all know that all know, and so on ad infinitum). In two-person games, *mutual knowledge* (i.e., all know) about the payoffs, the rationality and the conjectures of the players suffices to constitute a Nash equilibrium. Consequently, mutual knowledge about preferences leads to a Nash equilibrium whenever both individuals believe that their opponent is rational and know each other's conjectures.

We test this hypothesis by using two different subsets of our data. Recall that each subject played four games. Since there are a total of 164 subjects who participated in the experiment, we have data on 656 individual decisions, 336 in treatment baseline and 320 in treatment info. We exclude those decisions where both strategies are played with strictly positive probability in some Nash equilibrium, which leaves us with 425 decisions (213 in treatment baseline and 212 in treatment info). We also exclude those decisions where one pure strategy is strictly dominant. In such a situation, the best response does not depend on the other player's action and therefore, it should not matter whether or not the other players' preferences are known. This leaves us with 149 individual decisions, 77 in treatment baseline and 72 in treatment info. We test our main hypothesis using these 149 observations and will refer to the according subset of our data as "all subjects".

We run the same test a second time with a smaller subset of our data which no longer includes the decisions made by subjects who played a strictly dominant strategy in at least one of the four games. Either the preferences that these subjects stated in stage 1 do not reflect their true preferences or they are not rational in the sense that their choice in stage 2 is inconsistent with their stated preferences. Table 6.3 shows that approximately one third of our subjects violate strict dominance at least once. Just like in subset "all subjects" we also only use games where the subject has a unique equilibrium strategy that is not strictly dominant. Removing the choices made by inconsistent subjects therefore further reduces the number of observations to 110 individual decisions, 56 in treatment baseline and 54 in treatment info. We will refer to this subset of our data as "consistent subjects only".

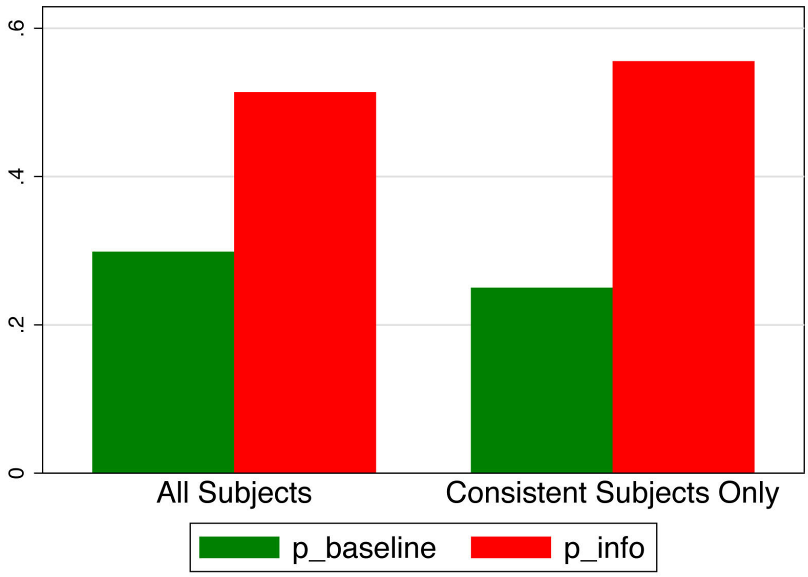


Figure 6.5: Probability unique equilibrium strategy is played

Figure 6.5 shows that subjects play an equilibrium strategy more often in treatment info than in treatment baseline, regardless of whether we use all subjects or only consistent subjects. To test whether these differences are significant, we run a probit regression. The dependent variable (eplay) assumes a value of 1 if a subject plays the unique equilibrium strategy and 0 otherwise. We include an intercept as well as a dummy variable, which assumes a value of 0 if the observation is generated in treatment info and 0 otherwise. The according results are shown in Table 6.5.

Table 6.5: Probit regression "eplay", robust standard errors clustered by subject

Dependent variable: eplay	All Subjects		Consistent subjects only	
	Coefficient	SE	Coefficient	SE
info	0.56**	0.22	0.81***	0.26
constant	−0.53***	0.16	−0.67***	0.19
n		149		110
Clusters		110		77
Pseudo R^2		0.036		0.074

** significant at 5% level, *** significant at 1% level

The treatment effect is significant using both subsets of our data. Therefore, informing subjects about their opponents preferences leads to a significantly higher frequency of equilibrium play.

Result 6.2. *Subjects are more likely to play a Nash equilibrium strategy when preferences are mutually known.*

6.2.4 Other results

Even though subjects in treatment info play a unique equilibrium strategy significantly more often than in treatment baseline, they still fail to do so almost half of the time. While the uncertainty with respect to the opponent's preferences is eliminated, there still is uncertainty about whether or not the other player will pick a strategy that is consistent with his reported preferences. Given that our subjects play a strictly dominated strategy roughly one fourth of the time, it seems plausible that some subjects may not be willing to rely on others to behave rationally. In the presence of such strategic uncertainty, subjects might try to avoid the lowest ranked payment-pair or try to reach the highest ranked payment-pair as an outcome. Such choices can be explained by models of strategic ambiguity (see Section 6.3). We therefore expect strategies that can yield the lowest ranked payment-pair as a possible outcome to be played relatively rarely while strategies that can lead to the highest ranked one should be played relatively often. Since subjects in the baseline treatment are uncertain not only about whether other players are rational but also about what their preferences are, these considerations should matter more in treatment baseline than in treatment info.

Table 6.6: Conditional logit regression "played", robust standard errors clustered by subject

Dependent variable: Played	Baseline		Info	
	Coefficient	SE	Coefficient	SE
equilibrium	−0.25	0.35	1.02***	0.34
maxmax	1.60***	0.31	1.07***	0.22
maxmin	1.53***	0.29	1.30***	0.21
n		456		424
Clusters		57		53
Pseudo R^2		0.42		0.42

*** significant at 1% level

We test these conjectures by running a conditional logit regression. An observation corresponds to a pure strategy. The dependent variable ("played") assumes a value of 1 if a strategy is played and 0 otherwise. Three independent variables are used to characterize each strategy: "equilibrium" indicates whether a strategy is a Nash equilibrium strategy. "maxmax" assumes a value of 1 if a strategy contains a most highly ranked payment-pair. "maxmin" indicates whether a strategy does NOT contain the lowest ranked payment-pair. We only use decisions made by subjects who never played a strictly dominated strategy. Table 6.6 shows that whether or not a strategy is a Nash equilibrium strategy only matters in treatment info when predicting which strategies subjects will play. In contrast, the coefficients of maxmax and maxmin are significant in both treatments.

Result 6.3. *In both treatments, a strategy is more likely to be played when it cannot lead to the lowest ranked payment-pair and when it can lead to the highest ranked payment-pair.*

While the coefficient estimate for the variable "equilibrium" differs significantly among the two treatments, the coefficients for "maxmax" and "maxmin" are not significantly different. The Nash equilibrium is only useful to predict play in treatment info but not in treatment baseline whereas the highest and the lowest possible payment seem to affect choices in both treatments.

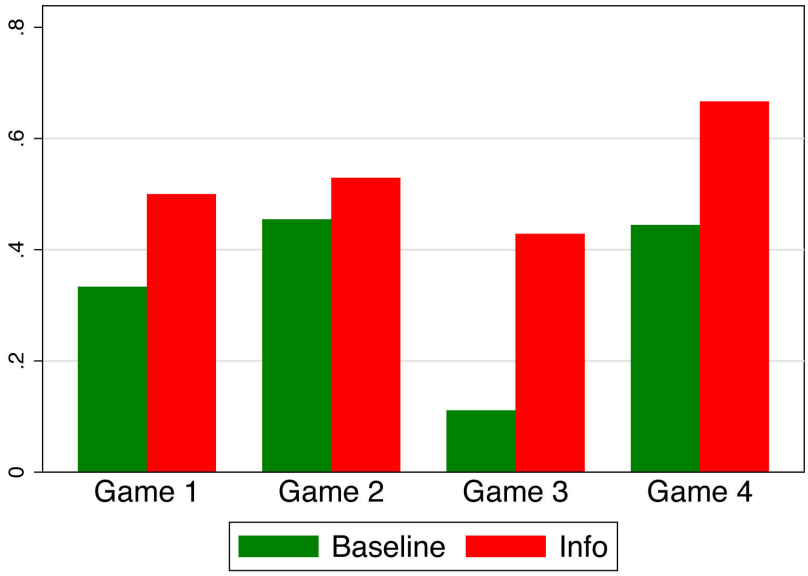


Figure 6.6: Freq. unique eq. strategy is played, all subjects

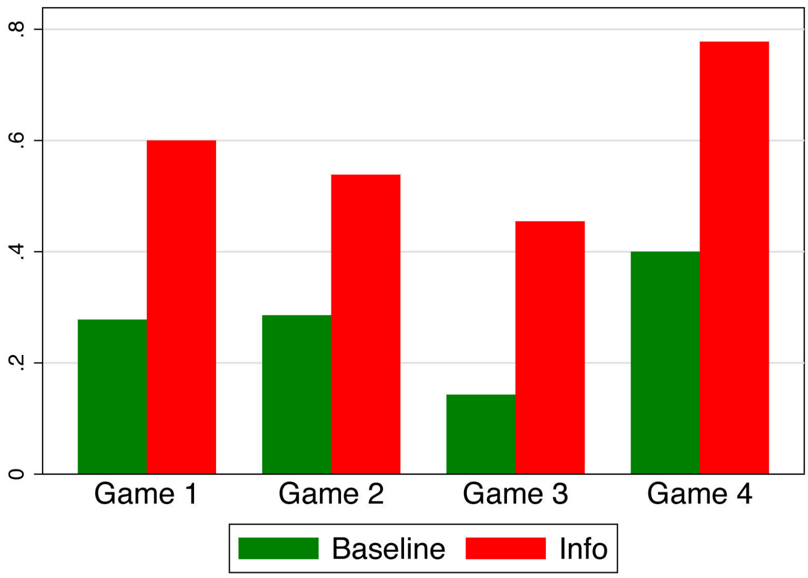


Figure 6.7: Freq. unique eq. strategy is played, consistent subjects only

As described in the introduction, the games played in the baseline treatment can also be considered as Bayesian games in which players with different preferences represent different types. It is then assumed that the prior distribution of types is commonly known. Recall that a substantial fraction of subjects played strategies that are strictly dominated and thus inconsistent with the preferences stated in the first stage (see Table 6.3). Therefore, we develop a quasi Bayesian model. In this model, we add a noisy type that randomly selects a pure strategy. All types other than this noisy type play a best response given the commonly known distribution of types. The model is described in detail in Section 6.3. While such a model is consistent with the behavior observed in the experiment, the according predictions are much less precise than those obtained under the assumption that players' payoff functions are mutually known, especially because we only elicit ordinal but not cardinal rankings of outcomes. However, in Section 6.3, we show that the predictions of Quasi-Bayesian Nash equilibrium could be falsified using our data and are not trivially consistent with our observations.

6.3 Possible explanations

In the following section, we first describe the strategic ambiguity model of Eichberger and Kelsey (2014), and show that this concept can rationalize maxmin and maxmax behavior. Subsequently, we specify the notion of Quasi-Bayesian Nash equilibrium, and show that the observed behavior in the baseline treatment is consistent with a QBNE.

The models discussed in this section, while more general, are also more difficult to analyze. Moreover, the assumption of the Bayesian model that type spaces and beliefs with respect to the distribution of these types are commonly known seems problematic in some applications. On the other hand, the predictions of these models would hopefully more often be consistent with observed behavior while still remaining falsifiable.

6.3.1 A non-Bayesian approach

The concept of Eichberger and Kelsey (2014) is called "equilibrium under ambiguity (EUA)". In their concept, player i's beliefs about the behavior of other players is represented by a capacity ν_i defined on $S_{-i} = \bigtimes_{j \in I \setminus \{i\}} S_j$, where S_j is the set of player j's pure strategies. Given her beliefs ν_i, player i's payoff from a pure strategy $s_i \in S_i$ corresponds to the Choquet integral of her payoff function $u_i(s_i, s_{-i})$ with respect to ν_i:

$$\begin{aligned} V_i(s_i,\nu_i) &= \int_{S_{-i}} u_i(s_i,s_{-i})\, d\nu_i \\ &= u_i(s_i,s^1_{-i})\nu(s^1_{-i}) + \sum_{r=2}^{R} u_i(s_i,s^r_{-i}\big[\nu(s^1_{-i},\dots,s^r_{-i}) - \nu(s^1_{-i},\dots,s^{r-1}_{-i})\big], \end{aligned}$$

where the strategy combinations in S_{-i} are numbered so that $u_i(s_i, s^1_{-i}) \geq u_i(s_i, s^2_{-i}) \geq \dots \geq u_i(s_i, s^R_{-i})$. Player i's best responses to her belief ν_i are defined in the usual way as

$$R_i(\nu_i) = \{s_i \mid s_i \in \arg\max_{s_i \in S_i} \; V_i\;(s_i,\nu_i)\}.$$

An essential ingredient of the model is the notion of support for a non-additive measure. Eichberger and Kelsey define the support of a convex capacity as the intersection of the supports of the probability measures in the core of the capacity:[39]

Definition 6.5. *The support of a convex capacity μ on S_{-i}* is defined as

$$\text{supp}(\mu) = \bigcap_{\pi \in core(\mu)} \text{supp}(\pi).$$

As described in Section 2.4, convex capacities represent ambiguity-aversion. To capture optimistic behavior, Eichberger and Kelsey use the class of capacities introduced by Jaffray and Philippe (1997) (JP-capacities). A JP-capacity has convex and concave parts. It is defined as a mixture of a convex capacity with its dual capacity.[40] Eichberger and Kelsey define the support of a JP-capacity ν, $\text{supp}_{JP}(\nu)$, as the support of its convex part according to Definition 6.5. This support definition has a useful implication for neo-additive capacities introduced in Chapter 4:

Proposition 6.1 (Eichberger and Kelsey, 2014)**.** *Let $\nu = \delta\alpha + (1-\delta)\pi$ be a neo-additive capacity on S_{-i}, where $\alpha, \delta \in [0,1]$, then* $\text{supp}_{JP}(\nu) = \text{supp}(\pi)$.

[39]For alternative support definitions and for arguments supporting Definition 6.5, see Eichberger and Kelsey (2014).

[40]*The dual capacity of capacity* μ is defined by $\bar{\mu}(E) = 1 - \mu(E^c)$. Hence, if μ is convex, then $\bar{\mu}$ is concave.

We will use neo-additive capacities to discuss the example in this section.

An equilibrium under ambiguity is a belief system in which, for each player i, the nonempty support of player i's belief about the opponents' behavior lies in the Cartesian product of the opponents best responses given their beliefs about the behavior of other players. To put it differently, in an equilibrium under ambiguity, the beliefs that agents hold are reasonable in the sense that neither player expects other players to play strategies that are not best responses given their beliefs.

Definition 6.6. A belief system (ν_i^*, ν_{-i}^*) is *an equilibrium under ambiguity* if for all $i \in I$

$$\text{supp}(\nu_i^*) \subseteq \bigtimes_{j \in I \setminus \{i\}} R_j(\nu_j^*) \text{ and } \text{supp}(\nu_i^*) \neq \varnothing.$$

In what follows, we show that an equilibrium under ambiguity can explain maxmin and maxmax strategy choices.

Example 6.2. *Consider Game 3 in stage 2 of treatment info. Suppose that the game is played by two subjects whose utility functions correspond to their own payment. That is, the game takes the following form:*

	L	R
U	$4,4$	$8,3$
D	$3,3$	$7,7$

Obviously, the row player has a strictly dominant strategy (U). If the column player believes that row will pick U, she will play L. The game has a unique Nash equilibrium (U, L). Whether or not the Nash equilibrium is played depends on the belief of the column player about whether the row player behaves rationally, i.e., whether row will play the strictly dominant strategy.

Denote the players by $I = \{r, c\}$, where r stands for row and c for column. If the column player is not sure whether row behaves rationally, she may try to reach the highest possible outcome (7) by playing strategy R. To show that this strategy choice is consistent with an equilibrium under ambiguity, suppose that the beliefs of the column player about row's behavior can be represented by a neo-additive capacity ν_c with reference prior

$\pi_c = (\pi_c(U), \pi_c(U)) = (1, 0)$. This can be viewed as a situation where column is uncertain about the prior π_c, i.e., whether row plays U. Furthermore, let column be an ambiguity-loving player, for simplicity, assume that $\alpha_c = 0$. We may interpret the parameter δ_c as the degree of ambiguity about π_c. The higher δ_c, the higher the degree of ambiguity. Given this belief, column's payoff from L equals

$$V_c(L, \nu_c) = (1 - \delta_c) \cdot 4 + \delta_c \cdot \left(\max\{u_c(L, s_r) \mid s_r \in S_r\} \right) = 4,$$

and column's payoff from R is

$$V_c(R, \nu_c) = (1 - \delta_c) \cdot 3 + \delta_c \cdot \left(\max\{u_c(R, s_r) \mid s_r \in S_r\} \right) = 3 + 4\delta_c.$$

Hence, if column is sufficiently uncertain about π_c ($\delta_c > \frac{1}{4}$), she will choose strategy R.

Suppose that the beliefs of the row player about column's behavior can also be represented by a neo-additive capacity ν_r with reference prior $\pi_r = (\pi_r(L), \pi_r(R)) = (0, 1)$. It is straightforward that the row player will play U given such a belief. Taken together, for $\delta_c > \frac{1}{4}$, we have that

$$R_r(\nu_r) = U \text{ and } R_c(\nu_c) = R,$$

and, by Proposition 6.1 it holds that

$$\text{supp}_{JP}(\nu_r) = \text{supp}(\pi_r) = R \text{ and } \text{supp}_{JP}(\nu_c) = \text{supp}(\pi_c) = U.$$

Consequently, the system (ν_r, ν_c) is an equilibrium under ambiguity in which the column player plays the maxmax strategy R. Similarly, one can show that the equilibrium under uncertainty concept can rationalize maxmin behavior if the players are ambiguity-averse.

6.3.2 A quasi-Bayesian approach

Recall that in a Bayesian game, a strategy σ_i of player i prescribes a mixed action for each possible type of player i, formally $\sigma_i : \Theta_i \to \Delta(A_i)$. As before, let Σ_i be the set of all strategies of player i and $\Sigma = \bigtimes_{i \in I} \Sigma_i$. The interim expected utility of player i with type $\theta_i \in \Theta_i$ from a mixed strategy profile $\sigma \in \Sigma$ is

$$EU_i(\sigma \mid \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \pi(\theta_{-i}|\theta_i) \Bigg(\sum_{a \in A} \Bigg(\prod_{j \in I} \sigma_j(a_j \mid \theta_j) \Bigg) u_i(a, \theta_i, \theta_{-i}) \Bigg).$$

where $\pi(\theta_{-i}|\theta_i)$ denotes the probability of θ_{-i} under the condition that i knows she is of type θ_i, and $\sigma_j(a_j \mid \theta_j)$ is the probability of action a_j that strategy σ_j prescribes for θ_j.

A situation in which the players do not know each other's preferences over the set of physical outcomes can be modeled as a Bayesian game where each player's type space corresponds to a set of potential preferences over X. In our quasi Bayesian approach, we add a noisy type $\tilde{\theta}_i$ to each player's type space that randomly selects a pure action. A Quasi-Bayesian Nash equilibrium is then defined as follows:

Definition 6.7. A *Quasi-Bayesian Nash equilibrium* for a quasi Bayesian game is a strategy profile $(\sigma_i^*, \sigma_{-i}^*) \in \Sigma$ such that, for each player $i \in I$,

$$\sigma_i^*(\theta_i) \;\in\; \arg\max_{\sigma_i \in \Sigma_i} \; EU_i(\sigma_i, \sigma_{-i}^* \mid \theta_i)$$

for all non-noisy types $\theta_i \in \Theta_i \setminus \{\tilde{\theta}_i\}$, and

$$\sigma_i^*(\tilde{\theta}_i) \in int(\Delta(A_i))$$

for the noisy type $\tilde{\theta}_i \in \Theta_i$, where $int(\Delta(A_i))$ denotes the interior of the set of player i's mixed actions.

Obviously, a Quasi-Bayesian Nash equilibrium is weaker than a Bayesian Nash equilibrium since it only requires that non-noisy types play mutual best responses. The existence of a QBNE follows from the standard fixed-point argument by Nash (1950, 1951).

Modeling the situation in the baseline treatment of our experiment as a quasi Bayesian game

At the first stage of treatment baseline, we elicit each subject $k's$ ordinal preferences $\succsim_k$ over eight payment-pairs.

Definition 6.8. Subject $k's$ ordinal preference ordering $\succsim_k$ on the set X defined in equation (6.1) is a function $f_k : X \to \{1, \ldots, 8\}$.

We do not know subjects $k's$ utility function $v_k(\cdot)$ exactly, but we know that $v_k(\cdot)$ is a representation of the ordinal ordering $\succsim_k$, i.e., for all $x, y \in X$ and all k, we have that $v_k(x) \geq v_k(y)$ if and only if $x \succsim_k y$.

At the second stage of the baseline treatment, subjects play four 2×2 games. Henceforth, given a game played by two subjects, the subject who is the row player is denoted by r and the column player by c. As described in the previous subsection, in our quasi Bayesian model, each subject's type space corresponds to a set of ordinal preference orderings and a noisy type.

We will assume that the observed fraction of row and column subjects who played a strictly dominated action at least once (see Table 6.3) is an estimator for the probability of the noisy row and column type, respectively.[41]

Assumption 6.1. The fraction of row and column subjects who played a strictly dominated action at least once corresponds to the probability of noisy types of row and column players, $\tilde{\theta}_r$ and $\tilde{\theta}_c$.

We shall assume that the set of non-noisy types corresponds to the set of ordinal rankings of all subjects, who never played a strictly dominated action. Again, the fraction of subjects with a specific ranking is used as an estimator for the probability of this type.

Assumption 6.2. The sets of non-noisy types of row and column players, $\bar{\Theta}_r$ and $\bar{\Theta}_c$, are $\bar{\Theta}_r = \{\succsim_k \mid \text{k is a non-noisy row player}\}$ and $\bar{\Theta}_c = \{\succsim_k \mid \text{k is a non-noisy column player}\}$.

As a consequence, we assume that the utility functions of all non-noisy subjects of the same type are identical:

Assumption 6.3. For any two subjects k and k', who are either both row or both column players and who never played a strictly dominated action, if $\succsim_k = \succsim_{k'}$, then $v_k(\cdot) = v_{k'}(\cdot)$.

The type spaces of row and column players, Θ_r and Θ_c, are the union of the set of non-noisy types and the noisy type, formally $\Theta_r = \bar{\Theta}_r \cup \tilde{\theta}_r$ and $\Theta_c = \bar{\Theta}_c \cup \tilde{\theta}_c$.

[41]Note that this is a biased estimator since "noisy subjects" may accidentally behave consistently. That is, the estimator systematically underestimates the probability of a noisy type.

As mentioned above, we take the observed relative frequencies as estimators for the probabilities of the types. For instance, the probability of the noisy row type equals

$$\pi[\tilde{\theta}_r] = \frac{\#\,\text{row subjects who violated strict dominance at least once}}{\#\,\text{row subjects}}. \tag{6.2}$$

Following Harsanyi (1967-68), we assume that the types and the prior distribution of types, i.e., the probabilities, are commonly known.

Assumption 6.4. Row and column subjects' beliefs over types, π_r and π_c, correspond to the relative frequencies of types in the experiment.

Subject behavior and Quasi-Bayesian Nash equilibrium

This section provides two propositions. The first one shows that the predictions of Quasi-Bayesian Nash equilibrium are falsifiable using our data. In the second proposition, we show that the behavior observed in the baseline treatment is consistent with a QBNE.

Before we state the propositions, we introduce some notation and definitions that will be used throughout this section. Consider stage 2 of the baseline treatment of our experiment.

Given Assumption 6.1, 6.2, 6.3, and 6.4, suppose that in each of the four interactive situations in stage 2, the subjects played a quasi Bayesian game. Fix a row or column type $\bar{\theta} \in \Theta_i$, $i \in \{r, c\}$, an action $a' \in A_i$ is said to be contained in the *support of a strategy* $\sigma_i(\bar{\theta})$ *given type* $\bar{\theta}$ if the strategy prescribes that the type $\bar{\theta}$ plays a' with strictly positive probability, formally $supp\big(\sigma_i(\bar{\theta})\big) = \{a' \in A_i \mid \sigma_i(a' \mid \bar{\theta}) > 0\}$. Recall that the action spaces of row and column subjects in each of the four interactions in stage 2 are $A_r = \{U, D\}$ and $A_c = \{L, R\}$ (see Figure 6.3). Hence, for each strategy of row, σ_r, given any fixed type $\theta_r \in \Theta_r$, it holds that $supp\big(\sigma_r(\theta_r)\big) \subseteq \{U, D\}$, and, accordingly, for column $supp\big(\sigma_c(\theta_c)\big) \subseteq \{L, R\}$. Then, *the support of a type-contingent strategy* of row or column, σ_i, equals the Cartesian product of the supports of the strategy for all given types, formally

$$supp(\sigma_i) = \bigtimes_{\theta_i \in \Theta_i} supp\big(\sigma_i(\theta_i)\big). \tag{6.3}$$

Consequently, for any strategy of row, we have that $supp(\sigma_r) \subseteq \{U, D\}^{|\Theta_r|}$ and for column $supp(\sigma_c) \subseteq \{L, R\}^{|\Theta_c|}$, where $|\Theta_r|$ and $|\Theta_c|$ denote the number of different row types and column types, respectively, in the baseline treatment of the experiment. Finally, the support of a type-contingent strategy profile, $\sigma \in \Sigma_r \times \Sigma_c$, is defined as follows.

Definition 6.9. The *support of a strategy profile* $(\sigma_r, \sigma_c) \in \Sigma_r \times \Sigma_c$ is the set

$$supp((\sigma_r, \sigma_c)) = supp(\sigma_r) \times supp(\sigma_c) \;\subseteq\; \{U, D\}^{|\Theta_r|} \times \{L, R\}^{|\Theta_c|}.$$

Let a_k^G be the action played by subject k in Game $G \in \{1, 2, 3, 4\}$ at the second stage of treatment baseline. The actions played in Game G that are associated with a given row or column type θ_i are denoted by $a_{\theta_i}^G$. These action sets are defined as the union of all action choices of subjects who are of type θ_i, formally

$$a_{\theta_i}^G = \bigcup_{k \text{ is of type } \theta_i \in \Theta_i} a_k^G \;\subseteq\; A_i. \tag{6.4}$$

The actions played in Game G that are associated with all types of row (resp. column) Θ_i, $i \in \{r, c\}$, are

$$a_{\Theta_i}^G = \bigtimes_{\theta_i \in \Theta_i} a_{\theta_i}^G \subseteq A_i^{|\Theta_i|}. \tag{6.5}$$

Based on these definitions, we can specify what an action set combination is.

Definition 6.10. A *potential action set combination in Game G* is a set

$$a^G \subseteq \{U, D\}^{|\Theta_r|} \times \{L, R\}^{|\Theta_c|} \text{ and}$$

the *observed action set combination in Game G* is

$$\hat{a}^G = (a_{\Theta_r}^G, a_{\Theta_c}^G).$$

The propositions in this section are based on a notion of consistency between action set combinations and QBNE. This consistency definition is the following.

Definition 6.11. An action set combination $a^G \subseteq \{U, D\}^{|\Theta_r|} \times \{L, R\}^{|\Theta_c|}$ in Game G *is said to be consistent with a QBNE* $\sigma^* \in \Sigma_r \times \Sigma_c$ for the quasi Bayesian game that results from G if $a^G \subseteq supp(\sigma^*)$.

Recall that a strategy $\sigma_i \in \Sigma_i$ prescribes a mixed action for each possible type of player $i \in \{r, c\}$. The rationale behind Definition 6.11 is that if $a^G \subseteq supp(\sigma^*)$, then each action that a^G associates with a particular type $\theta_i \in \Theta_i$ is a possible outcome of σ^* for all $\theta_i \in \Theta_i$ and all $i \in \{r, c\}$.

Under Assumption 6.1, 6.2, 6.3, and 6.4, given a quasi Bayesian game that results from Game G and a strategy profile $\sigma' = (\sigma'_r, \sigma'_c)$, all row subjects have the same belief, $\beta[L \mid \sigma'_c]$, that her opponent will play action L:

$$\beta[L \mid \sigma'_c] = \sum_{\theta_c \in \Theta_c} \pi[\theta_c] \cdot \sigma'_c(L \mid \theta_c), \tag{6.6}$$

where $\pi[\theta_c]$ denotes the probability of a column subject with type θ_c (i.e., according to Assumption 6.4, the relative frequency of θ_c types), and $\sigma'_c(L \mid \theta_c)$ the probability with which a column subject of type θ_c plays action L. Analogously, each column subject's belief that the opponent will play action U is

$$\beta[U \mid \sigma'_r] = \sum_{\theta_r \in \Theta_r} \pi[\theta_r] \cdot \sigma'_r(U \mid \theta_r). \tag{6.7}$$

Now, we are ready to state the propositions. The first proposition shows that the predictions of Quasi-Bayesian Nash equilibrium could be falsified using our data and are not trivially consistent with our observations.

Proposition 6.2. *Under Assumption 6.1, 6.2, 6.3, and 6.4, there exist action set combinations $a^1, a^3 \subseteq \{U, D\}^{|\Theta_r|} \times \{L, R\}^{|\Theta_c|}$ such that, if a^1 is consistent with a QBNE of the quasi Bayesian game that results from Game 1, then a_3 is not consistent with any QBNE of the quasi Bayesian game that results from Game 3, and vice versa.*

Proof. We prove the proposition by showing that two stated ordinal preferences and two action set combinations a^1 and a^3 in Game 1 and 3 imply the desired result.

In the baseline treatment of the experiment (see Table 6.7 at the end of this section), three column subjects stated the ordinal preference $\succsim_{\theta_7}$:

$$(5,8) \succ_{\theta_7} (7,7) \succ_{\theta_7} (3,8) \succ_{\theta_7} (4,4) \succ_{\theta_7} (8,3) \succ_{\theta_7} (3,3) \succ_{\theta_7} (6,2) \succ_{\theta_7} (2,2),$$

and two column subjects stated the preference $\succsim_{\theta_8}$:

$$(7,7) \succ_{\theta_8} (5,8) \succ_{\theta_8} (3,8) \succ_{\theta_8} (4,4) \succ_{\theta_8} (3,3) \succ_{\theta_8} (8,3) \succ_{\theta_8} (2,2) \succ_{\theta_8} (6,2),$$

where the second component of the payment-pair is the monetary payoff for column.

Consider the column types θ_7 and θ_8, and let v_7, v_8 be utility functions that represent $\succsim_{\theta_7}, \succsim_{\theta_8}$. Observe that the types θ_7 and θ_8 have no dominant action in Game 1 and 3. Let a_1, a_3 be action set combinations that satisfy

$$a^1_{\theta_7} = a^1_{\theta_8} = a^3_{\theta_7} = a^3_{\theta_8} = \{L, R\}. \tag{6.8}$$

Suppose to the contrary that a^1 is consistent with a QBNE σ^* for the quasi Bayesian game of Game 1 and, at the same time, a^3 is consistent with a QBNE σ^{**} for the one that results from Game 3. By (6.8), we know that a_1, a_3 are consistent with σ^*, σ^{**} only if σ^* and σ^{**} prescribe non-degenerate mixed actions for the types θ_7 and θ_8 in both quasi Bayesian games. Consequently, in the quasi Bayesian game of Game 1, it needs to hold that,

$$\beta[U \mid \sigma^*_r] = \frac{v_7(7,7) - v_7(3,8)}{(v_7(4,4) - v_7(8,3) + v_7(7,7) - v_7(3,8))} \quad \textit{and} \tag{6.9}$$

$$\beta[U \mid \sigma^*_r] = \frac{v_8(7,7) - v_8(3,8)}{(v_8(4,4) - v_8(8,3) + v_8(7,7) - v_8(3,8))}, \tag{6.10}$$

and in the quasi Bayesian game that results from Game 2:

$$\beta[U|\sigma^{**}_r] = \frac{v_7(7,7) - v_7(3,3)}{(v_7(4,4) - v_7(8,3) + v_7(7,7) - v_7(3,3))} \quad \textit{and} \tag{6.11}$$

$$\beta[U \mid \sigma^{**}_r] = \frac{v_8(7,7) - v_8(3,3)}{(v_8(4,4) - v_8(8,3) + v_8(7,7) - v_8(3,3))}. \tag{6.12}$$

Moreover, since v_7, v_8 represent $\succsim_{\theta_7}, \succsim_{\theta_8}$, we have that:

$$v_7(8,3) > v_7(3,3) \quad \textit{and} \tag{6.13}$$

$$v_8(8,3) < v_8(3,3). \tag{6.14}$$

Observe that the equations (6.9), (6.11), and (6.13) imply that $\beta[U \mid \sigma_r^{**}] > \beta[U \mid \sigma_r^*]$. Whereas, equations (6.10), (6.12), and (6.14) imply $\beta[U \mid \sigma_r^{**}] < \beta[U \mid \sigma_r^*]$ - a contradiction. Hence, if a^1 is consistent with σ^*, then a^3 is not consistent with any QBNE σ^{**} for the quasi Bayesian game 3, and vice versa. □

The second proposition shows that the observed action set combinations in the baseline treatment are consistent with a Quasi-Bayesian Nash equilibrium.

Proposition 6.3. *Under Assumption 6.1, 6.2, 6.3, and 6.4, there exists a QBNE σ^* of the quasi Bayesian game G such that the observed action set combination $\hat{a}^G$ is consistent with σ^* for all $G \in \{1, 2, 3, 4\}$.*

Proof. The proof is organized as follows. Suppose that row subjects' expectation that their opponent will play L in G equals $\beta_r^G \in (0, 1)$ and column players' expectation that their opponent will play U in G is $\beta_c^G \in (0, 1)$. Let $\succsim_{\theta_i}$ be the ordinal ranking that is associated with type $\theta_i \in \Theta_i$. In Lemma 6.1, we show that there exists a utility function u_{θ_i} for all non-noisy types $\theta_i \in \Theta_i$, which represents $\succsim_{\theta_i}$, such that if $\hat{a}^G_{\theta_i} = \{U, D\}$ ($\hat{a}^G_{\theta_i} = \{L, R\}$), then θ_i is indifferent between her pure actions, given the beliefs β_r^G (β_c^G). Subsequently, by using Lemma 6.1, we prove that there exists a QBNE σ_G^* for the quasi Bayesian game of $G \in \{1, 2, 3, 4\}$ such that if $\hat{a}^G_{\theta_i} = \{U, D\}$ ($\hat{a}^G_{\theta_i} = \{L, R\}$), then $\sigma_G^*(\theta_i)$ prescribes a proper mixed action for each non-noisy type θ_i.

Note that we do not have to consider noisy types, and non-noisy types who have a strictly dominant action. The played actions of both types are always consistent with a QBNE. Furthermore, the proof is trivial for non-noisy types, who have in only one of the four games no strictly dominant action. The remaining types are depicted in Table 6.7 at the end of this proof. Table 6.8 shows which games are relevant and the action sets associated with the types in each game.

Lemma 6.1. *Consider the types θ_j, $j = 1, \ldots, 11$, defined in Table 6.7 and 6.8 . Given Game G, let $\beta_r^G \in (0, 1)$ be row player's belief that her opponent will play L and $\beta_c^G \in (0, 1)$ be column player's belief that the opponent will play U. If $\beta_r^1 > \beta_r^3$ and $\beta_c^1 > \beta_c^3$, there exist utility functions v_j for all θ_j, which represent $\succsim_{\theta_j}$, such that if $\hat{a}^G_{\theta_j} = \{U, D\}$*

($\hat{a}^G_{\theta_j} = \{L, R\}$), then θ_j is indifferent between her pure actions in the quasi Bayesian game that results from G.

Proof. At first, consider the types θ_1-θ_3. The action sets of the types $j = 1, 2$ in Game 1 are $\hat{a}^G_{\theta_j} = \{U, D\}$. Given a belief $\beta^1_r \in (0, 1)$ in the quasi Bayesian game of Game 1, there exists utility functions, which represent $\succsim_1$ and $\succsim_2$, such that the types $j = 1, 2$ are indifferent if

$$\beta^1_r = \frac{v_j(7,7) - v_j(3,8)}{(v_j(4,4) - v_j(8,3) + v_j(7,7) - v_j(3,8))}. \tag{6.15}$$

In Game 3, the action set of all three types is $\{U, D\}$. Hence, given a belief $\beta^3_r \in (0, 1)$ in the quasi Bayesian game 3, there exists utility functions for $j = 1, 2, 3$ such that the types are indifferent if

$$\beta^3_r = \frac{v_j(7,7) - v_j(3,3)}{(v_j(4,4) - v_j(8,3) + v_j(7,7) - v_j(3,3))}. \tag{6.16}$$

Finally, in Game 2, only type 1 is a relevant type. The action set of type 1 in Game 2 is $\{U\}$. That means, given a belief $\beta^2_r \in (0, 1)$, a utility function u_{θ_1} that represents $\succsim_{\theta_1}$ needs to satisfy:

$$\beta^2_r u_{\theta_1}(5,8) + (1 - \beta^2_r) u_{\theta_1}(7,7) \geq \beta^2_r u_{\theta_1}(6,2) + (1 - \beta^2_r) u_{\theta_1}(3,3) \tag{6.17}$$

Now, choose $v_j(7,7), v_j(8,3)$, and $v_j(4,4)$ such that the utility values are consistent with the ordering $\succsim_{\theta_j}$ for $j = 1, 2, 3$. From equation (6.15) and (6.16), we obtain

$$v_j(3,8) - v_j(3,3) = \frac{(v_j(7,7) - v_j(8,3))(\beta^1_r - 1)(\beta^3_r - \beta^1_r)}{\beta^1_r q^3_r}.$$

Since $\beta^1_r > \beta^3_r$, and for all three types, $j = 1, 2, 3$, $v_j(7,7) > v_j(8,3)$, we have that $v_j(3,8) > v_j(3,3)$, which is consistent with the orderings $\succsim_{\theta_1}, \succsim_{\theta_2}$ and $\succsim_{\theta_3}$ given in Table 6.7. Note that it is now shown that the lemma holds for the types θ_2 and θ_3 since there are no further restrictions concerning their utility functions. For type θ_1, we may define a utility function v_1, which represents $\succsim_{\theta_1}$, such that the distance $v_1(6,2) - v_2(5,8)$ is arbitrary small. It follows immediately that the lemma is also true for θ_1.

Now, consider the types θ_4-θ_6. We omit the obvious proof for type θ_4, and turn to the types 5 and 6. In Game 4, the action set of both types is $\{U, D\}$. Given a belief $\beta_r^4 \in (0,1)$, the indifference condition for $j = 5, 6$ in the quasi Bayesian game 4 is

$$\beta_r^4 = \frac{v_j(3,8) - v_j(2,2)}{(v_j(3,8) - v_j(2,2) + v_j(8,3) - v_j(7,7))}. \tag{6.18}$$

In Game 2, the action set of both types is $\{U\}$. Hence, given $\beta_r^2 \in (0,1)$, for $j = 5, 6$,

$$\beta_r^2 v_j(5,8) + (1-\beta_r^2) u_{\theta_j}(7,7) \geq \beta_r^2 u_{\theta_j}(6,2) + (1-\beta_r^2) u_{\theta_j}(3,3). \tag{6.19}$$

If there is a strict inequality in equation (6.19), it is obvious that one can choose utility values that satisfy (6.18) and (6.19) and represent $\succsim_{\theta_5}$ and $\succsim_{\theta_6}$. Suppose that (6.19) holds with equality and choose the utility values $v_j(5,8)$, $v_j(6,2)$, $v_j(8,3), v_j(2,2)$,and $v_j(3,8)$ for $j = 5, 6$ such that the utility functions represent $\succsim_{\theta_5}$ and $\succsim_{\theta_6}$. Then, equation (6.18) and (6.19) imply that $v_j(7,7) > v_j(3,3)$. This is consistent with $\succsim_{\theta_5}$ and $\succsim_{\theta_6}$, which shows that the lemma holds for type 5 and 6.

For the column types θ_7-θ_{11}, the lemma can be proven similarly to the row types θ_1-θ_6. □

For the quasi Bayesian game of each Game G, consider a strategy profile σ_G^* such that

(i) $\sigma_G^*(\theta_j)$ is a proper mixed action for non-noisy row types θ_j where $\hat{a}_{\theta_j}^G = \{U, D\}$.

(ii) $\sigma_G^*(\theta_j)$ is a proper mixed action for non-noisy column types θ_j where $\hat{a}_{\theta_j}^G = \{L, R\}$.

(iii) $\beta[L \mid \sigma_1^*] > \beta[L \mid \sigma_3^*]$ and $\beta[U \mid \sigma_1^*] > \beta[U \mid \sigma_3^*]$.

It is obvious that such strategy profiles exist. By assumption, noisy types randomly select a pure action, i.e., they play a proper mixed action. Given any mixed action of noisy types in the quasi Bayesian game of each Game G, define utility functions for all non-noisy types such that these types have no incentive to deviate from σ_G^* in each game. By Lemma 6.1, we know that such utility functions always exist. Then, σ_G^* is a QBNE for the quasi Bayesian game of G for $G \in \{1, 2, 3, 4\}$. One can easily check that $\hat{a}^G \subseteq supp(\sigma_G^*)$ for all $G \in \{1, 2, 3, 4\}$, which proves the proposition. □

Table 6.7: Row and column types

Type	Ordinal preference ranking
Row types	
$\succsim_{\theta_1}$	(7,7)$\succ_{\theta_1}$(8,3)$\succ_{\theta_1}$(6,2)$\succ_{\theta_1}$(5,8)$\succ_{\theta_1}$(4,4)$\succ_{\theta_1}$ (3,8)$\succ_{\theta_1}$(3,3)$\succ_{\theta_1}$(2,2)
$\succsim_{\theta_2}$	(7,7)$\succ_{\theta_2}$(8,3)$\succ_{\theta_2}$(5,8)$\succ_{\theta_2}$(6,2)$\succ_{\theta_2}$(4,4)$\succ_{\theta_2}$ (3,8)$\succ_{\theta_2}$(3,3)$\succ_{\theta_2}$(2,2)
$\succsim_{\theta_3}$	(7,7)$\succ_{\theta_3}$(8,3)$\succ_{\theta_3}$(5,8)$\succ_{\theta_3}$(4,4)$\succ_{\theta_3}$(6,2)$\succ_{\theta_3}$ (3,8)$\succ_{\theta_3}$(3,3)$\succ_{\theta_3}$(2,2)
$\succsim_{\theta_4}$	(8,3)$\succ_{\theta_4}$(7,7)$\succ_{\theta_4}$(6,2)$\succ_{\theta_4}$(5,8)$\succ_{\theta_4}$(3,8)$\succ_{\theta_4}$ (4,4)$\succ_{\theta_4}$(3,3)$\succ_{\theta_4}$(2,2)
$\succsim_{\theta_5}$	(8,3)$\succ_{\theta_5}$(7,7)$\succ_{\theta_5}$(6,2)$\succ_{\theta_5}$(5,8)$\succ_{\theta_5}$(4,4)$\succ_{\theta_5}$ (3,8)$\succ_{\theta_5}$(3,3)$\succ_{\theta_5}$(2,2)
$\succsim_{\theta_6}$	(8,3)$\succ_{\theta_6}$(7,7)$\succ_{\theta_6}$(6,2)$\succ_{\theta_6}$(5,8)$\succ_{\theta_6}$(4,4)$\succ_{\theta_6}$ (3,3)$\sim_{\theta_6}$(3,8)$\succ_{\theta_6}$(2,2)
Column types	
$\succsim_{\theta_7}$	(5,8)$\succ_{\theta_7}$(7,7)$\succ_{\theta_7}$(3,8)$\succ_{\theta_7}$(4,4)$\succ_{\theta_7}$(8,3)$\succ_{\theta_7}$ (3,3)$\succ_{\theta_7}$(6,2)$\succ_{\theta_7}$(2,2)
$\succsim_{\theta_8}$	(7,7)$\succ_{\theta_8}$(5,8)$\succ_{\theta_8}$(3,8)$\succ_{\theta_8}$(4,4)$\succ_{\theta_8}$(8,3)$\succ_{\theta_8}$ (3,3)$\succ_{\theta_8}$(6,2)$\succ_{\theta_8}$(2,2)
$\succsim_{\theta_9}$	(5,8)$\succ_{\theta_9}$(3,8)$\succ_{\theta_9}$(7,7)$\succ_{\theta_9}$(4,4)$\succ_{\theta_9}$(8,3)$\succ_{\theta_9}$ (3,3)$\succ_{\theta_9}$(6,2)$\succ_{\theta_9}$(2,2)
$\succsim_{\theta_{10}}$	(3,8)$\succ_{\theta_{10}}$(5,8)$\succ_{\theta_{10}}$(7,7)$\succ_{\theta_{10}}$(4,4)$\succ_{\theta_{10}}$(3,3)$\succ_{\theta_{10}}$ (8,3)$\succ_{\theta_{10}}$(2,2)$\succ_{\theta_{10}}$(6,2)
$\succsim_{\theta_{11}}$	(3,8)$\sim_{\theta_{11}}$(5,8)$\succ_{\theta_{11}}$(7,7)$\succ_{\theta_{11}}$(4,4)$\succ_{\theta_{11}}$(8,3)$\succ_{\theta_{11}}$ (3,3)$\succ_{\theta_{11}}$(6,2)$\succ_{\theta_{11}}$(2,2)

Table 6.8: Types and associated action sets per game

Type	#Subjects	No str. dom. action in	Observed action sets
θ_1	6	G=1,2,3	$\hat{a}^1_{\theta_1} = \{U, D\}$, $\hat{a}^2_{\theta_1} = \{U\}$, $\hat{a}^3_{\theta_1} = \{U, D\}$
θ_2	4	G=1,3	$\hat{a}^1_{\theta_2} = \{U, D\}$, $\hat{a}^3_{\theta_2} = \{U, D\}$
θ_3	2	G=1,3	$\hat{a}^1_{\theta_3} = \{D\}$, $\hat{a}^3_{\theta_3} = \{U, D\}$
θ_4	2	G=1,2,4	$\hat{a}^1_{\theta_4} = \{U\}$, $\hat{a}^2_{\theta_4} = \{U\}$, $\hat{a}^4_{\theta_4} = \{D\}$
θ_5	12	G=2,4	$\hat{a}^2_{\theta_5} = \{U\}$, $\hat{a}^4_{\theta_5} = \{U, D\}$,
θ_6	5	G=2,4	$\hat{a}^2_{\theta_6} = \{U\}$, $\hat{a}^4_{\theta_6} = \{U, D\}$,
θ_7	5	G=1,2,3	$\hat{a}^1_{\theta_7} = \{L, R\}$, $\hat{a}^2_{\theta_7} = \{L, R\}$, $\hat{a}^3_{\theta_7} = \{L, R\}$,
θ_8	2	G=1,3	$\hat{a}^1_{\theta_8} = \{L, R\}$, $\hat{a}^3_{\theta_8} = \{L, R\}$,
θ_9	23	G=2,3,4	$\hat{a}^2_{\theta_9} = \{L, R\}$, $\hat{a}^3_{\theta_9} = \{L, R\}$ $\hat{a}^4_{\theta_9} = \{L, R\}$,
θ_{10}	2	G=2,3	$\hat{a}^2_{\theta_{10}} = \{L, R\}$, $\hat{a}^3_{\theta_{10}} = \{L, R\}$
θ_{11}	2	G=3,4	$\hat{a}^3_{\theta_{11}} = \{R\}$, $\hat{a}^4_{\theta_{11}} = \{L\}$

6.4 Summary

The assumption that payoffs are mutually known is often not satisfied in the laboratory. Healy (2011), for example, finds that subjects fail to accurately predict other subjects' preferences over payment-pairs. It seems plausible that similar difficulties exist in many real-world situations as well. Of course, such assumptions are strong idealizations but as Weibull (2004, p. 86) properly expressed it: "So what, then, can be tested? One can test whether the theoretical predictions are at least approximately correct in environments that approximate the assumptions. Such testing is important, because this is the way game theory is used in economics and the other social sciences." In this spirit, our experiment shows that mutual knowledge is a relevant assumption: making sure that payoffs are mutually known leads to significantly more equilibrium play.

When deciding what model to apply to a specific situation, whether or not agents can reasonably be expected to know other agents' payoff functions should therefore play an important role. At least in the simple 2×2 games we analyzed, subjects are very unlikely to play a Nash equilibrium strategy when payoffs are not mutually known. It might then be worthwhile to apply a more complex model such as a strategic ambiguity model or the Bayesian Nash equilibrium of Harsanyi (1967-68), even though such models tend to provide less precise predictions. We show that the strategic ambiguity concept of Eichberger and Kelsey (2014) and a noisy version of the Bayesian Nash equilibrium yield predictions that are more consistent with our data.

How the trade-off between tractability, precision and accuracy is resolved will depend on the specific situation. This chapter makes a contribution to improve the ex-ante assessment of the accuracy of a models' predictions.

Chapter 7

Conclusion

This thesis aims at examining how ambiguity-sensitive behavior affects strategic decision-making in interactive situations. We considered strategic interaction under payoff uncertainty, under strategic uncertainty, and with private information. To capture ambiguity-sensitive behavior, several alternatives to subjective expected utility theory of Savage (1954) have been proposed. For our investigations, we used strategic interaction models that are based on such alternative models of choice under uncertainty.

In Chapter 4, we develop and analyze a Hotelling duopoly game under demand ambiguity in which firms' beliefs are represented by neo-additive capacities introduced by Chateauneuf et al. (2007). Our model provides a unifying framework for the Hotelling models developed by Meagher and Zauner (2004) and Król (2012). We show that there exists a unique subgame-perfect pure strategy Nash equilibrium for the Hotelling game under ambiguity. Our capacity model incorporates a variety of different sources of uncertainty. For example, there is the variance of the reference probability, the confidence, or the degree of ambiguity, and the uncertain transportation cost parameter. We present comparative static results with respect to all model parameters. These results show that the effects of some parameters are interrelated. Therefore, one should be very cautious when it comes to drawing conclusions from real-world applications of Hotelling models under uncertainty. In fact, the conclusions from an observed increase or decrease in product differentiation might change in light of different sources of uncertainty. We illustrate that the capacity model offers additional explanations for observed real-world phenomena.

Chapter 5 investigates the extent to which we can distinguish expected and non-expected utility players on the basis of their behavior. A model of incomplete information games is used in which players can choose mixed strategies. The key assumption of this model is that players behave as expected utility maximizers with correct beliefs concerning mixed strategy combinations, yet they face ambiguous uncertainty about the environment. Expected and non-expected players sometimes cannot be distinguished by observing their equilibrium actions since they behave observationally equivalent. It is shown that uncertainty-averse non-expected utility players can often be identified by looking at their best responses. They may behave differently in the use of mixed strategies, called hedging behavior, and the response to mixed strategy combinations, called reversal behavior. It turns out that these are the sole behavioral differences between expected and non-expected utility players. Furthermore, the absence of hedging behavior is sufficient for observational equivalence. The chapter provides necessary and sufficient conditions for the existence of hedging and reversal behavior in terms of the payoff structure of a two-person two-strategies game. Finally, the underlying model is discussed, and an equilibrium concept is introduced that allows for players who are not uncertainty-averse.

Chapter 6 experimentally examines several two-person two-strategies games in which the Nash equilibrium prediction that results if players only care about their own payments is often not consistent with subject behavior. We test whether revealing players' preferences leads to more equilibrium play. For that purpose, we first elicit subjects' preferences over monetary payoff pairs. In one treatment, these preferences are then revealed to other players. We find that subjects are significantly more likely to play an equilibrium strategy when other players' preferences are revealed. Our results thus suggest that one should be careful in simply assuming that players' preferences are mutually known. If it is likely that players do not know each other's preferences, equilibrium concepts that are more general than Nash equilibrium might provide a more reliable prediction. We show that the observed strategy choices are consistent with a strategic ambiguity model.

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