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Reciprocity laws for smooth projective schemes

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Abstract. We prove that two natural isomorphisms between the first mod m Suslin homology and the mod m abelianized étale fundamental group agree for connected smooth projective schemes over algebraically closed fields.

Zusammenfassung. Wir zeigen, dass zwei natürliche Isomorphismen zwischen der ersten mod m Suslinhomologie und der mod m abelisierten étalen Fundamentalgruppe eines zusammenhängenden, glatten, projektiven Schemas über einem algebraisch abgeschlossenem Körper übereinstimmen.

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Contents

1. Introduction

This thesis lies at the intersection of three fields in mathematics: number theory, algebraic geometry and algebraic topology. The influence of number theory stems from class field theory. Class field theory was classically concerned with the study of finite abelian extensions of local and global fields, a problem which can be rephrased using A. Grothendieck's étale fundamental group as follows. For every connected, locally noetherian scheme X its étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ with respect to some geometric base point $\bar{x}$ classifies, by its very definition, all finite étale coverings of X. The aim of class field theory is then, in the case that X is the spectrum of a local or global field, to describe the abelianization $\pi_1^{\text{ét,ab}}(X)$ of $\pi_1^{\text{ét}}(X, \bar{x})$ in terms of data inherent to X, for example in terms of the idèle class group in the case that X is the spectrum of a global field. However, using the above reformulation the problem of class field theory can now be phrased for every connected, locally noetherian scheme X culminating in "higher-dimensional class field theory". This brings algebraic geometry into the picture. Thinking about abelianizations of fundamental groups from a topologist's point of view one is immediately led to the Hurewicz isomorphism

$$\pi_1(X, x)^{\text{ab}} \cong H_1(X, \mathbb{Z})$$

from the abelianization of the fundamental group $\pi_1(X, x)$ of a topological space X to its singular homology $H_1(X, \mathbb{Z})$. Searching for an analogue of the Hurewicz isomorphism for (separated) schemes of finite type over an algebraically closed field has turned out to be fruitful. First of all, A. Suslin and V. Voevodsky have proposed in [SV96] an analogue of singular homology for schemes over fields, namely Suslin homology $H_*^S(X, \mathbb{Z})$ (cf. section 5). For a scheme X, separated and of finite type over a field k, it is defined as the homology of a complex

$$\text{Cor}(\Delta^\bullet, X)$$

consisting of correspondences $\Delta^n \to X$ from the algebraic n-simplex

$$\Delta^n := \{(x_0, \cdots, x_n) \in \mathbb{A}_k^{n+1} \mid \sum_{i=0}^{n} x_i = 1\}$$

to X (thereby being a datum internal to X). It is not reasonable to expect an isomorphism between the abelianized étale fundamental group $\pi_1^{\text{ét,ab}}(X)$ and the first Suslin homology $H_1^S(X, \mathbb{Z})$ because the étale fundamental group is profinite while Suslin homology is discrete. The situation changes if Suslin homology with finite coefficients is considered. Then using the qfh-topology A. Suslin and V. Voevodsky proved the following remarkable theorem.

Theorem ([SV96]). *Let X be a scheme, separated and of finite type over an algebraically closed field k. Let $m \in \mathbb{Z}$ be a non-zero integer, invertible in k. Then for any $i \geq 0$ there is a natural isomorphism*

$$H^i_S(X, \mathbb{Z}/m) \cong H^i_{\text{ét}}(X, \mathbb{Z}/m)$$

between Suslin cohomology $H^i_S(X, \mathbb{Z}/m) := H_i^S(X, \mathbb{Z}/m)^\vee$ and étale cohomology $H^i_{\text{ét}}(X, \mathbb{Z}/m)$. In particular, for X connected there is a natural isomorphism

$$\Phi_{\text{qfh}} \colon H_1^S(X, \mathbb{Z}/m) \xrightarrow{\sim} H^1_{\text{ét}}(X, \mathbb{Z}/m)^\vee \cong \pi_1^{\text{ét,ab}}(X)/m,$$

called qfh reciprocity law.

The name "reciprocity law" is motivated by classical class field theory although the morphism Φ_{qfh} (or rather its inverse) deserves to be called "qfh Hurewicz isomorphism".
For $m \in \mathbb{Z}$ not necessarily invertible in k an important adjustment has to be made. While the Suslin homology $H_1^S(X, \mathbb{Z}/m)$ is homotopy invariant in any characteristic, the same is not true for the mod m abelianized étale fundamental group $\pi_1^{\text{ét,ab}}(X)/m$, but only for its tame version, i.e., the mod m abelianized tame étale fundamental group $\pi_1^{\mathrm{t,ab}}(X)/m$ (cf. section 6). If X is proper over the ground field k then the tame étale and étale fundamental group of X conincide. The same is also true for the mod m abelianized fundamental groups $\pi_1^{\mathrm{t,ab}}(X)/m$ and $\pi_1^{\text{ét,ab}}(X)/m$ if m is invertible in k. Thus imposing tameness is only a condition for non-proper schemes over fields of positive characteristic p and coefficients $\mathbb{Z}/p^r$. After having done this adjustment A. Schmidt and T. Geisser were able to prove the following theorem, partly generalising the Suslin-Voevodsky theorem.

Theorem ([GS])**.** *Let X be a separated scheme of finite type over an algebraically closed field k and let $m \in \mathbb{Z}$ be a non-zero integer. Then there exists a natural morphism, called geometric reciprocity law,*

$$\Phi_{\mathrm{geom}} \colon H_1^S(X, \mathbb{Z}/m) \to \pi_1^{\mathrm{t,ab}}(X)/m$$

from the first mod m Suslin homology to the abelianized tame étale fundamental group mod m. The morphism Φ_{geom} is an isomorphism if m is invertible in k or X is smooth projective or resolution of singularities holds over k. Moreover, for m invertible in k the morphisms Φ_{geom} and Φ_{qfh} agree.

Their construction of the morphism Φ_{geom} is of a geometric nature (cf. section 7) and a direct analogue of the classical Hurewicz morphism.
It is an open question whether the geometric reciprocity law can be extended to homology or cohomology of degree greater than one. A major problem is that there does not exist, at the moment, a definition of higher tame étale cohomology.
In the case of smooth projective schemes there exists – apart from the qfh reciprocity law Φ_{qfh} and the geometric reciprocity law Φ_{geom} – another reciprocity law Φ_{mot}, called motivic reciprocity law, relating Suslin homology and étale fundamental groups.

Theorem (cf. section 12)**.** *Let X be a connected smooth projective scheme over an algebraically closed field k and let $m \in \mathbb{Z}$ be an arbitrary non-zero integer. Then there exists a natural morphism, called motivic reciprocity law,*

$$\Phi_{\mathrm{mot}} \colon H_1^S(X, \mathbb{Z}/m) \xrightarrow{\sim} \pi_1^{\text{ét,ab}}(X)/m$$

obtained from general theory about motivic cohomology.

The precise definition of the motivic reciprocity law and the fact that it is an isomorphism is rather intricate and will occupy a large part of this thesis (namely, sections 5, 8, 9, 10, 11 and 12). We briefly present the definition of Φ_{mot}. The assumption that X is smooth and projective implies that its mod m Suslin homology is isomorphic to a higher Chow group in the sense of S. Bloch (cf. section 8):

$$H_1^S(X, \mathbb{Z}/m) \cong \mathrm{CH}^d(X, 1; \mathbb{Z}/m),$$

where $d := \dim X$ is the dimension of X.

General comparison theorems in motivic cohomology (cf. section 9) imply that this higher Chow group is isomorphic to some motivic cohomology group in the sense of V. Voevodsky:

$$\mathrm{CH}^d(X,1;\mathbb{Z}/m) \cong H^{2d-1}_{\mathcal{M}}(X,\mathbb{Z}/m(d)).$$

However, this isomorphism is not given by a morphism of underlying complexes, but a zig-zag of several isomorphisms, most notably comparison isomorphisms of both groups with hypercohomology of the Suslin-Friedlander weight n motivic complexes. Motivic cohomology with finite coefficients was shown to be isomorphic to étale cohomology, at least in a certain range. For m invertible in k this is the famous Beilinson-Lichtenbaum conjecture, now proven by V. Voevodsky, while for coefficients $\mathbb{Z}/p^r$ with $p > 0$ the characteristic of the ground field this follows from the Bloch-Gabber-Kato theorem (cf. section 10):

$$H^{2d-1}_{\mathcal{M}}(X,\mathbb{Z}/m(d)) \cong H^{2d-1}_{\text{ét}}(X,\mathbb{Z}/m(d)).$$

Finally Poincaré duality (cf. section 11) implies that

$$H^{2d-1}_{\text{ét}}(X,\mathbb{Z}/m(d)) \cong H^1_{\text{ét}}(X,\mathbb{Z}/m)^\vee \cong \pi_1^{\text{ét},\text{ab}}(X)/m.$$

Thus the motivic reciprocity law Φ_{mot} can be defined as the composition of all these isomorphisms above. We remark that we used Poincaré duality also for coefficients $\mathbb{Z}/p^r$ if $p = \mathrm{char}(k) > 0$, which is not as established as its counterpart with coefficients $\mathbb{Z}/m$ for m invertible in k. In fact, for $m = p^r$ we used the notation

$$\mathbb{Z}/p^r(d) = \nu_r(d)[-d]$$

for the shifted logarithmic de Rham-Witt sheaf $\nu_r(d)$ of weight d. The sheaf $\nu_r(d)$ is neither constructible nor locally constant, facts which cause a lot of issues we had to deal with.
It is by no means clear whether the motivic reciprocity law Φ_{mot} agrees for smooth projective schemes with the geometric reciprocity law Φ_{geom} (and thus with the qfh reciprocity law Φ_{qfh}).
The main theorem in this thesis will be the following comparison.

Theorem (cf. theorem 14.1)**.** *For every smooth projective scheme X the geometric reciprocity law Φ_{geom}, and hence also the qfh reciprocity law Φ_{qfh}, agrees with the motivic reciprocity law Φ_{mot}.*

We briefly describe the proof of this comparison theorem. First of all, there is the problem that the geometric reciprocity law is natural *only up to a sign.* The analogue problem for the classical Hurewicz morphism

$$\pi_1(X,x)^{\text{ab}} \to H_1(X,\mathbb{Z})$$

is its dependence on the choice a generator in $H_1(S^1,\mathbb{Z})$, i.e., on an orientation of the one-dimensional sphere S^1. Similarly, in the classical reciprocity laws for local and global fields uniformizers can be sent to arithmetic or geometric Frobenii – a choice analogous to fixing "orientations" on the "cirles" $\mathrm{Spec}(\mathcal{F}_p)$. In order to control the signs appearing in the comparison of Φ_{geom} and Φ_{mot} we have to carefully recall all isomorphisms occuring in their definition. This problem turns out to be delicate. In general, the first cohomology of an abelian sheaf F on a scheme X can be identified with isomorphism classes of F-torsors (cf. section 3).

Moreover, this identification can be chosen in such a way that for a short exact sequence

$$0 \to F \to G \to H \to 0$$

the boundary morphism maps a global section $s \in H(X)$ to the isomorphism class of the F-torsor of preimages of s in G (cf. lemma 3.1).[1] In the case of the short exact sequence

$$0 \to \mathbb{G}_m \to \mathcal{M}_X^\times \to \mathcal{M}_X^\times/\mathbb{G}_m \to 0$$

describing the sheaf of Cartier divisors $\mathcal{M}_X^\times/\mathbb{G}_m$ on the scheme X this has the following surprising and intricate consequence: The boundary map

$$\delta\colon H^0(X, \mathcal{M}_X^\times/\mathbb{G}_m) \to H^1(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$$

maps a Cartier divisor D to the class of the line bundle $\mathcal{O}_X(-D)$ and *not* to class of the line bundle $\mathcal{O}_X(D)$. In particular, it does not agree with the (inverse of the) first Chern class.

The proof that both reciprocity laws, Φ_{geom} and Φ_{mot}, agree can be reduced to the case that X is a smooth projective curve. In this case, there are isomorphisms, which again have to be made explicit to control possible signs,

$$\begin{aligned} H_1^S(X, \mathbb{Z}/m) &\cong H^1_{\text{ét}}(X, \mathbb{Z}/m(1)) \cong {}_m\mathrm{Pic}(X) \\ H^1_{\text{ét}}(X, \mathbb{Z}/m) &\cong \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m), \end{aligned}$$

where ${}_m\mathrm{Pic}(X)$ denotes the m-torsion in the Picard group $\mathrm{Pic}(X)$. The difficulty now lies in showing that the Poincaré duality pairing is given by the natural evaluation morphism under the above identifications. In the case that $m \in \mathbb{Z}$ is invertible in k, there is an argument of P. Deligne (cf. [Del77]) proving this. The main task in the proof of the comparison theorem is then to show that Deligne's argument can be adapted to the case of arbitrary non-zero integers $m \in \mathbb{Z}$ (cf. section 13). But for $m = p^r$ with $p = \mathrm{char}(k) > 0$ the appearing logarithmic de Rham-Witt cohomology is not as nice as its counterpart, the étale cohomology with coefficients given by tensor powers of the sheaf of m-th roots of unity $\mu_m^{\otimes n}$ for m invertible in k. For example, Deligne's argument in [Del77] uses two results from the full developed étale cohomology theory with constructible coefficients of order invertible on the scheme, which are not known in logarithmic de Rham-Witt cohomology (and appear to be not true in general): Firstly, the Künneth decomposition for products and secondly the description of the relative dualizing complex of a smooth morphism. Fortunately, we are able to prove the Künneth decomposition in degree 2 for the product $X \times X$ of a smooth projective curve X, which is the only case we need (cf. appendix A). We circumvent the use of relative duality for a morphism $f\colon Y \to Z$ by using absolute duality on Y and Z. As a result we have to do a rather tedious identification of the relative trace map (cf. appendix B). Finally, we can adapt Deligne's argument to accomodate all these difficulties and prove the desired identification of Poincaré duality for curves (cf. section 13), thereby finishing the proof of our main theorem.

[1] In fact, with this choice the identification of $H^1(X, F)$ with isomorphism classes of F-torsors extends to non-abelian sheaves of groups F.

2. Notations

We will use the following general notations.

- k denotes a perfect field, later assumed to be algebraically closed.
- Sch/k denotes the category of separated schemes of finite type over k.
- Sm/k denotes the category of (quasi-compact) smooth separated schemes over k.

3. Divisors and line bundles

In this section we briefly recall the well-known relation between Cartier divisors, Weil divisors, line bundles and $\mathbb{G}_m$-torsors on a scheme. For this let first $\mathfrak{X}$ be an arbitrary (Grothendieck) topos, e. g., $\mathfrak{X} = \widetilde{X_{\text{ét}}}$ or $\mathfrak{X} = \widetilde{X_{\mathrm{Zar}}}$ the (small) étale or Zariski topos of a scheme X. For every abelian group $F \in \mathfrak{X}$, i.e., every abelian group object in $\mathfrak{X}$, the first cohomology group $H^1(\mathfrak{X}, F)$ can naturally be identified with isomorphism classes of F-torsors in $\mathfrak{X}$, cf. [Sta16, Tag 03AG]. The following lemma is a direct consequence of this identification.

Lemma 3.1. *Let*

$$0 \to F \to H \to G \to 0$$

be a short exact sequence of abelian groups in $\mathfrak{X}$. Then the boundary map

$$\delta\colon H^0(\mathfrak{X}, G) \to H^1(\mathfrak{X}, F)$$

maps a section $s \in H^0(\mathfrak{X}, G)$ to the isomorphism class of the F-torsor $s \times_G H$ of sections in H mapping to s.

Proof. Let $H \hookrightarrow I$ be an embedding with I an injective abelian group in $\mathfrak{X}$ and consider the resulting diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & F \arrow[r] & I \arrow[r] & Q \arrow[r] & 0. \end{tikzcd}$$

with exact rows. Let $s \in H^0(\mathfrak{X}, G)$ be a section. Its image under the boundary map $\delta\colon H^0(\mathfrak{X}, G) \to H^1(\mathfrak{X}, F)$ agrees with the image of $\varphi(s)$ under the boundary map $\tilde{\delta}\colon H^0(\mathfrak{X}, Q) \to H^1(\mathfrak{X}, F)$ for the lower sequence. By [Sta16, Tag 03AG] this class is given by the isomorphism class of the F-torsor $\varphi(s) \times_Q I$ of sections in I mapping to $\varphi(s)$. But this torsor is naturally isomorphic to the F-torsor $s \times_G H$, hence the result. □

Let now X be an arbitrary scheme. Then the groupoid of $\mathbb{G}_m$-torsors for the Zariski topology on X is canonically equivalent to the groupoid of line bundles on X, hence $H^1_{\mathrm{Zar}}(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$. More precisely, the isomorphism is given by the map

$$H^1_{\mathrm{Zar}}(X, \mathbb{G}_m) \to \mathrm{Pic}(X), \mathcal{T} \mapsto \mathcal{T} \times^{\mathbb{G}_m} \mathcal{O}_X$$

sending (the isomorphism class of) a $\mathbb{G}_m$-torsor T to the (isomorphism class of the) contracted product $\mathcal{T} \times^{\mathbb{G}_m} \mathcal{O}_X$. The same also holds for the Nisnevich or étale topology on X due to "Hilbert's theorem 90"

$$H^1_{\mathrm{Zar}}(X, \mathbb{G}_m) \cong H^1_{\mathrm{Nis}}(X, \mathbb{G}_m) \cong H^1_{\text{ét}}(X, \mathbb{G}_m).$$

Let $\mathcal{M}_X$ be the sheaf of rational functions on X, i.e., the localisation

$$\mathcal{M}_X := \mathcal{S}^{-1}\mathcal{O}_X$$

of $\mathcal{O}_X$ at the multiplicative subsheaf $\mathcal{S}$ of regular elements

$$\mathcal{S} := \{ \ f \in \mathcal{O}_X \mid f \colon \mathcal{O}_X \to \mathcal{O}_X \text{ injective } \}.$$

Let $\mathcal{M}_X^\times$ be the sheaf of units in the sheaf of rings $\mathcal{M}_X$. The group $H^0(X, \mathcal{M}_X^\times/\mathbb{G}_m)$ of Cartier divisors on X is linked to the group of $\mathbb{G}_m$-torsors on X via the boundary map

$$\delta_{\mathrm{CaCl}} \colon H^0(X, \mathcal{M}_X^\times/\mathbb{G}_m) \to H^1(X_{\mathrm{Zar}}, \mathbb{G}_m)$$

of the short exact sequence

$$0 \to \mathbb{G}_m \to \mathcal{M}_X^\times \to \mathcal{M}_X^\times/\mathbb{G}_m \to 0.$$

Recall that to any Cartier divisor D there is associated the line bundle

$$\mathcal{O}_X(-D) \subseteq \mathcal{M}_X$$

which is locally generated by a defining equation for D. With this definition there exists a canonical isomorphism

$$\mathcal{O}_X(D) \cong \mathcal{O}_X(-D)^\vee.$$

We record the following lemma for later reference.

Lemma 3.2. *The composite map*

$$H^0(X, \mathcal{M}_X^\times/\mathbb{G}_m) \xrightarrow{\delta_{\mathrm{CaCl}}} H^1(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$$

sends a Cartier divisor D on X to the line bundle $\mathcal{O}_X(-D)$.

Proof. This is immediate from the description in lemma 3.1. The preimage of a Cartier divisor D under the map $\mathcal{M}_X^\times \to \mathcal{M}_X^\times/\mathbb{G}_m$ is precisely the $\mathbb{G}_m$-torsor $\mathcal{T}$ of defining equations for D, hence contracting this torsor with $\mathcal{O}_X$ defines the line bundle $\mathcal{O}_X(-D) \cong \mathcal{T} \times^{\mathbb{G}_m} \mathcal{O}_X$. □

Under very mild assumptions, the boundary map δ_{CaCl} is surjective, identifying $H^1_{\mathrm{Zar}}(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$ with the Cartier divisor class group

$$\mathrm{CaCl}(X) := H^0(X, \mathcal{M}_X^\times/\mathbb{G}_m)/H^0(X, \mathcal{M}_X^\times).$$

For example, X locally noetherian with associated points contained in some affine open subset or X reduced with locally finitely many irreducible components is sufficient (cf. [Gro67, Proposition (21.3.4)]). Moreover, to each Cartier divisor D on a scheme X (assumed to be noetherian) is associated the Weil divisor

$$D_W := \sum_{x \in X^1} v_x(f_x) \cdot x,$$

where $X^1 \subset X$ denotes the points of codimension 1, $v_x \colon \mathcal{M}_{X,x}^\times \to \mathbb{Z}$ the associated valuation[2] and $f_x \in \mathcal{M}_{X,x}^\times$ a defining equation for D in x. If X is locally factorial, then the map $D \mapsto D_W$ is an isomorphism between the groups of Cartier and Weil divisors and therefore the group $\mathrm{CaCl}(X)$ of Cartier divisor classes on X is isomorphic to the first Chow group $\mathrm{CH}^1(X)$. Finally, we introduce the first Chern

[2] for $g \in \mathcal{M}_{X,x}^\times \cap \mathcal{O}_{X,x}$ it satisfies $v_x(g) = \mathrm{length}(\mathcal{O}_{X,x}/g)$

class of a line bundle. Let X be a noetherian scheme and assume that its Cartier divisor class group and its Picard group are isomorphic via the map

$$\mathrm{CaCl}(X) \to \mathrm{Pic}(X), D \mapsto \mathcal{O}_X(-D).$$

Then the first Chern class (for line bundles) is defined as the map

$$c_1 \colon \mathrm{Pic}(X) \to \mathrm{CH}^1(X)$$

sending a line bundle $\mathcal{L} \cong \mathcal{O}_X(D)$ to the class of the Weil divisor D_W, which is easily seen to depend only on the isomorphism class of $\mathcal{L}$.
In the case that X is a smooth scheme over a field, and this is the case of our main interest, we will no longer distinguish between Weil and Cartier divisors and just speak about divisors.

4. Presheaves with transfers

In this section we recall the notion of (pre)sheaves with transfers. This definition is at the heart of Voevodsky's construction of a triangulated category $DM^-_{\mathrm{eff}}(k)$ of effective motivic complexes over a perfect field k. We will introduce some examples of (pre)sheaves with transfers and for later use we will present some constructions which can be performed with (pre)sheaves with transfers.
First we need the definition of a correspondence. Recall that Sch/k (resp. Sm/k) denotes the category of quasi-compact, separated, finite type (resp. smooth) schemes over a perfect field k.

Definition 4.1 ([MVW06, Lecture 1])**.** Let $X \in \mathrm{Sch}/k$ and $U \in \mathrm{Sm}/k$ be two schemes with U smooth over k. An elementary correspondence $\mathcal{Z} \colon U \to X$ is a closed, integral subscheme $V \subseteq U \times X$, which is finite and surjective over an irreducible component of U. A correspondence $\mathcal{Z} \colon U \to X$ is then a cycle $\mathcal{Z}$ on $U \times X$ which is a sum of elementary correspondences. We denote by

$$\mathrm{Cor}(U, X)$$

the abelian group of correspondences $\mathcal{Z} \colon U \to X$.

Clearly, as X is separated over k, associating to each homomorphism $f \colon U \to X$ its graph defines an embedding

$$\mathrm{Hom}_k(U, X) \hookrightarrow \mathrm{Cor}(U, X).$$

Correspondences can be composed (cf. [MVW06, Lemma 1.7.] for more details) extending the composition of morphisms. Namely, given correspondences $\mathcal{Z}' \colon U \to V$ and $\mathcal{Z} \colon V \to X$ with $U, V \in \mathrm{Sm}/k$ and $X \in \mathrm{Sch}/k$ the composition $\mathcal{Z} \circ \mathcal{Z}'$ is defined as the cycle

$$\mathcal{Z} \circ \mathcal{Z}' := p_{1,3_*}(p^*_{2,3}(\mathcal{Z}) \cdot p^*_{1,2}(\mathcal{Z}')) \tag{4.1}$$

on $U \times X$. Here the $p_{i,j}$ are the natural projections of the triple product $U \times V \times X$ and $\cdot$ denotes the intersection on $U \times V \times X$. Similarly, a correspondence $\mathcal{Z} \colon V \to X$ with $V \in \mathrm{Sm}/k$ and $X \in \mathrm{Sch}/k$ can be composed with a morphism $X \to X'$ in Sch/k, namely

$$f \circ \mathcal{Z} := p_{1,3_*}(p^*_{2,3}(\Gamma_f) \cdot p^*_{1,2}(\mathcal{Z})) \tag{4.2}$$

with $\Gamma_f \subseteq X \times X'$ the graph of $f \colon X \to X'$.
Therefore we can define now the category of correspondences.

Definition 4.2. We denote by

$$\mathrm{Cor}/k$$

the category of correspondences over k. This means that Cor/k has as objects the smooth schemes $U \in \mathrm{Sm}/k$ and for two objects $U, U' \in \mathrm{Sm}/k$ the morphisms are defined to be

$$\mathrm{Hom}_{\mathrm{Cor}/k}(U, U') := \mathrm{Cor}(U, U').$$

The composition in Cor/k is given by the composition of correspondences (4.1).

Clearly, the category Cor/k of correspondences is additive. The direct sum of two objects $U, U' \in \mathrm{Cor}/k$ is given by their disjoint union $U \coprod U'$.
We now record the following important definition.

Definition 4.3. An additive functor $F\colon (\mathrm{Cor}/k)^{\mathrm{op}} \to (\mathrm{Ab})$ is called a presheaf with transfers. A morphism of presheaves with transfers is a natural transformation of functors. We denote the category of presheaves with transfers over k by $\mathrm{PST}(k)$.

The category $\mathrm{PST}(k)$ is abelian (with kernel and cokernel defined pointwise). It is even a Grothendieck abelian category.
Clearly, there is a natural inclusion functor $\iota\colon \mathrm{Sm}/k \to \mathrm{Cor}/k$ which maps a morphism $f\colon U \to U'$ of smooth schemes over k to the graph $\Gamma_f \subseteq U \times_k U'$.
Let $\mathcal{T}$ be a Grothendieck topology on the category Sm/k, for example the Zariski, Nisnevich or étale topology. But $\mathcal{T}$ can also be taken to be the trivial topology, i.e., the topology all of whose coverings are isomorphisms.

Definition 4.4. A presheaf with transfers $F\colon \mathrm{Cor}/k \to (\mathrm{Ab})$ is called a $\mathcal{T}$-sheaf with transfers if the functor $F \circ \iota\colon \mathrm{Sm}/k \to (\mathrm{Ab})$ is a sheaf with respect to $\mathcal{T}$. We denote the category of $\mathcal{T}$-sheaves with transfers by $\mathrm{ST}_{\mathcal{T}}(k)$.

If $\mathcal{T}$ is the trivial topology, then a $\mathcal{T}$-sheaf with transfers is just a presheaf with transfers.
Again the category of $\mathrm{ST}_{\mathcal{T}}(k)$ is Grothendieck abelian for every Grothendieck topology $\mathcal{T}$ on Sm/k. Clearly, there are fully faithful inclusions

$$\mathrm{ST}_{\text{ét}}(k) \subseteq \mathrm{ST}_{\mathrm{Nis}}(k) \subseteq \mathrm{ST}_{\mathrm{Zar}}(k) \subseteq \mathrm{PST}(k)$$

admitting left adjoints (for formal reasons). But the question how to describe these left adjoints is rather subtle in general. First of all each Grothendieck topology $\mathcal{T}$ on Sm/k induces, via the inclusion $\mathrm{Sm}/k \to \mathrm{Cor}/k$, a Grothendieck topology $\tilde{\mathcal{T}}$ on Cor/k whose category of abelian sheaves is equivalent to $\mathrm{ST}_{\mathcal{T}}(\mathrm{k})$.[3] The sheafification with respect to the Grothendieck topologies $\tilde{\text{ét}}$, $\tilde{\mathrm{Nis}}$ or $\tilde{\mathrm{Zar}}$ will then be the searched for left adjoints. But for general $\mathcal{T}$ the underlying $\mathcal{T}$-sheaf of the $\tilde{\mathcal{T}}$-sheafification of some $F \in \mathrm{PST}(k)$ might not agree with the $\mathcal{T}$-sheafification of the underlying presheaf on Sm/k of F. The next proposition shows that this behaviour does not occur for the Nisnevich or étale topology (although it can possibly happen for the Zariski topology).

Proposition 4.5. *Let $F \in \mathrm{PST}(k)$ be a presheaf with transfers and denote by $\underline{F}$ the underlying presheaf on Sm/k. Then the Nisnevich (resp. étale) sheafification $\underline{F}_{\mathrm{Nis}}$ (resp. $\underline{F}_{\text{ét}}$) of $\underline{F}$ admits transfers, which are uniquely determined by requiring that the canonical morphism $F \to \underline{F}_{\mathrm{Nis}}$ (resp. $F \to \underline{F}_{\text{ét}}$) is a morphism of presheaves*

[3] at least if open-closed decompositions are coverings in $\mathcal{T}$. This ensures that abelian $\tilde{\mathcal{T}}$-sheaves on Cor/k are automatically additive functors.

with transfers. Moreover, sending F to $\underline{F}_{\mathrm{Nis}}$ (resp. $\underline{F}_{\text{ét}}$) provides a left adjoint for the inclusion $\mathrm{ST}_{\mathrm{Nis}}(k) \subseteq \mathrm{PST}(k)$ (resp. $\mathrm{ST}_{\text{ét}}(k) \subseteq \mathrm{PST}(k)$).

Proof. For the Nisnevich topology this is stated in [SV00a, Lemma 1.2], for the étale topology in [MVW06, Theorem 6.17, Corollary 6.18]. □

We now give some examples of sheaves with transfers.

Definition 4.6. Let $X \in \mathrm{Sch}/k$ be a scheme. Then we denote by $\mathbb{Z}_{\mathrm{tr}}(X)$ the presheaf with transfers whose group of sections over some $U \in \mathrm{Sm}/k$ is the group of correspondences $\mathrm{Cor}(U, X)$ with pullbacks induced by the composition of correspondences (4.1).

Clearly, we obtain a functor $\mathbb{Z}_{\mathrm{tr}}(-) \colon \mathrm{Sch}/k \to \mathrm{PST}(k)$ (by (4.2)). This functor is fully faithful only when restricted to smooth schemes, for example $\mathbb{Z}_{\mathrm{tr}}(X) = \mathbb{Z}_{\mathrm{tr}}(X_{\mathrm{red}})$ for every scheme $X \in \mathrm{Sch}/k$.

Proposition 4.7. *Let $X \in \mathrm{Sch}/k$ be a scheme. Then the presheaf $\mathbb{Z}_{\mathrm{tr}}(X)$ is an étale sheaf.*

Proof. Cf. [MVW06, Lemma 6.2] or [Ans, Corollary 4.9]. □

The following example includes in particular the sheaves $\mathbb{G}_m$ or $\mathbb{G}_a$.

Proposition 4.8. *Let G over k be an abelian group scheme of finite type. Then there exists a canonical presheaf with transfers $G' \in \mathrm{PST}(k)$ such that $G' \circ \iota = G$ is the étale sheaf of abelian groups represented by G. In particular, G' is a is an étale sheaf with transfers.*

Proof. Cf. [And04, Exemples 19.1.1.2)]. □

In other words, the étale sheaf G on Sm/k admits canonically transfers.
In the case $G = \mathbb{G}_m$ the transfers can be described explicitly as follows. Let $\mathcal{Z} \colon U \to U'$ be a correspondence and let $s \in \mathbb{G}_m(U')$ be a section. To construct the transfers we may assume that $\mathcal{Z}$ is an elementary correspondence and that U is connected. Then $\mathcal{Z} \to U$ is a finite, surjective morphism and thus admits a norm homomorphism $N_{Z/U} \colon \mathbb{G}_m(Z) \to \mathbb{G}_m(U)$ as U is regular. Denote by $s_{|\mathcal{Z}}$ the pullback of s to $\mathcal{Z}$ along the morphism $\mathcal{Z} \to U$. Then the transfer of s along the correspondence $\mathcal{Z}$ is finally defined as the norm of $s_{|\mathcal{Z}}$.
For G as in proposition 4.8 we will again write G for its canonical extension G' as a presheaf with transfers. Given G, the identity section $\mathrm{Id}_G \in G(G)$ defines a morphism of presheaves with transfers

$$\mathbb{Z}_{\mathrm{tr}}(G) \to G.$$

This morphism is always pointwise surjective.
The following example allows the definition of the "motive with compact supports" of a scheme $X \in \mathrm{Sch}/k$.
Let $X \in \mathrm{Sch}/k$ be a scheme and let $r \in \mathbb{N}$ be an integer. For $U \in \mathrm{Sm}/k$ we denote by $z_{\mathrm{equi}}(X, r)(U)$ the group of equidimensional cycles for the projection $X \times U \to U$ which are of relative dimension r. Together with the pullback of relative cycles [SV00b, Theorem 3.3.1] we obtain a presheaf $z_{\mathrm{equi}}(X, r)$ on Sm/k.

Proposition 4.9. *The presheaf $z_{\mathrm{equi}}(X, r)$ admits transfers, i.e., there is natural extension of $z_{\mathrm{equi}}(X, r)$ as an additive presheaf, still denoted $z_{\mathrm{equi}}(X, r)$, to Cor/k. Moreover, $z_{\mathrm{equi}}(X, r)$ is an étale sheaf (with transfers).*

Proof. The description of the transfers can be found in the beginning of [MVW06, Lecture 16]. The sheaf property can easily be checked. In fact, the property of being equidimensional of relative dimension r is étale local. Therefore the claim follows from [Ans, Theorem 4.8]. □

We now introduce a monoidal structure on the category $\mathrm{PST}(k)$ of presheaves with transfers (following [SV00a, Chapter 2]). Although the existence of this tensor product, which is a formal extension of the product of schemes, is easily established, it remains rather mysterious.

Proposition 4.10. *There exists a unique colimit preserving symmetric monoidal product*

$$\otimes^{\mathrm{tr}} \colon \mathrm{PST}(k) \times \mathrm{PST}(k) \to \mathrm{PST}(k)$$

such that $\mathbb{Z}_{\mathrm{tr}}(X) \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr}}(Y) = \mathbb{Z}_{\mathrm{tr}}(X \times Y)$ *for two smooth schemes* $X, Y \in \mathrm{Sm}/k$.

Proof. It is a general property that every presheaf with transfers $F \in \mathrm{PST}(k)$ is a canonical colimit of representable presheaves with transfers, namely

$$\varinjlim \mathbb{Z}_{\mathrm{tr}}(X) \cong F$$

where the colimit is taken over the category $(\mathrm{Cor}/k)/F$ of representable presheaves with transfers together with a morphism to F. Hence the existence and uniqueness of $\otimes^{\mathrm{tr}}$ is formal. More details can be found in [SV00a, Chapter 2]. □

It is clear that the tensor product $\otimes^{\mathrm{tr}}$ admits a unit given by the constant sheaf $\mathbb{Z} \cong \mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}(k))$. If $\mathcal{T}$ denotes the étale or Nisnevich topology we also obtain a tensor product on the cateogry of $\mathcal{T}$-sheaves with transfers $\mathrm{ST}_{\mathcal{T}}(k)$. Namely each presheaf with transfers admits a sheafification with respect to $\mathcal{T}$ (cf. proposition 4.5) and the tensor product $\otimes^{\mathrm{tr}}_{\mathcal{T}}$, or just $\otimes^{\mathrm{tr}}$, of two $\mathcal{T}$-sheaves F, G with transfers is defined as the $\mathcal{T}$-sheafification of the tensor product $F \otimes^{\mathrm{tr}} G$ from proposition 4.10.
The tensor product $\otimes^{\mathrm{tr}}$ on $\mathrm{PST}(k)$, $\mathrm{ST}_{\mathrm{Nis}}(k)$ or $\mathrm{ST}_{\mathrm{ét}}(k)$ admits a right adjoint $\underline{\mathrm{Hom}}$, which can be described as follows.

Proposition 4.11. *Let* $F, G \in \mathrm{ST}_{\mathcal{T}}(k)$ *be two sheaves with transfers, where* $\mathcal{T}$ *denotes the étale, Nisnevich or trivial topology. Then the presheaf with transfers, denoted by* $\underline{\mathrm{Hom}}(F, G)$ *or* $\underline{\mathrm{Hom}}_{\mathcal{T}}(F, G)$,

$$U \mapsto \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(U) \otimes^{\mathrm{tr}} F, G)$$

is again a sheaf and there is an adjunction

$$\mathrm{Hom}(F \otimes^{\mathrm{tr}}_{\mathcal{T}} G, H) \cong \mathrm{Hom}(F, \underline{\mathrm{Hom}}(G, H))$$

for every $F, G, H \in \mathrm{ST}_{\mathcal{T}}(k)$.

Proof. Writing $F = \varinjlim \mathbb{Z}_{\mathrm{tr}}(U)$ as a colimit of representables the adjunction is a formal consequence of the Yoneda lemma and the definition of $\underline{\mathrm{Hom}}$. To prove the sheaf property if $\mathcal{T}$ is the étale or Nisnevich property we note that by right exactness of $\otimes^{\mathrm{tr}}$ it suffices to proof that for a $\mathcal{T}$-covering $V \to U$ in Sm/k the sequence

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}(V \times_U V) \to \mathbb{Z}_{\mathrm{tr}}(V) \to \mathbb{Z}_{\mathrm{tr}}(U) \to 0$$

is exact. But this is proven in [MVW06, Proposition 6.12]. □

Let $\mathcal{T}$ be an arbitrary Grothendieck topology on Sm/k and denote the trivial topology on Sm/k by triv. Then we remark that for each $X \in \mathrm{Sch}/k$ and any $\mathcal{T}$-sheaf with transfers $G \in \mathrm{ST}_{\mathcal{T}}(k)$ the internal Hom $\underline{\mathrm{Hom}}_{\mathrm{triv}}(\mathbb{Z}_{\mathrm{tr}}(X), G)$ is again a $\mathcal{T}$-sheaf. Now let $\mathcal{T}$ be again the étale, Nisnevich or trivial topology. We record a compatibility between the tensor product $\otimes^{\mathrm{tr}}$ and the usual tensor product of abelian sheaves. First of all there is a natural homomorphism

$$F \otimes G \to F \otimes^{\mathrm{tr}} G$$

for every $F, G \in \mathrm{ST}_{\mathcal{T}}(k)$ coming from the universal property of the usual tensor product. More precisely, it suffices to construct this morphism in the case that $F = \mathbb{Z}_{\mathrm{tr}}(X)$ and $G = \mathbb{Z}_{\mathrm{tr}}(Y)$ are representable. For $U \in \mathrm{Sm}/k$, $\mathcal{Z} \in \mathbb{Z}_{\mathrm{tr}}(X)(U)$ and $\mathcal{W} \in \mathbb{Z}_{\mathrm{tr}}(Y)(U)$ we get the correspondence

$$U \xrightarrow{\Delta} U \times U \xrightarrow{\mathcal{Z}\times\mathcal{W}} X \times Y$$

which is an element in $\mathbb{Z}_{\mathrm{tr}}(X \times Y)(U)$. This defines the morphism

$$\mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathbb{Z}_{\mathrm{tr}}(Y) \to \mathbb{Z}_{\mathrm{tr}}(X \times Y).$$

of $\mathcal{T}$-sheaves with transfers.

Proposition 4.12. *Let $F, G \in \mathrm{ST}_{\acute{e}t}(k)$ be étale sheaves with transfers. Then the homomorphism*

$$F \otimes G \cong F \otimes^{\mathrm{tr}} G$$

is an isomorphism if F or G is locally constant, i.e., the pullback of a sheaf on the small étale site $\mathrm{Spec}(k)_{\acute{e}t}$ of $\mathrm{Spec}(k)$.

Proof. Assume that F is locally constant. Writing F, n as a discrete Galois module, as a colimit of its finitely generated submodules, we may assume that F is finitely generated and hence trivial after some finite base change $k \to k'$. The statement is local for the étale topology. In other words, we are allowed to restrict both sheaves to schemes in Sm/k'. Then F is constant and we can even assume that $F \cong \mathbb{Z} \cong \mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}(k))$. Then

$$F \otimes G \cong G \cong F \otimes^{\mathrm{tr}} G.$$

and the assertion follows. □

We now present some constructions for (pre)sheaves with transfers. We start by introducing the singular complex of a (pre)sheaf with transfers. Let $\mathcal{T}$ be an arbitrary Grothendieck topology on Sm/k (later assumed to be trivial, Nisnevich or étale).

Definition 4.13. For $n \geq 0$ we define

$$\Delta^n := \mathrm{Spec}(k[T_0, .., T_n]/\sum_{i=0}^{n} T_i - 1)$$

as the algebraic n-simplex. For $n, m \geq 0$ each nondecreasing morphism

$$\delta \colon [n] := \{0, \cdots, n\} \to [m] := \{0, \cdots, m\}$$

induces a morphism

$$\begin{array}{cccc} \delta_\Delta \colon & \Delta^n & \to & \Delta^m, \\ & (T_0, \cdots, T_n) & \mapsto & (\sum_{i \in \delta^{-1}(0)} T_i, \cdots, \sum_{i\in\delta^{-1}(m)} T_i) \end{array}$$

making $\Delta^\bullet$ into a cosimplicial scheme.

The singular complex is now defined, in principle, as in topology.

Definition 4.14. Let $F \in \mathrm{ST}_{\mathcal{T}}(k)$ be a $\mathcal{T}$-sheaf with transfers. Then the singular complex $C_\bullet(F)$ of F is defined as the complex associated with the simplicial $\mathcal{T}$-sheaf with transfers

$$\underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\Delta^\bullet), F).$$

Expanding the definitions yields (using the Yoneda lemma)

$$C_n(F)(U) = \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(\Delta^n) \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr}}(U), F) = F(\Delta^n \times U)$$

with differential defined as the alternating sum of the maps induced by the face maps of the cosimplicial scheme $\Delta^\bullet$.

Definition 4.15. A pointed $\mathcal{T}$-sheaf with transfers (F, a) consists of a sheaf $F \in \mathrm{ST}_{\mathcal{T}}(k)$ and a morphism $a\colon \mathbb{Z} \cong \mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}(k)) \to F$. A morphism $f\colon (F,a) \to (G,b)$ of pointed sheaves with transfers is a morphism $f\colon F \to G$ of $\mathcal{T}$-sheaves with transfers mapping the base point a of F to b. We denote the category of pointed $\mathcal{T}$-sheaves with transfers by $\mathrm{ST}_{\mathcal{T}}(k)_*$.

Clearly, every pointed scheme $(X, x) \in \mathrm{Sch}/k$, i.e., any pair (X, x) of a scheme $X \in \mathrm{Sch}/k$ and a point $x \in X(k)$, defines naturally a pointed sheaf with transfers $(\mathbb{Z}_{\mathrm{tr}}(X), x)$.
If $\mathcal{T}$ is either the Nisnevich, étale or trivial topology, then the category $\mathrm{ST}_{\mathcal{T}}(k)_*$ of pointed $\mathcal{T}$-sheaves with transfers admits a symmetric monoidal product, the smash product.

Definition 4.16. Let $(F, a), (G, b) \in \mathrm{ST}_{\mathcal{T}}(k)_*$ be two pointed sheaves with transfers. Then we define their smash product as

$$F \wedge G := \mathrm{coker}(\mathbb{Z} \otimes^{\mathrm{tr}} G \oplus F \otimes^{\mathrm{tr}} \mathbb{Z} \xrightarrow{(a,b)} F \otimes^{\mathrm{tr}} G).$$

By convention we set $F^{\wedge 1} = \mathrm{coker}(\mathbb{Z} \xrightarrow{a} F)$.

As a shorthand we use for a pointed scheme (X, x), like $(\mathbb{G}_m, 1)$, the notation

$$\mathbb{Z}_{\mathrm{tr}}(X^{\wedge n})$$

instead of $(\mathbb{Z}_{\mathrm{tr}}(X), x)^{\wedge n}$. Moreover, there is a canonical isomorphism

$$\mathbb{Z}_{\mathrm{tr}}(X^{\wedge n}) \cong \mathbb{Z}_{\mathrm{tr}}(X^{\wedge 1})^{\otimes^{\mathrm{tr}} n}$$

by right exactness of $\otimes^{\mathrm{tr}}$.

Definition 4.17. A sheaf with transfers $F \in \mathrm{ST}_{\mathcal{T}}(k)$ is called homotopy invariant if the natural pullback homomorphism

$$p^* : F \cong \underline{\mathrm{Hom}}(\mathbb{Z}, F) \to \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1), F)$$

is an isomorphism.

Said differently, a sheaf with transfers is homotopy invariant if and only if the pullback $F \to C_1(F)$ is an isomorphism, i.e., if for every $U \in \mathrm{Sm}/k$ the pullback $F(U) \to F(U \times \mathbb{A}^1)$ is an isomorphism. The étale sheaf with transfers $\mathbb{G}_m$ (cf. proposition 4.8) is an example of a homotopy invariant presheaf with transfers as for any reduced ring R the units in $R[T]^\times$ are the constants $R^\times$. Moreover, for any sheaf with transfers F the cohomology sheaves of its singular complex $C_\bullet(F)$ are homotopy invariant by [MVW06, Corollary 2.19].

Definition 4.18. We define the triangulated category $DM^-_{\text{eff}}(k)$ of effective motivic complexes over k as the full subcategory of $D^-(\mathrm{ST}_{\mathrm{Nis}}(k))$ consisting of complexes with homotopy invariant cohomology sheaves.

The basic example of an element in $DM^-_{\text{eff}}(k)$ is given by the singular complex $C_\bullet(F)$ of a Nisnevich sheaf with transfers $F \in \mathrm{ST}_{\mathrm{Nis}}(k)$ (cf. [MVW06, Corollary 2.19]). The special case that F is represented by a scheme $X \in \mathrm{Sch}/k$ yields "motives".

Definition 4.19. Let $X \in \mathrm{Sch}/k$ be a scheme. We define its (mixed) motive $M(X)$ as

$$M(X) := C_\bullet(\mathbb{Z}_{\mathrm{tr}}(X)) \in DM^-_{\text{eff}}(k).$$

Moreover, its motive with compact supports is defined as

$$M^c(X) := C_\bullet(z_{\mathrm{equi}}(X,0)) \in DM^-_{\text{eff}}(k).$$

Justification for the use of the (still) mysterious word "motive" can be found in [Voe00]. There several natural exact sequences in cohomology are lifted to "motives".
We add some remarks about the use of the Nisnevich topology in definition 4.18. Firstly, the Nisnevich topology, in contrast with the Zariski topology, admits a sheafification, cf. proposition 4.5. Secondly, Nisnevich and Zariski hypercohomology with coefficients in $DM^-_{\text{eff}}(k)$ agree (cf. [SV00a, Corollary 1.1.1]). And thirdly, the étale topology will define the "wrong" motivic cohomology groups.
Although the category $DM^-_{\text{eff}}(k)$ does hardly seem to be tractable, homomorphisms involving motives can be controlled by the following theorem.

Theorem 4.20. *Let $X \in \mathrm{Sm}/k$ be a smooth scheme over k and let $F^\bullet \in DM^-_{\text{eff}}(k)$ be a complex, i.e., $F^\bullet$ is a complex of Nisnevich sheaves with transfers having homotopy invariant cohomology sheaves. Then there is a canonical isomorphism*

$$\mathrm{Hom}_{DM^-_{\text{eff}}(k)}(M(X), F^\bullet) \xrightarrow{\sim} H^0_{\mathrm{Nis}}(X, F^\bullet).$$

Proof. This is proven in [SV00a, Theorem 1.5]. □

We remark that the canonical isomorphism in theorem 4.20 is induced by the Yoneda lemma.
The inclusion $DM^-_{\text{eff}}(k) \subseteq D^-(\mathrm{ST}_{\mathrm{Nis}}(k))$ admits a left adjoint, the singular complex $C_\bullet(-)$ extended to complexes in the usual way by taking a total complex.

Proposition 4.21. *Let $F^\bullet \in D^-(\mathrm{ST}_{\mathrm{Nis}}(k))$ and $G^\bullet \in DM^-_{\text{eff}}(k)$ be two complexes, in particular $G^\bullet$ has homotopy invariant cohomology sheaves. Then every homomorphism $F^\bullet \to G^\bullet$ factors uniquely through the canonical morphism $F^\bullet \to C_\bullet(F^\bullet)$, i.e., taking the singular complex $C_\bullet(-)$ is left adjoint to the inclusion $DM^-_{\text{eff}}(k) \subseteq D^-(\mathrm{ST}_{\mathrm{Nis}}(k))$.*

Proof. This is proven in [SV00a, Corollary 1.11.2]. □

An important property of the category $DM^-_{\text{eff}}(k)$ is its homotopy invariance, i.e., for every scheme $X \in \mathrm{Sch}/k$ the projection

$$M(X \times \mathbb{A}^1) \cong M(X)$$

is an isomorphism. In fact, this property is build into the definition of $DM^-_{\text{eff}}(k)$: the functor $C_\bullet(-)$ turns $\mathbb{A}^1$-equivalences into isomorphisms (cf. [MVW06, Lemma 9.10]).

The tensor product $\otimes^{\mathrm{tr}}$ of presheaves with transfers, or étale/Nisnevich sheaves with transfers, can formally be extended to complexes of such sheaves, thereby inducing a tensor structure – still denoted by $\otimes^{\mathrm{tr}}$ – on the categories $D^-(\mathrm{PST}(k))$, $D^-(\mathrm{ST}_{\text{ét}}(k))$ and $D^-(\mathrm{ST}_{\mathrm{Nis}}(k))$ (cf. [SV00a, Corollary 2.5]). However, the tensor product of two objects in $DM^-_{\mathrm{eff}}(k) \subseteq D^-(\mathrm{ST}_{\mathrm{Nis}}(k))$ need no longer have homotopy invariant cohomology sheaves. To get a tensor structure on $DM^-_{\mathrm{eff}}(k)$ the following definition has to be taken.

Definition 4.22. For $F, G \in DM^-_{\mathrm{eff}}(k)$ we set

$$F \otimes^{\mathrm{tr}} G := C_\bullet(F \otimes^{\mathrm{tr}} G)$$

where the right $\otimes^{\mathrm{tr}}$ means the above tensor structure on $D^-(\mathrm{ST}_{\mathrm{Nis}}(k))$.

We thus obtain an abuse of notation, but for $F, G \in DM^-_{\mathrm{eff}}(k)$ the tensor product $\otimes^{\mathrm{tr}}$ will always refer to the tensor product of 4.22.
Let $X, Y \in \mathrm{Sm}/k$ be two smooth schemes. Then by [SV00a, Proposition 2.8] the tensor product in definition 4.22 satisfies

$$M(X \times Y) \cong M(X) \otimes^{\mathrm{tr}} M(Y).$$

Assuming resolution of singularities the same also holds for their motives with compact supports $M^c(X \times Y) \cong M^c(X) \otimes^{\mathrm{tr}} M^c(Y)$ (cf. [Voe00, Proposition 4.1.7]).

5. Suslin homology

We now introduce Suslin homology, which has been defined by A. Suslin and V. Voevodsky in [SV96] under the name "singular homology". It plays the role of an algebraic analogue of singular homology for topological spaces.
We will continue to use notations as in the previous section. In particular, let $X \in \mathrm{Sch}/k$ be a scheme, separated and of finite type over our ground field k. Recall that we associated to X its motive or singular complex

$$M(X) = C_\bullet(\mathbb{Z}_{\mathrm{tr}}(X)) \in DM^-_{\mathrm{eff}}(k),$$

a complex of Nisnevich sheaves with transfers. The global sections, derived or not, of this complex are given by

$$M(X)(\mathrm{Spec}(k)) = C_\bullet(\mathbb{Z}_{\mathrm{tr}}(X))(\mathrm{Spec}(k)) = \mathrm{Cor}(\Delta^\bullet, X).$$

Let A be an abelian group.

Definition 5.1. The Suslin homology $H^S_\bullet(X, A)$ of X with coefficients in A is defined as the homology of the complex $\mathrm{Cor}(\Delta^\bullet, X) \otimes_{\mathbb{Z}} A$, i.e.,

$$H^S_i(X, A) := H_i(\mathrm{Cor}(\Delta^\bullet, X) \otimes_{\mathbb{Z}} A)$$

for $i \in \mathbb{Z}$.

It follows from the homotopy invariance of the complex $\mathrm{Cor}(\Delta^\bullet, X)$ that also the Suslin homology groups are homotopy invariant, i.e., for every scheme $X \in \mathrm{Sch}/k$ the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism.

$$H^S_i(X \times \mathbb{A}^1, A) \cong H^S_i(X, A).$$

Taking in definition 5.1 a naive approach replacing the complex $\mathrm{Cor}(\Delta^\bullet, X)$ of correspondences $\Delta^n \to X$ by the complex associated with the free simplicial abelian group $\mathbb{Z}[\mathrm{Hom}(\Delta^\bullet, X)]$ on plain homomorphisms $\Delta^n \to X$ will not yield sensible invariants. For example, if X is a curve of genus ≥ 1, then the simplicial set

$\mathrm{Hom}(\Delta^\bullet, X)$ is discrete, isomorphic to the set $X(k)$ of k-rational points of X and thus the complex $\mathbb{Z}[\mathrm{Hom}(\Delta^n, X)]$ has cohomology concentrated in degree 0.
By theorem 4.20 Suslin homology can also be expressed as cohomology of $\mathrm{Spec}(k)$ with coefficients in $M(X) \otimes_{\mathbb{Z}} A$. This yields the formula

$$H_i^S(X, A) \cong H^{-i}_{\mathrm{Nis}}(\mathrm{Spec}\, k, M(X) \otimes_{\mathbb{Z}} A) \cong \mathrm{Hom}_{DM^-_{\mathrm{eff}}(k)}(\mathbb{Z}[i], M(X) \otimes_{\mathbb{Z}} A)$$

which is reminiscent of expressing the usual singular homology $H_i(X, A)$ of a topological space X with coefficients in A by

$$H_i(X, A) = \mathrm{Hom}_{D(\mathbb{Z})}(\mathbb{Z}[i], C^{\mathrm{top}}_\bullet(X) \otimes_{\mathbb{Z}} A)$$

where $C^{\mathrm{top}}_\bullet(X)$ is the singular complex of X.
The residue field of every closed point of X is finite over k. Hence the elements in $\mathrm{Cor}(\Delta^0, X) = \mathrm{Cor}(\mathrm{Spec}(k), X)$ are precisely the 0-cycles on X. Therefore every element $[z] \in H_0^S(X, \mathbb{Z})$ in the 0-th Suslin homology is represented by a 0-cycle z on X. Recall that the Chow group $\mathrm{CH}_0(X)$ is defined as the group of 0-cycles on X modulo rational equivalence.

Proposition 5.2. *The canonical map*

$$\alpha\colon H_0^S(X, \mathbb{Z}) \to \mathrm{CH}_0(X), [z] \mapsto [z]$$

sending a 0-cycle $z \in \mathrm{Cor}(\Delta^0, X)$ to its class in $\mathrm{CH}^0(X)$ is well-defined. If X is proper, then α is an isomorphism.

Proof. We present the proof of this well-known result for the convenience of the reader. By [Ful98, Chapter 1.6] two cycles $\mathcal{Z}, \mathcal{Z}'$ on X are rationally equivalent if and only if there is a cycle $\mathcal{W}$ on $X \times \mathbb{P}^1$ which is dominant and quasi-finite over $\mathbb{P}^1$ satisfying $\mathcal{Z} = \mathcal{W}_{|X \times 0}, z' = \mathcal{W}_{|X \times \infty}$. Clearly, the point $\infty \in \mathbb{P}^1$ can also be replaced by the point $1 \in \mathbb{P}^1$. Moreover, restricting to $\mathbb{A}^1 \subseteq \mathbb{P}^1$ does not yield a difference. Hence, α is well-defined. If X is moreover proper, each cycle $\mathcal{W}$ on $X \times \mathbb{A}^1$ which is quasi-finite over $\mathbb{A}^1$ is already finite. Thus, α is an isomorphism in that case. □

We will end this section by illustrating Suslin homology in the case of curves. For this let k be an algebraically closed field and let X be a connected smooth projective curve over k. Moreover, let $m \in \mathbb{Z}$ be an arbitrary integer.

Proposition 5.3. *The Suslin homology of the curve X with integer coefficients is given by*

$$\begin{array}{rcll} H_0^S(X, \mathbb{Z}) & \overset{\alpha}{\cong} & \mathrm{CH}^1(X) & \\ H_1^S(X, \mathbb{Z}) & \cong & k^\times & \\ H_i^S(X, \mathbb{Z}) & = & 0, & \textit{for } \ i \geq 2, \end{array}$$

where α is the canonical map from proposition 5.2 The mod-m-Suslin homology of X is given by

$$\begin{array}{rcll} H_0^S(X, \mathbb{Z}/m) & \cong & \mathbb{Z}/m & \\ H_1^S(X, \mathbb{Z}/m) & \cong & {}_m\mathrm{CH}^1(X) & \\ H_2^S(X, \mathbb{Z}/m) & \cong & \mu_m(k) & \\ H_i^S(X, \mathbb{Z}/m) & = & 0, & \textit{for } \ i \geq 3. \end{array}$$

Proof. The statement about Suslin homology with integer coefficients can be found in [Lic93]. The statement about mod-m-coefficients can be derived from this using the coefficent sequence associated to

$$0 \to \mathbb{Z} \overset{m}{\to} \mathbb{Z} \to \mathbb{Z}/m \to 0.$$

Using the results about $H_0^S(X, \mathbb{Z})$ this coefficient sequence is given by

$$\begin{tikzcd}
0 \arrow[r] & H_2^S(X,\mathbb{Z}/m) \arrow[r] & k^\times \arrow[r, "m"] & k^\times \arrow[dlll, "0"'] & \\
H_1^S(X,\mathbb{Z}/m) \arrow[r] & \mathrm{CH}^1(X) \arrow[r, "m"] & \mathrm{CH}^1(X) \arrow[r] & H_0^S(X,\mathbb{Z}/m) \arrow[r] & 0
\end{tikzcd}$$

and the result follows. □

The above proposition 5.3 shows that for curves the Suslin homology with $\mathbb{Z}/m$-coefficients, m prime to the characteristic of k, has the expected behaviour, i.e., if g is the genus of X, then

$$\begin{array}{ll}
\mathrm{rk}_{\mathbb{Z}/m} H_0^S(X,\mathbb{Z}/m) & = 1 \\
\mathrm{rk}_{\mathbb{Z}/m} H_1^S(X,\mathbb{Z}/m) & = 2g \\
\mathrm{rk}_{\mathbb{Z}/m} H_2^S(X,\mathbb{Z}/m) & = 1 \\
H_i^S(X,\mathbb{Z}) & = 0, \quad \text{for } i \geq 3.
\end{array}$$

Over $k = \mathbb{C}$ there is even, for every scheme $X \in \mathrm{Sch}/\mathbb{C}$ and every non-zero integer $m \in \mathbb{Z}$, a canonical isomorphism

$$H_\bullet^S(X, \mathbb{Z}/m) \cong H_\bullet(X(\mathbb{C}), \mathbb{Z}/m)$$

of the mod-m-Suslin homology with the topological singular homology of the analytic space $X(\mathbb{C})$ associated with X (cf. [SV96]).

6. Tame étale cohomology

In this section we will present the definition of tame étale cohomology $H^1_{\mathrm{t}}(X, \mathbb{Z}/m)$ of a scheme $X \in \mathrm{Sch}/k$ following [GS]. If k is algebraically closed, then in section 7 a pairing

$$\Phi_{\mathrm{geom}} \colon H_1^S(X, \mathbb{Z}/m) \times H^1_{\mathrm{t}}(X, \mathbb{Z}/m) \to \mathbb{Z}/m$$

will be constructed. This pairing will turn out to be perfect in many cases (assuming resolution of singularities always). The tameness condition on étale cohomology, which is only a condition if $p = \mathrm{char}(k) > 0$ and $m = p^r$, is necessary as the following example shows. Take $X = \mathbb{A}^1_k$. Then by homotopy invariance of Suslin homology

$$H_1^S(\mathbb{A}^1_k, \mathbb{Z}/p^r) \cong H_1^S(\mathrm{Spec}(k), \mathbb{Z}/p^r) = 0$$

while

$$H^1_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/p^r) \neq 0$$

by the existence of Artin-Schreier covers.

Recall that an étale morphism $Y \to X$ of curves in Sch/k with X regular[4] of dimension 1 is called tamely ramified if the canonical extension $\bar{Y} \to \bar{X}$ to the regular compactifications of X and Y is an at most tamely ramified covering (in the sense of valuation theory). Note that non-trivial Artin-Schreier covers of $\mathbb{A}^1_k$ are not tamely ramified. However, the tame étale cohomology of $\mathbb{A}^1_k$ is trivial.

[4] equivalently smooth, as k is perfect

Definition 6.1 ([KS10])**.** Let $X \in \mathrm{Sch}/k$ be a scheme and $f\colon Y \to X$ a finite étale morphism. Then $f\colon Y \to X$ is called curve-tame, or just tame, if for every morphism $C \to X$ for a regular curve $C \in \mathrm{Sch}/k$ the base change $C \times_X Y \to C$ is a tame covering of C.

We can now give the definition of tame étale cohomology. Unfortunately, it is an ad-hoc definition working only in cohomological degree 1.
Let $X \in \mathrm{Sch}/k$ be a scheme and let A be a finite abelian group. Recall that cohomology classes in $H^1_{\text{ét}}(X, A)$ are represented by A-torsors over X (cf. section 3). Moreover, every A-torsor over X is represented by a scheme.

Definition 6.2. We define

$$H^1_{\mathrm{t}}(X, A) \subseteq H^1(X_{\text{ét}}, A)$$

as the subgroup generated by isomorphism classes of tame A-torsors, i.e., A-torsors which are tame coverings of X.

We list some properties of tame étale cohomology.

Proposition 6.3.
(1) *If k is algebraically closed every tame covering of the affine space $\mathbb{A}^n_k$ is trivial.*
(2) *Pullbacks, disjoint unions, compositions and fiber products of tame coverings are tame.*
(3) *If $f\colon X \to Y$, $g\colon Y \to Z$ are finite étale morphisms in* Sch/k *with f surjective and $g \circ f\colon X \to Z$ tame, then g is tame.*
(4) *Sums and inverses of tame A-torsors are again tame, in particular, every A-torsor whose isomorphism class lies in $H^1_{\mathrm{t}}(X, A)$ is tame.*
(5) *If A has order prime to the characteristic of k, then every A-torsor is tame.*
(6) *If $X \in$* Sch/k *is proper, then every finite étale morphism to X is tame. In particular,*

$$H^1_{\mathrm{t}}(X, A) \cong H^1_{\text{ét}}(X, A)$$

for X over k proper.

Proof. The statement (1) is proven in [GS, Corollary 2.11]. Statements (2) is easy. For proving (3) we may assume that Z is a regular curve. Then (3) follows from the observation that ramification indices are multiplicative for finite separable extensions of the function field of Z. Statement (4) follows from (2) and (3). If A has order prime to $p = \mathrm{char}(k)$ and $\mathcal{T}$ an A-torsor over some regular curve X, then $\mathcal{T}$ extends to a tamely ramified covering of the canonical compactification of X because every ramification group of this extension has order dividing the order of A. Hence, (5). If finally $f\colon Y \to X$ is a finite étale morphism with X proper, every morphism $C \to X$ with C a regular curve extends to a morphism $\overline{C} \to X$ of the canonical compactification $\overline{C}$ of C. Thus the pullback of f to C will extend to a finite étale covering of $\overline{C}$ and in particular it will be tamely ramified. □

For $X \in \mathrm{Sch}/k$ let $X_{\text{fét}}$ be the site of finite, étale X-schemes. For X connected the choice of a geometric base point $\bar{x}$ of X defines a fiber functor

$$F_{\bar{x}}\colon X_{\text{fét}} \to (\text{Sets}),\ Y \mapsto Y_{\bar{x}} := Y \times_X \bar{x}$$

making $X_{\text{fét}}$ into a Galois category (cf. [Gro71, Chapitre V.7]). In particular, there is an equivalence of categories

$$X_{\text{fét}} \cong \{\text{continuous, finite } \pi_1^{\text{ét}}(X, \bar{x})\text{-sets}\}$$

of $X_{\text{fét}}$ with the category of continuous, finite $\pi_1^{\text{ét}}(X,\bar{x})$-sets for Grothendieck's étale fundamental group

$$\pi_1^{\text{ét}}(X,\bar{x}) := \operatorname{Aut}(F_{\bar{x}}).$$

Let $X_{\text{tfét}} \subseteq X_{\text{fét}}$ be the full subcategory of finite étale morphisms $Y \to X$ which are tame.

Lemma 6.4. *If X is connected, the category $X_{\text{tfét}}$ is a Galois category. More precisely, every geometric base point $\bar{x} \in X$ defines a fiber functor*

$$F_{\bar{x}}^t \colon X_{\text{tfét}} \to (\text{Sets}),\ Y \mapsto Y_{\bar{x}}.$$

We denote by $\pi_1^{\text{t}}(X,\bar{x}) := \operatorname{Aut}(F_{\bar{x}}^t)$ the resulting tame étale fundamental group of X.

Proof. Using [Gro71, Chapitre V, Theoreme 4.1] and the mentioned fact that already $X_{\text{fét}}$ is a Galois category, it suffices to see that, in the notation of [Gro71, Chapitre V.4], the conditions (G1), (G2) and (G3) are satisfied for $X_{\text{tfét}}$. In fact, the fiber functor $F_{\bar{x}}$ for $X_{\text{fét}}$ will then restrict to the fiber functor $F_{\bar{x}}^t$ on $X_{\text{tfét}}$. But the conditions (G1)-(G3) follow easily from proposition 6.3 and the corresponding statement for $X_{\text{fét}}$ with the crucial property of tame finite étale morphisms being the point (3) in proposition 6.3. □

It is clear that for a chosen geometric base point $\bar{x}$ of the connected scheme X the étale fundamental group $\pi_1^{\text{ét}}(X,\bar{x})$ surjects onto its tame version $\pi_1^{\text{t}}(X,\bar{x})$ as the functor $X_{\text{tfét}} \to X_{\text{fét}}$ is fully faithful. For a site $\mathcal{C}$ we denote by $\widetilde{\mathcal{C}}$ its associated topos. Denoting the classifying topos of a profinite group Γ by $B\Gamma$ we then arrive at the following diagram of (morphisms of) topoi

$$\begin{tikzcd}
\widetilde{X_{\text{ét}}} \arrow[r] & \widetilde{X_{\text{fét}}} \arrow[r] \arrow[d, "\cong"] & \widetilde{X_{\text{tfét}}} \arrow[d, "\cong"] \\
& B\pi_1^{\text{ét}}(X,\bar{x}) \arrow[r] & B\pi_1^{\text{t}}(X,\bar{x}).
\end{tikzcd}$$

For every abelian group A there is thus a natural pullback morphism

$$H^i(\pi_1^{\text{ét}}(X,\bar{x}),A) \cong H^i(X_{\text{fét}},A) \to H^i(\widetilde{X_{\text{ét}}},A) = H^i_{\text{ét}}(X,A).$$

This morphism is independent of the base point $\bar{x}$ as inner automorphisms of $\pi_1^{\text{ét}}(X,\bar{x})$ act trivially on the cohomology $H^i(\pi_1^{\text{ét}}(X,\bar{x}),A)$. In degree 1 this morphism is an isomorphism by the description of the cohomology group $H^1_{\text{ét}}(X,A)$ in terms of A-torsors (cf. section 3) each of which is represented by a scheme finite, étale over X. Similarly, there exists a canonical pullback morphism

$$H^i(\pi_1^{\text{t}}(X,\bar{x}),A) \to H^i_{\text{ét}}(X,A).$$

The following lemma is a direct consequence of the definition of the tame étale fundamental group and tame étale cohomology.

Lemma 6.5. *For every finite abelian group A the natural pullback isomorphism*

$$H^1(\pi_1^{\text{ét}}(X,\bar{x}),A) \xrightarrow{\sim} H^1_{\text{ét}}(X,A)$$

restricts to an isomorphism

$$H^1(\pi_1^{\text{t}}(X,\bar{x}),A) \xrightarrow{\sim} H^1_{\text{t}}(X,A).$$

By $\pi_1^{\mathrm{t,ab}}(X)$ we will denote the abelianization $\pi_1^{\mathrm{t}}(X,\bar{x})^{\mathrm{ab}}$ of the tame étale fundamental group $\pi_1^{\mathrm{t}}(X,\bar{x})$ with respect to some geometric base point $\bar{x}$. As inner automorphisms act trivially on the abelianization the group $\pi_1^{\mathrm{t,ab}}(X)$ is independent of the choice of the geometric base point $\bar{x}$. We end this section by discussing tame étale cohomology of smooth curves. Assume that k is algebraically closed and let X be a smooth curve over k with canonical (smooth) compactification $X \subseteq \bar{X}$.

Proposition 6.6. *Let $D = \bar{X} \setminus X$ be the (reduced) divisor at infinity and denote by $\mathrm{Pic}_{X,D}$ the (connected) generalized Jacobian of X with modulus D (cf.* [Ser84]*). Then $\mathrm{Pic}_{X,D}$ is a semi-abelian variety and for a point $x \in X(k)$ the Albanese morphism*

$$\mathrm{alb}_x \colon X \to \mathrm{Pic}_{X,D}$$

induces an isomorphism

$$\mathrm{alb}\colon \pi_1^{\mathrm{t,ab}}(X) \xrightarrow{\sim} \pi_1^{\text{ét,ab}}(\mathrm{Pic}_{X,D}) = \pi_1^{\text{ét}}(\mathrm{Pic}_{X,D}, 0)$$

on abelianized tame étale fundamental groups independent of the chosen point $x \in X(k)$. Moreover, if $m \in \mathbb{Z}$ is non-zero, then

$$\mathrm{alb}\colon \pi_1^{\mathrm{t,ab}}(X)/m \cong \pi_1^{\text{ét,ab}}(\mathrm{Pic}_{X,D})/m$$

and $\pi_1^{\text{ét,ab}}(\mathrm{Pic}_{X,D})/m \cong {}_m\mathrm{Pic}(X)$. In particular,

$$\mathrm{alb}\colon H^1_{\mathrm{t}}(X,A) \xrightarrow{\sim} H^1(\pi_1^{\mathrm{t,ab}}(X),A) \xrightarrow{\sim} \mathrm{Hom}({}_m\mathrm{Pic}(X),A)$$

is an isomorphism for every finite abelian group A with $mA = 0$ by lemma 6.5.

Proof. This is proven in [Ser84, Chapter 5]. □

We remark that the universal ${}_m\mathrm{Pic}(X)$-torsor over X is given by the pullback of the multiplication $m\colon \mathrm{Pic}_{X,D} \to \mathrm{Pic}_{X,D}$ along the Albanese morphism $\mathrm{alb}\colon X \to \mathrm{Pic}_{X,D}$. For an abelian group A with $mA = 0$ any A-torsor will be the pushforward of this universal torsor along some morphism ${}_m\mathrm{Pic}(X) \to A$.

7. The geometric reciprocity law

Let X be as usual a separated scheme of finite type over our ground field k, i.e., $X \in \mathrm{Sch}/k$. In this section we assume moreover that k is algebraically closed. We will follow [GS] and present for an arbitrary non-zero integer $m \in \mathbb{Z}$ the construction of a pairing, called the geometric reciprocity pairing,

$$\Phi'_{\mathrm{geom}}\colon H_1^S(X,\mathbb{Z}/m) \times H_t^1(X,\mathbb{Z}/m) \to \mathbb{Z}/m,$$

between the first Suslin homology and tame étale cohomology. Using lemma 6.5 this pairing can equivalently be interpreted as a "Hurewicz" morphism, called the geometric reciprocity law,

$$\Phi_{\mathrm{geom}}\colon H_1^S(X,\mathbb{Z}/m) \to \pi_1^t(X,\bar{x}).$$

The construction of the pairing Φ'_{geom} will be a direct translation of its topological analogue: Let for the moment X be a "nice" topological space.[5] The pairing

$$\Phi'_{\mathrm{top}}\colon H_1(X,\mathbb{Z}) \times H^1(X,\mathbb{Z}) \to \mathbb{Z}$$

[5]connected, locally contractible, etc.

between the first singular homology $H_1(X,\mathbb{Z})$ of X and the sheaf cohomology $H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ associated with the usual Hurewicz morphism

$$\Phi_{\mathrm{top}}\colon \pi_1(X,x) \to H_1(X,\mathbb{Z})$$

can be described as follows. Pick a cycle $\gamma \in H_1(X,\mathbb{Z})$ and a $\mathbb{Z}$-torsor $\mathcal{T}$. Assume for simplicity that γ is represented by a map $\gamma\colon [0,1] \to X$ from the unit interval $[0,1]$ to X. As γ is a cycle the parallel transport in $\mathcal{T}$ along the closed path γ will induce an automorphism φ of the fiber $\mathcal{T}_{\gamma(0)}$ of the torsor $\gamma^*\mathcal{T}$ at 0. This automorphism commutes with the $\mathbb{Z}$-action on $\mathcal{T}$ and is thus given by some element $a \in \mathbb{Z}$. By definition

$$\Phi'_{top}(\gamma,\mathcal{T}) := a.$$

We remark that in order to speak about parallel transport in $\mathcal{T}$ it is necessary that the pullback of $\mathcal{T}$ to $[0,1]$ along γ is trivial. This is one problem which has to be resolved before the geometric construction of Φ_{top} can be translated to the case that $X \in \mathrm{Sch}/k$ is a scheme with singular homology $H_1(X,\mathbb{Z})$ replaced by Suslin homology $H_1(S,\mathbb{Z}/m)$ with finite coefficients and sheaf cohomology $H^1(X,\mathbb{Z})$ by tame étale cohomology $H^1_t(X,\mathbb{Z}/m)$. In fact, the following issues are adressed in [GS]:

(1) A pullback of torsors along correspondences has to be constructed.
(2) A pairing with $\mathbb{Z}/m$-coefficients will be constructed and it has to be shown that for a cycle in $H_1^S(X,\mathbb{Z}/m)$ the fibers over 0 and 1 are canonically isomorphic.
(3) For a $\mathbb{Z}/m$-torsor $\mathcal{T}$ over X its pullback along a correpondence $\gamma\colon \Delta^1 \to X$ need to be trivial, a condition shown to be implied by tameness of $\mathcal{T}$.

After resolving these issues the geometric reciprocity law can be defined. We (loosely) recall its definition. Let as in the beginning of this section X be a separated, finite type scheme over an algebraically closed field k. Moreover, let $m \in \mathbb{Z}$ be a non-zero integer and let A be a finite abelian group with $mA = 0$.

Definition 7.1 ([GS, Proposition 2.12])**.** The geometric reciprocity pairing for X

$$\Phi'_{\mathrm{geom}}\colon H_1^S(X,\mathbb{Z}/m) \times H^1_t(X,A) \to A$$

is defined as follows: For a cycle $z \in H_1^S(X,\mathbb{Z}/m)$ and a torsor $\mathcal{T} \in H^1_t(X,A)$ set

$$\Phi'_{\mathrm{geom}}(z,\mathcal{T}) := a$$

where $a \in \mathbb{Z}/m$ is the unique element which gives the parallel transport in $\mathcal{T}$ along the cycle z.

We denote by

$$\Phi_{\mathrm{geom}}\colon H_1^S(X,\mathbb{Z}/m) \to \mathrm{Hom}(H^1_t(X,\mathbb{Z}/m),\mathbb{Z}/m) \cong \pi_1^{t,\mathrm{ab}}(X)/m$$

the adjoint of Φ'_{geom} and call it the geometric reciprocity law. It could also be called "Hurewicz morphism" in analogy with its classical analogue. However, compared to its classical brother the morphism Φ_{geom} goes in the reverse direction. We remark that the parallel transport, and therefore the geometric reciprocity law, is only natural up to a sign. A similar problem also occurs in topology. For a topological space X the Hurewicz morphism

$$\pi_1(X,x) \to H_1(X,\mathbb{Z})$$

depends on the choice of an orientation of the sphere S^1.

The main result about the geometric reciprocity law Φ_{geom} is the following theorem [GS, Theorem 6.1].

Theorem 7.2. *Let $X \in \mathrm{Sch}/k$ be a scheme over an algebraically closed field k and let $m \in \mathbb{Z}$ be an integer.*

(1) *The morphism*
$$\Phi_{\mathrm{geom}} \colon H_1^S(X, \mathbb{Z}/m) \to \pi_1^{t,\mathrm{ab}}(X)/m$$
is surjective.

(2) *If m is prime to $p = \mathrm{char}(k)$, then Φ_{geom} is an isomorphism (of finite groups).*

(3) *If X is smooth and projective, then Φ_{geom} is an isomorphism (of finite groups) for general $m \in \mathbb{Z}$.*

(4) *If $m = p^r$ and resolution of singularities holds over k for schemes of dimension $\leq \dim X + 1$, then Φ_{geom} is an isomorphism (of finite groups).*

Proof. All statements except (3) are given in [GS, Theorem 6.1]. The third statement is only implicit in the proof. Namely, as Φ_{geom} is surjective in general it suffices to show that $H_1^S(X, \mathbb{Z}/m)$ and $H_t^1(X, \mathbb{Z}/m)^\vee \cong \pi_1^{t,\mathrm{ab}}(X)/m$ have the same order for X smooth and projective. But this is stated in step 3 of the proof of [GS, Theorem 6.1]. □

The proof of theorem 7.2 heavily rests on the following compatibility (which holds similarly without the assumption that X is projective). Let X be a smooth projective curve over k, which is assumed to be algebraically closed. Recall that there are isomorphisms

$$\begin{array}{lcl} \alpha \colon H_1^S(X, \mathbb{Z}/m) & \cong & \mathrm{CH}^1(X, 1; \mathbb{Z}/m) \cong {}_m\mathrm{CH}^1(X) \\ c_1 \colon {}_m\mathrm{CH}^1(X) & \cong & {}_m\mathrm{Pic}(X) \\ \mathrm{alb} \colon H_t^1(X, \mathbb{Z}/m) & \cong & \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m) \end{array}$$

by proposition 5.3, section 3 and proposition 6.6.

Theorem 7.3. *The diagram*

$$\begin{tikzcd} H_1^S(X, \mathbb{Z}/m) \times H_t^1(X, \mathbb{Z}/m) \arrow[r, "\Phi_{\mathrm{geom}}"] \arrow[d, "(c_1^{-1} \circ \alpha) \times \mathrm{alb}"] & \mathbb{Z}/m \arrow[d, equal] \\ {}_m\mathrm{Pic}(X) \times \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m) \arrow[r, "\mathrm{eval}"] & \mathbb{Z}/m \end{tikzcd}$$

commutes, where the bottom arrow denotes the canonical evaluation pairing.

Proof. Cf. [GS, Theorem 4.1]. □

8. Higher Chow groups

We will now define Bloch's higher Chow groups which have been a first instance of "motivic cohomology". In particular, they generalize the usual Chow groups. Recall that Δ^j denotes the algebraic j-simplex

$$\Delta^j = \{(x_0, \cdots, x_j) \in \mathbb{A}_k^{n+1} \mid \sum_{i=1}^{j} x_i = 1\}.$$

Let $X \in \mathrm{Sch}/k$ be a scheme. We assume that X is equidimensional. For $i \in \mathbb{Z}$ we will denote by $z^i(X, \bullet)$ Bloch's cycle complex in weight i. In other words, $z^i(X, j)$ is the group of cycles of codimension i on $X \times \Delta^j$ which intersect each face of $X \times \Delta^j$ properly (cf. [MVW06, Definition 17.1]). More precisely, a face of $X \times \Delta^j$ is a closed subscheme of the form $X \times \Delta^r$, $r < j$, embedded via a simplicial map $\Delta^r \to \Delta^j$ given by an injective non-decreasing map $[0, 1, \cdots, r] \to [0, 1, \cdots, j]$. Intersecting a face properly means that each component of the intersection cycle has codimension i in $X \times \Delta^r$, i.e., lies in $z^i(X, r)$. Therefore the differential of the complex $z^i(X, \bullet)$ can be defined by sending a cycle $\mathcal{Z} \in z^i(X, j)$ to the alternating sum of the intersection $\mathcal{Z} \cap X \times \Delta^{j-1} \in z^i(X, j-1)$ over all injective non-decreasing maps $[0, 1, \cdots, j-1] \to [0, 1, \cdots, j]$.
We remark that Bloch's cycle complex, as we defined it, is indexed homologically.
Let A be an abelian group.

Definition 8.1. For $i, j \in \mathbb{Z}$ the higher Chow group $\mathrm{CH}^i(X, j; A)$ of X with coefficients in A is defined as the j-th homology

$$\mathrm{CH}^i(X, j; A) := H_j(z^i(X, \bullet) \otimes_{\mathbb{Z}} A).$$

of Bloch's cycle complex $z^i(X, \bullet)$.

We list some properties of these higher Chow groups. Clearly, higher Chow groups are contravariantly functorial in X with respect to flat maps as those respect the condition on the intersection with faces.

Proposition 8.2. *Let $X \in \mathrm{Sch}/k$ be scheme, assumed to be equidimensional.*

(1) *The higher Chow group $\mathrm{CH}^i(X, 0; \mathbb{Z})$ is the usual Chow group $\mathrm{CH}^i(X)$ of codimension i cycles on X.*
(2) *Higher Chow groups are homotopy-invariant, i.e., the pullback $p\colon X \times \mathbb{A}^1 \to X$ yields an isomorphism*

$$p^*\colon \mathrm{CH}^i(X, j; A) \xrightarrow{\sim} \mathrm{CH}^i(X \times \mathbb{A}^1_k, j; A)$$

for every $i, j \in \mathbb{Z}$ and any abelian group A.

Proof. Cf. [Blo86] or [MVW06, Lecture 17]. □

We would like to "sheafify" Bloch's cycle complex $z^i(-, \bullet)$ to obtain a complex of presheaves with transfers (even étale sheaves with transfers) on Sm/k. But unfortunately the cycle complex $z^j(X, \bullet)$ lacks some functorial properties – it is not functorial with respect to *every* morphism $X \to Y \in \mathrm{Sm}/k$.

Definition 8.3. Let $X, Y \in \mathrm{Sch}/k$ be a two equidimensional schemes. We define for $i \in \mathbb{Z}$

$$z^i(- \times X, \bullet)$$

as the complex of presheaves on the *small* étale site $Y_{\text{ét}}$ by sending $U \in Y_{\text{ét}}$ to the complex

$$z^i(U \times X, \bullet).$$

As in the case of the sheaves $\mathbb{Z}_{\mathrm{tr}}(X)$ or $z_{\mathrm{equi}}(X, r)$ [Ans, Theorem 4.8] implies that $z^i(- \times X, \bullet)$ is actually a complex of étale sheaves.
By restricting $z^i(- \times X, \bullet)$ to open subsets of Y we obtain a complex of Zariski sheaves on Y and the higher Chow groups of Y can be expressed as Zariski hypercohomology of this complex.

Proposition 8.4. *For $X, Y \in \mathrm{Sch}/k$, $i \in \mathbb{Z}$ and any abelian group A the canonical map*

$$z^i(Y \times X, \bullet) \otimes_{\mathbb{Z}} A \to R\Gamma(Y_{\mathrm{Zar}}, z^i(- \times X, \bullet) \otimes_{\mathbb{Z}} A)$$

is a quasi-isomorphism, in particular

$$\mathrm{CH}^i(Y \times X, j; A) \xrightarrow{\sim} H^{-j}(Y_{\mathrm{Zar}}, z^i(- \times X, \bullet) \otimes_{\mathbb{Z}} A)$$

for all $i \geq 0$ and $j \in \mathbb{Z}$.

Proof. This is proven in [MVW06, Proposition 19.12] or [Blo86, 3.4]. □

Let $X \in \mathrm{Sch}/k$ be a scheme, equidimensional of dimension d. Let $\mathcal{Z}$ be a correspondence $\Delta^r \to X$. Then the cycle $\mathcal{Z}$ on $\Delta^r \times X$ meets each face $\Delta^i \times X \subseteq \Delta^r \times X$ properly and therefore defines a cycle in $z^d(X, r)$.

Definition 8.5. Let $Y \in \mathrm{Sm}/k$ be a smooth scheme over k. We set

$$\alpha \colon \mathrm{Cor}(\Delta^\bullet \times Y, X) \to z^d(X \times Y, \bullet),\ z \mapsto z$$

as the natural inclusion (which is well-defined by the above remark).

It is easily seen that α is in fact a map of complexes. Clearly, definition 8.5 lifts the morphism $\alpha \colon H_0^S(X, \mathbb{Z}) \to \mathrm{CH}_0(X)$ from proposition 5.2 to the level of complexes. To construct the motivic reciprocity law Φ_{mot} later we need the following statement.

Proposition 8.6. *Let $X \in \mathrm{Sm}/k$ be a connected smooth projective scheme of dimension d. Then the morphism*

$$\alpha \colon H_1^S(X, \mathbb{Z}/m) \to \mathrm{CH}^d(X, 1; \mathbb{Z}/m)$$

from definition 8.5 is an isomorphism for every $m \in \mathbb{Z}$.

Proof. Cf. [SS00, Theorem 2.7]. □

For proposition 8.6 to hold it is indeed necessary to assume that X is proper over k, because the left hand side $H_1^S(X, \mathbb{Z}/m)$ is covariantly functorial in X for all morphisms, while the right hand side $\mathrm{CH}^d(X, 1; \mathbb{Z}/m)$ only for proper ones.

9. Motivic cohomology

In this section we will present the definition of motivic cohomology in the sense of Voevodsky and its comparion with Bloch's higher Chow groups.
Motivic cohomology for a smooth scheme, in the sense of Voevodsky, will be defined as the Zariski hypercohomology of certain complexes $\mathbb{Z}(n)$ of sheaves with transfers, the motivic complexes for varying weight n. To motivate the construction we talk a bit about "motives".

Definition 9.1. Let (X, x) be a pointed scheme. Then its reduced motive $\widetilde{M}(X)$ is defined as

$$\widetilde{M}(X) := \mathrm{Cone}(M(x) \to M(X)) \in DM^-_{\mathrm{eff}}(k).$$

More concretely, the reduced motive $\widetilde{M}(X)$ is represented by the complex

$$\mathrm{coker}(C_\bullet(\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}(k)) \to C_\bullet(\mathbb{Z}_{\mathrm{tr}}(X))$$

or equivalently by

$$\ker(C_\bullet(\mathbb{Z}_{\mathrm{tr}}(X)) \to C_\bullet(\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}(k))).$$

In particular,

$$M(X) \cong \widetilde{M}(X) \oplus M(\mathrm{Spec}(k)) \cong \widetilde{M}(X) \oplus \mathbb{Z}.$$

In weight one the motivic complex $\mathbb{Z}(1)$ is, up to a shift, meant to be the “Tate motive“. In other words, $\mathbb{Z}(1)$ is meant to capture the first homology of the multiplicative group $\mathbb{G}_m$. In particular, $\mathbb{Z}(1)[1]$ must be the reduced motive $\widetilde{M}(\mathbb{G}_m)$ of the pointed scheme $(\mathbb{G}_m, 1)$. The motivic complexes $\mathbb{Z}(n)$ for higher weight are then expected to be obtained by taking tensor powers of $\mathbb{Z}(1)$, i.e., $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes^{\mathrm{tr}} n}$.

Definition 9.2. We define for $n \geq 0$ the Voevodsky weight n motivic complex by

$$\mathbb{Z}(n) := C_\bullet(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))[-n].$$

In particular, we obtain the equality

$$\mathbb{Z}(n) = \widetilde{M}(\mathbb{G}_m)[-1]^{\otimes^{\mathrm{tr}} n}$$

that was mentioned above. Moreover, for $n, m \geq 0$ the formula

$$\mathbb{Z}(n) \otimes^{\mathrm{tr}} \mathbb{Z}(m) \cong \mathbb{Z}(n+m)$$

holds.

It turns out that there are more natural candidates for the weight n motivic complexes. This is related to the fact that the “Tate motive” $\mathbb{Z}(1)$ can be constructed geometrically in different ways. Namely, there are these different incarnations of the “Tate motive” $\mathbb{Z}(1)$.

(1) As the first homology of $\mathbb{G}_m$, i.e.,

$$\mathbb{Z}(1) = \widetilde{M}(\mathbb{G}_m)[-1].$$

This was taken as a definition in definition 9.2.

(2) As the second homology with compact supports of $\mathbb{A}^1$, i.e.,

$$\mathbb{Z}(1) \cong \mathbb{Z}^{\mathrm{SF}}(1) := M^c(\mathbb{A}^1)[-2].$$

This motivates the definition of the Suslin-Friedlander weight n complexes $\mathbb{Z}^{\mathrm{SF}}(n)$ in definition 9.4.

(3) As the second homology of $\mathbb{P}^1$, i.e.,

$$\mathbb{Z}(1) \cong \widetilde{M}(\mathbb{P}^1)[-2].$$

A fourth and most concrete incarnation of the complex $\mathbb{Z}(1)$ is given by the following observation. Recall that by proposition 4.8 there is a canonical morphism

$$\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m) \to \mathbb{G}_m,$$

which defines, by homotopy invariance of $\mathbb{G}_m$ (cf. proposition 4.21), a morphism

$$M(\mathbb{G}_m) \to \mathbb{G}_m[-1].$$

Proposition 9.3. *The canonical morphism*

$$\widetilde{M}(\mathbb{G}_m) \to M(\mathbb{G}_m) \to \mathbb{G}_m$$

is a quasi-isomorphism of complexes of Nisnevich sheaves with transfers. In particular, there is a canonical isomorphism

$$\mathrm{cl}\colon \mathbb{Z}(1) \cong \mathbb{G}_m[-1].$$

Proof. Cf. [MVW06, Theorem 4.1] □

We will now define the Suslin-Friedlander weight n motivic complexes.

Definition 9.4. We define for $n \geq 0$ the Suslin-Friedlander weight n motivic complex by

$$\mathbb{Z}^{\mathrm{SF}}(n) := C_\bullet(z_{\mathrm{equi}}(\mathbb{A}^n, 0))[-2n]$$

In other words,

$$\mathbb{Z}^{\mathrm{SF}}(n) = M^c(\mathbb{A}^n)[-2n].$$

Assuming resolution of singularities we obtain

$$\mathbb{Z}^{\mathrm{SF}}(n) = M^c(\mathbb{A}^n)[-2n] \cong (M^c(\mathbb{A}^1)[-2])^{\otimes^{\mathrm{tr}} n}.$$

by the remark below definition 4.22.
Both complexes $\mathbb{Z}^{\mathrm{SF}}(n)$ and $\mathbb{Z}(n)$ are complexes of sheaves for the étale topology by proposition 4.7.
We will now recall how it can be shown that the different motivic complexes are quasi-isomorphic to each other. Later we will benefit from the following lemma.

Lemma 9.5. *Let G be either SL_n or assume that G is a semisimple algebraic group and k algebraically closed. Let $X \in \mathrm{Sch}/k$ be a scheme with an action of G. Then the induced action of the rational points $G(k)$ of G on $M(X)$ is trivial.*

Proof. In both cases the group $G(k)$ is generated by the images of points $\mathbb{G}_a(k)$ for morphisms $\mathbb{G}_a \to G$ from the additive group $\mathbb{G}_a \cong \mathbb{A}^1$. Thus it suffices to show that each element $x \in G(k)$ lying in the image of some morphism $\mathbb{G}_a \to G$ acts trivially on $M(X)$. Assume that this is the case and let $y \in \mathbb{G}_a(k)$ be a preimage of x under some morphism $\mathbb{G}_a \to G$. For a k-rational point $z \in Z(k)$ on a scheme $Z \in \mathrm{Sch}/k$ we will denote by

$$\tilde{z}\colon M(\mathrm{Spec}(k)) \to M(Z)$$

the associated morphism on motives. Then by homotopy invariance the morphism $\tilde{y}\colon M(\mathrm{Spec}(k)) \to M(\mathbb{G}_a)$ agrees with the morphism

$$\tilde{0}\colon M(\mathrm{Spec}(k)) \to M(\mathbb{G}_a)$$

induced by the unit $0 \in \mathbb{G}_a(k)$. Indeed, both are inverse to the canonical isomorphism $M(\mathbb{G}_a) \xrightarrow{\sim} M(\mathrm{Spec}(k))$. This implies that also the morphism

$$\tilde{x}\colon M(\mathrm{Spec}(k) \to M(G)$$

is given by the inclusion $\tilde{1}\colon M(\mathrm{Spec}(k)) \to M(G)$ of the unit $1 \in G(k)$ because both factor over $M(\mathbb{G}_a)$. The action of the element $x \in G(k)$ on the motive $M(X)$ can thus be factored as

$$M(X) \overset{(\tilde{x},\mathrm{Id}_X)}{\longrightarrow} M(G) \otimes^{\mathrm{tr}} M(X) \overset{\text{action}}{\longrightarrow} M(X)$$

with $\tilde{x} = \tilde{1}$. But $1 \in G(k)$ acts trivially on $M(X)$ and thus the composition

$$M(X) \overset{(\tilde{1},\mathrm{Id}_X)}{\longrightarrow} M(G) \otimes^{\mathrm{tr}} M(X) \overset{\text{action}}{\longrightarrow} M(X)$$

is the identity, i.e., x acts trivially. $\square$

It does not suffice to assume that G is connected (and k algebraically closed). For example, the natural translation action of $\mathbb{G}_m$ on $M(\mathbb{G}_m) \cong \mathbb{Z} \oplus \mathbb{G}_m$ is non-trivial.

Definition 9.6. We denote by
$$\tau\colon M(\mathbb{P}^n) \to \mathbb{Z}(1)[2]$$
the canonical morphism corresponding to $\mathcal{O}_{\mathbb{P}^n}(1)$ under the canonical isomorphism
$$\mathrm{Hom}_{DM^-_{\mathrm{eff}}(k)}(M(\mathbb{P}^n), \mathbb{Z}(1)[2]) \cong H^1_{\mathrm{Nis}}(\mathbb{P}^n, \mathbb{G}_m) \cong \mathrm{Pic}(\mathbb{P}^n)$$
obtained from theorem 4.20 and proposition 9.3.

In order to obtain a quasi-isomorphism $\mathbb{Z}(n) \cong \mathbb{Z}^{\mathrm{SF}}(n)$ in every weight we recall the decomposition of the motive $M(\mathbb{P}^n)$. Denote by
$$\tau^i\colon M(\mathbb{P}^n) \to \mathbb{Z}(i)[2i]$$
the i-th tensor power of τ, i.e., τ^i is the composition of
$$M(\underbrace{\mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{i \text{ times}}) \xrightarrow{\tau \otimes^{\mathrm{tr}} \cdots \otimes^{\mathrm{tr}} \tau} \mathbb{Z}(1)[2] \otimes^{\mathrm{tr}} \cdots \otimes^{\mathrm{tr}} \mathbb{Z}(1)[2] \cong \mathbb{Z}(i)[2i]$$
with the diagonal
$$\Delta\colon M(\mathbb{P}^n) \to M(\mathbb{P}^n \times \cdots \times \mathbb{P}^n).$$
In particular, $\tau^0\colon M(\mathbb{P}^n) \to \mathbb{Z}$ is the canonical morphism induced by the projection $\mathbb{P}^n \to \mathrm{Spec}(k)$.

Proposition 9.7. *The morphism*
$$(\tau^0, \tau, \cdots, \tau^n)\colon M(\mathbb{P}^n) \xrightarrow{\sim} \bigoplus_{i=0}^{n} \mathbb{Z}(i)[2i]$$
is an isomorphism.

Proof. This is proven in [SV00a, Proposition 4.4]. □

If $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ is a linear subspace, then the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ restricts to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. We thus obtain that the diagram
$$\begin{tikzcd}
M(\mathbb{P}^{n-1}) \arrow[r, "(\tau^0,\tau,\cdots,\tau^{n-1})"] \arrow[d] & \bigoplus_{i=0}^{n-1} \mathbb{Z}(i)[2i] \arrow[d, hook] \\
M(\mathbb{P}^n) \arrow[r, "(\tau^0,\tau,\cdots,\tau^n)"] & \bigoplus_{i=0}^{n} \mathbb{Z}(i)[2i]
\end{tikzcd}$$
commutes, where the right vertical morphism denotes the canonical inclusion of the first n summands. Moreover, by lemma 9.5 the morphism $M(\mathbb{P}^{n-1}) \to M(\mathbb{P}^n)$ is independent of the particular linear subspace as the group SL_{n+1} acts transitively on them.

Proposition 9.8. *Choose an inclusion $j\colon \mathbb{A}^n \to \mathbb{P}^n$ with complement a linear subspace $i\colon \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$. Then there exists a (split) distinguished triangle*
$$M(\mathbb{P}^{n-1}) \xrightarrow{i} M(\mathbb{P}^n) \xrightarrow{j^*} M^c(\mathbb{A}^n) \longrightarrow M(\mathbb{P}^{n-1})[1]$$
in $DM^-_{\mathrm{eff}}(k)$ and restricting the morphism j^* to the summand $\mathbb{Z}(n)[2n] \subseteq M(\mathbb{P}^n)$ from proposition 9.7 defines an isomorphism
$$\theta\colon \mathbb{Z}(n)[2n] \cong M^c(\mathbb{A}^n) = \mathbb{Z}^{\mathrm{SF}}(n)[2n]$$

independent of the chosen inclusion.

Proof. The last statement, except the independence, is proven in [MVW06, Theorem 16.8]. It then implies the existence of the distinguished triangle. The independence of θ follows from lemma 9.5. $\square$

We have finished the task of comparing the different motivic complexes $\mathbb{Z}(n)$ and $\mathbb{Z}^{\mathrm{SF}}(n)$. The following notation is motivated by proposition 9.3.

Definition 9.9. We set

$$\mathbb{Z}(1)_{\text{ét}} := \mathbb{G}_m[-1].$$

The benefits of this notation will become clear. We summarize our previous discussion about the weight n motivic complexes: The complexes $\mathbb{Z}(n)$ and $\mathbb{Z}^{\mathrm{SF}}(n)$ are quasi-isomorpic with a canonical quasi-isomorphism θ obtained in proposition 9.8 using the diagram

$$\begin{tikzcd} & M(\mathbb{P}^n) \arrow[dl, "\tau^n"'] \arrow[dr, "j^*"] & \\ \mathbb{Z}(n)[2n] & & \mathbb{Z}^{\mathrm{SF}}(n)[2n]. \end{tikzcd} \tag{9.1}$$

Moreover, in weight one both complexes $\mathbb{Z}(1)$ and $\mathbb{Z}^{\mathrm{SF}}(1)$ are canonically quasi-isomorphic to $\mathbb{Z}(1)_{\text{ét}} = \mathbb{G}_m[-1]$ (using the above isomorphism between $\mathbb{Z}(1)$ and $\mathbb{Z}^{\mathrm{SF}}(1)$ together with proposition 9.3). However, there is the following subtlety. There exists another natural construction of an isomorphism $\mathbb{Z}^{\mathrm{SF}}(1) \cong \mathbb{Z}(1)_{\text{ét}}$ based on a global sheaf $\mathcal{M}$ of rational functions. Unfortunately, both differ by a sign as we will see.

Definition 9.10. We define the presheaf $\mathcal{M}$ on Sm/k by sending $U \in \mathrm{Sm}/k$ to

$$\mathcal{M}(U) := \{f \in \mathcal{M}^{\times}_{U\times\mathbb{A}^1}(U \times \mathbb{A}^1) \mid \mathrm{div}(f) \text{ is quasi-finite over } U\}.$$

The norm of rational functions defines transfers for the presheaf $\mathcal{M}$. Moreover, $\mathcal{M}$ is a Zariksi sheaf. In fact, $\mathcal{M}$ is even an étale sheaf as we will see now.

Proposition 9.11. *Sending a rational function in $\mathcal{M}(U)$, $U \in \mathrm{Sm}/k$, to its divisor on $\mathbb{A}^1 \times U$ defines an exact sequence*

$$0 \to \mathbb{G}_m \to \mathcal{M} \overset{\mathrm{div}}{\to} z_{\mathrm{equi}}(\mathbb{A}^1, 0) \to 0 \tag{9.2}$$

of Zariski sheaves with transfers. Moreover, $\mathcal{M}$ is an étale sheaf.

Proof. As $\mathrm{CH}^1(U \times \mathbb{A}^1) \cong \mathrm{CH}^1(U)$ for every smooth scheme $U \in \mathrm{Sm}/k$, cf. proposition 8.2, the exactness follows from the fact that a divisor on U is Zariski locally principal. The kernel of the morphism $\mathrm{div}\colon \mathcal{M}(U) \to z_{\mathrm{equi}}(\mathbb{A}^1, 0)(U)$ is given by $\mathbb{G}_m(\mathbb{A}^1 \times U) \cong \mathbb{G}_m(U)$ and the exactness follows. As $\mathbb{G}_m$ and $z_{\mathrm{equi}}(\mathbb{A}^1, 0)$ are étale sheaves the same holds true for $\mathcal{M}$. In fact, denote by $F_{\text{ét}}$ the sheafification of a Zariski sheaf F on Sm/k for the étale topology. Forgetting transfers and sheafifying the exact sequence (9.2) for the étale topology yields a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & \mathcal{M} \arrow[r] \arrow[d] & z_{\mathrm{equi}}(\mathbb{A}^1, 0) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathcal{M}_{\text{ét}} \arrow[r] & z_{\mathrm{equi}}(\mathbb{A}^1, 0) \arrow[r] & 0 \end{tikzcd}$$

with exact rows and the claim follows from the 5-lemma. Hence the proposition is proven. □

The boundary map of (9.2) defines a morphism

$$\delta_{\mathrm{div}}\colon z_{\mathrm{equi}}(\mathbb{A}^1,0)\to\mathbb{G}_m[1]$$

and thus by homotopy invariance of $\mathbb{G}_m$ a morphism, still denoted δ_{div},

$$\delta_{\mathrm{div}}\colon \mathbb{Z}^{\mathrm{SF}}(1)[2]=M^c(\mathbb{A}^1)\to\mathbb{Z}(1)_{\text{ét}}[2]=\mathbb{G}_m[1].$$

We want to compare this morphism to our previous isomorphism

$$\mathbb{Z}^{\mathrm{SF}}(1)\overset{\theta}{\cong}\mathbb{Z}(1)\cong\mathbb{Z}(1)_{\text{ét}}$$

obtained via $\mathbb{P}^1$ (cf. (9.1)) which had the property that its image in $H^1_{\mathrm{Nis}}(\mathbb{P}^1,\mathbb{G}_m)$ is given by the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. More precisely, denoting the isomorphism from proposition 9.8 by

$$\theta\colon \mathbb{Z}^{\mathrm{SF}}(1)\to\mathbb{Z}(1)$$

the composition, with $j\colon\mathbb{A}^1\to\mathbb{P}^1$ some inclusion,

$$\theta\circ j^*\colon M(\mathbb{P}^1)\to\mathbb{Z}^{\mathrm{SF}}(1)[2]\to\mathbb{Z}(1)[2]\cong\mathbb{G}_m[1]$$

in $\mathrm{Hom}_{DM^-_{\mathrm{eff}}(k)}(M(\mathbb{P}^1),\mathbb{G}_m[1])\cong H^1_{\mathrm{Nis}}(\mathbb{P}^1,\mathbb{G}_m)$ is $\mathcal{O}_{\mathbb{P}^1}(1)$.
To describe the morphism $\delta_{\mathrm{div}}\colon z_{\mathrm{equi}}(\mathbb{A}^1,0)\to\mathbb{G}_m[1]$ resp. $\delta_{\mathrm{div}}\colon M^c(\mathbb{A}^1)\to\mathbb{G}_m[1]$ we choose an embedding $j\colon\mathbb{A}^1\hookrightarrow\mathbb{P}^1$ and use the diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d, equal] & \mathcal{M}' \arrow[r] \arrow[d] & \mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1) \arrow[r, "\delta'_{\mathrm{div}}"] \arrow[d, "j^*"] & \mathbb{G}_m[1] \arrow[d, equal] \\
\mathbb{G}_m \arrow[r] & \mathcal{M} \arrow[r] & z_{\mathrm{equi}}(\mathbb{A}^1,0) \arrow[r, "\delta_{\mathrm{div}}"] & \mathbb{G}_m[1]
\end{tikzcd}$$

in the derived category $D^-(\mathrm{ST}_{\mathrm{Nis}}(k))$. Here $\mathcal{M}'$ denotes the pullback of $\mathcal{M}$ along $j^*\colon\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1)\to z_{\mathrm{equi}}(\mathbb{A}^1,0)$ (in the abelian category $\mathrm{ST}_{\mathrm{Nis}}(k)$). The image of

$$\delta'_{\mathrm{div}}\colon\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1)\to\mathbb{G}_m[1]$$

under the isomorphism

$$\mathrm{Hom}_{D^-(\mathrm{ST}_{\mathrm{Nis}}(k))}(\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1),\mathbb{G}_m[1])\cong\mathrm{Hom}_{DM^-_{\mathrm{eff}}(k)}(M(\mathbb{P}^1),\mathbb{G}_m[1])\cong H^1_{\mathrm{Nis}}(\mathbb{P}^1,\mathbb{G}_m)$$

is the $\mathbb{G}_m$-torsor which is the preimage in $\mathcal{M}'$ of the diagonal $\Delta\in\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1)(\mathbb{P}^1)$. This torsor is isomorphic to the preimage $\mathcal{T}$ of the cycle $\Delta_{|\mathbb{P}^1\times\mathbb{A}^1}\in z_{\mathrm{equi}}(\mathbb{A}^1,0)(\mathbb{P}^1)$ in $\mathcal{M}$. The divisor $\Delta_{|\mathbb{P}^1\times\mathbb{A}^1}$ is rationally equivalent to the divisor $\infty\times\mathbb{A}^1$, $\infty\in\mathbb{P}^1(k)\setminus j(\mathbb{A}^1)(k)$ and the isomorphism class of $\mathcal{T}$ only depends on the rational equivalence class. By lemma 3.2 we see that

$$\mathcal{T}\cong\mathcal{O}_{\mathbb{P}^1}(-1).$$

In other words, we obtain the following proposition.

Proposition 9.12. *Denote by* $\theta\colon\mathbb{Z}(1)\cong\mathbb{Z}^{\mathrm{SF}}(1)$ *the isomorphism obtained in* (9.1), *by* $\mathrm{cl}\colon\mathbb{Z}(1)\cong\mathbb{Z}(1)_{\text{ét}}$ *the isomorphism from proposition 9.3 and by*

$$\delta_{\mathrm{div}}\colon\mathbb{Z}^{\mathrm{SF}}(1)\to\mathbb{Z}(1)_{\text{ét}}$$

the morphism considered above. Then the diagram

$$\begin{tikzcd} & \mathbb{Z}(1) \arrow[dl, "\theta"'] \arrow[dr, "\mathrm{cl}"] & \\ \mathbb{Z}^{\mathrm{SF}}(1) \arrow[rr, "\delta_{\mathrm{div}}"] & & \mathbb{Z}(1)_{\text{ét}} \end{tikzcd}$$

anticommutes, i.e., $\delta_{\mathrm{div}} = -\mathrm{cl} \circ \theta^{-1}$*. In particular,* δ_{div} *is an isomorphism and* $C_\bullet(\mathcal{M}) \cong 0$*.*

We will now define motivic cohomology following Voevodsky. For this let A be an abelian group. We denote by

$$A(n) := A \otimes_{\mathbb{Z}} \mathbb{Z}(n)$$
$$A^{\mathrm{SF}}(n) := A \otimes_{\mathbb{Z}} \mathbb{Z}^{\mathrm{SF}}(n)$$

the corresponding motivic complexes with coefficients.
Motivic cohomology is now defined as hypercohomology with coefficients in these motivic complexes.

Definition 9.13. The motivic cohomology of a smooth scheme $X \in \mathrm{Sm}/k$ with coefficients in A is defined as

$$H^i_{\mathcal{M}}(X, A(j)) := H^i_{\mathrm{Zar}}(X, A(j)) \cong H^i_{\mathrm{Zar}}(X, A^{\mathrm{SF}}(j))$$

for $i, j \in \mathbb{Z}$.

In definition 9.13 the Zariski topology may be replaced by the Nisnevich topology.

Proposition 9.14. *For* $X \in \mathrm{Sm}/k$ *and every* $i, j \in \mathbb{Z}$ *the canonical morphism*

$$H^i_{\mathcal{M}}(X, A(j)) = H^i_{\mathrm{Zar}}(X, A(j)) \to H^i_{\mathrm{Nis}}(X, A(j))$$

is an isomorphism.

Proof. This follows from [MVW06, Proposition 13.10] as the complex $A(j)$ has homotopy invariant cohomology sheaves with transfers. □

The same result is *not* true for the étale topology, cf. section 10. We will now recall how it can be shown that Bloch's higher Chow groups are isomorphic to Voevodsky's motivic cohomology. This comparison rests on the following relation between Bloch's cycle complexes and the Suslin-Friedlander complexes. Recall that for an equidimensional scheme $X \in \mathrm{Sch}/k$ we denote by $z^i(X, \bullet)$ Bloch's cycle complex (cf. section 8).

Definition 9.15. For $U \in \mathrm{Sm}/k$ we define the map

$$\beta \colon \mathbb{Z}^{\mathrm{SF}}(i)(U)[2i] \to z^i(U \times \mathbb{A}^i, \bullet), z \mapsto z.$$

The map β is a well-defined morphism of complexes as every equidimensional cycle in

$$\mathbb{Z}^{\mathrm{SF}}(i)(U)[2i] = z_{\mathrm{equi}}(\mathbb{A}^i, 0)(U \times \Delta^r)$$

intersects each face of $\mathbb{A}^i \times U \times \Delta^r$ properly and thus lies in Bloch's cycle complex.
Clearly, for every $Y \in \mathrm{Sm}/k$ the morphisms β for $U \in Y_{\text{ét}}$ constitute a morphism

$$\beta \colon \mathbb{Z}^{\mathrm{SF}}(i)[2i]_{|Y_{\text{ét}}} \to z^i(- \times \mathbb{A}^i, \bullet)$$

of complexes of sheaves (without transfers) on $Y_{\text{ét}}$.

The comparison between higher Chow groups and motivic cohomology begins with the following proposition.

Proposition 9.16. *For every $Y \in \mathrm{Sm}/k$ the map*

$$\beta : \mathbb{Z}^{\mathrm{SF}}(i)[2i]_{|Y_{\mathrm{Zar}}} \xrightarrow{\sim} z^i(- \times \mathbb{A}^i, \bullet)$$

is a quasi-isomorphism of complexes of Zariski sheaves, in particular

$$\beta\colon H^{j+2i}_{\mathrm{Zar}}(X, A^{\mathrm{SF}}(i)) \xrightarrow{\cong} H^j_{\mathrm{Zar}}(X, z^i(- \times \mathbb{A}^i, \bullet) \otimes_{\mathbb{Z}} A).$$

for any abelian group A.

Proof. This is proven in [MVW06, Theorem 19.8]. □

Combining proposition 9.16 with proposition 8.4 and homotopy invariance of higher Chow groups (cf. proposition 8.2) we derive an isomorphism

$$\mathrm{CH}^i(X, j; A) \cong H^{2i-j}_{\mathrm{Zar}}(X, A^{\mathrm{SF}}(i))$$

for every $i, j \in \mathbb{Z}$ and any abelian group A. To give it a name we make the following definition.

Definition 9.17. Let $X \in \mathrm{Sm}/k$ be a smooth scheme over k. We define for $i, j \in \mathbb{Z}$ and any abelian group A the isomorphism

$$\eta\colon \mathrm{CH}^i(X, j; A) \to H^{2i-j}_{\mathrm{Zar}}(X, A^{\mathrm{SF}}(i)) \cong H^{2i-j}_{\mathcal{M}}(X, A(i))$$

as the composition

$$\begin{aligned} &\mathrm{CH}^i(X, j; A) \xrightarrow{\sim} \mathrm{CH}^i(X \times \mathbb{A}^i, j; A) \\ \xrightarrow{\sim}\quad & H^{-j}_{\mathrm{Zar}}(X, z^i(- \times \mathbb{A}^i, \bullet)) \xrightarrow{\sim} H^{2i-j}_{\mathrm{Zar}}(X, A^{\mathrm{SF}}(i)) \cong H^{2i-j}_{\mathcal{M}}(X, A(i)). \end{aligned}$$

The occuring isomorphisms are the ones from 8.2, proposition 8.4, proposition 9.16 and proposition 9.8.
The next topic we adress is the compatibility of the isomorphism

$$\mathrm{CH}^1(X) \cong H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) \cong H^2(X, \mathbb{Z}(1)_{\text{ét}}) \cong \mathrm{Pic}(X)$$

with the first Chern class $c_1\colon \mathrm{Pic}(X) \to \mathrm{CH}^1(X)$ of line bundles. Surprisingly, our argument works only in the smooth projective case, but luckily this is all we need. Recall, cf. proposition 9.11, that there is triangle

$$\mathbb{G}_m \longrightarrow \mathcal{M} \longrightarrow z_{\mathrm{equi}}(\mathbb{A}^1, 0) \xrightarrow{\delta_{\mathrm{div}}} \mathbb{G}_m[1]$$

where $\mathcal{M}$ is the sheaf with transfers whose sections over $U \in \mathrm{Sm}/k$ are rational functions on $U \times \mathbb{A}^1$ with divisor quasi-finite over $\mathbb{A}^1$. Moreover, δ_{div} induces a morphism, denoted by the same name,

$$\delta_{\mathrm{div}}\colon M^c(\mathbb{A}^1) \to \mathbb{G}_m[1].$$

The following lemma was already noted in the proof of proposition 9.12.

Lemma 9.18. *Let $X \in \mathrm{Sm}/k$ be a smooth scheme and let*

$$D \in H^0(X, z_{\mathrm{equi}}(\mathbb{A}^1, 0)) = z_{\mathrm{equi}}(\mathbb{A}^1, 0)(X)$$

be a section, i.e., D is a divisor on $X \times \mathbb{A}^1$ equidimensional over X. Assume that E is a divisor on X whose pullback q^*E along $q\colon X \times \mathbb{A}^1 \to X$ is rationally equivalent to D. Then the image of D under the morphism

$$H^0(X, z_{\mathrm{equi}}(\mathbb{A}^1, 0)) \to H^0(X, M^c(\mathbb{A}^1)) \xrightarrow{\delta_{\mathrm{div}}} H^1(X, \mathbb{G}_m)$$

is given by $\mathcal{O}_X(-E)$.

Proof. We can argue as was done before proposition 9.12 with $\mathbb{P}^1$ replaced by X. Namely, the image of D in $H^1(X, \mathbb{G}_m)$ is given by the $\mathbb{G}_m$-torsor over X of trivialisations of the divisor D on $X \times \mathbb{A}^1$ by rational functions in $\mathcal{M}$. This $\mathbb{G}_m$-torsor is isomorphic to the $\mathbb{G}_m$-torsor of trivialisations of q^*E as D and q^*E are rationally equivalent. Finally, this torsor is isomorphic to torsor of trivialisations of $\mathcal{O}_X(-E)$. □

For a smooth scheme $X \in \mathrm{Sm}/k$ let $\eta\colon \mathrm{CH}^1(X) = \mathrm{CH}^1(X, 0; \mathbb{Z}) \cong H^2_{\mathrm{Zar}}(X, \mathbb{Z}^{\mathrm{SF}}(1))$ be the isomorphism of definition 9.17.

Proposition 9.19. *The diagram*

$$\begin{tikzcd} & \mathrm{CH}^1(X) \arrow[dl, "\eta"'] \arrow[dr, "c_1^{-1}"] & \\ H^2(X, \mathbb{Z}^{\mathrm{SF}}(1)) \arrow[rr, "\delta_{\mathrm{div}}"] & & H^1(X, \mathbb{G}_m) \cong \mathrm{Pic}(X) \end{tikzcd}$$

anticommutes, i.e.,

$$c_1 \circ \delta_{\mathrm{div}} \circ \eta = -\mathrm{Id}_{\mathrm{CH}^1(X)}.$$

Proof. Let E be a divisor on X and assume first that there exists a divisor $D \in H^0(X, z_{\mathrm{equi}}(\mathbb{A}^1, 0))$ rationally equivalent to the divisor q^*E, $q\colon X \times \mathbb{A}^1 \to X$. Then $\eta(E) = \mathrm{can}(D)$ where

$$\mathrm{can}\colon H^0(X, z_{\mathrm{equi}}(\mathbb{A}^1, 0)) \to H^0(X, M^c(\mathbb{A}^1)) = H^2(X, \mathbb{Z}^{\mathrm{SF}}(1))$$

is the canonical morphism induced by $z_{\mathrm{equi}}(\mathbb{A}^1, 0) \to C_\bullet(z_{\mathrm{equi}}(\mathbb{A}^1, 0)) = M^c(\mathbb{A}^1)$. By lemma 9.18 we can conclude that

$$\delta_{\mathrm{div}}(\eta(E)) = \mathcal{O}_X(-E) = -c_1^{-1}(E).$$

Therefore it suffices to show that for every divisor E on X we can find some divisor D as above. This will be the content of the next lemma. □

Lemma 9.20. *Let $X \in \mathrm{Sm}/k$ be a connected smooth projective scheme. Then every rational equivalence class of divisors on $X \times \mathbb{A}^1$ can be represented by some divisor D on $X \times \mathbb{A}^1$, which is equidimensional (of dimension 0) over X.*

Proof. As X is smooth, pullback along the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism $\mathrm{CH}^1(X) \cong \mathrm{CH}^1(X \times \mathbb{A}^1)$. Let E be a divisor on X. By [FV00, Theorem 7.1] the rational equivalence class of q^*E in $\mathrm{CH}^1(X \times \mathbb{P}^1)$, where $q\colon X \times \mathbb{P}^1 \to X$ is the canonical projection, can be represented by some divisor D' which is finite and surjective over X. Then the restriction $D := D'_{|X \times \mathbb{A}^1}$ will be equidimensional over X and rationally equivalent to $p^*E, p\colon X \times \mathbb{A}^1 \to X$, giving the claim. □

In the case that X is a smooth projective curve, the only case which matters to us, a direct argument replacing lemma 9.20 (and thus [FV00, Theorem 7.1]) can be given.

Lemma 9.21. *Let $X \in \mathrm{Sm}/k$ be a smooth projective curve. Then each rational equivalence class in $\mathrm{CH}^1(X \times \mathbb{A}^1)$ can be represented by some divisor equidimensional over X.*

Proof. The classes in $\mathrm{CH}^1(X \times \mathbb{A}^1)$ containing a divisor equidimensional over X form a subgroup. Moreover, $\mathrm{CH}^1(X) \cong \mathrm{CH}^1(X \times \mathbb{A}^1)$ is generated by divisors of the form mx where $x \in X$ is a point and $m \gg 0$ is an integer such that $\mathcal{O}_X(mx)$ is very ample. Hence it suffices to show that for every point $x \in X$ the divisor $m(x \times \mathbb{A}^1)$ is rationally equivalent to some divisor D equidimensional over X where $m \gg 0$ is any integer such that $\mathcal{O}_X(mx)$ is very ample. As $\mathcal{O}_X(mx)$ is very ample there exists an effective divisor $E \in \mathbb{P}(H^0(X, \mathcal{O}_X(mx))$ in the linear system of mx such that the support of E does not contain x. Let $f \in K(X)$ be a rational function with divisor $E - mx$. Let $t \in \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$ be a coordinate. We claim that the divisor

$$D := m(x \times \mathbb{A}^1) - \mathrm{div}(f - t),$$

which is clearly rationally equivalent to $m(x \times \mathbb{A}^1)$, is equidimensional over X. In fact, the zero divisor of $f - t$ is precisely the closure of the graph of the function $f\colon X \setminus \{x\} \to \mathbb{A}^1$ and this closure is equidimensional over X (it is even flat over X). On the other hand the polar divisor of $f - t$ is precisely $m(x \times \mathbb{A}^1)$ and the claim follows. $\square$

10. Motivic cohomology with finite coefficients

We want to relate motivic cohomology and étale cohomology via a cycle class map. For this the isomorphism

$$\mathrm{cl}\colon \mathbb{Z}(1) \to \mathbb{Z}(1)_{\text{ét}}$$

from proposition 9.3 will be crucial.

First of all we want to stress that motivic cohomology is defined using the Zariski topology (or equivalently the Nisnevich topology by proposition 9.14).

Definition 10.1. We define the morphism of sites

$$\varepsilon\colon (\mathrm{Sm}/k)_{\text{ét}} \to (\mathrm{Sm}/k)_{\mathrm{Zar}} \tag{10.1}$$

as the canonical change-of-topology map, i.e., the underlying functor

$$\varepsilon^{-1}\colon (\mathrm{Sm}/k)_{\mathrm{Zar}} \to (\mathrm{Sm}/k)_{\text{ét}}$$

is the identity.

It is time to give an account for which topologies we consider the various motivic complexes. The motivic complexes $\mathbb{Z}(n)$ (resp. $\mathbb{Z}^{\mathrm{SF}}(n)$ or $A(n)$, $A^{\mathrm{SF}}(n)$ for some abelian group A) will be considered as a complex of Zariski sheaves (with transfers), although each component is an étale sheaf. Contrary, the complex $\mathbb{Z}(1)_{\text{ét}} = \mathbb{G}_m[-1]$ will be considered as a complex of étale sheaves (with transfers).

Proposition 10.2. *Assume that $m \in \mathbb{Z}$ is non-zero and invertible in k. Then the Kummer triangle*

$$\mu_m \to \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m \to \mu_m[1]$$

defines an isomorphism

$$\mu_m \cong \mathbb{Z}/m(1)_{\text{ét}} := \mathbb{Z}/m \otimes^L_{\mathbb{Z}} \mathbb{Z}(1)_{\text{ét}},$$

where μ_m denotes the étale sheaf of m-th roots of unity. Moreover,

$$\mathbb{Z}/m(n)_{\text{ét}} := \mathbb{Z}/m(1)_{\text{ét}}{}^{\otimes^{\mathrm{tr}} n} \cong \mu_m^{\otimes n}.$$

Proof. The first statement is clear. The second statement follows from proposition 4.12 as μ_m is a locally constant sheaf because m is invertible in k. $\square$

For $m = p^r$ with $p := \mathrm{char}(k) > 0$ the quotient $\mathbb{Z}/p^r(1)_{\text{ét}} := \mathbb{Z}/p^r \otimes^L_{\mathbb{Z}} \mathbb{Z}(1)_{\text{ét}}$ is isomorphic to the logarithmic de Rham-Witt sheaf $\nu_r(1)[-1]$ (cf. [Gro85, Chapitre 1, 1.2]), a sheaf which is neither locally constant nor constructible. In particular, it is not clear how to describe the higher tensor powers $\mathbb{Z}/p^r(1)^{\otimes^{\mathrm{tr}} n}$.
We will be content with the following *ad-hoc* definition of higher weight motivic complexes $\mathbb{Z}/p^r(n)_{\text{ét}}$.

Definition 10.3. Let $m \in \mathbb{Z}$ be any integer. Factorize $m = m'p^r$ with $(m', p) = 1$. We define the mod m étale motivic complex of weight n as

$$\mathbb{Z}/m(n)_{\text{ét}} := \mu_{m'}^{\otimes n} \oplus \nu_r(n)[-n]$$

where $\mu_{m'} = \ker(\mathbb{G}_m \xrightarrow{m'} \mathbb{G}_m)$ is the sheaf of m'-roots of unity and $\nu_r(n)$ the logarithmic de Rham-Witt sheaf of weight n (cf. [Gro85, Chapitre 1, 1.2]).

By definition the logarithmic de Rham-Witt sheaf $\nu_r(n)$ is the étale subsheaf in $W_r(\Omega^n_X)$ generated by differential $\frac{d[x_1]}{[x_1]} \wedge \cdots \wedge \frac{d[x_n]}{[x_n]}$ where $[-]\colon \mathbb{G}_m \to W_r(\mathcal{O}_X)$ denotes the Teichmüller lift into the truncated Witt vectors of $\mathcal{O}_X$.
The wedge product $\nu_r(n) \otimes \nu_r(n') \to \nu_r(n+n')$ and the tensor product $\mu_{m'}^{\otimes n} \otimes \mu_{m'}^{\otimes n'} \cong \mu_{m'}^{\otimes n+n'}$ yield a multiplicative structure on the collection $\mathbb{Z}/m(n)_{\text{ét}}$, $n \geq 0$, i.e., products

$$\wedge : \mathbb{Z}/m(i)_{\text{ét}} \otimes \mathbb{Z}/m(j)_{\text{ét}} \to \mathbb{Z}/m(i+j)_{\text{ét}}$$

for $i, j \in \mathbb{Z}$.
If $n = 1$, then the complex $\mathbb{Z}/m(1)_{\text{ét}}$ sits naturally in a triangle, deserved to be called Kummer triangle,

$$\mathbb{Z}(1)_{\text{ét}} \xrightarrow{m} \mathbb{Z}(1)_{\text{ét}} \to \mathbb{Z}/m(1)_{\text{ét}} \to \mathbb{Z}(1)_{\text{ét}}[1]$$

of complexes of étale sheaves. For $m = p^r$ the morphism $\mathbb{Z}(1)_{\text{ét}} \to \mathbb{Z}/m(1)_{\text{ét}}$ is given by the dlog map

$$\mathrm{dlog}\colon \mathbb{G}_m \to v_r(1),\ x \mapsto \frac{d[x]}{[x]},$$

while for m invertible in k this morphism is the (shifted) connecting morphism of the Kummer sequence

$$0 \to \mu_m \to \mathbb{G}_m \to \mathbb{G}_m \to 0.$$

We recall that there is a canonical isomorphism

$$\mathrm{cl}\colon \varepsilon^*\mathbb{Z}(1) \to \mathbb{Z}(1)_{\text{ét}}$$

by proposition 9.3.

Definition 10.4. For $n \geq 0$ and any $m \in \mathbb{Z}$ invertible in k we define the cycle class map

$$\mathrm{cl} : \varepsilon^*\mathbb{Z}/m(n) \to \mathbb{Z}/m(n)_{\text{ét}}$$

as the isomorphism

$$\varepsilon^*\mathbb{Z}/m(n) \cong (\varepsilon^*\mathbb{Z}/m(1))^{\otimes^{\mathrm{tr}} n} \xrightarrow{\mathrm{cl}^{\otimes^{\mathrm{tr}} n}} (\mathbb{Z}/m(1)_{\text{ét}})^{\otimes n} \xrightarrow{\sim} \mathbb{Z}/m(n)_{\text{ét}}.$$

For $r \geq 0$ we define

$$\mathrm{cl}\colon \varepsilon^*\mathbb{Z}/p^r(n) \to \mathbb{Z}/p^r(n)_{\text{ét}}$$

as the isomorphism from [Gei10, Proposition 2.2] resp. [GL00, Theorem 8.5]. More precisely, for every $X \in \mathrm{Sm}/k$ we use the isomorphism

$$\begin{aligned}&\varepsilon^*\mathbb{Z}/p^r(n) \cong \varepsilon^*\mathbb{Z}/p^r{}^{\mathrm{SF}}(n)\\&\cong \mathbb{Z}/p^r \otimes z^n(- \times \mathbb{A}^n, \bullet)[-2n] \cong \mathbb{Z}/p^r \otimes z^n(-, \bullet)[-2n] \cong \mathbb{Z}/p^r(n)\end{aligned}$$

on the small étale site of X to obtain the cycle class map $\mathrm{cl}\colon \varepsilon^*\mathbb{Z}/p^r(n) \cong \mathbb{Z}/p^r(n)_{\text{ét}}$ on Sm/k.

For general $m \in \mathbb{Z}$ we obtain the cycle class map $\mathrm{cl}\colon \varepsilon^*\mathbb{Z}/m(n) \to \mathbb{Z}/m(n)_{\text{ét}}$ by splitting $m = m'p^r$ with $(m', p) = 1$.
We remark that also in the case $n = 1$ the cycle class map $\mathrm{cl}\colon \varepsilon^*\mathbb{Z}/p^r(1) \to \mathbb{Z}/p^r(1)_{\text{ét}}$ from 10.4 is given by tensoring the isomorphism $\varepsilon^*\mathbb{Z}(1) \cong \mathbb{Z}(1)_{\text{ét}}$ of proposition 9.3 with $\mathbb{Z}/m$ as both are given by the dlog map. In particular, for $X \in \mathrm{Sm}/k$ there exists a commutative diagram

$$\begin{tikzcd} H^1_{\mathcal{M}}(X, \mathbb{Z}/m(1)) \arrow[r, "\mathrm{cl}"] \arrow[d] & H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \arrow[d] \\ H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) \arrow[r, "\mathrm{cl}"] & H^2_{\text{ét}}(X, \mathbb{Z}(1)_{\text{ét}}), \end{tikzcd} \tag{10.2}$$

where the vertical arrows are the connecting morphims of the triangles

$$\mathbb{Z}(1) \xrightarrow{m} \mathbb{Z}(1) \to \mathbb{Z}/m(1) \to \mathbb{Z}(1)[1]$$

resp.

$$\mathbb{Z}(1)_{\text{ét}} \xrightarrow{m} \mathbb{Z}(1)_{\text{ét}} \to \mathbb{Z}/m(1)_{\text{ét}} \to \mathbb{Z}(1)_{\text{ét}}[1].$$

Formally, it is often not necessary to distinguish the cases $(m, p) = 1$ and $m = p^r$. Therefore the common notation $\mathbb{Z}/m(n)_{\text{ét}}$ simplifies the exposition:

$$\varepsilon^*\mathbb{Z}/m(n) \cong \mathbb{Z}/m(n)_{\text{ét}}$$

for *every* $m \in \mathbb{Z}$.
More interesting than the cycle class map

$$\mathrm{cl}\colon \varepsilon^*\mathbb{Z}/m(n) \cong \mathbb{Z}/m(n)_{\text{ét}}$$

is its adjoint, again called cycle class map and denoted by cl,

$$\mathrm{cl}\colon \mathbb{Z}/m(n) \to R\varepsilon_*(\mathbb{Z}/m(n)_{\text{ét}}).$$

By definition the complex $\mathbb{Z}/m(n)$ has zero cohomology sheaves in degrees (strictly) greater than n, therefore the above morphism cl factors through a morphism

$$\mathrm{cl}\colon \mathbb{Z}/m(n) \to \tau_{\leq n} R\varepsilon_*(\mathbb{Z}/m(n)_{\text{ét}}).$$

The following theorem, formerly called the Beilinson-Lichtenbaum conjecture in the case that m is prime to the characteristic of k, links motivic cohomology with finite coefficients to étale cohomology.

Theorem 10.5. *For every $m \in \mathbb{Z}$ the cycle class map*

$$\mathrm{cl} : \mathbb{Z}/m(n) \to \tau_{\leq n} R\varepsilon_*(\mathbb{Z}/m(n)_{\text{ét}})$$

is a quasi-isomorphism.

Proof. For m invertible in k [SV00a] (or [GL01]) reduces the theorem to the Bloch-Kato conjecture, which was then proven in [Voe11]. For $m = p^r$ cf. [GL00, Theorem 8.5.]. □

Theorem 10.5 is a very strong statement. It describes, at least in a certain range and with finite coefficients, algebraic cycles by cohomology. More precisely, we obtain an isomorphism between motivic cohomology and étale cohomology.

Corollary 10.6. *For any connected smooth scheme $X \in \mathrm{Sm}/k$ and any integer $m \in \mathbb{Z}$ the cycle class map*

$$\mathrm{cl} : H^i_{\mathcal{M}}(X, \mathbb{Z}/m(n)) \to H^i(X_{\text{ét}}, \mathbb{Z}/m(n)_{\text{ét}})$$

is an isomorphism if one the following conditions holds:

(1) $i \leq n$
(2) $n = \dim X$ *and* i *arbitrary*

Proof. The first statement is clear by theorem 10.5. By [Gei10, Theorem 3.1] there is a canonical isomorphism

$$z^{\dim X}(X, \bullet) \cong R\Gamma(X_{\text{ét}}, \mathbb{Z}(\dim X)).$$

Moreover, by proposition 8.4 the same holds for the Zariski topology. It follows that

$$R\Gamma(X_{\mathrm{Zar}}, \mathbb{Z}/m(\dim X)) \cong R\Gamma(X_{\text{ét}}, \mathbb{Z}/m(\dim X))$$

by (derived) modding out m. Using the fact that $\mathrm{cl}\colon \varepsilon^{\epsilon}\mathbb{Z}/m(n) \cong \mathbb{Z}/m(n)_{\text{ét}}$ yields the claim. □

11. Poincaré duality

In this section we recall Poincaré duality for proper smooth schemes over an algebraically closed field with $\mathbb{Z}/m$-coefficients for *any* $m \in \mathbb{Z}$, i.e., m not necessarily invertible on X.

For the whole section we assume that k is algebraically closed. Recall that for $X \in \mathrm{Sch}/k$ with structure morphism $f\colon X \to \mathrm{Spec}(k)$ there exists, because of the Brown representability theorem for triangulated categories, a right adjoint $f^!$ of the total derived functor of cohomology with compact supports $Rf_!$.

Theorem 11.1. *Let $m \in \mathbb{Z}$ be an integer. Let $f\colon X \to \mathrm{Spec}(k)$ be a connected smooth proper scheme over k of dimension d. Then the trace map*

$$\mathrm{Tr}\colon \mathbb{Z}/m(d)_{\text{ét}}[2d] \xrightarrow{\sim} f^!\mathbb{Z}/m$$

is an isomorphism.

Proof. Cf. [Gei10, Theorem 4.1] resp. [Gei10, Proposition 2.2, Corollary 4.7]. □

For the definition of the trace map cf. [Del77, Arcata, VI.4.] in the case $(m, p) = 1$ and [Gei10, Proposition 3.5] in general. One of its characteristic properties is that it sends the class of a point to 1, i.e.

$$\mathrm{Tr}(\mathrm{cl}(x)) = 1$$

for $x \in X(k)$ a rational point with class $\mathrm{cl}(x) \in H^{2d}(X_{\text{ét}}, \mathbb{Z}/m(d)_{\text{ét}})$. More precisely, $\mathrm{cl}(x)$ is the image of the 0-cycle $x \in \mathrm{CH}_0(X) \cong H^{2d}_{\mathcal{M}}(X, \mathbb{Z}(d))$ under the map

$$\mathrm{cl}\colon H^{2d}_{\mathcal{M}}(X, \mathbb{Z}(d) \to H^{2d}_{\mathcal{M}}(X, \mathbb{Z}/m(d)) \to H^{2d}_{\text{ét}}(X, \mathbb{Z}/m(d)_{\text{ét}}).$$

Here the isomorphism $\mathrm{CH}_0(X) \cong H^{2d}_{\mathcal{M}}(X, \mathbb{Z}(d))$ is the one of definition 9.17.
Theorem 11.1 now implies Poincaré duality in its usual form.

Corollary 11.2. *Let $m \in \mathbb{Z}$ be an integer. Let $X \in \mathrm{Sm}/k$ be a connected proper smooth scheme over k of dimension d. Then the composition*

$$\Phi'_{\mathrm{PD}} \colon H^i_{\text{ét}}(X, \mathbb{Z}/m(j)_{\text{ét}}) \times H^{2d-i}_{\text{ét}}(X, \mathbb{Z}/m(d-j)_{\text{ét}}) \overset{\cup}{\to} H^{2d}_{\text{ét}}(X, \mathbb{Z}/m(d)_{\text{ét}}) \overset{\mathrm{Tr}}{\to} \mathbb{Z}/m$$

defines a perfect pairing of $\mathbb{Z}/m$-modules for any $i \in \mathbb{Z}$ if one of the following two conditions is satisfied

(1) *$(m,p) = 1$ and $j \in \mathbb{Z}$ arbitrary*
(2) *$m = p^r$ and $j = 0$.*

Proof. First of all we note that

$$R\underline{\mathrm{Hom}}(\mathbb{Z}/m(j)_{\text{ét}}, \mathbb{Z}/m(d)_{\text{ét}}) \cong \mathbb{Z}/m(d-j)_{\text{ét}}$$

in both cases under consideration. Here $R\underline{\mathrm{Hom}}$ denotes the derived internal Hom of abelian sheaves on $X_{\text{ét}}$. It is then formal that the adjoint

$$\Phi_{\mathrm{PD}} \colon H^i(X_{\text{ét}}, \mathbb{Z}/m(j)_{\text{ét}}) \to \mathrm{Hom}_{\mathbb{Z}/m}(H^{2d-i}(X_{\text{ét}}, \mathbb{Z}/m(d-j)_{\text{ét}}), \mathbb{Z}/m)$$

is obtained by applying H^i to the isomorphism

$$\begin{aligned} & Rf_* R\underline{\mathrm{Hom}}(\mathbb{Z}/m(j)_{\text{ét}}, \mathbb{Z}/m(d)_{\text{ét}}[2d]) \\ \cong\ & Rf_* R\underline{\mathrm{Hom}}(\mathbb{Z}/m(j)_{\text{ét}}, f^!\mathbb{Z}/m) \\ \cong\ & R\mathrm{Hom}(Rf_*(\mathbb{Z}/m(j)_{\text{ét}}), \mathbb{Z}/m) \end{aligned}$$

coming from theorem 11.1 and the adjunction $Rf_* \dashv f^!$ for $f \colon X \to \mathrm{Spec}(k)$ (also note that $\mathbb{Z}/m$ is an injective $\mathbb{Z}/m$-module). In particular, Φ_{PD} is an isomorphism. □

12. The motivic reciprocity law

Let k be an algebraically closed field and let $X \in \mathrm{Sm}/k$ be a connected smooth *projective* variety over k of dimension d. Moreover, fix a non-zero integer $m \in \mathbb{Z}$.

Definition 12.1. Let

$$\nu \colon H_1^S(X, \mathbb{Z}/m) \xrightarrow{\sim} H^{2d-1}(X_{\text{ét}}, \mathbb{Z}/m(d)_{\text{ét}})$$

be the isomorphism obtained as the composition

$$\begin{aligned} & H_1^S(X, \mathbb{Z}/m) \\ \overset{\alpha}{\to}\ & \mathrm{CH}^d(X, 1; \mathbb{Z}/m) \\ \overset{\eta}{\to}\ & H^{2d-1}_{\mathcal{M}}(X, \mathbb{Z}/m(d)) \\ \overset{\mathrm{cl}}{\to}\ & H^{2d-1}_{\text{ét}}(X, \mathbb{Z}/m(d)_{\text{ét}}) \end{aligned}$$

with occuring isomorphisms defined in proposition 8.6, definition 9.17 and definition 10.4.

In the case of curves zero-cycles and divisors agree. This allows to lift the morphism ν to integral coefficients.

Definition 12.2. Assume $d = 1$, i.e., that X is a curve. We define

$$\nu' \colon H_0^S(X, \mathbb{Z}) \to H^2(X_{\text{ét}}, \mathbb{Z}(1)_{\text{ét}})$$

as the isomorphism obtained by the composition

$$H_0^S(X, \mathbb{Z}) \overset{\alpha}{\to} \mathrm{CH}^1(X, 0; \mathbb{Z}) \overset{\eta}{\to} H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) \overset{\mathrm{cl}}{\to} H^2(X_{\text{ét}}, \mathbb{Z}(1)_{\text{ét}}).$$

It is again an isomorphism.

Clearly, in the case $d = 1$ the morphisms ν and ν' are linked. Namely, the diagram

$$\begin{tikzcd} H_1^S(X, \mathbb{Z}/m) \arrow[r, "\nu"] \arrow[d] & H^1(X_{\text{ét}}, \mathbb{Z}/m(1)_{\text{ét}}) \arrow[d] \\ H_0^S(X, \mathbb{Z}) \arrow[r, "\nu'"] & H^2(X_{\text{ét}}, \mathbb{Z}(1)_{\text{ét}}) \end{tikzcd}$$

commutes where the vertical arrows are the connecting morphisms for the coefficient triangles

$$\mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to \mathbb{Z}[1]$$

resp.

$$\mathbb{Z}(1)_{\text{ét}} \xrightarrow{m} \mathbb{Z}(1)_{\text{ét}} \to \mathbb{Z}/m(1)_{\text{ét}} \to \mathbb{Z}(1)_{\text{ét}}[1].$$

We remark that both vertical arrows are injections as

$$H_1^S(X, \mathbb{Z}) \cong k^\times \cong H^1(X_{\text{ét}}, \mathbb{Z}(1)_{\text{ét}})$$

is m-divisible.

Definition 12.3. We define the motivic reciprocity law Φ_{mot} as the isomorphism

$$\Phi_{\text{mot}} \colon H_1^S(X, \mathbb{Z}/m) \xrightarrow{\nu} H_{\text{ét}}^{2d-1}(X_{\text{ét}}, \mathbb{Z}/m(d)_{\text{ét}}) \xrightarrow{\Phi_{\text{PD}}} H_{\text{ét}}^1(X, \mathbb{Z}/m)^\vee \cong \pi_1^{\text{ab}}(X)/m.$$

with Φ_{PD} denoting Poincaré duality (cf. corollary 11.2). Dually, the motivic reciprocity law Φ_{mot} can be seen as a pairing

$$\Phi'_{\text{mot}} \colon H_1^S(X, \mathbb{Z}/m) \times H^1(X_{\text{ét}}, \mathbb{Z}/m) \to \mathbb{Z}/m$$

which will be called the motivic reciprocity pairing.

The motivic reciprocity law Φ_{mot} is clearly functorial in X.

13. Poincaré duality for curves

Assume that k is algebraically closed and that $X \in \mathrm{Sm}/k$ is a smooth projective curve over k. Also fix an integer $m \in \mathbb{Z}$. In this section we will analyze Poincaré duality for curves more closely. We remark that, at least in the case $(m, p) = 1$, the considerations made here are actually a way of *proving* Poincaré duality (cf. [Del77, Dualité, §3]). Our interest lies in the case $m = p^r$ with $p = \mathrm{char}(k) > 0$ as the case m invertible in k has already been dealt with in [Del77, Dualité, §3]. However, the argument we are presenting, which is essentially Deligne's, works for every $m \in \mathbb{Z}$.

Proposition 13.1. *The Kummer triangle*

$$\mathbb{Z}(1)_{\text{ét}} \xrightarrow{m} \mathbb{Z}(1)_{\text{ét}} \longrightarrow \mathbb{Z}/m(1)_{\text{ét}} \longrightarrow \mathbb{Z}(1)_{\text{ét}}[1]$$

yields an isomorphism

$$\delta_{\text{Kum}} \colon H_{\text{ét}}^1(X, \mathbb{Z}/m(1)_{\text{ét}}) \to {}_m H_{\text{ét}}^2(X, \mathbb{Z}(1)_{\text{ét}}) \cong {}_m\mathrm{Pic}(X)$$

of $H_{\text{ét}}^1(X, \mathbb{Z}/m(1)_{\text{ét}})$ *with the* m*-torsion in* $\mathrm{Pic}(X)$.

Proof. Clear, as $H_{\text{ét}}^1(X, \mathbb{Z}(1)_{\text{ét}}) = H_{\text{ét}}^0(X, \mathbb{G}_m) = k^\times$ is m-divisible. □

The main statement in this section is the proof of the following compatibility. Recall that there is a canonical isomorphism

$$\mathrm{alb} \colon H_{\text{ét}}^1(X, \mathbb{Z}/m) \cong \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m)$$

(cf. proposition 6.6).

Theorem 13.2. *The diagram*

$$\begin{tikzcd} H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \times H^1_{\text{ét}}(X, \mathbb{Z}/m) \arrow[r, "\Phi_{\mathrm{PD}}"] \arrow[d, "\delta_{\mathrm{Kum}} \times \mathrm{alb}"] & \mathbb{Z}/m \arrow[d, equal] \\ {}_m\mathrm{Pic}(X) \times \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m) \arrow[r, "\mathrm{eval}"] & \mathbb{Z}/m \end{tikzcd}$$

commutes, where δ_{Kum} *is the isomorphism from proposition 13.1.*

Its proof, which will last this whole section, exploits the Künneth decomposition of the diagonal.

Definition 13.3. Let $u \in H^1_{\text{ét}}(X, \mathbb{Z}/m) \otimes H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$ be the preimage of $\mathrm{Id}_{{}_m\mathrm{Pic}(X)}$ under the isomorphism

$$\begin{aligned} & H^1_{\text{ét}}(X, \mathbb{Z}/m) \otimes H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \\ \xrightarrow{\mathrm{alb} \otimes \delta_{\mathrm{Kum}}} \; & \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m) \otimes {}_m\mathrm{Pic}(X) \\ \cong \; & \mathrm{Hom}({}_m\mathrm{Pic}(X), {}_m\mathrm{Pic}(X)). \end{aligned}$$

The following proposition is crucial for proving theorem 13.2. Its proof will be given in the appendix A.
In appendix A we will prove a Künneth decomposition for $X \times X$, namely we will prove that $H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}})$ is isomorphic to

$$H^2_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \oplus H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \otimes H^1_{\text{ét}}(X, \mathbb{Z}/m) \oplus H^2(X, \mathbb{Z}/m(1)_{\text{ét}})$$

via the exterior product. In particular, we can speak about Künneth decompositions $v = v^{2,0} + v^{1,1} + v^{0,2}$ of cohomology classes $v \in H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}})$.

Proposition 13.4. *Let* $u' \in H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}})$ *be the image of* u *under the exterior product*

$$\times \colon H^1_{\text{ét}}(X, \mathbb{Z}/m) \otimes H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \to H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}}).$$

Then u' *is the* $(1,1)$*-component of the class*

$$\mathrm{cl}(\Delta) \in H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}})$$

in the Künneth decomposition proposition A.2, i.e., $\mathrm{cl}(\Delta)^{1,1} = u'$*.*

Proof. This will be proven in appendix A. $\square$

Let

$$\begin{tikzcd} X \times X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X \\ X & \end{tikzcd}$$

be the natural projections. We will need the pushforward map

$$\mathrm{pr}_{2,*} \colon H^3_{\text{ét}}(X \times X, \mathbb{Z}/m(2)_{\text{ét}}) \to H^1_{\text{ét}}(X, \mathbb{Z}/m(1))$$

along the second projection $\mathrm{pr}_2 \colon X \times X \to X$ (described in more detail before proposition B.14).

Lemma 13.5. *Let $v \in H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}})$ be a class with Künneth decomposition $v = v^{2,0} + v^{1,1} + v^{0,2}$ and let*

$$\Phi_v \colon H^1_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}}) \to H^1_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}}),\ x \mapsto \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(x) \cup v)$$

be the associated correspondence (with $\mathrm{pr}_{2,*}$ as above). Then $\Phi_v = \Phi_{v^{1,1}}$.

Proof. It suffices to show that $\Phi_v = 0$ if v is of type $(2,0)$ or $(0,2)$. First consider the case that $v = \sum_i \mathrm{pr}_1^*(a_i) \cup \mathrm{pr}_2^*(b_i)$ is of type $(2,0)$, i.e., $a_i \in H^2_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$ and $b_i \in H^0_{\text{ét}}(X, \mathbb{Z}/m)$. Then

$$\mathrm{pr}_1^*(x) \cup v = \sum_i \mathrm{pr}_1^*(x \cup a_i) \cup \mathrm{pr}_2^*(b_i) = 0$$

is zero as $x \cup a_i \in H^3_{\text{ét}}(X, \mathbb{Z}/m(2)_{\text{ét}}) = 0$. Note that in the case $m = p$ with $p = \mathrm{char}(k) > 0$ the complex $\mathbb{Z}/p(2)_{\text{ét}}$ vanishes (and therefore also $\mathbb{Z}/p^r(2)_{\text{ét}}$ for $r \geq 0$) as $\mathbb{Z}/p(2)_{\text{ét}}$ embeds into $\Omega^2_X = 0$. Now assume that $v = \sum_i \mathrm{pr}_1^*(a_i) \cup \mathrm{pr}_2^*(b_i)$ is of type $(0,2)$, i.e., $a_i \in H^0_{\text{ét}}(X, \mathbb{Z}/m)$ and $b_i \in H^2_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$. We can compute

$$\Phi_v(x) = \sum_i \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(x) \cup \mathrm{pr}_1^*(a_i) \cup \mathrm{pr}_2^*(b_i))$$

using proposition B.14. Namely this proposition implies that

$$\mathrm{pr}_{2,*}(\mathrm{pr}_1^*(x \cup a_i) \cup \mathrm{pr}_2^*(b_i)) = \mathrm{Tr}(x \cup a_i) b_i = 0$$

as the trace

$$\mathrm{Tr} \colon H^*_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \to \mathbb{Z}/m$$

vanishes on $H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$ and the elements $x \cup a_i$ lie in $H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$. □

We are now prepared to proof theorem 13.2.

Proof of theorem 13.2. It has to be shown that for every $x \in {}_m\mathrm{Pic}(X)$ and any homomorphism $\varphi \colon {}_m\mathrm{Pic}\,(X) \to \mathbb{Z}/m$ the equality

$$\mathrm{Tr}(\delta_{\mathrm{Kum}}^{-1}(x) \cup \mathrm{alb}^{-1}(\varphi)) = \varphi(x)$$

holds. Using naturality in the coefficients $\mathbb{Z}/m$ it suffices to prove the universal case, i.e.,

$$\varphi = \mathrm{Id}_{{}_m\mathrm{Pic}(X)} \colon {}_m\mathrm{Pic}\,(X) \to {}_m\mathrm{Pic}(X).$$

More precisely, tensoring the diagram in theorem 13.2 with a $\mathbb{Z}/m$-module A from the right yields a diagram

$$\begin{tikzcd}
H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \otimes H^1_{\text{ét}}(X, \mathbb{Z}/m) \otimes A \arrow[r, "\Phi_{\mathrm{PD}} \otimes A"] \arrow[d, "\delta_{\mathrm{Kum}} \otimes \mathrm{alb} \otimes \mathrm{Id}_A"] & A \arrow[d, equal] \\
{}_m\mathrm{Pic}(X) \otimes \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m) \otimes A \arrow[r, "\mathrm{eval} \otimes A"] & A
\end{tikzcd}$$

natural in A. The universal case is then obtained by setting $A = {}_m\mathrm{Pic}(X)$ and $\varphi = \mathrm{Id}_{{}_m\mathrm{Pic}(X)} \in \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m) \otimes {}_m\mathrm{Pic}(X) \cong \mathrm{Hom}({}_m\mathrm{Pic}(X), {}_m\mathrm{Pic}(X))$. Consider the element

$$u \in H^1_{\text{ét}}(X, \mathbb{Z}/m) \otimes H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$$

from definition 13.3. Then u corresponds to $\mathrm{Id}_{{}_m\mathrm{Pic}(X)}$ under the isomorphism $\delta_{\mathrm{Kum}}\colon H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \cong {}_m\mathrm{Pic}(X)$ and we are left with showing that

$$\mathrm{Tr}'(x \cup' u) = x$$

for all $x \in H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$ where $\cup'$ denotes the morphism

$$a \otimes b \otimes c \mapsto (a \cup b) \otimes c$$

for $a \in H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$, $b \in H^1_{\text{ét}}(X, \mathbb{Z}/m)$, $c \in H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$ and Tr' the morphism

$$\mathrm{Tr}' = \mathrm{Tr} \otimes \mathrm{Id}_{H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})}$$

with $\mathrm{Tr}\colon H^2_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \to \mathbb{Z}/m$ the usual trace map. The exterior product

$$H^1_{\text{ét}}(X, \mathbb{Z}/m) \otimes H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \to H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}})$$

maps u, by definition, to the element $u' \in H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}})$ from proposition 13.4, and by proposition B.14 the map

$$x \mapsto \mathrm{Tr}'(x \cup' u)$$

is given by the (cohomological) correspondence

$$\Phi_{u'}(x) := \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(x) \cup u')$$

defined by u'. More precisely, write

$$u = \sum_i a_i \otimes b_i$$

with $a_i \in H^1(X_{\text{ét}}, \mathbb{Z}/m)$ and $b_i \in H^1(X_{\text{ét}}, \mathbb{Z}/m(1)_{\text{ét}})$. Then

$$u' = \sum_i a_i \times b_i$$

and

$$\begin{aligned}
\mathrm{Tr}'(x \cup' u) = & \sum_i \mathrm{Tr}'(x \cup a_i \otimes b_i) \\
= & \sum_i \mathrm{Tr}(x \cup a_i) b_i \\
\overset{B.14}{=} & \sum_i \mathrm{pr}_{2,*}((x \cup a_i) \times b_i) \\
= & \sum_i \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(x \cup a_i) \cup \mathrm{pr}_2^*(b_i)) \\
= & \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(x) \cup u').
\end{aligned}$$

Let $c := \mathrm{cl}(\Delta) \in H^2_{\text{ét}}(X \times X, \mathbb{Z}/m(1)_{\text{ét}})$ be the class of the diagonal. By lemma 13.5 the correspondence

$$\Phi_c(x) := \mathrm{pr}_{2,*}(\mathrm{pr}^1_*(x) \cup c)$$

with $x \in H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$ depends only on the $(1,1)$-component $c^{1,1}$ and therefore agrees with $\Phi_{u'}$ by proposition 13.4. Moreover the class $c = \Delta_*(1)$ is the pushforward of $1 \in H^0_{\text{ét}}(X, \mathbb{Z}/m)$ along the diagonal $\Delta\colon X \to X \times X$ (in the case $m = p^r$ cf. [Gro85, Chapitre 2, Proposition 2.5.1]) and we can use the projection formula (cf. [Gro85, Chapitre 2, Corollaire 2.2.5] if $m = p^r$) to conclude

$$\begin{aligned}
\mathrm{pr}_{2,*}(\mathrm{pr}^1_*(x) \cup c) = & \ \mathrm{pr}_{2,*}(\mathrm{pr}^1_*(x) \cup \Delta_*(1)) \\
= & \ \mathrm{pr}_{2,*}(\Delta_*(\Delta^*(\mathrm{pr}_1^*(x)) \cup 1)) \\
= & \ \mathrm{Id}_{X,*}\mathrm{Id}_X^*(x) \\
= & \ x
\end{aligned}$$

for $x \in H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}})$. $\square$

14. Comparison of the geometric and motivic reciprocity law

In this section we derive our main theorem. We assume for the whole section that our ground field k is algebraically closed. By now, we have for $X \in \mathrm{Sm}/k$ smooth, projective and any $m \in \mathbb{Z}$ presented the construction of two natural pairings

$$\Phi'_{\text{geom}} \colon H_1^S(X, \mathbb{Z}/m) \times H^1_{\text{ét}}(X, \mathbb{Z}/m) \to \mathbb{Z}/m,$$

proposition 7.1, and

$$\Phi'_{\text{mot}} \colon H_1^S(X, \mathbb{Z}/m) \times H^1_{\text{ét}}(X, \mathbb{Z}/m) \to \mathbb{Z}/m,$$

proposition 12.3, giving rise, by theorem 7.2 and definition 12.3, to "Hurewicz" isomorphisms

$$\Phi_{\text{geom}} \colon H_1^S(X, \mathbb{Z}/m) \to H^1_{\text{ét}}(X, \mathbb{Z}/m)^\vee \cong \pi_1^{\text{ab}}(X)/m$$

and

$$\Phi_{\text{mot}} \colon H_1^S(X, \mathbb{Z}/m) \to H^1_{\text{ét}}(X, \mathbb{Z}/m)^\vee \cong \pi_1^{\text{ab}}(X)/m,$$

again both natural in X. It is our aim in this section to prove that these two isomorphisms agree (up to a sign) for every X. We formulate this as a theorem.

Theorem 14.1. *For every smooth projective scheme X over k the reciprocity laws Φ_{geom} and Φ_{mot} coincide.*

The first step to prove theorem 14.1 is the reduction to the case that X is a smooth projective curve, cf. lemma 14.3. The curve case can then be analyzed directly, using the description of Φ_{geom} given in theorem 7.3.

For every $X \in \mathrm{Sch}/k$ the homology group $H_1^S(X, \mathbb{Z}/m)$ is defined using one-dimensional data, namely correspondences $\Delta^1 \to X$, so the following proposition is not suprising. However, its proof is unfortunately indirect.

Proposition 14.2. *For a smooth projective scheme $X \in \mathrm{Sm}/k$ the map*

$$\bigoplus_{f \colon C \to X} H_1^S(C, \mathbb{Z}/m) \xrightarrow{\sum f_*} H_1^S(X, \mathbb{Z}/m)$$

is surjective where the left sum is taken over all finite maps $f \colon C \to X$ of smooth projective curves $C \in \mathrm{Sm}/k$.

Proof. [GS, Proposition 5.6] implies that the map

$$H^1_{\text{ét}}(X, \mathbb{Z}/m) \to \prod_{f : C \to X} H^1_{\text{ét}}(C, \mathbb{Z}/m)$$

is injective. By naturality of either Φ_{geom} or Φ_{mot} it identifies with the dual of

$$\bigoplus_{f \colon C \to X} H_1^S(C, \mathbb{Z}/m) \overset{\sum f}{\to} H_1^S(X, \mathbb{Z}/m)$$

as $\mathbb{Z}/m$-modules. The functor $\mathrm{Hom}(-, \mathbb{Z}/m)$ is exact and faithful on the category of $\mathbb{Z}/m$-modules (in other words $\mathrm{Hom}(M, \mathbb{Z}/m) = 0$ only if $M = 0$), so we can conclude that the morphism

$$\bigoplus_{f \colon C \to X} H_1^S(C, \mathbb{Z}/m) \overset{\sum f}{\to} H_1^S(X, \mathbb{Z}/m)$$

is surjective. $\square$

We obtain the following lemma.

Lemma 14.3. *Assume that Φ_{geom} and Φ_{mot} agree for every smooth projective curve C, then they agree for all smooth projective X.*

Proof. By proposition 14.2 in the commutative diagram

$$\begin{tikzcd} \bigoplus_{f\colon C\to X} H_1^S(C,\mathbb{Z}/m) \arrow[r, "\Phi_{\text{geom}}-\Phi_{\text{mot}}=0"] \arrow[d] & \bigoplus_{f\colon C\to X} H^1_{\text{ét}}(C,\mathbb{Z}/m)^\vee \arrow[d] \\ H_1^S(X,\mathbb{Z}/m) \arrow[r, "\Phi_{\text{geom}}-\Phi_{\text{mot}}"] & H^1_{\text{ét}}(X,\mathbb{Z}/m)^\vee \end{tikzcd}$$

the left vertical arrow is an epimorphism. By assumption the top horizontal arrow is zero and so the composition around the left corner is zero, too. Hence $\Phi_{\text{geom}} = \Phi_{\text{mot}}$ for X because the left vertical arrow is surjective. $\square$

By lemma 14.3 we are left with showing that Φ_{geom} and Φ_{mot} agree in the case $X = C$ is a smooth projective curve, which we can even assume to be connected. Recall that in the case X is a smooth projective curve the diagram

$$\begin{tikzcd} H_1^S(X,\mathbb{Z}/m) \times H^1_{\text{ét}}(X,\mathbb{Z}/m) \arrow[r, "\Phi'_{\text{geom}}"] \arrow[d, "c_1^{-1}\circ\alpha\times\text{alb}"] & \mathbb{Z}/m \arrow[d, equal] \\ {}_m\text{Pic}(X) \times \text{Hom}({}_m\text{Pic}(X),\mathbb{Z}/m) \arrow[r, "\text{eval}"] & \mathbb{Z}/m \end{tikzcd} \tag{14.1}$$

commutes by theorem 7.3.
Moreover, by theorem 13.2 the diagram

$$\begin{tikzcd} H^1_{\text{ét}}(X,\mathbb{Z}/m(1)_{\text{ét}}) \times H^1_{\text{ét}}(X,\mathbb{Z}/m) \arrow[r, "\Phi'_{\text{PD}}"] \arrow[d, "\delta_{\text{Kum}}\times\text{alb}"] & \mathbb{Z}/m \arrow[d, equal] \\ {}_m\text{Pic}(X) \times \text{Hom}({}_m\text{Pic}(X),\mathbb{Z}/m) \arrow[r, "\text{eval}"] & \mathbb{Z}/m \end{tikzcd} \tag{14.2}$$

commutes.
Let $\nu\colon H_1^S(X,\mathbb{Z}/m) \to H^1(X_{\text{ét}},\mathbb{Z}/m(1)_{\text{ét}})$ be the isomorphism appearing in the the motivic reciprocity law, cf. definition 12.3.
We can now prove theorem 14.1.

Proof of theorem 14.1. By lemma 14.3 we only have to show the statement for X a smooth projective curve. Looking at the diagrams (14.1) and (14.2) one recognizes that in this case it is enough to prove that the morphisms

$$c_1^{-1}\circ\alpha\colon H_1^S(X,\mathbb{Z}/m) \to {}_m\text{Pic}(X)$$

with α induced from definition 8.5 and

$$\delta_{\text{Kum}}\circ\nu\colon H_1^S(X,\mathbb{Z}/m) \to {}_m\text{Pic}(X)$$

with δ_{Kum} from proposition 13.1 and ν from definition 12.3 agree, i.e., that the diagram

$$\begin{tikzcd} H_1^S(X, \mathbb{Z}/m) \arrow[r, "\nu"] \arrow[d, "\alpha"] & H^1_{\text{ét}}(X, \mathbb{Z}/m(1)_{\text{ét}}) \arrow[d, "\delta_{\text{Kum}}"] \\ {}_m\mathrm{CH}^1(X) \arrow[r, "c_1^{-1}"] & {}_m\mathrm{Pic}(X) \end{tikzcd}$$

commutes. Composing with the inclusion ${}_m\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X)$ shows that it suffices to show that the diagram

$$\begin{tikzcd} H_0^S(X, \mathbb{Z}) \arrow[r, "\nu'"] \arrow[d, "\alpha"] & H^2(X_{\text{ét}}, \mathbb{Z}(1)_{\text{ét}}) \arrow[d, "\cong"] \\ \mathrm{CH}^1(X) \arrow[r, "c_1^{-1}"] & \mathrm{Pic}(X) \end{tikzcd}$$

commutes, where ν' denotes the morphism from definition 12.2. We now factor the morphism ν' over the group $H^2_{\text{Zar}}(X, \mathbb{Z}^{\text{SF}}(1))$ and split our diagram into two. The resulting diagrams are

$$\begin{tikzcd} H_0^S(X, \mathbb{Z}) \arrow[r, "\eta\circ\alpha"] \arrow[d, "\alpha"] & H^2_{\text{Zar}}(X, \mathbb{Z}^{\text{SF}}(1)) \arrow[dl, "\eta"] \\ \mathrm{CH}^1(X) & \end{tikzcd}$$

where $\eta\colon \mathrm{CH}^1(X) \cong H^2_{\text{Zar}}(X, \mathbb{Z}^{\text{SF}}(1))$ denotes the isomorphism from definition 9.17 and

$$\begin{tikzcd} H^2_{\text{Zar}}(X, \mathbb{Z}^{\text{SF}}(1)) \arrow[r, "\mathrm{cl}\circ\theta"] \arrow[d, "\eta"] & H^2_{\text{ét}}(X, \mathbb{Z}(1)_{\text{ét}}) \arrow[d, "\cong"] \\ \mathrm{CH}^1(X) \arrow[r, "c_1^{-1}"] & \mathrm{Pic}(X) \end{tikzcd}$$

with $\theta\colon H^2_{\text{Zar}}(X, \mathbb{Z}^{\text{SF}}(1)) \cong H^2_{\text{Zar}}(X, \mathbb{Z}(1))$ the isomorphism obtained form proposition 9.8 and cl the cycle class map. The first diagram commutes trivially while the second one commutes by proposition 9.12 and proposition 9.19. □

In remains to prove proposition 13.4 and proposition B.12. This will be done in two appendices.

Appendix A. The Künneth components of the diagonal

We will now start proving proposition 13.4.
Fix $m \in \mathbb{Z}$. As a shorthand we denote by

$$H^{i,j}(-) := H^i_{\text{ét}}(-, \mathbb{Z}/m(j)_{\text{ét}})$$

the bigraded cohomology theory on Sm/k with coefficients in the étale sheaves $\mathbb{Z}/m(j)_{\text{ét}}$ from definition 10.3.
Let X be a connected smooth projective curve over the algebraically closed field k. Let p be the exponential characteristic of k, i.e., $p = 1$ if $\mathrm{char}(k) = 0$ and

$p = \mathrm{char}(k)$ otherwise. Consider the diagram

$$\begin{tikzcd} X \times X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & \mathrm{Spec}(k). \end{tikzcd}$$

We will first prove a Künneth decomposition

$$\begin{aligned} &H^{2,1}(X \times X) \\ &\cong H^{0,0}(X) \otimes H^{2,1}(X) \oplus H^{1,0}(X) \otimes H^{1,1}(X) \oplus H^{2,1}(X) \oplus H^{0,0}(X) \end{aligned}$$

for arbitrary $m \in \mathbb{Z}$. For $(m, p) = 1$ the statement is well-known and follows formally using base change from the isomorphism

$$\mathrm{pr}_1^*(\mathbb{Z}/m(i)_{\text{ét}}) \otimes^L_{\mathbb{Z}/m} \mathrm{pr}_2^*(\mathbb{Z}/m(j)) \cong \mathbb{Z}/m(i+j)_{\text{ét}}$$

and the projection formula (cf. the remark below definition B.7 for its abstract analogue). In the case $m = p^r$ (and $p > 1$) we are facing the problem that in this case

$$\mathrm{pr}_1^*(\mathbb{Z}/p^r(j)_{\text{ét}}) \otimes^L_{\mathbb{Z}/p^r} \mathrm{pr}_2^*(\mathbb{Z}/p^r(i)_{\text{ét}}) \ncong \mathbb{Z}/p^r(i+j)_{\text{ét}}$$

for $i, j \in \mathbb{Z}$.

Lemma A.1. *For every $m \in \mathbb{Z}$ the natural morphisms*

$$\mathbb{Z}/m(1)_{\text{ét}} \to R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})$$

and

$$R\mathrm{pr}_{1,*}(\mathrm{pr}_2^*(\mathbb{Z}/m(1)_{\text{ét}})) \to R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})$$

define an isomorphism

$$\mathbb{Z}/m(1)_{\text{ét}} \oplus R\mathrm{pr}_{1,*}(\mathrm{pr}_2^*(\mathbb{Z}/m(1)_{\text{ét}})) \cong R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}).$$

Proof. Results on the étale cohomological dimension of schemes imply that

$$R^i\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) = 0$$

for $i \geq 3$. We now use the distinguished triangle

$$R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) \xrightarrow{m} R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) \to R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \to R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}})[1].$$

Clearly, we have

$$\begin{aligned} R^0\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) &= 0 \\ R^1\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) &= \mathbb{G}_m \\ R^2\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) &= \mathrm{Pic}_X. \end{aligned}$$

Moreover

$$R^i\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) = 0, \text{ for } i \geq 3$$

by [Gro68, Corollaire 3.2]. Now the claim follows from the computation of $H^{*,1}(X)$ (cf. proposition 6.6) as

$$R\mathrm{pr}_{1,*}(\mathrm{pr}_2^*\mathbb{Z}/m(1)_{\text{ét}}) \cong f^*Rf_*(\mathbb{Z}/m(1)_{\text{ét}})$$

by the proper base change theorem. More precisely, there are short exact sequences (of sheaves on the small étale site of X)

$$\begin{gathered} 0 \to \mu_m \to \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m \to R^1\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \to {}_m\mathrm{Pic}(X) \to 0 \\ 0 \to \mathrm{Pic}_X/m\mathrm{Pic}_X \cong \mathbb{Z}/m \to R^2\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \to 0. \end{gathered}$$

We can conclude that $R^2\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \cong \mathbb{Z}/m$ and thus that the pullback

$$R^2\mathrm{pr}_{1,*}(\mathrm{pr}_2^*(\mathbb{Z}/m(1)_{\text{ét}})) \cong R^2\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})$$

is an isomorphism because a generator of $H^{2,1}(X)$, i.e., the class of a point $x \in X(k)$, is mapped under the pullback to the class of the line bundle $\mathcal{O}_X(x) \in \mathrm{Pic}(X)$. In degree 1 there is a short exact sequence

$$0 \to K \to R^1\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \to {}_m\mathrm{Pic}(X) \to 0$$

with $K = 0$ if m is invertible in k and $K \cong \nu_r(1)$ if $m = p^r$ (and $p > 1$). But

$$R^1\mathrm{pr}_{1,*}(\mathrm{pr}_2^*(\mathbb{Z}/m(1)_{\text{ét}})) \cong {}_m\mathrm{Pic}(X)$$

by proper base change with ${}_m\mathrm{Pic}(X)$ considered as a constant sheaf on $X_{\text{ét}}$,[6] thus the pullback

$$R^1\mathrm{pr}_{1,*}(\mathrm{pr}_2^*(\mathbb{Z}/m(1)_{\text{ét}})) \cong R^1\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})$$

splits this short exact sequence proving the claim because the morphism

$$\mu_m \to R^0\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})$$

is an isomorphism for every $m \in \mathbb{Z}$. $\square$

To prove the Künneth decomposition we note that there is a morphism

$$\begin{aligned}\times\colon & H^{0,0}(X) \otimes H^{2,1}(X) \oplus H^{1,0}(X) \otimes H^{1,1}(X) \oplus H^{2,1}(X) \oplus H^{0,0}(X)\\ & \to H^{2,1}(X \times X)\end{aligned}$$

given by the exterior product

$$a \times b = \mathrm{pr}_1^*(a) \cup \mathrm{pr}_2^*(b).$$

Proposition A.2. *The exterior product*

$$\begin{aligned}\times\colon & H^{0,0}(X) \otimes H^{2,1}(X) \oplus H^{1,0}(X) \otimes H^{1,1}(X) \oplus H^{2,1}(X) \oplus H^{0,0}(X)\\ & \to H^{2,1}(X \times X)\end{aligned}$$

is an isomorphism, i.e., the group $H^{2,1}(X \times X)$ admits a Künneth decomposition.

Proof. We will prove this proposition using the Leray spectral sequence

$$E_2^{pq} = H^p(X, R^q\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})) \Rightarrow H^{p+q,1}(X \times X) \tag{A.1}$$

for the first projection $\mathrm{pr}_1\colon X \times X \to X$. We may use lemma A.1. It implies that the complex $R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})$ is formal, i.e., isomorphic to the complex of its cohomology sheaves. In particular, the spectral sequence (A.1) degenerates defining a three step filtration $\mathrm{Fil}^\bullet$ on $H^{2,1}(X \times X)$ whose graded pieces are

$$\begin{aligned}\mathrm{gr}^0(H^{2,1}(X \times X)) &\cong H^0_{\text{ét}}(X, R^2\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}))\\ \mathrm{gr}^1(H^{2,1}(X \times X)) &\cong H^1_{\text{ét}}(X, R^1\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}))\\ \mathrm{gr}^2(H^{2,1}(X \times X)) &\cong H^2_{\text{ét}}(X, R^0\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})).\end{aligned}$$

Now we distinguish two cases. First assume that m is invertible in k. Then the sheaf $\mathbb{Z}/m(1)_{\text{ét}}$ on $X \times X$ is isomorphic to the pullback $\mathrm{pr}_2^*(\mathbb{Z}/m(1)_{\text{ét}})$ and we can use the proper base change theorem together with the freeness of the cohomology

[6] We remark that the constant sheaf ${}_m\mathrm{Pic}(X)$ coincides, as a sheaf on the *small* étale site of X, with the sheaf represented by the non-reduced group scheme ${}_m\mathrm{Pic}_X$ given by the m-torsion in Pic_X.

$H^{*,1}(X)$ as a $\mathbb{Z}/m$-module (this freeness follows from proposition 6.6 and the known structure of torsion in abelian varietes) to conclude that for $i+j=2$

$$H^i_{\text{ét}}(X, R^j\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})) \cong H^{i,0}(X) \otimes H^{j,1}(X).$$

The exterior product maps each direct summand in

$$H^{0,0}(X) \otimes H^{2,1}(X) \oplus H^{1,0}(X) \otimes H^{1,1}(X) \oplus H^{2,1}(X) \oplus H^{0,0}(X)$$

isomorphically to a graded piece of the filtration $\mathrm{Fil}^\bullet$ on $H^{2,1}(X \times X)$, thereby proving the Künneth decomposition (note that $H^{2,0}(X) \otimes H^{0,1}(X) \cong H^{2,1}(X) \otimes H^{0,0}(X)$ in this case). Now assume that $m = p^r$ and $p > 1$. Then

$$\begin{array}{rcl} R^0\mathrm{pr}_{1,*}(\mathbb{Z}/p^r(1)_{\text{ét}}) & \cong & 0 \\ R^1\mathrm{pr}_{1,*}(\mathbb{Z}/p^r(1)_{\text{ét}}) & \cong & \nu_r(1) \oplus f^*H^{1,1}(X) \\ R^2\mathrm{pr}_{1,*}(\mathbb{Z}/p^r(1)_{\text{ét}}) & \cong & f^*H^{2,1}(X) \end{array}$$

by lemma A.1. In particular, we see that the exterior product

$$\times\colon H^{2,1}(X) \otimes H^{0,0}(X) \oplus H^1(X,0) \otimes H^{1,1}(X) \to H^{2,1}(X \times X)$$

maps isomorphically onto $\mathrm{Fil}^1(H^{2,1}(X \times X))$. Note that $\mathrm{Fil}^2(H^{2,1}(X \times X)) = 0$ and that $H^{1,1}(X)$ is a free $\mathbb{Z}/p^r$-module and thus

$$H^1_{\text{ét}}(X, f^*H^{1,1}(X)) \cong H^{1,0}(X) \otimes H^{1,1}(X).$$

Moreover, the summand $H^{0,0}(X) \otimes H^{2,1}(X)$ maps under the exterior product isomorphically to the remaining graded piece $\mathrm{gr}^2(H^{2,1}(X \times X))$. This finishes the proof. □

Let now $m \in \mathbb{Z}$ be an arbitrary integer. Let $\Delta\colon X \to X \times X$ be the diagonal and let $c := \mathrm{cl}(\Delta) \in H^{2,1}(X \times X)$ be its class. We can write

$$c = c^{2,0} + c^{1,1} + c^{0,2}$$

in Künneth components. As c has degree 1 with respect to pr_1 and pr_2 the components $c^{2,0}$ resp. $c^{0,2}$ are given by the classes of the divisors $x \times X$ resp. $X \times x$ for some closed point $x \in X(k)$. Our goal is now to describe the remaining component

$$c^{1,1} \in H^{1,0}(X) \otimes H^{1,1}(X).$$

We have

$$\begin{gathered} H^{1,0}(X) \otimes H^{1,1}(X) \\ \cong \mathrm{Hom}({}_m\mathrm{Pic}(X), \mathbb{Z}/m) \otimes {}_m\mathrm{Pic}(X) \cong \mathrm{Hom}({}_m\mathrm{Pic}(X), {}_m\mathrm{Pic}(X)) \end{gathered}$$

and thus we get a class $u \in H^{1,0}(X) \otimes H^{1,1}(X)$ corresponding to the identity $\mathrm{Id}\colon {}_m\mathrm{Pic}\,(X) \to {}_m\mathrm{Pic}(X)$ under the above isomorphism. Let

$$f_0\colon X \to \mathrm{Pic}_X, x \mapsto \mathcal{O}_X(x - x_0)$$

be the Albanese morphism with respect to some fixed point $x_0 \in X(k)$. Then the element

$$u \in H^{1,0}(X) \otimes H^{1,1}(X) \cong H^1_{\text{ét}}(X, {}_m\mathrm{Pic}(X))$$

can also be described as the pullback of the ${}_m\mathrm{Pic}(X)$-torsor

$$\ker(\mathrm{Pic}^0_X \xrightarrow{m} \mathrm{Pic}^0_X)$$

along f_0. In other words, if

$${}_m\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}^0_X \xrightarrow{m} \mathrm{Pic}^0_X \xrightarrow{\delta_P} {}_m\mathrm{Pic}(X)[1]$$

denotes the natural distinguished triangle of abelian sheaves on the small étale site of X, then

$$u = \delta_P(f_0) \in H^1_{\text{ét}}(X, {}_m\mathrm{Pic}(X))$$

with $f_0 \in H^0(X, \mathrm{Pic}^0_X)$. Our final goal is now to prove that $c^{1,1} = u$. More precisely, $c^{1,1} = u'$ with u' the image of u under the exterior product

$$H^{1,0}(X) \otimes H^{1,1}(X) \to H^{2,1}(X \times X).$$

In other words, we prove proposition 13.4.

Proof of proposition 13.4. We have our fixed k-rational point $x_0 \in X(k)$ and define

$$\mathcal{O}_{X\times X}(E) := \mathcal{O}_{X\times X}(\Delta - X \times x_0) \in \mathrm{Pic}(X \times X)$$

with class $e := \mathrm{cl}(\mathcal{O}_{X\times X}(E))$. It has the same $(1,1)$-component as c, i.e.,

$$c^{1,1} = e^{1,1}.$$

Indeed, c and e differ by

$$\mathrm{cl}(\mathcal{O}_{X\times X}(X \times x_0)) = \mathrm{pr}_2^*(\mathrm{cl}(\mathcal{O}_X(x_0)),$$

which is of type $(0,2)$, i.e., lies in $H^{0,0}(X) \otimes H^{2,1}(X) \subseteq H^{2,1}(X \times X)$. The composition

$$H^2_{\text{ét}}(X \times X, \mathbb{Z}(1)_{\text{ét}}) \to H^{2,1}(X \times X) \to H^0(X, R^2\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})) \cong \mathbb{Z}/m$$

is given by sending a divisor $D \subseteq X$ to the relative degree mod m with respect to the projection $\mathrm{pr}_1 \colon X \times X \to X$. In particular, if Fil denotes the filtration induced by the Leray spectral sequence (cf. (A.1)) for pr_1, then e lies in $\mathrm{Fil}^1(H^{2,1}(X \times X))$, which is the image of

$$H^2_{\text{ét}}(X, \tau_{\leq 1} R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}))$$

in $H^{2,1}(X \times X) \cong H^2(X, R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}))$ under the canonical morphism

$$\tau_{\leq 1} R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \to R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}).$$

Fix a short exact sequence $0 \to A \to B \to C \to 0$ of complexes of abelian sheaves on $X_{\text{ét}}$ and an isomorphism

$$\begin{tikzcd}
R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})[-1] \arrow[r] \arrow[d, "\cong"] & R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) \arrow[r, "m"] \arrow[d, "\cong"] & R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) \arrow[r] \arrow[d, "\cong"] & R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \arrow[d, "\cong"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1].
\end{tikzcd}$$

As the truncated complex $\tau_{\leq 2} C$ surjects onto $R^2\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) \cong \mathrm{Pic}_X$ we can talk about the preimage C' of Pic^0_X in $\tau_{\leq 2} C$. Let B' be the preimage of C' in $\tau_{\leq 2} B$. By the divisibility of Pic^0_X we obtain a short exact sequence

$$0 \to \tau_{\leq 2} A \to B' \to C' \to 0$$

of complexes. Using the morphism

$$R^1\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \to {}_m\mathrm{Pic}(X)$$

from lemma A.1 (which is an isomorphism if $(m,p)=1$) we get a commutative diagram

$$\begin{tikzcd}
R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})[-1] \arrow[r] \arrow[d, "\cong"] & R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) \arrow[r, "m"] \arrow[d, "\cong"] & R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) \arrow[r, "\mathrm{cl}"] \arrow[d, "\cong"] & R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}}) \arrow[d, "\cong"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \\
\tau_{\leq 2}A \arrow[r] \arrow[u] \arrow[d] & B' \arrow[r] \arrow[u] \arrow[d] & C' \arrow[r] \arrow[u] \arrow[d] & \tau_{\leq 2}A[1] \arrow[u] \arrow[d] \\
{}_m\mathrm{Pic}(X)[-2] \arrow[r] & \mathrm{Pic}^0_X[-2] \arrow[r] & \mathrm{Pic}^0_X[-2] \arrow[r, "\delta_P"] & {}_m\mathrm{Pic}(X)[-1]
\end{tikzcd}$$

in the derived category of étale sheaves on X. Now, we apply $H^2_{\text{ét}}(X,-)$ and do a diagram chase in the resulting diagram

$$\begin{tikzcd}
\mathrm{Pic}(X\times X) \cong H^2(X, R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}})) \arrow[r, "\mathrm{cl}"] & H^{2,1}(X\times X) \\
H^2_{\text{ét}}(X, C') \arrow[r] \arrow[u] \arrow[d] & H^3_{\text{ét}}(X, \tau_{\leq 2}A) \arrow[u] \arrow[d] \\
H^0_{\text{ét}}(X, \mathrm{Pic}^0_X) = H^2_{\text{ét}}(X, \mathrm{Pic}^0_X[-2]) \arrow[r, "\delta_P"] & H^1_{\text{ét}}(X, {}_m\mathrm{Pic}(X)).
\end{tikzcd}$$

The lower right vertical arrow maps an element in

$$H^3_{\text{ét}}(X, \tau_{\leq 2}A) \cong H^2_{\text{ét}}(X, \tau_{\leq 1}R\mathrm{pr}_{1,*}(\mathbb{Z}/m(1)_{\text{ét}})) \cong \mathrm{Fil}^1(H^{2,1}(X\times X))$$

to its $(1,1)$-component in $H^1_{\text{ét}}(X, \mathrm{Pic}(X)_m) \cong H^{1,0}(X) \otimes H^{1,1}(X)$. The element

$$\mathcal{O}_{X\times X}(E) \in H^2_{\text{ét}}(X, R\mathrm{pr}_{1,*}(\mathbb{Z}(1)_{\text{ét}}) \cong H^2_{\text{ét}}(X\times X, \mathbb{Z}(1)_{\text{ét}})$$

has relative degree 0 along pr_1 and thus admits a preimage

$$z \in H^2(X, C')$$

by the definition of C'. Furthermore, the element z maps to f_0 in

$$H^2(X, \mathrm{Pic}^0_X[-2]) = H^0(X, \mathrm{Pic}^0_X)$$

by definition of $E = \Delta - X\times x_0$ and $f_0\colon X \to \mathrm{Pic}_X, x \mapsto \mathcal{O}_X(x-x_0)$. In particular, we obtain that the $(1,1)$-component of $e = \mathrm{cl}(\mathcal{O}_{X\times X}(E))$ is given by

$$\delta_P(f_0) = u,$$

which was our claim. Thus we have proven proposition 13.4. $\square$

We remark that our proof is basically Deligne's proof in [Del77, Dualité].

Appendix B. Formal properties of adjoints

The goal of this appendix is to prove the compatibility proposition B.14 of trace maps. The proof is entirely formal, except for the multiplicativity of the trace map (cf. proposition B.14), and can be seen as a discussion about cup–products in Grothendieck's six functor formalism. Therefore we put ourselves in an abstract context, namely adjoint functors between symmetric monoidal categories. We are going to use the following notations.

- We will write $F \star \eta$ for the composition of a functor F and a natural transformation η. Similarly, $\eta \star F$.
- In symmetric monoidal categories we write $\otimes$ for the monoidal product.
- By 1 we denote the identity (of an object, of a category etc.).
- We neglect the commutativity constraint $\sigma\colon A \otimes B \cong B \otimes A$. For example, if $a\colon A \otimes B \to C$ is a morphism, we use the same letter $a\colon B \otimes A \to C$ for $a \circ \sigma$.

First we will define the cup-product in this abstract setting. Consider a functor $f_*\colon \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories which has a monoidal left adjoint f^*. To save notations we suppress the domain and the range of the functor f_*. The basic example we have in mind is that f_* is the pushforward of a morphism of topoi $f\colon X \to Y$.
The equality, strictly speaking the isomorphism,

$$f^*(A \otimes B) = f^*(A) \otimes f^*(B)$$

has many consequences, as we will see. Let

$$\begin{aligned} \varepsilon_f &\colon 1 \to f_* f^* \\ \kappa_f &\colon f^* f_* \to 1 \end{aligned}$$

be the unit and counit of the adjunction $f^* \dashv f_*$.

Definition B.1. We define the cup-product $\cup\colon f_*(A) \otimes f_*(B) \to f_*(A \otimes B)$ as the morphism

$$\cup\colon f_*(A) \otimes f_*(B) \overset{\varepsilon_f}{\to} f_* f^*(f_* A \otimes f_* B) = f_*(f^* f_* A \otimes f^* f_* B) \overset{f_*(\kappa_f \otimes \kappa_f)}{\to} f_*(A \otimes B),$$

in other words, as the morphism adjoint to

$$\kappa_f \otimes \kappa_f\colon f^* f_* A \otimes f^* f_* B \overset{\kappa_f \otimes \kappa_f}{\to} A \otimes B$$

We will often consider adjoints of morphisms. The following fact, already usable above, might simplify the reading:
If

$$A \overset{\varphi}{\to} f_*(B) \overset{f_*(\psi)}{\to} f_*(C)$$

is a morphism and $\varphi'\colon f^* A \to B$ the adjoint of φ, then $\psi \circ \varphi'$ is the adjoint of $f_*(\psi) \circ \varphi$.

Definition B.2. Define now the morphism in the projection formula as

$$\alpha_f\colon f_*(A) \otimes B \overset{1 \otimes \varepsilon_f}{\to} f_*(A) \otimes f_* f^*(B) \overset{\cup}{\to} f_*(A \otimes f^*(B)).$$

We record some properties of α_f.

Lemma B.3. *The diagram*

$$\begin{tikzcd} f_*A \otimes B \arrow[r, "\varepsilon_f"] \arrow[dr, "\alpha_f"'] & f_*(f^*f_*A \otimes f^*B) \arrow[d, "f_*(\kappa_f \otimes 1)"] \\ & f_*(A \otimes f^*B) \end{tikzcd}$$

commutes, i.e., the adjoint morphism of α_f *is the morphism*

$$f^*f_*A \otimes f^*B \xrightarrow{\kappa_f \otimes 1} A \otimes f^*B$$

Proof. The adjoint of α_f is given as the composition

$$f^*f_*A \otimes f^*B \overset{1 \otimes f^* \star \varepsilon_f}{\longrightarrow} f^*f_*A \otimes f^*f_*f^*B \overset{\kappa_f \otimes \kappa_f \star f^*}{\longrightarrow} A \otimes f^*B$$

by the description of the adjoint of $\cup$. But

$$(\kappa_f \star f^*) \circ (f^* \star \varepsilon_f) = 1$$

by the triangle equality for adjunctions. □

Lemma B.4. *The diagram*

$$\begin{tikzcd} f_*A \otimes f_*B \arrow[r, "\cup"] \arrow[d, "\alpha_f"] & f_*(A \otimes B) \\ f_*(A \otimes f^*f_*B) \arrow[ur, "f_*(1 \otimes \kappa_f)"'] & \end{tikzcd}$$

commutes, i.e., α_f *determines the cup-product.*

Proof. Taking adjoints yields by definition B.1 and lemma B.3 the diagram

$$\begin{tikzcd} f^*f_*A \otimes f^*f_*B \arrow[r, "\kappa_f \otimes \kappa_f"] \arrow[d, "\kappa_f \otimes 1"] & A \otimes B \\ A \otimes f^*f_*B \arrow[ur, "1 \otimes \kappa_f"'] & \end{tikzcd}$$

which commutes. □

Lemma B.5. *The diagram*

$$\begin{tikzcd} f_*A \otimes B \arrow[r, "1 \otimes \varepsilon_f"] \arrow[d, "\alpha_f"] & f_*A \otimes f_*f^*B \arrow[d, "\alpha_f"] \arrow[dl, "\cup"'] \\ f_*(A \otimes f^*B) & f_*(A \otimes f^*f_*f^*B) \arrow[l, "f_*(1 \otimes \kappa_f \star f^*)"] \end{tikzcd}$$

commutes.

Proof. The upper triangle commutes by the definition B.2 of α_f and the lower one by lemma B.4. □

Now, we assume furthermore that f_* has a right adjoint $f^!$ and that

$$\alpha_f \colon f_*(A) \otimes B \xrightarrow{\sim} f_*(f^*(A) \otimes B)$$

is an isomorphism, i.e., the projection formula holds for f. We will use abuse of notations to write

$$\kappa_f\colon f_*f^! \to 1$$
$$\varepsilon_f\colon 1 \to f^! f_*$$

for the counit resp. unit of the adjunction $f_* \dashv f^!$.

Definition B.6. We define the cup product

$$\cup'\colon f^! A \otimes f^* B \to f^!(A \otimes B)$$

for $f^!$ as

$$\cup'\colon f^! A \otimes f^* B \xrightarrow{\varepsilon_f} f^! f_*(f^* A \otimes f^! B) \xrightarrow{f^! \star \alpha_f^{-1}} f^!(A \otimes f_* f^! B) \xrightarrow{f^!(1 \otimes \kappa_f)} f^!(A \otimes B).$$

In particular, it is adjoint to the morphism

$$f_*(f^* A \otimes f^! B) \xrightarrow{\alpha_f^{-1}} A \otimes f_* f^! B \xrightarrow{1 \otimes \kappa_f} A \otimes B.$$

Now, consider a diagram

$$\begin{tikzcd} \bullet \arrow[r, "q"] \arrow[d, "p"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$

is given. Each dot denotes a symmetric monoidal category and each letter denotes an adjoint triple, that is adjunctions

$$f^* \dashv f_* \dashv f^!$$
$$p^* \dashv p_* \dashv p^!$$
$$g^* \dashv g_* \dashv g^!$$
$$q^* \dashv q_* \dashv q^!$$

with f, p resp. g, q “composable” and f^*, g^*, p^*, q^* monoidal. The example that we have in mind is that of a diagram of morphisms of topoi and the pushforward resp. pullback functors between their derived categories. Again we will write $\kappa_f, \kappa_g, \kappa_{fp}, \ldots$ resp. $\varepsilon_f, \varepsilon_f, \varepsilon_{fp}, ..$ for the counits resp. the units of these adjunctions. For example,

$$\kappa_{fp} = \kappa_f \circ (f^* \star \kappa_p \star f_*)\colon f^* p^* p_* f_* \to f^* f_* \to 1$$

and

$$\kappa_{fp} = \kappa_f \circ (f_* \star \kappa_p \star f^!)\colon f_* p_* p^! f^! \to f_* f^! \to 1.$$

We hope that this abuse of notation will not cause any confusion. Sometimes we will denote by h the “composition” $f \circ p$, i.e.

$$h_* = f_* p_*, h^* = p^* f^*, \ldots.$$

Assume that

$$\eta\colon g_* q_* \to f_* p_*$$

is a natural transformation. We get a natural transformation, deserved to be called the base change morphism,

$$\overline{\eta}\colon f^* g_* \to p_* q^*$$

as the morphism

$$\overline{\eta}\colon f^* g_* \xrightarrow{f^* g_* \star \varepsilon_q} f^* g_* q_* q^* \xrightarrow{f^* \star \eta \star q^*} f^* f_* p_* q^* \xrightarrow{\kappa_f \star p_* q^*} p_* q^*.$$

Definition B.7. We define the exterior product

$$\times\colon f_*A \otimes g_*B \to h_*(p^*A \otimes q^*B)$$

as the composition

$$f_*A \otimes g_*B \xrightarrow{\alpha_f} f_*(A \otimes f^*g_*B) \xrightarrow{f_*(1\otimes\overline{\eta})} f_*(A \otimes p_*q^*B) \xrightarrow{f_*(\alpha_p)} h_*(p^*A \otimes q^*B).$$

In particular, the exterior product is an isomorphism if the projection formula holds for f, p and furthermore the base change map $\overline{\eta}\colon f^*g_* \to p_*q^*$ is an isomorphism.

Lemma B.8. *The adjoint of $\times$ with respect to the f-adjunction $f^* \dashv f_*$ is given by*

$$f^*f_*A \otimes f^*g_*B \xrightarrow{\kappa_f\otimes\overline{\eta}} A \otimes p_*q^*B \xrightarrow{\alpha_p} p_*(p^*A \otimes q^*B).$$

Proof. The adjoint of $\alpha_f\colon f_*A \otimes C \to f_*(A \otimes f^*C)$ is given by

$$\kappa_f \otimes 1\colon f^*f_*A \otimes f^*C \to A \otimes f^*C.$$

This implies the claim. □

Lemma B.9. *The adjoint of $\times$ with respect to the h-adjunction $h^* \dashv h_*$ is given by*

$$h^*f_*A \otimes h^*g_*B \xrightarrow{p^*\star\kappa_f\otimes\overline{\eta}} p^*A \otimes p^*p_*q^*B \xrightarrow{1\otimes\kappa_p\star q^*} p^*A \otimes q^*B.$$

Proof. By B.8 the h-adjoint of $\times$ is the p-adjoint of

$$f^*f_*A \otimes f^*g_*B \xrightarrow{\kappa_f\otimes\overline{\eta}} A \otimes p_*q^*B \xrightarrow{\alpha_p} p_*(p^*A \otimes q^*B).$$

The p-adjoint of α_p is $1 \otimes \kappa_p$ by lemma B.3, hence the p-adjoint of the above morphism is

$$h^*f_*A \otimes h^*g_*B \xrightarrow{p^*\star\kappa_f\otimes\overline{\eta}} p^*A \otimes p^*p_*q^*B \xrightarrow{1\otimes\kappa_p\star q^*} p^*A \otimes q^*B$$

as desired. □

From now on we assume that the morphisms $\alpha_p\colon A \otimes p_*B \to p_*(p^*A \otimes B)$ and $\alpha_f\colon A \otimes f_*B \to f_*(f^*A \otimes B)$ are isomorphisms. By definition B.6 we obtain a cup-product $\cup'\colon f^*A \otimes f^!B \to f^!(A \otimes B)$. Furthermore, we assume that the base change map

$$\overline{\eta}\colon f^*g_* \xrightarrow{\sim} p_*q^*$$

is an isomorphism. Thus the exterior product is an isomorphism, too (cf. definition B.7).

Definition B.10. We define the product

$$\cup''\colon p^*f^!A \otimes q^*g^!B \to h^!(A \otimes B)$$

as the adjoint of the composition

$$\begin{array}{lll} & p_*(p^*f^!A \otimes q^*g^!B) & \xrightarrow{\alpha_p^{-1}} \quad f^!A \otimes p_*q^*g^!B \\ \xrightarrow{1\otimes\overline{\eta}^{-1}} & f^!A \otimes f^*g_*g^!B & \xrightarrow{\cup'} \quad f^!(A \otimes g_*g^!B) \xrightarrow{f^!(1\otimes\kappa_g)} f^!(A \otimes B). \end{array}$$

Lemma B.11. *The h-adjoint of the product $\cup''\colon p^*f^!A \otimes q^*g^!B \to h^!(A \otimes B)$ is given by*

$$h_*(p^*f^!A \otimes q^*g^!B) \xrightarrow{\times^{-1}} f_*f^!A \otimes g_*g^!B \xrightarrow{\kappa_f\otimes\kappa_g} A \otimes B$$

Proof. The p-adjoint of $\cup''$ is given by

$$\begin{array}{rlcl} & p_*(p^*f^!A \otimes q^*g^!B) & \xrightarrow{\alpha_p^{-1}} & f^!A \otimes p_*q^*g^!B \\ \xrightarrow{1\otimes\overline{\eta}^{-1}} & f^!A \otimes f^*g_*g^!B & \xrightarrow{\cup} & f^!(A \otimes g_*g^!B) \xrightarrow{f^!(1\otimes\kappa_g)} f^!(A \otimes B). \end{array}$$

We have to calculate the f-adjoint of this morphism. The adjoint of

$$\cup' \colon f^!A \otimes f^*C \to f^!(A \otimes C)$$

is

$$f_*(f^!A \otimes f^*C) \overset{\alpha_f^{-1}}{\to} f_*f^!A \otimes C \overset{\kappa_f \otimes 1}{\to} A \otimes C$$

(cf. definition B.6). Setting $C = g_*g^!B$ and using $(1 \otimes \kappa_g) \circ (\kappa_f \otimes 1) = \kappa_f \otimes \kappa_g$ then yields the claim. $\square$

Proposition B.12. *The diagram*

$$\begin{tikzcd} f_*f^!A \otimes g_*g^!B \arrow[r, "1\otimes\kappa_g"] \arrow[d, "\times"] & f_*f^!A \otimes B \arrow[d, "\alpha_f"] \\ h_*(p^*f^!A \otimes q^*g^!B) \arrow[d, "h_*(\cup'')"] & f_*(f^!A \otimes f^*B) \arrow[d, "f_*(\cup')"] \\ h_*(h^!(A \otimes B) \arrow[r, "f_*\star\kappa_p\star f^!"] & f_*f^!(A \otimes B) \end{tikzcd}$$

commutes.

Proof. Passing to adjoints for $f^* \dashv f_*$ yields the diagram

$$\begin{tikzcd} f^*f_*f^!A \otimes f^*g_*g^!B \arrow[r, "1\otimes f^*\star\kappa_g"] \arrow[d, "b"] \arrow[dr, dashed, "c"] & f^*f_*f^!A \otimes f^*B \arrow[d, "\kappa_f\star f^!\otimes 1"] \\ p_*(p^*f^!A \otimes q^*g^!B) \arrow[d, "p_*(\cup'')"] \arrow[dr, "a"] & f^!A \otimes f^*B \arrow[d, "\cup'"] \\ p_*h^!(A \otimes B) \arrow[r, "\kappa_p\star f^!"] & f^!(A \otimes B) \end{tikzcd}$$

where a is the adjoint of the product $\cup'' \colon p^*f^!A \otimes q^*g^!B \to h^!(A \otimes B)$ as described in definition B.10 and b is the adjoint of $\times$ which was given in lemma B.8 as the composition

$$f^*f_*f^!A \otimes f^*g_*g^!B \overset{\kappa_f\otimes\overline{\eta}}{\to} f^!A \otimes p_*q^*g^!B \overset{\alpha_p}{\to} p_*(p^*f^!A \otimes q^*g^!B).$$

In particular, the morphism c given by

$$f^*f_*f^!A \otimes f^*g^*g^!B \xrightarrow{\kappa_f\star f^!\otimes 1} f^!A \otimes f^*g^*g^!B \xrightarrow{1\otimes f^*\star\kappa_g} f^!A \otimes f^*B$$

makes the diagram below c commute (using naturality of $\cup'$). But by definition the upper triangle commutes, hence the claim. $\square$

To finally prove proposition B.14 we put ourselves in the situation that we are given morphisms

$$\begin{aligned}\mathrm{Tr}\colon A' &\to f^!A\\ \mathrm{Tr}\colon B' &\to g^!B\\ \mathrm{Tr}\colon C' &\to h^!(A\otimes B).\end{aligned}$$

The adjoints of these morphisms will also be denoted by Tr. We want to investigate when a given product

$$\wedge\colon p^*A'\otimes q^*B' \to C'$$

makes the diagram

$$\text{(B.1)}\qquad \begin{tikzcd} p^*A'\otimes q^*B' \arrow[r, "p^*(\mathrm{Tr})\otimes q^*(\mathrm{Tr})"] \arrow[d, "\wedge"] & p^*f^!A\otimes q^*g^!B \arrow[d, "\cup"] \\ C' \arrow[r, "\mathrm{Tr}"] & h^!(A\otimes B) \end{tikzcd}$$

commutative.

Lemma B.13. *The diagram* (B.1) *commutes if and only if the Fubini-style formula*

$$\mathrm{Tr}(p^*a'\wedge q^*b') = \mathrm{Tr}(a')\otimes\mathrm{Tr}(b')$$

holds, i.e., the diagram

$$\text{(B.2)}\qquad \begin{tikzcd} f_*A'\otimes g_*B' \arrow[r, "\times"] \arrow[d, "\mathrm{Tr}\otimes\mathrm{Tr}"] & h_*(p^*A'\otimes q^*B') \arrow[d, "h_*(\wedge)"] \\ A\otimes B & h_*(C') \arrow[l, "\mathrm{Tr}"] \end{tikzcd}$$

commutes.

Proof. We will consider the diagram adjoint to (B.1). It is given by the lower square in the diagram

$$\begin{tikzcd} f_*A'\otimes g_*B' \arrow[r, "f_*(\mathrm{Tr})\otimes g_*(\mathrm{Tr})"] \arrow[d, "\times"] & f_*f^!A'\otimes g_*g^!B' \arrow[d, "\times"] \\ h_*(p^*A'\otimes q^*B') \arrow[r, "h_*(p^*(\mathrm{Tr})\otimes q^*(\mathrm{Tr}))"] \arrow[d, "h_*(\wedge)"] & h_*(p^*f^!A\otimes q^*g^!B) \arrow[d, "\text{adj. of } \cup''"] \\ h_*(C') \arrow[r, "\mathrm{Tr}"] & A\otimes B. \end{tikzcd}$$

By lemma B.11 the right vertical morphism is

$$\kappa_f\otimes\kappa_g\colon f_*f^!A\otimes g_*g^!B \to A\otimes B.$$

In particular, its composition with

$$f_*(\mathrm{Tr})\otimes g_*(\mathrm{Tr})\colon f_*A'\otimes g_*B' \to f_*f^!A'\otimes g_*g^!B'$$

is the morphism

$$\mathrm{Tr}\otimes\mathrm{Tr}\colon f_*A'\otimes g_*B' \to A\otimes B.$$

As the exterior product $\times$ is an isomorphism, we know that the lower square in the diagram above commutes if and only if the outer square commutes. But using the previous considerations this outer square commutes if and only if the diagram (B.2) commutes. $\square$

We will now specialize the abstract considerations made above to a concrete sitatuion.
For this we will denote for a morphism $f\colon X \to Y$ of relative dimension $r := \dim X - \dim Y$ of connected smooth proper schemes over an algebraically closed field k of exponential characteristic p by

$$f_*\colon H^i_{\text{ét}}(X, \mathbb{Z}/m(j)_{\text{ét}}) \to H^{i-2r}_{\text{ét}}(Y, \mathbb{Z}/m(j-r)_{\text{ét}})$$

the associated Gysin morphism. If $m = p^r$ this is the morphism from [Gro85, Chapitre 2, Definition 1.2.8]. If $(m,p) = 1$, then f_* is the morphism

$$H^i_{\text{ét}}(X, \mathbb{Z}/m(j)_{\text{ét}}) \to H^i_{\text{ét}}(X, f^!(\mathbb{Z}/m(j-r)_{\text{ét}})[-2r]) \to H^{i-2r}_{\text{ét}}(Y, \mathbb{Z}/m(j-r)_{\text{ét}})$$

coming from the relative trace morphism $\mathrm{Tr}\colon \mathbb{Z}/m(j)[2r] \to f^!(\mathbb{Z}/m(j-r)_{\text{ét}})$ (cf. [AGV71, Exposé XVIII, Théoreme 2.19]), which is even an isomorphism for f smooth, and the trace map

$$Rf_*f^! \to \mathrm{Id}.$$

We fix two connected smooth projective schemes $X, Y \in \mathrm{Sm}/k$ of dimensions $d := \dim X$ resp. $e := \dim Y$ over an algebraically closed field k and consider the cartesian diagram

$$\begin{tikzcd} X \times Y \arrow[r, "p"] \arrow[d, "q"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & \mathrm{Spec}(k). \end{tikzcd}$$

We can then apply the previous considerations to the total derived functors

$$\begin{aligned} f_*&\colon D(X_{\text{ét}}, \mathbb{Z}/m) \to D(\mathrm{Spec}(k)_{\text{ét}}, \mathbb{Z}/m) \\ f^*&\colon D(\mathrm{Spec}(k)_{\text{ét}}, \mathbb{Z}/m) \to D(X_{\text{ét}}, \mathbb{Z}/m) \\ f^!&\colon D(\mathrm{Spec}(k)_{\text{ét}}, \mathbb{Z}/m) \to D(X_{\text{ét}}, \mathbb{Z}/m) \end{aligned}$$

etc.

Proposition B.14. *For X, Y as above the following diagram commutes*

$$\begin{tikzcd} H^i_{\text{ét}}(X, \mathbb{Z}/m(d)_{\text{ét}}) \otimes H^j_{\text{ét}}(Y, \mathbb{Z}/m(e)_{\text{ét}}) \arrow[r, "\mathrm{Tr}\otimes\mathrm{Id}"] \arrow[d, "\times"] & H^{j+i-2d}(Y_{\text{ét}}, \mathbb{Z}/m(e)_{\text{ét}}) \\ H^{i+j}_{\text{ét}}(X \times Y, \mathbb{Z}/m(d+e)_{\text{ét}}) \arrow[ur, "q_*"] & \end{tikzcd}$$

for any $i, j \in \mathbb{Z}$. Here the trace morphism is the one from theorem 11.1, so in particular $\mathrm{Tr} \otimes \mathrm{Id}$ *is zero if $i \neq 2d$.*

Proof. This is proposition B.12 in the context of (derived categories of) étale sheaves. Namely, set $A = B = \mathbb{Z}/m$ and use that $f^!(\mathbb{Z}/m) \cong \mathbb{Z}/m(d)_{\text{ét}}[2d]$, $g^!(\mathbb{Z}/m) \cong \mathbb{Z}/m(e)_{\text{ét}}[2e]$, $h^!(\mathbb{Z}/m) \cong \mathbb{Z}/m(d+e)_{\text{ét}}[2(d+e)]$ via the trace morphism (cf. theorem 11.1). More precisely, under these isomorphisms the abstract product

$$\cup''\colon p^*f^!(\mathbb{Z}/m) \otimes q^*g^!(\mathbb{Z}/m) \to h^!(\mathbb{Z}/m)$$

becomes the usual multiplication in $\mathbb{Z}/m(*)_{\text{ét}}$. To see this we use lemma B.13. By this lemma it suffices to show that the "Fubini formula"

$$\mathrm{Tr}(p^*a \cup q^*b) = \mathrm{Tr}(a)\mathrm{Tr}(b)$$

holds. By [Gei10, Proposition 2.2]

$$\mathbb{Z}^c_Z/m \cong \mathbb{Z}/m(d_Z)[2d_Z]$$

for Z smooth of dimension d_Z over a perfect field. Moreover, this quasi-isomorphism identifies the exterior product for the complexes $\mathbb{Z}/m(*)_{\text{ét}}$ with the exterior product of cycles. The trace map [Gei10, Proposition 3.5] for proper schemes Z is (essentially) given by the degree map $\mathrm{CH}_0(Z) \to \mathbb{Z}$ of zero-cycles. Now the assertion of the lemma follows: Take a point $x \in X(k)$ and a point $y \in Y(k)$ (recall that k is algebraically closed). The exterior product of x and y is then given by the intersection $x \times Y \cap X \times y = (x, y)$. Therefore

$$\mathrm{Tr}(p^*x \cup q^*y) = \mathrm{Tr}(x)\mathrm{Tr}(y).$$

as desired. $\square$

We remark that the case that m is invertible in k is also treated in [AGV71, Exposé XVIII, Proposition 2.12].

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