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# Parahoric restriction for $\mathrm{GSp}(4)$ and the inner cohomology of Siegel modular threefolds

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## Abstract

For irreducible admissible representations of the group of symplectic similitudes $\mathrm{GSp}(4,F)$ of genus two over a $p$-adic number field $F$, we obtain the parahoric restriction with respect to an arbitrary parahoric subgroup $\mathscr{P}$. That means we determine the action of the Levi quotient $\mathscr{P}/\mathscr{P}^+$ on the invariants under the pro-unipotent radical $\mathscr{P}^+$ in terms of explicit character values. Especially, we get the parahoric restriction of local endoscopic $L$-packets in terms of lifting data.
The inner cohomology of the Siegel modular variety of genus two with an arbitrary $\ell$-adic local system admits an endoscopic and a Saito-Kurokawa part under spectral decomposition. For principal congruence subgroups of squarefree level $N$ they define simultaneous representations of the absolute Galois group $\Gamma_{\mathbb{Q}}$ and the Hecke action of $\mathrm{GSp}(4,\mathbb{Z}/N\mathbb{Z})$. We decompose them into irreducible constituents and give explicit character values. As an application, we prove the conjectures of Bergström, Faber and van der Geer on level two.



## Zusammenfassung

Für die Gruppe $\mathrm{GSp}(4,F)$ symplektischer Ähnlichkeitstransformationen über einem $p$-adischen Zahlkörper $F$ bestimmen wir die Parahori-Restriktion beliebiger irreduzibler zulässiger Darstellungen zu beliebigen Parahori-Gruppen. Das bedeutet, wir berechnen die Operation des Levi-Quotienten $\mathscr{P}/\mathscr{P}^+$ auf den Invarianten unter dem pro-unipotenten Radikal $\mathscr{P}^+$ und dessen Zerlegung in irreduzible Charaktere. Insbesondere erhalten wir auch die Parahori-Restriktion der lokalen endoskopischen $L$-Pakete von Tiefe Null für gegebene Liftungsdaten.
Die Spektralzerlegung der inneren Kohomologie der Siegelschen Modulvarietät vom Geschlecht zwei mit beliebigem lokalen Koeffizientensystem enthält einen schwach endoskopischen und einen Saito-Kurokawa Anteil. Für Hauptkongruenzgruppen quadratfreier Stufe $N$ zerlegen wir sie als simultane $\ell$-adische Darstellungen der absoluten Galoisgruppe $\Gamma_{\mathbb{Q}}$ und der Gruppe $\mathrm{GSp}(4,\mathbb{Z}/N\mathbb{Z})$ unter der Operation der Heckealgebra. In Stufe zwei liefert das einen Beweis für die Vermutungen von Bergström, Faber und van der Geer.

# Contents

# 1. Introduction

The cohomology of Siegel modular varieties encodes a wealth of information as a Hecke and Galois module. At least since Deligne used the case of genus one as a keystone in his proof of the Ramanujan conjecture [Del68], they have been a central focus of research. In this thesis we study the inner cohomology of Siegel modular threefolds for sufficiently large congruence subgroups, including all principal congruence subgroups of squarefree level.

We follow the notation of Weissauer [Wei09a]. The Siegel modular threefold is a Shimura variety

$$S(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash(X \times \mathbf{G}(\mathbb{A}_f))$$

attached to a Shimura Datum $(\mathbf{G}, X, h)$ for the group of symplectic similitudes $\mathbf{G} = \mathrm{GSp}(4)$ of genus two. Fix a local system $\mathcal{V}_\lambda$ on $S(\mathbb{C})$ attached to an algebraic representation of highest weight $\lambda$. For a finite set $S$ of non-archimedean places and parahoric subgroups $\mathscr{P}_v \subseteq \mathrm{GSp}(4, \mathbb{Q}_v)$, $v \in S$, with pro-unipotent radical $\mathscr{P}_v^+$, let $K \subseteq \mathrm{GSp}(4, \mathbb{A}_f)$ be the open congruence subgroup

$$K = \prod_{v \notin S} \mathrm{GSp}(4, \mathbb{Z}_v) \prod_{v \in S} \mathscr{P}_v^+.$$

We describe the weak endoscopic and the Saito-Kurokawa part of the inner cohomology $H^\bullet_!(S(\mathbb{C}), \mathcal{V}_\lambda)^K$ as an $\ell$-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}} : \mathbb{Q}) \times \prod_{v\in S} \mathscr{P}_v/\mathscr{P}_v^+$.

Our approach is based on the Matsushima-Murakami formula, which expresses the inner cohomology in terms of cuspidal automorphic representations. Important results on the classification of automorphic representations of GSp(4) have been obtained by Piatetski-Shapiro [PS83b], Schwermer [Sch95], Soudry [Sou88], Taylor [Tay93], Tsushima [Tsu83], Weissauer [Wei88], and others.

An alternative approach rests on a geometric description of Siegel modular threefolds. For example, the Shimura variety $S_K(\mathbb{C})$ attached to the modified principal congruence subgroup of level $N \in \mathbb{N}_{\geq 1}$

$$K = K'(N) = \{x \in \mathrm{GSp}(4, \hat{\mathbb{Z}})\,;\, x \equiv \mathrm{diag}(1, 1, *, *) \mod N\}.$$

is isomorphic to the moduli space $\mathcal{A}_{2,N}$ of principally polarized complex abelian surfaces with a level-$N$-structure. This approach dates back to Riemann and has been continued by Faltings and Chai [FC90], Lee and Weintraub [LW85], van Geemen and Nygaard [vGN95], van der Geer [vdG82], and also many others.

By counting rational points of hyperelliptic curves over finite fields and using the Lefschetz trace formula, Bergström, Faber and van der Geer [FvdG04], [BFvdG08] obtained a conjectural description of the compactly supported cohomology. This gave rise to explicit conjectural formulas about the $\ell$-adic representations of

$$\mathrm{GSp}(4, \mathbb{Z}/N\mathbb{Z}) \times \mathrm{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$$

on the motivic inner cohomology for level $N = 1, 2$. By recent work of Weissauer [Wei09b], Tehrani [Teh12] and Petersen [Pet15], these conjectures have been shown for level $N = 1$. As an application of our results, we prove the conjectures on level $N = 2$ in Section 5.5.

We also determine the Hodge numbers $h_!^{(p,q)} = \dim H_!^{(p,q)}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda)$ of the inner cohomology. For sufficiently regular local systems, it only remains to calculate $h_!^{(2,1)}$ by Faltings' result [Fal83] and Tsushima's formula [Tsu83]. We obtain

$$h_!^{(2,1)} = h_!^{(3,0)} + \tfrac{5}{4}(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2 - 2),$$

see Cor. 5.23. For the analogous result with irregular $\lambda$, see Cor. 5.24.

The main tool we employ in the local description of invariants is the parahoric restriction functor. Let $F/\mathbb{Q}_p$ be a non-archimedean local number field with integers $\mathfrak{o}$ and finite residue field $\mathfrak{o}/\mathfrak{p}$ of order $q$. Let $\mathbf{G}$ be a quasi-split connected reductive group defined over $F$ with $F$-rational points $G = \mathbf{G}(F)$. Parahoric groups are group schemes over $\mathfrak{o}$, whose $\mathfrak{o}$-rational points define subgroups $\mathscr{P}$ of $G$. They admit a Levi decomposition

$$0 \to \mathscr{P}^+ \to \mathscr{P} \to \underline{\mathscr{P}} \to 0$$

with respect to the pro-unipotent radical $\mathscr{P}^+$ and a reductive Levi quotient $\underline{\mathscr{P}}$ defined over the residue field $\mathfrak{o}/\mathfrak{p}$. Restricting an admissible representation $(\pi, V)$ of $G$ to $\mathscr{P}$ and taking invariants under $\mathscr{P}^+$ gives rise to a finite-dimensional representation of the Levi quotient $\underline{\mathscr{P}}$:

$$\mathbf{r}_{\mathscr{P}}(\pi) : \underline{\mathscr{P}} \to \mathrm{Aut}(V^{\mathscr{P}^+}).$$

This defines the parahoric restriction functor $\mathbf{r}_{\mathscr{P}}$ from admissible representations of $G$ to those of $\underline{\mathscr{P}}$, where the definition on morphisms is the obvious one. The construction is completely analogous to Jacquet's functor of parabolic restriction. We give a survey of the most important results in Section 2.4.

For the general linear group $\mathbf{G} = \mathrm{GL}(n)$ the parahoric restriction of some representations has been studied by Bushnell and Kutzko [BK93] and by Vignéras [Vig96]. We briefly discuss the cases $n = 1, 2$ at the end of Section 2.4.

For the group $G = \mathrm{GSp}(4, F)$ of symplectic similitudes of genus two, Moy [Moy88] has determined the parahoric restriction for certain cuspidal irreducible representations. Sally and Tadic [ST94] have classified the non-cuspidal irreducible representations and these have been studied extensively by Roberts and Schmidt [RS07]. For odd residue characteristic, Breeding-Allison [BA15] has determined the parahoric restriction with respect to the hyperspecial parahoric for parabolically induced representations. In Chapter 3 of this thesis, we complete this work and determine the character values of $\mathbf{r}_{\mathscr{P}}(\pi)$ explicitly for arbitrary parahoric subgroups $\mathscr{P}$ of $\mathrm{GSp}(4,F)$ and irreducible admissible representations in arbitrary residue characteristic. To this end, we make use of the classification of irreducible representations of finite group $\mathrm{GSp}(4,q)$ by Enomoto [Eno72] and Shinoda [Shi82]. A fortiori, we obtain a new proof of the classification of parahori-spherical vectors, see Cor. 3.8.

The non-generic cuspidal irreducible representations all occur in the anisotropic theta-lift and we also obtain their parahoric restriction as a special case of our results on the endoscopic lift. For generic depth-zero cuspidal irreducible admissible representations, the classification of Moy and Prasad [MP96, 6.8] and the work of deBacker and Reeder [DR09] implies that their parahoric restriction can only be non-zero for hyperspecial maximal parahorics, see Lemma 2.18.

Up to isomorphism, the only proper elliptic endoscopic datum for $\mathrm{GSp}(4,F)$ is attached to the group $M = \mathrm{GL}(2,F) \times \mathrm{GL}(2,F)/\mathrm{GL}(1,F)$ with respect to the antidiagonal embedding of $\mathrm{GL}(1,F)$. The endoscopic character lift attached to $M$ is a homomorphism $r$ between the Grothendieck groups of admissible representations of $M$ and $\mathrm{GSp}(4,F)$. For every irreducible representation $\sigma$ of $M$, we determine in Chapter 4 the parahoric restriction of its lift $r(\sigma)$. If $\sigma$ is unitary generic irreducible, the lift $r(\sigma)$ has one or two constituents, forming an endoscopic $L$-packet. We determine the parahoric restriction for each individual constituent [Wei09a] in terms of $\sigma$.

For example, hyperspecial parahoric subgroups $\mathscr{K}_M \subseteq M$ and $\mathscr{K}_G \subseteq G$ satisfy

$$\dim \mathbf{r}_{\mathscr{K}_G} \circ r(\sigma) = (q^2+1) \dim \mathbf{r}_{\mathscr{K}_M}(\sigma)$$

for arbitrary virtual representations $\sigma$ in the Grothendieck group of admissible representations of $M$. This implies that the matching condition of standard endoscopy is satisfied for the indicator functions $f^M = (q^2+1)\,\mathrm{char}_{\mathscr{K}_M^+}$ and $f^G = \mathrm{char}_{\mathscr{K}_G^+}$. To give another example, we can verify that an irreducible admissible representation $\sigma$ is of depth-zero if and only if $r(\sigma)$ has a depth-zero constituent. This is in compliance with the expected depth-preservation under the local Langlands correspondence [ABPS].

## Notation

The set of nonnegative integers is $\mathbb{N}_0 = \{0, 1, \dots\}$ and the positive integers are $\mathbb{N}_{>0} = \{1, 2, \dots\}$. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ have their usual meaning. Algebraic groups are denoted by boldface letters ($\mathbf{G}, \dots$), the corresponding groups of rational points over a fixed field $F$ in italics ($G = \mathbf{G}(F), \dots$). The finite field of order $q$ is $\mathbb{F}_q$. For the group of $\mathbb{F}_q$-rational points of an algebraic group $\mathbf{G}$ defined over $\mathbb{F}_q$ we write $\mathbf{G}(q)$ instead of $\mathbf{G}(\mathbb{F}_q)$. We will always assume that the prime number $\ell \in \mathbb{N}_{>0}$ is distinct from bad primes and any Frobenius primes.

The $n \times n$ identity matrix is $I_n$. Diagonal and antidiagonal matrices are

$$\operatorname{diag}(a_1, \dots, a_n) = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \text{ and } \operatorname{antidiag}(a_1, \dots, a_n) = \begin{pmatrix} & & a_1 \\ & \iddots & \\ a_n & & \end{pmatrix},$$

so the first entry of an antidiagonal matrix is in the upper right corner. The matrix with a single entry 1 in the $i$-th row and $j$-th column and 0 elsewhere is denoted by $E_{ij}$. Zeros in a matrix are usually omitted.

The disjoint union of sets is denoted by $\sqcup$. The characteristic function of a subset $A \subseteq B$ is

$$B \ni x \mapsto \operatorname{char}_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

For a finite cyclic group $(\mathcal{C}_m, \cdot)$ (multiplicative notation) of order $m$ each divisor $n$ of $m$ gives rise to a unique subgroup $\mathcal{C}_m[n] \subseteq \mathcal{C}_m$ of index $d = m/n$. This defines the projection $N_d : \mathcal{C}_m \to \mathcal{C}_m[n], x \mapsto x^d$ and the injection $i_d : \mathcal{C}_m[n] \to \mathcal{C}_m, x \mapsto x$.

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# 2. Preliminaries

## 2.1. Algebraic groups

An affine group scheme $\mathbf{G}$ of finite type over a ring $\mathfrak{o}$ defines a functor from the category of $\mathfrak{o}$-algebras to the category of groups. To an $\mathfrak{o}$-algebra $R$ with canonical morphism $\mathfrak{o} \to R$ the functor associates the group $\mathbf{G}(R)$ of $R$-rational points of $\mathbf{G} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(R)$. A morphism of $\mathfrak{o}$-algebras $R \to S$ gives rise to a unique morphism $\mathbf{G}(R) \to \mathbf{G}(S)$. It is a common abuse of notation to denote this functor by $\mathbf{G}$ again.

For an $\mathfrak{o}$-algebra $R$ the algebra $R[\epsilon] = R[X]/X^2$ is equipped with a natural ring homomorphism $e : R[\epsilon] \to R$ that sends $X$ to zero. The Lie algebra $\mathfrak{g}(R)$ of $\mathbf{G}(R)$ is the kernel of $\mathbf{G}(e)$. This gives rise to the *Lie functor* $\mathrm{Lie} : \mathbf{G}(R) \mapsto \mathfrak{g}(R)$. Every $x \in \mathbf{G}(R)$ defines a conjugation endomorphism $C_x : y \mapsto xyx^{-1}$ on $G(R)$, which gives rise to an automorphism $\mathrm{Ad}(x) = \mathrm{Lie}(C_x)$ of the Lie algebra $\mathfrak{g}(R)$, the adjoint representation. For details, see [Wat79].

**The symplectic group.** Fix an integer $g \geq 0$. The connected split reductive group scheme $\mathbf{G} = \mathrm{GSp}(2g)$ of *symplectic similitudes of genus* $g$ assigns to any $\mathbb{Z}$-algebra $A$ the group

$$\{(x, \nu) \in \mathrm{Mat}(2g \times 2g, A) \times \mathbb{G}_m(A) \,|\, xJx^t = \nu J\} \quad \text{for } J = \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}.$$

The *similitude character* is $\mathrm{sim} : (x, \nu) \mapsto \nu$. The Lie algebra of $\mathrm{GSp}(2g)$ is

$$\mathfrak{gsp}(2g) : K \mapsto \{(X, \nu) \in \mathrm{Mat}(2g \times 2g, K) \times \mathbb{G}_a(K) \,|\, XJ + JX^t = \nu J\}.$$

The *symplectic group* $\mathrm{Sp}(2g)$ is the kernel of $\mathrm{sim} : \mathrm{GSp}(2g) \to \mathbb{G}_m$. Its Lie algebra $\mathfrak{sp}(2g)$ fits into a split exact sequence

$$0 \to \mathfrak{sp}(2g) \longrightarrow \mathfrak{gsp}(2g) \xrightarrow{\nu} \mathbb{G}_a \to 0.$$

The splitting is given by $\mathbb{G}_a \to \mathfrak{gsp}(2g)$, $\nu \mapsto \mathrm{diag}(0, \ldots, 0, \nu, \ldots, \nu)$.

### 2.1.1. The root system

We review the root system of the split connected reductive group $\mathbf{G} = \mathrm{GSp}(2g)$. The torus $\mathbf{T}$ of diagonal matrices

$$t = \mathrm{diag}(t_1, \ldots, t_g, t_0/t_1, \ldots, t_0/t_g)$$

with $t_i \in \mathbb{G}_m$ for $i \in 0, \ldots, g$ is a split maximal torus, so the rank of $\mathrm{GSp}(2g)$ is $\mathrm{rk}(\mathbf{G}) = g + 1$. The Lie algebra of $\mathbf{T}$ is the subalgebra $\mathfrak{t} \subseteq \mathfrak{g} = \mathfrak{gsp}(2g)$ of diagonal matrices. Let $e_i : \mathbf{T} \to \mathbb{G}_m$ be the elementary character $e_i(t) = t_i$. The character group $\lambda : X^*(\mathbf{T}) = \{t \mapsto \prod_{i=0}^{g} t_i^{\lambda_i},\ \lambda_i \in \mathbb{Z}\}$ is abelian and we use additive notation:

$$(\lambda + \lambda')(t) = \lambda(t)\lambda'(t) \quad \text{ for } \lambda, \lambda' \in X^*(\mathbf{T}).$$

The character $\lambda = \sum_{i=1}^{g} \lambda_i e_i$ will be denoted by $(\lambda_0, \ldots, \lambda_g)$. The elementary cocharacters are the homomorphisms $f_j : \mathbb{G}_m \to \mathbf{T}$ with $j = 0, \ldots, g$ such that $e_i \circ f_j$ is the identity for $i = j$ and zero for $i \neq j$. They generate the cocharacter group $X_*(\mathbf{T}) = \bigoplus_{j=0}^{g} \mathbb{Z} f_j$. The canonical isomorphism $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ defines a bilinear form $X^*(\mathbf{T}) \times X_*(\mathbf{T}) \to \mathbb{Z}$. This bilinear pairing identifies $V^* = \mathbb{R} \otimes X^*(\mathbf{T})$ with the dual vector space of $V = \mathbb{R} \otimes X_*(\mathbf{T})$.

The torus $\mathbf{T}$ acts on $\mathfrak{g}$ via the adjoint representation and this gives rise to the decomposition $\mathfrak{g} = \bigoplus_{\lambda \in X^*(\mathbf{T})} \mathfrak{g}_\lambda$ into $\lambda$-eigenspaces

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(t)X = \lambda(t)X \ \forall t \in \mathbf{T}\}.$$

The non-zero characters $\lambda \in X^*(\mathbf{T})$ with $\mathfrak{g}_\lambda \neq 0$ form the root system $\Phi(\mathbf{G})$. Explicitly, these roots are

$$\begin{aligned} &\pm (e_i - e_j), && 1 \leq i < j \leq g, \\ &\pm (e_i + e_j - e_0), && 1 \leq i < j \leq g, \\ &\pm (2e_i - e_0) && 1 \leq i \leq g. \end{aligned}$$

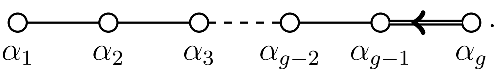
By lexicographic ordering we fix positive roots of $\mathrm{GSp}(2g)$

$$\Phi^+(\mathbf{G}) = \{e_i - e_j,\ e_i + e_j - e_0 \,|\, 1 \leq i < j \leq g\} \cup \{2e_i - e_0 \,|\, i = 1, \ldots, g\}.$$

The simple roots are

$$\Delta = \Delta(\mathbf{G}) = \{\alpha_i = e_i - e_{i+1} \,|\, i = 1, \ldots, g-1\} \cup \{\alpha_g = 2e_g - e_0\},$$

where $\alpha_g$ is the long root. They generate $\Phi^+$ and give rise to the Dynkin diagram

$C_g$ :


For each root $\alpha \in \Phi$ there is a *coroot* $\alpha^\vee \in X_*(\mathbf{T})$: Let $G_\alpha$ be the centralizer of the connected component of $\ker(\alpha)$, generated by $\mathbf{T}$ and $U_{\pm\alpha}$. Then there is a homomorphism $x_\alpha : \mathrm{SL}(2) \to G_\alpha$, such that $x_\alpha(\left(\begin{smallmatrix} 1 & c \\ & 1 \end{smallmatrix}\right)) \subseteq U_\alpha$. The coroot is the cocharacter $\alpha^\vee = x_\alpha \circ i$, where $i : \mathbb{G}_m \hookrightarrow \mathrm{SL}(2)$ is an embedding into the standard torus of SL(2), such that $\langle \alpha, \alpha^\vee \rangle = 2$. The set of coroots is denoted $\Phi^\vee$ and we get a bijection $\Phi \to \Phi^\vee$, $\alpha \mapsto \alpha^\vee$. The coroots of GSp($2g$) are

$$\begin{aligned} &\pm (e_i - e_j)^\vee = \pm(f_i - f_j), && 1 \le i < j \le g, \\ &\pm (e_i + e_j - e_0)^\vee = \pm(f_i + f_j), && 1 \le i < j \le g, \\ &\pm (2e_i - e_0)^\vee = \pm f_i, && 1 \le i \le g. \end{aligned}$$

The Weyl group $W_G = N_G(\mathbf{T})/\mathbf{T}$ of $\mathbf{G}$ is the finite quotient of the normalizer $N_G(\mathbf{T})$ by the standard split torus $\mathbf{T} \subseteq \mathbf{G}$. The action of $W_G$ on $\mathbf{T}$ by conjugation gives rise to a natural action on the character group $W_G \hookrightarrow \mathrm{Aut}(X^*(\mathbf{T}))$ that preserves $\Phi$. The reflections $s_\alpha$ for $\alpha \in \Phi\}$ at the hyperplanes $\alpha^{-1}(0) \subseteq V^*$

$$s_\alpha : V^* \to V^*, \quad x \mapsto x - \langle x, \alpha^\vee \rangle \, \alpha.$$

generate the Weyl group $W_G$ as a group of automorphisms of $X^*(\mathbf{T})$. It permutes the $e_i$ and changes their signs, so it is isomorphic to the semidirect product $\Sigma_g \ltimes \{\pm 1\}^g$.

A character $\lambda \in X^*(\mathbf{T})$ is *dominant* with respect to $\Phi^+(\mathbf{G})$ if $\langle \lambda, \alpha^\vee \rangle \ge 0$ for every positive root $\alpha \in \Phi^+(\mathbf{G})$, that means $\lambda_1 \ge \cdots \ge \lambda_g \ge 0$. Every Weyl-orbit of a character contains a dominant one. There is a unique dominant root $\beta = 2e_1 - e_0$ such that every other root is of the form $\beta - \sum_{\alpha \in \Delta} c_\alpha \alpha$ for non-negative integers $c_\alpha \ge 0$.

For the symplectic group $\mathbf{G} = \mathrm{Sp}(2g)$, the diagonal matrices

$$t = \mathrm{diag}(t_1, \ldots, t_g, t_1^{-1}, \ldots, t_g^{-1})$$

form a maximal split torus of rank $g$. The root system is similar to GSp($2g$), just drop the character $e_0$. For the general linear group $\mathbf{G} = \mathrm{GL}(g)$ a split maximal torus is given by the diagonal matrices $t = \mathrm{diag}(t_1, \ldots, t_g)$ with characters $e_i(t) = t_i$ for $i = 1, \ldots, g$ and the roots are $\alpha_{ij} = e_i - e_j$ for $i \ne j$.

The root datum of $\mathbf{G}$ is the quadruple $\Psi(\mathbf{G}) = (X^*(\mathbf{T}), \Phi, X_*(\mathbf{T}), \Phi^\vee)$. The dual root datum is $\Psi^\vee(\mathbf{G}) = (X_*(\mathbf{T}), \Phi^\vee, X^*(\mathbf{T}), \Phi)$.

For every split connected reductive group $G$ over a local number field $F$ with Weil group $W_F$ we fix a connected reductive complex group $\hat{G}$ together with an $L$-action $\rho_G$ of the absolute Galois group $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ on $\hat{G}$ and a $\Gamma_F$-bijection $\eta_G$ from the dual root datum $\Psi^\vee(G)$ to the root datum $\Psi(\hat{G})$. The $L$-datum is the triple $(\hat{G}, \rho_G, \eta_G)$. The $L$-group is the semidirect product ${}^L G = \hat{G} \rtimes W_F$ such that $\rho_G$ splits the exact sequence $1 \to \hat{G} \to {}^L G \to W_F \to 1$.

### 2.1.2. Parabolic subgroups

For each root $\alpha$ the corresponding eigenspace $\mathfrak{g}_\alpha$ is one-dimensional and corresponds to a unique one-dimensional root subgroup $U_\alpha \subseteq \mathbf{G}$ such that $\mathfrak{g}_\alpha$ is the Lie algebra of $U_\alpha$. The standard Borel subgroup is the subgroup $\mathbf{B}$ of $\mathbf{G}$ generated by $\mathbf{T}$ and $U_\alpha$ for all the positive roots $\alpha \in \Phi^+(\mathbf{G})$. The standard parabolic subgroups of $\mathbf{G}$ are the algebraic subgroups containing the standard Borel. For each subset $I \subseteq \Delta$ of the simple roots the standard parabolic subgroup $\mathbf{P} = \mathbf{P}_I$ is generated by $\mathbf{T}$ and $U_\alpha$ for $\alpha \in (-I) \cup \Phi^+$. The Levi subgroup $\mathbf{M}_I$ of $\mathbf{P}_I$ is generated by $\mathbf{T}$ and $U_\alpha$ for $\alpha \in I \cup -I$. The unipotent radical $\mathbf{U}_I$ is generated by the $U_\alpha$ with $\alpha \in \Phi^+ \setminus \langle I \rangle$. This yields the *Levi decomposition* $\mathbf{P}_I \cong \mathbf{M}_I \ltimes \mathbf{U}_I$.

The simple roots of the reductive group $\mathbf{M}_I$ are given by $I$. The standard Borel of $\mathbf{G} = \mathrm{GSp}(2g)$ is

$$\mathbf{B} = \mathbf{P}_\emptyset = \left\{ \begin{pmatrix} a & * \\ 0 & \nu(a^t)^{-1} \end{pmatrix} \in \mathrm{GSp}(2g) \,\middle|\, \nu \in \mathbb{G}_m,\ a \in \mathrm{Mat}(g \times g) \text{ with } a_{ij} = 0\ \forall i > j \right\}.$$

For a fixed subset $I \subseteq \Delta$ let $\Delta \backslash I =: \{\alpha_{m_1}, \ldots, \alpha_{m_k} \,|\, m_j < m_{j+1}\}$ be the set of simple roots not contained in $I$. For $j = 1, \ldots, k$ let $n_j = m_j - m_{j-1}$ (with $m_0 = 0$). Then the Levi subgroup $\mathbf{M}_I$ is the image of the embedding

$$\begin{aligned} &\mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k) \times \mathrm{GSp}(2g - 2m_k) \longrightarrow \mathbf{P}_I, \\ &(A_1, \ldots, A_k, M) \longmapsto \mathrm{blockdiag}(A_1, \ldots, A_k, M, \mathrm{sim}(M)(A_1^t)^{-1}, \ldots, \mathrm{sim}(M)(A_k^t)^{-1}). \end{aligned}$$

## 2.2. Bruhat-Tits theory

Fix a non-archimedean local number field $F$ with $\mathbb{Z}$-valuation $v$ and finite residue field $\mathfrak{o}/\mathfrak{p}$. For an split unramified connected reductive group $\mathbf{G}$ over $\mathfrak{o}$ fix a split maximal torus $\mathbf{T}$ and let $G$ and $T$ be their groups of $F$-rational points.

### 2.2.1. The affine root system

A split maximal torus $T = \mathbf{T}(F)$ in $G$ gives rise to a reduced root system $\Phi$. An apartment is an affine space $A$ over $V = \mathbb{R} \otimes X_*(\mathbf{T})$, which we identify with $V$ by the choice of an origin 0. The *affine roots* of $G$ are affine linear maps

$$\psi : A \to \mathbb{R}, \quad x \mapsto \alpha(x - 0) + m,$$

whose vector part $\mathbf{v}(\psi) = \alpha$ is a root of $G$ and whose constant part an integer $m \in \mathbb{Z}$. They form the affine root system $\Phi_{\mathrm{af}} = \{\psi = \alpha + m \,|\, \alpha \in \Phi(\mathbf{G}), m \in \mathbb{Z}\}$. For irreducible $\Phi(\mathbf{G})$, we fix a set of simple affine roots by

$$\Delta_{\mathrm{af}} = \{\psi = \alpha + 0 \,|\, \alpha \in \Delta\} \cup \{\psi_0 = -\beta + 1\},$$

where $\beta$ is the unique dominant root. A reducible $\Phi$ admits a unique decomposition into finitely many irreducible subsystems $\Phi = \bigsqcup_j \Phi_j$. For each $\Phi_j$ its simple affine roots $\Delta_{\mathrm{af},j}$ are constructed as above and the set of simple affine roots of $\Phi$ is $\Delta_{\mathrm{af}} = \bigsqcup_j \Delta_{\mathrm{af},j}$. The *affine Weyl group* $W_{\mathrm{af}}$ is generated by the reflections $s_\psi$ at the hyperplanes $\psi^{-1}(0)$ attached to the simple affine roots $\psi \in \Delta_{\mathrm{af}}$. The simple affine roots form the vertices of the affine Coxeter diagram, where two vertices $\psi_i = \alpha_i + m_i$ and $\psi_j = \alpha_j + m_j$ are joined by $4\langle\alpha_i, \alpha_j\rangle^2 / (\langle\alpha_i, \alpha_i\rangle \langle\alpha_j, \alpha_j\rangle)$ edges. Decorating the affine Coxeter diagram with arrows pointing from the long to the short roots produces the affine Dynkin diagram.

Each choice of a Chevalley basis determines such an origin 0 and defines for each root $\alpha$ a fixed isomorphism

$$\chi_\alpha : (F, +) \to U_\alpha(F)$$

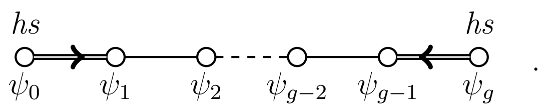
to the corresponding root subgroup $U_\alpha$ [Tit79, §1.1]. This defines a filtration of the root subgroups via $U_\psi = \chi_\alpha(\mathfrak{p}^m) \subseteq U_\alpha$ for the affine roots $\psi = \alpha + m$.

A point $x$ in the apartment $A$ is *special*[1] if every root $\alpha$ is proportional to the vector part of an affine root $\psi$ with $\psi(x) = 0$ [Tit79, §1.9]. Since the root system is reduced, this is equivalent to the condition

$$\psi(x) \in \mathbb{Z} \qquad \forall \psi \in \Phi_{\mathrm{af}}. \tag{2.1}$$

A vertex $\psi$ of an irreducible affine Dynkin diagram is special if and only if there is a special point $x$ with $\psi'(x) = 0$ for every other vertex $\psi' \neq \psi$ [Tit79, §1.9].

**Example 2.1.** *For* $\mathbf{G} = \mathrm{GSp}(2g)$ *(*$g \geq 2$*) we fix the simple affine roots* $\psi_i = \alpha_i + 0$ *for* $i = 1, \ldots, g$ *and* $\psi_0 = -\beta + 1$ *for the dominant root* $\beta$*. The isomorphisms* $\chi_{\alpha_i}$ *for the simple roots* $\alpha_i$ *are* $\chi_{\alpha_i}(x) = I_{2g} + xE_{i,i+1} - xE_{g+i+1,g+i}$ *for* $i = 1, \ldots, g-1$ *and* $\chi_{\alpha_g}(x) = I_{2g} + xE_{g,2g}$*. The vertices* $\psi_0$ *and* $\psi_g$ *are special. The affine Dynkin diagram is of type* $\mathscr{C}_g$*:*




For $\mathbf{G} = \mathrm{GL}(g)$ with $g \geq 2$ the affine Dynkin diagram of type $\mathscr{A}_g$ is a cycle of $g$ vertices, which are all hyperspecial [Tit79, 1.14].

[1] Since $G$ is split, special is equivalent to hyperspecial [Tit79, §1.10.2].

### 2.2.2. Parahoric subgroups

Fix a standard closed alcove $\mathcal{C} = \{x \in A \,|\, \psi(x) \geq 0 \ \forall \psi \in \Delta_{\text{af}}\}$. Attached to a proper subset $\Theta \subsetneq \Delta_{\text{af}}$ of simple affine roots is the facet

$$\mathcal{F} = \mathcal{C} \cap \bigcap_{\psi \in \Theta} \psi^{-1}(0).$$

Fix a point $x \in \mathcal{F}$ such that $\psi(x) > 0$ for $\psi \in \Delta_{\text{af}} \backslash \Theta$. Conversely, a point $x \in \mathcal{C}$ determines $\Theta = \{\psi \in \Phi_{\text{af}} \,|\, \psi(x) = 0\}$ uniquely and the corresponding facet $\mathcal{F}$ is the smallest facet that contains $x$. For each simple root $\alpha$ we have

$$0 \leq \alpha(x) \leq \beta(x) = 1 - \psi_0(x) \leq 1,$$

especially $-\alpha(x) + 1 \geq 0$.

**Definition 2.2.** The *standard parahoric subgroup* at $x$ is the group $\mathscr{P}_x = \mathscr{P}_{\mathcal{F}}$ generated by

$$\mathbf{T}(\mathfrak{o}) = \{t \in \mathbf{T}(F) \,|\, \lambda(t) \in \mathfrak{o}^\times \ \forall \lambda \in X^*(T)\}$$

and $U_\psi$ for every affine root $\psi$ with $\psi(x) \geq 0$. A standard parahoric subgroup is *(hyper-)special* if $\Delta_{\text{af}} \backslash \Theta$ contains a single (hyper-)special vertex, or (equivalently) if $x$ is (hyper-)special. The *standard Iwahori subgroup* is the parahoric subgroup attached to the facet $\mathcal{F} = \mathcal{C}$.

A parahoric subgroup only depends on the facet, but its filtration subgroups $\mathscr{P}_{x,r}$ depend on $x$, so the notation $\mathscr{P}_x$ is more appropriate.

**Levi decomposition.** Attached to every parahoric $\mathscr{P}_x$ is a smooth affine group scheme $\mathbf{G}_x$ over $\mathfrak{o}$ with generic fiber $\mathbf{G}_x(F) \cong G$ such that the group of $\mathfrak{o}$-rational points is $\mathscr{P}_x = \mathbf{G}_x(\mathfrak{o}) \subseteq G$. The *Levi decomposition* of $\mathscr{P}_x$ is the canonical exact sequence

$$1 \to \mathscr{P}_x^+ \to \mathscr{P}_x \to \underline{\mathscr{P}_x} \to 1,$$

where $\mathscr{P}_x^+$ is the pro-unipotent radical[2] and $\underline{\mathscr{P}_x}$ is isomorphic to the Levi quotient of the special fiber $\mathbf{G}_x(\mathfrak{o}/\mathfrak{p})$. The parahoric is special if and only if the special fiber $\mathbf{G}_x(\mathfrak{o}/\mathfrak{p})$ is reductive itself [Tit79, §3.8].

**Example 2.3.** *The pro-unipotent radical of the standard hyperspecial maximal parahoric* $\mathscr{P}_x = \mathrm{GSp}(2g, \mathfrak{o}) \subseteq \mathrm{GSp}(2g, F)$ *is the principle congruence subgroup*

$$\mathscr{P}_x^+ = \{X \in \mathrm{GSp}(2g, \mathfrak{o}) \,|\, X \equiv I_{2g} \mod \mathfrak{p}\} \quad \textit{with} \quad \mathscr{P}_x/\mathscr{P}_x^+ \cong \mathrm{GSp}(2g, \mathbb{F}_q).$$

[2]For facets $\mathcal{F}_1 \supseteq \mathcal{F}_2$, we have $\mathscr{P}_{\mathcal{F}_1} \subseteq \mathscr{P}_{\mathcal{F}_2}$, but $\mathscr{P}_{\mathcal{F}_1}^+ \supseteq \mathscr{P}_{\mathcal{F}_2}^+$ for their pro-unipotent radicals.

**Remark 2.4.** Let $P = M \ltimes U$ be a parabolic subgroup of $G$. Both $G$ and $M$ contain the standard split torus and therefore the standard apartment $A$ of $G$ is also an apartment of $M$. Let $\mathscr{P}_x$ be a parahoric subgroup of $G$ associated to a point $x \in A$, then

$$\mathscr{M}_x = M \cap \mathscr{P}_x \tag{2.2}$$

is the parahoric subgroup of $M$ associated to $x$ with pro-unipotent radical $\mathscr{M}_x^+ = M \cap \mathscr{P}_x^+$ [MP96, p.107]. If $x \in A$ is special for $G$, then $x$ is also special for $M$ (clear by condition (2.1)). Hence, for special $\mathscr{P}_x$ in $G$ the subgroup $\mathscr{M}_x$ is special in $M$.

**Lemma 2.5.** *For* $\mathbf{G} = \mathrm{GSp}(2g)$ *and* $\Theta \subsetneq \Delta_{\mathrm{af}}$ *let* $\Theta' = \{\psi_i \in \Delta_{\mathrm{af}} \mid \psi_{g-i} \in \Theta\}$. *Then the standard parahoric subgroup* $\mathscr{P}_x$ *attached to* $\Theta$ *is* $G$*-conjugate to the parahoric subgroup* $\mathscr{P}_{x'}$ *attached to* $\Theta'$. *Especially, all the hyperspecial parahoric subgroups of* $G$ *are pairwise* $G$*-conjugate.*

*Proof.* The Atkin-Lehner element $u_1 = \mathrm{antidiag}(1, \dots, 1, \varpi, \dots, \varpi) \in G$ gives rise to an element of the affine Weyl group that maps $e_i$ to $e_0 - e_{g+1-i}$ for $1 \le i \le g$ and fixes $e_0$. Therefore, it interchanges the root $\alpha_i$ with the root $\alpha_{g-i}$ for $1 \le i \le g-1$ and $\alpha_g$ with $-\beta$. The adjoint action of $u_1$ on the affine roots maps $\psi_i$ to $\psi_{g-i}$ for $0 \le i \le g$ and flips the affine Dynkin diagram [Tit79, §1.1]. Hence, $\mathscr{P}_x$ and $\mathscr{P}_{x'}$ are $G$-conjugate. Compare [Tit79, § 2.5]. □

**The Moy-Prasad filtration.** For each parahoric subgroup $\mathscr{P}_x \subseteq G$ Moy and Prasad [MP94, §2] have constructed an exhaustive filtration of open-compact normal subgroups $\mathscr{P}_{x,r} \subseteq \mathscr{P}_x$ for real $r \ge 0$. The group $\mathscr{P}_{x,r}$ is generated by

$$T_n = \{t \in \mathbf{T}(F) \mid \lambda(t) \in 1 + \mathfrak{p}^n \ \forall \lambda \in X^*(T)\}$$

for $n \ge r$ and the root subgroups $U_\psi$ for the affine roots $\psi(x) \ge r$. The subgroup $\mathscr{P}_{x,r^+} = \bigcup_{s>r} \mathscr{P}_{x,s}$ is normal in $\mathscr{P}_{x,r}$. Especially, the pro-unipotent radical of $\mathscr{P}_x = \mathscr{P}_{x,0}$ is $\mathscr{P}_x^+ = \mathscr{P}_{x,0^+}$. For $r, s \ge 0$ the commutator $[\mathscr{P}_{x,r}, \mathscr{P}_{x,s}]$ is contained in $\mathscr{P}_{x,r+s}$, so the quotient $\mathscr{P}_{x,r}/\mathscr{P}_{x,r^+}$ is abelian for $r > 0$.

Let $P = M \ltimes U$ be a standard parabolic subgroup of $G$ with opposite parabolic $P^- = M \ltimes U^-$. For every $x \in A$ and $r > 0$ the filtration subgroup $\mathscr{P}_{x,r} \subseteq \mathscr{P}_x$ admits the Iwahori decomposition [MP96, 4.2]

$$\mathscr{P}_{x,r} = (\mathscr{P}_{x,r} \cap U^-)(\mathscr{P}_{x,r} \cap M)(\mathscr{P}_{x,r} \cap U). \tag{2.3}$$

For every $x$ there is $r > 0$ such that the pro-unipotent radical is $\mathscr{P}_x^+ = \mathscr{P}_{x,r}$. Therefore $\mathscr{P}_x^+$ admits Iwahori decomposition, too.

## 2.3. Representations

Fix a totally real global number field $F$ with integers $\mathfrak{o}$ and adèle ring $\mathbb{A} = \mathbb{A}_\infty \times \mathbb{A}_f$. Let $\mathbf{G}$ be a connected reductive linear group scheme defined over $\mathfrak{o}$. For every place $v$ we equip $\mathbf{G}(F_v)$ with the inherited topology of $F_v$. For non-archimedean $v$, we fix the Haar measure $\mathrm{d}x$ such that the volume of a hyperspecial group $\mathbf{G}(\mathfrak{o}_v)$ is one. For archimedean places, the Haar measure on $\mathbf{G}(F_v)$ can be chosen canonically such that the product measure on the adelic group $\mathbf{G}(\mathbb{A})$ is the Tamagawa measure.

For locally compact topological groups $H$, the *modulus character* $\Delta_H : H \to \mathbb{R}_{>0}$ is defined by $\Delta_H(g) = \int_H f(xg)\,\mathrm{d}x / \int_H f(x)\,\mathrm{d}x$ for every $f \in C_c(H)$.

### 2.3.1. Real Lie groups

We fix an archimedean place $v$ of $F$ and write $\mathbb{R}$ instead of $F_v$. Let $G$ be the Lie group of $\mathbb{R}$-valued points $\mathbf{G}(\mathbb{R})$ with Lie algebra $\mathfrak{g}(\mathbb{R})$. The invariants under the Cartan involution form the maximal compact subgroup $K$ with corresponding Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$. Let $\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}$ denote the complexification.

**Admissible representations.** A *representation* $(\pi, H_\pi)$ of $G$ is a homomorphism $\pi : G \to \mathrm{Aut}(H_\pi)$ to the group of automorphisms on a complex Hilbert space $H_\pi$ such that the map $(x, v) \mapsto \pi(x)v$ is continuous. It is *unitary* if $\pi(x)$ is unitary for every $x \in G$. A vector $v \in H_\pi$ is *smooth* if for every $X$ in the Lie algebra $\mathfrak{g}$ the limit

$$Xv = \lim_{t \to 0} (\pi(\exp tX)v - v)/t$$

exists for real $t > 0$. The space $V^\infty$ of smooth vectors is dense in $V$ and is a $\mathfrak{g}$-module via $X \mapsto (v \mapsto Xv)$ for $X \in \mathfrak{g}$. The space of $K$*-finite* vectors

$$V_K = \{v \in V\,;\, \dim \operatorname{span} \pi(K)v < \infty\}$$

is dense in $V^\infty$. By Peter-Weyl's theorem, $V_K$ decomposes as a direct sum

$$V_K = \bigoplus_\tau n_\tau \tau$$

of finite-dimensional irreducible representations $\tau$ of $K$ with multiplicity $n_\tau \leq \infty$.

For smooth $\pi$ the representations $\tau$ with $n_\tau > 0$ are the $K$*-types*. When $n_\tau < \infty$ for every $K$-type of $\pi$, the representation $(\pi, V)$ is *admissible*. The category $\mathbf{Rep}(G)$ has isomorphism classes of admissible representations as objects and intertwining operators as morphisms.

Since $V_K$ is preserved under the action of $\mathfrak{g}$, it is a $(\mathfrak{g}, K)$-module. Representations of $G$ are *infinitesimally equivalent* if their $(\mathfrak{g}, K)$-modules are isomorphic.

**The induced representation.** Let $P \subseteq G$ be a parabolic subgroup and let $(\sigma, H_\sigma)$ be an admissible representation of $P$ such that $\sigma$ is unitary on the compact subgroup $K \cap P$.[3] The vector space of continuous functions $f : G \to H_\sigma$ with

$$f(px) = \Delta_P^{1/2}(p)\sigma(p)f(x) \qquad \text{for } p \in P \text{ and } x \in G$$

is a pre-Hilbert space $(H_\pi)_0$ with respect to the scalar product

$$\langle f_1, f_2 \rangle = \int_K \langle f_1(k), f_2(k) \rangle_\sigma \, \mathrm{d}k \qquad \text{for } f_1, f_2 \in (H_\pi)_0.$$

For $x, y \in G$ and $f \in (H_\pi)_0$ define $(\pi(y)f)(x) = f(xy)$, then $\pi(y)$ extends to a bounded operator on the completion $H_\pi$ of $(H_\pi)_0$. The *induced representation* from $\sigma$ is the $G$-representation is $\mathrm{Ind}_H^G(\sigma, H_\sigma) = (\pi, H_\pi)$. If $\sigma$ is unitary, then $\mathrm{Ind}_H^G(\sigma, H_\sigma)$ is also unitary.

**The infinitesimal character.** Fix an admissible irreducible representation $(\pi, V)$ of $G$. On its $K$-finite vectors every $z$ in the center $Z(\mathfrak{g}^\mathbb{C})$ of the universal enveloping algebra of $\mathfrak{g}^\mathbb{C}$ acts by multiplication with a complex scalar $\chi(z) \in \mathbb{C}$, $\chi(z)\mathrm{id}_V = \pi(z)$. This is the *infinitesimal character* $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ of $(\pi, V)$.

Every infinitesimal character is of the form $\chi_\lambda = \lambda \circ \gamma$, where $\lambda$ is a character of the universal enveloping algebra $\mathcal{U}(\mathfrak{t}^\mathbb{C})$ and $\gamma$ is the Harish-Chandra homomorphism

$$\gamma : Z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{t})^{W_G}$$

from the center of $\mathcal{U}(\mathfrak{g}^\mathbb{C})$ to the Weyl invariants in $\mathcal{U}(\mathfrak{t}^\mathbb{C})$ [Kna86, §VIII.6].

**The discrete series.** An irreducible continuous representation $\pi$ of a semisimple group $G$ on a complex Hilbert space $V$ is (in the) *discrete series* if it is equivalent to a direct summand of the regular representation of $G$ on $L^2(G, \mathbb{C})$ defined by right multiplication.

For semisimple $G$, the irreducible representations in the discrete series have been classified by Harish-Chandra [Kna86, Thms. 9.20, 12.21].

**Proposition 2.6** (Harish-Chandra)**.** *Let $G$ be a linear connected semisimple Lie group and $K$ a maximal closed subgroup with the same rank as $G$. Fix a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k} \subseteq \mathfrak{g}$. For every nonsingular character $\lambda : i\mathfrak{t} \to \mathbb{R}$, such that $\lambda + \delta_{\Delta_\lambda^+}$ is analytically integral, there is a discrete series representation $\pi_\lambda$ of $G$ such that*

- *i)* *$\pi_\lambda$ has infinitesimal character $\chi_\lambda$,*
- *ii)* *$\pi_\lambda$ has a minimal $K$-type $\tau_\Lambda$ with highest weight $\Lambda = \lambda + \delta_{\Delta_\lambda^+} - 2\delta_K$ (the* Blattner parameter*), where $\delta_{\Delta_\lambda^+}$ and $\delta_K$ are the half-sums of positive roots of $G$ and $K$, respectively,*

[3] Representations of compact groups are always unitarizable [Wal88, 1.4.8].

*iii) every other K-type has highest weight* $\Lambda' = \Lambda + \sum_{\alpha\in\Delta_\lambda^+} n_\alpha \alpha$ *with* $n_\alpha \in \mathbb{Z}_{\geq 0}$.

*Here* $\Delta_\lambda^+$ *is the set of roots* $\alpha$ *of* $G$ *with* $\langle\lambda,\alpha\rangle > 0$ *The half sum of (compact) roots in* $\Delta_\lambda^+$ *is* $\delta_{\Delta_\lambda^+}$ *(or* $\delta_K$*, respectively). Two such representations are infinitesimally equivalent if and only if their Harish-Chandra parameters* $\lambda$ *are conjugate under the compact Weyl group* $W_K$*. Up to infinitesimal equivalence, these are all the discrete series representations of* $G$.

*If the rank of* $K$ *is smaller than the rank of* $G$*, there are no discrete series representations.*

**The symplectic group.** For the semisimple Lie group $\mathbf{G} = \mathrm{Sp}(2g)$

$$K = \mathbf{K}(\mathbb{R}) = G \cap O(2g) = \left\{ x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in G \,|\, aa^t + bb^t = I_g,\ ab^t = ba^t \right\},$$

which is isomorphic to the unitary group $U(g)$ via $\kappa : x \mapsto a + \sqrt{-1}b \in U(g)$. The Lie algebra of $K$ is the space of invariants under the Cartan involution on Lie algebras $\mathfrak{g} \to \mathfrak{g}$, $X \mapsto -X^t$

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{Mat}(2g \times 2g, \mathbb{R}) \,|\, a = -a^t,\ b = b^t \right\} \subseteq \mathfrak{g}.$$

Together with the $-1$-eigenspace of the Cartan involution

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \mathrm{Mat}(2g \times 2g, \mathbb{R}) \,|\, a = a^t,\ b = b^t \right\} \subseteq \mathfrak{g}.$$

we obtain the decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$. Fix the compact torus $\widetilde{T} = \kappa^{-1}\{\mathrm{diag}(t_1, \ldots, t_g)); t_i \in \mathbb{C},\ |t_i| = 1\}$. The Cartan subalgebra $\widetilde{\mathfrak{t}} \subseteq \mathfrak{k}^{\mathbb{C}}$ is given by the set of $s = \sum_{j=1}^g s_j b_j$ for $s_j \in \mathbb{R}$ and $b_j = -\sqrt{-1}E_{j,g+j} + \sqrt{-1}E_{g+j,j}$. The group of complex linear characters of $\widetilde{\mathfrak{t}}^{\mathbb{C}}$ is generated by $\widetilde{e}_i : \widetilde{\mathfrak{t}}^{\mathbb{C}} \to \mathbb{C}$, $s \mapsto s_i$. The compact roots are the roots $\pm(\widetilde{e}_i - \widetilde{e}_j)$, $i < j$ of $(\mathfrak{k}, \mathfrak{t})$. Together with the non-compact roots $\pm(\widetilde{e}_i + \widetilde{e}_j)$, $1 \leq i, j \leq g$, they form the root system[4] of $(\mathfrak{g}, \mathfrak{t})$.

The Weyl group of $K$ is generated by the reflection $s_\alpha$ at the compact roots $\alpha \in \Phi(\mathbf{K})$. It permutes the generators $e_i$, so $W_K$ is isomorphic to $\Sigma_g$. The Weyl group of $G = \mathrm{Sp}(2g)$ is the semidirect product $\Sigma_g \ltimes \{\pm 1\}^g$.

The half sum of the positive roots of $\mathrm{Sp}(2g)$ in $\Delta_\lambda^+$ for $\lambda_1 > \lambda_2 > \cdots > 0$ is $\delta_{\Delta_\lambda^+} = \sum_{i=1}^{g}(g - i + 1)\widetilde{e}_i$ and the half sum of positive compact roots is $\delta_K = \frac{1}{2}\sum_{i=1}^{g}(g + 1 - 2i)\widetilde{e}_i$. For $g = 2$ this means $\delta_{\Delta_\lambda^+} = 2\widetilde{e}_1 + \widetilde{e}_2 = (2, 1)$ and $\delta_K = (\widetilde{e}_1 - \widetilde{e}_2)/2 = (\frac{1}{2}, -\frac{1}{2})$.

[4]This is not quite the same as the root system of Section 2.1.1. We write $\widetilde{e}_i$ instead of $e_i$ in order to emphasize this distinction.

**Example 2.7** $(\mathrm{SL}(2,\mathbb{R}))$**.** *Let $\nu(x) = |x|$ be the valuation and $\mathrm{sgn}(x) = x/|x|$ the sign character for $x \in \mathbb{R}^\times$. For $G = \mathrm{SL}(2,\mathbb{R})$ fix $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, $K = \mathrm{SO}(2,\mathbb{R})$ and the standard Borel $B$ of upper triangular matrices. Its roots $\pm 2\widetilde{e}_1$ for $\widetilde{e}_1$ are non-compact. The discrete series representations $\mathcal{D}^\pm(k) = \pi_\lambda$ for $\lambda \in \mathbb{Z}\backslash\{0\}$ and $k = |\lambda| + 1$ and $\pm = \mathrm{sgn}(\lambda)$ are the infinite dimensional subrepresentations of the normalized Borel induced representation*

$$\mathrm{Ind}_B^G \sigma = \{f : G \to \mathbb{R} \,|\, f(bx) = \delta_B^{1/2}(b)\sigma(b)f(x)\}$$

*where $\sigma(b) = \mathrm{sgn}^k \nu^{k-1}(a)$ and the modulus character is $\delta_B(b) = |a^2|$ for $b = \left(\begin{smallmatrix} a & * \\ & a^{-1} \end{smallmatrix}\right)$. The Harish-Chandra parameter is $\lambda(-\sqrt{-1})\widetilde{e}_1$ and the minimal $K$-type $\mathrm{SO}(2,\mathbb{R}) \to \mathbb{C}^\times$, $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mapsto (a+ib)^{\pm k}$ has Blattner parameter $\pm k(-\sqrt{-1})\widetilde{e}_1$.*

*The same construction for $k = 1$ gives the unitary* limit of discrete series $\mathcal{D}^\pm(1)$.

**Example 2.8** $(\mathrm{GL}(2,\mathbb{R}))$**.** *We denote by $\mathcal{D}_\omega(k)$ for $k \in \mathbb{N}_{>1}$ the essentially discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with central character $\omega : \mathbb{R}^\times \to \mathbb{C}^\times$ such that $\omega(-1) = (-1)^k$ and whose restriction to $\mathrm{SL}(2,\mathbb{R})$ is $\mathcal{D}^+(k) \oplus \mathcal{D}^-(k)$. This is the unique infinite-dimensional constituent of $\mu_1 \times \mu_2$ for smooth complex characters $\mu_1, \mu_2$ of $\mathbb{R}^\times$ with $\omega = \mu_1\mu_2$ and $\mu_1\mu_2^{-1} = \mathrm{sgn}^k\nu^{k-1}$. When the central character is trivial, we also write $\mathcal{D}(k)$.*

**Example 2.9** $(\mathrm{Sp}(4,\mathbb{R}))$**.** *Discrete series representations $\pi_\lambda$ of $G = \mathrm{Sp}(4,\mathbb{R})$ are attached to Harish-Chandra-parameters $(\lambda_1,\lambda_2) \in \mathbb{Z}^2$ with $0 \neq \lambda_1 \neq \pm\lambda_2 \neq 0$.*

*For $\lambda_1 > \lambda_2 > 0$, the representation $\pi_{(\lambda_1,\lambda_2)}$ is non-generic and holomorphic with Blattner parameter $\Lambda = \lambda + (1,2)$. For $0 > \lambda_1 > \lambda_2$ the representation $\pi_\lambda$ is non-generic and antiholomorphic with Blattner parameter $\Lambda = \lambda + (-2,-1)$.*

*For $\lambda_1 > 0 > \lambda_2$ or $\lambda_2 > 0 > \lambda_1$, the discrete series $\pi_\lambda$ is large and therefore generic, but not (anti-)holomorphic. The Blattner parameter is $\Lambda = \lambda + (1,0)$ or $\Lambda = \lambda + (0,-1)$, respectively.*

**Example 2.10** $(\mathrm{GSp}(4,\mathbb{R}))$**.** *Fix integers $\lambda_1 > \lambda_2 > 0$ and a character $\omega : \mathbb{R}^\times \to \mathbb{C}^\times$ with $\omega(-1) = (-1)^{\lambda_1+\lambda_2+1}$. Up to infinitesimal equivalence, there are two essentially discrete series representations of $\mathrm{GSp}(4,\mathbb{R})$ with infinitesimal character $\chi_\lambda$ and central character $\omega$. One of them is the non-generic representations $\pi^H_{\lambda,\omega}$ whose restriction to $\mathrm{Sp}(4,\mathbb{R})$ decomposes into the holomorphic and anti-holomorphic discrete series $\pi_{(\lambda_1,\lambda_2)} \oplus \pi_{(-\lambda_2,-\lambda_1)}$. The other one is the generic non-holomorphic representation $\pi^W_{\lambda,\omega}$ whose restriction to $\mathrm{Sp}(4,\mathbb{R})$ decomposes into the non-holomorphic discrete series $\pi_{(\lambda_1,-\lambda_2)} \oplus \pi_{(\lambda_2,-\lambda_1)}$.*

### 2.3.2. $p$-adic groups

The group $G = \mathbf{G}(F_v)$ of $F_v$-valued points of $\mathbf{G}$ for a non-archimedean place $v$ is a locally profinite topological group. A (complex linear) representation $(\pi, V)$ of $G$ is a homomorphism $\pi : G \to \mathrm{Aut}(V)$ to the automorphism group of a vector space $V$ over $\mathbb{C}$. For a subgroup $K \subseteq G$ the vector space $V^K$ of $K$-invariants is the space of $v \in V$ with $\pi(k)v = v$ for every $k \in K$. The representation $(\pi, V)$ is *smooth* if $V = \bigcup_K V^K$ for the compact open subgroups $K \subseteq G$. The representation is *admissible* if it is smooth and $V^K$ is finite-dimensional for every compact open subgroup $K \subseteq G$. Every irreducible smooth representation of $G$ is admissible.

An admissible irreducible representation of $G$ is called *parahori-spherical* (or $\mathscr{P}_x$-*spherical*) if it admits non-zero invariants under a parahoric subgroup $\mathscr{P}_x \subseteq G$. For hyperspecial $\mathscr{P}_x$, we say *spherical*.

**The Hecke algebra.** Let $\mathcal{H} = (C_c^\infty(G), *)$ be the Hecke algebra with the convolution product. Every admissible representation $(\pi, V)$ of $G$ defines a representation of the Hecke algebra $\mathcal{H}(G)$ via

$$\pi(f)v = \int_G f(x)\pi(x)v \, \mathrm{d}x, \qquad f \in C_c^\infty(G), \quad v \in V.$$

Every such $f$ is biinvariant with respect to some open compact subset $K_f \subseteq G$, so $\pi(f)v$ is contained in the finite-dimensional subspace $V^{K_f}$. The *character* of $\pi$ is the conjugation-invariant distribution

$$\chi_\pi : C_c^\infty(G) \longrightarrow \mathbb{C}, \qquad f \longmapsto \mathrm{tr}(\pi(f), V^{K_f}).$$

Two representations have the same character if and only if they are isomorphic up to semisimplification.

An intertwining operator $f : (\pi_1, V_1) \to (\pi_2, V_2)$ between admissible representations is a linear map $f : V_1 \to V_2$ such that $f \circ \pi_1(x) = \pi_2(x) \circ f$ for every $x \in G$. The category of isomorphism classes of admissible representations of $G$ will be denoted $\mathbf{Rep}(G)$. Its morphisms are the intertwining operators.

**Induced representation**. Let $(\sigma, H_\sigma)$ be a smooth representation of a closed subgroup $M \subseteq G$. Denote by $H_\pi$ be the vector space of functions $f : G \to H_\sigma$ such that

$$f(mxk) = \Delta_M^{1/2}(m)\sigma(m)f(x) \qquad \forall m \in M, x \in G, k \in K_f$$

for an open subgroup $K_f \subseteq G$ depending on $f$. The representation $\mathrm{Ind}_M^G(\sigma, H_\sigma) = (\pi, H_\pi)$ of $G$ is defined by

$$(\pi(y)f)(x) = f(xy) \qquad \forall x, y \in G.$$

This is the *induced representation* of $\sigma$. The subspace of those $f$ whose support is compact modulo $M$ is the compactly induced representation $\text{c-Ind}_H^G(\sigma, H_\sigma)$. If $M\backslash G$ is compact, induction preserves admissibility.

### 2.3.3. Parabolic induction and restriction

Let $\mathfrak{o} \to k$ be a nonarchimedean local or a finite field over $\mathfrak{o}$ and let $G = \mathbf{G}(k)$. For a parabolic subgroup $P = M \ltimes U$ of $G$ fix an admissible representation $\sigma$ of $M$. An admissible unitary representation $\sigma$ of $M$ gives rise to a representation $\sigma'$ of $P$ by inflation. Then

$$\mathbf{i}_{P,M}^{G}\,\sigma = \mathrm{Ind}_P^G(\sigma')$$

is the *parabolically induced representation* from $\sigma$. By the Iwasawa decomposition $P\backslash G$ is compact, so parabolic induction preserves admissibility.

Fix an admissible representation $(\pi, V)$ of $G$. The normalized restriction to a parabolic subgroup $P$ is the representation

$$P \to \mathrm{Aut}(V), \qquad p \mapsto \Delta_P^{-1/2} \cdot \pi(p).$$

The Levi quotient $M \cong P/U$ preserves the space of $U$-coinvariants

$$V_U = V / \langle \pi(u)v - v \,|\, u \in U, v \in V \rangle .$$

The *parabolical restriction* or *Jacquet module* of $\sigma$ is the admissible $M$-representation

$$\mathbf{r}_{P,M}^{G}(\pi, V) = (\Delta_P^{-1/2} \cdot \pi, V_U)$$

The parabolic induction functor $\mathbf{i}_{P,M}^{G} : \mathbf{Rep}(M) \to \mathbf{Rep}(G)$ is right adjoint to $\mathbf{r}_{P,M}^{G} : \mathbf{Rep}(G) \to \mathbf{Rep}(M)$. Both functors are exact and transitive. After semisimplification, they do not depend on $P$, only on $M$ and $G$. The normalization ensures that unitarity is preserved.

An irreducible admissible representation $\pi \in \mathbf{Rep}(G)$ is *cuspidal* if $\mathbf{r}_{P,M}^{G}(\pi) = 0$ for every proper parabolic subgroup $P \subsetneq G$.

**Tadic notation**. If $P_I = M_I \ltimes U_I$ is a standard parabolic subgroup of $G = \mathrm{GSp}(2g, k)$, the Levi quotient is of block diagonal form and admits a natural decomposition

$$M_I \cong \mathrm{GL}(n_1, k) \times \cdots \times \mathrm{GL}(n_m, k) \times \mathrm{GSp}(2g - 2|n|, k)$$

for $|n| = \sum_{i=1}^{m} n_i \leq g$. For $\sigma_i \in \mathbf{Rep}(\mathrm{GL}(n_i, k))$ and $\rho \in \mathbf{Rep}(\mathrm{GSp}(2g - 2|n|, k))$ Tadic [Tad91] has introduced the notation

$$\sigma_1 \times \cdots \times \sigma_m \rtimes \rho = \mathbf{i}_{P_I,M_I}^{G}(\sigma_1 \boxtimes \cdots \boxtimes \sigma_m \boxtimes \rho) \in \mathbf{Rep}(G). \tag{2.4}$$

### 2.3.4. Automorphic representations

Fix a totally real number field $F$ with adèle ring $\mathbb{A}$. Let $\omega : Z(F)\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ be a unitary character of the center $Z$ of $\mathbf{G}$.

A *square-integrable automorphic form with central character* $\omega$ is a smooth function $\phi : \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}) \to \mathbb{C}$ such that $\phi(xz) = \phi(x)\omega(z)$ for every $z \in Z(\mathbb{A})$ and such that $|\phi|^2$ is integrable over $(Z(\mathbb{A})\mathbf{G}(F))\backslash\mathbf{G}(\mathbb{A})$. As usual, we identify two such $\phi$ if their difference is non-zero only on a subset of measure zero. The completion of the vector space of these $\phi$ with respect to the $L^2$-norm is the Hilbert space

$$L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}), \omega).$$

Right-multiplication defines a unitary representation $R$ of $\mathbf{G}(\mathbb{A})$ on $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}), \omega)$

$$(R(g)\phi)(x) = \phi(xg) \text{ for } g, x \in \mathbf{G}(\mathbb{A}).$$

An *automorphic representation* of $\mathbf{G}(\mathbb{A})$ is an irreducible smooth representation that is isomorphic to a subquotient of $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}), \omega)$. A *Hecke character* is an automorphic representation of $\mathrm{GL}(1, \mathbb{A})$.

The *discrete spectrum* $L^2_d(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}), \omega)$ is the largest subspace $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}), \omega)$ that completely decomposes as a Hilbert direct sum. It orthogonal complement is the *continuous spectrum.*

An automorphic form $\phi$ is *cuspidal* if

$$\int_{N(F)\backslash N(\mathbb{A})} \phi(nx)\,\mathrm{d}n = 0$$

for the unipotent radical $N$ of every proper parabolic subgroup of $\mathbf{G}$ and (almost) every $x \in \mathbf{G}(\mathbb{A})$. The *cuspidal spectrum* $L^2_0(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}), \omega)$ consists of cuspidal $\phi$ and it is a closed subspace of the discrete spectrum. An automorphic representation is *cuspidal* if it occurs in the cuspidal spectrum.

By the tensor product theorem [Gel75, §4.C], every automorphic representation $\pi$ of $\mathbf{G}(\mathbb{A})$ is isomorphic to a restricted tensor product

$$\pi \cong \bigotimes_v \pi_v$$

of irreducible representations $\pi_v$ of $\mathbf{G}(F_v)$ for the non-archimedean places $v < \infty$ and $(\mathfrak{g}, K)$-modules $\pi_v$ for the real places $v|\infty$.

Two automorphic representations are *weakly equivalent* if their local factors are isomorphic at almost every place. This defines an equivalence relation. An automorphic representation is *cuspidal associated to a parabolic* or *CAP* if it cuspidal and weakly equivalent to a constituent of a representation globally parabolically induced from an automorphic representation.

## 2.4. The parahoric restriction functor

Let $G$ be the group of $F$-rational points of a split connected reductive linear algebraic group over a non-archimedean local number field $F$. Fix a split maximal torus $T$ and a basis of simple roots $\Delta$ generating a standard Borel subgroup $B$.

Let $\mathscr{P}_x \subseteq G$ be a parahoric subgroup with Levi decomposition

$$1 \to \mathscr{P}_x^+ \to \mathscr{P}_x \to \underline{\mathscr{P}_x} \to 1.$$

For an admissible (complex linear) representation $\pi : G \to \mathrm{Aut}(V)$, the action of $\mathscr{P}_x$ preserves the subspace $V^{\mathscr{P}_x^+}$ of $\mathscr{P}_x^+$-invariants in $V$. This defines a unique representation $(\pi|_{\mathscr{P}_x}, V^{\mathscr{P}_x^+})$ of $\mathscr{P}_x/\mathscr{P}_x^+ \cong \underline{\mathscr{P}_x}$.

An intertwining operator $V_1 \to V_2$ between admissible representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ of $G$ preserves $\mathscr{P}_x^+$-invariants and defines a canonical operator $V_1^{\mathscr{P}_x^+} \to V_2^{\mathscr{P}_x^+}$.

**Definition 2.11.** The *parahoric restriction functor* for $\mathscr{P}_x$ is the exact functor

$$\mathbf{r}_{\mathscr{P}_x} : \mathbf{Rep}(G) \to \mathbf{Rep}(\underline{\mathscr{P}_x}), \qquad \begin{cases} (\pi, V) \mapsto (\pi|_{\mathscr{P}_x}, V^{\mathscr{P}_x^+}), \\ (V_1 \to V_2) \mapsto (V_1^{\mathscr{P}_x^+} \to V_2^{\mathscr{P}_x^+}). \end{cases}$$

This is the parahoric analogoue of Jacquet's functor of parabolic restriction. Parahoric restriction has been studied by Morris [Mor93], Moy [Moy88], Vignéras [Vig01] and others.

### 2.4.1. Basic properties

For parahoric subgroups $\mathscr{P}_y \subseteq \mathscr{P}_x \subseteq G$, parahoric restriction is transitive with respect to Jacquet's parabolic restriction functor [Vig03, 4.1.3]

$$\mathbf{r}_{\mathscr{P}_y} \cong \mathbf{r}^{\mathscr{P}_x}_{\underline{\mathscr{P}_y}} \circ \mathbf{r}_{\mathscr{P}_x}. \tag{2.5}$$

Here we take $\underline{\mathscr{P}_y}$ as the Levi quotient of the parabolic subgroup $\mathscr{P}_y/\mathscr{P}_x^+ \subseteq \mathscr{P}_x/\mathscr{P}_x^+$.

The *depth* of an irreducible admissible representation $\pi$ of $G$ is the smallest real number $r \geq 0$ such that $\pi$ admits non-zero invariants under $\mathscr{P}_{x,r}^+$ for some parahoric subgroup $\mathscr{P}_x \subseteq G$. Especially, $\pi$ has depth zero if and only if it admits non-zero parahoric restriction with respect to some parahoric subgroup.

**Lemma 2.12.** *Twisting an admissible representation $\rho$ of $\mathrm{GSp}(2g, F)$ by a tamely ramified or unramified character $\mu$ of $F^\times$ commutes with parahoric restriction in the following sense:*

$$\mathbf{r}_{\mathscr{P}_x}((\mu \circ \mathrm{sim}) \otimes \rho) \cong (\widetilde{\mu} \circ \mathrm{sim}) \otimes \mathbf{r}_{\mathscr{P}_x}(\rho), \quad \textit{for} \quad \widetilde{\mu} = \mathbf{r}_{\mathfrak{o}^\times}(\mu). \tag{2.6}$$

*For $G = \mathrm{GL}(g, F)$ the analogous formula holds with respect to the determinant.*

*Proof.* The restriction of the similitude character (or the determinant, respectively) to $\mathscr{P}_x^+$ factors over $\mathscr{P}_x^+ \to 1 + \mathfrak{p} \subseteq \ker \mu$, so the subspace of $\mathscr{P}_x^+$-invariants is preserved under twisting. □

The corresponding statement for wildly ramified $\mu$ is not true.

### 2.4.2. Results of Moy and Prasad

Moy and Prasad [MP96, 3.4] have defined a *minimal $K$-type of depth zero* to be a pair $(\mathscr{P}_x, \sigma)$ of a parahoric subgroup $\mathscr{P}_x \subseteq G$ and a non-zero cuspidal irreducible admissible representation $\sigma$ of the finite reductive group $\underline{\mathscr{P}_x}$, inflated[5] to $\mathscr{P}_x$. A depth zero minimal $K$-type $(\mathscr{P}_x, \sigma)$ is *contained* in an irreducible admissible representation $(\pi, V)$ of $G$, if $\sigma$ is a subquotient of the parahoric restriction $\mathbf{r}_{\mathscr{P}_x}(\pi, V)$. Associativity of minimal $K$-types is defined as in [MP96].

**Lemma 2.13** ([MP96, 6.2])**.** *Suppose $(\sigma, \mathscr{P}_x)$ is a minimal $K$-type occuring in an irreducible admissible representation $(\pi, V)$. A minimal $K$-type $(\mathscr{P}_y, \rho)$ occurs in $(\pi, V)$ if and only if it is associate to $(\mathscr{P}_x, \sigma)$.*

Now let $P = M \ltimes U \subseteq G$ be a standard parabolic subgroup of $G$ with $T \subseteq M$. Let $\sigma$ be an admissible irreducible representation of $M$ and $\pi$ an irreducible subquotient of its parabolic induction $\mathbf{i}_{P,M}^G(\sigma)$.

**Corollary 2.14.** *Let $x \in A$ be a point in the standard apartment of $G$. If $\sigma$ has non-zero parahoric restriction with respect to $\mathscr{M}_x = M \cap \mathscr{P}_x$, then $\pi$ has non-zero parahoric restriction with respect to $\mathscr{P}_x$.*

*Proof.* By (2.3), the pro-unipotent radical $\mathscr{P}_x^+$ admits Iwahori decomposition. The statement is then implied by a theorem of Jacquet, see [Cas95, 3.3.6]. □

For special $x$ we have the equivalence $\mathbf{r}_{\mathscr{M}_x}(\sigma) \neq 0 \Leftrightarrow \mathbf{r}_{\mathscr{P}_x}(\pi) \neq 0$ by Thm. 2.19, but that does not hold for arbitrary $x$. However, the representations $\sigma$ and $\pi$ have the same depth [MP96, 5.2(1)].

[5]For the inflation of $\sigma$ to $\mathscr{P}_x$ we write $\sigma$ again.

### 2.4.3. Cuspidal irreducible smooth representations of depth zero

For a maximal parahoric subgroup $\mathscr{P}_x$ of $G$ and a cuspidal irreducible representation $\sigma$ of $\underline{\mathscr{P}_x}$ let $\tau$ be an irreducible representation of the normalizer $N(\mathscr{P}_x)$ whose restriction to $\mathscr{P}_x$ contains the inflation of $\sigma$ as a constituent.

**Theorem 2.15** (Classification). *The compactly induced representation* $\text{c-Ind}_{N(\mathscr{P}_x)}^G(\tau)$ *is irreducible, cuspidal, admissible, of depth zero, and contains the minimal $K$-type $(\mathscr{P}_x, \sigma)$. Every cuspidal admissible irreducible $G$-representation of depth zero is of this form for some minimal depth zero $K$-type $(\mathscr{P}_x, \sigma)$ with maximal parahoric $\mathscr{P}_x$.*

*Proof.* This has been shown by Moy and Prasad [MP96, 6.6, 6.8] and independently by Morris [Mor96]. □

Fix a non-trivial additive character $\psi : (F, +) \to \mathbb{C}^\times$ whose restriction to $\mathfrak{o}$ factors over a non-trivial additive character of the residue field $\widetilde{\psi} : \mathfrak{o}/\mathfrak{p} \to \mathbb{C}^\times$.

**Proposition 2.16.** *A depth zero supercuspidal irreducible admissible representation $\pi \cong \text{c-Ind}_{N(\mathscr{P}_x)}^G(\tau)$ of $G$ is $\psi$-generic if and only if $\mathscr{P}_x$ is a hyperspecial maximal parahoric and $\sigma$ is a $\widetilde{\psi}$-generic representation of $\mathscr{P}_x/\mathscr{P}_x^+$.*

*Proof.* This is a result of deBacker and Reeder [DR09, 6.1.1, 6.1.2]. □

**Corollary 2.17.** *For a depth zero cuspidal irreducible admissible $G$-representation $\pi = \text{c-Ind}_{N(\mathscr{P}_x)}^G(\tau)$ with maximal parahoric $\mathscr{P}_x$ the parahoric restriction $\mathbf{r}_{\mathscr{P}_{gx}}(\pi)$ for $g \in G$ is isomorphic to the restriction of $\tau$ to $\mathscr{P}_x/\mathscr{P}_x^+$. For another parahoric $\mathscr{P}_y \subseteq G$ whose normalizer is not $G$-conjugate to $N(\mathscr{P}_x)$ the parahoric restriction $\mathbf{r}_{\mathscr{P}_y}(\pi)$ is zero.*

*Proof.* Since $\mathscr{P}_{gx} = g\mathscr{P}_x g^{-1}$ we can assume $g = 1$ without loss of generality. By Mackey decomposition, the restriction of $\tau$ to $\mathscr{P}_x$ is a direct sum of $N(\mathscr{P}_x)$-conjugates of $\sigma$, which are all cuspidal. Since $\mathscr{P}_x^+$ is a normal subgroup of $N(\mathscr{P}_x)$, the space of $\pi^{\mathscr{P}_x^+}$-invariants in $V$ is preserved under the $\pi$-action of $N(\mathscr{P}_x)$ and this defines an admissible representation $(\rho, V^{\mathscr{P}_x^+})$ of $N(\mathscr{P}_x)$. Vignéras has shown that $\rho$ is isomorphic to $\tau$ [Vig01, Cor. 5.3]. Therefore $\mathbf{r}_{\mathscr{P}_x}(\pi)$ is isomorphic to the restriction of $\tau$ to $\mathscr{P}_x$.

If $\mathbf{r}_{\mathscr{P}_y}(\pi) \neq 0$ for a parahoric $\mathscr{P}_y$, then there is a minimal $K$-type $(\mathscr{P}_z, \chi)$ contained in $\pi$ with $\mathscr{P}_z \subseteq \mathscr{P}_y$. But $\mathscr{P}_z$ must be a maximal parahoric [MP96, 6.8] and therefore coincide with $\mathscr{P}_y$. Then the normalizer of $\mathscr{P}_y$ is $G$-conjugate to $N(\mathscr{P}_x)$ [Yu01, 3.3(ii)]. □

### 2.4.4. Hyperspecial parahoric restriction

Fix a hyperspecial parahoric subgroup $\mathscr{K}$ of $G$ and let $\mathbf{G}$ be the the associated reductive group scheme with $\mathbf{G}(F) = G$ and $\mathbf{G}(\mathfrak{o}) = \mathscr{K}$.

**Lemma 2.18.** *Let $\pi \cong \text{c-Ind}_{N(\mathscr{K})}^{G}(\tau)$ be a depth zero cuspidal irreducible admissible representation of $G$, where $\tau$ is an extension to $N(\mathscr{K})$ of an irreducible cuspidal admissible representation $\sigma$ of $\mathscr{K}/\mathscr{K}^{+}$. The hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}}(\pi)$ is irreducible and isomorphic to $\sigma$.*

*Proof.* The normalizer of the hyperspecial parahoric is $N(\mathscr{K}) = Z\mathscr{K}$, where $Z$ is the center of $G$, so $\tau$ is uniquely determined by $\sigma$ and a choice of a central character. By Cor. 2.17 the parahoric restriction $\mathbf{r}_{\mathscr{K}}(\pi)$ of $\pi$ is isomorphic to the restriction $\tau|_{\mathscr{K}}$. Since $Z$ commutes with $\mathscr{K}$, the restriction $\tau|_{\mathscr{K}}$ is irreducible and contains the irreducible constituent $\sigma$ by Frobenius reciprocity. □

Compare Vignéras [Vig96, 3.14a)] for $\mathbf{G} = \mathrm{GL}(g)$.

Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{N}$ be a standard parabolic subgroup scheme of $\mathbf{G}$ attached to $\mathscr{K}$ with corresponding group $P = M \ltimes U$ of $F$-rational points. Then $\mathscr{K}_M = \mathbf{M}(\mathfrak{o})$ naturally defines a hyperspecial parahoric subgroup of $M$.

**Theorem 2.19.** *Let $\sigma$ be an admissible representation of $M$. Then the hyperspecial parahoric restriction commutes with parabolic induction:*

$$\mathbf{i}_{\mathbf{P}(q),\mathbf{M}(q)}^{\mathbf{G}(q)} \circ \mathbf{r}_{\mathscr{K}_M}(\sigma) \cong \mathbf{r}_{\mathscr{K}} \circ \mathbf{i}_{P,M}^{G}(\sigma). \tag{2.7}$$

*For $\mathbf{G} = \mathrm{GL}(g), \mathrm{GSp}(2g)$ this implies*

$$\mathbf{r}_{\mathscr{K}}(\sigma_1 \times \cdots \times \sigma_m) \cong \mathbf{r}_{\mathrm{GL}(n_1,\mathfrak{o})}(\sigma_1) \times \cdots \times \mathbf{r}_{\mathrm{GL}(n_m,\mathfrak{o})}(\sigma_m),$$
$$\mathbf{r}_{\mathscr{K}}(\sigma_1 \times \cdots \times \sigma_m \rtimes \rho) \cong \mathbf{r}_{\mathrm{GL}(n_1,\mathfrak{o})}(\sigma_1) \times \cdots \times \mathbf{r}_{\mathrm{GL}(n_m,\mathfrak{o})}(\sigma_m) \rtimes \mathbf{r}_{\mathrm{GSp}(2g-|n|,\mathfrak{o})}(\rho)$$

*for admissible representations $\sigma_i$ of $\mathrm{GL}(n_i, F)$ with $i = 1, \ldots, m$ and representations $\rho$ of $\mathrm{GSp}(2g - |n|, F)$ using the Tadic notation.*

*Proof.* Hyperspecial parahoric subgroups admit Iwasawa decomposition [Tit79, 3.3.2], so one can apply Thm. 3.1.1 in Casselman's notes [Cas95]. □

For $\mathrm{GL}(g, F)$ an explicit proof has been given by Vignéras [Vig96, 3.14b)], the argument for the general case is completely analogous. Compare [MR].

| $\rho \in \mathbf{Irr}(G)$ | $\mathbf{r}_{\mathscr{K}_G}(\rho)$ of $\mathrm{GL}(2,\mathbb{F}_q)$ | $\mathbf{r}_{\mathscr{B}_G}(\rho)$ of $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ |
|---|---|---|
| $\mu \times \lambda$ | $\widetilde{\mu} \times \widetilde{\lambda}$ | $\widetilde{\mu} \boxtimes \widetilde{\lambda} + \widetilde{\lambda} \boxtimes \widetilde{\mu}$ |
| $\mu \cdot \mathbf{1}_G$ | $\widetilde{\mu} \cdot \mathbf{1}_{\mathrm{GL}(2,q)}$ | $\widetilde{\mu} \boxtimes \widetilde{\mu}$ |
| $\mu \cdot \mathrm{St}_G$ | $\widetilde{\mu} \cdot \mathrm{St}_{\mathrm{GL}(2,q)}$ | $\widetilde{\mu} \boxtimes \widetilde{\mu}$ |

Table 2.1.: Hyperspecial parahoric restriction and Iwahori restriction for non-cuspidal representations of $G = \mathrm{GL}(2, F)$.

### 2.4.5. Examples

**Example 2.20.** *For $\mathbf{G} = \mathrm{GL}(1)$ the irreducible admissible representations are the smooth characters $\mu : F^\times \to \mathbb{C}^\times$. The hyperspecial maximal parahoric subgroup of $F^\times$ is $\mathfrak{o}^\times$ with pro-unipotent radical $1 + \mathfrak{p}$. The hyperspecial restriction $\mathbf{r}_{\mathfrak{o}^\times}(\mu)$ of $\mu$ with $(1 + \mathfrak{p}) \subseteq \ker \mu$ (tamely ramified) is the character $\widetilde{\mu}$ of $\mathbb{F}_q^\times \cong \mathfrak{o}^\times/(1 + \mathfrak{p})$ over which $\mu$ factors by the homomorphism theorem. If $(1 + \mathfrak{p}) \nsubseteq \ker \mu$ (wildly ramified), then $\mathbf{r}_{\mathfrak{o}^\times}(\mu)$ is zero.*

**Example 2.21.** *For $G = \mathrm{GL}(2, F)$, the two conjugacy classes of parahoric subgroups are represented by the standard hyperspecial parahoric subgroup $\mathscr{K}_G = \mathrm{GL}(2, \mathfrak{o})$ and the standard Iwahori $\mathscr{B}_G = \mathscr{K} \cap \left(\begin{smallmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{smallmatrix}\right)$. The non-cuspidal irreducible representations $\rho$ of $G$ are*

1. *the principal series $\mu \times \lambda$,*
2. *the twists of the Steinberg representation $\mu\,\mathrm{St}$,*
3. *the one-dimensional representations $\mu\,\mathbf{1} = \mu \circ \det$,*

*for smooth characters $\mu$ and $\lambda$ of $F^\times$. The hyperspecial parahoric restriction and the Iwahori restriction of $\rho$ are given by Table 2.1.*

*Proof.* The hyperspecial parahoric restriction for parabolically induced representations $\mu \times \lambda$ is given by Thm. 2.19. Since parahoric restriction is exact, it preserves the exactness of the sequence

$$0 \longrightarrow \mu\,\mathbf{1}_{\mathrm{GL}(2,F)} \longrightarrow (|\cdot|^{-1/2}\mu) \times (|\cdot|^{1/2}\mu) \longrightarrow \mu\,\mathrm{St}_{\mathrm{GL}(2,F)} \longrightarrow 0.$$

Since $\mathbf{Rep}(\mathrm{GL}(2, \mathbb{F}_q))$ is a semisimple category, this implies

$$\mathbf{r}_{\mathfrak{o}^\times}(|\cdot|^{-1/2}\mu) \times \mathbf{r}_{\mathfrak{o}^\times}(|\cdot|^{1/2}\mu) \cong \mathbf{r}_{\mathscr{K}_G}(\mu\,\mathbf{1}_{\mathrm{GL}(2,F)}) + \mathbf{r}_{\mathscr{K}_G}(\mu\,\mathrm{St}_{\mathrm{GL}(2,F)}).$$

For at most tamely ramified $\mu$ we have $\det(\mathscr{K}_G^+) = 1+\mathfrak{p} \subseteq \widetilde{\mu}$, so $\mathbf{r}_{\mathscr{K}_G}(\mu\circ\det) = \widetilde{\mu}\circ\det$.[6] For the Iwahori restriction use (2.5) and apply the Jacquet functor to $\mathbf{r}_{\mathscr{K}_G}(\rho)$. □

[6] Alternatively, we can use $0 < \dim \mathbf{r}_{\mathscr{K}_G}(\mu\,\mathbf{1}_{\mathrm{GL}(2,F)}) \leq 1$ and $\dim \widetilde{\mu}\,\mathrm{St}_{\mathrm{GL}(2,q)} > 1$ and employ Cor. 2.14.

## 2.5. $\ell$-adic Galois representations

Let $F/\mathbb{Q}$ be a global number field with a fixed non-archimedean place $v$. Fix a prime $\ell$. An $\ell$-adic representation of the absolute Galois group $\Gamma_F$ is a continuous morphism from $\Gamma_F$ to the automorphism group of a finite dimensional vector field over the algebraic closure $\overline{\mathbb{Q}_\ell}$. Such a representation is always defined over a finite extension of $\mathbb{Q}_\ell$.

For a non-archimedean place $v$ let $F_v$ be the corresponding local field with residue field $\mathbb{F}_q$. The arithmetic Frobenius is $\mathrm{Frob}_q : \overline{\mathbb{F}_q} \to \overline{\mathbb{F}_q}$, $x \mapsto x^q$. A fixed element in the preimage of $\mathrm{Frob}_q$ under the canonical map $\Gamma_{F_v} \to \Gamma_{\mathbb{F}_q}$ is denoted $\mathrm{Frob}_q$ again. Its image under the embedding $\Gamma_{F_v} \hookrightarrow \Gamma_F$ is also denoted $\mathrm{Frob}_q$.

**The Tate twist.** The absolute Galois group $\Gamma_F$ acts on the group $\mu_{\ell^m}$ of $\ell^m$-th of roots of unity. Any $\gamma \in \Gamma_F$ raises them to a power $\zeta \mapsto \zeta^a$ for $a = a(\gamma) \in (\mathbb{Z}/\ell^m\mathbb{Z})^\times$. This defines a homomorphism $\Gamma_F \to (\mathbb{Z}/\ell^m\mathbb{Z})^\times$. Varying $m$ defines the *$\ell$-adic cyclotomic character* $\chi_\ell : \Gamma_F \to \mathbb{Z}_\ell^\times$ over the projective limit $\varprojlim_m (\mathbb{Z}/\ell^m\mathbb{Z})^\times \cong \mathbb{Z}_\ell^\times$. For every integer $k$, the *$k$-th Tate twist* of an $\ell$-adic representation $V$ of $\Gamma_F$ is the twisted representation

$$V(k) = V \otimes_{\mathbb{Z}_\ell} \chi_\ell^k.$$

When $q$ is coprime to $\ell$, the image of the arithmetic Frobenius is $\chi_\ell(\mathrm{Frob}_q) = q$.

# 3. Parahoric Restriction for GSp(4)

Let $\mathbf{G} = \mathrm{GSp}(4)$ be the group of symplectic similitudes of genus two. Fix a nonarchimedean local number field $F$ with finite residue field $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$ of order $q$. The valuation character $\nu = |\cdot|$ of $F^\times$ is normalized such that $|\varpi| = q^{-1}$ for a uniformizing element $\varpi \in \mathfrak{p}$. In this chapter we determine the parahoric restriction for irreducible admissible representations and arbitrary parahoric subgroups of $G = \mathbf{G}(F)$.

The standard torus $\mathbf{T}$ in $\mathbf{G}$ is the group of diagonal matrices $t = \mathrm{diag}(t_1, t_2, t_0/t_1, t_0/t_2)$. Its character group $X^*(\mathbf{T}) \cong \mathbb{Z}^3$ is generated by $e_i : t \mapsto t_i$ for $i = 0, 1, 2$ and the simple roots are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2 - e_0$. The reflections $s_1$, $s_2$ at $\alpha_1$, $\alpha_2$ generate the Weyl group $N(\mathbf{T})/\mathbf{T}$, explicitly given by representatives modulo $\mathbf{T}$

$$I_4 = \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 &&& \\ &&& 1 \\ && 1 & \\ & -1 && \end{pmatrix}, \quad s_1 s_2 = \begin{pmatrix} &&& 1 \\ 1 &&& \\ & -1 && \\ && 1 & \end{pmatrix}, \quad s_2 s_1 s_2 = \begin{pmatrix} &&& 1 \\ && 1 & \\ & -1 && \\ -1 &&& \end{pmatrix},$$

$$s_1 = \begin{pmatrix} & 1 && \\ 1 &&& \\ &&& 1 \\ && 1 & \end{pmatrix}, \; s_2 s_1 = \begin{pmatrix} & 1 && \\ && 1 & \\ &&& 1 \\ -1 &&& \end{pmatrix}, \; s_1 s_2 s_1 = \begin{pmatrix} && 1 & \\ & 1 && \\ -1 &&& \\ &&& 1 \end{pmatrix}, \; s_1 s_2 s_1 s_2 = \begin{pmatrix} && 1 & \\ &&& 1 \\ -1 &&& \\ & -1 && \end{pmatrix}.$$

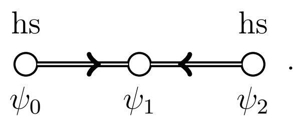
The Weyl group is isomorphic to the dihedral group of eight elements. We fix the standard Borel, Siegel and Klingen parabolic subgroups

$$\mathbf{B} = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ && * & \\ && * & * \end{pmatrix} \cap \mathbf{G}, \qquad \mathbf{P} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ && * & * \\ && * & * \end{pmatrix} \cap \mathbf{G}, \qquad \mathbf{Q} = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ && * & \\ & * & * & * \end{pmatrix} \cap \mathbf{G}$$

and write $B = M_B \ltimes U_B$, $P = M_P \ltimes U_P$, $Q = M_Q \ltimes U_Q$ for their $F$-rational points.

A non-trivial additive character $\psi : F \to \mathbb{C}^\times$ gives rise to a generic character of the unipotent radical $U_B$ of the standard Borel subgroup of $\mathrm{GSp}(4, F)$ via $\psi_U : U_B \to \mathbb{C}$, $u \mapsto \psi(a u_{12} + b u_{24})$ for $a, b \in \mathbb{C}^\times$. An admissible representation $\rho$ of $\mathrm{GSp}(4, F)$ is *generic* if it admits a non-trivial $U_B$-intertwining operator $(\rho|_{U_B}, V) \to (\psi_U, \mathbb{C})$. This does not depend on the choice of $\psi$ or $a, b$. Therefore we fix $a = b = 1$ and assume that $\psi$ has conductor one, so the restriction to $\mathfrak{o}$ factors over a non-trivial additive character $\widetilde{\psi}$ of $\mathfrak{o}/\mathfrak{p}$.

We briefly review the classification of standard parahoric subgroups of $G$, compare [Sch05a], [Moy88]. For $T = \mathbf{T}(F)$, fix the simple affine roots $\psi_0 = -(2e_1 - e_0) + 1, \psi_1 = e_1 - e_2$ and $\psi_2 = 2e_2 - e_0$ in the apartment of $T$. The affine Dynkin diagram is of type $\mathscr{C}_2$:



Let $N(T)$ be the normalizer of the standard torus $T = \mathbf{T}(F)$ in $G$. The affine Weyl group $W_{\mathrm{af}} = N(T)/\mathbf{T}(\mathfrak{o})$ is generated by the root reflections $s_i$ at $\psi_i$ for $i = 0, 1, 2$. A further non-trivial symmetry of the standard apartment is given by the Atkin-Lehner element $u_1$:

$$s_0 = \begin{pmatrix} & & \varpi^{-1} & \\ & 1 & & \\ -\varpi & & & \\ & & & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}, \quad u_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \varpi & & \\ \varpi & & & \end{pmatrix},$$

The closed standard alcove $\mathcal{C}$ is the set of points where $\psi_i(x) \geq 0$ for $i = 0, 1, 2$. To each facet in $\mathcal{C}$ there is attached a standard parahoric subgroup. Explicitly, the seven standard parahoric subgroups of $\mathrm{GSp}(4, F)$ with pro-unipotent radicals are

1. the standard Iwahori subgroup $\mathscr{B}$, attached to $\mathcal{C}$

$$\mathscr{B} = \mathrm{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathscr{B}^+ = \mathscr{B} \cap \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & 1+\mathfrak{p} \end{pmatrix},$$

2. the standard Siegel parahoric $\mathscr{P}$, attached to the facet $\psi_1^{-1}(0) \cap \mathcal{C}$,

$$\mathscr{P} = \mathrm{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathscr{P}^+ = \mathscr{P} \cap \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix},$$

3. the standard Klingen parahoric $\mathscr{Q}$, attached to the facet $\psi_2^{-1}(0) \cap \mathcal{C}$,

$$\mathscr{Q} = \mathrm{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathscr{Q}^+ = \mathscr{Q} \cap \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & 1+\mathfrak{p} \end{pmatrix},$$

4. the standard hyperspecial parahoric subgroup $\mathscr{K} = \mathrm{GSp}(4, \mathfrak{o})$, attached to the facet $\psi_1^{-1}(0) \cap \psi_2^{-1}(0) \cap \mathcal{C}$,

$$\mathscr{K} = \mathrm{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathscr{K}^+ = \mathscr{K} \cap \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix},$$

5. the standard paramodular subgroup $\mathscr{J}$, with facet $\psi_0^{-1}(0) \cap \psi_2^{-1}(0) \cap \mathcal{C}$,

$$\mathscr{J} = \mathrm{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathscr{J}^+ = \mathscr{J} \cap \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & 1+\mathfrak{p} \end{pmatrix},$$

6. the parahoric $u_1^{-1}\mathscr{Q}u_1$ attached to the facet $\psi_0^{-1}(0) \cap \mathcal{C}$,

7. the hyperspecial parahoric $u_1^{-1}\mathscr{K}u_1$ attached to the facet $\psi_0^{-1}(0) \cap \psi_1^{-1}(0) \cap \mathcal{C}$.

The simple affine roots $\psi_0$ and $\psi_2$ are conjugate under the Atkin Lehner element $u_1$ by Lemma 2.5. This non-trivial automorphism preserves $\mathscr{B}$, $\mathscr{Q}$ and $\mathscr{J}$. The standard maximal parahorics are the conjugates of $\mathscr{K}$, $\mathscr{J}$ and $u_1^{-1}\mathscr{K}u_1$.

## 3.1. Main result

The parahoric restriction functor factors over semisimplification, because admissible representations of finite groups form a semisimple category. Therefore it is sufficient to determine parahoric restriction of *irreducible* admissible representations $\rho$ of $G$.

The *non-cuspidal* irreducible admissible representations of $G$ have been classified by Sally and Tadic [ST94]. We use the notation of Roberts and Schmidt [RS07].

The *cuspidal* irreducible admissible representations of $G$ with depth zero have been classified by Moy and Prasad [MP96, 6.8], see Thm. 2.15. For positive depth, the parahoric restriction is zero by definition.

### 3.1.1. Hyperspecial parahoric restriction

In this section, we determine the parahoric restriction of the irreducible admissible representations $(\rho, V)$ of $\mathrm{GSp}(4, F)$ with respect to the standard hyperspecial parahoric subgroup $\mathscr{K} \subseteq G$.

**Lemma 3.1.** *Let $\pi$ be a cuspidal irreducible admissible representation of $G$. If $\pi$ is isomorphic to $\text{c-Ind}_{Z\mathscr{K}}^{G}\tau$ where $\tau$ is an irreducible extension of some cuspidal irreducible representation $\sigma$ of $\mathscr{K}/\mathscr{K}^+$ to the normalizer $Z\mathscr{K}$ of $\mathscr{K}$, then $\mathbf{r}_{\mathscr{K}}(\pi) \cong \sigma$. Otherwise $\mathbf{r}_{\mathscr{K}}(\pi) = 0$.*

*Proof.* This is a special case of Lemma 2.18. □

**Theorem 3.2.** *The hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}}(\rho)$ of non-cuspidal admissible irreducible representations $\rho$ of $\mathrm{GSp}_4(F)$ with depth-zero induction data is given by Table 3.1. For induction data of depth $> 0$, the hyperspecial parahoric restriction of $\rho$ is zero.*

The proof is in Section 3.2.

| type | $(\rho,V)\in\mathbf{Irr}(\mathrm{GSp}(4,F))$ | $\mathbf{r}_{\mathscr{K}}(\rho)\vert_{\mathrm{Sp}(4)}$ (even $q$) | $\mathbf{r}_{\mathscr{K}}(\rho)$ (odd $q$) | central char. | dimension |
|---|---|---|---|---|---|
| I | $\mu_1\times\mu_2\rtimes\mu$ | $\chi_1(k_1,k_2)$ | $X_1(\widetilde{\mu_1},\widetilde{\mu_2},\widetilde{\mu})$ | $\widetilde{\mu_1}\widetilde{\mu_2}\widetilde{\mu}^2$ | $(q+1)^2(q^2+1)$ |
| IIa | $\mu_1\,\mathrm{St}\rtimes\mu$ | $\chi_{10}(k_1)$ | $\chi_4(\widetilde{\mu_1},\widetilde{\mu})$ | $\widetilde{\mu_1}^2\widetilde{\mu}^2$ | $q(q+1)(q^2+1)$ |
| IIb | $\mu_1\,\mathbf{1}\rtimes\mu$ | $\chi_6(k_1)$ | $\chi_3(\widetilde{\mu_1},\widetilde{\mu})$ | $\widetilde{\mu_1}^2\widetilde{\mu}^2$ | $(q+1)(q^2+1)$ |
| IIIa | $\mu_1\rtimes\mu\,\mathrm{St}$ | $\chi_{11}(k_1)$ | $\chi_2(\widetilde{\mu_1},\widetilde{\mu})$ | $\widetilde{\mu_1}\widetilde{\mu}^2$ | $q(q+1)(q^2+1)$ |
| IIIb | $\mu_1\rtimes\mu\,\mathbf{1}$ | $\chi_7(k_1)$ | $\chi_1(\widetilde{\mu_1},\widetilde{\mu})$ | $\widetilde{\mu_1}\widetilde{\mu}^2$ | $(q+1)(q^2+1)$ |
| IVa | $\mu\,\mathrm{St}_{\mathrm{GSp}(4,F)}$ | $\theta_4$ | $\theta_5(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q^4$ |
| IVb | $L(\nu^2,\nu^{-1}\mu\,\mathrm{St})$ | $\theta_1\oplus\theta_2$ | $\theta_1(\widetilde{\mu})\oplus\theta_3(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q^3+q^2+q$ |
| IVc | $L(\nu^{3/2}\,\mathrm{St},\nu^{-3/2}\mu)$ | $\theta_1\oplus\theta_3$ | $\theta_1(\widetilde{\mu})\oplus\theta_4(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q^3+q^2+q$ |
| IVd | $\mu\,\mathbf{1}_{\mathrm{GSp}(4,F)}$ | $\theta_0$ | $\theta_0(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $1$ |
| Va | $\delta([\xi_u,\nu\xi_u],\nu^{-1/2}\mu)$ | $\theta_3\oplus\theta_4$ | $\theta_4(\widetilde{\mu})\oplus\theta_5(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q^4+\frac{1}{2}q(q^2+1)$ |
| | $\delta([\xi_t,\nu\xi_t],\nu^{-1/2}\mu)$ | $-$ | $\tau_3(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q^2(q^2+1)$ |
| Vb | $L(\nu^{1/2}\xi_u\,\mathrm{St},\nu^{-1/2}\mu)$ | $\theta_1$ | $\theta_1(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $\frac{1}{2}q(q+1)^2$ |
| | $L(\nu^{1/2}\xi_t\,\mathrm{St},\nu^{-1/2}\mu)$ | $-$ | $\tau_2(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q(q^2+1)$ |
| Vc | $L(\nu^{1/2}\xi_u\,\mathrm{St},\nu^{-1/2}\xi_u\mu)$ | $\theta_1$ | $\theta_1(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $\frac{1}{2}q(q+1)^2$ |
| | $L(\nu^{1/2}\xi_t\,\mathrm{St},\nu^{-1/2}\xi_t\mu)$ | $-$ | $\tau_2(\widetilde{\mu}\lambda_0)$ | $\widetilde{\mu}^2$ | $q(q^2+1)$ |
| Vd | $L(\nu\xi_u,\xi_u\rtimes\nu^{-1/2}\mu)$ | $\theta_0\oplus\theta_2$ | $\theta_0(\widetilde{\mu})\oplus\theta_3(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $1+\frac{1}{2}q(q^2+1)$ |
| | $L(\nu\xi_t,\xi_t\rtimes\nu^{-1/2}\mu)$ | $-$ | $\tau_1(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q^2+1$ |
| VIa | $\tau(S,\nu^{-1/2}\mu)$ | $\theta_1\oplus\theta_4$ | $\theta_1(\widetilde{\mu})\oplus\theta_5(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q^4+\frac{1}{2}q(q+1)^2$ |
| VIb | $\tau(T,\nu^{-1/2}\mu)$ | $\theta_2$ | $\theta_3(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $\frac{1}{2}q(q^2+1)$ |
| VIc | $L(\nu^{1/2}\,\mathrm{St},\nu^{-1/2}\mu)$ | $\theta_3$ | $\theta_4(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $\frac{1}{2}q(q^2+1)$ |
| VId | $L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\mu)$ | $\theta_0\oplus\theta_1$ | $\theta_0(\widetilde{\mu})\oplus\theta_1(\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $1+\frac{1}{2}q(q+1)^2$ |
| VII | $\mu_1\rtimes\Pi$ | $\chi_3(k_1,l')$ | $X_3(\Lambda,\widetilde{\mu_1})$ | $\widetilde{\mu_1}\cdot\Lambda\vert_{\mathbb{F}_q^\times}$ | $q^4-1$ |
| VIIIa | $\tau(S,\Pi)$ | $\chi_{13}(l')$ | $\chi_8(\Lambda)$ | $\Lambda\vert_{\mathbb{F}_q^\times}$ | $q(q-1)(q^2+1)$ |
| VIIIb | $\tau(T,\Pi)$ | $\chi_9(l')$ | $\chi_7(\Lambda)$ | $\Lambda\vert_{\mathbb{F}_q^\times}$ | $(q-1)(q^2+1)$ |
| IXa | $\delta(\nu\xi_u,\nu^{-1/2}\Pi)$ | $\chi_{13}(l')$ | $\chi_8(\Lambda)$ | $\Lambda\vert_{\mathbb{F}_q^\times}$ | $q(q-1)(q^2+1)$ |
| | $\delta(\nu\xi_t,\nu^{-1/2}\Pi)$ | $-$ | $\tau_5(\lambda')$ | $\lambda_0\cdot\Lambda\vert_{\mathbb{F}_q^\times}$ | $q^2(q^2-1)$ |
| IXb | $L(\nu\xi_u,\nu^{-1/2}\Pi)$ | $\chi_9(l')$ | $\chi_7(\Lambda)$ | $\Lambda\vert_{\mathbb{F}_q^\times}$ | $(q-1)(q^2+1)$ |
| | $L(\nu\xi_t,\nu^{-1/2}\Pi)$ | $-$ | $\tau_4(\lambda')$ | $\lambda_0\cdot\Lambda\vert_{\mathbb{F}_q^\times}$ | $q^2-1$ |
| X | $\Pi\rtimes\mu$ | $\chi_2(l)$ | $X_2(\Lambda,\widetilde{\mu})$ | $\widetilde{\mu}^2\cdot\Lambda\vert_{\mathbb{F}_q^\times}$ | $(q^4-1)$ |
| XIa | $\delta(\nu^{1/2}\Pi,\nu^{-1/2}\mu)$ | $\chi_{12}(l'')$ | $\chi_6(\omega_\Lambda,\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $q(q-1)(q^2+1)$ |
| XIb | $L(\nu^{1/2}\Pi,\nu^{-1/2}\mu)$ | $\chi_8(l'')$ | $\chi_5(\omega_\Lambda,\widetilde{\mu})$ | $\widetilde{\mu}^2$ | $(q-1)(q^2+1)$ |

Table 3.1.: Hyperspecial parahoric restriction for non-cuspidal irreducible admissible representations of $\mathrm{GSp}(4,F)$.

**Notation 3.3** (Table 3.1)**.** Let $\mu, \mu_1, \mu_2 : F^\times \to \mathbb{C}^\times$ be tamely ramified or unramified characters which restrict to characters $\widetilde{\mu}$, $\widetilde{\mu_1}$, $\widetilde{\mu_2}$ of $(\mathfrak{o}/\mathfrak{p})^\times$.

Let $\Pi$ be a cuspidal irreducible admissible representation of $\mathrm{GL}(2, F)$ of depth zero with hyperspecial restriction $\pi_\Lambda$ attached to a character $\Lambda$ of $\mathbb{F}_{q^2}^\times$ in the notation of Table A.1. The non-trivial unramified quadratic character of $F^\times$ is $\xi_u$. For odd $q$ let $\xi_t$ be either one of the two tamely ramified quadratic characters which reduce to the non-trivial quadratic character $\lambda_0$ over the residue field $\mathbb{F}_q^\times$.

The irreducible representations of $\mathrm{GSp}(4, q)$ for odd $q$ have been classified by Shinoda [Shi82], see Section A.3.

For even $q$ there is an isomorphism $\mathrm{GSp}(4, q) \cong \mathrm{Sp}(4, q) \times \mathbb{F}_q^\times$ and the irreducible representations of $\mathrm{Sp}(4, q)$ have been classified by Enomoto [Eno72], see Section A.4. Fix a generator $\hat{\theta}$ of the character group of $\mathbb{F}_{q^2}^\times$. Denote its restrictions to $\mathbb{F}_{q^2}^\times[q-1] = \mathbb{F}_q^\times$ and to $\mathbb{F}_{q^2}^\times[q+1]$ by $\hat{\gamma}$ and $\hat{\eta}$, respectively. Let $k, k_1, k_2 \in \mathbb{Z}/(q-1)\mathbb{Z}$ be such that $\hat{\gamma}^{k_i} = \widetilde{\mu_i}$. Let $l \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ be such that $\Lambda = \hat{\theta}^l$ and let $l'$ be the image of $l$ under the canonical projection $\mathbb{Z}/(q^2-1)\mathbb{Z} \twoheadrightarrow \mathbb{Z}/(q+1)\mathbb{Z}$ so that the restriction of $\Lambda$ to $\mathbb{F}_{q^2}^\times[q+1]$ is $\hat{\eta}^{l'}$.

If $\Lambda^{q+1} = 1$, then $\Lambda$ factors over a character $\omega_\Lambda$ of $\mathbb{F}_{q^2}^\times[q+1]$ so that $\Lambda = \omega_\Lambda \circ N_{q-1}$. For even $q$ there is a unique preimage $l''$ of $l$ under the canonical injection $\mathbb{Z}/(q+1)\mathbb{Z} \hookrightarrow \mathbb{Z}/(q^2-1)\mathbb{Z}$ with $\omega_\Lambda = \eta^{l''}$. If $\Lambda^{q-1} = \Lambda_0$ is the quadratic character, let $\lambda'$ be the character of $\mathbb{F}_{q^2}^\times[2(q-1)]$ with $\Lambda = \lambda' \circ N_{(q+1)/2}$.

### 3.1.2. Parahoric restriction at non-maximal parahorics

Every non-maximal parahoric subgroup of $G$ is conjugate to either the Iwahori subgroup $\mathscr{B}$, the standard Klingen parahoric $\mathscr{Q}$, or the standard Siegel parahoric $\mathscr{P}$. Corollary 2.17 implies that for cuspidal irreducible admissible representations of $G$, the parahoric restriction at non-maximal parahoric subgroups is zero.

**Theorem 3.4.** *For non-cuspidal irreducible admissible representations* $(\rho, V)$ *of* $\mathrm{GSp}(4, F)$ *with depth-zero inducing data, the parahoric restriction with respect to the standard non-maximal parahoric subgroups* $\mathscr{B}$*,* $\mathscr{P}$ *and* $\mathscr{Q}$ *is given by Table 3.2.*

*Proof.* According to (2.5) it is sufficient to determine the parabolic restriction of the hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}}(\rho)$. This is given by Tables A.4 and A.7. □

In Table 3.2, $\mu$, $\mu_1$, $\mu_2$ are smooth characters of $F^\times$ and $\xi$ is either the unramified or a tamely ramified quadratic character of $F^\times$. We denote by $\Pi$ a cuspidal irreducible admissible representation of $\mathrm{GL}(2, F)$ with hyperspecial parahoric restriction $\mathbf{r}_{\mathrm{GL}(2,\mathfrak{o})}(\Pi) =: \widetilde{\pi}$.

| type | $(\rho,V)\in\mathbf{Irr}(\mathrm{GSp}(4,F))$ | $\mathbf{r}_{\mathscr{B}}(\rho)\in\mathbf{Rep}((\mathbb{F}_q^\times)^3)$ | $\mathbf{r}_{\mathscr{Q}}(\rho)\in\mathbf{Rep}(\mathbb{F}_q^\times\times\mathrm{GSp}(2,q))$ | $\mathbf{r}_{\mathscr{P}}(\rho)\in\mathbf{Rep}(\mathrm{GL}(2,q)\times\mathbb{F}_q^\times)$ |
|---|---|---|---|---|
| I | $\mu_1\times\mu_2\rtimes\mu$ | $A[\widetilde{\mu_1},\widetilde{\mu_2},\tilde{\mu}]+A[\widetilde{\mu_2},\widetilde{\mu_1},\tilde{\mu}]$ | $B[\widetilde{\mu_1},\widetilde{\mu_2},\tilde{\mu}]+B[\widetilde{\mu_2},\widetilde{\mu_1},\tilde{\mu}]$ | $C[\widetilde{\mu_1},\widetilde{\mu_2},\tilde{\mu}]+C[\widetilde{\mu_1},\widetilde{\mu_2}^{-1},\widetilde{\mu_2}\tilde{\mu}]$ |
| IIa | $\mu_1\,\mathrm{St}\rtimes\mu$ | $A[\widetilde{\mu_1},\widetilde{\mu_1},\tilde{\mu}]$ | $B[\widetilde{\mu_1},\widetilde{\mu_1},\tilde{\mu}]$ | $\widetilde{\mu_1}\,\mathrm{St}\boxtimes\tilde{\mu}+\widetilde{\mu_1}^{-1}\,\mathrm{St}\boxtimes\tilde{\mu}\widetilde{\mu_1}^2$ $+(\widetilde{\mu_1}\times\widetilde{\mu_1}^{-1})\boxtimes\tilde{\mu}\widetilde{\mu_1}$ |
| IIb | $\mu_1\,\mathbf{1}\rtimes\mu$ | $A[\widetilde{\mu_1},\widetilde{\mu_1},\tilde{\mu}]$ | $B[\widetilde{\mu_1},\widetilde{\mu_1},\tilde{\mu}]$ | $\widetilde{\mu_1}\,\mathbf{1}\boxtimes\tilde{\mu}+\widetilde{\mu_1}^{-1}\,\mathbf{1}\boxtimes\tilde{\mu}\widetilde{\mu_1}^2$ $+(\widetilde{\mu_1}\times\widetilde{\mu_1}^{-1})\boxtimes\tilde{\mu}\widetilde{\mu_1}$ |
| IIIa | $\mu_1\rtimes\mu\,\mathrm{St}$ | $\widetilde{\mu_1}\boxtimes 1\boxtimes\tilde{\mu}+\widetilde{\mu_1}^{-1}\boxtimes 1\boxtimes\widetilde{\mu_1}\tilde{\mu}$ $+1\boxtimes\widetilde{\mu_1}\boxtimes\tilde{\mu}+1\boxtimes\widetilde{\mu_1}^{-1}\boxtimes\widetilde{\mu_1}\tilde{\mu}$ | $\widetilde{\mu_1}\boxtimes\tilde{\mu}\,\mathrm{St}+1\boxtimes\widetilde{\mu_1}\rtimes\tilde{\mu}$ $+\widetilde{\mu_1}^{-1}\boxtimes\widetilde{\mu_1}\tilde{\mu}\,\mathrm{St}$ | $C[\widetilde{\mu_1},1,\tilde{\mu}]$ |
| IIIb | $\mu_1\rtimes\mu\,\mathbf{1}$ | $\widetilde{\mu_1}\boxtimes 1\boxtimes\tilde{\mu}+\widetilde{\mu_1}^{-1}\boxtimes 1\boxtimes\widetilde{\mu_1}\tilde{\mu}$ $+1\boxtimes\widetilde{\mu_1}\boxtimes\tilde{\mu}+1\boxtimes\widetilde{\mu_1}^{-1}\boxtimes\widetilde{\mu_1}\tilde{\mu}$ | $\widetilde{\mu_1}\boxtimes\tilde{\mu}\,\mathbf{1}+1\boxtimes\widetilde{\mu_1}\rtimes\tilde{\mu}$ $+\widetilde{\mu_1}^{-1}\boxtimes\widetilde{\mu_1}\tilde{\mu}\,\mathbf{1}$ | $C[\widetilde{\mu_1},1,\tilde{\mu}]$ |
| IVa | $\mu\,\mathrm{St}_{\mathrm{GSp}(4)}$ | $1\boxtimes 1\boxtimes\tilde{\mu}$ | $1\boxtimes\tilde{\mu}\,\mathrm{St}$ | $\mathrm{St}\boxtimes\tilde{\mu}$ |
| IVb | $L(\nu^2,\nu^{-1}\mu\,\mathrm{St})$ | $3(1\boxtimes 1\boxtimes\tilde{\mu})$ | $1\boxtimes\tilde{\mu}\,\mathbf{1}+2(1\boxtimes\tilde{\mu}\,\mathrm{St})$ | $\mathrm{St}\boxtimes\tilde{\mu}+2(\mathbf{1}\boxtimes\tilde{\mu})$ |
| IVc | $L(\nu^{3/2}\,\mathrm{St},\nu^{-3/2}\mu)$ | $3(1\boxtimes 1\boxtimes\tilde{\mu})$ | $2(1\boxtimes\tilde{\mu}\,\mathbf{1})+1\boxtimes\tilde{\mu}\,\mathrm{St}$ | $2(\mathrm{St}\boxtimes\tilde{\mu})+\mathbf{1}\boxtimes\tilde{\mu}$ |
| IVd | $\mu\,\mathbf{1}_{\mathrm{GSp}(4)}$ | $1\boxtimes 1\boxtimes\tilde{\mu}$ | $1\boxtimes\tilde{\mu}\,\mathbf{1}$ | $\mathbf{1}\boxtimes\tilde{\mu}$ |
| Va | $\delta([\xi,\nu\xi],\nu^{-1/2}\mu)$ | $\tilde{\xi}\boxtimes\tilde{\xi}\boxtimes\tilde{\mu}+\tilde{\xi}\boxtimes\tilde{\xi}\boxtimes\tilde{\xi}\tilde{\mu}$ | $\tilde{\xi}\boxtimes(\tilde{\xi}\rtimes\tilde{\mu})$ | $\tilde{\xi}\,\mathrm{St}\boxtimes\tilde{\mu}+\tilde{\xi}\,\mathrm{St}\boxtimes\tilde{\xi}\tilde{\mu}$ |
| Vb | $L(\nu^{1/2}\xi\,\mathrm{St},\nu^{-1/2}\mu)$ | $\tilde{\xi}\boxtimes\tilde{\xi}\boxtimes\tilde{\mu}+\tilde{\xi}\boxtimes\tilde{\xi}\boxtimes\tilde{\xi}\tilde{\mu}$ | $\tilde{\xi}\boxtimes(\tilde{\xi}\rtimes\tilde{\mu})$ | $\tilde{\xi}\,\mathrm{St}\boxtimes\tilde{\mu}+\tilde{\xi}\,\mathbf{1}\boxtimes\tilde{\xi}\tilde{\mu}$ |
| Vc | $L(\nu^{1/2}\xi\,\mathrm{St},\nu^{-1/2}\xi\mu)$ | $\tilde{\xi}\boxtimes\tilde{\xi}\boxtimes\tilde{\mu}+\tilde{\xi}\boxtimes\tilde{\xi}\boxtimes\tilde{\xi}\tilde{\mu}$ | $\tilde{\xi}\boxtimes(\tilde{\xi}\rtimes\tilde{\mu})$ | $\tilde{\xi}\,\mathbf{1}\boxtimes\tilde{\mu}+\tilde{\xi}\,\mathrm{St}\boxtimes\tilde{\xi}\tilde{\mu}$ |
| Vd | $L(\nu\xi,\xi\rtimes\nu^{-1/2}\mu)$ | $\tilde{\xi}\boxtimes\tilde{\xi}\boxtimes\tilde{\mu}+\tilde{\xi}\boxtimes\tilde{\xi}\boxtimes\tilde{\xi}\tilde{\mu}$ | $\tilde{\xi}\boxtimes(\tilde{\xi}\rtimes\tilde{\mu})$ | $\tilde{\xi}\,\mathbf{1}\boxtimes\tilde{\mu}+\tilde{\xi}\,\mathbf{1}\boxtimes\tilde{\xi}\tilde{\mu}$ |
| VIa | $\tau(S,\nu^{-1/2}\mu)$ | $3(1\boxtimes 1\boxtimes\tilde{\mu})$ | $1\boxtimes\tilde{\mu}\,\mathbf{1}+2(1\boxtimes\tilde{\mu}\,\mathrm{St})$ | $2(\mathrm{St}\boxtimes\tilde{\mu})+\mathbf{1}\boxtimes\tilde{\mu}$ |
| VIb | $\tau(T,\nu^{-1/2}\mu)$ | $1\boxtimes 1\boxtimes\tilde{\mu}$ | $1\boxtimes\tilde{\mu}\,\mathrm{St}$ | $\mathbf{1}\boxtimes\tilde{\mu}$ |
| VIc | $L(\nu^{1/2}\,\mathrm{St},\nu^{-1/2}\mu)$ | $1\boxtimes 1\boxtimes\tilde{\mu}$ | $1\boxtimes\tilde{\mu}\,\mathbf{1}$ | $\mathrm{St}\boxtimes\tilde{\mu}$ |
| VId | $L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\mu)$ | $3(1\boxtimes 1\boxtimes\tilde{\mu})$ | $2(1\boxtimes\tilde{\mu}\,\mathbf{1})+1\boxtimes\tilde{\mu}\,\mathrm{St}$ | $\mathrm{St}\boxtimes\tilde{\mu}+2(\mathbf{1}\boxtimes\tilde{\mu})$ |
| VII | $\mu_1\rtimes\Pi$ | 0 | $\widetilde{\mu_1}\boxtimes\tilde{\pi}+\widetilde{\mu_1}^{-1}\boxtimes\widetilde{\mu_1}\tilde{\pi}$ | 0 |
| VIIIa | $\tau(S,\Pi)$ | 0 | $1\boxtimes\tilde{\pi}$ | 0 |
| VIIIb | $\tau(T,\Pi)$ | 0 | $1\boxtimes\tilde{\pi}$ | 0 |
| IXa | $\delta(\nu\xi,\nu^{-1/2}\Pi)$ | 0 | $\tilde{\xi}\boxtimes\tilde{\pi}$ | 0 |
| IXb | $L(\nu\xi,\nu^{-1/2}\Pi)$ | 0 | $\tilde{\xi}\boxtimes\tilde{\pi}$ | 0 |
| X | $\Pi\rtimes\mu$ | 0 | 0 | $\tilde{\pi}\boxtimes\tilde{\mu}+(\tilde{\pi})^*\boxtimes\widetilde{\omega_\pi}\tilde{\mu}$ |
| XIa | $\delta(\nu^{1/2}\Pi,\nu^{-1/2}\mu)$ | 0 | 0 | $\tilde{\pi}\boxtimes\tilde{\mu}$ |
| XIb | $L(\nu^{1/2}\Pi,\nu^{-1/2}\mu)$ | 0 | 0 | $\tilde{\pi}\boxtimes\tilde{\mu}$ |

Table 3.2.: Parahoric restriction of non-cuspidal irreducible admissible representations with respect to non-maximal parahoric subgroups of $\mathrm{GSp}(4,F)$ .

### 3.1.3. Paramodular restriction

For the standard paramodular subgroup $\mathscr{J}$ we identify $\mathscr{J}/\mathscr{J}^+$ with

$$(\mathrm{GL}(2,q)^2)^0 = \{(\mathrm{x},\mathrm{y}) \in \mathrm{GL}(2,q) \times \mathrm{GL}(2,q) \mid \det \mathrm{x} = \det \mathrm{y}\}.$$

via

$$\mathscr{J} \ni \begin{pmatrix} x_{1,1} & * & \varpi^{-1}x_{1,2} & * \\ * & y_{1,1} & * & y_{1,2} \\ \varpi x_{2,1} & * & x_{2,2} & * \\ * & y_{2,1} & * & y_{2,2} \end{pmatrix} \mapsto \left( \begin{pmatrix} \mathrm{x}_{1,1} & \mathrm{x}_{1,2} \\ \mathrm{x}_{2,1} & \mathrm{x}_{2,2} \end{pmatrix}, \begin{pmatrix} \mathrm{y}_{1,1} & \mathrm{y}_{1,2} \\ \mathrm{y}_{2,1} & \mathrm{y}_{2,2} \end{pmatrix} \right)$$

where $\mathrm{x}_{ij}$, $\mathrm{y}_{ij}$ is the image of $x_{i,j}$, $y_{i,j}$ under the canonical map $\mathfrak{o} \to \mathfrak{o}/\mathfrak{p}$. The parahoric restriction functor with respect to $\mathscr{J}$ is the *paramodular restriction* $\mathbf{r}_{\mathscr{J}}$ from admissible representations of $G$ to those of $(\mathrm{GL}(2,q)^2)^0$. The irreducible representations of $(\mathrm{GL}(2,q)^2)^0$ have been classified in Lemma A.6.

**Lemma 3.5.** *Let $\rho$ be an irreducible admissible representation of* $\mathrm{GSp}(4,F)$*. For every irreducible constituent $\sigma$ of $\mathbf{r}_{\mathscr{J}}(\rho)$ the opposite $\sigma^*$ is also a constituent.*

*Proof.* The Atkin Lehner involution $u_1$ preserves $\mathscr{J}$ and maps $\sigma$ to $\sigma^*$. □

**Proposition 3.6.** *Let $\pi$ be a cuspidal irreducible admissible representation of $G$. The paramodular restriction $\mathbf{r}_{\mathscr{J}}(\pi)$ is non-zero if and only if $\pi$ is isomorphic to a compactly induced representation* $\text{c-Ind}_{N(\mathscr{J})}^G(\tau)$ *where $\tau$ is an irreducible extension of some cuspidal irreducible representation $\sigma$ of $\mathscr{J}/\mathscr{J}^+$. In that case $\pi$ is non-generic.*

*If $\sigma \cong \sigma^*$, then $\mathbf{r}_{\mathscr{J}}(\pi) \cong \sigma$. If $\sigma \not\cong \sigma^*$, then $\mathbf{r}_{\mathscr{J}}(\pi) \cong \sigma \oplus \sigma^*$.*

*Proof.* This is clear by Frobenius reciprocity, 3.35 and Corollary 2.17. The Atkin-Lehner involution preserves $\tau$. If $\pi$ was generic, then it would be induced from the normalizer of a hyperspecial parahoric subgroup [DR09, §6.1.2]. □

The main result in this section is:

**Theorem 3.7.** *For non-cuspidal irreducible admissible representations $\rho$ of* $\mathrm{GSp}(4,F)$ *with depth-zero induction data, the paramodular restriction $\mathbf{r}_{\mathscr{J}}(\rho)$ is given by Table 3.3. For induction data of depth $> 0$, the paramodular restriction is zero.*

The proof is given in Section 3.3.

| type | $\rho \in \mathbf{Irr}(\mathrm{GSp}(4,F))$ | $\mathbf{r}_{\mathscr{J}}(\rho) \in \mathbf{Rep}((\mathrm{GL}(2,q)^2)^0)$ | $\dim \mathbf{r}_{\mathscr{J}}(\rho)$ |
|---|---|---|---|
| I | $\mu_1 \times \mu_2 \rtimes \mu$ | $\widetilde{\mu}[1 \times \widetilde{\mu_1}, 1 \times \widetilde{\mu_2}] + \widetilde{\mu}[1 \times \widetilde{\mu_2}, 1 \times \widetilde{\mu_1}]$ | $2(q+1)^2$ |
| IIa | $\mu_1 \mathrm{St} \rtimes \mu$ | $\widetilde{\mu}[1 \times \widetilde{\mu_1}, 1 \times \widetilde{\mu_1}]$ | $(q+1)^2$ |
| IIb | $\mu_1 \mathbf{1} \rtimes \mu$ | $\widetilde{\mu}[1 \times \widetilde{\mu_1}, 1 \times \widetilde{\mu_1}]$ | $(q+1)^2$ |
| IIIa | $\mu_1 \rtimes \mu \mathrm{St}$ | $\widetilde{\mu}[1 \times \widetilde{\mu_1}, \mathrm{St}] + \widetilde{\mu}[\mathrm{St}, 1 \times \widetilde{\mu_1}]$ | $2q(q+1)$ |
| IIIb | $\mu_1 \rtimes \mu \mathbf{1}$ | $\widetilde{\mu}[1 \times \widetilde{\mu_1}, \mathbf{1}] + \widetilde{\mu}[\mathbf{1}, 1 \times \widetilde{\mu_1}]$ | $2(q+1)$ |
| IVa | $\mu \mathrm{St}_{\mathrm{GSp}(4,F)}$ | $\widetilde{\mu}[\mathrm{St}, \mathrm{St}]$ | $q^2$ |
| IVb | $L(\nu^2, \nu^{-1}\mu \mathrm{St})$ | $\widetilde{\mu}[\mathrm{St}, \mathrm{St}] + \widetilde{\mu}[\mathbf{1}, \mathrm{St}] + \widetilde{\mu}[\mathrm{St}, \mathbf{1}]$ | $q^2+2q$ |
| IVc | $L(\nu^{3/2} \mathrm{St}, \nu^{-3/2}\mu)$ | $\widetilde{\mu}[\mathbf{1}, \mathbf{1}] + \widetilde{\mu}[\mathbf{1}, \mathrm{St}] + \widetilde{\mu}[\mathrm{St}, \mathbf{1}]$ | $2q+1$ |
| IVd | $\mu \mathbf{1}_{\mathrm{GSp}(4,F)}$ | $\widetilde{\mu}[\mathbf{1}, \mathbf{1}]$ | $1$ |
| Va | $\delta([\xi_u, \nu\xi_u], \nu^{-1/2}\mu)$ | $\widetilde{\mu}[\mathbf{1}, \mathrm{St}] + \widetilde{\mu}[\mathrm{St}, \mathbf{1}]$ | $2q$ |
| | $\delta([\xi_t, \nu\xi_t], \nu^{-1/2}\mu)$ | $\widetilde{\mu}[1 \times \lambda_0, 1 \times \lambda_0]_{\pm}$ | $(q+1)^2/2$ |
| Vb | $L(\nu^{1/2}\xi_u \mathrm{St}, \nu^{-1/2}\mu)$ | $\widetilde{\mu}[\mathbf{1}, \mathbf{1}] + \widetilde{\mu}[\mathrm{St}, \mathrm{St}]$ | $q^2+1$ |
| | $L(\nu^{1/2}\xi_t \mathrm{St}, \nu^{-1/2}\mu)$ | $\widetilde{\mu}[1 \times \lambda_0, 1 \times \lambda_0]_{\mp}$ | $(q+1)^2/2$ |
| Vc | $L(\nu^{1/2}\xi_u \mathrm{St}, \nu^{-1/2}\xi_u\mu)$ | $\widetilde{\mu}[\mathbf{1}, \mathbf{1}] + \widetilde{\mu}[\mathrm{St}, \mathrm{St}]$ | $q^2+1$ |
| | $L(\nu^{1/2}\xi_t \mathrm{St}, \nu^{-1/2}\xi_t\mu)$ | $\widetilde{\mu}[1 \times \lambda_0, 1 \times \lambda_0]_{\mp}$ | $(q+1)^2/2$ |
| Vd | $L(\nu\xi_u, \xi_u \rtimes \nu^{-1/2}\mu)$ | $\widetilde{\mu}[\mathbf{1}, \mathrm{St}] + \widetilde{\mu}[\mathrm{St}, \mathbf{1}]$ | $2q$ |
| | $L(\nu\xi_t, \xi_t \rtimes \nu^{-1/2}\mu)$ | $\widetilde{\mu}[1 \times \lambda_0, 1 \times \lambda_0]_{\pm}$ | $(q+1)^2/2$ |
| VIa | $\tau(S, \nu^{-1/2}\mu)$ | $\widetilde{\mu}[\mathrm{St}, \mathrm{St}] + \widetilde{\mu}[\mathrm{St}, \mathbf{1}] + \widetilde{\mu}[\mathbf{1}, \mathrm{St}]$ | $q^2+2q$ |
| VIb | $\tau(T, \nu^{-1/2}\mu)$ | $\widetilde{\mu}[\mathrm{St}, \mathrm{St}]$ | $q^2$ |
| VIc | $L(\nu^{1/2} \mathrm{St}, \nu^{-1/2}\mu)$ | $\widetilde{\mu}[\mathbf{1}, \mathbf{1}]$ | $1$ |
| VId | $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\mu)$ | $\widetilde{\mu}[\mathbf{1}, \mathbf{1}] + \widetilde{\mu}[\mathrm{St}, \mathbf{1}] + \widetilde{\mu}[\mathbf{1}, \mathrm{St}]$ | $2q+1$ |
| VII | $\mu_1 \rtimes \Pi$ | $[1 \times \widetilde{\mu_1}, \widetilde{\pi}] + [\widetilde{\pi}, 1 \times \widetilde{\mu_1}]$ | $2(q^2-1)$ |
| VIIIa | $\tau(S, \Pi)$ | $[\mathbf{1}, \widetilde{\pi}] + [\widetilde{\pi}, \mathbf{1}]$ | $2(q-1)$ |
| VIIIb | $\tau(T, \Pi)$ | $[\mathrm{St}, \widetilde{\pi}] + [\widetilde{\pi}, \mathrm{St}]$ | $2(q-1)q$ |
| IXa | $\delta(\nu\xi_u, \nu^{-1/2}\Pi)$ | $[\mathrm{St}, \widetilde{\pi}] + [\widetilde{\pi}, \mathrm{St}]$ | $2(q-1)q$ |
| | $\delta(\nu\xi_t, \nu^{-1/2}\Pi)$ | $[\widetilde{\pi}, 1 \times \lambda_0]_{\mp} + [1 \times \lambda_0, \widetilde{\pi}]_{\mp}$ | $q^2-1$ |
| IXb | $L(\nu\xi_u, \nu^{-1/2}\Pi)$ | $[\mathbf{1}, \widetilde{\pi}] + [\widetilde{\pi}, \mathbf{1}]$ | $2(q-1)$ |
| | $L(\nu\xi_t, \nu^{-1/2}\Pi)$ | $[\widetilde{\pi}, 1 \times \lambda_0]_{\pm} + [1 \times \lambda_0, \widetilde{\pi}]_{\pm}$ | $q^2-1$ |
| X | $\Pi \rtimes \mu$ | $0$ | $0$ |
| XIa | $\delta(\nu^{1/2}\Pi, \nu^{-1/2}\mu)$ | $0$ | $0$ |
| XIb | $L(\nu^{1/2}\Pi, \nu^{-1/2}\mu)$ | $0$ | $0$ |

Table 3.3.: Paramodular restriction for non-cuspidal irreducible admissible representations of $\mathrm{GSp}(4,F)$. The index is determined by $\xi_t(\varpi) = \pm 1$.

| type | $\rho \in \mathbf{Irr}(\mathrm{GSp}(4,F))$ | $\dim \rho^{\mathscr{K}}$ | $\dim \rho^{\mathscr{J}}$ | $\dim \rho^{\mathscr{P}}$ | $\dim \rho^{\mathscr{Q}}$ | $\dim \rho^{\mathscr{B}}$ |
|---|---|---|---|---|---|---|
| I | $\mu_1 \times \mu_2 \rtimes \mu$ | 1 | 2 | 4 | 4 | 8 |
| IIa | $\mu_1 \mathrm{St} \rtimes \mu$ | 0 | 1 | 1 | 2 | 4 |
| IIb | $\mu_1 \mathbf{1} \rtimes \mu$ | 1 | 1 | 3 | 2 | 4 |
| IIIa | $\mu_1 \rtimes \mu \mathrm{St}$ | 0 | 0 | 2 | 1 | 4 |
| IIIb | $\mu_1 \rtimes \mu \mathbf{1}$ | 1 | 2 | 2 | 3 | 4 |
| IVa | $\mu \mathrm{St}_{\mathrm{GSp}(4,F)}$ | 0 | 0 | 0 | 0 | 1 |
| IVb | $L(\nu^2, \nu^{-1}\mu \mathrm{St})$ | 0 | 0 | 2 | 1 | 3 |
| IVc | $L(\nu^{3/2} \mathrm{St}, \nu^{-3/2}\mu)$ | 0 | 1 | 1 | 2 | 3 |
| IVd | $\mu \mathbf{1}_{\mathrm{GSp}(4,F)}$ | 1 | 1 | 1 | 1 | 1 |
| Va | $\delta([\xi_u, \nu\xi_u], \nu^{-1/2}\mu)$ | 0 | 0 | 0 | 1 | 2 |
| Vb | $L(\nu^{1/2}\xi_u \mathrm{St}, \nu^{-1/2}\mu)$ | 0 | 1 | 1 | 1 | 2 |
| Vc | $L(\nu^{1/2}\xi_u \mathrm{St}, \nu^{-1/2}\xi_u\mu)$ | 0 | 1 | 1 | 1 | 2 |
| Vd | $L(\nu\xi_u, \xi_u \rtimes \nu^{-1/2}\mu)$ | 1 | 0 | 2 | 1 | 2 |
| VIa | $\tau(S, \nu^{-1/2}\mu)$ | 0 | 0 | 1 | 1 | 3 |
| VIb | $\tau(T, \nu^{-1/2}\mu)$ | 0 | 0 | 1 | 0 | 1 |
| VIc | $L(\nu^{1/2} \mathrm{St}, \nu^{-1/2}\mu)$ | 0 | 1 | 0 | 1 | 1 |
| VId | $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\mu)$ | 1 | 1 | 2 | 2 | 3 |

Table 3.4.: Dimensions of parahori-spherical vectors in non-cuspidal representations of $\mathrm{GSp}(4,F)$ for unramified characters $\mu_1$, $\mu_2$, $\mu$, $\xi$ of $F^\times$.

### 3.1.4. Parahori-spherical representations

As a corollary, we obtain the dimensions of parahori-spherical vectors.

**Corollary 3.8.** *An irreducible admissible representation $\rho$ of $\mathrm{GSp}(4,F)$ is parahori-spherical if and only if it is a subquotient of $\mu_1 \times \mu_2 \rtimes \mu_0$ for unramified characters $\mu_1, \mu_2, \mu_0$ of $F^\times$. The dimension of invariants under the standard parahoric subgroups are given in Table 3.4.*

*Proof.* The dimension of spherical vectors is the multiplicity of the trivial representation in the parahoric restriction of $\rho$. For non-cuspidal $\rho$, this is given by the tables above. If $\rho$ is cuspidal, the parahoric restriction is either zero or a sum of cuspidal representations by Cor. 2.17 and Mackey's theorem, so parahoric restriction with respect to the Iwahori subgroup $\mathscr{B}$ gives zero by transitivity (2.5). A fortiori, there are no non-zero parahori-spherical vectors. □

**Remark 3.9.** The non-cuspidal case has already been determined by Roberts and Schmidt [RS07, Table A.15].

## 3.2. The proof for hyperspecial parahoric restriction

We now prove the result on hyperspecial parahoric restriction of non-cuspidal irreducible admissible representations of $\mathrm{GSp}(4, F)$.

*Proof of Thm. 3.2.* Irreducible representations $\rho$ of type I, IIa, IIb, IIIa, IIIb, VII and X are parabolically induced, so the result is clear by Thm. 2.19 and Examples 2.20 and 2.21. In the other cases, $\rho$ is a non-trivial subquotient of a parabolically induced representation $\kappa$. If $\mathbf{r}_{\mathscr{K}}(\kappa) = 0$, then $\mathbf{r}_{\mathscr{K}}(\rho) = 0$ by exactness of the parahoric restriction functor, otherwise $\mathbf{r}_{\mathscr{K}}(\rho) \neq 0$ by Cor. 2.14. Furthermore, $\mathbf{r}_{\mathscr{K}}(\rho)$ is a non-zero subquotient of the parabolically induced $\mathbf{r}_{\mathscr{K}}(\kappa)$. It remains to determine the correct constituents of $\mathbf{r}_{\mathscr{K}}(\kappa)$ case by case.

We only discuss the case of odd $q$. For even $q$ one can determine the restriction of $\mathbf{r}_{\mathscr{K}}(\rho)$ to $\mathrm{Sp}(4, q)$ by the analogous arguments, see [Rös12, Thm. 2.33]. The central character of $\mathbf{r}_{\mathscr{K}}(\rho)$ is the restriction $\widetilde{\omega_\rho}$ of the central character $\omega_\rho$ of $\rho$.

For $\rho = \mu\,\mathbf{1}_{\mathrm{GSp}(4,F)}$ of type IVd, the hyperspecial parahoric restriction is at most one-dimensional, so it must be $\widetilde{\mu}\,\mathbf{1}_{\mathrm{GSp}(4,q)} = \theta_0(\widetilde{\mu})$ (for at most tamely ramified $\mu$) or zero (for wildly ramified $\mu$). By [RS07, (2.9)] and Table A.5

$$\mathbf{r}_{\mathscr{K}}(L(\nu^2, \nu^{-1}\mu\,\mathrm{St}_{\mathrm{GSp}(2,F)})) + \mathbf{r}_{\mathscr{K}}(\mu\,\mathbf{1}_{\mathrm{GSp}(4,F)}) \equiv \mathbf{1}_{\mathrm{GL}(2,q)} \rtimes \widetilde{\mu} = \chi_3(1, \widetilde{\mu})$$

decomposes as $\chi_3(1, \widetilde{\mu}) = \widetilde{\mu}(\theta_0 + \theta_1 + \theta_3)$. For type IVb we have

$$\mathbf{r}_{\mathscr{K}}(L(\nu^2, \nu^{-1}\mu\,\mathrm{St}_{\mathrm{GSp}(2,F)})) = \widetilde{\mu}(\theta_1 + \theta_3).$$

By the same argument we determine the parahoric restriction for type IVa and IVc as constituents of $\chi_1(1, \widetilde{\mu}) = 1 \rtimes \widetilde{\mu}\,\mathbf{1}_{\mathrm{GSp}(2,q)}$ and $\chi_2(1, \widetilde{\mu}) = 1 \rtimes \widetilde{\mu}\,\mathrm{St}_{\mathrm{GSp}(2,q)}$.

The representation $\rho = L(\nu^{1/2}\xi\,\mathrm{St}, \nu^{-1/2}\mu)$ of type Vb is a constituent of both $\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2,F)} \rtimes \nu^{-1/2}\mu$ and $\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2,F)} \rtimes \nu^{-1/2}\xi\mu$ by [RS07, (2.10)]. Therefore the hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}}(\rho)$ must be a constituent of both $\widetilde{\xi}\,\mathrm{St}_{\mathrm{GL}(2,q)} \rtimes \widetilde{\mu}$ and $\widetilde{\xi}\,\mathrm{St}_{\mathrm{GL}(2,q)} \rtimes \widetilde{\xi}\widetilde{\mu}$. By Table A.2, the only common constituent is $\theta_1(\widetilde{\mu})$ for unramified $\xi$ and $\tau_2(\widetilde{\mu})$ for tamely ramified $\xi$. By exactness and [RS07, (2.10)], types Va, Vc and Vd are clear.

The representation $\rho = \tau(T, \nu^{-1/2}\mu)$ of type VIb is a constituent of $1 \rtimes \mu\,\mathrm{St}_{\mathrm{GL}(2,q)}$ and $\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2,q)} \rtimes \nu^{-1/2}\mu$ by [RS07, (2.11)]. By exactness the hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}}(\rho)$ must be a constituent of both $\chi_2(1, \widetilde{\mu}) = \widetilde{\mu}(\theta_1 + \theta_3 + \theta_5)$ and of $\chi_3(1, \widetilde{\mu}) = \widetilde{\mu}(\theta_0 + \theta_1 + \theta_3)$. Therefore $\mathbf{r}_{\mathscr{K}}(\rho)$ can only be $\theta_3(\widetilde{\mu})$ or $\theta_1(\widetilde{\mu})$ or $\theta_1(\widetilde{\mu}) + \theta_3(\widetilde{\mu})$. By (2.6) we can assume that $\mu$ is unitary, then Thm. 4.29 and [Wei09a, Thm. 4.5] imply $\dim \mathbf{r}_{\mathscr{K}}(\tau(T, \nu^{-1/2}\mu)) = q(q^2+1)/2$. This implies $\mathbf{r}_{\mathscr{K}}(\tau(T, \nu^{-1/2}\mu)) = \widetilde{\mu}\,\theta_3$. By exactness the restrictions of type VIa, VIc and VId are clear.

The representation $\rho = \tau(T, \Pi)$ of type VIIIb is the non-generic constituent of $1 \rtimes \Pi$ for an irreducible cuspidal admissible representation $\Pi$ of $\mathrm{GL}(2, F)$ whose non-zero parahoric restriction $\mathbf{r}_{\mathrm{GL}(2,\mathfrak{o})}(\Pi) = \widetilde{\pi}$ is the cuspidal irreducible representation of $\mathrm{GL}(2, q)$ attached to a character $\Lambda$ of $\mathbb{F}_{q^2}^\times$. Its hyperspecial parahoric restriction is one of the two irreducible constituents of $\mathbf{r}_{\mathscr{K}}(1 \rtimes \Pi) \cong 1 \rtimes \widetilde{\pi} = X_3(\Lambda, 1) = \chi_7(\Lambda) + \chi_8(\Lambda)$. By a suitable character twist we can assume that $\Pi$ is unitary. Then $\tau(T, \Pi) = \mathrm{Wp}_-(\Pi)$ is the anisotropic theta-lift of $(\Pi, \Pi)$ by [Wei09a, 4.5]. The dimension of the hyperspecial parahoric restriction of $\tau(T, \Pi)$ is $(q^2 + 1)(q - 1)$, so it must be $\chi_7(\Lambda)$. The rest is clear.

For $\rho$ of type IXa or IXb, this is the statement of Thms. 3.18 and 3.21.

Let $\rho = \delta(\nu^{1/2}\Pi, \nu^{-1/2}\mu)$ be an irreducible representation of type XIa where $\Pi$ is a cuspidal irreducible admissible representation of $\mathrm{GL}(2, F)$ with trivial central character. By (2.6) we can assume without loss of generality that $\mu = 1$. The hyperspecial restriction of $\rho$ must be one of the two irreducible subquotients of

$$\mathbf{r}_{\mathscr{K}}(\nu^{1/2}\Pi, \nu^{-1/2}\mu) = \widetilde{\pi} \rtimes 1 = X_2(\omega_\Lambda \circ N_{q-1}, 1) = \chi_5(\omega_\Lambda, 1) + \chi_6(\omega_\Lambda, 1).$$

By [RS07, Table A.12], $\rho$ has paramodular level $\geq 3$ and therefore does not admit non-zero invariants under the subgroup

$$\mathscr{L} = \begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{o} \end{pmatrix} \cap \mathscr{K}.$$

By character theory, $\chi_5(\omega_\Lambda, 1)$ admits non-zero invariants under the image of $\mathscr{L}$ in $\mathscr{K}/\mathscr{K}^+$, so $\mathbf{r}_{\mathscr{K}}(\rho)$ cannot be $\chi_5(\omega_\Lambda, 1)$. The rest is clear. $\square$

### 3.2.1. Type IX

For non-cuspidal irreducible admissible representations $(\rho, V)$ of type IXa and IXb we need to determine the hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}}(\rho)$ for $\mathscr{K} = \mathrm{GSp}(4, \mathfrak{o})$ by hand. We consider two operators $T$ and $H$, which preserve a subspace $W_1$, and compare their eigenvectors in $W_1$.

Recall the setting: Fix a non-trivial additive character $\psi : (F, +) \to \mathbb{C}^\times$ whose restriction to $\mathfrak{o}$ factors over a non-trivial additive character $\widetilde{\psi}$ of $\mathfrak{o}/\mathfrak{p}$, fix a non-trivial tamely ramified or unramified quadratic character $\xi$ of $F^\times$ and a depth zero cuspidal irreducible admissible representation $(\Pi, V_\Pi)$ of $\mathrm{GL}(2, F)$ with central character $\omega_\Pi$ and complex multiplication $\xi\Pi = \Pi$. The hyperspecial parahoric restriction of $\Pi$ is a cuspidal and irreducible representation $\widetilde{\pi} = \mathbf{r}_{\mathrm{GL}(2,\mathfrak{o})}(\Pi)$ of $\mathrm{GL}(2, q)$ by Lemma 2.18.

This defines an irreducible representation $\nu\xi \boxtimes \nu^{-1/2}\Pi$ of the Levi subgroup of the Klingen parabolic $Q$. The normalized Klingen induced representation $(\rho, V) := \nu\xi \rtimes \nu^{-1/2}\Pi$ has an explicit model as the space of smooth functions

$$V = \left\{ f : \mathrm{GSp}(4,F) \to V_\Pi;\ f(pg) = \delta_Q^{1/2}(p)(\nu\xi \boxtimes \nu^{-1/2}\Pi)(p) f(g)\ \forall p \in Q \right\}, \quad (3.1)$$

where $\mathrm{GSp}(4,F)$ acts by right multiplication. It decomposes into an exact sequence

$$0 \to \delta(\nu\xi, \nu^{-1/2}\Pi) \to \nu\xi \rtimes \nu^{-1/2}\Pi \to L(\nu\xi, \nu^{-1/2}\Pi) \to 0 \quad (3.2)$$

with two irreducible constituents.

As a $\mathscr{K}$-module, $V^{\mathscr{K}^+}$ is naturally isomorphic to the representation $(\widetilde{\rho}, \widetilde{V}) = \widetilde{\xi} \rtimes \widetilde{\pi}$ of $\mathrm{GSp}(4,\mathbb{F}_q) \cong \mathscr{K}/\mathscr{K}^+$ by right-multiplication on the function space

$$\widetilde{V} = \left\{ \widetilde{f} : \mathscr{K}/\mathscr{K}^+ \to V_{\widetilde{\pi}};\ \widetilde{f}(pg) = (\widetilde{\xi} \boxtimes \widetilde{\pi})(p)\widetilde{f}(g)\ \forall p \in \mathbf{Q}(\mathbb{F}_q) \right\}.$$

This natural isomorphism is given by the restriction $V^{\mathscr{K}^+} \ni f \mapsto \widetilde{f} = f|_{\mathscr{K}}$ to $\mathscr{K}$. Each $\widetilde{f}$ extends to a unique $f$ by Iwasawa decomposition $G = Q\mathscr{K}$.

The representation $(\widetilde{\rho}, \widetilde{V})$ admits two irreducible constituents by Tables A.2 and A.5. For each subquotient of $\rho$, the hyperspecial parahoric restriction is one of the two constituents of $(\widetilde{\rho}, \widetilde{V}) \cong \widetilde{\xi} \rtimes \widetilde{\pi}$ by Cor. 2.14 and we have to determine the correct one.

We will use the following two vector subspaces of $V^{\mathscr{K}^+}$:

$$W_1 = \left\{ f \in V;\ f(gu) = \psi(u_{24}) f(g) \forall u \in \mathscr{B}^+ \right\} \text{ and}$$
$$W_2 = \left\{ f \in V;\ f(gu) = \psi(u_{12} + u_{24}) f(g) \forall u \in \mathscr{B}^+ \right\}.$$

Let $0 \neq v_0 \in V_{\widetilde{\pi}}$ be a fixed vector with $\widetilde{\pi}\left(\begin{smallmatrix} 1 & c \\ & 1 \end{smallmatrix}\right) v_0 = \widetilde{\psi}(c) v_0$ for all $c \in \mathbb{F}_q$. Since $\widetilde{\pi}$ is cuspidal, $v_0$ exists and is unique up to multiples.

For $w = 1$ and $w = s_1 s_2 s_1$ let $f_w \in V$ be defined by

$$f_w(x) = \begin{cases} \psi(u_{24})(\nu\xi \boxtimes \nu^{-1/2}\pi)(p) v_0 & \text{if } x = pwu \in Qw\mathscr{B}^+, \\ 0 & \text{else.} \end{cases}$$

Likewise, let $f_{\mathrm{Wh}} \in V$ be given by

$$f_{\mathrm{Wh}}(x) = \begin{cases} \psi(u_{12} + u_{24})(\nu\xi \boxtimes \nu^{-1/2}\pi)(p) v_0 & \text{if } x = ps_1 s_2 s_1 u \in Qs_1 s_2 s_1 \mathscr{B}^+, \\ 0 & \text{else.} \end{cases}$$

The choice of $v_0$ ensures that $f_w$ and $f_{\mathrm{Wh}}$ do not depend on the decomposition of $x$.

**Lemma 3.10.** *$W_1 = \mathbb{C} f_1 \oplus \mathbb{C} f_{s_1 s_2 s_1}$ and $W_2 = \mathbb{C} f_{\mathrm{Wh}}$.*

*Proof.* It is clear by construction that $f_1, f_{s_1s_2s_1} \in W_1$ and $f_{\mathrm{Wh}} \in W_2$. The three functions are non-trivial because $f_1(1) = f_{s_1s_2s_1}(s_1s_2s_1) = f_{\mathrm{Wh}}(s_1s_2s_1) = v_0$ is non-zero. By (3.14) there is the disjoint double coset decomposition

$$\mathrm{GSp}(4,F) = \bigsqcup_w Qw\mathscr{B}^+ \qquad\qquad w \in \{1, s_1, s_1s_2, s_1s_2s_1\} \tag{3.3}$$

and therefore $\mathrm{supp}(f_1) \cap \mathrm{supp}(f_{s_1s_2s_1}) = \emptyset$, so $f_1$ and $f_{s_1s_2s_1}$ are linearly independent. The subspace $W_2$ is the space of Whittaker vectors in the $\mathscr{K}$-representation $\widetilde{\xi} \rtimes \widetilde{\pi}$ on $\widetilde{V}$, so $W_2$ is one-dimensional. It remains to be shown that $W_1$ is two-dimensional. Indeed, an arbitrary $f \in W_1$ is uniquely determined by its values on $1, s_1s_2s_1, s_1, s_1s_2$ because of (3.3). For every $f \in W_1$ and arbitrary $c \in \mathfrak{o}$ we have

$$\widetilde{\pi}\left(\begin{smallmatrix}1 & c\\ & 1\end{smallmatrix}\right) f(i) = f\left(\begin{pmatrix}1 &&&\\ & 1 & & \\ && 1 & c\\ &&& 1\end{pmatrix} i\right) = f\left(i \begin{pmatrix}1 &&&\\ & 1 & & \\ && 1 & c\\ &&& 1\end{pmatrix}\right) = \psi(c) f(i),$$

and this implies $f(1), f(s_1s_2s_1) \in \mathbb{C}v_0$, by definition of $v_0$. Furthermore,

$$\begin{aligned}
\widetilde{\pi}\left(\begin{smallmatrix}1 & c\\ & 1\end{smallmatrix}\right) f(s_1) &= f\left(s_1 \begin{pmatrix}1 & & c &\\ & 1 & & \\ && 1 & \\ &&& 1\end{pmatrix}\right) = f(s_1) \text{ and}\\
\widetilde{\pi}\left(\begin{smallmatrix}1 & c\\ & 1\end{smallmatrix}\right) f(s_1s_2) &= f\left(s_1s_2 \begin{pmatrix}1 & & c &\\ & 1 & & \\ && 1 & \\ &&& 1\end{pmatrix}\right) = f(s_1s_2),
\end{aligned}$$

so $f(s_1)$ and $f(s_1s_2)$ are invariant under $\widetilde{\pi}\left(\begin{smallmatrix}1 & *\\ & 1\end{smallmatrix}\right)$. But $\widetilde{\pi}$ is cuspidal, so $f(s_1) = f(s_1s_2) = 0$. Now $W_1 = \mathbb{C}f_1 \oplus \mathbb{C}f_{s_1s_2s_1}$ is clear by definition of $W_1$. □

**Lemma 3.11.** *The $\mathscr{K}$-intertwining operator*

$$T : \widetilde{V} \to \widetilde{V}, \quad (T\widetilde{f})(g) = \sum_{u \in \mathscr{Q}^+/\mathscr{K}^+} \widetilde{f}(s_1s_2s_1ug) \quad \forall g \in \mathscr{K},$$

*is well-defined and stabilizes the subspaces $W_1$ and $W_2$.*

*Proof.* It has to be shown that $T\widetilde{f} \in \widetilde{V}$ for every $\widetilde{f} \in \widetilde{V}$. Let $g \in \mathscr{K}/\mathscr{K}^+ \cong \mathrm{GSp}(4,q)$ be arbitrary. By construction, $T\widetilde{f}(ug) = T\widetilde{f}(g)$ for any $u$ in the unipotent radical $\mathscr{Q}^+/\mathscr{K}^+$ of the finite Klingen parabolic $\mathscr{Q}/\mathscr{K}^+ \cong \mathbf{Q}(q)$. For every $m$ in the Levi quotient $\mathscr{Q}/\mathscr{Q}^+ \cong \mathbb{F}_q^\times \times \mathrm{GSp}(2,\mathbb{F}_q)$, we have

$$\begin{aligned}
T\widetilde{f}(mg) &= (\widetilde{\xi} \boxtimes \widetilde{\pi})(s_1s_2s_1m(s_1s_2s_1)^{-1}) T\widetilde{f}(g)\\
&= \widetilde{\xi}(m_{11})^{-1} (\widetilde{\xi}\,\widetilde{\pi}) \left(\begin{smallmatrix} m_{22} & m_{24}\\ m_{42} & m_{44}\end{smallmatrix}\right) T\widetilde{f}(g)\\
&= (\widetilde{\xi} \boxtimes \widetilde{\pi})(m) T\widetilde{f}(g),
\end{aligned}$$

so $T\widetilde{f} \in \widetilde{V}$. Since $T$ is $\mathscr{K}$-intertwining, it preserves $\widetilde{W_1}$ and $\widetilde{W_2}$. □

**Lemma 3.12.** *i) For* $A := \mathscr{Q}^+ \operatorname{diag}(\varpi, 1, \varpi^{-1}, 1)\mathscr{Q}^+$ *the Hecke operator*

$$H : V \longrightarrow V^{\mathscr{Q}^+}, \quad f \longmapsto \operatorname{vol}(\mathscr{Q}^+)^{-1}\rho(\operatorname{char}_A)f.$$

*acts on* $f \in V^{\mathscr{Q}^+}$ *by*

$$Hf = \sum_{\substack{a,b\in\mathfrak{o}/\mathfrak{p}\\ c\in\mathfrak{o}/\mathfrak{p}^2}} \rho\begin{pmatrix} \varpi & a & c\varpi^{-1} & b \\ & 1 & b\varpi^{-1} & \\ & & \varpi^{-1} & \\ & & -a\varpi^{-1} & 1 \end{pmatrix} f. \tag{3.4}$$

*ii) The Hecke operator* $H$ *stabilises the subspace* $W_1 \subseteq V^{\mathscr{Q}^+}$.

*Proof.* i) The double-coset $A$ is the disjoint union

$$A = \bigsqcup_{\substack{a,b\in\mathfrak{o}/\mathfrak{p}\\ c\in\mathfrak{o}/\mathfrak{p}^2}} \begin{pmatrix} \varpi & a & c\varpi^{-1} & b \\ & 1 & b\varpi^{-1} & \\ & & \varpi^{-1} & \\ & & -a\varpi^{-1} & 1 \end{pmatrix} \mathscr{Q}^+.$$

Since $\rho(\operatorname{char}_{\mathscr{Q}^+})f = \operatorname{vol}(\mathscr{Q}^+)f$ for every $f \in V^{\mathscr{Q}^+}$, we have (3.4).

ii) For $u \in \mathscr{B}^+$ there is $\widetilde{u} \in \mathscr{Q}^+$ such that $u = (I_4 + u_{24}E_{24})\widetilde{u}$. Since $\mathscr{Q}^+$ is a normal subgroup of $\mathscr{B}^+$, we have $uA = Au$ and that implies $\rho(u) \circ H = H \circ \rho(u)$. For $f \in W_1$ this means $\rho(u)H(f) = \psi(u_{24})H(f)$, so $Hf \in W_1$. □

**Type IX for unramified $\xi$.** For the unramified quadratic character $\xi(x) = (-1)^{v_F(x)}$ of $F^\times$, we will now determine the action of the endomorphisms $T : W_1 \to W_1$ and $H : W_1 \to W_1$ explicitly on the generators $f_1$, $f_{s_1s_2s_1}$. The intersection of the subrepresentation $V_\delta = \delta(\nu\xi, \nu^{-1/2}\Pi)$ of $(\rho, V)$ with the subspace $W_1$ must be preserved by both $H$ and $T$.

Since $T$ is $\mathscr{K}$-intertwining on $(\widetilde{\rho}, \widetilde{V})$, one of the eigenvalues of $T : W_1 \to W_1$ must coincide with the eigenvalue of $T : W_2 \to W_2$. The corresponding eigenvectors must belong to the generic constituent of $(\widetilde{\rho}, \widetilde{V})$. If this eigenvector $f_a \in W_1$ is also an eigenvector of $H$ and the other $T$ eigenvector $f_b \in W_1$ is not an eigenvector of $H$, then $\mathbf{r}_{\mathscr{K}}(\delta(\nu\xi, \nu^{-1/2}\Pi))$ must contain the space of Whittaker vectors $W_2$ and thus be generic.

**Lemma 3.13.** *For any cuspidal admissible representation* $(\pi, V_\pi) \in \mathbf{Rep}(\mathrm{GL}(2, q))$

$$\sum_{\substack{a,b,c\in\mathbb{F}_q\\ c\neq 0}} \pi\begin{pmatrix} 1 - ab/c & -a^2/c \\ b^2/c & 1 + ab/c \end{pmatrix} = (q - q^2)\pi(I_2). \tag{3.5}$$

*Proof.* Fix $(a,b)\neq(0,0)$. Since $\left(\begin{smallmatrix}-ab & -a^2\\ b^2 & ab\end{smallmatrix}\right)$ is nilpotent, there is $A\in\mathrm{GL}(2,q)$ such that $\left(\begin{smallmatrix}-ab & -a^2\\ b^2 & ab\end{smallmatrix}\right)=A^{-1}\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)A$ and this implies:

$$\sum_{c\in\mathbb{F}_q^\times}\pi\begin{pmatrix}1-ab/c & -a^2/c\\ b^2/c & 1+ab/c\end{pmatrix}=\pi(A^{-1})\underbrace{\sum_{c'\in\mathbb{F}_q}\pi\begin{pmatrix}1 & c'\\ & 1\end{pmatrix}}_{=0\ (\pi\text{ is cuspidal})}\pi(A)-\pi(A^{-1}A)=-\pi(I_2).$$

with $1/c=c'$ for $c'\neq 0$. The sum over $a,b\in\mathbb{F}_q$ and $c\in\mathbb{F}_q^\times$ is then clear. $\square$

**Lemma 3.14.** *The action of $T$ on $W_1$ and $W_2$ is given by*

$$Tf_1=f_{s_1s_2s_1},\qquad Tf_{s_1s_2s_1}=q^3f_1+(q-q^2)f_{s_1s_2s_1},\qquad Tf_{\mathrm{Wh}}=qf_{\mathrm{Wh}}.\tag{3.6}$$

*Proof.* For $f\in W_1$ we have

$$Tf(1)=\sum_{u\in\mathscr{Q}^+/\mathscr{K}^+}f(s_1s_2s_1u)=\sum_{u\in\mathscr{Q}^+/\mathscr{K}^+}f(s_1s_2s_1)=q^3f(s_1s_2s_1),$$

since $f$ is right invariant under $\mathscr{Q}^+$ by definition of $W_1$. Further,

$$\begin{aligned}
Tf(s_1s_2s_1)&=\sum_{a,b,c\in\mathbb{F}_q}f\left(\left(\begin{smallmatrix}-1 & & & \\ -b & 1 & & \\ c & -a & -1 & -b\\ a & & & 1\end{smallmatrix}\right)\right)=\sum_{\substack{a,b,c\in\mathbb{F}_q\\ a=b=0\text{ or }c\neq0}}f\left(\left(\begin{smallmatrix}-1 & & & \\ -b & 1 & & \\ c & -a & -1 & -b\\ a & & & 1\end{smallmatrix}\right)\right)\\
&=f(1)+\sum_{\substack{a,b,c\in\mathbb{F}_q\\ c\neq0}}f\left(\left(\begin{smallmatrix}-1/c & -a/c & 1 & -b/c\\ & 1-ab/c & b & -b^2/c\\ & & -c & \\ & a^2/c & -a & 1+ab/c\end{smallmatrix}\right)s_1s_2s_1\left(\begin{smallmatrix}1 & -a/c & -1/c & -b/c\\ & 1 & -b/c & \\ & & 1 & \\ & & a/c & 1\end{smallmatrix}\right)\right)\\
&=f(1)+\sum_{\substack{a,b,c\in\mathbb{F}_q\\ c\neq0}}\widetilde{\pi}\left(\left(\begin{smallmatrix}1-ab/c & -b^2/c\\ a^2/c & 1+ab/c\end{smallmatrix}\right)\right)f(s_1s_2s_1)\\
&=f(1)+(q-q^2)f(s_1s_2s_1)\qquad\text{by Lemma 3.13.}
\end{aligned}$$

Since $\operatorname{supp}Tf\subseteq Q\sqcup Qs_1s_2s_1\mathscr{B}^+$, the image $Tf\in W_1$ is uniquely determined by the values on 1 and $s_1s_2s_1$. For $W_2$ we have $\operatorname{supp}f_{\mathrm{Wh}}\subseteq Qs_1s_2s_1\mathscr{B}^+$, hence

$$\begin{aligned}
Tf_{\mathrm{Wh}}(s_1s_2s_1)&=\sum_{\substack{a,b,c\in\mathbb{F}_q\\ c\neq0}}f_{\mathrm{Wh}}\left(\left(\begin{smallmatrix}-1/c & -a/c & 1 & -b/c\\ & 1-ab/c & b & -b^2/c\\ & & -c & \\ & a^2/c & -a & 1+ab/c\end{smallmatrix}\right)s_1s_2s_1\left(\begin{smallmatrix}1 & -a/c & -1/c & -b/c\\ & 1 & -b/c & \\ & & 1 & \\ & & a/c & 1\end{smallmatrix}\right)\right)\\
&=\sum_{\substack{a,b,c\in\mathbb{F}_q\\ c\neq0}}\widetilde{\psi}(-a/c)\widetilde{\pi}\left(\left(\begin{smallmatrix}1-ab/c & -b^2/c\\ a^2/c & 1+ab/c\end{smallmatrix}\right)\right)f_{\mathrm{Wh}}(s_1s_2s_1)\\
&=\sum_{a',b'\in\mathbb{F}_q}\widetilde{\psi}(-a')\sum_{c\in\mathbb{F}_q^\times}\widetilde{\pi}\left(\left(\begin{smallmatrix}1-a'b'c & -b'^2c\\ a'^2c & 1+a'b'c\end{smallmatrix}\right)\right)f_{\mathrm{Wh}}(s_1s_2s_1)\\
&\overset{(3.6)}{=}\widetilde{\psi}(0)\sum_{c\in\mathbb{F}_q^\times}\widetilde{\pi}\left(I_2\right)f_{\mathrm{Wh}}(s_1s_2s_1)+\sum_{\substack{a',b'\in\mathbb{F}_q\\ (a',b')\neq(0,0)}}\widetilde{\psi}(-a')(-\widetilde{\pi}(I_2))f_{\mathrm{Wh}}(s_1s_2s_1)\\
&=(q-1)f_{\mathrm{Wh}}(s_1s_2s_1)+f_{\mathrm{Wh}}(s_1s_2s_1)=qf_{\mathrm{Wh}}(s_1s_2s_1).
\end{aligned}$$

In the third line the substitution $a' := a/c$ and $b' := b/c$ was made. □

**Lemma 3.15.** *The Hecke operator $H : W_1 \to W_1$ is given by*

$$Hf_1 = -qf_1 + (q - q^{-1})f_{s_1s_2s_1} \qquad \textit{and} \qquad Hf_{s_1s_2s_1} = -q^3 f_{s_1s_2s_1}. \tag{3.7}$$

*Proof.* We begin by calculating $Hf(1)$ for $f \in W_1$. The modulus character of the standard Klingen parabolic $Q$ is $\delta_Q(p) = |p_{11}|^4|p_{22}p_{44} - p_{24}p_{42}|^{-2}$. Therefore

$$\begin{aligned} Hf(1) &= \sum_{\substack{a,b\in\mathfrak{o}/\mathfrak{p}\\ c\in\mathfrak{o}/\mathfrak{p}^2}} f\begin{pmatrix} \varpi & a & c\varpi^{-1} & b \\ & 1 & b\varpi^{-1} & \\ & & \varpi^{-1} & \\ & & -a\varpi^{-1} & 1 \end{pmatrix} = \sum_{\substack{a,b\in\mathfrak{o}/\mathfrak{p}\\ c\in\mathfrak{o}/\mathfrak{p}^2}} \delta_Q^{1/2}\begin{pmatrix} \varpi & a & c\varpi^{-1} & b \\ & 1 & b\varpi^{-1} & \\ & & \varpi^{-1} & \\ & & -a\varpi^{-1} & 1 \end{pmatrix} \xi(\varpi)\nu(\varpi)f(1) \\ &= \sum_{\substack{a,b\in\mathfrak{o}/\mathfrak{p}\\ c\in\mathfrak{o}/\mathfrak{p}^2}} q^{-2}(-q^{-1})f(1) = -qf(1). \end{aligned}$$

In order to calculate $Hf(s_1s_2s_1)$, recall $\mathrm{supp}(f) \subseteq Q\mathscr{Q}^+ \sqcup Qs_1s_2s_1\mathscr{Q}^+$. By Table 3.5 we have $Hf_1(s_1s_2s_1) = (q - q^{-1})v_0$ and $Hf_{s_1s_2s_1}(s_1s_2s_1) = -q^3v_0$. Functions in $W_1$ are uniquely determined by their values on 1 and $s_1s_2s_1$, so (3.7) follows. □

The representation

$$\mathbf{r}_{\mathscr{K}}(\xi \rtimes \Pi) = \mathbf{1}_{\mathbb{F}_q^\times} \rtimes \widetilde{\pi}$$

of $\mathscr{K}/\mathscr{K}^+$ has two irreducible constituents. The generic one is $\chi_g = \chi_{13}(l_2) \boxtimes \omega_{\widetilde{\pi}}$ for even $q$ and $\chi_g = \chi_8(\Lambda)$ for odd $q$ and the non-generic one is $\chi_n = \chi_9(l_2) \boxtimes \omega_{\widetilde{\pi}}$ for even $q$ and $\chi_n = \chi_7(\Lambda)$ for odd $q$.[1]

Lemma 3.14 implies that $T : W_1 \to W_1$ admits the eigenvector $f_b := qf_1 - f_{s_1s_2s_1}$ with eigenvalue $-q^2$ and the eigenvector $f_a := q^2f_1 + f_{s_1s_2s_1}$ with eigenvalue $q$. But the Whittaker vector $f_{\mathrm{Wh}}$ has eigenvalue $q$ and belongs to the generic representation $\chi_g$, so Schur's Lemma implies:

**Corollary 3.16.** *The vector $f_a$ generates the constituent $\chi_g$, while $f_b$ generates $\chi_n$.*

**Corollary 3.17.** *The $T$-eigenvector $f_a \in W_1$ is an $H$-eigenvector with $H$-eigenvalue $-q$, but the $T$-eigenvector $f_b \in W_1$ is not an $H$-eigenvector.*

*Proof.* Lemma 3.15 implies $Hf_a = -qf_a$ and $Hf_b = -q^3f_b + (q^2 - 1)f_a$. □

**Theorem 3.18.** *If $\xi : F^\times \to \mathbb{C}^\times$ is the non-trivial unramified quadratic character of $F^\times$ and $\Pi$ is a depth zero cuspidal irreducible admissible representation of* $\mathrm{GL}(2, F)$ *with $\xi\Pi = \Pi$, then the hyperspecial parahoric restriction for type IX is*

$$\mathbf{r}_{\mathscr{K}}(\delta(\nu\xi, \nu^{-1/2}\Pi)) = \chi_g \qquad \textit{and} \qquad \mathbf{r}_{\mathscr{K}}(L(\nu\xi, \nu^{-1/2}\Pi)) = \chi_n.$$

[1] The character $\Lambda$ of $\mathbb{F}_{q^2}^\times$ is the one that determines $\widetilde{\pi}$ by Thm. A.1 with $l_2$ like in Not. 3.3.

$$
\begin{aligned}
Hf(s_1s_2s_1) &= \sum_{\substack{a,b\in\mathfrak{o} \mod \mathfrak{p}\\ c\in\mathfrak{o} \mod \mathfrak{p}^2}} f\left(s_1s_2s_1\begin{pmatrix}\varpi & a & c\varpi^{-1} & b\\ & 1 & b\varpi^{-1} & \\ & & \varpi^{-1} & \\ & & -a\varpi^{-1} & 1\end{pmatrix}\right) = \sum_{\substack{a,b\in\mathfrak{o} \mod \mathfrak{p}\\ c\in\mathfrak{o} \mod \mathfrak{p}^2}} f\begin{pmatrix} & & \varpi^{-1} & \\ & 1 & b\varpi^{-1} & \\ -\varpi & -a & -c\varpi^{-1} & -b\\ & & -a\varpi^{-1} & 1\end{pmatrix}\\
&= \sum_{\substack{a,b\in\mathfrak{p} \mod \mathfrak{p}\\ c\in\mathfrak{p}-\mathfrak{p}^2 \mod \mathfrak{p}^2}} f\Bigg(\underbrace{\begin{pmatrix}-\varpi/c & -a/c & \varpi^{-1} & -b/c\\ & 1-ab/c & b\varpi^{-1} & -b^2/c\\ & & -c\varpi^{-1} & \\ & a^2/c & -a\varpi^{-1} & 1+ab/c\end{pmatrix}}_{\in Q}\underbrace{\begin{pmatrix}1 & & & \\ -\varpi b/c & 1 & & \\ \varpi^2/c & \varpi a/c & 1 & \varpi b/c\\ \varpi a/c & & & 1\end{pmatrix}}_{\in \mathscr{Q}^+}\Bigg)\\
&+ \sum_{\substack{a,b\in\mathfrak{o} \mod \mathfrak{p}\\ c\in\mathfrak{o}-\mathfrak{p} \mod \mathfrak{p}^2}} f\Bigg(\underbrace{\begin{pmatrix}-\varpi 1/c & -a/c & \varpi^{-1} & -b/c\\ & 1-ab/c & b\varpi^{-1} & -b^2/c\\ & & -c\varpi^{-1} & \\ & a^2/c & -a\varpi^{-1} & 1+ab/c\end{pmatrix}}_{\in Q}\underbrace{\begin{pmatrix}1 & & & \\ -\varpi b/c & 1 & & \\ \varpi^2/c & \varpi a/c & 1 & \varpi b/c\\ \varpi a/c & & & 1\end{pmatrix}}_{\in \mathscr{Q}^+}\Bigg)\\
&+ \sum_{\substack{a,b\in\mathfrak{p} \mod \mathfrak{p}\\ c\in\mathfrak{p}^2 \mod \mathfrak{p}^2}} f\Bigg(\underbrace{\begin{pmatrix}\varpi^{-1} & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1\end{pmatrix}}_{\in Q}\underbrace{s_1s_2s_1\begin{pmatrix}1 & a\varpi^{-1} & c\varpi^{-2} & b\varpi^{-1}\\ & 1 & b\varpi^{-1} & \\ & & 1 & \\ & & -a\varpi^{-1} & 1\end{pmatrix}}_{\in \mathscr{Q}^+}\Bigg)\\
&+ \sum_{\substack{a,b\in\mathfrak{o} \mod \mathfrak{p}\\ (a,b)\neq(0,0) \mod (\mathfrak{p},\mathfrak{p})\\ c\in\mathfrak{p} \mod \mathfrak{p}^2}} f\underbrace{\begin{pmatrix} & & \varpi^{-1} & \\ & 1 & b\varpi^{-1} & \\ -\varpi & -a & -c\varpi^{-1} & -b\\ & & -a\varpi^{-1} & 1\end{pmatrix}}_{\notin \operatorname{supp}(f)} \quad \text{by Lemma 3.32}\\
&= \sum_{\substack{a,b\in\mathfrak{p} \mod \mathfrak{p}\\ c\in\mathfrak{p}-\mathfrak{p}^2 \mod \mathfrak{p}^2}} |-\varpi c^{-1}|^2\nu\xi(-\varpi c^{-1})(\nu^{-1/2}\Pi)(I_2)f(1)\\
&+ \sum_{\substack{a,b\in\mathfrak{o} \mod \mathfrak{p}\\ c\in\mathfrak{o}-\mathfrak{p} \mod \mathfrak{p}^2}} |-\varpi c^{-1}|^2\nu\xi(-\varpi c^{-1})(\nu^{-1/2}\Pi)\begin{pmatrix}1-ab/c & -b^2/c\\ a^2/c & 1+ab/c\end{pmatrix}f(1)\\
&+ \sum_{\substack{a,b\in\mathfrak{p} \mod \mathfrak{p}\\ c\in\mathfrak{p}^2 \mod \mathfrak{p}^2}} |\varpi^{-1}|^2\nu\xi(\varpi^{-1})f(s_1s_2s_1)\\
&= \sum_{\substack{a,b\in\mathfrak{p} \mod \mathfrak{p}\\ c\in\mathfrak{p}-\mathfrak{p}^2 \mod \mathfrak{p}^2}} f(1) + \sum_{\substack{a,b\in\mathfrak{o} \mod \mathfrak{p}\\ c\in\mathfrak{o}-\mathfrak{p} \mod \mathfrak{p}^2}} q^{-2}(-q^{-1})\Pi\begin{pmatrix}1-ab/c & -b^2/c\\ a^2/c & 1+ab/c\end{pmatrix}f(1)\\
&+ \sum_{\substack{a,b\in\mathfrak{p} \mod \mathfrak{p}\\ c\in\mathfrak{p}^2 \mod \mathfrak{p}^2}} q^2(-q)f(s_1s_2s_1)\\
&= (q-1)f(1) + q(q-q^2)q^{-2}(-q^{-1})f(1) - q^3f(s_1s_2s_1) \quad \text{(Lemma 3.13)}\\
&= (q-q^{-1})f(1) - q^3f(s_1s_2s_1).
\end{aligned}
$$

Table 3.5.: The calculation of $Hf(s_1s_2s_1)$ in the proof of Lemma 3.15.

*Proof.* By definition, $\delta(\nu\xi, \nu^{-1/2}\Pi))$ is a subrepresentation of $\nu\xi \rtimes \nu^{-1/2}\Pi$, defined on a vector subspace $V_\delta \subseteq V_\rho$. The subspace $W_1 \cap V_\delta^{\mathscr{K}+}$ is one-dimensional, because $W_1$ contains $T$-eigenvectors with two different eigenvalues. The operators $T$ and $H$ both stabilize $W_1 \cap V_\delta^{\mathscr{K}+}$, so this space must contain a common eigenvector of $T$ and $H$. We have already seen that $f_b$ is not an $H$-eigenvector, so $W_1 \cap V_\delta = \mathbb{C} f_a$. Therefore $f_a \in V_\delta^{\mathscr{K}+}$ and Cor. 3.16 implies that $\chi_g$ is a constituent of $\mathbf{r}_{\mathscr{K}}(\delta(\nu\xi, \nu^{-1/2}\Pi)))$. Then the parahoric restriction of $L(\nu\xi, \nu^{-1/2}\Pi)$ can only be zero or $\chi_n$. It cannot be zero because of Cor. 2.14. □

**Type IX for tamely ramified quadratic character.** Let $q$ be an odd prime power and fix a tamely ramified quadratic character $\xi = \xi_t$ of $F^\times$ such that the restriction to $\mathfrak{o}^\times$ factors over the nontrivial quadratic character $\lambda_0$ of $\mathfrak{o}^\times/(1+\mathfrak{p}) \cong \mathbb{F}_q^\times$. In this section, we determine the hyperspecial parahoric restriction of the irreducible admissible constituents of the Klingen induced representation $(\rho, V) = \nu\xi \rtimes \nu^{-1/2}\Pi$. The Gauß sum

$$\mathfrak{G} = \sum_{x \in \mathbb{F}_q^\times} \lambda_0(x)\widetilde{\psi}(x) = \sum_{x \in \mathbb{F}_q} \widetilde{\psi}(x^2) \tag{3.8}$$

has square $\mathfrak{G}^2 = q\lambda_0(-1) = (-1)^{(q-1)/2}q$, but $\mathfrak{G}$ depends on the choice of $\widetilde{\psi}$.

The argument is analogous to the unramified case. For any cuspidal irreducible representation $\widetilde{\pi}$ of $\mathrm{GL}(2, q)$, a tedious calculation with the model of Prop. A.2 yields

$$\sum_{\substack{a,b,c \in \mathbb{F}_q \\ c \neq 0}} \xi(-\tfrac{1}{c})\psi(\tfrac{a}{c})\widetilde{\pi}\begin{pmatrix} 1-ab/c & -b^2/c \\ a^2/c & 1+ab/c \end{pmatrix} = -\mathfrak{G} \cdot \widetilde{\pi}(I_2), \tag{3.9}$$

$$\sum_{\substack{a,b,c \in \mathbb{F}_q \\ c \neq 0}} \xi(-\tfrac{1}{c})\widetilde{\pi}\begin{pmatrix} 1-ab/c & -b^2/c \\ a^2/c & 1+ab/c \end{pmatrix} = (q^2-1)\mathfrak{G} \cdot \widetilde{\pi}(I_2). \tag{3.10}$$

**Lemma 3.19.** *For $f_a = q^3\widetilde{f}_1 - \mathfrak{G}\widetilde{f}_{s_1s_2s_1}$ and $f_b = q\widetilde{f}_1 + \mathfrak{G}\widetilde{f}_{s_1s_2s_1}$ in $W_1$ we have*

$$Tf_a = -\mathfrak{G} \cdot f_a, \qquad Tf_b = q^2 \cdot f_b, \qquad Tf_{\mathrm{Wh}} = -\mathfrak{G} \cdot f_{\mathrm{Wh}}.$$

*Sketch of proof.* For every $f \in W_1$ and $f_{\mathrm{Wh}} \in W_2$ we have

$$Tf(1) = q^3 f(s_1s_2s_1), \quad Tf(s_1s_2s_1) = \xi(-1)f(1) + (q^2-1)\mathfrak{G} \cdot f(s_1s_2s_1),$$
$$T\widetilde{f}_{\mathrm{Wh}}(s_1s_2s_1) = -\mathfrak{G} \cdot \widetilde{f}_{\mathrm{Wh}}(s_1s_2s_1)$$

by a standard calculation using (3.9) and (3.10). □

**Lemma 3.20.** *The Hecke operator $H$ acts on $W_1$ via*

$$Hf_a = q\xi(\varpi)f_a \qquad Hf_b = \xi(\varpi)((\tfrac{1}{q} - q)f_a + q^3 f_b).$$

*Proof.* A standard calculation as in Lemma 3.15 yields $Hf(1) = \xi(\varpi)qf(1)$ and

$$Hf(s_1s_2s_1) = \xi(\varpi)(1-q^{-2})\mathfrak{G}f(1) + \xi(\varpi)q^3 f(s_1s_2s_1).$$

□

By the same argument as in Thm. 3.18 we obtain

**Theorem 3.21.** *For a tamely ramified quadratic character $\xi$ of $F^\times$ and a depth zero cuspidal irreducible admissible representation $\Pi$ of $\mathrm{GL}(2,F)$ with complex multiplication $\xi\Pi \cong \Pi$, the hyperspecial parahoric restriction is*

$$\mathbf{r}_{\mathscr{K}}(\delta(\nu^{1/2}\xi, \nu^{-1/2}\Pi)) = \tau_5(\lambda'), \qquad \mathbf{r}_{\mathscr{K}}(L(\nu^{1/2}\xi, \nu^{-1/2}\Pi)) = \tau_4(\lambda'),$$

*with a character $\lambda'$ of $\mathbb{F}_{q^2}^\times[2(q-1)]$ such that the cuspidal irreducible representation $\widetilde{\pi} = \mathbf{r}_{\mathrm{GL}(2,\mathfrak{o})}(\Pi)$ is generated by the character $\Lambda = \lambda' \circ N_{(q+1)/2}$.*

## 3.3. The proof for paramodular restriction

We now prove the result on paramodular restriction of non-cuspidal irreducible admissible representations of $\mathrm{GSp}(4,F)$.

*Proof of Thm. 3.7.* The strategy is similar to the hyperspecial case by combining Lemma 3.22, Lemma 3.5 and Cor. 2.14. If a parabolically induced representation $\kappa$ of $G$ has non-zero paramodular restriction, then for every irreducible constituent $\rho$ of $\kappa$ the paramodular restriction $\mathbf{r}_{\mathscr{J}}(\rho)$ is a non-zero (not necessarily irreducible) subquotient of $\mathbf{r}_{\mathscr{J}}(\kappa)$. For each case we have to determine the correct constituent(s) of the representations $\mathbf{r}_{\mathscr{J}}(\kappa)$ given by Lemmas 3.22 and 3.23. We can assume without loss of generality that the inducing data are of depth zero [MP96, 5.2(1)].

The irreducible admissible representations $\rho$ of type I, IIIa, IIIb and VII are Klingen induced and the statement is implied by Lemma 3.22. For representations of type IIa, IIb, X, XIa and XIb there is Lemma 3.23.

The irreducible admissible representation $\rho = \mu\,\mathrm{St}_{\mathrm{GSp}(4,F)}$ of type IVa is the irreducible subrepresentation of $\nu^2 \times \nu \rtimes \nu^{-3/2}\mu$. By Table 3.2, the paramodular restriction of $\rho$ is irreducible and either $\widetilde{\mu}[\mathbf{1}, \mathrm{St}]$ or $\widetilde{\mu}[\mathrm{St}, \mathrm{St}]$. By Lemma 3.5 it must be $\widetilde{\mu}[\mathrm{St}, \mathrm{St}]$. By [RS07, (2.9)], IVa and IVb are the constituents of the Klingen induced representation $\nu^2 \times \nu\mu\,\mathrm{St}$, so case IVb is also clear by Lemma 3.22. Lemma 3.23 gives the paramodular restriction for representations of type IVc and IVd. The proof for type VI is analogous [RS07, (2.11)].

We begin with $\rho$ of type Vc with an unramified quadratic character $\xi_u$ of $F^\times$. By Thm. 3.30, the paramodular restriction $\mathbf{r}_{\mathscr{J}}(\rho)$ has a generic subquotient, which must be $\widetilde{\mu}[\,\mathrm{St},\,\mathrm{St}]$. Since the Klingen parahoric restriction of type Vc contains two constituents, there must be another irreducible constituent of $\widetilde{\mu}[1\times 1, 1\times 1]$ in $\mathbf{r}_{\mathscr{J}}(\rho)$. By the symmetry argument of Lemma 3.5, this can only be $\widetilde{\mu}[\,\mathbf{1},\,\mathbf{1}]$. For type Vc with tamely ramified quadratic character $\xi_t$, the paramodular restriction is the generic constituent of $[1\times\lambda_0, 1\times\lambda_0]$ if and only if $\xi_t(\varpi) = -1$ (and the non-generic one otherwise) by Thm. 3.30. The paramodular restriction for types Va,Vb and Vd is then clear by Lemma 3.23 and [RS07, (2.10)].

Representations of type VIII are irreducible subquotients of $1 \rtimes \Pi$. Without loss of generality let $\Pi$ be of depth zero. Their paramodular restriction is either $[\widetilde{\pi},\,\mathbf{1}]+[\,\mathbf{1},\widetilde{\pi}]$ or $[\widetilde{\pi},\,\mathrm{St}] + [\,\mathrm{St},\widetilde{\pi}]$ by Lemma 3.22 and Lemma 3.5. Corollary 4.15 implies that the character value of $\mathbf{r}_{\mathscr{J}}(\tau(S,\Pi) - \tau(T,\Pi)) = \mathbf{r}_{\mathscr{J}} \circ r(\Pi,\Pi)$ on the conjugacy class $\mathrm{E_G}(\alpha\beta, \alpha\beta^q)$ stably conjugate to $(\mathrm{diag}(\alpha\beta, \alpha^q\beta^q), \mathrm{diag}(\alpha\beta^q, \alpha^q\beta))$ is given by

$$-2(-\Lambda(\alpha)-\Lambda(\alpha^q))(-\Lambda(\beta)-\Lambda(\beta^q)) = 2(-\Lambda(\alpha\beta)-\Lambda^q(\alpha\beta)) + 2(-\Lambda(\alpha\beta^q)-\Lambda^q(\alpha\beta^q))$$

for $\alpha,\beta\in\mathbb{F}_{q^2}^\times$ with $\alpha,\beta,\alpha\beta,\alpha\beta^q\notin\mathbb{F}_q^\times$. The correct choice is therefore given by $\mathbf{r}_{\mathscr{J}}(\tau(T,\Pi)) = [\widetilde{\pi},\,\mathrm{St}] + [\,\mathrm{St},\widetilde{\pi}]$ and $\mathbf{r}_{\mathscr{J}}(\tau(S,\Pi)) = [\widetilde{\pi},\,\mathbf{1}] + [\,\mathbf{1},\widetilde{\pi}]$.

For representations of type IX, there is Thm. 3.25 in Subsection 3.3.2. □

### 3.3.1. Klingen and Siegel induced representations

As an analogue of Thm. 2.19, the following theorem describes the paramodular restriction of Klingen parabolically induced representations of $\mathrm{GSp}(4,F)$.

**Lemma 3.22.** *For admissible representations $(\sigma, V_\sigma)$ of $\mathrm{GL}(2,F)$ and $(\mu, V_\mu)$ of $F^\times$, the paramodular restriction of the Klingen induced representation $\mu\rtimes\sigma$ is*

$$\mathbf{r}_{\mathscr{J}}(\mu\rtimes\sigma) \cong [\widetilde{\mu}\times 1, \widetilde{\sigma}] \oplus [\widetilde{\sigma}, \widetilde{\mu}\times 1] \tag{3.11}$$

*for the hyperspecial parahoric restrictions $\widetilde{\mu} = \mathbf{r}_{\mathfrak{o}^\times}(\mu)$ and $\widetilde{\sigma} = \mathbf{r}_{\mathrm{GL}(2,\mathfrak{o})}(\sigma)$.*

The proof is similar to that of Thm. 2.19.

*Proof.* An explicit model of $\mathbf{r}_{\mathscr{J}}(\mu\rtimes\sigma)$ is given by the right-action of $\mathscr{J}$ on

$$\widetilde{V} = \{f : G\to V_\mu\otimes V_\sigma \,|\, f(pgk) = \delta_Q^{1/2}(p)(\mu\boxtimes\sigma)(p)f(g)\ \forall p\in Q, g\in G, k\in\mathscr{J}^+\}.$$

By (3.13), any $f\in\widetilde{V}$ is uniquely determined by its restriction to $\mathscr{J}$ and $s_1\mathscr{J}$, so the $\mathscr{J}$-representation $\widetilde{V}$ is isomorphic to the direct sum

$$\{f|_{\mathscr{J}} : \mathscr{J}\to V_\mu\otimes V_\sigma \,|\, f\in\widetilde{V}\} \oplus \{f|_{s_1\mathscr{J}} : s_1\mathscr{J}\to V_\mu\otimes V_\sigma \,|\, f\in\widetilde{V}\}.$$

We claim that the first summand factors over $\mathscr{J}/\mathscr{J}^+ \cong (\mathrm{GL}(2,q)^2)^0$. Indeed, every element of it is right invariant under $\mathscr{J}^+$ and factors over a unique function

$$\widetilde{f}: \mathscr{J}/\mathscr{J}^+ \to V_\mu^{1+\mathfrak{p}} \otimes V_\sigma^{\mathrm{GL}(2,\mathfrak{o})^+} \qquad \text{with} \qquad \widetilde{f}(qg) = (\widetilde{\mu}(q_{11}) \otimes \widetilde{\sigma}\left(\begin{smallmatrix} q_{22} & q_{24} \\ q_{42} & q_{44} \end{smallmatrix}\right))\widetilde{f}(g)$$

for every $g \in \mathscr{J}/\mathscr{J}^+ \cong (\mathrm{GL}(2,q)^2)^0$ and every

$$q \in (\mathscr{J} \cap Q)\mathscr{J}^+/\mathscr{J}^+ \cong (\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \times \left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right))^0 \subseteq (\mathrm{GL}(2,q)^2)^0.$$

By definition, the action of $(\mathrm{GL}(2,q)^2)^0$ on the space of these $\widetilde{f}$ is the induced representation $[\widetilde{\mu} \times 1, \widetilde{\sigma}]$.

For the second summand the argument is analogous after conjugation with $s_1$, it yields the representation $[\widetilde{\sigma}, \widetilde{\mu} \times 1]$. $\square$

**Lemma 3.23.** *Let $\mu$ and $\mu_1$ be a smooth characters of $F^\times$ and let $\sigma$ be an irreducible admissible representation of $\mathrm{GL}(2,F)$. The paramodular restriction of the Siegel induced representation $\sigma \rtimes \mu$ is*

$$\mathbf{r}_{\mathscr{J}}(\sigma \rtimes \mu) \cong \widetilde{\mu}[1 \times \widetilde{\mu_1}, 1 \times \widetilde{\mu_1}] \qquad \textit{for } \sigma = \mu_1 \mathrm{St}, \mu_1 \mathbf{1},$$

*and it is $\mathbf{r}_{\mathscr{J}}(\sigma \rtimes \mu) \cong 0$ for cuspidal $\sigma$.*

*Proof.* An explicit model $\widetilde{V}$ of $\mathbf{r}_{\mathscr{J}}(\mu_1 \mathrm{St} \rtimes \mu)$ is given by right-multiplication with elements of $\mathscr{J}$ on the space of smooth functions $f: G \to V_{\mu_1 \mathrm{St}}$ with

$$f(pgk) = \delta_P^{1/2}(p)\mu(\mathrm{sim}(p))(\mu_1 \mathrm{St})\left(\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{smallmatrix}\right) f(g)$$

for $p \in P$, $g \in G$ and $k \in \mathscr{J}^+$. By the decomposition $G = P\mathscr{J}$ of Prop. 3.31 every such $f$ is uniquely determined by its restriction to $\mathscr{J}$. Therefore $\widetilde{V}$ is isomorphic to the vector space of $\mathscr{J}^+$ invariant functions

$$\widetilde{f}: \mathscr{J} \to V_{\mu_1 \mathrm{St}}$$

which satisfy for every $g \in \mathscr{J}$ and every $p \in P \cap \mathscr{J}$ the condition

$$\widetilde{f}(pg) = \mu(\mathrm{sim}(p)) \cdot (\mu_1 \mathrm{St})\left(\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{smallmatrix}\right) \widetilde{f}(g).$$

Especially, $\widetilde{f}(g)$ is invariant under every $p \in P \cap \mathscr{J}^+$ with the property

$$\left(\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \mod \mathfrak{p} \qquad \text{and} \qquad \mathrm{sim}(p) \in 1 + \mathfrak{p},$$

so $\widetilde{f}(g)$ must be contained in the parahoric restriction of $\mu_1 \mathrm{St}$ with respect to the standard Iwahori subgroup[2] of $\mathrm{GL}(2,F)$. The above condition is equivalent to

$$\widetilde{f}(pg) = \widetilde{\mu}(\mathrm{sim}(p)) \cdot \widetilde{\mu_1}(p_{11})\, \widetilde{\mu_1}(p_{22})\widetilde{f}(g).$$

for $g \in \mathscr{J}/\mathscr{J}^+$ and $p \in P \cap \mathscr{J}/P \cap \mathscr{J}^+$. By construction of the isomorphism to $(\mathrm{GL}(2,q)^2)^0$, this is the induced representation $\widetilde{\mu}[1 \times \widetilde{\mu_1}, 1 \times \widetilde{\mu_1}]$.

The proof for the other cases is analogous. $\square$

[2]That means $\widetilde{f}(g)$ is in the Jacquet module $\widetilde{\mu_1} \boxtimes \widetilde{\mu_1}$ of the hyperspecial parahoric restriction $\mathbf{r}_{\mathrm{GL}(2,\mathfrak{o})}(\mu_1 \mathrm{St}) = \widetilde{\mu_1} \mathrm{St}_{\mathrm{GL}(2,q)}$.

### 3.3.2. Type IX

Let $\xi = \xi_u, \xi_t$ be an unramified or tamely ramified non-trivial quadratic character of $F^\times$ and $\Pi$ be a depth zero irreducible cuspidal admissible representation of $\mathrm{GL}(2,F)$ with hyperspecial parahoric restriction $\widetilde{\pi}$. We determine the paramodular restriction of the irreducible constituents in the representation $(\rho, V) = \xi \rtimes \Pi$ with $\xi\Pi \cong \Pi$ using the explicit model (3.1). The argument is analogous to the hyperspecial case in the previous section with minor modifications, so the proof will only be sketched.

For every $\mathscr{J}^+$-invariant $f \in V$ with support in $Q\mathscr{J}$, we denote by $\widetilde{f}$ its restriction to $\mathscr{J}$. By the proof of Lemmma 3.22, these $\widetilde{f}$ generate the representation $\widetilde{V} = [\widetilde{\xi} \times 1, \widetilde{\pi}]$. We consider the $\mathscr{J}$-intertwining operator

$$T : \widetilde{V} \to \widetilde{V}, \quad Tf(g) = \sum_{u \in (\mathscr{J} \cap U_2)\mathscr{J}^+/\mathscr{J}^+} f(s_0 u g) = \sum_{c \in \mathfrak{o}/\mathfrak{p}} f(s_0 u(c) g)$$

for $u(c) = \begin{pmatrix} 1 & & c\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $s_0 = \mathrm{diag}(\varpi^{-1}, 1, \varpi, 1) s_1 s_2 s_1 \in \mathscr{J}$.

The space $W_1 \subseteq V^{\mathscr{Q}^+} \subseteq V^{\mathscr{J}^+}$ is the same as in the hyperspecial case and it is generated by the functions

$$f_w(x) = \begin{cases} \psi(u_{24})(\nu\xi \boxtimes \nu^{-1/2}\Pi)(p)v_0 & \text{for } x = pwu \in Qw\mathscr{B}^+ \\ 0 & \text{else.} \end{cases}$$

for $w \in \{1, s_1 s_2 s_1\}$. Both have support in $Q\mathscr{J}$, so the space $W_1$ belongs to $\widetilde{V}$. The space

$$W_2 = \{f \in V \mid f(xu) = f(x)\psi(-u_{31}\varpi^{-1} + u_{24}) \forall u \in \mathscr{B}^+\} \subseteq \widetilde{V} \subseteq V^{\mathscr{J}^+}$$

is $(\mathrm{GL}(2,q)^2)^0$-conjugate to the one-dimensional space of standard Whittaker vectors in in the paramodular restriction $\widetilde{V}$. It is generated by

$$f_{\mathrm{Wh}} = \begin{cases} \nu\xi \boxtimes \nu\Pi(-\varpi^{-1}u_{31} + u_{24})v_0 & x = qu \in Q\mathscr{B}^+, \\ 0 & \text{else,} \end{cases}$$

by the same argument as in the hyperspecial case.

**Lemma 3.24.** *For unramified $\xi = \xi_u$, eigenvectors of $T$ in $W_1$ are given by*

$$f_a = q^2 f_1 + f_{s_1 s_2 s_1} \qquad Tf_a = (-1) \cdot f_a$$
$$f_b = q^3 f_1 - f_{s_1 s_2 s_1} \qquad Tf_b = q \cdot f_b.$$

*and for tamely ramified $\xi = \xi_t$ by*

$$f_\pm = q^3 f_1 \pm \xi(-1)\mathfrak{G} f_{s_1 s_2 s_1} \qquad Tf_\pm = \pm\, \xi(-\varpi)\mathfrak{G} T f_\pm.$$

*The operator $T$ acts on $W_2$ by $Tf_{\mathrm{Wh}} = -1 \cdot f_{\mathrm{Wh}}$ for unramified $\xi = \xi_u$ and $Tf_{\mathrm{Wh}} = \mathfrak{G} \cdot f_{\mathrm{Wh}}$ for tamely ramified $\xi = \xi_t$.*

*Proof.* For $\widetilde{f} \in W_1$ we have

$$T\widetilde{f}(1) = \sum_{c \in \mathfrak{o}/\mathfrak{p}} \widetilde{f}(s_0 u(c)) = \widetilde{f}(s_0) + \sum_{c \in (\mathfrak{o}/\mathfrak{p})^\times} \widetilde{f}(\underbrace{\begin{pmatrix} -c^{-1} & & \varpi^{-1} & \\ & 1 & & \\ & & -c & \\ & & & 1 \end{pmatrix}}_{\in Q} \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ \varpi c^{-1} & & 1 & \\ & & & 1 \end{pmatrix}}_{\in \mathscr{P}_\emptyset^+})$$

$$= \widetilde{f}(s_0) + \sum_{c \in (\mathfrak{o}/\mathfrak{p})^\times} |-c^{-1}|^3 \xi(-c^{-1}) \widetilde{f}(1)$$

$$= \xi(\varpi) q^3 \widetilde{f}(s_1 s_2 s_1) + \begin{cases} (q-1)\widetilde{f}(1) & \xi = \xi_u, \\ 0 & \xi = \xi_t. \end{cases}$$

and

$$T\widetilde{f}(s_1 s_2 s_1) = q^{-3} \xi(\varpi) T\widetilde{f}(s_0) = q^{-3}\xi(\varpi) \sum_{c \in \mathfrak{o}/\mathfrak{p}} \widetilde{f}(s_0 u(c) s_0)$$

$$= q^{-3}\xi(\varpi) \sum_{c \in \mathfrak{o}/\mathfrak{p}} \widetilde{f}(\underbrace{\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}}_{\in Q} \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ -\varpi c & & 1 & \\ & & & 1 \end{pmatrix}}_{\in \mathscr{P}_\emptyset^+}) = q^{-2}\xi(-\varpi)\widetilde{f}(1).$$

The action of $T$ on $W_1$ is therefore given by

$$Tf_1 = q^{-2}\xi(-\varpi) f_{s_1 s_2 s_1} + \begin{cases} (q-1) f_1 & \xi = \xi_u, \\ 0 & \xi = \xi_t, \end{cases} \quad \text{and} \quad Tf_{s_1 s_2 s_1} = q^3 \xi(\varpi) f_1.$$

The calculation for $T\widetilde{f}_{\mathrm{Wh}}(1)$ is analogous to $T\widetilde{f}(1)$. □

Therefore the eigenvectors $f_a$ and $f_{\xi(-\varpi)}$ belong to generic constituents of $\widetilde{V}$.

The Hecke operator $H$ is the same as in the hyperspecial case. For unramified $\xi$ we have $Hf_1 = -qf_1 + (q + q^{-1}) f_{s_1 s_2 s_1}$ and $Hf_{s_1 s_2 s_1} = -q^3 f_{s_1 s_2 s_1}$; the other eigenvector is $f_a = q^2 f_1 + f_{s_1 s_2 s_1}$ with eigenvalue $-q$. For tamely ramified $\xi$ it is given by $Hf_1 = q\xi(\varpi) f_1 + \xi(\varpi)(1 - q^{-2})\mathfrak{G} f_{s_1 s_2 s_1}$ and $Hf_{s_1 s_2 s_1} = \xi(\varpi) q^3 f_{s_1 s_2 s_1}$; the other eigenvector is $q^3 f_1 - \mathfrak{G} f_{s_1 s_2 s_1}$ with eigenvalue $\xi(\varpi) q$.

**Theorem 3.25.** *Fix an at most tamely ramified quadratic character $\xi = \xi_u, \xi_t$ of $F^\times$ and a depth zero irreducible cuspidal admissible representation $\Pi$ of $\mathrm{GL}(2, F)$ with $\xi\Pi \cong \Pi$.*

*The paramodular restriction of the subrepresentation $\delta(\nu\xi, \nu^{-1/2}\Pi)$ of type IXa is*

$$\mathbf{r}_{\mathscr{J}}(\delta(\nu\xi, \nu^{-1/2}\Pi)) = \begin{cases} [\mathrm{St}, \widetilde{\pi}] + [\widetilde{\pi}, \mathrm{St}] & \text{if } \xi = \xi_u, \\ [1 \times \lambda_0, \widetilde{\pi}]_\mp + [\widetilde{\pi}, 1 \times \lambda_0]_\mp & \text{if } \xi = \xi_t \text{ and } \xi(\varpi) = \pm 1. \end{cases}$$

*The paramodular restriction of the quotient $L(\nu\xi, \nu^{-1/2}\Pi)$ of type IXb is*

$$\mathbf{r}_{\mathscr{J}}(L(\nu\xi, \nu^{-1/2}\Pi) = \begin{cases} [\mathbf{1}, \widetilde{\pi}] + [\widetilde{\pi}, \mathbf{1}] & \text{if } \xi = \xi_u, \\ [1 \times \lambda_0, \widetilde{\pi}]_\pm + [\widetilde{\pi}, 1 \times \lambda_0]_\pm & \text{if } \xi = \xi_t \text{ and } \xi(\varpi) = \pm 1. \end{cases}$$

*Proof.* For the subrepresentations of $\widetilde{V} = [\widetilde{\xi} \times 1, \widetilde{\pi}]$ the argument is completely analogous to the hyperspecial case. The rest is implied by Lemma 3.5. □

### 3.3.3. Type V

We determine the paramodular restriction for non-cuspidal irreducible admissible representations of type Vc and Vd. By definition, these are the irreducible constituents of the Siegel induced representation $(\rho, V) = \nu^{1/2}\xi\,\mathbf{1} \boxtimes \nu^{-1/2}\mu$, which is given by right multiplication on the function space

$$V = \{f : G \to \mathbb{C} \mid f(pg) = \delta_P^{1/2}(p)(\nu^{1/2}\xi\,\mathbf{1} \boxtimes \nu^{-1/2}\mu)(p)f(g) \forall p \in P\}.$$

The modulus character of the Siegel parabolic is $\delta_P(p) = |\det\left(\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22}\end{smallmatrix}\right)|^3 \cdot |\operatorname{sim}(p)|^{-3}$. By the double coset decomposition (3.13), every $f \in V$ is uniquely determined by its restriction to $\mathscr{J}$. The vector subspaces

$$W_1 = \{f \in \widetilde{V} \mid f(gu) = f(g)\psi(u_{24})\ \forall u \in \mathscr{B}\},$$
$$W_2 = \{f \in \widetilde{V} \mid f(gu) = f(g)\psi(-\varpi^{-1}u_{31} + u_{24})\ \forall u \in \mathscr{B}\},$$

are contained in the space of $\mathscr{J}^+$-invariants in $V$. For $x \in G$ and $w \in \{s_0s_2, s_2\}$ let

$$f_w(x) = \begin{cases} \delta_P^{1/2}(p)(\nu^{1/2}\xi\,\mathbf{1} \rtimes \nu^{-1/2}\mu)(p)\psi(u_{31}\varpi^{-1}) & x = pwu \in Pw\mathscr{B}^+, \\ 0 & \text{else,} \end{cases}$$

and let

$$f_{\mathrm{Wh}}(x) = \begin{cases} \delta_P^{1/2}(p)(\nu^{1/2}\xi\,\mathbf{1} \rtimes \nu^{-1/2}\mu)(p)\psi(-u_{31}\varpi^{-1} + u_{24}) & x = ps_2u \in Ps_2\mathscr{B}^+, \\ 0 & \text{else.} \end{cases}$$

This definition does not depend on the choice of double coset decomposition of $x$.

**Lemma 3.26.** $W_1 = \mathbb{C}f_{s_0s_2} \oplus \mathbb{C}f_{s_2}$ *and* $W_2 = \mathbb{C}f_{\mathrm{Wh}}$.

*Proof.* Eq. (3.13) implies the disjoint double coset decomposition

$$G = \bigsqcup_w Pw\mathscr{B}^+ \qquad w \in \{1, s_2, s_0, s_0s_2\}, \tag{3.12}$$

so any $f$ in $W_1$ and $W_2$ is uniquely determined by its values on $w \in \{1, s_2, s_0, s_0 s_2\}$. For every $c \in \mathfrak{o}$, the element $I_4 + cE_{24} \in \mathrm{GSp}(4, F)$ is in $\mathscr{B}^+$ and in the unipotent radical of $P$. For every $f \in W_1$ we have

$$f(1)\psi(c) = f(I_4 + cE_{24}) = f(1),$$
$$f(s_0)\psi(c) = f(s_0(I_4 + cE_{24})) = f((I_4 + cE_{24})s_0) = f(s_0),$$

so $f(1) = f(s_0) = 0$. Therefore $W_1$ is at most two-dimensional. Since the supports of $f_{s_0 s_2}$ and $f_{s_2}$ are disjoint, the are linearly independent and generate $W_1$. The space $W_2$ is $(\mathrm{GL}(2,q)^2)^0$-conjugate to the one-dimensional space of Whittaker vectors in $\widetilde{\mu}[1 \times \widetilde{\xi}, 1 \times \widetilde{\xi}]$ and it is generated by $f_{\mathrm{Wh}}$. □

**Lemma 3.27.** *The $\mathscr{J}$-intertwining operator $T : \widetilde{V} \to \widetilde{V}$*

$$T\widetilde{f}(g) = \sum_{u \in (P \cap \mathscr{J})\mathscr{J}^+/\mathscr{J}^+} \widetilde{f}(s_0 s_2 u g) = \sum_{a,b \in \mathfrak{o}/\mathfrak{p}} \widetilde{f}(s_0 s_2 \begin{pmatrix} 1 & & b\varpi^{-1} & \\ & 1 & & a \\ & & 1 & \\ & & & 1 \end{pmatrix} g)$$

*for $g \in \mathscr{J}$ preserves $W_1$ and $W_2$.*

*Proof.* That $T : \widetilde{V} \to \widetilde{V}$ is well-defined is shown in the same way in Lemma 3.11. It is $\mathscr{J}$-intertwining and therefore preserves $W_1$ and $W_2$. □

**Lemma 3.28.** *For unramified $\xi = \xi_u$ with $\widetilde{\xi} = 1$, $T$ is given on $W_1$ and $W_2$ by*

$$Tf_{s_2} = -(q-1)f_{s_2} - qf_{s_0 s_2}, \qquad Tf_{s_0 s_2} = -f_{s_2}, \qquad Tf_{\mathrm{Wh}} = f_{\mathrm{Wh}},$$

*with eigenvalue equations*

$$T(f_a) = f_a, \qquad f_a = f_{s_2} - qf_{s_0 s_2},$$
$$T(f_b) = -qf_b, \qquad f_b = f_{s_2} + f_{s_0 s_2}.$$

*For tamely ramified $\xi = \xi_t$ we have*

$$Tf_{s_2} = q\widetilde{\xi}(-1)\mathfrak{G} f_{s_0 s_2}, \qquad Tf_{s_0 s_2} = \mathfrak{G} f_{s_0}, \qquad Tf_{\mathrm{Wh}} = \mathfrak{G}^2 f_{\mathrm{Wh}},$$

*with eigenvalue equations $T(f_\pm) = \pm\mathfrak{G}^2 f_\pm$ for $f_\pm = f_{s_2} \pm \mathfrak{G} f_{s_0 s_2}$.*

*Proof.* For $f \in W_1$ the value of $Tf(s_2)$ is the following sum over $a, b \in \mathfrak{o}/\mathfrak{p}$.

$$Tf(s_2) = \sum_{a,b \in \mathfrak{o}/\mathfrak{p}} f(\begin{pmatrix} & & \varpi^{-1} & \\ & -1 & & \\ -\varpi & & -b & \\ & a & & -1 \end{pmatrix})$$

$$= \sum_{\substack{a \neq 0 \\ b \neq 0}} f(\begin{pmatrix} -b^{-1} & & \varpi^{-1} & \\ & -a^{-1} & & 1 \\ & & -b & \\ & & & -a \end{pmatrix} s_2 \begin{pmatrix} 1 & & & \\ & 1 & & -a^{-1} \\ \varpi b^{-1} & & 1 & \\ & & & 1 \end{pmatrix})$$

$$+ \sum_{\substack{a \neq 0 \\ b = 0}} f(\begin{pmatrix} 1 & & & \\ & -a^{-1} & & 1 \\ & & 1 & \\ & & & -a \end{pmatrix} s_0 s_2 \begin{pmatrix} 1 & & & \\ & 1 & & -a^{-1} \\ & & 1 & \\ & & & 1 \end{pmatrix})$$

$$= \sum_{\substack{a \neq 0 \\ b \neq 0}} \widetilde{\xi}(ab)\psi(-a^{-1}) f(s_2) + \sum_{\substack{a \neq 0 \\ b = 0}} \widetilde{\xi}(-a^{-1})\psi(-a^{-1}) f(s_0 s_2)$$

$$= \begin{cases} -(q-1)f(s_2) - f(s_0 s_2) & \xi = \xi_u, \\ \mathfrak{G} f(s_0 s_2) & \xi = \xi_t. \end{cases}$$

The terms for $a = 0$ vanish because $f(1) = f(s_0) = 0$.

$$Tf(s_0 s_2) = \sum_{a,b} f\begin{pmatrix} -1 & & & \\ & -1 & & \\ \varpi b & & -1 & \\ & a & & -1 \end{pmatrix}$$

$$= \sum_{\substack{a \neq 0 \\ b}} f(\begin{pmatrix} -1 & & & \\ & -a^{-1} & & 1 \\ & & -1 & \\ & & & -a \end{pmatrix} s_2 \begin{pmatrix} 1 & & & \\ & 1 & & -a^{-1} \\ -\varpi b & & 1 & \\ & & & 1 \end{pmatrix}) + \sum_b f(-I_4 \begin{pmatrix} 1 & & & \\ & 1 & & \\ -\varpi b & & 1 & \\ & & & 1 \end{pmatrix})$$

$$= \sum_{\substack{a \neq 0 \\ b}} \widetilde{\xi}((-1)(-a^{-1})) f(s_2) \psi(-a^{-1}) + \sum_b \xi((-1)^2) f(1) = \begin{cases} -q f(s_2) & \xi = \xi_u, \\ q\widetilde{\xi}(-1)\mathfrak{G} f(s_2) & \xi = \xi_t. \end{cases}$$

The calculation for $f_{\mathrm{Wh}}$ is analogous. The eigenvalue equations are clear. □

**Lemma 3.29.** *The Hecke operator $H : V \to V$ from Lemma 3.12 is given for $f \in V^{\mathscr{Q}^+}$ by the finite sum (3.4). It preserves $W_1$ with*

$$Hf_{s_2} = \xi(\varpi)q^2 f_{s_2} + \begin{cases} -(q^4 - q^3) f_{s_0 s_2} & \xi = \xi_u, \\ \mathfrak{G}(q^3 - q^2) f_{s_0 s_2} & \xi = \xi_t, \end{cases} \quad \text{and} \quad Hf_{s_0 s_2} = \xi(\varpi) q^3 f_{s_0 s_2}.$$

*and satisfies the eigenvalue equations $Hf_a = -q^2 \cdot f_a$ for unramified $\xi = \xi_u$ and $Hf_{-\xi(\varpi)} = \xi(\varpi)q^2 \cdot f_{-\xi(\varpi)}$ for tamely ramified $\xi = \xi_t$ with the $T$-eigenvectors from Lemma 3.28.*

*Proof.* That $H$ preserves $W_1$ and that (3.4) holds is shown the same way as in

Lemma 3.12. For every $f \in W_1$ we have

$$
\begin{aligned}
Hf(s_2) &= \sum_{\substack{a,b \in \mathfrak{o}/\mathfrak{p} \\ c \in \mathfrak{o}/\mathfrak{p}^2}} f(s_2 \begin{pmatrix} \varpi & a & c\varpi^{-1} & b \\ & 1 & b\varpi^{-1} & \\ & & \varpi^{-1} & \\ & & -a\varpi^{-1} & 1 \end{pmatrix}) = \sum_{\substack{a,b \in \mathfrak{o}/\mathfrak{p} \\ c \in \mathfrak{o}/\mathfrak{p}^2}} f(\begin{pmatrix} \varpi & b & c\varpi^{-1} & -a \\ & 1 & -a\varpi^{-1} & \\ & & \varpi^{-1} & \\ & & -b\varpi^{-1} & 1 \end{pmatrix} s_2) \\
&= \sum_{\substack{a,b \in \mathfrak{o}/\mathfrak{p} \\ c \in \mathfrak{o}/\mathfrak{p}^2}} (\nu^{3/2}\nu^{1/2}\xi)(\det \left( \begin{smallmatrix} \varpi & b \\ & 1 \end{smallmatrix} \right)) f(s_2) = \xi(\varpi) q^2 f(s_2)
\end{aligned}
$$

and $Hf(s_0 s_2)$ is given by Table 3.6. This determines $Hf$ uniquely. The eigenvalue equations are clear. □

**Theorem 3.30.** *Let $\xi$ and $\mu$ be at most tamely ramified characters of $F^\times$, where $\xi$ is non-trivial quadratic. For unramified $\xi = \xi_u$, the paramodular restriction of the irreducible admissible representation $\rho_{\mathrm{sub}} = L(\nu^{1/2}\xi\,\mathrm{St}, \nu^{-1/2}\xi\mu)$ of type Vc admits a generic subquotient. For tamely ramified $\xi = \xi_t$ the parahoric restriction of $\rho_{\mathrm{sub}}$ is $\widetilde{\mu}[1 \times \lambda_0, 1 \times \lambda_0]_\mp$ for $\xi(\varpi) = \pm 1$.*

*Proof.* The representation $\rho_{\mathrm{sub}}$ is the unique subrepresentation of the Siegel induced representation $\nu^{1/2}\xi\,\mathbf{1} \rtimes \nu^{-1/2}\mu$ [ST94, 3.6]. For unramified $\xi$, both $W_1$ and $W_2$ are contained in the subrepresentation

$$\widetilde{\mu}[1 \times 1, \mathrm{St}] \subseteq \widetilde{\mu}[1 \times 1, 1 \times 1] = \mathbf{r}_{\mathscr{J}}(\nu^{1/2}\xi\,\mathbf{1} \rtimes \nu^{-1/2}\mu),$$

of $(\mathrm{GL}(2,q)^2)^0$ because the action of the subgroup $\{(I_2, \left( \begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix} \right))\} \subseteq (\mathrm{GL}(2,q)^2)^0$ is non-trivial. This subrepresentation $\widetilde{\mu}[1 \times 1, \mathrm{St}]$ has two irreducible constituents and by Schur's lemma applied to $T$ the subspace $W_1$ has non-zero intersection with both of these constituents. Any $v \in W_1 \cap \mathbf{r}_{\mathscr{J}}(\rho_{\mathrm{sub}})$ must be an eigenvector for both $H$ and $T$. The $T$-eigenvalue of $v$ coincides with the $T$-eigenvalue for $f_{\mathrm{Wh}}$ if and only if $\mathbf{r}_{\mathscr{J}}(\rho_{\mathrm{sub}})$ is generic.

For tamely ramified $\xi$, the representation $\mathbf{r}_{\mathscr{J}}(\nu^{1/2}\xi\,\mathbf{1} \rtimes \nu^{-1/2}\mu) = \widetilde{\mu}[1 \times \lambda_0, 1 \times \lambda_0]$ has one generic constituent $\widetilde{\mu}[1 \times \lambda_0, 1 \times \lambda_0]_+$ and one non-generic constituent $\widetilde{\mu}[1 \times \lambda_0, 1 \times \lambda_0]_-$ by Lemma A.6. The same argument works here as well. □

## 3.4. Double-coset decompositions

Let $\mathscr{K}$ be the standard hyperspecial and $\mathscr{J}$ be the standard paramodular subgroup of $G = \mathrm{GSp}(4,F)$.

**Proposition 3.31.** *There are disjoint double coset decompositions*

$$\mathrm{GSp}(4,F) = B\mathscr{K} = \bigsqcup_{w \in W_G} Bw\mathscr{B}^+ = Q\mathscr{J} \sqcup Qs_1\mathscr{J} = P\mathscr{J} \tag{3.13}$$

*for the standard parabolics $B$, $P$, $Q$ and the Weyl group $W_G$.*

$$
\begin{aligned}
Hf(s_0 s_2) &= \sum_{\substack{a,b\in\mathfrak{o}/\mathfrak{p}\\ c\in\mathfrak{o}/\mathfrak{p}^2}} f\begin{pmatrix} & & \varpi^{-2} & \\ & & -a\varpi^{-1} & 1 \\ -\varpi^2 & -a\varpi & -c & -b\varpi \\ & -1 & -b\varpi^{-1} & \end{pmatrix} \\
&= \sum_{\substack{a\in\mathfrak{o}/\mathfrak{p}\\ b\in(\mathfrak{o}-\mathfrak{p})/\mathfrak{p}}} \sum_{y\in(\mathfrak{o}-\mathfrak{p})/\mathfrak{p}^2} (\nu^2\xi)(y^{-1}) f(s_2)\psi(b^2y^{-1}) \\
&\quad + \sum_{\substack{a\in\mathfrak{o}/\mathfrak{p}\\ b\in\mathfrak{p}/\mathfrak{p}}} \sum_{y\in\mathfrak{p}^2/\mathfrak{p}^2} (\nu^2\xi)(\varpi^{-1}) f(s_0 s_2) \\
&\quad + \sum_{\substack{a\in\mathfrak{o}/\mathfrak{p}\\ b\in\mathfrak{p}/\mathfrak{p}}} \sum_{y\in(\mathfrak{o}-\mathfrak{p}^2)/\mathfrak{p}^2} (\nu^2\xi)(y^{-1}) f(s_2) \underbrace{\psi(b^2y^{-1})}_{=1} \\
&\quad + \sum_{\substack{a\in\mathfrak{o}/\mathfrak{p}\\ b\in(\mathfrak{o}-\mathfrak{p})/\mathfrak{p}}} \sum_{y\in\mathfrak{p}/\mathfrak{p}^2} (\nu^2\xi)(-b^{-2}) \underbrace{f(1)}_{=0} \qquad \text{by Lemma 3.33} \\
&= q^2 \sum_{b\in(\mathfrak{o}-\mathfrak{p})/\mathfrak{p}} \underbrace{\sum_{y\in(\mathfrak{o}-\mathfrak{p})/\mathfrak{p}} \xi(y^{-1})\psi(b^2y^{-1})}_{=-1 \text{ if } \widetilde{\xi}=1,\ =\mathfrak{G} \text{ if } \widetilde{\xi}=\lambda_0} f(s_2) \\
&\quad + q^3\xi(\varpi^{-1}) f(s_0s_2) + q \sum_{y\in(\mathfrak{o}-\mathfrak{p}^2)/\mathfrak{p}^2} (\nu^2\xi)(y^{-1}) f(s_2) \\
&= \begin{cases} -(q^3-q^2)\cdot f(s_2) - q^3 f(s_0s_2) & \widetilde{\xi}=1, \\ (q^3-q^2)\cdot \mathfrak{G} f(s_2) + \xi(\varpi) q^3 f(s_0s_2) & \widetilde{\xi}=\lambda_0 \end{cases} \\
&\quad + q^2 \sum_{y\in(\mathfrak{o}-\mathfrak{p})/\mathfrak{p}} \xi(y) f(s_2) + q^3 \sum_{y\in(\mathfrak{p}-\mathfrak{p}^2)/\mathfrak{p}^2} \xi(y) f(s_2) \\
&= \begin{cases} -(q^4-q^3) f(s_2) - q^3 f(s_0s_2) & \widetilde{\xi}=1 \\ (q^3-q^2)\mathfrak{G} f(s_2) + \xi(\varpi) q^3 f(s_0s_2) & \widetilde{\xi}=\lambda_0. \end{cases}
\end{aligned}
$$

Table 3.6.: The calculation of $Hf(s_0s_2)$ for the proof of Lemma 3.29.

*Proof.* Iwasawa decomposition and Bruhat decomposition imply

$$G = B\mathscr{K} = B(\bigsqcup_{w \in W_G} \mathscr{B} w \mathscr{B}^+) = \bigsqcup_{w \in W_G} B\mathscr{K}^+ w \mathscr{B}^+ = \bigsqcup_{w \in W_G} Bw\mathscr{B}^+. \tag{3.14}$$

The last equality follows because $w^{-1}\mathscr{K}^+ w = \mathscr{K}^+ \subseteq \mathscr{B}^+$. It is disjoint because Iwasawa decomposition is unique up to elements in $B \cap \mathscr{K} \subseteq \mathscr{B}$.

The element $s_2$ is contained in the paramodular group $\mathscr{J}$ and in the Klingen parabolic $Q$. Furthermore, $s_1 s_2 s_1 = \mathrm{diag}(\varpi, 1, \varpi^{-1}, 1)s_0 \in Q\mathscr{J}$, so $G = Q\mathscr{J} \cup Qs_1\mathscr{J}$. For any $k \in \mathscr{J}$ at least one of the matrix entries $k_{31}$ and $k_{33}$ is non-zero, since $\det(k) \in \mathfrak{o}^\times$. Then $(pk)_{31} \neq 0$ or $(pk)_{33} \neq 0$ for $p \in Q$, so $s_1 \notin Q\mathscr{P}_{\{02\}}$ and the decomposition is disjoint. For the Siegel parabolic $P$ the proof is analogous. $\square$

In the proof of Lemma 3.15 we need the following matrix decompositions:

**Lemma 3.32.** *Let $a, b, c \in \mathfrak{o}$ be arbitrary. The disjoint decomposition* (3.3) *of*

$$\alpha := \begin{pmatrix} & & \varpi^{-1} & \\ & 1 & b\varpi^{-1} & \\ -\varpi & -a & -c\varpi^{-1} & -b \\ & & -a\varpi^{-1} & 1 \end{pmatrix}$$

*takes the following explicit form*

*i) For $c \in \mathfrak{o} - \mathfrak{p}^2$ and $a, b \in c \cdot \mathfrak{o}$ we have $\alpha \in\in Q\mathscr{B}^+$ with*

$$\alpha = \begin{pmatrix} -\varpi c^{-1} & -c^{-1}a & \varpi^{-1} & -c^{-1}b \\ & 1 - ab/c & b\varpi^{-1} & -b^2/c \\ & & -c\varpi^{-1} & \\ & a^2/c & -a\varpi^{-1} & 1 + ab/c \end{pmatrix} \begin{pmatrix} 1 & & & \\ -\varpi b/c & 1 & & \\ \varpi^2/c & \varpi a/c & 1 & \varpi b/c \\ \varpi a/c & & & 1 \end{pmatrix},$$

*ii) for $c \in \mathfrak{p}^2$ and $a, b \in \mathfrak{p}$ we have $\alpha \in Qs_1 s_2 s_1 \mathscr{B}^+$ with*

$$\alpha = \begin{pmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{pmatrix} s_1 s_2 s_1 \begin{pmatrix} 1 & a\varpi^{-1} & c\varpi^{-2} & b\varpi^{-1} \\ & 1 & \varpi^{-1}b & \\ & & 1 & \\ & & -\varpi^{-1}a & 1 \end{pmatrix}.$$

*iii) For $a, c \in \mathfrak{p}$ and $b \in \mathfrak{o}^\times$ we get $\alpha \in Qs_1\mathscr{B}^+$ with*

$$\alpha = \begin{pmatrix} -b^{-1} & & & \varpi^{-1} \\ & & & b\varpi^{-1} \\ & & -b & \\ & -\varpi b^{-1} & 1 & (\frac{c}{b} - a)\varpi^{-1} \end{pmatrix} s_1 \begin{pmatrix} 1 & c(b\varpi)^{-1} & & \\ & 1 & & \\ & \varpi b^{-1} & 1 & \\ \varpi b^{-1} & ab^{-1} & -c(b\varpi)^{-1} & 1 \end{pmatrix},$$

*iv) Finally, $c \in \mathfrak{p}$ and $a \in \mathfrak{o}^\times$ implies $\alpha \in Qs_1s_2\mathscr{B}^+$ with*

$$\alpha = \begin{pmatrix} -a^{-1} & & & \varpi^{-1} \\ & \varpi a^{-1} & -1 & (b-\frac{c}{a})\varpi^{-1} \\ & & -a & \\ & & & \varpi^{-1}a \end{pmatrix} s_1s_2 \begin{pmatrix} 1 & & & c(a\varpi)^{-1} \\ -\varpi a^{-1} & 1 & c(a\varpi)^{-1} & -a^{-1}b \\ & & 1 & \varpi a^{-1} \\ & & & 1 \end{pmatrix}.$$

*Proof.* This can be checked directly. □

**Lemma 3.33.** *For $a, b, c \in \mathfrak{o}$, the decomposition* (3.12) *of*

$$\alpha = \begin{pmatrix} & & \varpi^{-2} & \\ & & -a\varpi^{-1} & 1 \\ -\varpi^2 & -a\varpi & -c & -b\varpi \\ & -1 & -b\varpi^{-1} & \end{pmatrix} \in \mathrm{GSp}(4, F)$$

*admits the following form:*

1. *If $y = ab - c \in \mathfrak{o} - \mathfrak{p}^2$ and $y^{-1}b \in \mathfrak{o}$, then $\alpha \in Ps_2\mathscr{B}^+$ with*

$$\alpha = \begin{pmatrix} y^{-1} & by^{-1}\varpi^{-1} & \varpi^{-2} & \\ -ay^{-1}\varpi & -cy^{-1} & -a\varpi^{-1} & \\ & & -c & a\varpi \\ & & -b\varpi^{-1} & 1 \end{pmatrix} s_2 \begin{pmatrix} 1 & & & \\ by^{-1}\varpi & 1 & & b^2y^{-1} \\ -y^{-1}\varpi^2 & & 1 & -by^{-1}\varpi \\ & & & 1 \end{pmatrix}.$$

2. *If $y = ab - c \in \mathfrak{p}$ and $b \in \mathfrak{o}^\times$, then $\alpha \in P\mathscr{B}^+$ with*

$$\alpha = \begin{pmatrix} & -b^{-1}\varpi^{-1} & \varpi^{-2} & \\ -b^{-1}\varpi & cb^{-2} & -a\varpi^{-1} & 1 \\ & & -c & -b\varpi \\ & & -b\varpi^{-1} & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & & b^{-1}\varpi & 1 \\ b^{-1}\varpi & yb^{-2} & & 1 \end{pmatrix}.$$

3. *If $y = ab - c \in \mathfrak{p}^2$ and $b \in \mathfrak{p}$, then $\alpha \in Ps_0s_2\mathscr{B}^+$ with*

$$\alpha = \begin{pmatrix} \varpi^{-1} & & & \\ -a & 1 & & \\ & & \varpi & a\varpi \\ & & & 1 \end{pmatrix} s_0s_2 \begin{pmatrix} 1 & & -y\varpi^{-2} & b\varpi^{-1} \\ & 1 & b\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

4. *There are no $a, b, c \in \mathfrak{o}$ with $\alpha \in Ps_0\mathscr{B}^+$.*

*Proof.* The first three cases can be checked directly. These are all the $a, b, c \in \mathfrak{o}$. □

**Proposition 3.34.** *The group* $\mathrm{GSp}(4,F)$ *admits the following disjoint double coset decompositions with respect to the maximal parahorics*

$$\mathrm{GSp}(4,F) = \bigsqcup_{n_1 \le n_2 \le n_0 - n_2} \mathscr{K}\lambda^\vee(\varpi)\mathscr{K}, \tag{3.15}$$

$$\mathrm{GSp}(4,F) = \bigsqcup_{\substack{n_1 \le n_0 - n_1 \\ n_2 \le n_0 - n_2}} \mathscr{J}\lambda^\vee(\varpi)\mathscr{J} \sqcup \mathscr{J}u_1\lambda^\vee(\varpi)\mathscr{J} \tag{3.16}$$

*for cocharacters* $\lambda^\vee = n_0 f_0 + n_1 f_1 + n_2 f_2$ *with integer coefficients.*

Here $\lambda^\vee(\varpi) = \mathrm{diag}(\varpi^{n_1}, \varpi^{n_2}, \varpi^{n_0-n_1}, \varpi^{n_0-n_2})$ and $u_1$ is the Atkin-Lehner element.

*Proof.* Associating to $w$ in the normalizer of the standard torus the double coset $\mathscr{B}w\mathscr{B}$ furnishes a bijection between the affine Weyl group $\widetilde{W}$ and the double coset decomposition $\mathscr{B}\backslash G/\mathscr{B}$ [BT72, 7.3.4]. For every parahoric $\mathscr{P}_I$ generated by $\mathscr{B}$ and reflections $s_i$ with $i \in I$, this gives a bijection from $\mathscr{P}_I\backslash G/\mathscr{P}_I$ to $\widetilde{W}_I\backslash\widetilde{W}/\widetilde{W}_I$, where $\widetilde{W}_I$ is generated by the $s_i$ [Moy88, p.258]. For the hyperspecial parahoric $\mathscr{K}$, such representatives are $\lambda^\vee(\varpi)$ for the cocharacters $\lambda^\vee$ with $\langle\alpha_i, \lambda^\vee\rangle \ge 0$ for the simple affine roots $\alpha_1$, $\alpha_2$, cp. [Tit79, 3.3.3]. For the paramodular group $\mathscr{J}$, representatives are $\lambda^\vee(\varpi)$ and $u_1\lambda^\vee(\varpi)$ where $\langle\alpha_i, \lambda^\vee\rangle \ge 0$ for $\alpha_0$, $\alpha_2$. □

**Corollary 3.35.** *The normalizer of* $\mathscr{K}$ *is* $N_G(\mathscr{K}) = Z\mathscr{K}$ *and the normalizer of* $\mathscr{J}$ *is* $N_G(\mathscr{J}) = Z\mathscr{J} \cup u_1 Z\mathscr{J}$*.*

# 4. Depth-Zero Endoscopy for $\mathrm{GSp}(4)$

Fix a local non-archimedean number field $F$ with finite residue field $\mathbb{F}_q = \mathfrak{o}_F/\mathfrak{p}_F$ of order $q$, unramified closure $F^{\mathrm{un}}$, absolute Galois group $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$, and Weil group $W_F$. Let $\nu(x) = |x|$ for $x \in F$ be the valuation character normalized such that $\nu(\varpi) = q^{-1}$ for a fixed uniformizer $\varpi \in \mathfrak{p}$. Fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$ whose restriction to $\mathfrak{o}_F$ comes from a non-trivial character of $\mathfrak{o}_F/\mathfrak{p}_F$.

## 4.1. Local Endoscopy

**The endoscopic datum.** Let $\mathbf{G}$ be a quasisplit connected reductive group over $F$ with center $Z(\mathbf{G})$. An $L$-group datum $(\hat{\mathbf{G}}, \rho_G, \eta_G)$ defines a Langlands dual group ${}^L G$. A (standard) *endoscopic datum* attached to $\mathbf{G}$ as given by Langlands and Shelstad [LS87, p. 224] is a quadruple $(\mathbf{H}, \mathcal{H}, s, \xi)$, composed of

1. a quasisplit reductive group $\mathbf{H}$ over $F$ with $L$-datum $(\hat{\mathbf{H}}, \rho_H, \eta_H)$,
2. a split extension $\mathcal{H}$ of $\hat{\mathbf{H}}$ by $W_F$ which gives rise to an exact sequence
$$1 \to \hat{\mathbf{H}} \to \mathcal{H} \to W_F \to 1,$$
such that the splitting $\rho_{\mathcal{H}} : W_F \to \mathrm{Aut}(\hat{\mathbf{H}})$ coincides with $\rho_H$,
3. a semisimple element $s \in \hat{\mathbf{G}}$,
4. an $L$-homomorphism $\xi : \mathcal{H} \to {}^L G$ such that
    a) $\xi$ defines an isomorphism from $\hat{\mathbf{H}}$ to the connected component of the centralizer $\mathrm{Cent}(s, \hat{\mathbf{G}})$,
    b) $\mathrm{Int}(s) \circ \xi \cong a \otimes \xi$ where $a$ is a trivial 1-cocycle of $W_F$ in $Z(\hat{\mathbf{G}})$ and $(a \otimes \xi)(h) = a(w)\xi(h)$ for $h \in \mathcal{H}$ with image $w \in W_F$.

The trivial endoscopic datum is $(\mathbf{G}, {}^L G, 1, \mathrm{id})$. There is a natural definition of equivalence between endoscopic data [LS87, § 1.2].

An endoscopic datum is *elliptic* if the connected component of $\xi(Z(\hat{\mathbf{H}})^\Gamma)$ is contained in $Z(\hat{\mathbf{G}})$. That means $\xi(\mathcal{H})$ is not contained in a proper parabolic subgroup of ${}^L G$.

**Point correspondences.** Fix a quasisplit connected reductive group $\mathbf{G}$ and an elliptic endoscopic datum $(\mathbf{H}, \mathcal{H}, s, \xi)$. As usual, $G$ and $H$ denote the groups of $F$-rational points. Borel pairs[1] $(B_{\mathbf{G}}, T_{\mathbf{G}})$ of $\mathbf{G}$ and $(B_{\mathbf{H}}, T_{\mathbf{H}})$ of $\mathbf{H}$ define tori $\mathcal{T}_{\mathbf{H}} \subseteq \mathcal{H}$ and $\mathcal{T}_{\mathbf{G}} \subseteq {}^L G$ in the $L$-groups. If $\xi(\mathcal{T}_{\mathbf{H}}) = \mathcal{T}_{\mathbf{G}}$ and $s \in \mathcal{T}_{\mathbf{G}}$, there is a naturally defined isomorphism

$$\psi : T_{\mathbf{H}} \to T_{\mathbf{G}}.$$

This $\psi$ is an *admissible embedding* if it is defined over $F$ [LS87, (1.3)]. For every maximal torus $T_{\mathbf{H}} \subseteq \mathbf{H}$ over $F$ there is a maximal torus $T_{\mathbf{G}} \subseteq \mathbf{G}$ over $F$ with an admissible embedding $\psi : T_{\mathbf{H}} \to T_{\mathbf{G}}$ [KS99, Lemma 3.3.B]. This gives rise to a $\Gamma$-invariant canonical map $\mathcal{A}_{H/G}$ from semisimple conjugacy classes of $\mathbf{H}(\bar{F})$ to semisimple conjugacy classes of $\mathbf{G}(\bar{F})$ [KS99, Lemma 3.3A].

A semisimple $\delta \in \mathbf{H}(\bar{F})$ is *strongly regular* if its centralizer $\mathrm{Cent}(\delta, \mathbf{H}(\bar{F}))$ is a torus. It is *strongly $G$-regular* if the image of its conjugacy class under $\mathcal{A}_{H/G}$ consists of strongly regular elements in $\mathbf{G}(\bar{F})$. A semisimple strongly $G$-regular $\gamma_H \in \mathbf{H}(F)$ is an *image of* $\gamma_G \in \mathbf{G}(F)$ if $\gamma_G \in \mathcal{A}_{H/G}(\mathrm{Int}(\mathbf{H}(\bar{F}))(\gamma_H))$, i.e. if $\gamma_G$ is a semisimple strongly regular element in the image of the $\mathbf{H}(\bar{F})$-conjugacy class of $\gamma_H$ under $\mathcal{A}_{H/G}$.

**Orbital integrals.** Strongly regular semisimple $\delta, \delta' \in \mathbf{H}(F)$ are *stably conjugate* $\delta \sim \delta'$ if they are conjugate in $\mathbf{H}(\bar{F})$. The stable conjugacy class of $\delta$ is a disjoint union of finitely many $\mathbf{H}(F)$-conjugacy classes. The *orbital integral* of a compactly supported smooth function $f \in C_c^\infty(G)$ at an element $\delta \in G$ is the integral over the $G$-conjugacy class of $\delta \in G$

$$O_\delta(f) = \int_{\mathrm{Cent}(\delta,G)\backslash G} f(g^{-1}\delta g)\,\mathrm{d}t\backslash\,\mathrm{d}g, \tag{4.1}$$

where $\mathrm{Cent}(\delta, G)$ is the centralizer of $\delta$ in $G$. The *stable orbital integral* $SO_\delta(f)$ is the integral over the stable conjugacy class of $\delta$. It coincides with the sum

$$SO_\delta(f) = \sum_{\delta' \sim \delta} O_{\delta'}(f) \tag{4.2}$$

over representatives $\delta'$ of the $G$-conjugacy classes stably conjugate to $\delta$.

**Matching.** The Langlands-Shelstad-transfer factor $\Delta(\gamma_H, \gamma_G) \in \mathbb{C}$ for strongly $G$-regular semisimple $\gamma_H \in H$ and strongly regular semisimple $\gamma_G \in G$ depends only on the conjugacy class of $\gamma_G$ and the stable conjugacy classes of $\gamma_H$ [LS87, §1.4, §3.7]. It is zero whenever $\gamma_H$ is not an image of $\gamma_G$.

**Definition 4.1.** A pair of functions $f \in C_c^\infty(G)$ and $f^H \in C_c^\infty(H)$ satisfies the *matching condition for standard endoscopy* if

$$SO_{\gamma_H}(f^H) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G) O_{\gamma_G}(f) \tag{4.3}$$

[1] A Borel pair is a choice of a Borel subgroup $B_{\mathbf{G}}$ of $\mathbf{G}$ with maximal torus $T_{\mathbf{G}}$ in $B_{\mathbf{G}}$.

for every strongly $G$-regular semisimple $\gamma_H \in H$. The sums runs over representatives $\gamma_G \in G$ for conjugacy classes of strongly regular semisimple elements in $G$ and only finitely many terms are non-zero. Such a pair $(f, f^H)$ is called a *transfer* $f \to f^H$.

The Fundamental Lemma for Standard Endoscopy [Ngô10] asserts that for every $f \in C_c^\infty(G)$ there is a (non-unique) $f^M \in C_c^\infty(M)$ such that $f \to f^M$ is a transfer.

A distribution $\alpha : C_c^\infty(H) \to \mathbb{C}$ is *invariant* if $\alpha(f) = \alpha(f^h)$ for every $h \in H$ where $f^h(x) = f(hxh^{-1})$. It is *stably invariant* if it factors over stable equivalence, that means over the orbital integrals $O_\delta(f)$ for all regular semisimple $\delta \in M$. This permits the definition of the *endoscopic lift* of a stably invariant distribution $\alpha : C_c^\infty(H) \to \mathbb{C}$ as the distribution

$$\alpha^G : C_c^\infty(G) \to \mathbb{C}, \qquad \alpha^G(f) = \alpha(f^H)$$

for every transfer $f \to f^H$. Thus, every stably invariant distribution $\alpha$ can be lifted to an invariant distribution $\alpha^G$ on $G$.

### 4.1.1. Local Endoscopy for GSp(4)

Let $\mathbf{G} = \mathrm{GSp}(4)$ be group of symplectic similitudes in genus two with respect to the symplectic element $\left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$. Its dual is $\hat{\mathbf{G}} = \mathrm{GSp}(4, \mathbb{C})$ and the $L$-group is ${}^L G \cong \hat{\mathbf{G}} \rtimes W_F$. Endoscopy for $\mathbf{G}$ has been studied by Hales [Hal89, Hal97] and Weissauer [Wei09a]. Up to equivalence, there is only one proper elliptic endoscopic datum $(\mathbf{M}, {}^L M, s, \xi)$. The endoscopic group is

$$\mathbf{M} = \mathrm{GSO}(2,2) \cong (\mathrm{GL}(2) \times \mathrm{GL}(2))/\Delta\,\mathrm{GL}(1),$$

where the quotient is formed with respect to the antidiagonal embedding

$$\Delta : \mathrm{GL}(1) \to \mathrm{GL}(2) \times \mathrm{GL}(2), \quad t \mapsto (tI_2, t^{-1}I_2).$$

The $L$-group is ${}^L M = \hat{\mathbf{M}} \rtimes W_F$ for

$$\hat{\mathbf{M}} = (\mathrm{GL}(2,\mathbb{C})^2)^0 = \{(x, x') \in \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})) \mid \det x = \det x'\}. \tag{4.4}$$

The embedding $\xi$ is defined on the dual groups as

$$\xi : \hat{\mathbf{M}} \longrightarrow \mathrm{GSp}(4,\mathbb{C}) \qquad (x,y) \longmapsto \begin{pmatrix} x_{11} & & x_{12} & \\ & y_{11} & & y_{12} \\ x_{21} & & x_{22} & \\ & y_{21} & & y_{22} \end{pmatrix} \tag{4.5}$$

into the connected component of the centralizer of $s = \mathrm{diag}(1,-1,1,-1)$. The affine Dynkin diagram of $M = \mathbf{M}(F)$ is composed of two disjoint copies of the affine Dynkin diagram of $\mathrm{GL}(2,F)$. There are seven classes of maximal tori $T_G$ in $\mathrm{GSp}(4,F)$; four of them admit an admissible embedding from a torus $T_M \subseteq M$ [Wei09a, 4.4.2]. The embedding $\mathcal{A}_{M/G}$ of semisimple conjugacy classes in $\mathbf{G}(\bar{F})$ is given on representatives in the split diagonal tori by [Wei09a, Lemma 8.1]

$$(\mathrm{diag}(t_1,t_1'),\mathrm{diag}(t_2,t_2')) \mapsto \mathrm{diag}(t_1't_2', t_2't_1, t_2t_1, t_2t_1'). \tag{4.6}$$

The transfer factors for $\mathrm{GSp}(4,F)$ are only unique up to a scalar. We use the normalization of [Wei09a, p.212].

**Representations of $M$.** By inflation, the irreducible admissible representations $\sigma$ of $M$ are in one to one correspondence with pairs $(\sigma_1,\sigma_2)$ of irreducible representations of $\mathrm{GL}(2,F)$ with equal central character $\omega_{\sigma_1}=\omega_{\sigma_2}$. Indeed, $\sigma_1\boxtimes\sigma_2(tI_2,t^{-1}I_2)=\mathrm{id}$ for every $t\in F^\times\}$, so the representation $\sigma_1\boxtimes\sigma_2$ of $\mathrm{GL}(2,F)\times\mathrm{GL}(2,F)$ factors over a unique representation $\sigma$ of $M$. Every irreducible admissible representations $\sigma$ of $M$ defines a unique pair $(\sigma_1,\sigma_2)$ of representations of $\mathrm{GL}(2,F)$ such that $\sigma$ pulls back to $\sigma_1\boxtimes\sigma_2$ under the natural projection

$$\mathrm{GL}(2,F)\times\mathrm{GL}(2,F)\twoheadrightarrow M.$$

We will write $(\sigma_1,\sigma_2)$ for $\sigma$. An irreducible admissible representation $\sigma$ of $M$ is generic, cuspidal, or discrete series if and only if both $\sigma_1$ and $\sigma_2$ are generic, cuspidal, or discrete series, respectively. For each irreducible admissible representation $\sigma$ of $M$, the character $\chi_\sigma$ pulls back to an invariant distribution of $\mathrm{GL}(2,F)\times\mathrm{GL}(2,F)$ and is therefore stably invariant.

**The local endoscopic character lift for** $\mathrm{GSp}(4)$**.** Let $\sigma$ be an irreducible admissible representation of $M$. Then its character $\chi_\sigma$ is a stably invariant distribution on $M$, so there is a well defined lift to an invariant distribution $\chi_\sigma^G$ on $G$. This lift is a finite linear combination

$$\chi_\sigma^G=\sum_\pi n(\sigma,\pi)\chi_\pi$$

of characters of irreducible admissible representations $\pi$ of $G$ with integer coefficients $n(\sigma,\pi)\in\mathbb{Z}$ [Wei09a, Cor. 4.5]. The endoscopic character lift gives rise to a homomorphism of Grothendieck groups of finitely generated admissible representations

$$r:\mathbf{R}_{\mathbb{Z}}(M)\longrightarrow\mathbf{R}_{\mathbb{Z}}(G) \tag{4.7}$$

such that $\chi_\sigma^G=\chi_{r(\sigma)}$.

**Proposition 4.2** ([Wei09a, §4.11])**.** *For irreducible admissible representations $\sigma$ of $M$ with central character $\omega_\sigma$, the endoscopic lift $r(\sigma)$ satisfies*

1. $r(\sigma_1,\sigma_2)=r(\sigma_2,\sigma_1)$,

2. $r(\mu\sigma_1, \mu\sigma_2) = (\mu \circ \mathrm{sim}) \otimes r(\sigma_1, \sigma_2)$ *for smooth characters* $\mu$ *of* $F^\times$,

3. *each constituent* $\pi$ *of* $r(\sigma)$ *has central character* $\omega_\pi = \omega_\sigma$.

**Lemma 4.3** ([Wei09a, Lemma 4.23])**.** *Let* $\sigma$ *be an admissible representation of* $M$ *such that* $\sigma_2 = \mu_1 \times \mu_2$ *for a pair of smooth characters* $\mu_1$, $\mu_2$ *of* $F^\times$ *with central character* $\mu_1\mu_2 = \omega_{\sigma_1}$. *Then the endoscopic lift is the semisimplified Siegel induced representation*

$$r(\sigma) = \mu_1^{-1}\sigma_1 \rtimes \mu_1. \tag{4.8}$$

**Lemma 4.4.** *For every essentially discrete series irreducible admissible representation* $\sigma$ *of* $M$*, the endoscopic lift has two irreducible constituents*

$$r(\sigma) = \pi_+(\sigma) - \pi_-(\sigma). \tag{4.9}$$

*Proof.* After a character twist we can assume $\sigma$ to be unitary. By [Wei09a, Thm. 4.5] $\pi_+(\sigma) = \theta_+(\sigma)$ is the generic irreducible isotropic theta lift [PSS81], [GT11, §8.2] and $\pi_-(\sigma) = \theta_-(\sigma)$ is the nongeneric irreducible anisotropic theta lift [GT11, §8.1]. □

Every generic irreducible admissible representations of $M$, which is not in the essential discrete series, is parabolically induced and the endoscopic character lift is given by 4.3. The endoscopic lift of a non-generic admissible representation $\sigma$ of $M$ can be determined by linear combinations of generic representations $\sigma$ [Wei09a, §4.11].

**The local endoscopic $L$-packets.** For every irreducible representation $\sigma$ of $M$, the *local endoscopic packet* attached to $\sigma$ is the finite set of irreducible representations of $G$ that occur in $r(\sigma) = \sum_\pi n(\sigma,\pi)\pi$ with nonzero multiplicity $n(\sigma,\pi)$ [Wei09a, Def. 4.5]. For preunitary generic $\sigma$ this is called the *local endoscopic L-packet.* The packet attached to preunitary *non-generic* $\sigma$ is the *Arthur-packet.*

**Lemma 4.5.** *Let* $\sigma$ *be a unitary generic irreducible admissible representation of* $M$. *If* $\sigma$ *is in the discrete series, the L-packet attached to* $\sigma$ *has exactly two unitary constituents* $\pi_\pm(\sigma)$. *If* $\sigma$ *is not in the discrete series, then the endoscopic lift* $r(\sigma)$ *is irreducible, and the local L-packet has exactly one unitary constituent* $\pi_+(\sigma)$. *The non-cuspidal constituents are explicitly given by Table 4.1.*

*Proof.* The discrete series case is [Wei09a, Thm. 4.5], so we assume $\sigma$ is not in the discrete series. Without loss of generality let $\sigma_2 = \mu_1 \times \mu_2$ be parabolically induced from a pair of smooth characters $\mu_1, \mu_2$ of $F^\times$. Then $r(\sigma)$ is the Siegel induced representation $\mu_1^{-1}\sigma_1 \rtimes \mu_1$ by [Wei09a, Lemma 4.23]. Either $\sigma_2$ is in the tempered principal series (with unitary $\mu_i$) or it is in the unitary complementary series. For both cases irreducibility is shown in [Wei09a, p. 156]. The conditions for unitarity of $\pi_+(\sigma) = r(\sigma)$ apply, cp. [RS07, Table A.2]. □

**Notation 4.6** (Table 4.1)**.** Let $\xi, \mu, \mu_1, \dots, \mu_4$ be smooth characters of $F^\times$ with $\xi^2 = 1$. Let $\Pi_1$ and $\Pi_2$ be two non-isomorphic cuspidal irreducible representations of $\mathrm{GL}(2, F)$. The central characters are $\omega_{\Pi_1} = \omega_{\Pi_2} = \mu_1\mu_2 = \mu_3\mu_4 = \mu^2$.

| $\sigma_1$ | $\sigma_2$ | $\pi_+(\sigma)$ | type | $\pi_-(\sigma)$ | type |
|---|---|---|---|---|---|
| $\xi\mu \cdot \mathrm{St}$ | $\mu \cdot \mathrm{St}$ | $\delta(\xi\nu^{1/2} \cdot \mathrm{St} \rtimes \mu\nu^{-1/2})$ | Va | cuspidal | |
| $\mu \cdot \mathrm{St}$ | $\mu \cdot \mathrm{St}$ | $\tau(S, \nu^{-1/2}\mu)$ | VIa | $\tau(T, \nu^{-1/2}\mu)$ | VIb |
| $\Pi_1$ | $\mu \cdot \mathrm{St}$ | $\delta(\mu^{-1}\nu^{1/2} \cdot \Pi_1 \rtimes \mu\nu^{-1/2})$ | XIa | cuspidal | |
| $\Pi_1$ | $\Pi_1$ | $\tau(S, \Pi_1)$ | VIIIa | $\tau(T, \Pi_1)$ | VIIIb |
| $\Pi_1$ | $\Pi_2$ | cuspidal | | cuspidal | |
| $\mu_3 \times \mu_4$ | $\mu_1 \times \mu_2$ | $\mu_3\mu_1^{-1} \times \mu_4\mu_1^{-1} \rtimes \mu_1$ | I | — | |
| $\mu \cdot \mathrm{St}$ | $\mu_1 \times \mu_2$ | $\mu\mu_1^{-1} \cdot \mathrm{St}_{\mathrm{GL}(2,F)} \rtimes \mu_1$ | IIa | — | |
| $\Pi_1$ | $\mu_1 \times \mu_2$ | $\mu_3^{-1} \cdot \Pi_1 \rtimes \mu_3$ | X | — | |

Table 4.1.: Constituents of the local endoscopic $L$-packet attached to generic unitary irreducible $\sigma$.

## 4.2. Main result on parahoric restriction of endoscopic lifts

Parahoric restriction with respect to a parahoric subgroup $\mathscr{P}_x \subseteq G$ is an exact functor between categories of admissible representations, so it defines a homomorphism between Grothendieck groups of finitely generated admissible representations

$$\mathbf{r}_{\mathscr{P}_x} : R_{\mathbb{Z}}(G) \to R_{\mathbb{Z}}(\mathscr{P}_x/\mathscr{P}_x^+). \tag{4.10}$$

In this section, we will determine the parahoric restriction of the endoscopic lift $r(\sigma)$ for every irreducible admissible representation $\sigma$ of $M$. We will also determine the parahoric restriction of the local endoscopic $L$-packets attached to unitary generic irreducible admissible $\sigma$. By (2.5), it is sufficient to study maximal parahoric subgroups.

### 4.2.1. Hyperspecial parahoric restriction

Let $\mathbf{r}_{\mathscr{K}} : \mathbf{Rep}(\mathrm{GSp}(4,F)) \to \mathbf{Rep}(\mathrm{GSp}(4,q))$ be the parahoric restriction functor with respect to the standard hyperspecial parahoric subgroup $\mathscr{K} = \mathscr{K}_G = \mathrm{GSp}(4, \mathfrak{o}_F)$ with Levi quotient $\underline{\mathscr{K}} \cong \mathrm{GSp}(4,q)$.

**Theorem 4.7.** *Fix a unitary generic irreducible representation $\sigma$ of $M$ and let $\pi_\pm = \pi_\pm(\sigma)$ be an irreducible constituent of the attached local endoscopic $L$-packet. If $\sigma$ has depth zero, then $\mathbf{r}_{\mathscr{K}}(\pi_\pm)$ is given by Table 4.2. If $\sigma$ has depth $> 0$, then $\mathbf{r}_{\mathscr{K}}(\pi) = 0$.*

*Proof.* The non-cuspidal $\pi$ are explicitly given by Table 4.1; for $\mathbf{r}_{\mathscr{K}}(\pi)$ see Thm. 3.2. If $\pi$ is cuspidal, then $\mathbf{r}_{\mathscr{K}}(\pi)$ is either zero or cuspidal irreducible by Lemma 2.18.

| $\sigma$ | $\mathbf{r}_{\mathscr{K}}(\pi_+)\vert_{\mathrm{Sp}(4,q)}$ (even $q$) | $\mathbf{r}_{\mathscr{K}}(\pi_+)$ (odd $q$) | dimension |
|---|---|---|---|
| $(\mu_1 \times \mu_2, \mu_3 \times \mu_4)$ | $\chi_1(k_1 - k_3, k_2 - k_3)$ | $X_1(\tilde{\mu}_1/\tilde{\mu}_3, \tilde{\mu}_2/\tilde{\mu}_3, \tilde{\mu}_1)$ | $(q+1)^2(q^2+1)$ |
| $(\mu_1 \times \mu_2, \mu \cdot \mathrm{St})$ | $\chi_{10}(k - k_1)$ | $\chi_4(\tilde{\mu}/\tilde{\mu}_1, \tilde{\mu}_1)$ | $(q^2+q)(q^2+1)$ |
| $(\mu_1 \times \mu_2, \mu_1 \cdot \Pi_1)$ | $\chi_2(l_1)$ | $X_2(\Lambda_1, \tilde{\mu}_1)$ | $q^4 - 1$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \mathrm{St})$ | $\theta_1 + \theta_4$ | $\theta_1(\tilde{\mu}) + \theta_5(\tilde{\mu})$ | $q^4 + q(q+1)^2/2$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_u \cdot \mathrm{St}$ | $\theta_3 + \theta_4$ | $\theta_4(\tilde{\mu}) + \theta_5(\tilde{\mu})$ | $q^4 + q(q^2+1)/2$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_t \cdot \mathrm{St})$ | — | $\tau_3(\tilde{\mu})$ | $q^4 + q^2$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \Pi_1)$ | $\chi_{12}(l_1'')$ | $\chi_6(\omega_{\Lambda_1}, \tilde{\mu})$ | $(q^2+1)(q^2-q)$ |
| $(\Pi_1, \Pi_1)$ | $\chi_{13}(l_1')$ | $\chi_8(\Lambda_1)$ | $(q^2+1)(q^2-q)$ |
| $(\Pi_1, \Pi_2)$ | $\chi_4(\tilde{k}_+, \tilde{k}_-)$ | $X_5(\Lambda_1, \omega_{\Lambda_2/\Lambda_1})$ | $(q^2+1)(q-1)^2$ |
| $\sigma$ | $\mathbf{r}_{\mathscr{K}}(\pi_-)\vert_{\mathrm{Sp}(4,q)}$ (even $q$) | $\mathbf{r}_{\mathscr{K}}(\pi_-)$ (odd $q$) | dimension |
| $(\mu \cdot \mathrm{St}, \mu \cdot \mathrm{St})$ | $\theta_2$ | $\theta_3(\tilde{\mu})$ | $q(q^2+1)/2$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_u \cdot \mathrm{St})$ | $\theta_5$ | $\theta_2(\tilde{\mu})$ | $q(q-1)^2/2$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_t \cdot \mathrm{St})$ | — | 0 | 0 |
| $(\mu \cdot \mathrm{St}, \mu \cdot \Pi_1)$ | 0 | 0 | 0 |
| $(\Pi_1, \Pi_1)$ | $\chi_9(l_1')$ | $\chi_7(\Lambda_1)$ | $(q^2+1)(q-1)$ |
| $(\Pi_1, \Pi_2)$ | 0 | 0 | 0 |

Table 4.2.: Hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}}(\pi_\pm)$ of $\pi_\pm(\sigma)$ in the endoscopic $L$-packet for depth zero preunitary generic irreducible admissible $\sigma$.

If $\pi = \theta_-(\sigma)$ is non-generic and cuspidal, we must have $\sigma_1 \ncong \sigma_2$. By Thm. 4.29, the hyperspecial parahoric restriction of $\pi$ is zero unless $\sigma \cong (\mu\,\mathrm{St}, \mu\xi_u\,\mathrm{St})$ for an at most tamely ramified character $\mu$ and the unramified quadratic character $\xi_u$. For even $q$, the only irreducible cuspidal representation of $\mathrm{GSp}(4,q)$ with dimension $q(q^2-1)/2$ and central character $\widetilde{\mu}^2$ is $\theta_5 \boxtimes \widetilde{\mu}^2$. For odd $q$ it is either $\theta_2(\widetilde{\mu})$ or $\theta_2(\lambda_0\widetilde{\mu})$ where $\lambda_0$ is the non-trivial quadratic character of $\mathbb{F}_q^\times$. The character value of $\mathbf{r}_{\mathscr{K}} \circ r(\sigma)$ on the conjugacy class $L_0$ is given by (4.20), and we have $\theta_-(\sigma) = \theta_+(\sigma) - r(\sigma)$ in the Grothendieck group. Hence $\mathbf{r}_{\mathscr{K}}(\theta_-(\sigma))$ is the twist of the unipotent cuspidal non-generic representation $\theta_2$ by the character $\widetilde{\mu}$.

If $\pi = \theta_+(\sigma)$ is generic and cuspidal, we must have $\sigma \cong (\Pi_1, \Pi_2)$ for a pair of non-isomorphic cuspidal irreducible representations $\Pi_1$, $\Pi_2$ of $\mathrm{GL}(2,F)$ with equal central character. Thm. 4.29 implies $\mathbf{r}_{\mathscr{K}}(\theta_-(\sigma)) = 0$, so $\mathbf{r}_{\mathscr{K}}(\pi) = \mathbf{r}_{\mathscr{K}} \circ r(\sigma)$. This is either zero or an irreducible cuspidal representation of $\mathrm{GSp}(4,q)$. For odd $q$, the only cuspidal irreducible representations of $\mathrm{GSp}(4,q)$ are of type $X_4$, $X_5$ and $\theta_2$. But the character values of $\mathbf{r}_{\mathscr{K}} \circ r(\sigma)$ on the anisotropic semisimple conjugacy classes $K_0$ and $L_0$ are given by (4.22) and (4.20) and completely determine $\mathbf{r}_{\mathscr{K}}(\pi)$. For even $q$ the proof is analogous. □

**Notation 4.8** (Table 4.2 and 4.3)**.** Let $\mu$, $\mu_1, \mu_2, \mu_3, \mu_4$ be tamely ramified or unramified characters of $F^\times$ which restrict to non-zero characters $\widetilde{\mu}, \widetilde{\mu_1}, \widetilde{\mu_2}, \widetilde{\mu_3}, \widetilde{\mu_4}$ of $(\mathfrak{o}/\mathfrak{p})^\times$. Fix two non-isomorphic cuspidal irreducible depth zero representations $\Pi_1$ and $\Pi_2$ of

| $\sigma \in \mathbf{Irr}(M)$ | $(\mathbf{r}_{\mathscr{K}} \circ r(\sigma))\vert_{\mathrm{Sp}(4)}$ (even $q$) | $\mathbf{r}_{\mathscr{K}} \circ r(\sigma)$ (odd $q$) | dimension |
|---|---:|---:|---:|
| $(\mu_1 \times \mu_2, \mu_3 \times \mu_4)$ | $\chi_1(k_1 - k_3, k_2 - k_3)$ | $X_1(\widetilde{\mu}_1/\widetilde{\mu}_3, \widetilde{\mu}_2/\widetilde{\mu}_3, \widetilde{\mu}_3)$ | $(q+1)^2(q^2+1)$ |
| $(\mu_1 \times \mu_2, \mu \cdot \mathrm{St})$ | $\chi_{10}(k - k_1)$ | $\chi_4(\widetilde{\mu}/\widetilde{\mu}_1, \widetilde{\mu}_1)$ | $q(q+1)(q^2+1)$ |
| $(\mu_1 \times \mu_2, \mu \cdot \mathbf{1})$ | $\chi_6(k - k_1)$ | $\chi_3(\widetilde{\mu}/\widetilde{\mu}_1, \widetilde{\mu}_1)$ | $(q+1)(q^2+1)$ |
| $(\mu_1 \times \mu_2, \mu_1 \cdot \Pi_1)$ | $\chi_2(l_1)$ | $X_2(\Lambda_1, \widetilde{\mu}_1)$ | $(q^2-1)(q^2+1)$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \mathrm{St})$ | $\theta_1 + \theta_4 - \theta_2$ | $\theta_1(\widetilde{\mu}) + \theta_5(\widetilde{\mu}) - \theta_3(\widetilde{\mu})$ | $q^2(q^2+1)$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_u \cdot \mathrm{St})$ | $\theta_3 + \theta_4 - \theta_5$ | $\theta_4(\widetilde{\mu}) + \theta_5(\widetilde{\mu}) - \theta_2(\widetilde{\mu})$ | $q^2(q^2+1)$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_t \cdot \mathrm{St})$ | — | $\tau_3(\widetilde{\mu})$ | $q^2(q^2+1)$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \Pi_1)$ | $\chi_{12}(l_1'')$ | $\chi_6(\omega_{\Lambda_1}, \widetilde{\mu})$ | $q(q-1)(q^2+1)$ |
| $(\Pi_1, \Pi_1)$ | $\chi_{13}(l_1') - \chi_9(l_1')$ | $\chi_8(\Lambda_1) - \chi_7(\Lambda_1)$ | $(q-1)^2(q^2+1)$ |
| $(\Pi_1, \Pi_2)$ | $\chi_4(\tilde{k}_+, \tilde{k}_-)$ | $X_5(\Lambda_1, \omega_{\Lambda_2/\Lambda_1})$ | $(q-1)^2(q^2+1)$ |
| $(\mu \cdot \mathbf{1}, \mu \cdot \mathrm{St})$ | $\theta_2 + \theta_3$ | $\theta_3(\widetilde{\mu}) + \theta_4(\widetilde{\mu})$ | $q(q^2+1)$ |
| $(\mu \cdot \mathbf{1}, \mu\xi_u \cdot \mathrm{St})$ | $\theta_1 + \theta_5$ | $\theta_1(\widetilde{\mu}) + \theta_2(\widetilde{\mu})$ | $q(q^2+1)$ |
| $(\mu \cdot \mathbf{1}, \mu\xi_t \cdot \mathrm{St})$ | — | $\tau_2(\widetilde{\mu})$ | $q(q^2+1)$ |
| $(\mu \cdot \mathbf{1}, \mu \cdot \mathbf{1})$ | $\theta_0 + \theta_1 - \theta_3$ | $\theta_0(\widetilde{\mu}) + \theta_1(\widetilde{\mu}) - \theta_4(\widetilde{\mu})$ | $(q^2+1)$ |
| $(\mu \cdot \mathbf{1}, \mu\xi_u \cdot \mathbf{1})$ | $\theta_0 + \theta_2 - \theta_5$ | $\theta_0(\widetilde{\mu}) + \theta_3(\widetilde{\mu}) - \theta_2(\widetilde{\mu})$ | $(q^2+1)$ |
| $(\mu \cdot \mathbf{1}, \mu\xi_t \cdot \mathbf{1})$ | — | $\tau_1(\widetilde{\mu})$ | $(q^2+1)$ |
| $(\mu \cdot \mathbf{1}, \mu \cdot \Pi_1)$ | $\chi_8(l_1'')$ | $\chi_5(\omega_{\Lambda_1}, \widetilde{\mu})$ | $(q-1)(q^2+1)$ |

Table 4.3.: Hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}} \circ r(\sigma)$ of the endoscopic lift $r(\sigma)$ of depth zero irreducible admissible representations $\sigma$ of $M$.

$\mathrm{GL}(2, F)$ with common central character. Their hyperspecial parahoric restriction is a cuspidal irreducible representation $\pi_{\Lambda_i}$ of $\mathrm{GL}(2, q)$ attached to a character $\Lambda_i$ of $\mathbb{F}_{q^2}^\times$ as in Thm. A.1 with $\Lambda_1\vert_{\mathbb{F}_q^\times} = \Lambda_2\vert_{\mathbb{F}_q^\times}$. A character $\Lambda$ of $\mathbb{F}_{q^2}^\times$ with $\Lambda\vert_{\mathbb{F}_q^\times} = 1$ factors over a character $\omega_\Lambda$ of $\mathbb{F}_q^\times[q+1]$ with $\Lambda(x) = \omega_\Lambda(x^{q-1})$. The nontrivial unramified quadratic character of $F$ is $\xi_u$. Either one of the tamely ramified quadratic characters is denoted $\xi_t$. Equality of central characters of $\sigma_1$ and $\sigma_2$ is tacitly assumed.

For irreducible representations of the finite group $\mathrm{GSp}(4, q)$ with odd $q$ we use the notation of Shinoda [Shi82].

For even $q$ a representation of $\mathrm{GSp}(4, q)$ is uniquely determined by its central character and its restriction to $\mathrm{Sp}(4, q)$. The irreducible representations of $\mathrm{Sp}(4, q)$ have been classified by Enomoto [Eno72]. Fix a primitive character $\hat{\theta} : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ and let $\hat{\gamma}$ and $\hat{\eta}$ be its restriction to $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times[q+1]$, respectively. Let $k_j \in \mathbb{Z}/(q-1)\mathbb{Z}$ be such that $\hat{\gamma}^{k_j} = \tilde{\mu_j}$. Let $l_i \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ be such that $\hat{\theta}^{l_i} = \Lambda_i$. Denote by $l_i'$ the image of $l_i$ under the canonical projection $\mathbb{Z}/(q^2-1)\mathbb{Z} \twoheadrightarrow \mathbb{Z}(q+1)\mathbb{Z}$. If $\Lambda_i\vert_{\mathbb{F}_q^\times} = 1$, there is a unique preimage $l_i''$ of $l_i$ under the canonical injection $\mathbb{Z}/(q+1)\mathbb{Z} \hookrightarrow \mathbb{Z}(q^2-1)\mathbb{Z}$ so that $\omega_{\Lambda_i} = \hat{\eta}^{l_i''}$. Finally, for $\Lambda_1\vert_{\mathbb{F}_q^\times} = \Lambda_2\vert_{\mathbb{F}_q^\times}$ let $\tilde{k}_\pm = \frac{q+2}{2}(l_1' \pm l_2')$.

**Theorem 4.9.** *Let $\sigma$ be an irreducible admissible representation of $M$. If $\sigma$ has depth zero, the hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}}(r(\sigma))$ of the endoscopic lift is given by Table 4.3. If $\sigma$ has positive depth, then $\mathbf{r}_{\mathscr{K}}(r(\sigma)) = 0$.*

*Proof.* If $\sigma$ is essentially discrete series, we can assume $\sigma$ to be in the unitary discrete series after a character twist. Then the result is implied by Thm. 4.7 and $r(\sigma) = \pi_+(\sigma) - \pi_-(\sigma)$. If $\sigma$ is parabolically induced, the result is implied by Lemma 4.3 and Thm. 3.2. Lemma 4.3 implies $\nu^{1/2}\mu^{-1}\sigma_2 \rtimes \nu^{-1/2}\mu \equiv r(\mu\,\mathrm{St}, \sigma_2) + r(\mu\,\mathbf{1}, \sigma_2)$ in the Grothendieck group, so $\mathbf{r}_{\mathscr{K}} \circ r(\mu\,\mathbf{1}, \sigma_2)$ can be determined by linear combinations using the previous results. □

Since $r(\sigma_1, \sigma_2) = r(\sigma_2, \sigma_1)$, Table 4.3 determines $\mathcal{F} \circ r(\sigma)$ for every depth zero irreducible admissible representation $\sigma$ of the endoscopic group.

**Corollary 4.10.** *For every admissible representation $\sigma$ of $M$, the endoscopic character lift satisfies the equation*

$$\dim \mathbf{r}_{\mathscr{K}_G}(r(\sigma)) = (q^2+1) \dim \mathbf{r}_{\mathscr{K}_M}(\sigma). \tag{4.11}$$

*for hyperspecial parahoric subgroups $\mathscr{K}_G \subseteq G$, $\mathscr{K}_M \subseteq M$.*

### 4.2.2. Paramodular restriction

For the standard paramodular subgroup $\mathscr{J}$ of $G = \mathrm{GSp}(4,F)$ we fix the isomorphism $\mathscr{J}/\mathscr{J}^+ \to (\mathrm{GL}(2,q)^2)^0 = \{(x,x') \in \mathrm{GL}(2,q)^2 \mid \det x = \det x'\}$

$$\begin{pmatrix} x_{1,1} & * & \varpi^{-1}x_{1,3} & * \\ * & x_{2,2} & * & x_{2,4} \\ \varpi x_{3,1} & * & x_{3,3} & * \\ * & x_{4,2} & * & x_{4,4} \end{pmatrix} \mapsto \left( \begin{pmatrix} \mathrm{x}_{1,1} & \mathrm{x}_{1,3} \\ \mathrm{x}_{3,1} & \mathrm{x}_{3,3} \end{pmatrix}, \begin{pmatrix} \mathrm{x}_{2,2} & \mathrm{x}_{2,4} \\ \mathrm{x}_{4,2} & \mathrm{x}_{4,4} \end{pmatrix} \right), \tag{4.12}$$

where $\mathrm{x}_{ij}$ is the image of $x_{ij} \in \mathfrak{o}_F$ under the projection $\mathfrak{o}_F \to \mathfrak{o}_F/\mathfrak{p}_F$. The representations of $(\mathrm{GL}(2,q)^2)^0$ have been classified in Lemma A.6. The paramodular restriction functor is discussed in Subsection 3.1.3.

**Theorem 4.11.** *Let $\sigma$ be a preunitary discrete series generic irreducible admissible representation of $M$ and let $\pi = \pi_\pm(\sigma)$ be an irreducible constituent of the local endoscopic L-packet attached to $\sigma$. If $\sigma$ has depth zero, then the paramodular restriction $\mathbf{r}_{\mathscr{J}}(\pi)$ is the $(\mathrm{GL}(2,q)^2)^0$-representation given by Table 4.4. If $\sigma$ has positive depth, then the paramodular restriction $\mathbf{r}_{\mathscr{J}}(\pi)$ is zero.*

*Proof.* For the non-cuspidal representations $\pi$, the paramodular restriction is determined by Table 4.1 and Thm.3.7.

If $\pi$ is cuspidal and generic, then $\pi$ is compactly induced from an extension of $\tau$ to the normalizer $Z_G\mathscr{K}_G$ of a hyperspecial parahoric subgroup $\mathscr{K}_G$ [DR09, 6.2.1]. The paramodular restriction of $\pi$ is then zero by Corollary 2.17.

If $\pi$ is non-generic and cuspidal, then $\pi = \theta_-(\sigma)$ for $\sigma_1 \ncong \sigma_2$ in the discrete series. The paramodular restriction is either zero or a sum of one or two cuspidal irreducible representations of $(\mathrm{GL}(2,q)^2)^0$ by Prop. 3.6. The character values on the anisotropic conjugacy classes $\mathrm{E_G}$ are given by (4.21). We discuss each case separately:

For $\sigma = (\Pi_1, \Pi_2)$ of depth zero with $\Pi_1 \ncong \Pi_2$ we have $\theta_+(\sigma) = 0$, so the character value of $\mathbf{r}_{\mathscr{J}}(\pi)$ at $\mathrm{E_G} = \mathrm{E_G}(\alpha\beta, \alpha\beta^q)$ is

$$\begin{aligned}\mathrm{tr}(\mathbf{r}_{\mathscr{J}}(\pi); \mathrm{E_G}) = -\,\mathrm{tr}(\mathbf{r}_{\mathscr{J}} \circ r(\sigma); \mathrm{E_G}) &\overset{(4.21)}{=} \mathrm{tr}(\mathbf{r}_{\mathscr{K}} \circ r(\sigma); \mathrm{E_M}(\alpha,\beta)) \\ &= (\Lambda_1(\alpha) + \Lambda_1^q(\alpha)) \cdot (\Lambda_2(\beta) + \Lambda_2^q(\beta)) \\ &\quad + (\Lambda_2(\alpha) + \Lambda_2^q(\alpha)) \cdot (\Lambda_1(\beta) + \Lambda_1^q(\beta)) \\ &= (\Lambda_a(\alpha\beta) + \Lambda_a^q(\alpha\beta)) \cdot (\Lambda_b(\alpha\beta^q) + \Lambda_b^q(\alpha\beta^q)) \\ &\quad + (\Lambda_b(\alpha\beta) + \Lambda_b^q(\alpha\beta)) \cdot (\Lambda_a(\alpha\beta^q) + \Lambda_a^q(\alpha\beta^q))\end{aligned}$$

for $\Lambda_a\Lambda_b = \Lambda_1$ and $\Lambda_a\Lambda_b^q = \Lambda_2$. Therefore the paramodular restriction of $\pi$ must be the representation $[\pi_{\Lambda_a}, \pi_{\Lambda_b}] + [\pi_{\Lambda_b}, \pi_{\Lambda_a}]$. The characters $\Lambda_a$ and $\Lambda_b$ are only unique up to a character twist, but by (A.1) this does not affect $\mathbf{r}_{\mathscr{J}}(\pi)$.

Consider $\sigma = (\mu \cdot \mathrm{St}, \mu \cdot \Pi_1)$ where $\Pi_1$ is of depth zero and has trivial central character. Since $\theta_+(\sigma) = 0$, the character value of $\pi = \theta_-(\sigma)$ at $\mathrm{E_G} = \mathrm{E_G}(\alpha\beta, \alpha\beta^q)$ is

$$\begin{aligned}\mathrm{tr}(\mathbf{r}_{\mathscr{J}} \circ \pi; \mathrm{E_G}) = -\,\mathrm{tr}(\mathbf{r}_{\mathscr{J}} \circ r(\sigma); \mathrm{E_G}) &\overset{(4.21)}{=} \mathrm{tr}(\mathbf{r}_{\mathscr{K}_M}(\sigma); \mathrm{E_M}(\alpha,\beta)) \\ &= \widetilde{\mu}((\alpha\beta)^{q+1}) \cdot (\Lambda_1(\alpha) + \Lambda_1(\alpha^q) + \Lambda_1(\beta) + \Lambda_1(\beta^q)).\end{aligned}$$

Hence $\mathbf{r}_{\mathscr{J}}(\pi)$ is the irreducible representation $\widetilde{\mu} \cdot [\pi_{\Lambda_1'}, \pi_{\Lambda_1'^{-1}}]$ with $(\Lambda_1')^{q-1} = \Lambda_1$. Like before, $\pi_{\Lambda_1'}$ is only unique up to a twist with an $\mathbb{F}_q^\times$-character, but $\mathbf{r}_{\mathscr{J}}(\pi)$ is uniquely determined.

If $\sigma = (\mu \cdot \mathrm{St}, \mu\xi_t \cdot \mathrm{St})$ for a tamely ramified quadratic character $\xi_t$, then the character value of $\mathbf{r}_{\mathscr{J}} \circ \theta_+(\sigma)$ at the conjugacy class $\mathrm{E_G}(\alpha\beta, \alpha\beta^q)$ is zero. Then (4.21) implies

$$\begin{aligned}\mathrm{tr}(\mathbf{r}_{\mathscr{J}} \circ \pi; \mathrm{E_G}) = -\,\mathrm{tr}(\mathbf{r}_{\mathscr{J}} \circ r(\sigma); \mathrm{E_G}) &\overset{(4.21)}{=} \mathrm{tr}(\mathbf{r}_{\mathscr{K}_M}(\sigma); \mathrm{E_M}(\alpha,\beta)) \\ &= \widetilde{\mu}((\alpha\beta)^{q+1})(\Lambda_0(\alpha) + \Lambda_0(\beta)).\end{aligned}$$

Therefore the paramodular restriction of $\pi = \theta_-(\sigma)$ is one of the two irreducible constituents in $\widetilde{\mu} \cdot [\pi_{\Lambda_0'}, \pi_{\Lambda_0'^{-1}}]$ with $\Lambda_0'$ such that ${\Lambda_0'}^{q-1} = \Lambda_0$ is the nontrivial quadratic character of $\mathbb{F}_{q^2}^\times$. The correct choice depends on $\xi_t(\varpi_F)$ and is identified by the character value of $\mathbf{r}_{\mathscr{J}}(\pi)$ at $\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}\right) \in (\mathrm{GL}(2,q)^2)^0$ by (4.31) and (A.2).

Finally, for $\sigma = (\mu \cdot \mathrm{St}, \mu\xi_u \cdot \mathrm{St})$ we have $\mathbf{r}_{\mathscr{J}} \circ \theta_-(\sigma) = 0$ because the character value of $\mathbf{r}_{\mathscr{J}}(\theta_+(\sigma) - r(\sigma))$ at $\mathrm{E_G}$ is zero. $\square$

| $\sigma \in \mathbf{Irr}(M)$ | $\mathbf{r}_{\mathscr{J}} \circ \pi_+(\sigma)$ | dimension |
|---|---|---|
| $(\mu_1 \times \mu_2, \mu_3 \times \mu_4)$ | $\widetilde{\mu_1}^{-1} \cdot [\widetilde{\mu_1} \times \widetilde{\mu_3}, \widetilde{\mu_1} \times \widetilde{\mu_4}]$ $+\widetilde{\mu_1}^{-1} \cdot [\widetilde{\mu_1} \times \widetilde{\mu_4}, \widetilde{\mu_1} \times \widetilde{\mu_3}]$ | $2(q+1)^2$ |
| $(\mu_1 \times \mu_2, \mu \cdot \mathrm{St})$ | $\widetilde{\mu_1}^{-1} \cdot [\widetilde{\mu_1} \times \widetilde{\mu}, \widetilde{\mu_1} \times \widetilde{\mu}]$ | $(q+1)^2$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [\,\mathbf{1},\, \mathrm{St}_{\mathrm{GL}(2,q)}] + \widetilde{\mu} \cdot [\,\mathrm{St}_{\mathrm{GL}(2,q)},\, \mathbf{1}]$ $+\widetilde{\mu} \cdot [\,\mathrm{St}_{\mathrm{GL}(2,q)},\, \mathrm{St}_{\mathrm{GL}(2,q)}]$ | $q^2 + 2q$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_u \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [\,\mathbf{1},\, \mathrm{St}_{\mathrm{GL}(2,q)}] + \widetilde{\mu} \cdot [\,\mathrm{St}_{\mathrm{GL}(2,q)},\, \mathbf{1}]$ | $2q$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_t \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [1 \times \lambda_0, 1 \times \lambda_0]_\pm$ | $(q+1)^2/2$ |
| $(\mu_1 \times \mu_2, \mu_1 \cdot \Pi_1)$ | $0$ | $0$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \Pi_1)$ | $0$ | $0$ |
| $(\Pi_1, \Pi_1)$ | $[\pi_{\Lambda_1},\, \mathbf{1}] + [\,\mathbf{1}, \pi_{\Lambda_1}]$ | $2(q-1)$ |
| $(\Pi_1, \Pi_2)$ | $0$ | $0$ |
| $\sigma \in \mathbf{Irr}(M)$ | $\mathbf{r}_{\mathscr{J}} \circ \pi_-(\sigma)$ | dimension |
| $(\mu \cdot \mathrm{St}, \mu \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [\,\mathrm{St}_{\mathrm{GL}(2,q)},\, \mathrm{St}_{\mathrm{GL}(2,q)}]$ | $q^2$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_u \cdot \mathrm{St})$ | $0$ | $0$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_t \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [\pi_{\Lambda_0'}, \pi_{\Lambda_0'^{-1}}]_\pm$ | $(q-1)^2/2$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \Pi_1)$ | $\widetilde{\mu} \cdot [\pi_{\Lambda_1'}, \pi_{\Lambda_1'^{-1}}]$ | $(q-1)^2$ |
| $(\Pi_1, \Pi_1)$ | $[\,\mathrm{St}_{\mathrm{GL}(2,q)}, \pi_{\Lambda_1}] + [\pi_{\Lambda_1},\, \mathrm{St}_{\mathrm{GL}(2,q)}]$ | $2q(q-1)$ |
| $(\Pi_1, \Pi_2)$ | $[\pi_{\Lambda_a}, \pi_{\Lambda_b}] + [\pi_{\Lambda_b}, \pi_{\Lambda_a}]$ | $2(q-1)^2$ |

Table 4.4.: Paramodular restriction of $\pi_+(\sigma)$ and $\pi_-(\sigma)$ in the endoscopic $L$-packet attached to preunitary generic depth zero irreducible admissible representations $\sigma$ of $M$. The index is determined by the parity of $\xi_t(\varpi) = \pm 1$.

| $\sigma \in \mathbf{Irr}(M)$ | $\mathbf{r}_{\mathscr{J}} \circ r(\sigma)$ | dimension |
|---|---|---|
| $(\mu_1 \times \mu_2, \mu_3 \times \mu_4)$ | $\widetilde{\mu_1}^{-1} \cdot [\widetilde{\mu_1} \times \widetilde{\mu_3}, \widetilde{\mu_1} \times \widetilde{\mu_4}]$ $+\widetilde{\mu_1}^{-1} \cdot [\widetilde{\mu_1} \times \widetilde{\mu_4}, \widetilde{\mu_1} \times \widetilde{\mu_3}]$ | $2(q+1)^2$ |
| $(\mu_1 \times \mu_2, \mu \cdot \mathrm{St})$ | $\widetilde{\mu_1}^{-1} \cdot [\widetilde{\mu_1} \times \widetilde{\mu}, \widetilde{\mu_1} \times \widetilde{\mu}]$ | $(q+1)^2$ |
| $(\mu_1 \times \mu_2, \mu \cdot \mathbf{1})$ | $\widetilde{\mu_1}^{-1} \cdot [\widetilde{\mu_1} \times \widetilde{\mu}, \widetilde{\mu_1} \times \widetilde{\mu}]$ | $(q+1)^2$ |
| $(\mu_1 \times \mu_2, \mu_1 \cdot \Pi_1)$ | $0$ | $0$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [\mathbf{1}, \mathrm{St}_{\mathrm{GL}(2,q)}] + \widetilde{\mu} \cdot [\mathrm{St}_{\mathrm{GL}(2,q)}, \mathbf{1}]$ | $2q$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_u \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [\mathbf{1}, \mathrm{St}_{\mathrm{GL}(2,q)}] + \widetilde{\mu} \cdot [\mathrm{St}_{\mathrm{GL}(2,q)}, \mathbf{1}]$ | $2q$ |
| $(\mu \cdot \mathrm{St}, \mu\xi_t \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [1 \times \lambda_0, 1 \times \lambda_0]_\pm - \widetilde{\mu} \cdot [\pi_{\Lambda_0'}, \pi_{\Lambda_0'^{-1}}]_\pm$ | $2q$ |
| $(\mu \cdot \mathrm{St}, \mu \cdot \Pi_1)$ | $-\widetilde{\mu} \cdot [\pi_{\Lambda_1'}, \pi_{\Lambda_1'^{-1}}]$ | $-(q-1)^2$ |
| $(\Pi_1, \Pi_1)$ | $[\pi_{\Lambda_1}, \mathbf{1} - \mathrm{St}] + [\mathbf{1} - \mathrm{St}, \pi_{\Lambda_1}]$ | $-2(q-1)^2$ |
| $(\Pi_1, \Pi_2)$ | $-[\pi_{\Lambda_a}, \pi_{\Lambda_b}] - [\pi_{\Lambda_b}, \pi_{\Lambda_a}]$ | $-2(q-1)^2$ |
| $(\mu \cdot \mathbf{1}, \mu \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [\mathrm{St}_{\mathrm{GL}(2,q)}, \mathrm{St}_{\mathrm{GL}(2,q)}] + \widetilde{\mu} \cdot [\mathbf{1}, \mathbf{1}]$ | $q^2+1$ |
| $(\mu \cdot \mathbf{1}, \mu\xi_u \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [\mathrm{St}_{\mathrm{GL}(2,q)}, \mathrm{St}_{\mathrm{GL}(2,q)}] + \widetilde{\mu} \cdot [\mathbf{1}, \mathbf{1}]$ | $q^2+1$ |
| $(\mu \cdot \mathbf{1}, \mu\xi_t \cdot \mathrm{St})$ | $\widetilde{\mu} \cdot [1 \times \lambda_0, 1 \times \lambda_0]_\mp + \widetilde{\mu} \cdot [\pi_{\Lambda_0'}, \pi_{\Lambda_0'^{-1}}]_\pm$ | $q^2+1$ |
| $(\mu \cdot \mathbf{1}, \mu \cdot \mathbf{1})$ | $\widetilde{\mu} \cdot [\mathbf{1}, \mathrm{St}_{\mathrm{GL}(2,q)}] + \widetilde{\mu} \cdot [\mathrm{St}_{\mathrm{GL}(2,q)}, \mathbf{1}]$ | $2q$ |
| $(\mu \cdot \mathbf{1}, \mu\xi_u \cdot \mathbf{1})$ | $\widetilde{\mu} \cdot [\mathbf{1}, \mathrm{St}_{\mathrm{GL}(2,q)}] + \widetilde{\mu} \cdot [\mathrm{St}_{\mathrm{GL}(2,q)}, \mathbf{1}]$ | $2q$ |
| $(\mu \cdot \mathbf{1}, \mu\xi_t \cdot \mathbf{1})$ | $\widetilde{\mu} \cdot [1 \times \lambda_0, 1 \times \lambda_0]_\pm - \widetilde{\mu} \cdot [\pi_{\Lambda_0'}, \pi_{\Lambda_0'^{-1}}]_\pm$ | $2q$ |
| $(\mu \cdot \mathbf{1}, \mu \cdot \Pi_1)$ | $\widetilde{\mu} \cdot [\pi_{\Lambda_1'}, \pi_{\Lambda_1'^{-1}}]$ | $(q-1)^2$ |

Table 4.5.: Paramodular restriction $\mathbf{r}_{\mathscr{J}} \circ r(\sigma)$ of the endoscopic lift $r(\sigma)$ in the Grothendieck group for depth zero irreducible admissible representations $\sigma$ of $M$. The index is determined by $\xi_t(\varpi) = \pm 1$.

**Notation 4.12** (Tables 4.4 and 4.5)**.** Irreducible representations $\sigma$ of $M$ are denoted as before. Irreducible representations of $(\mathrm{GL}(2,q)^2)^0$ are denoted as in Lemma A.6. The pair of characters $(\Lambda_a, \Lambda_b)$ is an arbitrary solution of $\Lambda_a\Lambda_b = \Lambda_1$ and $\Lambda_a\Lambda_b^q = \Lambda_2$. For every character $\Lambda$ of $\mathbb{F}_{q^2}^\times$ with $\Lambda^{q+1} = 1$ let $\Lambda'$ be an arbitrary solution of $(\Lambda')^{q-1} = \Lambda$.

**Theorem 4.13.** *Let $\sigma$ be a generic irreducible admissible representation of $M$. If $\sigma$ has depth zero, then the paramodular restriction $\mathbf{r}_{\mathscr{J}} \circ r(\sigma)$ of the endoscopic lift $r(\sigma)$ is the virtual $(\mathrm{GL}(2,q)^2)^0$-representation given by Table 4.5. If $\sigma$ has positive depth, then the paramodular restriction of the endoscopic lift of $\sigma$ is zero.*

*Proof.* The proof is completely analogous to Thm. 4.9. □

**Corollary 4.14.** *For a generic preunitary irreducible representations $\sigma$ of $M$ let $\pi = \pi_\pm(\sigma)$ be in the local L-packet attached to $\sigma$. Then $\sigma$ is depth zero or $\mathscr{K}_M$-spherical if and only if $\pi$ is depth zero or $\mathscr{K}_G$-spherical, respectively.*

*$\pi$ has non-zero hyperspecial parahoric reduction if and only if $\pi$ admits non-zero invariants under the modified principle congruence subgroup.*

*Proof.* The representation $\pi$ has depth zero if and only if it admits non-zero parahoric restriction with respect to at least one maximal parahoric subgroup. Up to conjugacy, the only parahoric subgroups are the hyperspecial parahoric and the paramodular subgroup. The statement is implied by Thms. 4.7 and 4.13.

A representation is $\mathscr{K}$-spherical if its hyperspecial parahoric restriction admits a trivial constituent. This occurs exactly for $\sigma = (\mu_1 \times \mu_2, \mu_3 \times \mu_4)$ with unramified characters $\mu_i$ of $F^\times$, compare Table 4.2.

The space of invariants under the modified principal congruence subgroup $\mathscr{K}_G'$ in $\pi$ is the subspace of $\{\mathrm{diag}(1,1,*,*)\}$-invariants in $\mathbf{r}_{\mathscr{K}}(\pi)$. But for every occuring representation $\pi$, character theory gives

$$\dim \mathrm{Hom}_{\mathrm{diag}(1,1,*,*)}(\pi, 1) = \sum_{a\in\mathbb{F}_q^\times} \mathrm{tr}\, \mathbf{r}_{\mathscr{K}}(\pi)(\mathrm{diag}(1,1,a,a)) > 0.$$

□

Preservation of depth zero under the endoscopic lift for generic pre-unitary irreducible representations of $M$ is a special case of depth preservation under the local theta correspondence [Pan02]. It complies with depth preservation under the local Langlands correspondence [ABPS].

## 4.3. Matchings

Fix $\mathbf{G} = \mathrm{GSp}(4)$ and its unique proper elliptic endoscopic group

$$\mathbf{M} = \mathrm{GL}(2)^2/\Delta\, \mathrm{GL}(1).$$

Let $G, M$ be the groups of $F$-rational points with center $Z_G$ and $Z_M$. In order to identify the depth zero cuspidal irreducible constituents in the endoscopic character lift we need certain character formulas on the anisotropic conjugacy classes. To this end we determine three pairs of matching functions, which determine character identities by the following Lemma.

**Lemma 4.15.** *Let $\mathscr{P} \subseteq G$ be an arbitrary parahoric subgroup with pro-unipotent radical $\mathscr{P}^+$. For a conjugacy class $\mathrm{C} \subseteq \mathscr{P}/\mathscr{P}^+$ let $\mathrm{char}_C \in C_c^\infty(G)$ be the indicator function of the preimage $C = p^{-1}(\mathrm{C})$ under the projection $p : \mathscr{P} \to \mathscr{P}/\mathscr{P}^+$. Then we have*

$$\mathrm{tr}(\mathbf{r}_{\mathscr{P}}(\pi); \mathrm{C}) = \mathrm{vol}(C)^{-1}\chi_\pi(\mathrm{char}_C). \tag{4.13}$$

*Proof.* This is clear by unraveling the definitions. □

We show two matchings $f \to f^M$ between functions with support in the maximal tori of elliptic case I [Wei09a, 4.4.2] for the unramified quadratic extension $L/F$. Together with a third matching, this will provide the necessary character identities (4.20), (4.21) and (4.22).

### 4.3.1. Maximal tori of unramified elliptic case I

Let $L$ be the unramified quadratic field extension of $F$. Fix $\zeta \in \mathfrak{o}_F^\times$ such that $L$ is the splitting field of the irreducible Artin-Schreier polynomial[2] $X^2 - X - \zeta$. As an $F$-vector space, $L \cong F[X]/(X^2 - X - \zeta)$ is generated by 1 and $X$. The image of the regular representation

$$\phi_\zeta : L^\times \to \mathrm{GL}(2, F), \quad a + bX \mapsto \begin{pmatrix} a & b\zeta \\ b & a+b \end{pmatrix}, \tag{4.14}$$

is an anisotropic torus of $\mathrm{GL}(2, F)$.

Let the anisotropic maximal torus $T_M$ in $M$ be the image of the canonical morphism

$$(\phi_\zeta, \phi_\zeta) : L^\times \times L^\times \to \mathrm{GL}(2, F) \times \mathrm{GL}(2, F) \to M$$

Every other $M$-torus that is isomorphic to $T_M$ is conjugate to $T_M$ over $M$.

The non-trivial Galois automorphism of $L$ is $a + bX \mapsto \overline{a + bX} = a + b - bX$. The identity $(a+bX)(\overline{a + bX}) = a(a+b) - b^2\zeta = \det \phi_\zeta(a+bX)$ gives rise to an embedding $(\phi_\zeta, \phi_\zeta) : (L^\times \times L^\times)^0 \longrightarrow (\mathrm{GL}(2, F)^2)^0$, where the exponent 0 indicates the subset of pairs with equal norm or determinant. With the embedding

$$\phi : (\mathrm{GL}(2, F)^2)^0 \longrightarrow G, \quad (x, y) \longmapsto \begin{pmatrix} x_{11} & & x_{12} & \\ & y_{11} & & y_{12} \\ x_{21} & & x_{22} & \\ & y_{21} & & y_{22} \end{pmatrix}, \tag{4.15}$$

the maximal torus $T_G$ is the image of $(L^\times \times L^\times)^0$ under $\phi \circ (\phi_\zeta, \phi_\zeta)$. The alternative regular representation

$$\phi'_\zeta : L^\times \to \mathrm{GL}(2, F), \quad a + bX \mapsto \begin{pmatrix} a & b\zeta\varpi^{-1} \\ b\varpi & a+b \end{pmatrix} \tag{4.16}$$

gives rise to a different torus $T'_G$ as the image of $(L^\times \times L^\times)^0$ under $\phi \circ (\phi'_\zeta, \phi_\zeta)$. The image of $(\mathfrak{o}_L^\times \times \mathfrak{o}_L^\times)^0$ is $T_G \cap \mathscr{K}_G$ or $T'_G \cap \mathscr{J}$, respectively. The tori $T_G$ and $T'_G$ generate the two $G$-conjugacy classes of embeddings of $(L^\times \times L^\times)^0$ [Wei09a, Lemma 6.1].

[2] This is a small deviation from [Wei09a, §6.2] in order to include even residue characteristic.

The canonical map (4.6) over $L$

$$^{(L^\times \times L^\times)}\!/_{\{(t,t^{-1})\,|\,t\in F^\times\}} \longrightarrow (L^\times \times L^\times)^0, \quad (t,t') \longmapsto (tt', t\overline{t'}). \tag{4.17}$$

defines admissible embeddings $T_M \to T_G$ and $T_M \to T'_G$ [Wei09a, §6.2-3].

The endoscopic matching condition (4.3) for $\gamma_M \in T_M$ becomes

$$SO_{\gamma_M}(f^M) = \Delta(\gamma_G, \gamma_M) O^\kappa_{\gamma_G}(f) \quad \text{for} \quad O^\kappa_{\gamma_G} = O_{\gamma_G}(f) - O_{\gamma'_G}(f) \tag{4.18}$$

with $\gamma_G \in T_G$ and $\gamma'_G \in T'_G$ stably related to $\gamma_M$ [Wei09a, 6.2.1].

In the following sections, let $\mathscr{K}_M = \mathbf{M}(\mathfrak{o}_F)$ and $\mathscr{K}_G = \mathbf{G}(\mathfrak{o}_F)$ be the standard hyperspecial parahoric subgroups. Fix Haar measures on $M$, $G$, $T_M$, $T_G$, and $T'_G$, such that the volume of $\mathscr{K}_G$, $\mathscr{K}_M$, and $\mathscr{K}_M \cap T_M$ is one and such that the admissible embeddings $T_M \to T_G$ and $T_M \to T'_G$ preserve the measure. These groups are reductive and therefore unimodular, so there are well-defined quotient measures on $T_G\backslash G$, $T'_G\backslash G$ and $T_M\backslash M$.

#### 4.3.2. First matching

Fix $\alpha, \beta \in \mathbb{F}^\times_{q^2} \cong \mathfrak{o}_L/\mathfrak{p}_L$ with $\alpha, \beta, \alpha\beta, \alpha\beta^q \notin \mathbb{F}^\times_q$.

Let $\mathrm{L}_0 = \mathrm{L}_0(\alpha\beta, \alpha\beta^q)$ be the conjugacy class[3] in $\mathbf{G}(q)$ of elements stably conjugate to $\mathrm{diag}(\alpha\beta, \alpha\beta^q, \alpha^q\beta^q, \alpha^q\beta) \in \mathbf{G}(q^2)$ and let $L_0 = p^{-1}(\mathrm{L}_0)$ be its preimage under the projection $p : \mathscr{K}_G \twoheadrightarrow \mathscr{K}_G/\mathscr{K}_G^+$.

**Lemma 4.16.** *Let $f_1 \in C_c^\infty(G)$ be the indicator function of $L_0$. Fix a strongly regular semisimple $\gamma_G \in G$ that comes from an admissible embedding of $M$. The orbital integral is $O_{\gamma_G}(f_1) = 1$ if $\gamma_G$ is $G$-conjugate to an element of $L_0$, and it is $O_{\gamma_G}(f_1) = 0$ else.*

*Proof.* If $\gamma_G$ is conjugate to an element of $L_0$, we can assume $\gamma_G \in L_0$, since orbital integrals are conjugation invariant. The eigenvalues of $\gamma_G \in \mathscr{K}_G$ are integers in the unramified quadratic field extension $L$, so the centralizer $C_G(\gamma_G)$ is isomorphic to $(L^\times \times L^\times)^0$ of unramified elliptic case I. Up to conjugation in $\mathscr{K}_G$, we can assume that $\gamma_G \in T_G$, so the centralizer is $C_G(\gamma_G) = T_G$.

We now claim that for $t \in G$, we have $t^{-1}\gamma_G t \in L_0$ if and only if $t \in Z_G\mathscr{K}_G$. Indeed, $t = k_1\,\mathrm{diag}(\varpi^{n_1}, \varpi^{n_2}, \varpi^{n_0-n_1}, \varpi^{n_0-n_2})k_2$ for certain $n_i \in \mathbb{Z}$ and $k_1, k_2 \in \mathscr{K}_G$ by Cartan decomposition (3.15). Since $L_0$ is preserved under $\mathscr{K}_G$-conjugacy it is

[3] For odd $q$, this is Shinoda's $\mathrm{L}_0$ [Shi82]. For even $q$, it is a twist of Enomoto's $B_4$ [Eno72].

sufficient to look at the case $k_2 = 1$. Up to $\mathscr{K}_G$-conjugacy of $\gamma_G$ we can assume $k_1^{-1}\gamma_G k_1 \in T_G$ for the embedding (4.15), so

$$k_1^{-1}\gamma_G k_1 = \begin{pmatrix} a & & b\zeta & \\ & a' & & b'\zeta \\ b & & a+b & \\ & b' & & a'+b' \end{pmatrix}$$

for integers $a, b, a', b' \in \mathfrak{o}_F$. Since the image of $k_1^{-1}\gamma_G k_1$ in $\mathscr{K}_G/\mathscr{K}_G^+$ does not admit eigenvalues in $\mathbb{F}_q^\times$, we have $b, b' \in \mathfrak{o}_F^\times$. If $t^{-1}\gamma_G t \in L_0$, then $t^{-1}\gamma_G t$ must have integer matrix entries, so $n_1 = n_2 = n_0/2$ implies $t \in Z_G\mathscr{K}_G$. Conversely, it is clear that $t^{-1}\gamma_G t \in L_0$ for every $t \in Z_G\mathscr{K}_G$, because $L_0$ is a conjugacy class of $\mathscr{K}_G/\mathscr{K}_G^+$.

This implies that the support of $t \mapsto f_1(t^{-1}\gamma_G t)$ is $Z_G\mathscr{K}_G = T_G\mathscr{K}_G$. Now we have

$$O_{\gamma_G}(f_1) = \int_{T_G\backslash G} f_1(t^{-1}\gamma_G t)\,\mathrm{d}t = \int_{T_G\backslash G} \mathrm{char}_{T_G\backslash T_G\mathscr{K}_G}(t)\,\mathrm{d}t = 1.$$

If $\gamma_G$ is not $G$-conjugate to an element of $L_0$, the orbital integral is clearly zero. □

Let $\mathrm{E}_M(\alpha,\beta) \subseteq \mathbf{M}(q)$ be the image of $\mathrm{E}(\alpha) \times \mathrm{E}(\beta) \subseteq \mathrm{GL}(2,q)^2$, where $\mathrm{E}(\alpha)$ denotes the anisotropic conjugacy class in $\mathrm{GL}(2,q)$ with eigenvalues $\alpha, \alpha^q$ as in Section A.1. Denote by $E_M(\alpha,\beta)$ the preimage of $\mathrm{E}_M(\alpha,\beta)$ under the projection $\mathscr{K}_M \twoheadrightarrow \mathscr{K}_M/\mathscr{K}_M^+$.

**Lemma 4.17.** *Let $f_1^M \in C_c^\infty(M)$ be the indicator function of $E_M(\alpha,\beta) \sqcup E_M(\beta,\alpha)$. Let $\gamma_M \in M$ be a strongly regular semisimple element. The stable orbital integral is $SO_{\gamma_M}(f_1^M) = 1$ if $\gamma_M$ is $M$-conjugate to an element of $E_M(\alpha,\beta) \sqcup E_M(\beta,\alpha)$, else it is $SO_{\gamma_M}(f_1^M) = 0$.*

*Proof.* For $\gamma_M \in E_M(\alpha,\beta)$, the eigenvalues generate of $\gamma_M$ generate $L$, so the centralizer of $\gamma_M$ conjugate to the torus $T_M$.

For every $s \in M$, we claim that $s^{-1}\gamma_M s \in E_M(\alpha,\beta)$ if and only if $s \in T_M\mathscr{K}_M$ and that $s^{-1}\gamma_M s \notin \mathscr{K}_M$ otherwise. By Cartan decomposition

$$s = k_1\mathbf{t}k_2 \mod \{(x, x^{-1}) \mid x \in F^\times\}$$

for $\mathbf{t} = (\mathrm{diag}(\varpi^{n_1}, \varpi^{n_2}), \mathrm{diag}(\varpi^{n_3}, \varpi^{n_4}))$ with $n_i \in \mathbb{Z}$ and $k_1, k_2 \in \mathscr{K}_M$. Let $k_2 = 1$ without loss of generality. By replacing $\gamma_M$ with a $\mathscr{K}_M$-conjugate we can assume

$$k_1^{-1}\gamma_M k_1 = \left(\left(\begin{smallmatrix} a_1 & b_1\zeta \\ b_1 & a_1+b_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_1 & b_1\zeta \\ b_1 & a_1+b_1 \end{smallmatrix}\right)\right) \in T_M \cap \mathscr{K}_M$$

for $a_1, b_1, a_2, b_2 \in \mathfrak{o}$. Since $\alpha, \beta \notin \mathbb{F}_q^\times$, we have $b_i \in \mathfrak{o}^\times$ as before.

If $s^{-1}\gamma_M s \in \mathscr{K}_M$, then $n_1 = n_2$ and $n_3 = n_4$, therefore $s \in Z_M\mathscr{K}_M \subseteq T_M\mathscr{K}_M$. It is clear that the orbit of $\gamma_M$ under the conjugation action with $T_M\mathscr{K}_M$ preserves $E_M(\alpha,\beta)$. The orbital integral for $\gamma_M \in E_M(\alpha,\beta)$ is therefore

$$O_{\gamma_M}(f^M) = \int_{T_M\backslash M} f_1^M(s^{-1}\gamma_M s)\,\mathrm{d}s = \int_{T_M\backslash M} \mathrm{char}_{T_M\backslash T_M\mathscr{K}_M}(s) = 1.$$

The elements of $E_M(\alpha, \beta)$ have stable conjugation orbit, so $SO_{\gamma_M}(f_1^M) = O_{\gamma_M}(f_1^M)$ is the stable orbital integral. When the stable $G$-conjugation orbit of $\gamma_M$ is disjoint to $E_M(\alpha, \beta)$, the stable orbital integral is zero. □

**Lemma 4.18.** *For semisimple strongly $G$-regular $\gamma_M \in E_M$ and $\gamma_G \in T_G$ stably related by the admissible embedding $T_M \to T_G$, the transfer factor is $\Delta(\gamma_G, \gamma_M) = 1$.*

*Proof.* Without loss of generality let $\gamma_G \in T_G$ for the $T_G$ constructed above. Let $(x, x') \in (L^\times \times L^\times)^0$ be the preimage of $\gamma_G \in T_G$ under the embedding (4.15). The character $\xi_{L/F}$ attached to the unramified quadratic field extension $L/F$ by class field theory is the unramified quadratic character. By assumption, the image of $x - \overline{x} \in \mathfrak{o}_L$ under the projection $\mathfrak{o}_L \to \mathfrak{o}_L/\mathfrak{p}_L$ is $\alpha\beta - \alpha\beta^q \neq 0$. Since $x, x', x - \overline{x}, x' - \overline{x'}$ are invertible in $\mathfrak{o}_L^\times$, every factor in the following expression [Wei09a, Cor. 8.1] is trivial:

$$\Delta(\gamma_M, \gamma_G) = \frac{\xi_L(x - \overline{x})\xi_L(x' - \overline{x'}) \cdot |x - \overline{x}| \cdot |x' - \overline{x'}|}{|xx'|} = 1. \tag{4.19}$$

□

**Proposition 4.19.** *The pair $(f_1, f_1^M)$ satisfies the matching condition* (4.3).

*Proof.* For a semisimple strongly $G$-regular $\gamma_M \in E_M(\alpha, \beta) \sqcup E_M(\beta, \alpha) \subseteq G$, the stable orbital integral is $SO_{\gamma_M}(f_1^M) = 1$ by Lemma 4.17. There are two conjugacy classes in $G$ stably related to $\gamma_M$ with representatives $\gamma_G \in L_0 \cap T_G$ and $\gamma'_G \in T'_G$. Since $\gamma'_G$ is not conjugate to an element of $L_0$, only $\gamma_G$ gives a non-zero orbital integral $O_{\gamma_G}(f_1) = 1$ by Lemma 4.16. This implies

$$O^\kappa_{\gamma_G} = O_{\gamma_G}(f_1) - O_{\gamma'_G}(f_1) = 1 - 0 = 1.$$

The transfer factor is $\Delta(\gamma_M, \gamma_G) = 1$ by Lemma 4.18, so (4.18) holds.

If $\gamma_M$ is not conjugate to an element of $E_M(\alpha, \beta) \sqcup E_M(\beta, \alpha)$, the orbital integrals are all zero. □

**Corollary 4.20.** *Let $\alpha, \beta \in \mathbb{F}_{q^2}^\times$ with $\alpha, \beta, \alpha\beta, \alpha\beta^q \notin \mathbb{F}_q^\times$ be arbitrary. The hyperspecial parahoric restriction of the endoscopic lift of any admissible representation $\sigma$ of $M$ satisfies*

$$\mathrm{tr}(\mathbf{r}_{\mathscr{K}_G} \circ r(\sigma); \mathrm{L}_0(\alpha\beta, \alpha\beta^q)) = \mathrm{tr}(\mathbf{r}_{\mathscr{K}_M}(\sigma); \mathrm{E}(\alpha, \beta)) + \mathrm{tr}(\mathbf{r}_{\mathscr{K}_M}(\sigma); \mathrm{E}(\beta, \alpha)). \tag{4.20}$$

*Proof.* Lemma 4.15 implies

$$\mathrm{tr}(\mathbf{r}_{\mathscr{K}_G} \circ r(\sigma); \mathrm{L}_0(\alpha\beta, \alpha\beta^q)) = \mathrm{vol}(L_0(\alpha\beta, \alpha\beta^q))^{-1} \chi_{r(\sigma)}(f_1) = (q-1)(q+1)^2 \chi_{r(\sigma)}(f_1)$$

and $\mathrm{tr}(\mathbf{r}_{\mathscr{K}_G}(r(\sigma)); E_M(\alpha, \beta)) = (q-1)(q+1)^2 \chi_\sigma(f_1^M)$. By definition of the endoscopic character lift we have $\chi_{r(\sigma)}(f_1) = \chi_\sigma(f_1^M)$. □

### 4.3.3. Second matching

The second matching is the analogue of the first matching with respect to the standard paramodular subgroup $\mathscr{J} = \mathscr{P}_{\{0,2\}} \subseteq G$. Fix $\alpha, \beta \in \mathbb{F}_{q^2}^\times$ with $\alpha, \beta, \alpha\beta, \alpha\beta^q \notin \mathbb{F}_q^\times$ as before. Let $E_G = E_G(\alpha\beta, \alpha\beta^q) \subseteq \mathscr{J}$ be the preimage of $\mathrm{E}(\alpha\beta) \times \mathrm{E}(\alpha\beta^q) \subseteq (\mathrm{GL}(2,q)^2)^0$ under the embedding (4.12) and set

$$f_2 \in C_c^\infty(G), \quad f_2 = -\operatorname{vol}(\mathscr{J})^{-1} \operatorname{char}_{E_G(\alpha\beta,\alpha\beta^q)}.$$

**Lemma 4.21.** *Let $\gamma'_G \in G$ be strongly regular semisimple. The orbital integral of $f_2$ is $O_{\gamma'_G}(f_2) = -1$ if $\gamma'_G$ is $G$-conjugate to an element of $E_G$, and zero else.*

*Proof.* Without loss of generality, let $\gamma'_G \in E_G$. Up to conjugation in $G$, the centralizer of $\gamma_G$ is $G$-conjugate to the torus $T'_G$, so we can assume $\gamma_G \in T'_G \cap E_G$.

For $t \in G$ we claim that $t^{-1}\gamma_G t \in E_G(\alpha\beta, \alpha\beta^q)$ if and only if $t \in Z_G\mathscr{J}$. Indeed, by (3.16) $t$ is either $t = k_1\mathbf{t}k_2$ or $t = k_1u_1\mathbf{t}k_2$ for $\mathbf{t} = \operatorname{diag}(\varpi^{n_1}, \varpi^{n_2}, \varpi^{n_0-n_1}, \varpi^{n_0-n_2})$ with $n_i \in \mathbb{Z}$ and $k_1, k_2 \in \mathscr{J}$. Since $E_G$ is preserved under $\mathscr{J}$-conjugacy, it is sufficient to assume $k_2 = 1$. Since

$$\gamma_G = \begin{pmatrix} a & & \varpi^{-1}b\zeta & \\ & a' & & b'\zeta \\ \varpi b & & a+b & \\ & b' & & a'+b' \end{pmatrix}$$

for integers $a, b, a', b' \in \mathfrak{o}_F$ does not split under projection to $\mathscr{J}/\mathscr{J}^+$, we must have $b, b' \in \mathfrak{o}_F^\times$, so for $\mathbf{t}^{-1}\gamma'_G\mathbf{t} \in \mathscr{J}$ it is necessary that $\mathbf{t} \in Z_G$. Conjugation by $u_1$ preserves $\mathscr{J}$, but maps $E_G(\alpha\beta, \alpha\beta^q)$ to $E_G(\alpha\beta^q, \alpha\beta)$. Therefore $t^{-1}\gamma'_G t \in E_G(\alpha\beta, \alpha\beta^q)$ if and only if $t = \varpi^n k$ for $n \in \mathbb{Z}$ and $k \in \mathscr{J}$ if and only if $t \in Z_G\mathscr{J}$. Since $Z_G\mathscr{J} = T'_G\mathscr{J}$, the orbital integral is

$$O_{\gamma'_G}(f_2) = \int_{T'_G\backslash G} f_2(t^{-1}\gamma'_G t)\,\mathrm{d}t = -\operatorname{vol}(\mathscr{J})^{-1} \int_{T'_G\backslash G} \operatorname{char}_{T'_G\backslash T'_G\mathscr{J}}(t) = -\operatorname{vol}(T'_G \cap \mathscr{J})^{-1} = -1.$$

The last equation follows from $\operatorname{vol}(T'_G \cap \mathscr{J}) = \operatorname{vol}(T'_G(\mathfrak{o}_G)) = \operatorname{vol}(T_M \cap \mathscr{K}_M)$, since the admissible embeding preserves the measure.

If $\gamma'_G$ is not conjugate to an element of $E_G$, the orbital integral is zero. □

Let $f_2^M = f_1^M \in C_c^\infty(M)$ be the indicator function of $E_M(\alpha, \beta) \sqcup E_M(\beta, \alpha)$ as before.

**Proposition 4.22.** *The pair $(f_2, f_2^M)$ satisfies the matching condition* (4.3)*.*

*Proof.* Fix some semisimple strongly $G$-regular $\gamma_M \in E_M$. Up to conjugation in $\mathscr{K}_M$ we can assume $\gamma_M \in E_M \cap T_M$. Let $\gamma_G \in L_0 \cap T_G$ and $\gamma'_G \in E_G \cap T_G$ be semisimple strongly regular and stably related to $\gamma_M$. The orbital integral $O_{\gamma_M}(f_2^M) = SO_{\gamma_M}(f_2^M) = 1$ has been determined in Lemma 4.17. The $\kappa$-orbital integral is

$$O^\kappa_{\gamma_G}(f_2) = O_{\gamma_G}(f_2) - O_{\gamma'_G}(f_2) = 0 - (-1) = 1$$

by Lemma 4.21, since $\gamma_G \in T_G$ is not conjugate to an element in $E_G$. The transfer factor $\Delta(\gamma_G, \gamma_M) = 1$ has been determined in Lemma 4.18, so (4.18) holds.

If $\gamma_M$ is not conjugate to an element in $E_M$, then the orbital integrals are zero. □

**Corollary 4.23.** *Fix $\alpha, \beta \in \mathbb{F}_{q^2}^\times$ with $\alpha, \beta, \alpha\beta, \alpha\beta^q \notin \mathbb{F}_q^\times$. Then the paramodular restriction $\mathbf{r}_{\mathscr{J}} \circ r(\sigma)$ of the endoscopic lift of every admissible representation $\sigma$ of $M$ satisfies*

$$\mathrm{tr}(\mathbf{r}_{\mathscr{J}} \circ r(\sigma); \mathrm{E}_G(\alpha\beta, \alpha\beta^q)) = -\,\mathrm{tr}(\mathbf{r}_{\mathscr{K}_M}(\sigma); \mathrm{E}_M(\alpha, \beta)) - \mathrm{tr}(\mathbf{r}_{\mathscr{K}_M}(\sigma); \mathrm{E}_M(\beta, \alpha)). \tag{4.21}$$

*Proof.* Lemma 4.15 implies

$$\mathrm{tr}(\mathbf{r}_{\mathscr{J}} \circ r(\sigma); \mathrm{E}_G(\alpha\beta, \alpha\beta^q)) = \tfrac{\mathrm{vol}(J)}{\mathrm{vol}(E_G)} \chi_{r(\sigma)}(-f_2) = -(q-1)(q+1)^2 \chi_{r(\sigma)}(f_2)$$

and by the proof of Corollary 4.20

$$\mathrm{tr}(\mathbf{r}_{\mathscr{K}_G}(r(\sigma)); E_M(\alpha, \beta)) + \mathrm{tr}(\mathbf{r}_{\mathscr{K}_G}(r(\sigma)); E_M(\beta, \alpha)) = (q-1)(q+1)^2 \chi_\sigma(f_2^M).$$

The endoscopic character lift gives $\chi_{r(\sigma)}(f_1) = \chi_\sigma(f_1^M)$. □

### 4.3.4. Third matching

Fix $\tau \in \mathbb{F}_{q^4}^\times - \mathbb{F}_{q^2}^\times$ with $\tau^{(q^2+1)(q-1)} = 1$. Let $\mathrm{K}_0(\tau)$ be the conjugacy class[4] of elements in $G$ which are stably conjugate to $\mathrm{diag}(\tau, \tau^q, \tau^{q^2}, \tau^{q^3})$ and let $K_0(\tau) = p^{-1}(\mathrm{K}_0(\tau))$ be the preimage of $\mathrm{K}_0(\tau)$ under the projection $p : \mathscr{K}_G \to \mathscr{K}_G/\mathscr{K}_G^+$.

**Lemma 4.24.** *Let $f_3 \in C_c^\infty(G)$ be the characteristic function of $K_0(\tau)$ and let $f_3^M \in C_c^\infty(M)$ be zero. Then $(f_3, f_3^M)$ satisfies the matching condition* (4.3)*.*

*Proof.* Every element in $K_0(\tau)$ has an eigenvalue that generates the unramified field extension $E/F$ of order four. For every semisimple regular $\gamma_G$ that comes from an admissible embedding, the eigenvalues generate at most quadratic field extensions [Wei09a, 4.4.2]. Therefore, $\gamma_G$ is not conjugate to an element in the support of $f_3$. □

**Corollary 4.25.** *For every such $\tau$ and every admissible representation $\sigma$ of $M$*

$$\mathrm{tr}(\mathbf{r}_{\mathscr{K}_G} \circ r(\sigma); \mathrm{K}_0(\tau)) = 0. \tag{4.22}$$

*Proof.* This is analogous to Corollary 4.20. □

[4]For odd $q$, this is the conjugacy class denoted $\mathrm{K}_0$ by Shinoda [Shi82]. For even $q$, it is a twist of Enomoto's class $B_5$ [Eno72].

## 4.4. The anisotropic theta lift

For the proof of Thm. 4.11, we need to find the paramodular restriction of the anisotropic theta-lift $\theta_-(\mathrm{St}_{\mathrm{GL}(2,F)}, \xi_t\,\mathrm{St}_{\mathrm{GL}(2,F)})$ with tamely ramified quadratic character $\xi_t$. The only two candidates are $[\pi_{\Lambda_0'}, \pi_{\Lambda_0'^{-1}}]_\pm$, where $\Lambda_0'$ is a regular character of $\mathbb{F}_{q^2}^\times$ such that $(\Lambda_0')^{q-1}$ is the non-trivial quadratic character. In Prop. 4.32 below we distinguish these representations by their character values on the unipotent conjugacy classes. Eq. (A.2) implies then that the sign is given by $\xi_t(\varpi)$.

At first, we briefly review the anisotropic theta lift [Wei09a, §4.12]. Let $D$ be the unique quaternion division algebra over the local non-archimedean number field $F$ with $F$-linear conjugation map $D \to D$, $d \mapsto \overline{d}$. This defines the (surjective) reduced norm and reduced trace homomorphisms

$$\mathrm{nrd}_{D/F} : D^\times \to F^\times,\ d \mapsto d \cdot \overline{d}, \qquad \mathrm{trd}_{D/F} : D \to F,\ d \mapsto d + \overline{d}.$$

The natural bilinear form $B(x,y) = \frac{1}{2}\,\mathrm{trd}_{D/F}(x\overline{y})$ on $D \times D$ is normalized so that $B(x,x) = x\bar{x} = \mathrm{nrd}_{D/F}(x)$. For every symmetric $2 \times 2$ matrix $T = T^t$ over $F$ let $Q(T,\cdot,\cdot)$ be the $F$-bilinear form $Q : D^2 \times D^2 \to F$

$$Q(T,X,Y) = \tfrac{1}{2}\,\mathrm{trd}_{D/F}\big((x_1,x_2)T\left(\begin{smallmatrix}\overline{y_1}\\ \overline{y_2}\end{smallmatrix}\right)\big) \in F \qquad X = (x_1,x_2),\ Y = (y_1,y_2).$$

The valuation $v_F$ of $F$ defines a valuation $v_D = v_F \circ \mathrm{nrd}_{D/F}$ on $D^\times$ and gives rise to an $\mathfrak{o}_F$-algebra $\mathfrak{o}_D = \{d \in D \,|\, v_D(d) \geq 0\}$ with two-sided principal ideal $\mathfrak{P} = \{d \in D \,|\, v_D(d) > 0\}$ and residue field $\mathfrak{o}_D/\mathfrak{p}_D \cong \mathbb{F}_{q^2}$. Fix Haar-measures $\mathrm{d}x$ and $\mathrm{d}^\bullet t$ on $D$ and $F^\times$ such that $\mathrm{vol}(\mathcal{O}_D) = 1$ and $\mathrm{vol}^\bullet(\mathcal{O}_F^\times) = 1$. Let $\mathrm{d}X = \mathrm{d}x_1\,\mathrm{d}x_2$ be the corresponding product measure on $D \times D$. Let $M_c$ be the inner form of $M$

$$M_c = \mathrm{GSO}(D) = (D^\times \times D^\times)/\Delta F^\times$$

for the antidiagonal embedding $\Delta : F^\times \to D^\times \times D^\times,\ t \mapsto (t, t^{-1})$. For the coset $(d_1,d_2)\Delta F^\times$ in $M_c$ we write $(d_1,d_2)$. The pre-Hilbert space of complex Schwarz-Bruhat-functions $\mathcal{S}(D \times D \times F^\times)$ with scalar product

$$\langle \varphi_1, \varphi_2 \rangle = \int_{D^2 \times F^\times} \varphi_1(X,t)\overline{\varphi_2(X,t)}\,\mathrm{d}X\,\mathrm{d}^\bullet t. \tag{4.23}$$

for $\varphi_1, \varphi_2 \in \mathcal{S}(D \times D \times F^\times)$ is a dense subspace of the Hilbert space $L^2(D \times D \times F^\times)$.
The complex *Weil constant* $\epsilon$ is independent of $\psi$ and given by

$$\epsilon = \lim_{i \to \infty} \frac{c_i}{|c_i|} \quad \text{for} \quad c_i = \int\limits_{\mathfrak{p}_D^{-i}} \psi(\mathrm{nrd}_{D/F}(x))\,\mathrm{d}x.$$

For fixed $t \in F^\times$, the Haar measure $\mathrm{d}Y_{\psi_t} = |t|^4\,q^2\,\mathrm{d}Y$ with normalization

$$\int\limits_{\mathcal{O}_D \times \mathcal{O}_D} \int\limits_{D \times D} \psi_t(2Q(I_2,X,Y))\,\mathrm{d}X_{\psi_t}\,\mathrm{d}Y_{\psi_t} = 1$$

gives rise to the *Fourier transform* $\widehat{\varphi}(\cdot,t)$ of $\varphi(\cdot,t) \in \mathcal{S}(D \times D)$ via

$$\widehat{\varphi}(X,t) = \int\limits_{D\times D} \varphi(Y,t)\psi_t(2Q(I_2,X,Y))\,\mathrm{d}Y_{\psi_t}. \tag{4.24}$$

such that $\widehat{\widehat{\varphi}}(X,t) = \varphi(-X,t)$.

**Definition 4.26.** The action of $G = \mathrm{GSp}(4,F)$ on $\varphi \in \mathcal{S}(D \times D \times F^\times)$ by

$$\pi\begin{pmatrix} I_2 & T \\ & I_2 \end{pmatrix}\varphi(X,t) = \psi(tQ(T,X,X)) \cdot \varphi(X,t), \tag{4.25}$$

$$\pi\begin{pmatrix} A & \\ & A^{-t} \end{pmatrix}\varphi(X,t) = |\det(A)|^2 \cdot \varphi(XA,t), \tag{4.26}$$

$$\pi\begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}\varphi(X,t) = \epsilon\widehat{\varphi}(X,t), \tag{4.27}$$

$$\pi\begin{pmatrix} I_2 & \\ & \lambda I_2 \end{pmatrix}\varphi(X,t) = \varphi(X,t\lambda^{-1}). \tag{4.28}$$

gives rise to a well-defined unique preunitary representation of $G$. Let $M_c$ act from the right on $\varphi$ via

$$(\varphi\,\pi_{M_c}(d_1,d_2))(X,t) = \left|\mathrm{nrd}_{D/F}(d_1 d_2)\right|^2 \cdot \varphi(d_1 X \overline{d_2}, \mathrm{nrd}_{D/F}(d_1 d_2)^{-1} t). \tag{4.29}$$

The extension to unique unitary representations $\pi$, $\pi_{M_c}$ on the closure $L^2(D \times D \times F^\times)$ is the *Weil Representation* of $G$ and $M_c$. The actions of $G$ and $M_c$ commute.

The center of $G$ and $M_c$ operates via

$$(\varphi\,\pi_{M_c}(s,1))(X,t) = |s|^4\,\varphi(sX,s^{-2}t) = \pi(sI_4)\varphi(X,t) \qquad \forall s \in F^\times.$$

Fix a unitary irreducible admissible representation $\sigma$ of $M$ in the discrete series. It gives rise to a unitary irreducible representation $\hat{\sigma}$ of $M_c$ by applying the Jacquet-Langlands correspondence to $\sigma_1$ and $\sigma_2$. The $\pi$-action of $G$ preserves the $\hat{\sigma}$-isotypic quotient $\mathcal{S}(D^2 \times F^\times, \hat{\sigma})$ [Wei09a, §4.12.2]. The big Theta-lift is the $G$-representation $\Theta(\hat{\sigma})$ so that the $M_c \times G$-representation on $\mathcal{S}(D^2 \times F^\times, \hat{\sigma})$ is isomorphic to $\sigma \boxtimes \Theta(\sigma)$. The maximal semisimple quotient of $\Theta(\hat{\sigma})$ is the *anisotropic theta lift* $\theta_-(\sigma)$.

**Lemma 4.27.** *For every unitary irreducible admissible representation $\sigma = (\sigma_1, \sigma_2)$ of $M$ in the discrete series, the lift $\Theta(\hat{\sigma})$ of $\sigma$ is non-zero, unitary, irreducible, has the same central character as $\sigma$ and is not generic. Especially, $\Theta(\hat{\sigma}) = \theta_-(\sigma)$. It is invariant under the outer automorphism $\sigma \mapsto \sigma^* = (\sigma_2, \sigma_1)$.*

*The image of the anisotropic theta-lift $\theta_-$ is precisely the set of non-generic tempered irreducible admissible representations of $\mathrm{GSp}(4,F)$. If $\sigma_1 \cong \sigma_2$, then $\theta_-(\sigma)$ is the unique non-generic irreducible subrepresentation of the Klingen induced representation $1 \rtimes \sigma_1$. For $\sigma_1 \not\cong \sigma_2$ the lift $\theta_-(\sigma)$ is cuspidal.*

*Proof.* See Gan and Takeda [GT11, Thm. 8.1] and Weissauer [Wei09a, §4.12]. □

### 4.4.1. Parahoric restriction for the Weil representation

Fix the additive character $\psi : F \to \mathbb{C}^\times$ such that it factors over a non-trivial character of the residue field. Let $\omega$ be an at most tamely ramified unitary character. Then we have the following result on the parahoric restriction of the Weil representation.

**Proposition 4.28.** *The subspace of $\mathscr{K}^+$-invariants in the Weil representation on $L^2(D^2 \times F^\times, \omega)$ is representated by the space of smooth $\varphi : D^2 \times F^\times \to \mathbb{C}$ with*

1. $\varphi(X,t) = 0$ *for every* $(\mathrm{nrd}(x_1)t, \mathrm{nrd}(x_2)t) \notin \mathfrak{o}_F \times \mathfrak{o}_F$,
2. $\varphi(X+Y,t) = \varphi(X,t)$ *for every* $Y \in D^2$ *with* $(\mathrm{nrd}(y_1)t, \mathrm{nrd}(y_2)t) \in \mathfrak{p}_F \times \mathfrak{p}_F$,
3. $\varphi(X,st) = \varphi(X,t)$ *for every* $s \in 1 + \mathfrak{p}_F$.

*The subspace of $\mathscr{J}^+$-invariants in $L^2(D^2 \times F^\times, \omega)$ is representated by the smooth functions $\varphi : D^2 \times F^\times \to \mathbb{C}$ that satisfy*

1. $\varphi(X,t) = 0$ *for every* $(\mathrm{nrd}(x_1)t, \mathrm{nrd}(x_2)t) \notin \mathfrak{p}_F \times \mathfrak{o}_F$,
2. $\varphi(X+Y,t) = \varphi(X,t)$ *for every* $Y \in D^2$ *with* $(\mathrm{nrd}(y_1)t, \mathrm{nrd}(y_2)t) \in \mathfrak{p}_F^2 \times \mathfrak{p}_F$,
3. $\varphi(X,st) = \varphi(X,t)$ *for every* $s \in 1 + \mathfrak{p}_F$.

The case for $\mathscr{K}^+$ is [Rös12, Prop. 3.20]. The valuation there is off by one because a different $\psi$ was chosen. The paramodular case is analogous.

### 4.4.2. Hyperspecial parahoric restriction

**Theorem 4.29** (Dimension Formula)**.** *Let $\sigma$ be a unitary irreducible admissible representation of $M$ in the discrete series. Then the dimension of the hyperspecial parahoric restriction of the anisotropic theta lift $\theta_-(\sigma)$ is*

$$\dim \mathbf{r}_{\mathscr{K}}(\theta_-(\sigma)) = \begin{cases} (q^2+1)(q-1), & \text{if } \sigma_1 \cong \sigma_2 \text{ is cuspidal of depth zero,} \\ q(q^2+1)/2, & \text{if } \sigma \cong (\mu\,\mathrm{St}, \mu\,\mathrm{St}), \\ q(q-1)^2/2, & \text{if } \sigma \cong (\xi\mu\,\mathrm{St}, \mu\,\mathrm{St}), \\ 0 & \text{else.} \end{cases} \tag{4.30}$$

*Here $\xi$ is the unramified quadratic character of $F^\times$ and $\mu$ runs through the unramified or tamely ramified unitary characters of $F^\times$.*

*Proof.* This is the main result of the author's diploma thesis [Rös12, Thm. 3.41]. □

### 4.4.3. Character values on unipotent conjugacy classes

Let $F$ be a non-archimedean local field with odd residue characteristic. Let $\theta_-(\sigma)$ be the anisotropic theta lift of $\sigma = (\mu\,\mathrm{St}, \xi\mu\,\mathrm{St})$, where $\mu$ is a unitary at most tamely ramified character and $\xi = \xi_t$ is a tamely ramified quadratic character of $F^\times$. We will construct an explicit basis and calculate the trace of $\theta_-(\sigma)$ on the unipotent conjugacy class generated by $u = \left(\begin{smallmatrix} I_2 & T \\ & I_2 \end{smallmatrix}\right)$ for $T = \mathrm{diag}(\varpi^{-1}u_1, u_2)$ with $u_1, u_2 \in \mathfrak{o}^\times$.

Under the (generalized) Jacquet-Langlands correspondence [Wei09a, §4.12.3], $\sigma$ corresponds to the representation $\hat\sigma = (\mu \circ \mathrm{nrd}, (\mu\xi)\circ \mathrm{nrd})$ of $M_c$. Since $\dim\hat\sigma = 1$, the paramodular restriction $\mathbf{r}_{\mathscr{J}} \circ \Theta(\hat\sigma)$ is isomorphic to the action of $\mathscr{J}/\mathscr{J}^+$ on the $\hat\sigma$-isotypic subspace[5] of $L^2(D^2 \times F^\times, \mu^2)^{\mathscr{J}^+}$.

**Lemma 4.30.** *For $\alpha, \beta \in \mathfrak{o}_F^\times$ with $\xi(\alpha^{-1}\beta) = \xi(\mathrm{nrd}(\varpi_D))$ let $N_{\alpha,\beta}$ be the set of $(X, t) \in D^2 \times F^\times$ with $\mathrm{nrd}(x_2)t \equiv \beta \mod 1+\mathfrak{p}_F$ and $\mathrm{nrd}(x_1\varpi_D^{-1})t \equiv \alpha \mod 1+\mathfrak{p}_F$. Fix a character $\Lambda : (\mathfrak{o}_D/\mathfrak{p}_D)^\times \to \mathbb{C}^\times$ of order $2(q-1)$. Then the function*

$$\varphi_{\alpha,\beta}(X,t) = \begin{cases} |t|^2\mu^{-1}(t)\Lambda(\varpi_D x_1^{-1}x_2) & (X,t) \in N_{\alpha,\beta}, \\ 0 & \text{else.} \end{cases}$$

*is in $L^2(D^2 \times F^\times, \mu^2)^{\mathscr{J}^+}$. The subspace $\mathbb{C}\varphi_{\alpha,\beta}$ is $\hat\sigma$-isotypic under $\pi_{M_c}$.*

*Proof.* The first statement is Prop. 4.28. It is clear that the action of $D^\times \times D^\times$ via $\pi_{M_c}$ preserves the support $N_{\alpha,\beta}$. For $(X,t) \in N_{\alpha,\beta}$ and $(d_1, d_2) \in D^\times \times D^\times$ we have

$$\begin{aligned}(\varphi_{\alpha,\beta}\pi_{M_c}(d_1,d_2))(X,t) &= |\,\mathrm{nrd}(d_1d_2)|^2\varphi_{\alpha,\beta}(d_1X\overline{d_2}, \mathrm{nrd}(d_1d_2)^{-1}t) \\ &= \mu\circ\mathrm{nrd}(d_1d_2)\cdot\frac{\Lambda(\varpi_D(\overline{d_2})^{-1}x_1^{-1}x_2\overline{d_2})}{\Lambda(\varpi_D x_1^{-1}x_2)}\cdot\varphi_{\alpha,\beta}(X,t).\end{aligned}$$

It remains to show that the quotient is $\xi\circ\mathrm{nrd}(d_2)$. Indeed, for $d_2 \in \mathfrak{o}_D^\times$ we have[6]

$$\Lambda(\varpi_D\overline{d_2}^{-1}x_1^{-1}x_2\overline{d_2}) = \Lambda(\varpi_D x_1^{-1}x_2d_2^{-1}\overline{d_2}) = \Lambda(\varpi_D x_1^{-1}x_2)\cdot\Lambda(d_2^{-1}\overline{d_2})$$

and $\Lambda(d_2^{-1}\overline{d_2}) = \Lambda(d_2^{q-1}) = \xi\circ\mathrm{nrd}(d_2)$. For $d_2 = \varpi_D$ write $h = \varpi_D x_1^{-1}x_2 \in \mathfrak{o}_D^\times$, then by the analogous argument

$$\Lambda(\varpi_D\overline{d_2}^{-1}x_1^{-1}x_2\overline{d_2}) = \Lambda(x_1^{-1}x_2\varpi_D) = \Lambda(\varpi_D^{-1}h\varpi_D) = \Lambda(\overline{h}) = \Lambda(\varpi_D x_1^{-1}x_2)\Lambda(h^{-1}\overline{h})$$

and $\Lambda(h^{-1}\overline{h}) = \Lambda(h^{q-1}) = \xi(\mathrm{nrd}(h)) = \xi(\alpha^{-1}\beta) = \xi(\mathrm{nrd}(\varpi_D))$. □

**Lemma 4.31.** *The functions $\varphi_{\alpha,\beta}$ for coset representatives $\alpha, \beta$ in $(\mathfrak{o}_F/\mathfrak{p}_F)^\times$ with $\xi(\alpha\beta) = \xi(\mathrm{nrd}(\varpi_D))$ form a basis of the $\hat\sigma$-isotypic subspace of $L^2(D^2 \times F^\times, \mu^2)^{\mathscr{J}^+}$.*

[5] It is sufficient to study the isotypic subspace instead of the quotient, because $M_c$ is compact modulo center, so $\pi_{M_c}$ on $L^2(D^2 \times F^\times, \mu^2)$ is semisimple.

[6] We use $x\varpi_D = \varpi_D\overline{x}$ for $x \in \mathfrak{o}_D^\times$ and the fact that $\mathfrak{o}_D^\times/(1+\mathfrak{p}_D) \cong \mathbb{F}_{q^2}$ is commutative.

*Sketch of proof.* They are linearly independent because their supports are mutually disjoint. Any $\hat{\sigma}$-isotypic $\varphi \in L^2(D^2 \times F^\times, \mu^2)^{\mathscr{I}^+}$ must satisfy the conditions of Prop. 4.28. The restriction of $\varphi$ to $N_{\alpha,\beta}$ is then a constant multiple of $\varphi_{\alpha,\beta}$ by the $\hat{\sigma}$-action and Hilbert's Theorem 90. The condition on $\alpha, \beta$ comes from $\varphi(X,t) = (\varphi\pi_{M_c}(x_1 x_2^{-1}, \overline{x_1^{-1} x_2}))(X,t) = \xi(\operatorname{nrd}(x_1^{-1}x_2))\varphi(X,t)$ for nonzero $x_1, x_2 \in D^\times$. The values of $\varphi(0, x_2, t)$ and $\varphi(x_1, 0, t)$ are invariant under $\pi_{M_c}(d, d^{-1})$ for every $d \in D^\times$, and therefore zero. Hence $\varphi$ is a linear combination of the $\varphi_{\alpha,\beta}$. □

Especially, the dimension of the paramodular restriction of $\theta_-(\sigma)$ is $(q-1)^2/2$.

**Proposition 4.32.** *The trace of* $\mathbf{r}_{\mathscr{J}} \circ \theta_-(\sigma)$ *at* $u = \left(\begin{smallmatrix} I_2 & T \\ & I_2 \end{smallmatrix}\right)$ *for* $T = \operatorname{diag}(\varpi^{-1}u_1, u_2)$ *with* $u_1, u_2 \in \mathfrak{o}_F^\times$ *is*

$$\operatorname{tr}(\mathbf{r}_{\mathscr{J}} \circ \theta_-(\sigma); u) = \tfrac{1}{2}(1 + \xi(-\varpi_F^{-1}u_1u_2)q). \tag{4.31}$$

*Proof.* The Weil action of $u$ is $\pi(u)\varphi(X,t) = \psi(tQ(T,X,X))\varphi(X,t)$ by (4.25). For $(X,t) \in N_{\alpha,\beta}$ this factor is

$$\psi(tQ(T,X,X)) = \psi(\varpi_F^{-1}u_1 t \operatorname{nrd}(x_1) + u_2 t \operatorname{nrd}(x_2)) = \psi(\varpi_F^{-1}u_1 \operatorname{nrd}(\varpi_D)\alpha + u_2\beta).$$

The trace is then calculated with respect to the basis constructed in Lemma 4.31:

$$\begin{aligned}
\operatorname{tr}(\mathbf{r}_{\mathscr{J}} \circ \theta_-(\sigma); u) &= \sum_{\substack{\alpha,\beta \in (\mathfrak{o}_F/\mathfrak{p}_F)^\times \\ \xi(\alpha\beta) = \xi(\operatorname{nrd}\varpi_D)}} \psi(u_1\varpi_F^{-1}\alpha \operatorname{nrd}(\varpi_D) + u_2\beta) \\
&= \sum_{\alpha \in (\mathfrak{o}_F/\mathfrak{p}_F)^\times} \psi(u_1\varpi_F^{-1}\alpha \operatorname{nrd}(\varpi_D)) \sum_{\beta \in (\mathfrak{o}_F/\mathfrak{p}_F)^\times} \psi(u_2\beta)\tfrac{1}{2}(\xi(\operatorname{nrd}(\varpi_D)\alpha\beta) + 1) \\
&= \tfrac{1}{2} \sum_{\alpha \in (\mathfrak{o}_F/\mathfrak{p}_F)^\times} \psi(u_1\varpi_F^{-1}\alpha \operatorname{nrd}(\varpi_D))\,(-1 + \xi(\operatorname{nrd}(\varpi_D)\alpha u_2)\mathfrak{G}) \\
&= \tfrac{1}{2}\Big(1 + \sum_{\alpha \in (\mathfrak{o}_F/\mathfrak{p}_F)^\times} \psi(u_1\varpi_F^{-1}\alpha \operatorname{nrd}(\varpi_D))\xi(\operatorname{nrd}(\varpi_D)\alpha u_2)\mathfrak{G}\Big) = \tfrac{1}{2}(1 + \xi(\varpi_F^{-1}u_1u_2)\mathfrak{G}^2)
\end{aligned}$$

with the Gauß sum $\mathfrak{G} = \sum_{\alpha \in (\mathfrak{o}_F/\mathfrak{p}_F)^\times} \psi(\alpha)\xi(\alpha) = \sum_{\beta \in (\mathfrak{o}_F/\mathfrak{p}_F)^\times} \psi(\beta)\xi(\beta)$ and $\mathfrak{G}^2 = \xi(-1)q$. □

# 5. Cohomology of Siegel modular threefolds

We employ our results on parahoric restriction in order to describe the weak endoscopic (Thm. 5.8) and the Saito-Kurokawa part (Thm. 5.4) of the inner cohomology $H_!^\bullet(S_K(\mathbb{C}), \mathcal{V}_\lambda)$ as an $\ell$-adic representation of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ and the Hecke algebra. We then prove the conjectures of Bergström, Faber and van der Geer [BFvdG08].

## 5.1. Preliminaries

Let $\mathbf{G} = \mathrm{GSp}(2g)$ be the group scheme of symplectic similitudes of genus $g \geq 1$ with real Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathbf{G}(\mathbb{R}))$ and center $Z \cong \mathbb{G}_m$.[1] Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the ring of rational adèle. For every prime $p$, fix the Haar measure on $\mathbf{G}(\mathbb{Q}_p)$ normalized on the hyperspecial subgroups by $\mathrm{vol}(\mathbf{G}(\mathbb{Z}_p)) = 1$.

**The adelic Siegel modular variety.** The Siegel modular variety admits a description as a Shimura variety. We recall Milne's exposition [Mil04, §6]:

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ be the Deligne torus. Fix a symplectic form $\psi : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}$ preserved by $G = \mathbf{G}(\mathbb{R})$. A complex structure $J$ preserving $\psi$ (i.e. $\psi(J\,\cdot\,, J\,\cdot\,) = \psi$) defines a Hodge structure $h_J : \mathbb{S}(\mathbb{R}) \to G,\ a + ib \mapsto a + Jb$. Let $X^+$ and $X^-$ be the set of $J$ such that $\psi(\,\cdot\,, J\,\cdot\,)$ is positive or negative definite, respectively. For $X = X^+ \sqcup X^-$ this defines a map $h : X \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}(\mathbb{R}), G), J \mapsto h_J$. The triple $(\mathbf{G}, X, h)$ is a Shimura datum in the sense of Pink [Pin92, §3], cp. Deligne [Del79, §2.1.1]. For the conjugation action of $G$ on $X$ let $K'_\infty = \mathrm{Cent}_G(h) \subseteq G$ be the stabilizer of $h$, so $X$ is diffeomorphic to $G/K'_\infty$.[2] For every open compact $K_f \subseteq \mathbf{G}(\mathbb{A}_f)$, the Shimura variety $S_{K_f}$ attached to $(\mathbf{G}, X, h)$ is defined over the reflex field $\mathbb{Q}$. It is a quasi-projective variety whose complex points are diffeomorphic to the orbifold

$$S_{K_f}(\mathbb{C}) \cong \mathbf{G}(\mathbb{Q}) \backslash (X \times \mathbf{G}(\mathbb{A}_f)/K_f).$$

For neat congruence subgroups $K_f$ of $\mathbf{G}(\mathbb{A}_f)$, this orbifold is a smooth analytic variety. Every compact open $K_f$ contains a neat compact open subgroup of finite index [Bor69, 17.4].

[1] Mutatis mutandis, the following holds for arbitrary Shimura varieties with quasisplit connected reductive groups $\mathbf{G}$ over a totally real global number field $F/\mathbb{Q}$.

[2] Under this diffeomorphism, $X^+ \cong \mathrm{GSp}^+(2g, \mathbb{R})/K'_\infty$ corresponds to the Siegel upper half space of genus $g$, see (B.3).

**The local system.** Let $\lambda : \mathbf{G}(\mathbb{Q}) \to \mathrm{Aut}_{\mathbb{Q}}(V_\lambda)$ be an algebraic finite dimensional irreducible linear representation over a $\mathbb{Q}$-vector space $V_\lambda$. For every $\mathbb{Q}$-algebra $A$ and every compact open subgroup $K_f$ of $\mathbf{G}(\mathbb{A})$ this defines a vector bundle

$$\mathbf{G}(\mathbb{Q})\backslash(V_\lambda(A) \times \mathbf{G}(\mathbb{A})/K'_\infty K_f) \longrightarrow \mathbf{G}(\mathbb{Q})\backslash(\mathbf{G}(\mathbb{A})/K'_\infty K_f) \cong S_{K_f}(\mathbb{C}),$$

with fiber $V_\lambda(A) = V_\lambda \otimes_{\mathbb{Q}} A$. Here $\mathbf{G}(\mathbb{Q})$ acts both on $V_\lambda(A)$ via $\lambda$ and on $\mathbf{G}(\mathbb{A})$ via the diagonal embedding. The *local system* $\mathcal{V}_\lambda(A)$ on $S_{K_f}(\mathbb{C})$ attached to $\lambda$ is the locally constant sheaf of locally constant sections of this vector bundle.

We fix an arbitrary isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$ and write $\mathcal{V}_\lambda$ for $\mathcal{V}_\lambda(\mathbb{C})$ or $\mathcal{V}_\lambda(\overline{\mathbb{Q}_\ell})$. This should not lead to confusion as no other $\mathbb{Q}$-algebra will occur.

**The $L^2$-cohomology.** The $L^2$-cohomology is defined via square-integrable differential forms [Zuc88, §1.6], [Sap05]. Fix a compact open subgroup $K = K_f \subseteq \mathbf{G}(\mathbb{A}_f)$. A differential form $\omega$ on an orbifold[3] is square-integrable if both $\omega \wedge *\omega$ and $\mathrm{d}\omega \wedge * \mathrm{d}\omega$ are Lebesgue-integrable. Let $\Omega^\bullet_{(2)}$ be the sheaf of complex-valued square-integrable smooth differential forms on $S_K(\mathbb{C})$. It gives rise to a complex of square-integrable differential forms with cofficients in $\mathcal{V}_\lambda(\mathbb{C})$ via the global section functor

$$L^\bullet_2(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C})) = H^0(S_K(\mathbb{C}), \Omega^\bullet_{(2)} \otimes \mathcal{V}_\lambda(\mathbb{C})).$$

The differential forms in this complex are square-integrable on $S_K(\mathbb{C})$ with respect to the fiber metric on $\mathcal{V}_\lambda$. The $L^2$-cohomology $H^\bullet_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C}))$ is the cohomology of this complex. On the pro-variety $S(\mathbb{C}) = \varprojlim_K S_K(\mathbb{C})$ the $L^2$-cohomology is the direct limit over the compact open subgroups $K$ of $\mathbf{G}(\mathbb{A}_f)$

$$H^\bullet_{(2)}(S(\mathbb{C}), \mathcal{V}_\lambda) = \varinjlim_K H^\bullet_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\lambda). \tag{5.1}$$

The canonical Hermitian complex structure gives rise to a Hodge decomposition, i.e. a bigrading $H^\bullet_{(2)} = \bigoplus_{p,q} H^{(p,q)}_{(2)}$, for $0 \leq p, q \leq \dim_{\mathbb{C}} X = g(g+1)/2$. This $L^2$-cohomology of $S_K(\mathbb{C})$ is finite-dimensional [BC83, Thm. A] and by the $L^2$-product it enjoys Poincaré duality.

**The Hecke algebra.** For $h \in \mathbf{G}(\mathbb{A}_f)$ and compact open subgroups $K_1, K_2 \subseteq \mathbf{G}(\mathbb{A}_f)$ with $h^{-1}K_2 h \subseteq K_1$, there is a natural morphism

$$T_h : S_{K_2}(\mathbb{C}) \to S_{K_1}(\mathbb{C}), \qquad \mathbf{G}(\mathbb{Q})(x_\infty K'_\infty, x_f K_2) \mapsto \mathbf{G}(\mathbb{Q})(x_\infty K'_\infty, x_f h K_1).$$

On $S(\mathbb{C}) = \varprojlim_K S_K(\mathbb{C})$ the morphism $T_h$ is well-defined for every $h$ and defines a right action of $\mathbf{G}(\mathbb{A}_f)$. By abuse of notation, the operator $H^\bullet_{(2)}(T_h)$ acting from the left on $H^\bullet_{(2)}(S(\mathbb{C}), \mathcal{V}_\lambda)$ is also denoted $T_h$. For every open compact subgroup $K$, the $L^2$-cohomology of the orbifold $S_K(\mathbb{C})$ is isomorphic to the subspace of invariants under $T_h$ for $h \in K$

$$H^\bullet_{(2)}(S(\mathbb{C}), \mathcal{V}_\lambda)^K = H^\bullet_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\lambda).$$

[3]For differential forms on orbifolds, compare Satake [Sat57, §1.5].

Recall that the Hecke algebra $\mathcal{H}$ is the algebra of compactly supported smooth functions $f : \mathbf{G}(\mathbb{A}_f) \to \mathbb{C}$ with the convolution product. It acts on $H^\bullet_{(2)}(S(\mathbb{C}), \mathcal{V}_\lambda)$ from the left by $\int_{\mathbf{G}(\mathbb{A}_f)} f(h) T_h \, \mathrm{d}h$ for $f \in \mathcal{H}$. The subalgebra $\mathcal{H}_K \subseteq \mathcal{H}$ of $K$-biinvariant functions preserves the $K$-invariant subspace.

**The Galois representation.** The Baily-Borel-Satake compactification $\overline{S_K}$ of $S_K$ is a normal projective variety defined over $\mathbb{Q}$ with an embedding $j : S_K \hookrightarrow \overline{S_K}$. In general, this compactification is highly singular. For a non-archimedean valuation $\ell$ of $\mathbb{Q}$ we fix a noncanonical field isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$. Then the intersection cohomology of $\overline{S_K}$ is determined by the Zucker isomorphism[4]

$$H^\bullet_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C})) \cong IH^\bullet(\overline{S_K} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\overline{\mathbb{Q}}); j_{!*}\mathcal{V}_\lambda(\overline{\mathbb{Q}_\ell})).$$

This defines a canonical action of the absolute Galois group $\Gamma_\mathbb{Q}$ on $H^\bullet_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C}))$, which commutes with the action of the Hecke algebra $\mathcal{H}_K$.

**The Matsushima-Murakami formula.** By a well-known extension of a result of Borel and Casselman [BC83, Prop. 5.6] there is a Hecke-equivariant isomorphism to the relative Lie algebra cohomology

$$H^\bullet_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C})) \cong \bigoplus_\omega H^\bullet(\mathfrak{g}, K'_\infty; L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \omega)^\infty \otimes V_\lambda)^K. \tag{5.2}$$

The sum runs over unitary central characters $\omega = \omega_\infty \omega_f$, trivial on $Z(\mathbb{A}) \cap K'_\infty K$, and $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \omega)^\infty$ denotes the space of smooth automorphic forms with central character $\omega$.

The Hilbert direct sum of the irreducible subrepresentations in $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \omega)$ constitutes the discrete spectrum $L^2_d(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \omega)$. Its orthocomplement is the continuous spectrum; it does not contribute to the $L^2$-cohomology [BC83, Thm. 4.5]. The $L^2$-cohomology of $S_K(\mathbb{C})$ is therefore determined by the $K$-invariant subspace $L^2_d(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \omega)^K$ of the discrete spectrum. Inside the discrete spectrum is the cuspidal spectrum $L^2_{\mathrm{cusp}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \omega)$, the closed subspace of cuspidal automorphic forms. The *cuspidal cohomology* is the subspace of (5.2) with

$$H^\bullet_{\mathrm{cusp}}(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C})) \cong \bigoplus_\omega H^\bullet(\mathfrak{g}, K'_\infty; L^2_{\mathrm{cusp}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \omega)^K \otimes V_\lambda). \tag{5.3}$$

The relative Lie algebra cohomology admits a natural bigrading $H^n = \bigoplus_{p+q=n} H^{(p,q)}$ compatible with the bigrading on $H^\bullet_{(2)}(S(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C}))$ and with the “filtration bête“ of Faltings and Chai [FC90, Thm. 5.5].

[4] It was conjectured by Zucker and proved by Saper and Stern [SS90] and independently by Looijenga [Loo88], compare [Zuc88].

**The spectral decomposition.** The regular representation of $\mathbf{G}(\mathbb{A}_f)$ on the smooth discrete spectrum decomposes as a direct sum $L^2_d(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}),\omega) \cong \bigoplus_\pi m(\pi)\pi$ over isomorphism classes of irreducible admissible representations $\pi$ with finite multiplicity $m(\pi)$. Those $\pi$ with $m(\pi) > 0$ are by definition the *automorphic representations in the discrete spectrum.* Every irreducible automorphic representation is a restricted tensor product $\pi = \bigotimes_v \pi_v = \pi_\infty \otimes \pi_f$ over irreducible admissible representations $\pi_v$ of $\mathbf{G}(\mathbb{Q}_v)$ for $v < \infty$ and an irreducible $(\mathfrak{g}, K'_\infty)$-module $\pi_\infty$, see [Gel75, §4.C]. The *spectral decomposition* is the Hecke-equivariant isomorphism

$$H^\bullet_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C})) \quad \cong \quad \bigoplus_\pi m(\pi) H^\bullet(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)\pi_f^K \tag{5.4}$$

to the direct sum over isomorphism classes of automorphic representations $\pi$, compare [Art89, (2.2)]. The $(\mathfrak{g}, K'_\infty)$-cohomology $H^\bullet(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$ has been determined by Vogan and Zuckerman [VZ84]. By Wigner's Lemma, it vanishes unless the central character and the infinitesimal character of $\pi_\infty \otimes V_\lambda(\mathbb{C})$ are trivial. The Hecke algebra acts on the $\mathcal{H}_K$-modules $\pi_f^K$. Since the Hecke action commutes with the Galois action, each $\pi_f$ is preserved by action of the absolute Galois group $\Gamma_\mathbb{Q}$ and the cohomology decomposes as a direct sum of $\Gamma_\mathbb{Q} \times \mathcal{H}_K$-modules

$$H^\bullet_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C})) \quad \cong \quad \bigoplus_{\pi_f} \rho_{\pi_f} \boxtimes \pi_f^K$$

where $\rho_{\pi_f}$ is the associated $\ell$-adic Galois representation. The cuspidal cohomology admits the same decomposition as a sum over cuspidal $\pi$.

**The inner cohomology.** Let us temporarily drop $(S_K(\mathbb{C}), \mathcal{V}_\lambda(\mathbb{C}))$ from the notation. The image of the natural map

$$H^\bullet_c \longrightarrow H^\bullet \tag{5.5}$$

from the cohomology with compact support to the cohomology is the *inner cohomology* $H^\bullet_!$ and the kernel is the *compactly supported Eisenstein cohomology* $H^\bullet_{c,Eis}$. There are well-known natural Hecke-equivariant morphisms

$$H^\bullet_{\text{cusp}} \longrightarrow H^\bullet_c \longrightarrow H^\bullet_{(2)} \longrightarrow H^\bullet. \tag{5.6}$$

Borel [Bor81, Cor. 5.5] has shown that the composition $H^\bullet_{\text{cusp}} \longrightarrow H^\bullet_!$ is an injection. At least for $g = 2$ the cuspidal cohomology is actually isomorphic to the inner cohomology. Weissauer has shown this for the trivial local system [Wei88, 10.4], but the proof for arbitrary local systems is analogous.

## 5.2. Cohomological automorphic representations for GSp(4)

Let $\mathbf{G} = \mathrm{GSp}(4)$ be the group of symplectic similitudes of genus two. From now on, we fix a pair of integers $\lambda_1 \geq \lambda_2 \geq 0$ with even sum $\lambda_1 + \lambda_2$. Let $\lambda$ be the irreducible algebraic representation of $\mathbf{G}(\mathbb{Q})$ with trivial central character and whose restriction to $\mathrm{Sp}(4, \mathbb{Q})$ has highest weight $(\lambda_1, \lambda_2)$. This choice of $\lambda$ is self-dual and corresponds to the unitary normalization in the sense of [Wei09a, p.3]. We say $\lambda$ is *regular* if $\lambda_1 > \lambda_2 > 0$. For regular $\lambda$ the inner cohomology $H^i_!(S_K(\mathbb{C}), \mathcal{V}_\lambda)$ vanishes for $i \neq 3$ by a result of Faltings [Fal83].

### 5.2.1. $(\mathfrak{g}, K'_\infty)$-modules with non-zero cohomology

By the spectral decomposition, an automorphic representation $\pi = \pi_\infty \otimes \pi_f$ contributes to the $L^2$-cohomology if and only if $H^\bullet(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$ is non-zero. The irreducible $(\mathfrak{g}, K'_\infty)$-modules with non-zero cohomology have been determined by Vogan and Zuckerman [VZ84]. For the case of GSp(4), compare [SO90, §2], [Tay93].

**Theorem 5.1** (Vogan-Zuckerman)**.** *For irreducible admissible pre-unitary (unitary up to twist) $(\mathfrak{g}, K'_\infty)$-modules $\pi$, the cohomology $H^{(i,j)}(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$ with Hodge type $(p, q)$ is one-dimensional in the following cases and zero otherwise.*

1. *For every $\lambda$ as above, the holomorphic non-generic discrete series $\pi_\infty = \pi^H_{\lambda+(2,1)}$ with trivial central character and infinitesimal character $\chi_{\lambda+(2,1)}$ contributes with Hodge types $(3, 0)$, $(0, 3)$.*

2. *For every $\lambda$ as above, the generic non-holomorphic discrete series $\pi_\infty = \pi^W_{\lambda+(2,1)}$ with trivial central character and infinitesimal character $\chi_{\lambda+(2,1)}$ contributes with Hodge types $(2, 1)$, $(1, 2)$.*

3. *For $\lambda_1 = \lambda_2 \geq 0$, the $(\mathfrak{g}, K'_\infty)$-module of the non-tempered Langlands quotient*
$$\pi_\infty = L(\nu^{1/2}\mathcal{D}(2\lambda_1 + 4), \xi\nu^{-1/2}) \tag{5.7}$$
*with $\xi \in \{1, \mathrm{sgn}\}$ contributes with Hodge types $(1, 1)$ and $(2, 2)$. It is denoted $(\xi \circ \mathrm{sim}) \otimes \sigma^-_{\lambda_1+3}$ by Schmidt [Sch05b] and $\pi^{2,\pm}_\lambda$ by Taylor [Tay93].*

4. *For $\lambda_1 \geq \lambda_2 = 0$, the $(\mathfrak{g}, K'_\infty)$-module of the non-tempered Langlands quotient*
$$\pi_\infty = L(\mathrm{sgn} \cdot \nu, \nu^{-1/2}\mathcal{D}_{\mathrm{sgn}}(\lambda_1 + 3)) \tag{5.8}$$
*with trivial central character contributes with Hodge types $(0, 2)$, $(2, 0)$, $(1, 3)$, $(3, 1)$. This is $\pi^1_\lambda$ with $c = 0$ in Taylor's notation.*

5. *For $\lambda_1 = \lambda_2 = 0$, the $(\mathfrak{g}, K'_\infty)$-modules of the one-dimensional representations $\pi_\infty = \mathrm{sgn} \circ \mathrm{sim}$ and $\pi_\infty = 1$ with trivial central character contribute with Hodge types $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$.*

Notation: On $\mathbb{R}^\times$ the valuation character is $\nu(x) = |x|$ and the sign character is $\mathrm{sgn}(x) = x/|x|$. The discrete series representations of $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{GSp}(4,\mathbb{R})$ are described in Examples 2.8 and 2.10.

### 5.2.2. Cohomological discrete spectrum

According to Langland's principle of functoriality every L-homomorphism ${}^LH \to {}^LG$ should give rise to a correspondence between automorphic representations. For $\mathbf{G} = \mathrm{GSp}(4)$ we have the following classification of cohomological automorphic representations $\pi$ of $\mathbf{G}(\mathbb{A})$ in the discrete spectrum.

1. The *Soudry lifts* [Sou84] are strongly associated[5] to the Klingen parabolic $Q = L_Q \ltimes U_Q$. On the $L$-group side they correspond to the embedding ${}^LL_Q \to {}^LG$. Their degree five $L$-function has a simple pole at $s = 2$. They are attached to automorphic representations $\sigma$ of GL(2). Their archimedean factor is the Langlands quotient (5.8) for $\lambda_2 = 0$. The cuspidal Soudry lifts are the *Soudry-CAP*s.[6]

2. Piatetski-Shapiro's *Saito-Kurokawa lifts* [PS83b] are strongly associated[7] to the Siegel parabolic $P = L_P \ltimes U_P$. On the $L$-group side they correspond to the standard embedding ${}^LL_P \to {}^LG$. Up to twists, these are the automorphic representations which have a pole in the degree four spinor $L$-function. Their archimedean factors are the holomorphic discrete series $\pi^H_{\lambda+(2,1)}$ and the Langlands quotients (5.7) for $\lambda_1 = \lambda_2$. The condition for cuspidality has been explicitly determined, see Section 5.3.

3. The *weak endoscopic lifts* are the cuspidal automorphic representations $\pi$ whose degree four spinor $L$-function is a product $L(s,\sigma_{1,v})L(s,\sigma_{2,v})$ at almost every local place for distinct cuspidal automorphic representation $\sigma_1, \sigma_2$ of GL(2) with equal central character. Their archimedean factors are the discrete series $\pi_\infty = \pi^H_{\lambda+(2,1)}$ and $\pi_\infty = \pi^W_{\lambda+(2,1)}$. On the $L$-group side they correspond to the embedding $\xi : {}^LM \to {}^LG$ of (4.5). See Section 5.4.

4. The *stable spectrum* consists of the cuspidal automorphic representations $\pi$ which are neither CAP nor weak endoscopic lifts. The archimedean component is either $\pi^H_{\lambda+(2,1)}$ or $\pi^W_{\lambda+(2,1)}$ in the discrete series. The four-dimensional Galois representations $\rho_{\pi_f}$ are irreducible [Wei05].

---

[5] There are also Soudry lifts strongly associated to the Borel, but they are not cohomological.

[6] Not every Soudry lift is cuspidal, compare [Wei88, §8] and [Sou84, Lemma 1.3]. This is a misprint in [Tay93, p.293].

[7] Piatetski-Shapiro's construction yields lifts strongly associated to the Borel, but they are not cohomological.

5. The *one-dimensional automorphic representations* $\pi = \chi \circ \mathrm{sim}$ factor over the similitude character with a Hecke character $\chi$. They are never cuspidal.

This gives rise to a decomposition[8] of the $L^2$-cohomology:

$$H^\bullet_{(2)} = H^\bullet_{(2),\mathrm{Soudry}} \oplus H^\bullet_{(2),\mathrm{SK}} \oplus H^\bullet_{\mathrm{endo}} \oplus H^\bullet_{\mathrm{stab}} \oplus H^\bullet_{\mathrm{sim}} \tag{5.9}$$

where each summand is the subspace of the spectral decomposition (5.2) generated by the corresponding automorphic representations in the above list. The inner cohomology decomposes as the subspace generated by the cuspidal spectrum:

$$H^\bullet_! = H^\bullet_{!,\mathrm{Soudry}} \oplus H^\bullet_{!,\mathrm{SK}} \oplus H^\bullet_{\mathrm{endo}} \oplus H^\bullet_{\mathrm{stab}}. \tag{5.10}$$

In the remaining part of this chapter we will discuss the Saito-Kurokawa part and the weak endoscopic part of the $L^2$-cohomology for certain $K \subseteq \mathrm{GSp}(4, \mathbb{A}_f)$. In order to apply results about parahoric restriction, we consider only those compact open subgroups $K$ that satisfy

$$\prod_{v<\infty, v\notin S} \mathrm{GSp}(4, \mathbb{Z}_v) \prod_{v<\infty, v\in S} \mathscr{P}_v^+ \subseteq K \subseteq \mathbf{G}(\mathbb{A}_f), \tag{5.11}$$

for a finite set $S$ of places and parahoric subgroups $\mathscr{P}_v \subseteq \mathrm{GSp}(4, \mathbb{Q}_v)$ for each $v \in S$. This includes the principal congruence subgroups of squarefree level.

## 5.3. Saito-Kurokawa Lift

The classical Saito-Kurokawa Lift has been constructed by Maaß, Andrianov and Zagier [Zag81] for the full modular group. To an elliptic cuspidal eigenform $f$ for the full modular group $\mathrm{SL}(2, \mathbb{Z})$ and weight $2k-2$ for even $k \geq 10$ it attaches a scalar-valued genus two Siegel cuspform for the full Siegel modular group $\mathrm{Sp}(4, \mathbb{Z})$ and weight $\mathrm{Sym}^0 \otimes \det^k$, such that its degree four spinor $L$-function equals

$$\zeta(s-k+1)\zeta(s-k+2)L(f, s),$$

where $\zeta$ denotes the Riemann Zeta Function and $L(f, s)$ has central value at $s = k-1$. Piatetski-Shapiro [PS83b] and Schmidt [Sch05b] have generalized the classical Saito-Kurokawa-lift from automorphic representations $\sigma$ of $\mathrm{PGL}(2, \mathbb{A})$ to automorphic representations of the twofold covering of $\mathrm{SL}(2)$ and from there to automorphic representations $\pi$ of $\mathrm{PGSp}(4, \mathbb{A})$. Every automorphic representation strongly associated to the Siegel parabolic (or the Borel) is a twist of such a lift by a Hecke character $\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$.

[8] Since $S_K(\mathbb{C})$ and $\lambda$ are clear from the context, we write $H^\bullet_?$ instead of $H^\bullet_?(S(\mathbb{C}), \mathcal{V}_\lambda)$.

Let $\sigma$ be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character. Fix a finite set $S$ of places including $\infty$, such that $\sigma_v$ is spherical for $v \notin S$. For a set $\Sigma \subseteq S$ of places let $\sigma_\Sigma$ be the irreducible subquotient of the parabolically induced representation $|\cdot|_{\mathbb{A}}^{1/2} \times |\cdot|_{\mathbb{A}}^{-1/2}$ of $\mathrm{GL}(2,\mathbb{A})$ such that $\sigma_{\Sigma,v}$ is locally in the discrete series exactly at the places $v \in \Sigma$. The non-cuspidal automorphic representation $\sigma_\Sigma$ of $\mathrm{PGL}(2,\mathbb{A})$ is locally given by

$$\sigma_{\Sigma,v} = \begin{cases} \mathbf{1}_{\mathrm{GL}(2,\mathbb{Q}_v)} & v \notin \Sigma, \\ \mathrm{St}_{\mathrm{GL}(2,\mathbb{Q}_v)} & v \in \Sigma, \end{cases}$$

where $\mathrm{St}_{\mathrm{GL}(2,\mathbb{R})}$ is the discrete series representation $\mathcal{D}(2)$. We assume that $\sigma_v$ is in the discrete series for every $v \in \Sigma$. Then there is an irreducible admissible representation $\pi = \pi(\sigma, \sigma_\Sigma)$ of $\mathrm{GSp}(4,\mathbb{A})$ with trivial central character such that at almost every place $v$ the degree four $L$-factor is

$$L(\pi_v, s) = L(\sigma_v, s)L(\sigma_{\Sigma,v}, s).$$

For an explicit construction, see Piatetski-Shapiro [PS83b, §4–6]. The lift is local in the sense that $\pi_v = \pi_v(\sigma_v, \sigma_{\Sigma,v})$ only depends on $\sigma_v$ and $\sigma_{\Sigma,v}$. Every local factor $\pi_v$ is unitary and non-generic.

**The local non-archimedean lift.** Fix a non-archimedean place $v$ of $\mathbb{Q}$ and let $\nu = |\cdot|_v$. The non-cuspidal local factors $\pi_v$ have been determined explicitly by Schmidt [Sch05b, p. 239], this is reprinted in the third column of Table 5.1. Cuspidal local factors $\pi_v$ can only occur at $v \in \Sigma$. But then $\pi_v$ coincides with the anisotropic theta-lift $\pi_v(\sigma_v, \mathrm{St}_v) = \theta_-(\sigma_v, \mathrm{St}_v)$ of Section 4.4 and Table 4.1 by [Sch05b, Prop. 5.8].

**The local archimedean lift.** The archimedean factors have also been determined by Schmidt [Sch05b, §4].

If $\infty \notin \Sigma$ and $\sigma_\infty$ is the irreducible principal series representation $\chi \times \chi^{-1}$ for a unitary character $\chi$ of $\mathbb{R}^\times$, then the lift is the unitary non-generic irreducible representation $\pi_\infty(\sigma_\infty, \mathbf{1}_{\mathrm{GL}(2,\mathbb{R})}) = \chi\, \mathbf{1}_{\mathrm{GL}(2,\mathbb{R})} \rtimes \chi^{-1}$. Its $(\mathfrak{g}, K)$-module does not contribute to cohomology.

If $\infty \notin \Sigma$ and $\sigma_\infty$ is the discrete series [9] representation $\mathcal{D}(2k-2)$, $k \geq 2$, of $\mathrm{GL}(2,\mathbb{R})$ with trivial central character, then the lift is the non-tempered Langlands quotient

$$\pi_\infty(\sigma_\infty, \mathbf{1}_{\mathrm{GL}(2,\mathbb{R})}) = L(|\cdot|_\infty^{1/2}\mathcal{D}(2k-2), |\cdot|_\infty^{-1/2})$$

of (5.7), has minimal $K$-type of weight $(k-1, 1-k)$ and contributes to cohomology with local system $\mathcal{V}_\lambda$ for $\lambda = (k-3, k-3)$ and with Hodge-type $(1,1)$ and $(2,2)$.

If $\infty \in \Sigma$, then by assumption $\sigma_\infty = \mathcal{D}(2k-2)$, $k \geq 2$, is in the discrete series. For $k \geq 3$, the lift $\pi_\infty(\mathcal{D}(2k-2), \mathrm{St}_v)$ is the holomorphic discrete series representation

[9] Schmidt [Sch05b] denotes the discrete series of $\mathrm{PGL}(2,\mathbb{R})$ with Blattner parameter $2k-2$ by $\mathcal{D}(2k-3)$ instead of our $\mathcal{D}(2k-2)$.

| $\sigma_v$ | $\sigma_{\Sigma,v}$ | $\pi_v$ | $\mathbf{r}_{\mathscr{K}_v}(\pi_v)$ | | $\mathbf{r}_{\mathscr{J}_v}(\pi_v)$ |
|---|---|---|---|---|---|
| | | | even $q$ | odd $q$ | |
| $\mu \times \mu^{-1}$ | $\mathbf{1}$ | $(\mu \cdot \mathbf{1}) \rtimes \mu^{-1}$ | $\chi_6(k)$ | $\chi_3(\tilde{\mu}, \tilde{\mu}^{-1})$ | $[1 \times \widetilde{\mu}, 1 \times \widetilde{\mu}^{-1}]$ |
| St | $\mathbf{1}$ | $L(\nu^{1/2} \cdot \mathrm{St}, \nu^{-1/2})$ | $\theta_3$ | $\theta_4(1)$ | $[\mathbf{1}, \mathbf{1}]$ |
| | St | $\tau(T, \nu^{-1/2})$ | $\theta_2$ | $\theta_3(1)$ | [St, St] |
| $\xi_u \cdot$ St | $\mathbf{1}$ | $L(\nu^{1/2}\xi_u \cdot \mathrm{St}, \nu^{-1/2})$ | $\theta_1$ | $\theta_1(1)$ | $[\mathbf{1}, \mathbf{1}]$ + [St, St] |
| | St | $\theta_-(\xi_u \cdot \mathrm{St}, \mathrm{St})$ cusp. | $\theta_5$ | $\theta_2(1)$ | 0 |
| $\xi_t \cdot$ St | $\mathbf{1}$ | $L(\nu^{1/2}\xi_t \cdot \mathrm{St}, \nu^{-1/2})$ | – | $\tau_2(1)$ | $[1 \times \lambda_0, 1 \times \lambda_0]_\mp$ |
| | St | $\theta_-(\xi_t \cdot \mathrm{St}, \mathrm{St})$ cusp. | – | 0 | $[\pi_{\Lambda'_0}, \pi_{\Lambda'^{-1}_0}]_\pm$ |
| cuspidal | $\mathbf{1}$ | $L(\nu^{1/2} \cdot \sigma_v, \nu^{-1/2})$ | $\chi_8(l'')$ | $\chi_5(\omega_\Lambda, 1)$ | 0 |
| | St | $\theta_-(\sigma_v, \mathrm{St})$ cusp. | 0 | 0 | $[\pi_{\Lambda'}, \pi_{\Lambda'^{-1}}]$ |

Table 5.1.: The local Saito-Kurokawa lifts at a non-archimedean place $v$. For depth zero $\sigma_v$, the right hand side shows the parahoric restriction of $\pi_v$ at the standard hyperspecial parahoric $\mathscr{K}_v$ and the paramodular $\mathscr{J}_v$. The sign depends on $\xi_t(\varpi) = \pm 1$.

$\pi^H_{k-1,k-2}$ with infinitesimal character $\chi_{k-1,k-2}$ and Blattner parameters $\pm(k,k)$. It contributes to the cohomology with local system $\mathcal{V}_\lambda$ for $\lambda = (k-3, k-3)$ and with Hodge types $(3,0)$ and $(0,3)$. For $k = 2$, the lift $\pi_\infty(\mathrm{St}_{\mathrm{GL}(2,\mathbb{R})}, \mathrm{St}_{\mathrm{GL}(2,\mathbb{R})})$ is only in the limit of the discrete series.

**The global Saito-Kurokawa lift.** Attached to a cuspidal automorphic representation $\sigma$ of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character and a subset $\Sigma \subseteq S$ of places where $\sigma_v$ is in the discrete series, is the restricted tensor product $\pi = \bigotimes_v \pi_v(\sigma_v, \sigma_{\Sigma,v})$ of the local lifts. It occurs in the discrete spectrum of $\mathrm{GSp}(4,\mathbb{A})$ with trivial central character and multiplicity

$$m(\pi) = \tfrac{1}{2}(1 + (-1)^{\#\Sigma}\epsilon(\sigma, 1/2)). \tag{5.12}$$

For the ground field $\mathbb{Q}$, an argument analogous to Prop. 5.7 shows that the degree four Euler factors are

$$L(\pi_v, s) = L(\sigma_v, s)L(\sigma_{\Sigma,v}, s) \quad \text{and} \quad \epsilon(\pi_v, s) = \epsilon(\sigma_v, s)\epsilon(\sigma_{\Sigma,v}, s) \tag{5.13}$$

at *every* place $v$. The global lift $\pi$ is cuspidal if and only if $L(\sigma, 1/2) = 0$ or $\Sigma \neq \emptyset$, compare Schmidt [Sch05b, Thm. 3.1] and Piateski-Shapiro [PS83b, Thm. 2.6]. It is weakly equivalent to the globally Siegel induced representation $|\cdot|^{1/2}_{\mathbb{A}}\sigma \rtimes |\cdot|^{-1/2}_{\mathbb{A}}$.

Piatetski-Shapiro makes the additional assumption $L(\chi\sigma, 1/2) \neq 0$ for a certain quadratic Hecke character $\chi$, but for $\epsilon(\sigma, 1/2) = 1$ that property is always satisfied [FH95, Thm. B.1].

**Parahoric restriction**. Fix a non-archimedean place $v$ of $\mathbb{Q}$ with residue field $\mathbb{F}_q$. In order to determine the parahoric restriction of the local Saito-Kurokawa lift, it is sufficient to do this for the standard hyperspecial parahoric $\mathscr{K}_v = \mathrm{GSp}(4, \mathbb{Z}_v)$ and the standard paramodular group $\mathscr{J}_v \subseteq \mathrm{GSp}(4, \mathbb{Q}_v)$ by (2.5).

**Theorem 5.2.** *Let $\pi_v = \pi_v(\sigma_v, \sigma_{\Sigma,v})$ be a local Saito-Kurokawa lift of a generic irreducible admissible representation $\sigma_v$ of $\mathrm{GL}(2, \mathbb{Q}_v)$ with trivial central character. Then $\sigma_v$ has depth zero if and only if $\pi_v$ has depth zero. In that case, the hyperspecial parahoric restriction and the paramodular restriction are given by the corresponding columns of Table 5.1.*

*Proof.* For the non-cuspidal local factors $\pi_v$, the hyperspecial parahoric restriction $\mathbf{r}_{\mathscr{K}_v}(\pi_v)$ is given by Table 3.1. If $\pi_v$ is cuspidal, it is isomorphic to the anisotropic theta-lift $\theta_-(\sigma_v, \mathrm{St}) = \pi_v$, see Table 4.2. □

For the hyperspecial parahoric restriction at non-archimedean $v$, we obtain

$$\dim \mathbf{r}_{\mathscr{K}_v} \pi_v(\sigma_v, \mathbf{1}) + \dim \mathbf{r}_{\mathscr{K}_v} \pi_v(\sigma_v, \mathrm{St}_v) = (q^2+1) \dim \mathbf{r}_{\mathrm{GL}(2,\mathbb{Z}_v)} \sigma_v, \tag{5.14}$$

where the second summand is zero for $v \notin \Sigma$.

**Notation 5.3** (Table 5.1)**.** The notation is analogous to Section 4.2. For a smooth character $\mu$ of $\mathbb{Q}_v^\times$ we write $\tilde{\mu} = \mathbf{r}_{\mathbb{Z}_v^\times} \mu$. If $\sigma_v$ is cuspidal irreducible of depth zero, its hyperspecial parahoric restriction $\mathbf{r}_{\mathrm{GL}(2,\mathbb{Z}_v)}(\sigma_v)$ is also cuspidal irreducible and attached to a regular character $\Lambda$ of $\mathbb{F}_{q^2}^\times$ as in Table A.1. Since $\Lambda^{q+1} = 1$, there is a character $\Lambda'$ of $\mathbb{F}_{q^2}^\times$ with $(\Lambda')^{q-1} = \Lambda$. Let $\omega_\Lambda$ be the restriction of $\Lambda'$ to $\mathbb{F}_{q^2}^\times[q+1]$. As usual, $\xi_u$ and $\xi_t$ are the unramified and either one of the tamely ramified quadratic characters. Let $\lambda_0$ and $\Lambda_0$ denote the non-trivial quadratic characters of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$, respectively.

For even $q$ the canonical homomorphism $\mathrm{Sp}(4, q) \to \mathrm{PGSp}(4, q)$ is an isomorphism and we can use Enomoto's notation [Eno72] as in Section A.4. Let $k \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $l'' \in \mathbb{Z}/(q+1)\mathbb{Z}$ be such that $\tilde{\mu} = \hat{\gamma}^k$ and $\omega_\Lambda = \hat{\eta}^{l''}$. For $F = \mathbb{Q}$, this means $k = 0$ and $l'' = 1$ or 2.

**The cohomology $H^\bullet_{\mathrm{SK}}$.** We have shown:

**Theorem 5.4.** *Fix a finite set $S$ of places of $\mathbb{Q}$, including $\infty \in S$, and let $K \subseteq \mathrm{GSp}(4, \mathbb{A}_f)$ be a subgroup of the form (5.11). For $\lambda = (k-3, k-3)$, $k \geq 3$, the Hecke action of $\prod_{v<\infty, v\in S} \mathscr{P}_v / \mathscr{P}_v^+$ on the Saito-Kurokawa part of the cohomology is the representation*

$$H^{(p,q)}_{\mathrm{SK}}(S_K(\mathbb{C}), \mathcal{V}_\lambda) \cong \bigoplus_\sigma \bigoplus_\omega \bigoplus_\Sigma H^{(p,q)}(\mathfrak{g}, K'_\infty; \omega_\infty \pi_\infty \otimes V_\lambda) \bigotimes_{v\in S, v<\infty} \mathbf{r}_{\mathscr{P}_v}(\omega_v \pi_v). \tag{5.15}$$

*The first sum runs over cuspidal automorphic representations $\sigma$ of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character, with $\sigma_\infty = \mathcal{D}(2k-2)$, spherical outside of $S$ and $\sigma_v$ of depth* *zero for $v \in S$, $v < \infty$. The second sum runs over locally tamely ramified unitary Hecke characters $\omega$ of $\mathbb{A}^\times$ with $\omega_\infty \in \{1, \mathrm{sgn}\}$, which are unramified outside of $S$. The third sum runs over subsets $\Sigma \subseteq S$ of places where $\sigma_v$ is in the discrete series such that $(-1)^{\#\Sigma} = \epsilon(\sigma, 1/2)$.*

**Example 5.5.** *Fix a principal congruence subgroup $K$ of prime level $p$ and let $\lambda = (k-3, k-3)$, $k \geq 3$. The Saito-Kurokawa cohomology with local coefficients in $\mathcal{V}_\lambda$ defines a representation of $\mathrm{GSp}(4, \hat{\mathbb{Z}})/K \cong \mathrm{GSp}(4, \mathbb{Z}/p\mathbb{Z})$, which is non-zero only in the following cases. For Hodge type $(*,*) = (3,0)$ and $(0,3)$ it is*

$$H^{*,*}_{SK}(S_K(\mathbb{C}), \mathcal{V}_\lambda) = \bigoplus_{\sigma,\omega} \begin{cases} \widetilde{\omega} \cdot \mathbf{r}_{\mathscr{K}_p}\, \pi_p(\sigma_p, \mathrm{St}), & \epsilon(\sigma_p, 1/2) = (-1)^{k-1}, \\ \widetilde{\omega} \cdot \mathbf{r}_{\mathscr{K}_p}\, \pi_p(\sigma_p, \mathbf{1}), & \epsilon(\sigma_p, 1/2) = -(-1)^{k-1}. \end{cases} \tag{5.16}$$

*For Hodge type $(*) = (1,1)$ and $(2,2)$ it is*

$$H^{*,*}_{SK}(S_K(\mathbb{C}), \mathcal{V}_\lambda) = \bigoplus_{\sigma,\omega} \begin{cases} \widetilde{\omega} \cdot \mathbf{r}_{\mathscr{K}_p}\, \pi_p(\sigma_p, \mathrm{St}), & \epsilon(\sigma_p, 1/2) = -(-1)^{k-1}, \\ \widetilde{\omega} \cdot \mathbf{r}_{\mathscr{K}_p}\, \pi_p(\sigma_p, \mathbf{1}), & \epsilon(\sigma_p, 1/2) = (-1)^{k-1}. \end{cases} \tag{5.17}$$

*The sum runs over automorphic representations $\sigma$ of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character $\sigma_\infty \cong \mathcal{D}(2k-2)$. The second sum runs over Hecke characters $\omega$ corresponding to characters $\widetilde{\omega}$ of $(\mathbb{Z}/p\mathbb{Z})^\times$. The hyperspecial parahoric restriction is given by Table 5.1.*

*Proof.* The four cases correspond to the subsets $\Sigma = \{\infty, p\}, \{\infty\}, \{p\}, \emptyset$ of $S = \{p, \infty\}$. The central archimedean epsilon factor is $\epsilon(\mathcal{D}(2k-2), 1/2) = (-1)^{k-1}$, so the automorphy condition $\epsilon(\sigma, s) = \epsilon(\sigma_\Sigma, s)$ of the global Saito-Kurokawa lift depends on the parity of $k$. □

The non-cuspidal Saito-Kurokawa lifts are those with $\Sigma = \emptyset$ and $L(\sigma, 1/2) \neq 0$. Therefore $H^{3,0}_{SK} \subseteq H^{3,0}_{\mathrm{cusp}}$ is always cuspidal and analogous for $(0,3)$. For Hodge types $(1,1)$ and $(2,2)$, the cuspidal part $H_{SK} \cap H_{\mathrm{cusp}}$ is given by excluding those $\sigma$ with $\epsilon(\sigma_p, 1/2) = (-1)^{k-1}$ and $L(\sigma, 1/2) \neq 0$ from the above sum.

**The Galois representation.** Let $\rho_\sigma$ be the $\ell$-adic Galois representation attached to a cohomological representation $\sigma$ of $\mathrm{GL}(2, \mathbb{A})$ with $\sigma_\infty = \mathcal{D}(2k-2), k \geq 3$ in the discrete series. Then the $\ell$-adic Galois representation $\rho_\pi$ attached to the cohomological Saito-Kurokawa lift $\pi = \pi(\sigma, \sigma_\Sigma)$ occuring with $\lambda_1 = \lambda_2 = k-3$ is

$$\rho_{\pi_{fin}} \cong \begin{cases} \rho_\sigma & \pi_\infty = \pi^H_{\lambda+(2,1)}, \\ \overline{\mathbb{Q}_\ell}(-\lambda_1 - 1) \oplus \overline{\mathbb{Q}_\ell}(-\lambda_1 - 2), & \pi_\infty = \pi^{2\pm}_\lambda. \end{cases} \tag{5.18}$$

## 5.4. Weak endoscopic lift

The weak endoscopic lift is the correspondence that belongs to the embedding of $L$-groups ${}^L M \to {}^L G$ for the proper elliptic endoscopic group

$$M = (\mathrm{GL}(2) \times \mathrm{GL}(2)) / \mathrm{GL}(1)$$

of $G = \mathrm{GSp}(4, F)$ in (4.5) under Langlands functoriality. To a cuspidal automorphic representation $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A})$ with $\sigma_1 \not\cong \sigma_2$ it attaches an endoscopic $L$-packet of automorphic representations $\pi$ of GSp(4) with local degree four spinor $L$-factor

$$L(\pi, v, s) = L(\sigma_{1,v}, s) L(\sigma_{2,v}, s)$$

at almost every place $v$. Since we are interested in contributions to the cohomology with a local coefficient system, we fix integers $\lambda_1 \geq \lambda_2 \geq 0$.

**The local endoscopic $L$-packets.** Let $v$ be a local place of $\mathbb{Q}$. The local endoscopic $L$-packet attached to a unitary generic irreducible admissible representation $\sigma_v$ of $M(\mathbb{Q}_v)$ contains one or two unitary irreducible admissible representations of $\mathrm{GSp}(4, F_v)$. If $\sigma_v$ is in the discrete series, the local $L$-packet is $\{\Pi_+(\sigma_v), \Pi_-(\sigma_v)\}$, otherwise it is a singleton $\Pi_+(\sigma_v)$.

For non-archimedean places the local endoscopic $L$-packet is the one in Lemma 4.5.

For the archimedean place $v = \infty$ let $\omega$ be a unitary character $\omega$ of $\mathbb{R}^\times$ with $\omega(-1) = (-1)^{\lambda_1 + \lambda_2 + 1}$. To the generic discrete series representation $\sigma_\infty = (\mathcal{D}_\omega(r_1), \mathcal{D}_\omega(r_2))$ with central character $\omega$ and weights

$$r_1 = \lambda_1 + \lambda_2 + 4, \qquad r_2 = \lambda_1 - \lambda_2 + 2 \tag{5.19}$$

is attached the local endoscopic $L$-packet $\{\Pi_+(\sigma_\infty), \Pi_-(\sigma_\infty)\}$. It contains the holomorphic non-generic discrete series irreducible representation $\Pi_-(\sigma_\infty) = \pi^H_{\lambda,\omega}$ and the non-holomorphic generic irreducible representation $\Pi_+(\sigma_\infty) = \pi_\infty$ described in Example 2.10. Compare [Wei09a, Cor.4.2].

If $\sigma_\infty$ is not in the discrete series, then without loss of generality $\sigma_{\infty,1} = \mu_1 \times \mu_2$ is in the principal or in the complementary series. The archimedean local endoscopic $L$-packet contains the single representation $\Pi_+(\sigma_\infty) = \mu_1^{-1}\sigma_2 \rtimes \mu_1$ [Wei09a, Lemma 4.27]. It does not contribute to cohomology.

**The global lift.** Fix a generic cuspidal automorphic representation $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A})$ and let $S$ be the finite set of local places $v$ where $\sigma_v$ is in the discrete series. We assume that $\infty \in S$. A cuspidal automorphic representation $\pi$ of $\mathrm{GSp}(4, \mathbb{A})$ is a *weak endoscopic lift attached to $\sigma$* if it is not CAP and has local degree four spinor $L$-factor

$$L(\pi_v, s) = L(\sigma_{1,v}, s) L(\sigma_{2,v}, s). \tag{5.20}$$

at almost every place $v$. The cuspidal automorphic representations $\pi$ that occur as weak endoscopic lifts form the *global $L$-packet attached to $\sigma$*. The global $L$-packet contains a cuspidal automorphic representation if and only if $\sigma_1 \not\cong \sigma_2$. A weak endoscopic lift attached to $\sigma$ is also attached to $\sigma^* = (\sigma_2, \sigma_1)$, but this is the only equivalence between global $L$-packets by the strong multiplicity one theorem for GL(2) [Wei09a, Prop. 5.2].

**Theorem 5.6** ([Wei09a, Thm. 5.2])**.** *Suppose the generic automorphic cuspidal representation $\sigma$ satisfies $\sigma_1 \not\cong \sigma_2$. A restricted tensor product of irreducible admissible representations $\pi = \bigotimes_v' \pi_v$ is a weak endoscopic lift of $\sigma$ if and only if there is a subset $\Sigma \subseteq S$ of finite even cardinality such that*

$$\pi_v \cong \begin{cases} \Pi_+(\sigma_v) & v \notin \Sigma, \\ \Pi_-(\sigma_v) & v \in \Sigma. \end{cases}$$

*In that case $\pi$ is cuspidal automorphic, not CAP, and has multiplicity one.*

The automorphic representations in the global $L$-packet form an equivalence class under weak equivalence. If $\sigma_v$ is in the discrete series at $d \geq 2$ places, then the global $L$-packet contains $2^{d-1}$ automorphic representations, otherwise there is only the globally generic automorphic representation $\pi_+(\sigma) = \bigotimes_v' \Pi_+(\sigma_v)$.

**The local degree four Euler factors.** Suppose $\sigma = (\sigma_1, \sigma_2)$ is a cuspidal automorphic representation of $M(\mathbb{A})$ with $\sigma_1 \not\cong \sigma_2$.

**Proposition 5.7.** *For every weak endoscopic lift $\pi$ of $\sigma$, the local degree four spinor factors are*

$$L(\sigma_{1,v}, s)L(\sigma_{2,v}, s) = L(\pi_v, s) \quad \text{and} \quad \epsilon(\sigma_{1,v}, s)\epsilon(\sigma_{2,v}, s) = \epsilon(\pi_v, s) \tag{5.21}$$

*at* every *nonarchimedean place $v$.*

*Proof.* Each non-cuspidal factor is given explicitly by Table 4.1 and the local spinor factors are given by [RS07, Tables A.8 and A.9]. If $\pi_v$ is cuspidal generic, then both $\sigma_{1,v}$ and $\sigma_{2,v}$ are also cuspidal generic, so $L(\pi_v, s) = 1 = L(\sigma_1, s)L(\sigma_2, s)$. The corresponding equation of $\gamma$-factors [PSS81, Thm. 3.1] implies (5.21) by the local functional equation. If $\pi_v$ is non-generic cuspidal and if $\mathbb{Q}_v$ has odd residue characteristic, (5.21) is implied by a result of Danisman [Dan11, Cor. 4.5] and the fact that the local Jacquet-Langlands correspondence preserves $L$- and $\epsilon$-factors.

It remains to discuss the case of non-generic cuspidal $\pi_v$ at the place $v = 2$. Choose a cuspidal automorphic representations $\sigma'$ of $M(\mathbb{A})$ with $\sigma'_1 \not\cong \sigma'_2$, unramified at $v = 2$, in the discrete series at at least some other non-archimedean place, with the same archimedean factors $\sigma'_{i,\infty} \cong \sigma_{i,\infty}$ [Wei09a, p. 100]. For any weak endoscopic lift $\pi'$ of $\sigma'$ with the same archimedean factor, the previous arguments imply

(5.21). By the local functional equation, the archimedean $\gamma$-factors must satisfy $\gamma(\sigma_{1,\infty},s)\gamma(\sigma_{2,\infty},s)=\gamma(\pi_\infty,s)$. But then the same functional equation applied to $\sigma$ and $\pi$ implies $\gamma(\sigma_{1,v},s)\gamma(\sigma_{2,v},s)=\gamma(\pi_v,s)$ for $v=2$. By the same argument as in the proof of [Dan11, Cor. 4.5], the local $\gamma$-factors uniquely determine the $L$- and $\epsilon$-factors at $v=2$. □

**The cohomology.** By the above arguments, we have shown:

**Theorem 5.8.** *Fix a finite set $S$ of places of $\mathbb{Q}$, including $\infty\in S$, and a compact open subgroup $K\subseteq \mathrm{GSp}(4,\mathbb{A}_f)$ of the form* (5.11)*. For every $\lambda_1\geq\lambda_2\geq 0$, the Hecke action of $\prod_{v<\infty,v\in S}\mathscr{P}_v/\mathscr{P}_v^+$ on the weak endoscopic part of the cohomology is the representation*

$$H^\bullet_{\mathrm{endo}}(S_K(\mathbb{C}),\mathcal{V}_\lambda)\cong\bigoplus_\sigma\bigoplus_\Sigma H^\bullet(\mathfrak{g},K'_\infty;\pi_\infty\otimes V_\lambda)\bigotimes_{v\in S,v<\infty}\mathbf{r}_{\mathscr{P}_v}(\pi_v). \tag{5.22}$$

*The first sum runs over cuspidal automorphic representations $\sigma$ of $M(\mathbb{A})$, with archimedean factor $\sigma_\infty=(\mathcal{D}(r_1),\mathcal{D}(r_2))$ as in* (5.19)*, spherical for non-archimedean $v\notin S$ and of depth zero at the non-archimedean $v\in S$. The second sum runs over subsets $\Sigma\subseteq S$ of finite even cardinality. The parahoric restriction of $\Pi_+(\sigma_v)$ and $\Pi_-(\sigma_v)$ at the maximal standard parahorics is given by Tables 4.2 and 4.4.*

**Example 5.9.** *Suppose $K\subseteq\mathrm{GSp}(4,\mathbb{A}_f)$ is a principal congruence subgroup of prime level $p$ corresponding to the standard hyperspecial parahoric $\mathscr{K}_p$ of $\mathrm{GSp}(\mathbb{Q}_p)$. The action of $\mathrm{GSp}(4,\hat{\mathbb{Z}})$ on the weak endoscopic part of the cohomology of $S_K(\mathbb{C})$ with local coefficients in $\mathcal{V}_\lambda$ defines a representation of $\mathrm{GSp}(4,\mathbb{Z}/p\mathbb{Z})$, which is non-zero only in the following cases:*

$$H^{(3,0)}_{\mathrm{endo}}(S_K(\mathbb{C}),\mathcal{V}_\lambda)\ \cong\ H^{(0,3)}_{\mathrm{endo}}(S_K(\mathbb{C}),\mathcal{V}_\lambda)\ \cong\ \bigoplus_\sigma\mathbf{r}_{\mathscr{K}_p}\Pi_-(\sigma_p), \tag{5.23}$$

$$H^{(2,1)}_{\mathrm{endo}}(S_K(\mathbb{C}),\mathcal{V}_\lambda)\ \cong\ H^{(1,2)}_{\mathrm{endo}}(S_K(\mathbb{C}),\mathcal{V}_\lambda)\ \cong\ \bigoplus_\sigma\mathbf{r}_{\mathscr{K}_p}\Pi_+(\sigma_p). \tag{5.24}$$

*The sums run over cuspidal automorphic representations $\sigma$ of $M(\mathbb{A})$, unramified at $v\neq p$, with archimedean factor $\sigma_\infty\cong(\mathcal{D}(r_1),\mathcal{D}(r_2))$ for the weights* (5.19)*. The hyperspecial parahoric restriction is given by Table 4.2.*

**The Galois representations.** Let $\sigma=(\sigma_1,\sigma_2)$ be a cuspidal automorphic representation of $M(\mathbb{A})$ with $\ell$-adic Galois representations $\rho_{\sigma_1}$ and $\rho_{\sigma_2}$ attached to $\sigma_1$ and $\sigma_2$. We assume $\sigma_\infty$ is the discrete series $\mathcal{D}_\omega(r_1),\mathcal{D}_\omega(r_2)$ with weights $r_1>r_2$, this distinguishes $\sigma_1$ from $\sigma_2$. Then the (semisimplified) $\ell$-adic Galois representation attached to a weak endoscopic lift $\pi$ of $\sigma$ is calculated by the cohomological trace formula [Wei09a, Cor. 4.2, 4.4]

$$\rho_{\pi_f}=\begin{cases}\rho_{\sigma_1} & \pi_\infty=\pi^H_\lambda,\\ \rho_{\sigma_2}(-\lambda_2-1) & \pi_\infty=\pi^W_\lambda.\end{cases} \tag{5.25}$$

By a result of Ribet these representations are irreducible.

## 5.5. Conjectures of Bergström, Faber and van der Geer

Bergström, Faber and van der Geer [BFvdG08] made explicit conjectures on the inner cohomology in level two. We will now prove these conjectures.

At first, we adjust our notation to their situation. The moduli space $\mathcal{A}_{2,N}$ of principally polarized abelian surfaces with a level $N$-structure is a Deligne-Mumford stack defined over $\mathrm{Spec}(\mathbb{Z}[1/N])$. Its analytification over the category of complex analytic spaces is isomorphic to the orbifold $\mathbb{H}_2/\Gamma[N]$, the quotient of the Siegel upper half space by the principal congruence subgroup $\Gamma[N] \subseteq \mathrm{Sp}(4,\mathbb{Z})$. The cohomology is preserved by Serre's GAGA theorems. Under strong approximation, $\mathbb{H}_2/\Gamma[N]$ is isomorphic to $S_{K'(N)}(\mathbb{C})$ for the modified principal congruence subgroup

$$K'(N) = \{x \in \mathrm{GSp}(4,\hat{\mathbb{Z}}) \,|\, x \equiv \mathrm{diag}(1,1,*,*) \mod N\}. \tag{5.26}$$

The local system $\mathcal{V}_\lambda$ is parametrized by integers $(l,m) = (\lambda_1, \lambda_2)$ with $l \geq m \geq 0$.

The holomorphic part $H_!^{(3,0)}(\mathcal{A}_{2,N}, \mathcal{V}_\lambda)$ of the inner cohomology is Hecke-isomorphic to the space of holomorphic Siegel cuspforms of weight $\mathrm{Sym}^{l-m} \otimes \det^{m+3}$ for the principal congruence subgroup $\Gamma[N] \subseteq \mathrm{Sp}(4,\mathbb{Z})$.

Let $\tau_{N,k} = \# S_k(\Gamma_0[N])^{\mathrm{new}}$ denote the cardinality of the finite set of normalized elliptic cuspidal newforms[10] of weight $k$ and level $N$. We write $\tau'_{N,k}$ for the cardinality of the subset with vanishing central $L$-value.[11] In level $N=2$, we denote by $\tau_k^{\pm}$ the subspace of forms with Atkin-Lehner eigenvalue $\pm 1$.

Scholl [Sch90] has constructed a motive for the space of elliptic cuspidal newforms with weight $k$ and level $N$. Since we need only the underlying semisimple $\ell$-adic Galois representation of $\Gamma_\mathbb{Q}$, we write

$$\mathrm{S}[\Gamma_0(N), k]^{\mathrm{new}} = \bigoplus_{f \in S_k(\Gamma_0[N])^{\mathrm{new}}} \rho_f$$

where $\rho_f$ is the Galois representation attached to $f$ by Deligne [Del68]. We use the corresponding notation also for the subspaces $S_k^{\pm}(\Gamma_0[N])$ with Atkin-Lehner eigenvalue $\pm 1$ and the subspace $S'_k(\Gamma_0[N])$ with vanishing central $L$-value. We write $L^m = \overline{\mathbb{Q}_\ell}(-m)$ for the $m$-power of the dual cyclotomic character as in [BFvdG08], and the trivial character $L^0$ is dropped from the notation.

### 5.5.1. Previous results on level one

For level $N=1$, Faber and van der Geer [FvdG04, §2], [vdG11, Cor. 10.2] and Petersen [Pet15] have determined the Eisenstein cohomology of $\mathcal{A}_{2,1}$ explicitly. We

[10] By definition, newforms are eigenforms of the Hecke algebra.

[11] For $k \equiv 2 \mod 4$, we have $\tau'_{N,k} = \tau_{N,k}$ by the functional equation. For $k \equiv 0 \mod 4$, it can be verified numerically that $\tau'_{N,k} = 0$ for small values of $k$ and $N$.

only give the regular case here to simplify the notation. For irregular $\lambda$, see Petersen [Pet15, Thm. 2.1].

**Theorem 5.10.** *Let $\mathcal{V}_\lambda$ be a regular local system with $l+m$ even. The compactly supported Eisenstein cohomology $H^i_{\mathrm{c,eis}}(\mathcal{A}_{2,1}, \mathcal{V}_\lambda(\overline{\mathbb{Q}_\ell}))$ decomposes as an $\ell$-adic Galois representation for $i=2$ as*

$$\mathrm{S}[\mathrm{Sp}(4,\mathbb{Z}), m+2] + \tau_{1,l-m+2}L^0 + \begin{cases} L^0 & l,m \text{ odd,} \\ 0 & l,m \text{ even,} \end{cases}$$

*for $i=3$ as*

$$\mathrm{S}[\mathrm{Sp}(4,\mathbb{Z}), l+3] + \tau_{1,l+m+4}L^{m+1},$$

*and is zero for $i = 0,1,4,5,6$.*

By counting points on finite fields [vdG], Faber and van der Geer obtained conjectural results on the Euler characteristic of the cohomology with compact support and this led to precise conjectures about the inner cohomology [FvdG04]. Indeed, the inner cohomology can be described explicitly as follows.

**Theorem 5.11.** *Fix an arbitrary local system $\mathcal{V}_\lambda$ with $l+m$ even. The inner cohomology $H^\bullet_!(\mathcal{A}_2, \mathcal{V}_\lambda)$ of the moduli space of principally polarized abelian varieties is the direct sum of the following contributions. The endoscopic part*

$$H^i_{\mathrm{endo}}(\mathcal{A}_2, \mathcal{V}_\lambda) = \begin{cases} \tau_{1,l+m+4}\mathrm{S}[\mathrm{Sp}(4,\mathbb{Z}), l-m+2] \otimes L^{m+1} & i=3 \\ 0 & i \neq 3. \end{cases}$$

*is concentrated in Hodge types $(2,1)$ and $(1,2)$. The Saito-Kurokawa part*

$$H^i_{!,\mathrm{SK}}(\mathcal{A}_2, \mathcal{V}_\lambda) = \begin{cases} \tau'_{1,2l-2}L^{m+1} & \text{if } i=2 \text{ and } l=m \text{ even,} \\ \mathrm{S}[\mathrm{Sp}(4,\mathbb{Z}), l+m+4] & \text{if } i=3 \text{ and } l=m \text{ odd,} \\ \tau'_{1,2l-2}L^{m+2} & \text{if } i=4 \text{ and } l=m \text{ even,} \\ 0 & \text{else.} \end{cases}$$

*is concentrated in Hodge types $(1,1)$, $(3,0)$, $(0,3)$ and $(2,2)$, respectively. The contribution from the Soudry part is*

$$H^i_{!,\mathrm{Soudry}}(\mathcal{A}_2, \mathcal{V}_\lambda) = 0.$$

*The stable spectrum*[12] *decomposes into irreducible four-dimensional Galois representations*

$$H^i_{\mathrm{stable}}(\mathcal{A}_2, \mathcal{V}_\lambda) = \begin{cases} \bigoplus_{\pi_f} \rho_{\pi_f} & i=3 \\ 0 & i \neq 3, \end{cases}$$

*attached to cuspidal automorphic representations, that are neither CAP nor weak endoscopic, and which are spherical at every finite place. It occurs with Hodge-type $(3,0)$, $(2,1)$, $(1,2)$, $(3,0)$.*

[12] This is $\mathrm{S}[l-m, m+3]$ in the notation of [FvdG04] and $\hat{S}[l-m, m+3]$ in [BFvdG14].

This has been shown by Weissauer [Wei09b], Tehrani [Teh12], and Petersen [Pet15]. In our notation, the endoscopic part is the special case of (5.22) with $S = \{\infty\}$ and $\Sigma = \emptyset$ and the Galois representation given by (5.25). The Saito-Kurokawa part is (5.15) with $S = \{\infty\}$, $\omega = 1$ and $\Sigma = \emptyset$ or $\{\infty\}$ and the Galois representation of (5.18). The Soudry part contributes to cohomology when $m = 0$, but there are no non-zero elliptic cuspforms of odd weight $l + 3$ for the full modular group.

### 5.5.2. Level two

The case of level $N = 2$ has been studied extensively [vdG82], [LW85, §8], [BFvdG08], [BFvdG14], [CvdGG15]. The Hecke action of $\mathrm{GSp}(4, \hat{\mathbb{Z}})$ on the invariants under the principal congruence subgroup $K(2) = K'(2) \subseteq \mathrm{GSp}(4, \hat{\mathbb{Z}})$ of level two gives rise to a representation of $\mathrm{GSp}(4, \hat{\mathbb{Z}})/K(2) \cong \mathrm{Sp}(4, \mathbb{F}_2)$. After semisimplification, the cohomology of $\mathcal{A}_{2,2}$ decomposes into $\ell$-adic representations of $\mathrm{Sp}(4, \mathbb{F}_2) \times \Gamma_{\mathbb{Q}}$.

Since $\mathrm{Sp}(4, \mathbb{F}_2)$ is isomorphic to the symmetric group in six letters, the irreducible representations of $\mathrm{Sp}(4, \mathbb{F}_2)$ can be classified by partitions of six. We fix parabolically induced representations $A, B, C, A', B', C'$ as in Section 5.5.5. We write $r_1 = l+m+4$ and $r_2 = l - m + 2$ as in (5.19).

For the compactly supported Eisenstein cohomology of $\mathcal{A}_{2,2}$, the Euler characteristic has been determined as before [BFvdG08, Thms. 4.2, 4.4] using the BGG-complex of Faltings and Chai [FC90].

**Theorem 5.12** (Bergström, Faber, van der Geer)**.** *For regular $\lambda$, the Euler characteristic of the compactly supported Eisenstein cohomology decomposes as a representation of* $\mathrm{Sp}(4, \mathbb{F}_2) \times \Gamma_{\mathbb{Q}}$ *as*

$$
\begin{aligned}
e_{c,Eis}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda) = {} & \tau_{1,r_2}(A' + B') + \tau_{2,r_2} B' + \tau_{4,r_2} C' - (\tau_{1,r_1}(A' + B') + \tau_{2,r_1} B' + \tau_{4,r_1} C') L^{m+1} \\
& + (A + B) \boxtimes \mathrm{S}[\Gamma_0(1), m + 2] + B \boxtimes \mathrm{S}[\Gamma_0(2), m + 2]^{\mathrm{new}} + C \boxtimes \mathrm{S}[\Gamma_0(4), m + 2]^{\mathrm{new}} \\
& - (A + B) \boxtimes \mathrm{S}[\Gamma_0(1), l + 3] - B \boxtimes \mathrm{S}[\Gamma_0(2), l + 3]^{\mathrm{new}} - C \boxtimes \mathrm{S}[\Gamma_0(4), l + 3]^{\mathrm{new}} \\
& + \tfrac{1}{2}(1 + (-1)^m)(A + B).
\end{aligned}
$$

We can explicitly describe the endoscopic part of the inner cohomology:

**Theorem 5.13.** *For $l \geq m \geq 0$, the semisimplified endoscopic part $H^{\bullet}_{\mathrm{endo}}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda)$ of the inner cohomology decomposes under the action of* $\mathrm{Sp}(4, \mathbb{F}_2) \times \Gamma_{\mathbb{Q}}$ *as*

$$
\begin{aligned}
H^{(3,0)}_{\mathrm{endo}} \oplus H^{(0,3)}_{\mathrm{endo}} \cong {} & \tau_{4,r_2} \cdot s[2, 1^4] \boxtimes \mathrm{S}[\Gamma_0(4), r_1]^{\mathrm{new}} \\
& + \left(\tau^+_{r_2} \cdot s[2^3] + \tau^-_{r_2} \cdot s[1^6]\right) \boxtimes \mathrm{S}^+[\Gamma_0(2), r_1]^{\mathrm{new}} \\
& + \left(\tau^+_{r_2} \cdot s[1^6] + \tau^-_{r_2} \cdot s[2^3]\right) \boxtimes \mathrm{S}^-[\Gamma_0(2), r_1]^{\mathrm{new}}
\end{aligned}
$$

*and*

$$H^{(2,1)}_{\text{endo}} \oplus H^{(1,2)}_{\text{endo}} \cong L^{m+1}\Big(\big(\tau_{4,r_1}\cdot s[3,1^3] + \tau_{2,r_1}\cdot s[4,1^2] + \tau_{1,r_1}\cdot C'\big)\boxtimes \mathrm{S}[\Gamma_0(4),r_2]^{\text{new}}$$
$$+\big(\tau_{2,r_1}\cdot s[3,2,1] + \tau_{4,r_1}\cdot s[4,1^2] + \tau_{1,r_1}\cdot B'\big)\boxtimes \mathrm{S}[\Gamma_0(2),r_2]^{\text{new}}$$
$$+\big(\tau^+_{r_1}\cdot s[4,2] + \tau^-_{r_1}\cdot s[5,1]\big)\boxtimes \mathrm{S}^+[\Gamma_0(2),r_2]^{\text{new}}$$
$$+\big(\tau^-_{r_1}\cdot s[4,2] + \tau^+_{r_1}\cdot s[5,1]\big)\boxtimes \mathrm{S}^-[\Gamma_0(2),r_2]^{\text{new}}$$
$$+\big(\tau_{4,r_1}\cdot C' + \tau_{2,r_1}\cdot B' + \tau_{1,r_1}\cdot (A'+B')\big)\boxtimes \mathrm{S}[\Gamma_0(1),r_2]\Big)$$

*with the notation of Section 5.5.5. The other Hodge numbers are zero.*

The proof is given in the next subsection.

**Remark 5.14.** This solves Conjectures 6.4 and 7.1 of Bergström, Faber and van der Geer [BFvdG08]. They call the non-holomorphic part with Hodge types $(2,1)$ and $(1,2)$ the "middle endoscopic part". The holomorphic part with Hodge types $(3,0)$ and $(0,3)$ is the "leading part" and corresponds to Yoshida lifts.

**Corollary 5.15.** *For $l \geq m \geq 0$ we have*

$$\dim H^{(2,1)}_{\text{endo}}(\mathcal{A}_{2,2},\mathcal{V}_\lambda) - \dim H^{(3,0)}_{\text{endo}}(\mathcal{A}_{2,2},\mathcal{V}_\lambda) = 5\cdot \dim S_{r_1}(\Gamma_0[4])\cdot \dim S_{r_2}(\Gamma_0[4]).$$

*Proof.* The left hand side is given by Thm. 5.13. It equals the right hand side by Atkin-Lehner theory. □

**Remark 5.16.** This is a special case of (4.11). It approximates Conjecture 7.2 of [BFvdG08] for regular $l > m > 0$, but the conjecture does not hold literally, because "trailing terms" do not appear in the cohomology $H^{(3,0)}_{\text{endo}} \oplus H^{(0,3)}_{\text{endo}}$.

**Theorem 5.17.** *For $l = m \geq 0$ the semisimplified Saito-Kurokawa part of the inner cohomology $H^\bullet_!(\mathcal{A}_{2,2},\mathcal{V}_\lambda)$ decomposes under the action of $\mathrm{Sp}(4,\mathbb{F}_2)\times\Gamma_\mathbb{Q}$ as*

$$H^{(3,0)}_{!,\text{SK}}(\mathcal{A}_{2,2},\mathcal{V}_\lambda)\oplus H^{(0,3)}_{!,\text{SK}}(\mathcal{A}_{2,2},\mathcal{V}_\lambda)\cong$$
$$\begin{cases} \mathrm{S}^+[\Gamma_0(2),r_1]^{\text{new}}\boxtimes s[4,2] + \mathrm{S}^-[\Gamma_0(2),r_1]^{\text{new}}\boxtimes s[2^3] + \mathrm{S}[\Gamma_0(1),r_1]\boxtimes A' & l \text{ odd},\\ \mathrm{S}[\Gamma_0(4),r_1]^{\text{new}}\boxtimes s[3^2] + \mathrm{S}^+[\Gamma_0(2),r_1]^{\text{new}}\boxtimes s[1^6] + \mathrm{S}^-[\Gamma_0(2),r_1]^{\text{new}}\boxtimes s[5,1] & l \text{ even}\end{cases}$$

*and*

$$H^{(1,1)}_{!,\text{SK}}(\mathcal{A}_{2,2},\mathcal{V}_\lambda)\cong$$
$$L^{m+1}\begin{cases} \mathrm{S}'[\Gamma_0(4),r_1]^{\text{new}}\boxtimes s[3^2] + \mathrm{S}'^+[\Gamma_0(2),r_1]^{\text{new}}\boxtimes s[1^6] + \mathrm{S}^-[\Gamma_0(2),r_1]^{\text{new}}\boxtimes s[5,1] & l \text{ odd},\\ \mathrm{S}^+[\Gamma_0(2),r_1]^{\text{new}}\boxtimes s[4,2] + \mathrm{S}'^-[\Gamma_0(2),r_1]^{\text{new}}\boxtimes s[2^3] + \mathrm{S}'[\Gamma_0(1),r_1]\boxtimes A' & l \text{ even},\end{cases}$$

*and $H^{(2,2)}_{!,\text{SK}}(\mathcal{A}_{2,2},\mathcal{V}_\lambda)\cong L\otimes H^{(1,1)}_{!,\text{SK}}(\mathcal{A}_{2,2},\mathcal{V}_\lambda)$. The other Hodge numbers are zero.*

The proof is given in the next subsection.

**Remark 5.18.** This solves Conjectures 6.6 and 7.4 of Bergström, Faber and van der Geer [BFvdG08]. A Saito-Kurokawa lift is cuspidal if and only if it comes from an automorphic representation $(\sigma, \sigma_\Sigma)$ with $\Sigma = \emptyset$ or $L(\sigma, 1/2) = 0$; this explains the S′ components. Replacing S′ by S gives the Saito-Kurokawa part $H^\bullet_{(2),\mathrm{SK}}$ of the $L^2$-cohomology.

**Corollary 5.19.** $\dim H^{(1,1)}_{(2),\mathrm{SK}}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda) + \dim H^{(3,0)}_{(2),\mathrm{SK}}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda) = 5 \cdot \dim S_{r_1}(\Gamma_0[4])$.

The proof is analogous to Cor. 5.15. This is the closest approximation we can give to Conjecture 7.2 in [BFvdG08] for the case $l = m$. The conjecture is not literally true because the Galois representations on both sides are different.

**Corollary 5.20.** *The inner cohomology of* $\mathcal{A}_{2,2}$ *with* $l \geq m \geq 0$ *and* $l + m \equiv 0 \mod 2$ *decomposes as a direct sum*

$$H^\bullet_!(\mathcal{A}_{2,2}, \mathcal{V}_\lambda) = H^\bullet_{!,\mathrm{endo}}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda) \oplus H^\bullet_{!,\mathrm{SK}}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda) \oplus H^\bullet_{!,\mathrm{stable}}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda),$$

*where the first two terms are given above and the last term is the stable part that decomposes into four-dimensional irreducible Galois representations and contributes equally to Hodge types* $(3,0)$, $(2,1)$, $(1,2)$, $(0,3)$.

*Proof.* This is decomposition (5.10). The Soudry lift does not contribute to the inner cohomology in level two by the same argument as in level one: For $m = 0$ we have even $l$, so the central character $\omega$ of a Soudry lift must satisfy $\omega(-1) = 1$. But the Soudry lift preserves central characters and there are no non-zero elliptic modular forms of odd weight $3 + m$ whose congruence group contains $-1$. □

Together with Theorem 5.12 this determines the compactly supported cohomology with regular weight up to semisimplification.

### 5.5.3. The proof

We will now prove the above theorems about the endoscopic and the Saito-Kurokawa part of the inner cohomology in level two. At first, we establish the correspondence between modular forms and automorphic representation.

It is well-known that every cuspidal automorphic representation $\sigma$ of $\mathrm{GL}(2, \mathbb{A})$ with discrete series archimedean factor $\sigma_\infty \cong \mathcal{D}(r)$ and trivial central character is generated by a unique elliptic newform of weight $r$ and level $\Gamma_0(N)$, compare e.g. [Gel75]. At the unramified places $p$ not dividing $N$ the local factor $\sigma_p$ is spherical and determined by the Satake parameters. If $p$ divides $N$ exactly once, $\sigma_p$ is determined by the Atkin-Lehner eigenvalue. Otherwise it is not known in general how to describe $\sigma_p$ in terms of $f$. However, for $p = 2$ the situation simplifies.

**Lemma 5.21.** *Let $\sigma$ be an irreducible smooth representation of $G = \mathrm{GL}(2, \mathbb{Q}_2)$ of depth zero, but not Iwahori-spherical. Then $\sigma$ is cuspidal, its hyperspecial parahoric restriction is cuspidal, and $\sigma$ is uniquely determined by its central character. Its local Euler factors are $\epsilon(\sigma, s) = -1$ and $L(\sigma, s) = 1$ for every $s \in \mathbb{C}$.*

*Proof.* An Iwahori subgroup of $\mathrm{GL}(2, \mathbb{Q}_2)$ is its own pro-unipotent radical, so the parahoric restriction with respect to an Iwahori subgroup is zero. The hyperspecial parahoric restriction $\tilde{\sigma} = \mathbf{r}_{\mathrm{GL}(2,\mathbb{Z}_2)}(\sigma)$ as a representation of $\mathrm{GL}(2, \mathbb{F}_2)$ is non-zero by assumption and it is cuspidal by (2.5). By Example 2.21, $\sigma$ itself is cuspidal. By Lemma 2.18, the hyperspecial parahoric restriction $\tilde{\sigma}$ is irreducible. There is only one isomorphism class of cuspidal irreducible representations $\mathrm{GL}(2, \mathbb{F}_2)$. Now Thm. 2.15 implies that $\sigma$ is uniquely determined by the central character $\omega_\sigma$. It remains to determine the Euler factors.

Fix an additive character $\psi : \mathbb{Q}_2 \to \mathbb{C}^\times$ of conductor one. Its restriction to $\mathbb{Z}_2$ factors over the non-trivial additive character $\tilde{\psi} : \mathbb{Z}_2/2\mathbb{Z}_2 \to \mathbb{C}^\times$. The $\epsilon$-factor of $\sigma$ is given by §25.2 of [BH06] (their notation)

$$\epsilon(\sigma, s, \psi) = 2^{2\ell(\sigma)(\frac{1}{2}-s)} \frac{\tau(\Xi, \psi)}{(\mathfrak{A} : \mathfrak{P}^{n+1})^{1/2}}.$$

Depth zero implies that $n$ and $\ell(\sigma)$ are zero. We have $\mathfrak{A} = \mathrm{Mat}_2(\mathbb{Z}_2)$ and $\mathfrak{P} = I_2 + \mathrm{Mat}_2(2\mathbb{Z}_2)$ and therefore $(\mathfrak{A} : \mathfrak{P}) = 16$. In the classification of Thm. A.1, the cuspidal irreducible $\tilde{\sigma}$ corresponds to a non-trivial regular character of $\mathbb{F}_4^\times$, denoted by $\theta$ in §6.4 of [BH06]. By the first equation in §23.7 of [BH06]

$$\tau(\Xi, \psi) = -2 \sum_{x \in \mathbb{F}_4^\times} \overline{\Lambda(x)} \tilde{\psi}(x + x^2) = -4.$$

The local $L$-factor is trivial for every cuspidal representation of $\mathrm{GL}(2, \mathbb{Q}_2)$. □

The analogue of Lemma 5.21 for odd residue characteristic includes principal series representations induced from tamely ramified characters.

**Lemma 5.22.** *For a cuspidal elliptic newform $f$ of level $N$ and weight $r$ let $\sigma$ be the automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ generated by $f$. The local factor $\sigma_2$ is of depth zero if and only if it belongs to one of the following:*

1. *The spherical principal series $\sigma_2 = \mu \times \mu^{-1}$ for an unramified character $\mu$ of $\mathbb{Q}_2^\times$ with $\mu^2 \neq |\cdot|^{\pm 1}$. This occurs if and only if $N$ is odd.*
2. *the Steinberg representation $\sigma_2 = \mathrm{St}_{\mathrm{GL}(2,\mathbb{Q}_2)}$. This occurs if and only if $N \cong 2 \bmod 4$ and the Atkin-Lehner eigenvalue at two is $\epsilon_2 = -1$,*

3. *the twist* $\sigma_2 = \xi_u \operatorname{St}_{\mathrm{GL}(2,\mathbb{Q}_2)}$ *of the Steinberg by the unramified quadratic character* $\xi$ *of* $\mathbb{Q}_2^\times$*. This occurs if and only if* $f$ *has level* $N \equiv 2 \mod 4$ *and Atkin-Lehner eigenvalue* $\epsilon_2 = +1$*,*

4. *the unique cuspidal representation of* $\mathrm{GL}(2,\mathbb{Q}_2)$ *with depth zero and trivial central character. This occurs if and only if* $N \equiv 4 \mod 8$*.*

*Proof.* If $\sigma_2$ is Iwahori-spherical, it belongs to one of the first three cases. The level is given by [Gel75, Prop. 5.21]. Otherwise it belongs to the last case by Lemma 5.21 and because $\Gamma_0(4)$ is conjugate to the principal congruence subgroup $\Gamma(2) \subseteq \mathrm{SL}(2,\mathbb{Z})$. □

*Proof of Theorem 5.13.* This is Example 5.9 for $p = 2$. The central characters of the automorphic representations $\sigma$ must factor over $(\mathbb{Z}/2\mathbb{Z})^\times$ and are therefore trivial. The sum runs over pairs of automorphic representations $\sigma = (\sigma_1, \sigma_2)$ of $\mathrm{PGL}(2,\mathbb{A}) \times \mathrm{PGL}(2,\mathbb{A})$ with archimedean part in the discrete series of weight $r_1$ and $r_2$.

For a fixed $\sigma$, only the endoscopic lifts $\pi$ with generic non-holomorphic archimedean factor $\pi_\infty = \Pi_+(\sigma_\infty)$ contribute to cohomology with Hodge numbers $(2,1)$ and $(1,2)$. At the non-archimedean places $v \neq 2$, the local lift $\pi_v$ is spherical if and only if $\sigma_v$ is spherical.

At $v = 2$ the local lift must be the generic $\pi_v = \Pi_+(\sigma_v)$ by the multiplicity formula in Thm. 5.6. It is depth zero if and only if $\sigma_v$ is depth zero at $v = 2$ by Cor. 4.14. For each local factor $\sigma_{i,v}$ there are four possible cases by Lemma 5.22 and they occur with cardinalities $\tau_{1,r_i}$, $\tau^-_{2,r_i}$, $\tau^+_{2,r_i}$, $\tau_{4,r_i}$, respectively. The corresponding local lifts $\Pi_+(\sigma_v)$ are given in Table 4.1 and their hyperspecial parahoric restriction is given in Table 4.2 in the language of Enomoto's characters. The translation into irreducible representation of the symmetric group $\Sigma_6$ is given in Table 5.2. The $\ell$-adic Galois representation is $L^{m+1}\rho_{\sigma_2}$ by (5.25).

For Hodge types $(3,0)$, $(0,3)$ the proof is analogous with the local lift $\pi_v = \Pi_-(\sigma_v)$ at the place $v = 2$. This is only possible when $\sigma_v$ is in the discrete series at $v = 2$, so there are no contributions with level $N = 1$. □

*Proof of Theorem 5.17.* This is Example 5.5 for $p = 2$. The central character $\omega$ must factor over $(\mathbb{Z}/2\mathbb{Z})^\times$ and hence be trivial. The argument is analogous to the endoscopic case. Contributions to Hodge type $(3,0)$ and $(0,3)$ come from Saito-Kurokawa lifts whose archimedean factor is the holomorphic discrete series representation $\pi_\infty = \pi^H_\lambda = \pi(\mathcal{D}(r_1), \mathcal{D}(2))$.

Non-zero contributions to the cohomology can only come from automorphic representations $\sigma$ of $\mathrm{GL}(2,\mathbb{A})$ which are locally spherical at $v \neq 2$ and of depth zero at $v = 2$. Their lifts $\pi_v$ are spherical at $v \neq 2, \infty$ and the local factor $\sigma_2$ belongs to one of the four cases in Lemma 5.22. The local lifts and their parahoric restrictions are given by Table 5.1. A global lift is automorphic if and only if the $\epsilon$-factor satisfies $\epsilon(\sigma, 1/2) = (-1)^{\#\Sigma}$. Here the right hand side is $(-1)^{r_1}\epsilon(\sigma_v, 1/2)$ for $v = 2$, so automorphy depends on the parity of $l$. The lifts are all cuspidal, because $\Sigma$ is not empty. Summing over a the possible $\sigma$ gives the contributions with corresponding multiplicities like in the endoscopic case. The Galois representation is given by (5.18).

Contributions to Hodge types $(1,1)$ and $(2,2)$ come from Saito-Kurokawa lifts whose archimedean factor is the non-holomorphic Langlands quotient $L(\nu^{1/2}\mathcal{D}(r_1), \xi\nu^{-1/2})$. The proof is analogous, but we have to exclude any $\sigma$ with $\epsilon(\sigma, 1/2) = 1$ and $L(\sigma, 1/2) \neq 0$, because its lift does not contribute to the cuspidal spectrum. □

### 5.5.4. Hodge numbers

We determine the Hodge numbers $h_!^{(p,q)} = \dim H_!^{(p,q)}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda)$ of the inner cohomology attached to a local system of highest weight $(l, m)$.

**Corollary 5.23.** *For regular local systems with $l > m \geq 2$ and even sum $l + m$ the non-zero Hodge numbers of the inner cohomology are*

$$h_!^{(3,0)} = h_!^{(0,3)} = \tfrac{1}{24}(l-m+1)(l+2)(m+1)(l+m+3) \\ -\tfrac{5}{8}(l-m+1)(l+m) - (-1)^m\tfrac{5}{8}(m-2)(l-1),$$

$$h_!^{(2,1)} = h_!^{(1,2)} = h_!^{(3,0)} + \tfrac{5}{4}(l+m)(l-m-2).$$

*Proof.* Tsushima [Tsu83, Thm. 3] has calculated the dimension of $H_!^{(3,0)}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda)$. The Hecke modules $H^{(3,0)}_{\text{stable}}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda)$ and $H^{(2,1)}_{\text{stable}}(\mathcal{A}_{2,2}, \mathcal{V}_\lambda)$ are isomorphic [Wei05] and we obtain $h^{(2,1)}_{\text{stable}} = h^{(3,0)}_{\text{stable}}$. For regular $\lambda$ there is no contribution from CAP-representations. Now Cor. 5.15 implies

$$h_!^{(2,1)} = h_!^{(3,0)} + 5\dim S_{r_1}(\Gamma_0[4]) \cdot \dim S_{r_2}(\Gamma_0[4]) = h_!^{(3,0)} + 5 \cdot \frac{l+m}{2} \cdot \frac{l-m-2}{2}.$$

Complex conjugation gives $h_!^{(0,3)} = h_!^{(3,0)}$ and $h_!^{(1,2)} = h_!^{(2,1)}$. By Falting's result [Fal83] the other Hodge numbers are zero. □

**Corollary 5.24.** *For local systems with $l = m \geq 1$, the non-zero Hodge numbers of*

| $\mathrm{Sp}(4,\mathbb{F}_2)$ | $\theta_0$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\chi_5(1)$ | $\chi_8(1)$ | $\chi_9(1)$ | $\chi_{12}(1)$ | $\chi_{13}(1)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Sigma_6$ | $[6]$ | $[4,2]$ | $[2^3]$ | $[5,1]$ | $[3,2,1]$ | $[1^6]$ | $[2^2,1^2]$ | $[3^2]$ | $[2,1^4]$ | $[4,1^2]$ | $[3,1^3]$ |
| dim | 1 | 9 | 5 | 5 | 16 | 1 | 9 | 5 | 5 | 10 | 10 |

Table 5.2.: Irreducible representations of $\mathrm{Sp}(4,\mathbb{F}_2) \cong \Sigma_6$.

*the inner cohomology are*

$$\begin{aligned} h_!^{(3,0)} = h_!^{(0,3)} &= \tfrac{1}{24}(l-m+1)(l+2)(m+1)(l+m+3) \\ &\quad -\tfrac{5}{8}(l-m+1)(l+m) - (-1)^m \tfrac{5}{8}(m-2)(l-1), \end{aligned}$$

$$h_!^{(1,2)} = h_!^{(2,1)} = h_!^{(3,0)} - \begin{cases} 9\tau_{r_1}^{+} + 5\tau_{r_1}^{-} + 15\tau_{1,r_1} & l \text{ odd}, \\ 5\tau_{4,r_1} + 1\tau_{r_1}^{+} + 5\tau_{r_1}^{-} & l \text{ even}, \end{cases}$$

$$h_!^{(1,1)} = h_!^{(2,2)} = \begin{cases} 5\tau'_{4,r_1} + 1\tau'^{+}_{r_1} + 5\tau_{r_1}^{-} & l \text{ odd}, \\ 9\tau'^{+}_{r_1} + 5\tau'^{-}_{r_1} + 15\tau'_{1,r_1} & l \text{ even}. \end{cases}$$

*Proof.* The non-zero terms come from the Saito-Kurokawa lift and from the stable part. The argument is analogous. □

### 5.5.5. The isomorphism $\mathrm{GSp}(4,\mathbb{F}_2) \cong \Sigma_6$

Permuting the six Weierstraß points on a hyperelliptic curve of genus two defines an action of the symmetric group $\Sigma_6$ on the moduli space of their Jacobians and thus on $\mathcal{A}_{2,2}$. It preserves the Weil pairing and thus gives rise to a morphism from $\Sigma_6$ to the finite symplectic group $\mathrm{Sp}(4,\mathbb{F}_2)$. This defines an isomorphism of finite groups.

Let us give an alternative. The symmetric group $\Sigma_6$ acts naturally on the vector space $\mathbb{F}_2^6$ and preserves the inner product $\langle v,w\rangle = \sum_{i=1}^6 v_i w_i$. The isotropic vector $u = \mathrm{diag}(1,1,1,1,1,1)$ is fixed under action. Since the inner product is symplectic on the four-dimensional space $u^\perp/u$, this defines a group homomorphism $\Sigma_6 \to \mathrm{Sp}(4,\mathbb{F}_2)$, which is actually an isomorphism.

The irreducible representations of $\Sigma_6$ are classified by partitions of six. The dictionary between Enomoto's characters [Eno72] and the partitions of six is given in Table 5.2. The finite group $\mathrm{GL}(2,\mathbb{F}_2)$ admits three isomorphism classes of irreducible representations, the trivial $\mathbf{1}$, the Steinberg St and the cuspidal representation $\sigma$. Parabolic induction via the Klingen parabolic (respectively, the Siegel parabolic) in $\mathrm{Sp}(4,\mathbb{F}_2)$ gives rise to the following representations:

$A = \chi_7(0) \cong s[6] + s[4,2] + s[5,1]$, $\quad A' = \chi_6(0) \cong s[6] + s[4,2] + s[2^3]$,
$B = \chi_{11}(0) \cong s[3,2,1] + s[4,2] + s[2^3]$, $\quad B' = \chi_{10}(0) \cong s[3,2,1] + s[4,2] + s[5,1]$,
$C = \chi_3(0,1) \cong s[3,1^3] + s[2,1^4]$, $\quad C' = \chi_2(1) \cong s[4,1^2] + s[3^2]$.

# 6. Conclusion

The parahoric restriction of irreducible admissible representations $\pi$ of $\mathrm{GSp}(4,F)$ over a non-archimedean local number field is now completely known. For non-cuspidal $\pi$, this is given by our results in Chapter 3. The non-generic cuspidal $\rho$ occur in the anisotropic theta-lift. Up to possible character twists, their parahoric restriction is given by the lower halfs of Table 4.2 and Table 4.4. For generic cuspidal $\rho$ of depth zero, the parahoric restriction with respect to hyperspecial parahorics is irreducible cuspidal and it is zero for every other parahoric. This is implied by a result of deBacker and Reeder [DR09, 6.1.1] and our Lemma 2.18. For positive depth, the parahoric restriction is zero by definition.

Our description of the endoscopic character lift in depth zero has provided further evidence for the expected depth preservation under the local Langlands correspondence. The non-cuspidal case is already a result of Moy and Prasad [MP96, 5.2(1)].

The description of the inner cohomology of the Siegel modular threefold with principal congruence subgroup level two is now complete. In principle, we can also calculate the Hodge numbers for arbitrary open compact subgroups of $\mathrm{GSp}(4,\mathbb{A}_{\mathrm{fin}})$, that contain a principal congruence subgroup of squarefree level, in terms of automorphic representations of $\mathrm{GL}(2)$. This would depend on an explicit local description of Soudry lifts and the analogues to Tsushimua's dimension formulas [Tsu83]. The Eisenstein cohomology has been described precisely by Harder [Har12] and can be used to determine the compactly supported cohomology in the analogous fashion to Petersen's results [Pet15].

**Outlook**. In a recent article, Clery, van der Geer and Grushevsky [CvdGG15] have begun a detailed description of the $\mathrm{Sp}(4,\mathbb{F}_2)$-representations on the vector space of Siegel modular forms invariant under the principal congruence subgroup of level two. It would be interesting to understand the explicit correspondence with our results.

Bergström, Faber and van der Geer have extended their numerical calculations to genus three [BFvdG14]. If there was a classification of automorphic representations of $\mathrm{GSp}(6)$, the analogous techniques could be employed to prove their conjectures. To the author's knowledge, such a classification seems absent.

We have obtained some evidence that the hyperspecial parahoric restriction of a depth zero generic irreducible admissible representation $\pi$ of $\mathrm{GSp}(4,F)$ contains a generic constituent. Under certain technical assumptions on $\pi$ we could show this for arbitrary unramified connected reductive groups [MR].

# A. Representations of finite groups

Let $\mathbb{F}_q$ be the finite field of order $q$. For certain linear groups $\mathbf{G}$ over $\mathbb{F}_q$ we recall the irreducible representations of $\mathbf{G}(q)$. Fix an non-trivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$. For odd $q$ let $\lambda_0$ and $\Lambda_0$ be the non-trivial quadratic characters of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$.

The character $\chi_\sigma$ of a representation $\sigma$ is denoted by $\sigma$ again, when the meaning is clear from the context.

## A.1. $\mathrm{GL}(2,q)$ and $\mathrm{SL}(2,q)$

An extensive survey of the representation theory of $\mathrm{GL}(2,q)$ has been given by Piatetski-Shapiro [PS83a]. The conjugacy classes of $G = \mathrm{GL}(2,q)$ are those which admit an $\mathbb{F}_q$-rational Jordan normal form and the anisotropic classes $E(\alpha)$ of elements with eigenvalues $\alpha, \alpha^q$ for $\alpha \in \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times$ like $\left(\begin{smallmatrix} 0 & -\alpha^{q+1} \\ 1 & \alpha+\alpha^q \end{smallmatrix}\right) \in \mathrm{E}(\alpha)$.

**Theorem A.1.** *Up to isomorphism, the irreducible representations of $G$ are*

1. *twists of the trivial representation $\mu \cdot \mathbf{1}_G = \mu \circ \det$ for $\mu : \mathbb{F}_q^\times \to \mathbb{C}^\times$,*
2. *twists of the Steinberg representation $\mu \cdot \mathrm{St}_G$ for $\mu : \mathbb{F}_q^\times \to \mathbb{C}^\times$,*
3. *principal series $\mu_1 \times \mu_2$ for characters $\mu_1, \mu_2 : \mathbb{F}_q^\times \to \mathbb{C}^\times$ with $\mu_1 \neq \mu_2$,*
4. *cuspidal representations $\pi_\Lambda$ for a character $\Lambda : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ with $\Lambda \neq \Lambda^q$.*

*They are pairwise inequivalent except for $\mu_1 \times \mu_2 \cong \mu_2 \times \mu_1$ and $\pi(\Lambda) \cong \pi(\Lambda^q)$. The character values are listed in Table A.1.*

A representation $(\sigma, V)$ of $\mathrm{GL}(2,q)$ is *generic* if it admits a non-zero $v \in V$ with $\sigma\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) v = \psi(x)v$ for $x \in \mathbb{F}_q$. This does not depend on the choice of $\psi$. Except for the twists of the trivial representation, all the irreducible representations are generic.

For odd $q$ there is an explicit model of the cuspidal irreducible representations:

**Proposition A.2.** *For an odd prime power $q$ fix a character $\Lambda$ of $\mathbb{F}_{q^2}^\times$ with $\Lambda \neq \Lambda^q$. Let $V = \{f : \mathbb{F}_q^\times \to \mathbb{C}\}$ and let $\rho : G \to \mathrm{Aut}(V)$ be the homomorphism given on the standard Borel of $G$ by*

$$(\rho\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) f)(x) = \Lambda(d)\psi(\tfrac{b}{d}x)f(\tfrac{a}{d}x)$$

| Conj. class | $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ $a \in \mathbb{F}_q^\times$ | $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ $a \in \mathbb{F}_q^\times$ | $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ $a, d \in \mathbb{F}_q^\times, a \neq d$ | $\mathrm{E}(\alpha)$ $\alpha \in \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times$ |
|---|---|---|---|---|
| # of classes | $q-1$ | $q-1$ | $\frac{1}{2}(q-1)(q-2)$ | $\frac{1}{2}(q-1)q$ |
| $\mu\,\mathbf{1}_{\mathrm{GL}(2,q)}$ | $\mu(a)^2$ | $\mu(a)^2$ | $\mu(ad)$ | $\mu \circ \mathrm{nr}(\alpha)$ |
| $\mu \cdot \mathrm{St}_{\mathrm{GL}(2,q)}$ | $q\mu(a)^2$ | $0$ | $\mu(ad)$ | $-\mu \circ \mathrm{nr}(\alpha)$ |
| $\mu \times \nu$ | $(q+1)\mu(a)\nu(a)$ | $\mu(a)\nu(a)$ | $\mu(a)\nu(d) + \mu(d)\nu(a)$ | $0$ |
| $\pi_\Lambda$ | $(q-1)\Lambda(a)$ | $-\Lambda(a)$ | $0$ | $-\Lambda(\alpha) - \Lambda(\alpha^q)$ |

Table A.1.: Character table of $\mathrm{GL}(2, q)$.

*and on the non-trivial Weyl group element $w' = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ by*

$$(\rho(w')f)(x) = -q^{-1} \sum_{y \in \mathbb{F}_q^\times} \Lambda(y^{-1}) \sum_{\substack{\alpha \in \mathbb{F}_{q^2}^\times \\ \alpha^{q+1} = xy}} \psi(\alpha + \alpha^q)\Lambda(\alpha) f(y).$$

*for $a, d, x \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$. This gives rise to a well-defined cuspidal irreducible representation $(\rho, V)$ of $G$ isomorphic to $\pi_\Lambda$.*

*Proof.* For the proof, see Piatetski-Shapiro [PS83a, §13].[1] □

**Remark A.3.** In the special case $q = 2$ there is an isomorphism $\mathrm{GL}(2, 2) \cong \Sigma_3$ to the symmetric group in three letters given by the natural action of $\mathrm{GL}(2, 2)$ on the projective space $\mathbb{P}^1\mathbb{F}_2$. Therefore irreducible representations of $\mathrm{GL}(2, 2)$ can be classified by partitions of three as in [BFvdG08].

Irreducible representations of $\mathrm{SL}(2, q)$ can be obtained by restricting representations of $\mathrm{GL}(2, q)$. The definition of *generic* is the same as for $\mathrm{GL}(2, q)$, but depends on $\psi$.

**Corollary A.4** (Irreducible Representations of $\mathrm{SL}(2, q)$)**.** *Let $\sigma$ be an irreducible representation of $\mathrm{GL}(2, q)$. If $q$ is even or if $\lambda_0\sigma \not\cong \sigma$, then its restriction $[\sigma]$ to $\mathrm{SL}(2, q)$ is irreducible. Otherwise the restriction has two irreducible equidimensional constituents, a $\psi$-generic $[\sigma]_+$ and a non-$\psi$-generic $[\sigma]_-$. These are all the irreducible representations of $\mathrm{SL}(2, q)$.*

For non-square $a \in \mathbb{F}_q^\times$ the representation $[\sigma]_-$ is $\psi_a$-generic and $[\sigma]_+$ is not $\psi_a$-generic with respect to the additive character $\psi_a : x \mapsto \psi(ax)$.

**Remark A.5.** For odd $q$, the only representations $\sigma$ of $\mathrm{GL}(2, q)$ with $\sigma \cong \lambda_0\sigma$ are $\sigma = \mu \times \lambda_0\mu$ and $\sigma = \pi_{\Lambda'_0}$ such that ${\Lambda'_0}^{q-1} = \Lambda_0$ is the nontrivial quadratic character of $\mathbb{F}_{q^2}^\times$. The character values of $[\sigma]_\pm$ on $u_x = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$ are $\mathrm{tr}((\mu \times \lambda_0\mu)_\pm; u_x) = \frac{1}{2}(1 \pm \lambda_0(x)\mathfrak{G})$ and $\mathrm{tr}((\pi_\Lambda)_\pm; u_x) = \frac{1}{2}(-1 \pm \lambda_0(x)\mathfrak{G})$ with the Gauß sum $\mathfrak{G} = \sum_{x \in \mathbb{F}_q^\times} \lambda_0(x)\psi(x)$.

[1] There is a minus sign missing in the definition of $j$ in loc. cit.

## A.2. $(\mathrm{GL}(2,q)^2)^0$

Let $G = (\mathrm{GL}(2,q)^2)^0$ be the group of $(x,y) \in \mathrm{GL}(2,q)^2$ with equal determinant $\det x = \det y$. We call an irreducible representation $(\rho, V)$ of $G$ *generic*, if it contains a non-zero $v \in V$ with $\rho(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right))v = \psi(x+y)v$. This condition does not depend on the choice of $\psi$.

**Lemma A.6** (Irreducible representations)**.** *For an irreducible representation $\sigma = \sigma_1 \boxtimes \sigma_2$ of $\mathrm{GL}(2,q)^2$ let $[\sigma]$ denote its restriction to $G$. If $q$ is even or if $\sigma_1 \not\cong \lambda_0\sigma_1$ or $\sigma_2 \not\cong \lambda_0\sigma_2$, then $[\sigma]$ is irreducible. Otherwise*[2] *$[\sigma]$ is a direct sum of two equidimensional irreducible representations $[\sigma]_+$ and $[\sigma]_-$, where $[\sigma]_+$ is generic and $[\sigma]_-$ is not generic.*[3] *These are all the irreducible representations of $G$.*

*Proof.* For an irreducible representation $\sigma$ of $\mathrm{GL}(2,q)^2$, Frobenius reciprocity implies

$$\begin{aligned}
&\dim \mathrm{Hom}_G([\sigma],[\sigma]) \\
&= \sum_{\mu \in \widehat{\mathbb{F}_q^\times}} \dim \mathrm{Hom}_{\mathrm{GL}(2,q)}(\sigma_1, \mu \cdot \sigma_1) \cdot \dim \mathrm{Hom}_{\mathrm{GL}(2,q)}(\sigma_2, \mu^{-1} \cdot \sigma_2) \\
&= \begin{cases} 1 & q \text{ even}, \\ 1 + \dim \mathrm{Hom}_{\mathrm{GL}(2,q)}(\sigma_1, \lambda_0 \cdot \sigma_1) \cdot \dim \mathrm{Hom}_{\mathrm{GL}(2,q)}(\sigma_2, \lambda_0 \cdot \sigma_2) & q \text{ odd}. \end{cases}
\end{aligned}$$

The sum runs over characters $\mu : \mathbb{F}_q^\times \to \mathbb{C}^\times$ and for $\mu^2 \neq 1$ the factors must be zero by comparing central characters. For odd $q$, this term is two if and only if $\sigma_i \cong \lambda_0\sigma_i$ and in that case there are two irreducible constituent in $[\sigma]$. Each of these irreducible subquotients decomposes into two irreducible representations of $\mathrm{SL}(2,q) \times \mathrm{SL}(2,q)$ by the analogous argument. Let $[\sigma]_+$ be the $G$-constituent that becomes $[\sigma_1]_+ \boxtimes [\sigma_2]_+ \oplus [\sigma_1]_- \boxtimes [\sigma_2]_-$ upon restriction to $\mathrm{SL}(2,q)^2$, then for any choice of $\psi$ exactly one of these $\mathrm{SL}(2,q)^2$-constituents contains a Whittaker-vector. By Frobenius reciprocity every irreducible representation of $G$ is contained in some irreducible representation of $\mathrm{GL}(2,q)^2$. □

**Remark A.7.** The twist of $[\sigma]$ by a character $\mu$ of $\mathbb{F}_q^\times$ is

$$\mu \cdot [\sigma_1, \sigma_2] = [\mu\sigma_1, \sigma_2] = [\sigma_1, \mu\sigma_2]. \tag{A.1}$$

This is the only equivalence between representations of $G$.

For an irreducible representation $\sigma = \sigma_1 \boxtimes \sigma_2$ of $\mathrm{GL}(2,q)^2$ we call the involution $\sigma \mapsto \sigma^* = \sigma_2 \boxtimes \sigma_1$ the *opposite*. For the corresponding representations of $G$ write

[2]This happens exactly when $\sigma_1$ and $\sigma_2$ belong to the cases described in Remark A.5.

[3]If $q$ is odd and $\lambda_0\sigma \cong \sigma$ and if we replace $(x,y) \mapsto \psi(x+y)$ by $\psi' : (x,y) \mapsto \psi(ax+by)$, where $ab$ is not a square in $\mathbb{F}_q$, then $[\sigma]_-$ is $\psi'$-generic and $[\sigma]_+$ is not $\psi'$-generic.

$[\sigma]^* := [\sigma^*]$ and $[\sigma]^*_\pm := [\sigma^*]_\pm$. The generic representations are preserved under this involution, so the notation is justified.

On the unipotent element $u = ((\begin{smallmatrix} 1 & u_1 \\ & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & u_2 \\ & 1 \end{smallmatrix})) \in G$ with $u_1, u_2 \in \mathbb{F}_q^\times$ Remark A.5 implies that the character value of $[\sigma_1, \sigma_2]_\pm$ is

$$\operatorname{tr}([1 \times \lambda_0, 1 \times \lambda_0]_\pm; u) = \operatorname{tr}([\pi_{\Lambda_0'}, \pi_{\Lambda_0'}]_\pm; u) = \tfrac{1}{2}(1 \pm \lambda_0(-u_1 u_2) q), \tag{A.2}$$
$$\operatorname{tr}([1 \times \lambda_0, \pi_{\Lambda_0'}]_\pm; u) = \operatorname{tr}([\pi_{\Lambda_0'}, 1 \times \lambda_0]_\pm; u) = \tfrac{1}{2}(-1 \pm \lambda_0(-u_1 u_2) q). \tag{A.3}$$

## A.3. $\mathrm{GSp}(4, q)$ for odd $q$

The irreducible representations of the finite group $G = \mathrm{GSp}(4, q)$ with odd $q$ have been classified by Shinoda [Shi82, Table 5].

Characters of $\mathbb{F}_q^\times$ are denoted $\lambda$ or $\nu$, while $\Lambda, \omega, \lambda'$ and $\Theta$ will always be characters of $\mathbb{F}_{q^2}^\times, \mathbb{F}_{q^2}^\times[q+1], \mathbb{F}_{q^2}^\times[2(q-1)]$ and $\mathbb{F}_{q^4}^\times[(q-1)(q^2+1)]$, respectively. We only consider the case where $\lambda'$ does not factor over some $\lambda$, which means $\lambda'(-1) = -1$. The non-trivial quadratic character of $\mathbb{F}_{q^2}^\times[q+1]$ is $\omega_0$. For typesetting reasons, we write

$$\begin{aligned} A[\lambda_1, \lambda_2, \nu] =& \lambda_1 \boxtimes \lambda_2 \boxtimes \nu + \lambda_1 \boxtimes \lambda_2^{-1} \boxtimes \lambda_2 \nu \\ & + \lambda_1^{-1} \boxtimes \lambda_2 \boxtimes \lambda_1 \nu + \lambda_1^{-1} \boxtimes \lambda_2^{-1} \boxtimes \lambda_1 \lambda_2 \nu, \\ B[\lambda_1, \lambda_2, \nu] =& \lambda_1 \boxtimes (\lambda_2 \rtimes \nu) + \lambda_1^{-1} \boxtimes (\lambda_2 \rtimes \lambda_1 \nu), \\ C[\lambda_1, \lambda_2, \nu] =& (\lambda_1 \times \lambda_2) \boxtimes \nu + (\lambda_1^{-1} \times \lambda_2) \boxtimes \lambda_1 \nu. \end{aligned}$$

Shinoda denotes the conjugacy classes of $G$ by $A_0,\ A_1,\ \ldots,\ L_1$ in his Table 2. He gives explicit values for certain virtual characters in his Table 5. For their decomposition into irreducibles see Table A.2. This gives all the irreducible representations of $G$ [Shi82, §5].

**Lemma A.8** (Shinoda [Shi82, p.1403]). *The decomposition of the virtual characters $X_1, \ldots, X_5$ and $\chi_1, \ldots, \chi_8$ into irreducible components is given in Table A.2. The characters $\theta_1(\nu), \ldots, \theta_5(\nu)$ and $\tau_1(\lambda), \ldots, \tau_5(\lambda')$ for $\lambda'(-1) = -1$ are always irreducible.*

**Corollary A.9.** *The parabolic restriction (Jacquet functor) of the irreducible representations of* $\mathrm{GSp}(4, q)$ *is given by Table A.4.*

*Proof.* This is directly implied by Frobenius reciprocity and the decompositions in Table A.2. □

**Lemma A.10.** *Shinoda's virtual characters are pairwise inequivalent except for the following identities.*

$$X_4(\Theta) = X_4(\Theta^q) = X_4(\Theta^{q^2}) = X_4(\Theta^{q^3})$$
$$X_5(\Lambda,\omega) = X_5(\Lambda\tilde{\omega},\omega^{-1}) = X_5(\Lambda^q,\omega^{-1}) = X_5(\Lambda^q\tilde{\omega}^q,\omega) = X_5(\Lambda,(\Lambda\circ i_{q-1})\cdot\omega^{-1})$$
$$= X_5(\Lambda^q,(\Lambda^q\circ i_{q-1})\cdot\omega) = X_5(\Lambda\tilde{\omega},(\Lambda\circ i_{q-1})\cdot\omega^{-1}) = X_5(\Lambda^q\tilde{\omega}^q,(\Lambda^q\circ i_{q-1})\cdot\omega),$$

$$\begin{aligned}
&\chi_1(\lambda,\nu) = \chi_1(\lambda^{-1},\lambda\nu), && \chi_2(\lambda,\nu) = \chi_2(\lambda^{-1},\lambda\nu),\\
&\chi_3(\lambda,\nu) = \chi_3(\lambda^{-1},\lambda^2\nu), && \chi_4(\lambda,\nu) = \chi_4(\lambda^{-1},\lambda^2\nu),\\
&\chi_5(\omega,\nu) = \chi_5(\omega^{-1},\nu), && \chi_6(\omega,\nu) = \chi_6(\omega^{-1},\nu),\\
&\chi_7(\Lambda,\nu) = \chi_7(\Lambda^q,\nu), && \chi_8(\Lambda,\nu) = \chi_8(\Lambda^q,\nu),\\
&\tau_1(\lambda) = \tau_1(\lambda\lambda_0), && \tau_3(\lambda) = \tau_3(\lambda\lambda_0),\\
&\tau_4(\lambda') = \tau_4(\lambda'^q), && \tau_5(\lambda') = \tau_5(\lambda'^q).
\end{aligned}$$

*The identities between $X_1, X_2, X_3$ are those from the action of the Weyl group and parabolic induction.*

*Proof.* This is a simple calculation. □

**Lemma A.11.** *Let $\rho$ be an irreducible representation of* GSp(4, $q$) *for odd $q$. Table A.3 lists the central character $\omega_\rho$ of $\rho$; the cuspidal $\rho$ (a "•" in the c-column); the generic $\rho$ (a "•" in the g-column); and the dimension of invariants under the subgroup* $\{\mathrm{diag}(1,1,*,*)\} \subseteq \mathrm{GSp}(4,q)$.

*Proof.* The central characters are given by $\omega_\rho(a) = \rho(A_0(a))/\dim\rho$ for $a \in \mathbb{F}_q^\times$. The cuspidal irreducible representations are those that do not occur as subrepresentations of parabolically induced proper representations in Table A.2. The generic irreducible ones are exactly those whose dimension is a fourth degree polynomial in $q$ [Shi82, p. 1405]. The dimension of invariants under $\{\mathrm{diag}(1,1,*,*)\}$ is given by $\frac{1}{q-1}(\rho(A_0(1)) + \sum_{1\neq b\in\mathbb{F}_q^\times} \rho(D_0(1,b)))$. The calculations are straightforward. □

| $\rho \in \mathbf{R}_{\mathbb{Z}}(G)$ | induced | Decomposition | for |
|---|---|---|---|
| $X_1(\lambda_1, \lambda_2, \nu)$ | $\lambda_1 \times \lambda_2 \rtimes \nu$ | (irreducible) | $1 \neq \lambda_1 \neq \lambda_2^{\pm 1} \neq 1$ |
| | | $\chi_1(\lambda_1, \nu) + \chi_2(\lambda_1, \nu)$ | $\lambda_1 \neq \lambda_2 = 1$ |
| | | $\chi_3(\lambda_1, \nu) + \chi_4(\lambda_1, \nu)$ | $\lambda_1 = \lambda_2 \neq 1, \lambda_0$ |
| | | $\nu(\tau_1 + \tau_2 + \lambda_0\tau_2 + \tau_3)$ | $\lambda_1 = \lambda_2 = \lambda_0$ |
| | | $\nu(\theta_0 + 2\theta_1 + \theta_3 + \theta_4 + \theta_5)$ | $\lambda_1 = \lambda_2 = 1$ |
| $X_2(\Lambda, \nu)$ | $\pi(\Lambda) \rtimes \nu$ | (irreducible) | $\Lambda^q \neq \Lambda^{\pm 1}$ |
| | | $\chi_4(\lambda, \nu) - \chi_3(\lambda, \nu)$ | $\Lambda = \lambda \circ N_{q+1}$ |
| | | $\chi_5(\omega_\Lambda, \nu) + \chi_6(\omega_\Lambda, \nu)$ | $\Lambda = \omega_\Lambda \circ N_{q-1}$ |
| | | $\nu(-\tau_1 - \lambda_0\tau_2 + \tau_2 + \tau_3)$ | $\Lambda = \Lambda_0$ |
| | | $\nu(-\theta_0 - \theta_3 + \theta_4 + \theta_5)$ | $\Lambda = 1$ |
| $X_3(\Lambda, \lambda)$ | $\lambda \rtimes \pi(\Lambda)$ | (irreducible) | $\begin{cases} \Lambda^q \neq \Lambda,\ \lambda \neq 1 \text{ and} \\ \Lambda^{q-1} \neq \Lambda_0 \text{ if } \lambda = \lambda_0 \end{cases}$ |
| | | $\chi_7(\Lambda) + \chi_8(\Lambda)$ | $\lambda = 1$ |
| | | $\chi_2(\lambda, \nu) - \chi_1(\lambda, \nu)$ | $\Lambda = \nu \circ N_{q+1}$ |
| | | $\tau_4(\lambda') + \tau_5(\lambda')$ | $\lambda = \lambda_0, \Lambda = \lambda' \circ N_{(q+1)/2}$ |
| | | $\nu(-\theta_0 + \theta_3 - \theta_4 + \theta_5)$ | $\lambda = 1, \Lambda = \nu \circ N_{q+1}$ |
| $X_4(\Theta)$ | | (irreducible) | $\Theta^{2(q-1)} \neq 1$ |
| | | $\tau_5(\lambda') - \tau_4(\lambda')$ | $\Theta = \lambda' \circ N_{(q^2+1)/2}$ |
| | | $\nu(\theta_0 - \theta_1 + \theta_2 + \theta_5)$ | $\Theta = \nu \circ N_{q^2+1}$ |
| $X_5(\Lambda, \omega)$ | | (irreducible) | $\begin{cases} \Lambda^q \neq \Lambda,\ \omega \neq 1 \text{ and} \\ \Lambda \circ i_{q-1} \neq \omega, \omega^2 \end{cases}$ |
| | | $\chi_6(\omega, \nu) - \chi_5(\omega, \nu)$ | $\Lambda = \nu \circ N_{q+1}, \omega^2 \neq 1$ |
| | | $\chi_8(\Lambda) - \chi_7(\Lambda)$ | $\omega = 1$ or $\omega = \Lambda \circ i_{q-1}$ |
| | | $\nu(\tau_1 - \tau_2 - \lambda_0\tau_2 + \tau_3)$ | $\Lambda = \nu \circ N_{q-1}$, $\omega = \omega_0$ |
| | | $\nu(\theta_0 - 2\theta_2 - \theta_3 - \theta_4 + \theta_5)$ | $\Lambda = \nu \circ N_{q+1}, \omega = 1$ |
| $\chi_1(\lambda, \nu)$ | $\lambda \rtimes \nu\, \mathbf{1}_{\mathrm{GSp}(2,q)}$ | (irreducible) | $\lambda \neq 1$ |
| | | $\nu(\theta_0 + \theta_1 + \theta_4)$ | $\lambda = 1$ |
| $\chi_2(\lambda, \nu)$ | $\lambda \rtimes \nu\, \mathrm{St}_{\mathrm{GSp}(2,q)}$ | (irreducible) | $\lambda \neq 1$ |
| | | $\nu(\theta_1 + \theta_3 + \theta_5)$ | $\lambda = 1$ |
| $\chi_3(\lambda, \nu)$ | $\lambda\, \mathbf{1}_{\mathrm{GL}(2,q)} \rtimes \nu$ | (irreducible) | $\lambda^2 \neq 1$ |
| | | $\nu(\tau_1 + \lambda_0\tau_2)$ | $\lambda = \lambda_0$ |
| | | $\nu(\theta_0 + \theta_1 + \theta_3)$ | $\lambda = 1$ |
| $\chi_4(\lambda, \nu)$ | $\lambda\, \mathrm{St}_{\mathrm{GL}(2,q)} \rtimes \nu$ | (irreducible) | $\lambda^2 \neq 1$ |
| | | $\nu(\tau_2 + \tau_3)$ | $\lambda = \lambda_0$ |
| | | $\nu(\theta_1 + \theta_4 + \theta_5)$ | $\lambda = 1$ |
| $\chi_5(\omega, \nu)$ | | (irreducible) | $\omega^2 \neq 1$ |
| | | $\nu(\tau_2 - \tau_1)$ | $\omega = \omega_0$ |
| | | $\nu(-\theta_0 + \theta_2 + \theta_4)$ | $\omega = 1$ |
| $\chi_6(\omega, \nu)$ | | (irreducible) | $\omega^2 \neq 1$ |
| | | $\nu(\tau_3 - \lambda_0\tau_2)$ | $\omega = \omega_0$ |
| | | $\nu(-\theta_2 - \theta_3 + \theta_5)$ | $\omega = 1$ |
| $\chi_7(\Lambda)$ | | (irreducible) | $\Lambda^{q-1} \neq 1$ |
| | | $\lambda(-\theta_0 + \theta_2 + \theta_3)$ | $\Lambda = \lambda \circ N_{q+1}$ |
| $\chi_8(\Lambda)$ | | (irreducible) | $\Lambda^{q-1} \neq 1$ |
| | | $\lambda(-\theta_2 - \theta_4 + \theta_5)$ | $\Lambda = \lambda \circ N_{q+1}$ |

Table A.2.: Decomposition of virtual representations $\rho$ of $\mathrm{GSp}(4, q)$ for odd $q$.

| $\rho \in \mathbf{Irr}(G)$ | $\omega_\rho$ | c | g | $\dim \rho^{\{\mathrm{diag}(1,1,*,*)\}}$ |
|---|---|---|---|---|
| $X_1(\lambda_1, \lambda_2, \nu)$ | $\lambda_1\lambda_2\nu^2$ | | • | $\begin{cases} q^3+3q^2+6q+4 & \nu^{-1}=1,\lambda_1,\lambda_2,\lambda_1\lambda_2 \\ q^3+3q^2+5q+3 & \text{else} \end{cases}$ |
| $X_2(\Lambda, \nu)$ | $(\Lambda \circ i_{q+1})\nu^2$ | | • | $\begin{cases} q^3+q^2+2q-2 & \nu^{-1}=1,\Lambda\vert_{\mathbb{F}_q^\times} \\ q^3+q^2+q-1 & \text{else} \end{cases}$ |
| $X_3(\Lambda, \nu)$ | $(\Lambda \circ i_{q+1})\nu$ | | • | $q^3+q^2+q+1$ |
| $X_4(\Theta)$ | $\Theta \circ i_{q^2+1}$ | • | • | $q^3+q^2-q-1$ |
| $X_5(\Lambda, \omega)$ | $\Lambda \circ i_{q+1}$ | • | • | $q^3-q^2+q-1$ |
| $\chi_1(\lambda, \nu)$ | $\lambda\nu^2$ | | | $\begin{cases} q^2+3q+2 & \nu^{-1}=\lambda,1 \\ q^2+2q+1 & \text{else} \end{cases}$ |
| $\chi_2(\lambda, \nu)$ | $\lambda\nu^2$ | | • | $\begin{cases} q^3+2q^2+4q+3 & \nu^{-1}=\lambda,1 \\ q^3+2q^2+3q+2 & \text{else} \end{cases}$ |
| $\chi_3(\lambda, \nu)$ | $\lambda^2\nu^2$ | | | $\begin{cases} q^2+3q+3 & \nu^{-1}=\lambda \\ q^2+2q+3 & \nu^{-1}=\lambda^2,1 \\ q^2+2q+2 & \text{else} \end{cases}$ |
| $\chi_4(\lambda, \nu)$ | $\lambda^2\nu^2$ | | • | $\begin{cases} q^3+2q^2+4q+2 & \nu^{-1}=\lambda \\ q^3+2q^2+4q+1 & \nu^{-1}=\lambda^2,1 \\ q^3+2q^2+3q+1 & \text{else} \end{cases}$ |
| $\chi_5(\omega, \nu)$ | $\nu^2$ | | | $\begin{cases} q^2+q-1 & \nu=1 \\ q^2 & \nu\neq 1 \end{cases}$ |
| $\chi_6(\omega, \nu)$ | $\nu^2$ | | • | $\begin{cases} q^3+2q-2 & \nu=1 \\ q^3+q-1 & \nu\neq 1 \end{cases}$ |
| $\chi_7(\Lambda, \nu)$ | $\Lambda \circ i_{q+1}$ | | | $q^2+1$ |
| $\chi_8(\Lambda, \nu)$ | $\Lambda \circ i_{q+1}$ | | • | $q^3+q$ |
| $\tau_1(\lambda)$ | $\lambda^2$ | | | $\begin{cases} q+2 & \lambda=1,\lambda_0 \\ q+1 & \lambda\neq 1,\lambda_0 \end{cases}$ |
| $\tau_2(\lambda)$ | $\lambda^2$ | | | $\begin{cases} q^2+2q+1 & \lambda=1 \\ q^2+q+2 & \lambda=\lambda_0 \\ q^2+q+1 & \lambda\neq 1,\lambda_0 \end{cases}$ |
| $\tau_3(\lambda)$ | $\lambda^2$ | | • | $\begin{cases} q^3+q^2+3q & \lambda=1,\lambda_0 \\ q^3+q^2+2q & \lambda\neq 1,\lambda_0 \end{cases}$ |
| $\tau_4(\lambda')$ | $\lambda' \circ i_2$ | | | $q+1$ |
| $\tau_5(\lambda')$ | $\lambda' \circ i_2$ | | • | $q^2(q+1)$ |
| $\theta_0(\lambda)$ | $\lambda^2$ | | | $\begin{cases} 1 & \lambda=1 \\ 0 & \lambda\neq 1 \end{cases}$ |
| $\theta_1(\lambda)$ | $\lambda^2$ | | | $\begin{cases} \frac{1}{2}(q^2+5q+4) & \lambda=1 \\ \frac{1}{2}(q^2+3q+2) & \lambda\neq 1 \end{cases}$ |
| $\theta_2(\lambda)$ | $\lambda^2$ | • | | $\frac{1}{2}q(q-1)$ |
| $\theta_3(\lambda)$ | $\lambda^2$ | | | $\begin{cases} \frac{1}{2}(q^2+q+4) & \lambda=1 \\ \frac{1}{2}(q^2+q+2) & \lambda\neq 1 \end{cases}$ |
| $\theta_4(\lambda)$ | $\lambda^2$ | | | $\begin{cases} \frac{1}{2}(q^2+3q) & \lambda=1 \\ \frac{1}{2}(q^2+q) & \lambda\neq 1 \end{cases}$ |
| $\theta_5(\lambda)$ | $\lambda^2$ | | • | $\begin{cases} (q^3+q^2+2q) & \lambda=1 \\ (q^3+q^2+q) & \lambda\neq 1 \end{cases}$ |

Table A.3.: The irreducible representations $\rho$ of $\mathrm{GSp}(4,q)$ for odd $q$.

| $\rho \in \mathbf{Irr}(G)$ | $\mathbf{r}^G_{P_\emptyset}(\rho) \in \mathbf{Rep}((\mathbb{G}_m(q))^3)$ | $\mathbf{r}^G_{P_2}(\rho) \in \mathbf{Rep}(\mathbb{G}_m(q) \times \mathrm{GL}(2,q))$ | $\mathbf{r}^G_{P_1}(\rho) \in \mathbf{Rep}(\mathrm{GL}(2,q) \times \mathbb{G}_m(q))$ |
|---|---|---|---|
| $X_1(\lambda_1,\lambda_2,\nu)$ | $A[\lambda_1,\lambda_2,\nu] + A[\lambda_2,\lambda_1,\nu]$ | $B(\lambda_1,\lambda_2,\nu) + B(\lambda_2,\lambda_1,\nu)$ | $C(\lambda_1,\lambda_2,\nu) + C(\lambda_1,\lambda_2^{-1},\nu)$ |
| $X_2(\Lambda,\nu)$ | 0 | 0 | $\pi(\Lambda) \boxtimes \nu + \pi(\Lambda^{-1}) \boxtimes \Lambda\|_{\mathbb{F}_q^\times}\nu$ |
| $X_3(\Lambda,\nu)$ | 0 | $\lambda \boxtimes \pi(\Lambda) + \lambda^{-1} \boxtimes \lambda\pi(\Lambda)$ | 0 |
| $\chi_1(\lambda,\nu)$ | $1 \boxtimes \lambda \boxtimes \nu + 1 \boxtimes \lambda^{-1} \boxtimes \lambda\nu$ $+\lambda \boxtimes 1 \boxtimes \nu + \lambda^{-1} \boxtimes 1 \boxtimes \lambda\nu$ | $\lambda \boxtimes \nu\,\mathbf{1} + \lambda^{-1} \boxtimes \lambda\nu\,\mathbf{1}$ $+1 \boxtimes (\lambda \rtimes \nu)$ | $C(\lambda,1,\nu)$ |
| $\chi_2(\lambda,\nu)$ | $1 \boxtimes \lambda \boxtimes \nu + 1 \boxtimes \lambda^{-1} \boxtimes \lambda\nu$ $+\lambda \boxtimes 1 \boxtimes \nu + \lambda^{-1} \boxtimes 1 \boxtimes \lambda\nu$ | $\lambda \boxtimes \nu\,\mathrm{St} + \lambda^{-1} \boxtimes \lambda\nu\,\mathrm{St}$ $+1 \boxtimes (\lambda \rtimes \nu)$ | $C(\lambda,1,\nu)$ |
| $\chi_3(\lambda,\nu)$ | $A[\lambda,\lambda,\nu]$ | $B(\lambda,\lambda,\nu)$ | $\lambda\,\mathbf{1} \boxtimes \nu + \lambda^{-1}\,\mathbf{1} \boxtimes \lambda^2\nu$ $+(\lambda \times \lambda^{-1}) \boxtimes \lambda\nu$ |
| $\chi_4(\lambda,\nu)$ | $A[\lambda,\lambda,\nu]$ | $B(\lambda,\lambda,\nu)$ | $\lambda\,\mathrm{St} \boxtimes \nu + \lambda^{-1}\,\mathrm{St} \boxtimes \lambda^2\nu$ $+(\lambda \times \lambda^{-1}) \boxtimes \lambda\nu$ |
| $\chi_5(\omega,\nu)$ | 0 | 0 | $\pi(\omega \circ N_{q-1}) \boxtimes \nu$ |
| $\chi_6(\omega,\nu)$ | 0 | 0 | $\pi(\omega \circ N_{q-1}) \boxtimes \nu$ |
| $\chi_7(\Lambda,\nu)$ | 0 | $1 \boxtimes \pi(\Lambda)$ | 0 |
| $\chi_8(\Lambda,\nu)$ | 0 | $1 \boxtimes \pi(\Lambda)$ | 0 |
| $\tau_1(\nu)$ | $\lambda_0 \boxtimes \lambda_0 \boxtimes \nu + \lambda_0 \boxtimes \lambda_0 \boxtimes \lambda_0\nu$ | $\lambda_0 \boxtimes (\lambda_0 \rtimes \nu)$ | $\lambda_0\,\mathbf{1} \boxtimes \nu + \lambda_0\,\mathbf{1} \boxtimes \lambda_0\nu$ |
| $\tau_2(\nu)$ | $\lambda_0 \boxtimes \lambda_0 \boxtimes \nu + \lambda_0 \boxtimes \lambda_0 \boxtimes \lambda_0\nu$ | $\lambda_0 \boxtimes (\lambda_0 \rtimes \nu)$ | $\lambda_0\,\mathrm{St} \boxtimes \nu + \lambda_0\,\mathbf{1} \boxtimes \lambda_0\nu$ |
| $\tau_3(\nu)$ | $\lambda_0 \boxtimes \lambda_0 \boxtimes \nu + \lambda_0 \boxtimes \lambda_0 \boxtimes \lambda_0\nu$ | $\lambda_0 \boxtimes (\lambda_0 \rtimes \nu)$ | $\lambda_0\,\mathrm{St} \boxtimes \nu + \lambda_0\,\mathrm{St} \boxtimes \lambda_0\nu$ |
| $\tau_4(\lambda')$ | 0 | $\lambda_0 \boxtimes \pi(\lambda' \circ N_{(q+1)/2})$ | 0 |
| $\tau_5(\lambda')$ | 0 | $\lambda_0 \boxtimes \pi(\lambda' \circ N_{(q+1)/2})$ | 0 |
| $\theta_0(\nu)$ | $1 \boxtimes 1 \boxtimes \nu$ | $1 \boxtimes \nu\,\mathbf{1}$ | $\mathbf{1} \boxtimes \nu$ |
| $\theta_1(\nu)$ | $1 \boxtimes 1 \boxtimes \nu + 1 \boxtimes 1 \boxtimes \nu$ | $1 \boxtimes \nu\,\mathbf{1} + 1 \boxtimes \nu\,\mathrm{St}$ | $\mathbf{1} \boxtimes \nu + \mathrm{St} \boxtimes \nu$ |
| $\theta_3(\nu)$ | $1 \boxtimes 1 \boxtimes \nu$ | $1 \boxtimes \nu\,\mathrm{St}$ | $\mathbf{1} \boxtimes \nu$ |
| $\theta_4(\nu)$ | $1 \boxtimes 1 \boxtimes \nu$ | $1 \boxtimes \nu\,\mathbf{1}$ | $\mathrm{St} \boxtimes \nu$ |
| $\theta_5(\nu)$ | $1 \boxtimes 1 \boxtimes \nu$ | $1 \boxtimes \nu\,\mathrm{St}$ | $\mathrm{St} \boxtimes \nu$ |

Table A.4.: Parabolic restriction for non-cuspidal irreducible representations $\rho$ of $\mathrm{GSp}(4,q)$ with odd $q$.

## A.4. $\mathrm{GSp}(4,q)$ for even $q$

For even $q$ the Frobenius gives rise to an isomorphism $\mathrm{GSp}(4,q) \cong \mathbb{F}_q^\times \times \mathrm{Sp}(4,q)$, so it is sufficient to know the irreducible representations of $\mathrm{Sp}(4,q)$. The irreducible representations of $\mathrm{Sp}(4,q)$ with even $q$ have been classified by Enomoto [Eno72]. Enomoto's notation is different from Shinoda's. Fix a generator $\hat{\kappa}$ of the Pontrjagin dual of $\mathbb{F}_{q^4}^\times$. Denote by $\hat{\gamma}$ its restriction to $\mathbb{F}_q^\times$, by $\hat{\theta}$ its restriction to $\mathbb{F}_{q^2}^\times$ and by $\hat{\eta}$ its restriction to $\mathbb{F}_{q^2}^\times[q+1]$. Enomoto defines certain virtual characters of $\mathrm{Sp}(4,q)$, which depend on integers $(k,l)$. For example, the character $\chi_1(k,l)$ for $k,l \in \mathbb{Z}/(q-1)\mathbb{Z}$ corresponds to the principal series representation $\hat{\gamma}^k \times \hat{\gamma}^l \rtimes 1$. These virtual characters are pairwise inequivalent except for the following identities:

1. the eight characters generated by the equivalence $\chi_s(k,l) = \chi_s(l,k) = \chi_s(-l,k)$ are equal for $s \in \{1,4\}$,
2. $\chi_s(k) = \chi_s(-k) = \chi_s(qk) = \chi_s(-qk)$ for $s = 2,5$,
3. $\chi_3(k,l) = \chi_3(k,-l) = \chi_3(-k,l) = \chi_3(-k,-l)$,
4. and $\chi_s(k) = \chi_s(-k)$ for $s = 6,7,8,9,10,11,12,13$.

The decomposition into irreducible components is given in Table A.5. The properties of the irreducible representations are given in Table A.6. The parabolic restriction is described in Table A.7. Again, for typesetting reasons, we write

$$A'[\hat{\gamma}^{k_1}, \hat{\gamma}^{k_2}] = \hat{\gamma}^{k_1} \boxtimes \hat{\gamma}^{k_2} + \hat{\gamma}^{k_1} \boxtimes \hat{\gamma}^{-k_2} + \hat{\gamma}^{-k_1} \boxtimes \hat{\gamma}^{k_2} + \hat{\gamma}^{-k_1} \boxtimes \hat{\gamma}^{-k_2}.$$

| $\rho$ | Parameters | Name | Decomposition | for |
|---|---|---|---|---|
| $\chi_1(k,l)$ | $k,l \in \mathbb{Z}/(q-1)\mathbb{Z}$ | $\hat{\gamma}^k \times \hat{\gamma}^l \rtimes 1$ | (irreducible) | $0 \neq l \neq \pm k \neq 0$ |
| | | | $\chi_6(k) + \chi_{10}(k)$ | $l = \pm k$ |
| | | | $\chi_7(k) + \chi_{11}(k)$ | $l = 0$ |
| | | | $\theta_0 + 2\theta_1 + \theta_2 + \theta_3 + \theta_4$ | $l = k = 0$ |
| $\chi_2(l)$ | $l \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ | $\pi(\hat{\theta}^l) \rtimes 1$ | (irreducible) | $(q \pm 1)l \neq 0$ |
| | | | $\chi_8(k) + \chi_{12}(k)$ | $l = (q-1)k$ |
| | | | $\chi_{10}(k) - \chi_6(k)$ | $l = (q+1)k$ |
| | | | $-\theta_0 - \theta_2 + \theta_3 + \theta_4$ | $l = 0$ |
| $\chi_3(k,l)$ | $k \in \mathbb{Z}/(q-1)\mathbb{Z}$ | $\hat{\gamma}^k \rtimes \pi(\hat{\eta}^l)$ | (irreducible) | $k \neq 0$, $l \neq 0$ |
| | $l \in \mathbb{Z}/(q+1)\mathbb{Z}$ | | $\chi_9(l) + \chi_{13}(l)$ | $k = 0$ |
| | | | $\chi_{11}(k) - \chi_7(k)$ | $l = 0$ |
| | | | $-\theta_0 + \theta_2 - \theta_3 + \theta_4$ | $k = 0$, $l = 0$ |
| $\chi_4(k,l)$ | $k,l \in \mathbb{Z}/(q+1)\mathbb{Z}$ | | (irreducible) | $0 \neq k \neq \pm l \neq 0$ |
| | | | $\chi_{12}(k) - \chi_8(k)$ | $l = \pm k$ |
| | | | $\chi_{13}(k) - \chi_9(k)$ | $l = 0$ |
| | | | $\theta_0 - \theta_2 - \theta_3 + \theta_4 - 2\theta_5$ | $l = k = 0$ |
| $\chi_5(k)$ | $k \in \mathbb{Z}/(q^2+1)\mathbb{Z}$ | | (irreducible) | $k \neq 0$ |
| | | | $\theta_0 - \theta_1 + \theta_4 + \theta_5$ | $k = 0$ |
| $\chi_6(k)$ | $k \in \mathbb{Z}/(q-1)\mathbb{Z}$ | $\hat{\gamma}^k\, \mathbf{1}_{\mathrm{GL}(2,q)} \rtimes 1$ | (irreducible) | $k \neq 0$ |
| | | | $\theta_0 + \theta_1 + \theta_2$ | $k = 0$ |
| $\chi_7(k)$ | $k \in \mathbb{Z}/(q-1)\mathbb{Z}$ | $\hat{\gamma}^k \rtimes \mathbf{1}_{\mathrm{Sp}(2,q)}$ | (irreducible) | $k \neq 0$ |
| | | | $\theta_0 + \theta_1 + \theta_3$ | $k = 0$ |
| $\chi_8(k)$ | $k \in \mathbb{Z}/(q+1)\mathbb{Z}$ | | (irreducible) | $k \neq 0$ |
| | | | $-\theta_0 + \theta_3 + \theta_5$ | $k = 0$ |
| $\chi_9(k)$ | $k \in \mathbb{Z}/(q+1)\mathbb{Z}$ | | (irreducible) | $k \neq 0$ |
| | | | $-\theta_0 + \theta_2 + \theta_5$ | $k = 0$ |
| $\chi_{10}(k)$ | $k \in \mathbb{Z}/(q-1)\mathbb{Z}$ | $\hat{\gamma}^k\, \mathrm{St}_{\mathrm{GL}(2,q)} \rtimes 1$ | (irreducible) | $k \neq 0$ |
| | | | $\theta_1 + \theta_3 + \theta_4$ | $k = 0$ |
| $\chi_{11}(k)$ | $k \in \mathbb{Z}/(q-1)\mathbb{Z}$ | $\hat{\gamma}^k \rtimes \mathrm{St}_{\mathrm{Sp}(2,q)}$ | (irreducible) | $k \neq 0$ |
| | | | $\theta_1 + \theta_2 + \theta_4$ | $k = 0$ |
| $\chi_{12}(k)$ | $k \in \mathbb{Z}/(q+1)\mathbb{Z}$ | | (irreducible) | $k \neq 0$ |
| | | | $-\theta_2 + \theta_4 - \theta_5$ | $k = 0$ |
| $\chi_{13}(k)$ | $k \in \mathbb{Z}/(q+1)\mathbb{Z}$ | | (irreducible) | $k \neq 0$ |
| | | | $-\theta_3 + \theta_4 - \theta_5$ | $k = 0$ |
| $\theta_0$ | | $\mathbf{1}_{\mathrm{Sp}(4,q)}$ | (irreducible) | |
| $\theta_1$ | | | (irreducible) | |
| $\theta_2$ | | | (irreducible) | |
| $\theta_3$ | | | (irreducible) | |
| $\theta_4$ | | $\mathrm{St}_{\mathrm{Sp}(4,q)}$ | (irreducible) | |
| $\theta_5$ | | | (irreducible) | |

Table A.5.: Decomposition of Enomoto's virtual characters $\rho$ of $\mathrm{Sp}(4,q)$ for even $q$.

| $\rho = \rho\|_{\mathrm{Sp}(4,q)} \boxtimes \omega_\rho$ | c | g | $\dim \rho^{\{\mathrm{diag}(1,1,*,*)\}}$ | |
|---|---|---|---|---|
| $\chi_1(k,l) \boxtimes \hat{\gamma}^r$ | | • | $\begin{cases} q^3+3q^2+6q+4 \\ q^3+3q^2+5q+3 \end{cases}$ | $r = \pm(k+l), \pm(k-l)$<br>$r \neq \pm(k+l), \pm(k-l)$ |
| $\chi_2(l) \boxtimes \hat{\gamma}^r$ | | • | $\begin{cases} q^3+q^2+2q-2 \\ q^3+q^2+q-1 \end{cases}$ | $r = \pm l \mod (q-1)$<br>$r \neq \pm l \mod (q-1)$ |
| $\chi_3(k,l) \boxtimes \hat{\gamma}^r$ | | • | $q^3+q^2+q+1$ | |
| $\chi_4(k,l) \boxtimes \hat{\gamma}^r$ | • | • | $q^3-q^2+q-1$ | |
| $\chi_5(k) \boxtimes \hat{\gamma}^r$ | • | • | $q^3+q^2-q-1$ | |
| $\chi_6(k) \boxtimes \hat{\gamma}^r$ | | | $\begin{cases} q^2+3q+3 \\ q^2+2q+3 \\ q^2+2q+2 \end{cases}$ | $r = 0$<br>$r = \pm 2k$<br>$r \neq \pm 2k, 0$ |
| $\chi_7(k) \boxtimes \hat{\gamma}^r$ | | | $\begin{cases} q^2+3q+2 \\ q^2+2q+1 \end{cases}$ | $r = \pm k$<br>$r \neq \pm k$ |
| $\chi_8(k) \boxtimes \hat{\gamma}^r$ | | | $\begin{cases} q^2+q-1 \\ q^2 \end{cases}$ | $r = 0$<br>$r \neq 0$ |
| $\chi_9(k) \boxtimes \hat{\gamma}^r$ | | | $q^2+1$ | |
| $\chi_{10}(k) \boxtimes \hat{\gamma}^r$ | | • | $\begin{cases} q^3+2q^2+4q+2 \\ q^3+2q^2+4q+1 \\ q^3+2q^2+3q+1 \end{cases}$ | $r = 0$<br>$r = \pm 2k$<br>$r \neq \pm 2k, 0$ |
| $\chi_{11}(k) \boxtimes \hat{\gamma}^r$ | | • | $\begin{cases} q^3+2q^2+4q+3 \\ q^3+2q^2+3q+2 \end{cases}$ | $r = \pm k$<br>$r \neq \pm k$ |
| $\chi_{12}(k) \boxtimes \hat{\gamma}^r$ | | • | $\begin{cases} q^3+2q-2 \\ q^3+q-1 \end{cases}$ | $r = 0$<br>$r \neq 0$ |
| $\chi_{13}(k) \boxtimes \hat{\gamma}^r$ | | • | $q^3+q$ | |
| $\theta_0 \boxtimes \hat{\gamma}^r$ | | | $\begin{cases} 1 \\ 0 \end{cases}$ | $r = 0$<br>$r \neq 0$ |
| $\theta_1 \boxtimes \hat{\gamma}^r$ | | | $\begin{cases} \frac{1}{2}(q^2+5q+4) \\ \frac{1}{2}(q^2+3q+2) \end{cases}$ | $r = 0$<br>$r \neq 0$ |
| $\theta_2 \boxtimes \hat{\gamma}^r$ | | | $\begin{cases} \frac{1}{2}(q^2+q+4) \\ \frac{1}{2}(q^2+q+2) \end{cases}$ | $r = 0$<br>$r \neq 0$ |
| $\theta_3 \boxtimes \hat{\gamma}^r$ | | | $\begin{cases} \frac{1}{2}(q^2+3q) \\ \frac{1}{2}(q^2+q) \end{cases}$ | $r = 0$<br>$r \neq 0$ |
| $\theta_4 \boxtimes \hat{\gamma}^r$ | | • | $\begin{cases} q^3+q^2+2q \\ q^3+q^2+q \end{cases}$ | $r = 0$<br>$r \neq 0$ |
| $\theta_5 \boxtimes \hat{\gamma}^r$ | • | | $\frac{1}{2}(q^2-q)$ | |

Table A.6.: The irreducible representations $\rho$ of $\mathrm{GSp}(4,q)$ for even $q$. The conditions for irreducibility of $\rho|_{\mathrm{Sp}(4,q)}$ are tacitly imposed on the parameters. The central character of $\rho$ is $\omega_\rho = \hat{\gamma}^r$ for some $r \in \mathbb{Z}/(q-1)\mathbb{Z}$.

| $\rho$ | $\mathbf{r}^{\mathrm{Sp}(4,q)}_{P_\emptyset \cap \mathrm{Sp}(4,q)}(\rho) \in \mathbf{Rep}(\mathbb{F}_q^\times \times \mathbb{F}_q^\times)$ | $\mathbf{r}^{\mathrm{Sp}(4,q)}_{P_2 \cap \mathrm{Sp}(4,q)}(\rho) \in \mathbf{Rep}(\mathbb{F}_q^\times \times \mathrm{Sp}(2,q))$ | $\mathbf{r}^{\mathrm{Sp}(4,q)}_{P_1 \cap \mathrm{Sp}(4,q)}(\rho) \in \mathbf{Rep}(\mathrm{GL}(2,q))$ |
|---|---|---|---|
| $\theta_0$ | $1 \boxtimes 1$ | $1 \boxtimes \mathbf{1}_{\mathrm{Sp}(2,q)}$ | $\mathbf{1}_{\mathrm{GL}(2,q)}$ |
| $\theta_1$ | $1 \boxtimes 1 + 1 \boxtimes 1$ | $1 \boxtimes \mathrm{St}_{\mathrm{Sp}(2,q)} + 1 \boxtimes \mathbf{1}_{\mathrm{Sp}(2,q)}$ | $\mathrm{St}_{\mathrm{GL}(2,q)} + \mathbf{1}_{\mathrm{GL}(2,q)}$ |
| $\theta_2$ | $1 \boxtimes 1$ | $1 \boxtimes \mathrm{St}_{\mathrm{Sp}(2,q)}$ | $\mathbf{1}_{\mathrm{GL}(2,q)}$ |
| $\theta_3$ | $1 \boxtimes 1$ | $1 \boxtimes \mathbf{1}_{\mathrm{Sp}(2,q)}$ | $\mathrm{St}_{\mathrm{GL}(2,q)}$ |
| $\theta_4$ | $1 \boxtimes 1$ | $1 \boxtimes \mathrm{St}_{\mathrm{Sp}(2,q)}$ | $\mathrm{St}_{\mathrm{GL}(2,q)}$ |
| $\chi_1(k,l)$ | $A'[\hat\gamma^k, \hat\gamma^l] + A'[\hat\gamma^l, \hat\gamma^k]$ | $\hat\gamma^k \boxtimes (\hat\gamma^l \rtimes 1) + \hat\gamma^{-k} \boxtimes (\hat\gamma^l \rtimes 1)$ $+\hat\gamma^l \boxtimes (\hat\gamma^k \rtimes 1) + \hat\gamma^{-l} \boxtimes (\hat\gamma^k \rtimes 1)$ | $(\hat\gamma^k \times \hat\gamma^l) + (\hat\gamma^k \times \hat\gamma^{-l})$ $+(\hat\gamma^{-k} \times \hat\gamma^l) + (\hat\gamma^{-k} \times \hat\gamma^{-l})$ |
| $\chi_2(l)$ | 0 | 0 | $\pi(\hat\theta^l) + \pi(\hat\theta^{-l})$ |
| $\chi_3(k,l)$ | 0 | $\hat\gamma^k \boxtimes \pi(\hat\eta^l) + \hat\gamma^{-k} \boxtimes \pi(\hat\eta^l)$ | 0 |
| $\chi_6(k)$ | $A'[\hat\gamma^k, \hat\gamma^k]$ | $\hat\gamma^k \boxtimes (\hat\gamma^k \rtimes 1) + \hat\gamma^{-k} \boxtimes (\hat\gamma^k \rtimes 1)$ | $\hat\gamma^k\, \mathbf{1} + \hat\gamma^{-k}\, \mathbf{1} + (\hat\gamma^k \times \hat\gamma^{-k})$ |
| $\chi_7(k)$ | $\hat\gamma^k \boxtimes 1 + \hat\gamma^{-k} \boxtimes 1 + 1 \boxtimes \hat\gamma^k + 1 \boxtimes \hat\gamma^{-k}$ | $\hat\gamma^k \boxtimes \mathbf{1} + \hat\gamma^{-k} \boxtimes \mathbf{1} + 1 \boxtimes (\hat\gamma^k \rtimes 1)$ | $(\hat\gamma^k \times 1) + (\hat\gamma^{-k} \times 1)$ |
| $\chi_8(k)$ | 0 | 0 | $\pi(\hat\theta^{(q-1)k})$ |
| $\chi_9(l)$ | 0 | $1 \boxtimes \pi(\hat\eta^l)$ | 0 |
| $\chi_{10}(k)$ | $A'[\hat\gamma^k, \hat\gamma^k]$ | $\hat\gamma^k \boxtimes (\hat\gamma^k \rtimes 1) + \hat\gamma^{-k} \boxtimes (\hat\gamma^k \rtimes 1)$ | $\hat\gamma^k\, \mathrm{St} + \hat\gamma^{-k}\, \mathrm{St} + (\hat\gamma^k \times \hat\gamma^{-k})$ |
| $\chi_{11}(k)$ | $\hat\gamma^k \boxtimes 1 + \hat\gamma^{-k} \boxtimes 1 + 1 \boxtimes \hat\gamma^k + 1 \boxtimes \hat\gamma^{-k}$ | $\hat\gamma^k \boxtimes \mathrm{St} + \hat\gamma^{-k} \boxtimes \mathrm{St} + 1 \boxtimes (\hat\gamma^k \rtimes 1)$ | $(\hat\gamma^k \times 1) + (\hat\gamma^{-k} \times 1)$ |
| $\chi_{12}(k)$ | 0 | 0 | $\pi(\hat\theta^{(q-1)k})$ |
| $\chi_{13}(l)$ | 0 | $1 \boxtimes \pi(\hat\eta^l)$ | 0 |

Table A.7.: Parabolic restriction for non-cuspidal irreducible representations $\rho$ of $G = \mathrm{Sp}(4,q)$ with even $q$.

# B. Vector-valued Siegel modular forms

For the convenience of the reader, we give a brief survey on the correspondence between Siegel modular forms and automorphic representations (without proofs). A good reference is the book of Gelbart [Gel75] and also the survey article of Schmidt and Asgari [AS01]. We then review the results of Chapter 5 for classical Siegel modular forms.

**Classical Siegel modular forms.** The real symplectic group of genus $g \geq 1$

$$\mathrm{Sp}(2g, \mathbb{R}) = \{x \in \mathrm{GL}(2g, \mathbb{C}) \,|\, xJx^t = J\} \quad \text{for} \quad J = \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$$

acts on the Siegel upper half space

$$\mathbb{H}_g = \{\tau = \tau^t \in \mathrm{Mat}(g \times g, \mathbb{C}) \,|\, \mathrm{Im}(\tau) > 0\}$$

via modular substitutions

$$x \cdot \tau = (a\tau + b)(c\tau + d)^{-1}, \qquad x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R}).$$

The automorphic factor

$$j : \mathrm{Sp}(2g, \mathbb{R}) \times \mathbb{H}_g \to \mathrm{GL}(g, \mathbb{C}), \qquad j(x, \tau) = (c\tau + d).$$

satisfies the cocycle condition $j(x_1 x_2, \tau) = j(x_1, x_2\tau) j(x_2, \tau)$ for $x_1, x_2 \in \mathrm{Sp}(2g, \mathbb{R})$.

Fix a holomorphic finite-dimensional irreducible complex representation $(\rho, V_\rho)$ of $\mathrm{GL}(g, \mathbb{C})$ and a hermitian scalar product $\langle \cdot, \cdot \rangle_\rho$ on $V_\rho$ with

$$\langle \rho(x)v, w \rangle_\rho = \langle v, \rho(\overline{x}^t)w \rangle_\rho \qquad \forall x \in \mathrm{GL}(g, \mathbb{C}),$$

so that $\rho|_{U(g)}$ is unitary. The symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ acts from the right on the space of holomorphic functions $f : \mathbb{H}_g \to V_\rho$ via the Petersson operator

$$(f \,|_\rho\, x)(\tau) = \rho(j(x, \tau))^{-1} f(x \cdot \tau).$$

A *vector-valued Siegel modular form* of weight $\rho$ with congruence subgroup $\Gamma \subseteq \mathrm{Sp}(2g, \mathbb{Z})$ is a holomorphic function $f : \mathbb{H}_g \to V_\rho$ that is invariant under the right action of $\Gamma$ and holomorphic at the cusps.

By the Köcher principle, the second condition is redundant for $g > 1$. The Siegel modular forms of type $\rho$ and level $\Gamma$ form a vector space $S_\rho(\Gamma)$.

The imaginary part $\mathrm{Im}(\tau)$ of $\tau \in \mathbb{H}_g$ transforms via

$$\overline{j(x,\tau)}^t \cdot \mathrm{Im}(x\cdot\tau)\cdot j(x,\tau) = \mathrm{Im}(\tau) \qquad \text{for } x \in \mathrm{Sp}(2n,\mathbb{R}), \tag{B.1}$$

so $\tau \mapsto \langle \rho(\mathrm{Im}(\tau)) f_1(\tau), f_2(\tau)\rangle_\rho$ is invariant under $\Gamma$ for Siegel modular forms $f_1, f_2 \in S_\rho(\Gamma)$. The Petersson scalar product for square-integrable[1] $f_1, f_2$ is the hermitian product

$$\langle f_1, f_2\rangle = \frac{1}{[\mathrm{Sp}(2g,\mathbb{Z}) : \Gamma]} \int_{\Gamma\backslash\mathbb{H}_n} \langle \rho(\mathrm{Im}(\tau)) f_1(\tau), f_2(\tau)\rangle \,\mathrm{d}\tau, \tag{B.2}$$

where $\mathrm{d}\tau = \det \mathrm{Im}(\tau)^{-n-1}\cdot \prod_{i\le j} \mathrm{d}x_{ij}\,\mathrm{d}y_{ij}$ is a fixed $\mathrm{Sp}(2g,\mathbb{Z})$-invariant measure.

**Real automorphic forms.** The stabilizer of $iI_g = \mathrm{diag}(i,\dots,i) \in \mathbb{H}_g$ is the compact group

$$K_\infty = \left\{ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid aa^t + bb^t = I_g, ab^t = ba^t \right\} \subseteq \mathrm{Sp}(2g,\mathbb{R}),$$

which is isomorphic to the unitary group $U(g) \subseteq \mathrm{GL}(g,\mathbb{C})$ via $x \mapsto j(x, iI_g)$. For a convenient normalization of the Haar measure this gives rise to a homeomorphism

$$\mathrm{Sp}(2g,\mathbb{R})/K_\infty \cong \mathbb{H}_g, \qquad x \mapsto x\cdot iI_g. \tag{B.3}$$

For a Siegel modular form $f \in S_\rho(\Gamma)$, the smooth function

$$\phi_f : \mathrm{Sp}(2n,\mathbb{R}) \to V_\rho, \qquad \phi_f(x) = (f\mid_\rho x)(iI_n), \tag{B.4}$$

satisfies

$$\rho(j(k,iI_g))^{-1}\phi_f(x) = \phi_f(\gamma x k), \qquad k \in K_\infty, \gamma \in \Gamma. \tag{B.5}$$

Since $\rho(\mathrm{Im}(iI_g)) = \mathrm{id}$, (B.1) implies

$$\langle \phi_{f_1}(x), \phi_{f_2}(x)\rangle_\rho = \langle \rho(\mathrm{Im}(\tau)) f_1(\tau), f_2(\tau)\rangle_\rho, \qquad f_1, f_2 \in S_\rho(\Gamma)$$

for $x \in \mathrm{Sp}(2g,\mathbb{R})$ and $\tau = x\cdot iI_g$. Hence, up to normalization of the Haar measure, the Petersson scalar product for square-integrable $f_1, f_2$ equals

$$\langle \phi_{f_1}, \phi_{f_2}\rangle = \frac{1}{[\mathrm{Sp}(2g,\mathbb{Z}) : \Gamma]} \int_{\Gamma\backslash \mathrm{Sp}(2g,\mathbb{R})/K_\infty} \langle \phi_{f_1}(x), \phi_{f_2}(x)\rangle \,\mathrm{d}x. \tag{B.6}$$

This defines an isometric embedding $f \mapsto \phi_f$ from square-integrable vector-valued Siegel modular forms to vector-valued $K_\infty$-finite automorphic forms.

[1]By definition, a Siegel modular form $f$ is square-integrable if and only if $\langle f, f\rangle$ exists.

**Strong approximation.** The rational idele $\mathbb{A}^\times = \mathbb{R}^\times \times \mathbb{A}_{\text{fin}}^\times$ admit a decomposition $\mathbb{A}^\times = \mathbb{Q}^\times(\mathbb{R}_{>0} \times \hat{\mathbb{Z}}^\times)$, where $\mathbb{Q}^\times$ is diagonally embedded and

$$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}_p \subseteq \mathbb{A}_{\text{fin}}$$

is the Prüfer ring. Strong approximation is the generalization of this result to reductive groups.

Denote by $\mathrm{GSp}^+(2g, \mathbb{R}) \subseteq \mathrm{GSp}(2g, \mathbb{R})$ the connected subgroup with similitude character in $\mathbb{R}_{>0}$. Fix a compact open subgroup $K \subseteq \mathrm{GSp}(2g, \mathbb{A}_{\text{fin}})$. If the similitude character $\mathrm{sim} : K \to \hat{\mathbb{Z}}^\times$ is surjective, every $g \in \mathrm{GSp}(2g, \mathbb{A})$ admits a decomposition $g_\mathbb{Q}(g_\infty k)$ with $g_\mathbb{Q} \in \mathrm{GSp}(2g, \mathbb{Q})$, $g_\infty \in \mathrm{GSp}^+(2g, \mathbb{R})$ and $k \in K$. In other words, there is a decomposition

$$\mathrm{GSp}(2g, \mathbb{A}) = \mathrm{GSp}(2g, \mathbb{Q})(\mathrm{GSp}^+(2g, \mathbb{R}) \times K) \tag{B.7}$$

for the diagonally embedded $\mathrm{GSp}(2g, \mathbb{Q})$. We obtain a homeomorphism

$$\Gamma \backslash \mathrm{Sp}(2g, \mathbb{R}) \cong \mathrm{GSp}(2g, \mathbb{Q}) \backslash \mathrm{GSp}(2g, \mathbb{A})/(\mathbb{R}^\times \times K) \tag{B.8}$$

for $\Gamma = (\mathrm{GSp}^+(2g, \mathbb{R})K) \cap \mathrm{Sp}(2n, \mathbb{Q})$. For example, the modified principal congruence subgroup

$$K'(N) = \{x \in \mathrm{GSp}(2g, \hat{\mathbb{Z}}) \,|\, \exists \lambda \in \hat{\mathbb{Z}} \,:\, x \equiv \mathrm{diag}(I_g, \lambda I_g) \mod N\}$$

satisfies $\mathrm{sim}(K'(N)) = \hat{\mathbb{Z}}^\times$ and $\Gamma = K'(N) \cap \mathrm{Sp}(2g, \mathbb{Q})$ is the principal congruence subgroup of level $N$.

**Adelic automorphic forms.** By strong approximation, a real automorphic form $\phi : \Gamma \backslash \mathrm{Sp}(2g, \mathbb{R}) \to V_\rho$ gives rise to a function $\phi_\mathbb{A} : \mathrm{GSp}(2g, \mathbb{Q}) \backslash \mathrm{GSp}(2g, \mathbb{A}) \to V_\rho$. This defines an isometric Hecke-equivariant embedding from square-integrable Siegel modular forms to the subspace of $V_\rho$-valued adelic automorphic forms $\phi_\mathbb{A}$ with

1. $\phi_\mathbb{A}(\gamma x k) = \phi_\mathbb{A}(x)$ for $\gamma \in \mathrm{GSp}(2n, \mathbb{Q}), k \in K$,
2. $\phi_\mathbb{A}(x k_\infty) = \rho(j(k_\infty, iI_g))\phi_\mathbb{A}(x)$ for $k_\infty \in K_\infty$,
3. $\phi_\mathbb{A}(zx) = \phi_\mathbb{A}(x)$ for $z \in Z(\mathbb{A})$ the center of $\mathrm{GSp}(2g)$,
4. $\phi_\mathbb{A}$ is square-integrable over $\mathrm{GSp}(2g, \mathbb{Q}) \backslash \mathrm{GSp}(2g, \mathbb{A})/Z(\mathbb{R})K_\infty$.

**Automorphic representations.** Fix an arbitrary non-zero linear projection $V_\rho \to \mathbb{C}$. For every $V_\rho$-valued automorphic form $\phi_{\mathbb{A}}$, its scalar-valued image belongs to the space of automorphic forms $L^2(\mathrm{GSp}(2g,\mathbb{Q})\backslash \mathrm{GSp}(2g,\mathbb{A}),1)$ with trivial central character. Every automorphic representation $\pi$ occuring in this space decomposes as a tensor product $\pi \cong \pi_\infty \otimes \pi_{\mathrm{fin}}$ of representations of $\mathrm{GSp}(2g,\mathbb{R})$ and $\mathrm{GSp}(2g,\mathbb{A}_{\mathrm{fin}})$. This decouples the congruence condition from the weight condition: The weight is encoded by the Langlands-parameter of the discrete series representation $\pi_\infty$. Invariance under a compact open subgroup $K$ of $\mathrm{GSp}(2g,\mathbb{A}_{\mathrm{fin}})$ is a statement about non-archimedean factor $\pi_{\mathrm{fin}}$.

For $g=1$ the theory of newforms provides a one-to-one correspondence between normalized cuspidal elliptic newforms, eigenforms under the Hecke algebra, and the automorphic representations they generate. For $g=2$ an analogous theory has been developed by Roberts and Schmidt [RS07] for locally generic representations, but the general case is still open.

We now translate the results about local parahoric restrictions into the language of Siegel modular forms.

**On the Saito-Kurokawa lift.**

**Corollary B.1.** *Fix an elliptic cuspidal newform $f$ of level $\Gamma_0[N] \subseteq \mathrm{SL}(2,\mathbb{Z})$ for squarefree $N$ and weight $2k-2$, $k\geq 3$, with Atkin-Lehner eigenvalues $\epsilon_p$ at $p \mid N$, which is an eigenform of the Hecke algebra. For any divisor $M$ of $N$ with Möbius $\mu$-function*

$$\mu(M) = (-1)^{\#\{\text{primes dividing } M\}} = (-1)^k \prod_{p|N} \epsilon_p, \tag{B.9}$$

*there is a scalar-valued genus two Siegel cuspform $F$ with weight $\det^k$, invariant under the principle congruence subgroup $\Gamma[N] \subseteq \mathrm{Sp}(4,\mathbb{Z})$, whose spinor $L$-function is*

$$L(F,s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s)\prod_{p|M}\frac{(1-p^{-s+k-1})(1-p^{-s+k-2})}{(1+\epsilon_p p^{-s+k-2})},$$

*where $\zeta$ denotes the Riemann zeta function.*

*Proof.* Let $S=\{p \mid M\}\cup\{\infty\}$ and let $\sigma$ be the cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ generated by $f$. The Saito-Kurokawa lift $\pi=\pi(\sigma,\sigma_S)$ is automorphic by (5.12) and

$$\epsilon(\sigma,1/2) = \epsilon(f,k-1) = (-1)^{k-1}\prod_{p|N}\epsilon_p.$$

It is cuspidal because $S$ is not empty. Every local factor $\pi_v$, $v \mid N$, has non-zero hyperspecial parahoric restriction and one can show show that $\pi_f$ admits non-zero invariants under the modified principal congruence subgroup

$$K'(N) = \prod_{v<\infty} \{x_v \in \mathrm{GSp}(4, \mathbb{Z}_v) \,|\, x_v \equiv \mathrm{diag}(1, 1, *, *) \mod N\}\,.$$

Strong approximation defines an automorphic form on $\mathrm{Sp}(4, \mathbb{R})$, left-invariant under $\mathrm{Sp}(4, \mathbb{Q}) \cap K'(N) = \Gamma[N]$. This gives rise to a Siegel cuspform $F$ as above. The spinor $L$-function is given at every non-archimedean place by (5.13). □

Schmidt [Sch07, Thm. 5.2.ii)] has already shown this with the restriction to the case of even $k$ and where $M$ is the product over the primes with Atkin-Lehner eigenvalue $\epsilon_p = -1$.

**On the Yoshida lift.**

**Corollary B.2.** *Fix cuspidal elliptic newforms $f_1, f_2$, eigenforms under the Hecke-algebra, with weights $r_1 > r_2 \geq 2$ such that $r_1 + r_2 \equiv 0 \mod 2$ and with level $\Gamma_0(N_i)$, $i = 1, 2$, such that $N_1$ and $N_2$ are squarefree, but not coprime. Then there is a genus two Siegel cuspform $F$ with weight*

$$\rho = \mathrm{Sym}^{r_2-2}(\mathrm{std}) \otimes \det{}^{(r_1-r_2)/2+2},$$

*invariant under the principal congruence group $\Gamma^2(N)$ for the least common multiple $N = \mathrm{lcm}(N_1, N_2)$ and with spinor $L$-function*

$$L(F, s) = L(f_1, s)L(f_2, s + \tfrac{1}{2}(r_2 - r_1)). \tag{B.10}$$

*Proof.* Let $\sigma_i$ be the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ generated by $f_i$. For every non-archimedean place $v$ dividing $N_i$, the local factor $\sigma_{i,v}$ is a twist of the Steinberg representation by an unramified character. Fix a prime $p_0$ dividing $N_1$ and $N_2$, then $\sigma_{i,p_0}$ are both in the discrete series. By Thm. 5.6 there is a weak endoscopic lift $\pi$, attached to $\sigma$, that is locally generic at every place except $p_0$ and $\infty$. The archimedean factor $\pi_\infty = \Pi_-(\sigma_\infty)$ is the non-generic holomorphic discrete series representations. By strong approximation $\sigma_{i,v}$ is Iwahori-spherical for every $v$ dividing $N_i$, so Thm. 4.7 implies that $\pi_v \neq 0$ admits non-zero invariants under the modified principal congruence subgroup of level $p$ for every prime $p$ dividing $N_1$ or $N_2$. Pick a non-zero adelic automorphic form $\phi$ in $\pi$ invariant under the modified principal congruence $K'(N)$ subgroup of level $N = \mathrm{lcm}(N_1, N_2)$ and whose archimedean component corresponds to a lowest weight vector. By strong approximation $\phi$ gives a Siegel modular form $F$ of weight $\rho$ invariant under the principal congruence subgroup $\Gamma^2[N]$.

The equation of $L$-factors holds at every non-archimedean place by Prop. 5.7. □

The result on the Yoshida lift has already been shown by Schmidt and Saha [SS13, Prop. 3.1], but under the restriction that the Atkin-Lehner eigenvalues of $f_1$, $f_2$ coincide at every common divisor of $N_1$ and $N_2$.

The case $N_1 = N_2 = 2$ proves the first part of Conjecture 6.1 of Bergström, Faber and van der Geer [BFvdG08]. By similar arguments using Lemma 5.21 one can also prove the second part.

# Bibliography


[ABPS] A.-M. Aubert, P. Baum, R. Plymen, and M. Solleveld. Depth and the Local Langlands Correspondence. arXiv 1311.1606.

[Art89] J. Arthur. The $L^2$-Lefschetz numbers of Hecke operators. *Inv. Math.*, 97:257–290, 1989.

[AS01] M. Asgari and R. Schmidt. Siegel Modular Forms and representations. *Man. Math.*, 104:173–200, 2001.

[BA15] J. Breeding-Allison. Irreducible Characters of $GSp(4, \mathbb{F}_q)$. *Ramanujan J.*, 36(3):305–354, 2015.

[BC83] A. Borel and W. Casselman. $L^2$-cohomology of locally symmetric manifolds of finite volume. *Duke Math. J.*, 50:625–647, 1983.

[BFvdG08] J. Bergström, C. Faber, and G. van der Geer. Siegel Modular Forms of Genus 2 and Level 2: Cohomological Computations and Conjectures. *International Mathematics Research Notices*, 2008. Article ID: rnn100.

[BFvdG14] J. Bergström, C. Faber, and G. van der Geer. Siegel Modular forms of degree three and the cohomology of local systems. *Selecta Math.*, 20(1):83–124, 2014. arXiv:1108.3731.

[BH06] C. J. Bushnell and G. Henniart. *The Local Langlands Conjecture for $GL(2)$*. Grundlehren der Mathematischen Wissenschaften. Springer, 2006.

[BK93] C. J. Bushnell and P. C. Kutzko. *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, volume 129 of *Annals of mathematics studies.* Princeton University Press, 1993.

[Bor69] A. Borel. *Introduction aux Groupes Arithmétiques.* Hermann, Paris, 1969.

[Bor81] A. Borel. Stable real cohomology of arithmetic groups. II. In J. Hano, editor, *Manifolds and Lie groups*, volume 14 of *Progr. Math.*, pages 21–56. Birkhäuser, 1981.

[BT72] F. Bruhat and J. Tits. Groupes Reductifs sur un Corps Local. I. Donnees Radicielles Valuees. *Publications Mathematiques*, 41:5–251, 1972.

[Cas95] W. Casselman. Introduction to the Theory of admissible representations of p-adic reductive groups. Unpublished, 1995.

[CvdGG15] F. Cléry, G. van der Geer, and S. Grushevsky. Siegel Modular Forms of Genus 2 and Level 2. *Int. J. Math.*, 26(3), 2015.

[Dan11] Y. Danisman. *L-Factors of Supercuspidal Representations of p-adic $GSp(4)$*. PhD thesis, Ohio State University, 2011.

[Del68] P. Deligne. Formes modulaires et représentations $l$-adiques. *Séminaire N. Bourbaki*, 355:139–172, 1968.

[Del79] P. Deligne. Variétés de Shimura: Interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions*, volume 33 of *Proceedings of Symposia in Pure Mathematics*, pages 247–290. AMS, 1979.

[DR09] S. DeBacker and M. Reeder. Depth zero supercuspidal $L$-packets and their stability. *Ann. of Math.*, 169(3):795–901, 2009.

[Eno72] H. Enomoto. The characters of the finite symplectic group $Sp(4, q)$, $q = 2^f$. *Osaka J. Math.*, 9:75–94, 1972.

[Fal83] G. Faltings. On the cohomology of locally symmetric hermitian spaces. In *Séminaire d'Algèbre Paul Dubrail and Marie-Paule Malliavin*, volume 1029 of *Lecture Notes in Mathematics*, pages 55–98. Springer, 1983.

[FC90] G. Faltings and C.-L. Chai. *Degeneration of Abelian Varieties*, volume 22 of *Ergebnisse der Mathematik und Ihrer Grenzgebiete.* Springer, 1990.

[FH95] S. Friedberg and J. Hoffstein. Nonvanishing Theorems for Automorphic $L$-functions on GL(2). *Ann. of Math.*, 142(2):385–423, 1995.

[Fli05] Y. Z. Flicker. *Automorphic Forms and Shimura Varieties of PGSp(2).* World Scientific, 2005.

[FvdG04] C. Faber and G. van der Geer. Sur la Cohomologie des Systèmes Locaux sur les espaces des modules des Courbes de Genre 2 et des Surface Abéliennes. *Comptes Rendu*, 338:381–384,467–470, 2004.

[Gel75] S. S. Gelbart. *Automorphic Forms on Adele Groups.* Number 83 in Annals of mathematics studies. Princeton University Press, 1975.

[GT11] W. T. Gan and S. Takeda. Theta correspondences for GSp(4). *Representation Theory*, 15:670–718, 2011.

[Hal89] T.C. Hales. Shalika Germes on $GSp(4)$. In *Orbites Unipotentes et Représentations*, volume 171-172 of *Astérisque*, pages 195–256. Soc. math. Fr., 1989.

[Hal97] T.C. Hales. The Fundamental Lemma for $Sp(4)$. *Proc. Amer. Math. Soc.*, 125:301–308, 1997.

[Har12] G. Harder. The Eisenstein motive for the cohomology of $GSp_2(\mathbb{Z})$. In C. Faber, G. Farkas, and R. De Jong, editors, *Geometry and Arithmetic*, pages 143–164. European Mathematical Society, 2012.

[Kna86] A. Knapp. *Representation Theory of Semisimple Groups.* Princeton Landmarks in Mathematics. Princeton University Press, 1986.

[KS99] R.E. Kottwitz and D. Shelstad. *Foundations of Twisted Endoscopy*, volume 255 of *Asterisque.* Soc. math. Fr., 1999.

[Loo88] E. Looijenga. $L^2$-cohomology of locally symmetric varieties. *Comp. Math.*, 67(1):3–20, 1988.

[LS87] R. P. Langlands and D. Shelstad. On the definition of transfer factors. *Math. Ann.*, 278:219–271, 1987.

[LW85] R. Lee and S. H. Weintraub. Cohomology of a Siegel Modular Variety of Degree 2. In R. Schultz, editor, *Group Actions on Manifolds*, volume 36 of *Contemporary Mathematics*, pages 433–488. AMS, 1985.

[Mil04] J. S. Milne. Introduction to Shimura Varieties, 2004. `www.jmilne.org`.

[Mor93] L. Morris. Tamely ramified intertwining algebras. *Inv. Math.*, 114:1–54, 1993.

[Mor96] L. Morris. Tamely ramified supercuspidal representations. *Annales scientifiques de l'École Normale Superieure*, 29(5):639–667, 1996.

[Moy88] A. Moy. Representations of GSp(4) over a $p$-adic field: part 1. *Comp. Math.*, 66(3):237–284, 1988.

[MP94] A. Moy and G. Prasad. Unrefined minimal $K$-types for $p$-adic groups. *Inv. Math.*, 116:393–408, 1994.

[MP96] A. Moy and G. Prasad. Jacquet functors and unrefined minimal K-types. *Comment. Math. Helvetici*, 71:98–121, 1996.

[MR] M. Mishra and M. Rösner. Genericity under parahoric restriction. arXiv: 1512.06204.

[Ngô10] B. C. Ngô. Le lemme fondamental pour les algébres de Lie. *Publications Mathématiques de L'Institut des Hautes Études Scientifiques*, 111(1):1–169, 2010.

[Pan02] S.-Y. Pan. Depth preservation in local theta correspondence. *Duke Math. J.*, 113(3):531–591, 2002.

[Pet15] D. Petersen. Cohomology of local systems on the moduli of principally polarized abelian surfaces . *Pacific J. Math.*, 275(1):39–61, 2015.

[Pin92] R. Pink. On $\ell$-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification. *Math. Ann.*, 292:197–240, 1992.

[PS83a] I. Piatetski-Shapiro. *Complex Representations of $GL(2, K)$ for Finite Fields $K$*, volume 16 of *Contemporary Mathematics*. AMS, 1983.

[PS83b] I. Piatetski-Shapiro. On the Saito-Kurokawa-Lifting. *Inv. Math.*, 71:309–338, 1983.

[PSS81] I. Piatetski-Shapiro and D. Soudry. The $L$ and $\epsilon$ Factors for $GSp(4)$. *J. Fac. Sci. Univ. Tokyo, Sect. 1a*, 28:505–530, 1981.

[Rös12] M. Rösner. The anisotropic Theta lift for $\mathrm{GSp}(4, F)$. Master's thesis, Universität Heidelberg, 2012.

[RS07] B. Roberts and R. Schmidt. *Local Newforms for $GSp(4)$*, volume 1918 of *Lecture Notes in Mathematics*. Springer, 1 edition, October 2007.

[Sap05] L. Saper. $L^2$-cohomology of locally symmetric spaces. I. *Pure and Applied Mathematics Quarterly*, 1(4):889–937, 2005.

[Sat57] I. Satake. The Gauss-Bonnet Theorem for $V$-manifolds. *J. Math. Soc. Japan*, 9(4):464–492, 1957.

[Sch90] A. J. Scholl. Motives for modular forms. *Inv. Math.*, 100(2):419–430, 1990.

[Sch95] J. Schwermer. On Euler products and residual Eisenstein cohomology classes for Siegel modular varieties. *For. Math.*, 71:1–28, 1995.

[Sch05a] R. Schmidt. Iwahori-spherical representations of $GSp(4)$ and Siegel modular forms of degree 2 with square-free level. *J. Math. Soc. Japan*, 57(1), 2005.

[Sch05b] R. Schmidt. The Saito-Kurokawa lifting and functoriality. *Amer. J. Math.*, 127:209–240, 2005.

[Sch07] R. Schmidt. On classical Saito-Kurokawa-Liftings. *J. Reine Angew. Math.*, 604:211–236, 2007.

[Shi82] K.-I. Shinoda. The characters of the finite conformal symplectic group $CSp(4, q)$. *Communications in Algebra*, 10(13):1369–1419, 1982.

[SO90] J. Schwermer and T. Oda. Mixed Hodge structures and automorphic forms for Siegel modular varieties of degree two. *Math. Ann.*, 286:481–509, 1990.

[Sou84] D. Soudry. The $L$ and $\gamma$ factors for generic representations of $GSp(4, k) \times GL(2, k)$ over a local nonarchimedean field $k$. *Duke Math. J.*, 51(2):355–394, 1984.

[Sou88] D. Soudry. The CAP representations of $GSp(4)$. *J. Reine Angew. Math.*, 383:97–108, 1988.

[SS90] L. Saper and M. Stern. $L^2$-cohomology of arithmetic varieties. *Ann. of Math.*, 132:1–69, 1990.

[SS13] R. Schmidt and A. Saha. Yoshida Lifts and Simultaneous non-vanishing of dihedral twists of modular $L$-functions. *J. London Math. Soc*, 2013.

[ST94] P. Sally and M. Tadić. Induced Representations and classifications for $GSp(2, F)$ and $Sp(2, F)$. *Mémoires Société Mathématique de France*, 52:75–133, 1994.

[Tad91] M. Tadić. On Jacquet Modules of Induced representations of p-adic symplectic groups. In *Harmonic Analysis on Reductive Groups*, volume 101 of *Progress in Mathematics*, pages 305–314. Birkhäuser, 1991.

[Tay93] R. Taylor. On the $l$-adic cohomology of Siegel threefolds. *Inv. Math.*, 114:289–310, 1993.

[Teh12] S.S. Tehrani. *Non-holomorphic cuspidal automorphic forms of* $GSp(4, \mathbb{A})$ *and the Hodge structure of Siegel threefolds.* PhD thesis, University of Toronto, 2012.

[Tit79] J. Tits. Reductive Groups over local Fields. In *Automorphic forms, representations and L-functions*, volume 33 of *Proceedings of Symposia in pure Mathematics*, pages 29–69. AMS, 1979.

[Tsu83] R. Tsushima. An Explicit Dimension Formula for the Spaces of Generalized Automorphic Forms with Respect to $\mathrm{Sp}(2, \mathbb{Z})$. *Proc. Japan Acad.*, 59(4):139–142, 1983.

[vdG] G. van der Geer. Counting Curves over finite fields. arXiv: 1409.6090.

[vdG82] G. van der Geer. On the geometry of a Siegel modular Threefold. *Math. Ann.*, 260:317–350, 1982.

[vdG11] G. van der Geer. Rank one Eisenstein cohomology of local systems on the moduli of abelian varieties. *Sci. China. Math.*, 54(83):1621–1634, 2011.

[vGN95] B. van Geemen and N.O. Nygaard. On the Geometry and Arithmetic of Some Siegel Modular threefolds. *J. of Number Theory*, 53:45–87, 1995.

[Vig96] M.-F. Vignéras. *Représentations $l$-modulaires d'un groupe réductif $p$-adique avec $l \neq p$*, volume 137 of *Progress in Mathematics*. Birkhäuser, 1996.

[Vig01] M.-F. Vignéras. Irreducible Modular Representations of a reductive $p$-adic group and simple Modules for Hecke Algebras. In C. Casacuberta et al., editors, *European Congress of Mathematics, Barcelona*, volume 201 of *Progress in Mathematics*, pages 117–133. Birkhäuser, 2001.

[Vig03] M.-F. Vignéras. Schur Algebras of reductive $p$-adic groups. *Duke Math. J.*, 116(1):35–75, 2003.

[VZ84] D.A. Vogan and G.J. Zuckerman. Unitary representations with non-zero cohomology. *Comp. Math.*, 53(1):51–90, 1984.

[Wal88] N. R. Wallach. *Real reductive groups I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, 1988.

[Wat79] W.C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer, 1979.

[Wei88] R. Weissauer. Differentialformen zu Untergruppen der Siegelschen Modulgruppe zweiten Grades. *J. Reine Angew. Math.*, 391:100–156, 1988.

[Wei05] R. Weissauer. Four dimensional Galois representations. In J. Tilouine, H. Carayol, M. Harris, and M.-F. Vignéras, editors, *Formes automorphes (II), Le cas du groupe GSp(4)*, volume 302 of *Asterisque*, pages 67–149. Soc. math. Fr., 2005.

[Wei09a] R. Weissauer. *Endoscopy for $GSp(4)$ and the Cohomology of Siegel Modular Threefolds*, volume 1968 of *Lecture notes in Mathematics*. Springer, 1. edition, 2009.

[Wei09b] R. Weissauer. The Trace of Hecke operators on the space of classical holomorphic Siegel modular forms of genus two. arXiv:0909.1744v1, 2009.

[Yu01] J.-K. Yu. Construction of Tame Supercuspidal Representations. *Journal of the American Mathematical Society*, 14(3):579–622, 2001.

[Zag81] D. Zagier. Sur la Conjecture de Saito-Kurokawa (d'aprés Hans Maass). *PM*, 1219:371–394, 1981.

[Zuc88] S. Zucker. $L^2$-cohomology of Shimura varieties. In L. Clozel and J. S. Milne, editors, *Automorphic Forms, Shimura Varieties, and L-functions*, volume 2 of *Perspectives in Mathematics*, pages 377–391. Academic Press, 1988.

# Index

# Symbols