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The spectra of reducible matrices over completed commutative idempotent semifields and their spectral lattices

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Previous work has shown a relation between L-valued extensions of Formal Concept Analysis and the spectra of some matrices related to L-valued contexts. To clarify this relation we investigated elsewhere the nature of the spectra of irreducible matrices over idempotent semifields in the framework of dioids, naturally-ordered semirings, that encompass several of those extensions. This initial work already showed many differences with respect to their counterparts over incomplete idempotent semifields in what concerns the definition of the spectrum and the eigenvectors. Considering special sets of eigenvectors also brought out complete lattices in the picture and we argue that such structure may be more important than standard eigenspace structure for matrices over completed idempotent semifields. In this paper we complete that investigation in the sense that we consider the spectra of reducible matrices over completed idempotent semifields and dioids, giving, as a result, a constructive solution to the all-eigenvectors problem in this setting. This solution shows that the relation of complete lattices to eigenspaces is even tighter than suspected.

Keywords: dioids; complete idempotent semifields; all-eigenvectors problem; spectral order lattices; eigenlattices.

1. Motivation

The eigenvectors and eigenspaces over certain naturally ordered semirings called *dioids* seem to be intimately related to multi-valued extensions of Formal Concept Analysis (Ganter and Wille 1999). For instance (Belohlavek and Vychodil 2010) and (Belohlavek 2012) prove that formal concepts are optimal factors for decomposing a matrix with entries in complete residuated semirings over $[0, 1]$. In those papers there is a strong formal analogy with the Singular Value Decomposition, with formal concepts taking the role of pairs of left and right eigenvectors. Indeed, (Valverde-Albacete and Peláez-Moreno 2008) proved that, at least on a particular kind of dioids, the idempotent semifields, formal concepts are related to the eigenvectors of the unit in the semiring. These results, however, cannot be unified both for theoretical reasons—since idempotent semifields are incomplete (see below)—as well as for practical reasons—since the carrier set of idempotent semifields is almost never included in $[0, 1]$.

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A possible way forward is to develop these theories under the common framework of the L-fuzzy sets, where L is any complete lattice (Goguen 1967). Such an endeavour has already been called for (Gondran and Minoux 2007), although it remains unfulfilled. Therefore, this paper has a two-fold aim:

(1) to clarify the spectral theory over *completed* idempotent semifields, and
(2) to take steps towards a common framework for the interpretation of *L-Formal Concept Analysis as a spectral construction.*

First steps have been taken in this direction with the development of a spectral theory of irreducible matrices (Valverde-Albacete and Peláez-Moreno 2014) over complete idempotent semifields, whose main results are included below, but the general case, here presented, shows a more intimate relation to lattice theory, as well as representing a constructive solution to the all-eigenvectors problem for matrices over complete idempotent semifields.

1.1. *Dioids and their spectral theory*

A *semiring* is an algebra $\mathcal{S} = \langle S, \oplus, \otimes, \epsilon, e\rangle$ whose additive structure, $\langle S, \oplus, \epsilon\rangle$, is a commutative monoid and whose multiplicative structure, $\langle S\backslash\{\epsilon\}, \otimes, e\rangle$, is a monoid with multiplication distributing over addition from right and left and with additive neutral element absorbing for $\otimes$, i.e. $\forall a \in S,\ \epsilon \otimes a = \epsilon$.

Given $A \in S^{n\times n}$ the *right (left) eigenproblem* is the task of finding the *right eigenvectors* $v \in S^{n\times 1}$ and *right eigenvalues* $\rho \in S$ (respectively *left eigenvectors* $u \in S^{1\times n}$ and *left eigenvalues* $\lambda \in S$) satisfying:

$$u \otimes A = \lambda \otimes u \qquad\qquad A \otimes v = v \otimes \rho \tag{1}$$

The left and right eigenspaces and spectra are the sets of these solutions:

$$\mathcal{U}_\lambda(A) = \{u \in S^{1\times n} \mid u \otimes A = \lambda \otimes u\} \quad \mathcal{V}_\rho(A) = \{v \in S^{n\times 1} \mid A \otimes v = v \otimes \rho\} \tag{2}$$

$$\Lambda(A) = \{\lambda \in S \mid \mathcal{U}_\lambda(A) \neq \{\epsilon^n\}\} \qquad \mathrm{P}(A) = \{\rho \in S \mid \mathcal{V}_\rho(A) \neq \{\epsilon^n\}\} \tag{3}$$

$$\mathcal{U}(A) = \bigcup_{\lambda\in\Lambda(A)} \mathcal{U}_\lambda(A) \qquad \mathcal{V}(A) = \bigcup_{\rho\in\mathrm{P}(A)} \mathcal{V}_\rho(A) \tag{4}$$

Since $\Lambda(A) = \mathrm{P}(A^{\mathrm{T}})$ and $\mathcal{U}_\lambda(A) = \mathcal{V}_\lambda(A^{\mathrm{T}})$, from now on we will omit references to left eigenvalues, eigenvectors and spectra, unless we want to emphasize differences. With so little structure it might seem hard to solve (1), but a very generic solution based in the concept of transitive closure of a matrix $A^+ = \sum_{i=1}^{\infty} A^i$ and transitive-reflexive closure $A^* = \sum_{i=0}^{\infty} A^i$ is given by the following theorem:

Theorem 1.1. *(Gondran and Minoux 1977, Theorem 1) Let $A \in \mathcal{S}^{n\times n}$. If A^* exists, the following two conditions are equivalent:*
(1) $A^+_{\cdot i} \otimes \mu = A^*_{\cdot i} \otimes \mu$ for some $i \in \{1 \dots n\}$, and $\mu \in S$.
(2) $A^+_{\cdot i} \otimes \mu$ (and $A^*_{\cdot i} \otimes \mu$) is an eigenvector of A for e, $A^+_{\cdot i} \otimes \mu \in \mathcal{V}_e(A)$.

In (Valverde-Albacete and Peláez-Moreno 2014) this result was made more specific in two directions: on the one hand, by focusing on particular types of completed idempotent semirings—semirings with a natural order where infinite additions of elements exist so transitive closures are guaranteed to exist and sets of generators can be found for the eigenspaces—and, on the other hand, by considering more easily visualizable subsemimodules than the whole eigenspace—a better choice for exploratory data analysis.

Specifically, every commutative semiring accepts a canonical preorder, $a \leq b$ if and only if there exists $c \in D$ with $a \oplus c = b$. A *dioid* is a semiring $\mathcal{D}$ where this relation is actually an order. Dioids are zerosumfree and entire, that is they have no non-null additive or multiplicative factors of zero. Commutative complete dioids are already complete residuated lattices. Similarly, semimodules over complete commutative dioids are also complete lattices.

We will make occasional use of the following proposition,

Proposition 1.2. *(Golan 1999, p. 150) If $f : \mathcal{R} \to \mathcal{S}$ is a morphism of semirings, and if $\mathcal{X}$ is a right $\mathcal{S}$-semimodule then it is also canonically a right $\mathcal{R}$-semimodule, with scalar multiplication defined by $rx = f(r)x$ for all $r \in R$ and $x \in X$. In particular, if $\mathcal{X}$ is a right $\mathcal{S}$-semimodule then $\mathcal{X}$ is a left $\mathcal{R}$-semimodule for every subsemiring $\mathcal{R}$ of $\mathcal{S}$, by the inclusion map, $\hookrightarrow_{\mathcal{S}}(r) = r$.*

An *idempotent semiring* is a dioid whose addition is idempotent, and a *selective semiring* one where the arguments attaining the value of the additive operation can be identified.

Example 1. *Examples of idempotent dioids are*

(1) The Boolean lattice $\mathbb{B} = \langle \{0,1\}, \vee, \wedge, 0, 1 \rangle$
(2) All fuzzy semirings, e.g. $\langle [0,1], \max, \min, 0, 1 \rangle$
(3) The min-plus algebra $\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{\infty\}, \min, +, \infty, 0 \rangle$
(4) The max-plus algebra $\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$ □

Of the semirings above, only the boolean lattice and the fuzzy semirings are complete dioids, since the rest lack the *top* element $\top$ as an adequate inverse for the bottom in the order.

1.2. *Completed idempotent semifields and their spectral theory for irreducible matrices*

A semiring is a *semifield* if there exists a multiplicative inverse for every element $a \in S$, notated as a^{-1}, and *radicable* if the equation $a^b = c$ can be solved for a. As exemplified above, idempotent semifields are incomplete in their natural order. Luckily, there are procedures for *completing* such structures (Valverde-Albacete and Peláez-Moreno 2011) and we will not differentiate between *complete or completed* structures,

Example 2. *The max-plus $\mathbb{R}_{\max,+}$ and min-plus $\mathbb{R}_{\min,+}$ semifields can be completed as:*

(1) The complete min-plus semifield $\overline{\mathbb{R}}_{\min,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \min, \dot{+}, -\cdot, \infty, 0 \rangle$.
(2) The complete max-plus semifield $\overline{\mathbb{R}}_{\max,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \max, \dot{+}, -\cdot, -\infty, 0 \rangle$.

In this notation we have $\forall c, -\infty + c = -\infty$ and $\infty \dot{+} c = \infty$, which solves several issues in dealing with the separately completed dioids. These two completions are inverses $\overline{\mathbb{R}}_{\min,+} = \overline{\mathbb{R}}^{-1}_{\max,+}$, hence order-dual lattices. □

In fact, idempotent semifields $\mathcal{K} = \langle K, \oplus, \dot{\oplus}, \otimes, \dot{\otimes}, \cdot^{-1}, \bot, e, \top \rangle$, appear as enriched structures, the advantage of working with them being that meets can be expressed by means of joins and inversion as $a \wedge b = (a^{-1} \dot{\oplus} b^{-1})^{-1}$. On a practical note, residuation in complete commutative idempotent semifields can be expressed in terms of inverses, and this extends to eigenspaces.

As proven in (Valverde-Albacete and Peláez-Moreno 2014), the set of eigenvalues on complete dioids is extended with respect to the incomplete case, so it makes sense

to distinguish between the *proper eigenvalues* $\mathrm{P}^{\mathrm{p}}(A)$, associated with eigenvectors with finite coordinates, and the *improper eigenvalues* $\mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{p}}(A)$ associated with eigenvectors with non-finite coordinates.

The eigenspaces of matrices over complete dioids have the structure of a complete lattice. But since these lattices may be continuous and difficult to visualize we introduce the more easily-represented *(right) eigenlattices* $\mathcal{L}_\rho(A)$ which are complete *finite* sublattices of the eigenspaces to be used as scaffolding in visual representations[1].

The basic building block is the spectrum of irreducible matrices: for a matrix $A \in \mathcal{M}_n(\mathcal{S})$, the *network or weighted digraph induced by* A, $N_A = (V_A, E_A, w_A)$, consists of a set of vertices V_A, a set of arcs , $E_A = \{(i,j) \mid A_{ij} \neq \epsilon_S\}$, and a weight $w_A : V_A \times V_A \to S,\ (i,j) \mapsto w_A(i,j) = a_{ij}$. Then matrix A is *irreducible* if every node of V_A is connected to every other node in V_A though a path, otherwise it is *reducible*.

This allows us to apply intuitively all notions from networks to matrices and vice versa, like the underlying graph $G_A = (V_A, E_A)$, the set of paths $\Pi_A^+(i,j)$ between nodes i and j or the set of cycles C_A^+. In particular, if $l(c)$ is the length of a cycle $c \in C_A^+$ and $w(c)$ its weight, then the mean of the cycle is $\mu_\oplus(c) = \sqrt[l(c)]{w(c)}$, and the *aggregate cycle mean* of A is $\mu_\oplus(A) = \sum\{\mu_\oplus(c) \mid c \in C_A^+\}$. If the semiring is idempotent and selective, the nodes in the circuits that attain this mean are called the *critical nodes of* A, $V_A^c = \{i \in c \mid \mu_\oplus(c) = \mu_\oplus(A)\}$.

For a finite $\rho = \mu_\oplus(A)$, let $\tilde{A}^{\rho^+} = (A \,/\, \rho)^+$ be the *normalized transitive closure of* A. Then the critical nodes are $V_A^c = \{i \in V_A \mid \tilde{A}_{ii}^+ = e\}$, and we define the set of (right) *fundamental eigenvectors of* A *for* ρ as

$$\mathrm{FEV}_\rho(A) = \{\tilde{A}_{\cdot i}^+ \mid i \in V_A^c\} = \{\tilde{A}_{\cdot i}^+ \mid \tilde{A}_{ii}^+ = e\}.$$

Theorem 1.3 ((**Right**) **spectral theory for irreducible matrices**, (Valverde-Albacete and Peláez-Moreno 2014))**.** *Let* $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ *be an irreducible matrix over a complete commutative selective radicable semifield. Then:*

(1) The right spectrum of the matrix includes the whole semiring but the zero:

$$\mathrm{P}(A) = \overline{\mathcal{K}} \setminus \{\bot\}$$

(2) The right proper spectrum only comprises the aggregate cycle mean:

$$\mathrm{P}^{\mathrm{p}}(A) = \{\mu_\oplus(A)\}$$

(3) If an eigenvalue is improper $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{p}}(A)$*, then its eigenspace (and eigenlattice) is reduced to the two vectors:*

$$\mathcal{V}_\rho(A) = \{\bot^n, \top^n\} = \mathcal{L}_\rho(A)$$

(4) The eigenspace for a finite proper eigenvalue $\rho = \mu_\oplus(A) < \top$ *is generated from its fundamental eigenvectors over the whole semifield, while the eigenlattice is generated by* $\mathbb{3}$*:*

$$\mathcal{V}_\rho(A) = \langle \mathrm{FEV}_\rho(A) \rangle_{\overline{\mathcal{K}}} \supset \mathcal{L}_\rho(A) = \langle \mathrm{FEV}_\rho(A) \rangle_{\mathbb{3}}$$

[1]Right and left eigenlattices will only be distinguished by the indexing them with the standard notation for a right and left eigenvalue, respectively.

Refer to (Valverde-Albacete and Peláez-Moreno 2014) for further details.

1.3. *Reading guide*

In this paper we try and find analogue results to Theorem 1.3 for the reducible case, and in doing so solve the all-eigenvectors problem for matrices over completed idempotent semifields. First, we present in Section 2.3 a recursive scheme to render matrices over idempotent semifields into specialized Upper Frobenius Normal Forms (UFNF), thus providing in Sections 3.2-3.4 a bottom-up construction of the spectra of reducible matrices from that of their irreducible ones. By defining particular sets of *fundamental eigenvectors*, $\mathrm{FEV}_\rho(A)$ for each particular UFNF form we present in Section 3.5 an overarching formulation of our results for eigenspaces and eigenlattices. Finally, we discuss our findings and approach in Section 4 and relate them to previous attempts at describing such structures.

2. Preliminaries

2.1. *Partial orders and lattices*

A *(partially) ordered set* [2] is an algebra $\mathcal{P} = \langle P, \leq \rangle$ where $\leq$ is a reflexive, antisymmetric and transitive relation on a carrier set P. Every ordered $\mathcal{P}$ has a dual $\mathcal{P}^{\mathrm{d}} = \langle P, \leq^{\mathrm{d}} \rangle$ where the converse relation holds, $y \leq^{\mathrm{d}} x \equiv y \geq x \Leftrightarrow x \leq y$. We may use $x \leq y$ and $y \geq x$ for $x, y \in P$ interchangeably, and we use $x \parallel y$ to denote non-comparability: $x \parallel y \Leftrightarrow x \not\leq y$ and $y \not\leq x$. Low-complexity partial orders are practically drawn using *order (or Hasse) diagrams* [3].

Example 3. *Every set V with $|V| = n$ elements and the reflexive identity relation $I = \{(v, v) \mid v \in V\}$ is called an* anti-chain of n elements*, and we notate them as $\langle V, I \rangle \cong \overline{\mathbf{n}}$. Anti-chains are (vacuously transitive, antisymmetric) partial orders, one natural transposition of sets to order theory.* □

Let $\mathcal{P} = \langle P, \geq \rangle$ be an ordered set and $Q \subseteq P$. Then Q is an *order ideal or downset* if for $x \in Q, y \in P$ whenever $y \leq x$ then $y \in Q$. Dually, Q is an *order filter or upset* if for $x \in Q, y \in P$ whenever $y \geq x$ then $y \in Q$. For arbitrary $Q \subseteq P$, $\downarrow Q = \{y \in P \mid \exists x \in Q, y \leq x\}$ (read 'down Q'), and dually $\uparrow Q = \{y \in P \mid \exists x \in Q, y \geq x\}$ (read 'up Q'). Downsets (upsets) of the form $\downarrow x = \downarrow \{x\}$ ($\uparrow x = \uparrow \{x\}$) are called *principal order ideals (filters)*. The family of all downsets of $\mathcal{P}$ (or upsets) is denoted by $\mathcal{O}(P)$ ($\mathcal{F}(P)$) and is ordered by set inclusion.

Let $\mathcal{P}$ be an ordered set and $Q \subseteq P$. An element $x \in P$ is an *upper bound* of Q if $y \leq x$ for all $y \in Q$. A *lower bound* is defined dually. The set of all upper bounds is written Q^u and the set of all lower bounds as Q^l. Since $\leq$ is transitive, Q^l is always a downset and Q^u an upset. For $Q \subseteq P$, $x \in P$ is the *least upper bound*, or *supremum* or *join*, of Q if x is an upper bound of Q such that $x \leq y$ for all upper bounds y of Q. Dually if Q^l has a greatest element this is the *greatest upper bound*, or *infimum*, of Q.

Let $\mathcal{L} = \langle L, \leq \rangle$ be an ordered set. If the supremum exists for every pair $x, y \in L$–we write $x \vee y$–then $\mathcal{L}$ is a $\vee$*-semilattice* or *join semilattice*. Dually, if the infimum exists for every pair $x, y \in L$–we write $x \wedge y$–then $\mathcal{L}$ is a $\wedge$*-semilattice* or *meet*

[2] Essentially, this introductory material follows (Davey and Priestley 2002).
[3] Recall that these actually depicts the irreflexive, transitive reduction of the orders they represent.

semilattice. When both suprema and infima exist, then $\mathcal{L} = \langle L, \vee, \wedge \rangle$ is a *lattice*. The order and algebraic operations can be related by the connecting Lemma:

Lemma 2.1. *Let $\mathcal{L}$ be a lattice and $a, b \in L$. Then $a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$.*

Example 4. *(1) Every set V with $|V| = n$ elements and a total order $\leq \subseteq V \times V$ is isomorphic to a lattice called the* chain of n elements, *$\langle V, \leq \rangle \cong \mathbf{n}$. Lattice $\mathbb{1} \cong \mathbf{1}$ is the vacuously-ordered singleton. Lattice $\mathbb{2}$ is the boolean lattice isomorphic to chain $\mathbf{2}$. Lattice $\mathbb{3}$ is the lattice lying at the heart of completed semifields, the $\mathbb{3}$-blog, isomorphic to chain $\mathbf{3}$.*

(2) Anti-chains are not lattices, except for $n = 1$. □

When the supremum exists for every subset $Q \subseteq L$, then $\mathcal{L}$ is a *complete $\vee$-semilattice*. Similarly, when the infimum exists for every subset $Q \subseteq L$, then $\mathcal{L}$ is a *complete $\wedge$-semilattice*. When $\mathcal{L}$ is both a complete join- and meet-semilattice, then it is a *complete lattice*. Complete lattices have *top* $\top = \bigvee S$ and bottom elements $\bot = \bigwedge L$. The following are two important results:

Proposition 2.2. *(1) If $\mathcal{L}$ is a complete $\vee$-semilattice with bottom element $\bot$ then it is also a complete lattice (dually for complete $\wedge$-semilattices with $\top$).*

(2) Finite lattices are complete.

For an element $a \in L$ in a complete lattice, we say that a is *join-irreducible* if it cannot be obtained as the join of its strictly lower bounds, $a \in \mathcal{J}(L) \Leftrightarrow a \neq \bigvee\{x \in L \mid x < a\}$. *Meet-irreducibles* are defined dually, $b \in \mathcal{M}(L) \Leftrightarrow b \neq \bigwedge\{x \in L \mid b < x\}$. Next call a subset $Q \subseteq L$ *join-dense (supremum-dense)* if every element of L can be obtained as a join of a subset of Q, and dually for a *meet-dense (infimum-dense)*. The result below is basic:

Proposition 2.3. *(Ganter and Wille 1999, Proposition 2) If L is a finite lattice,*

(1) $a \in L$ is join-irreducible if and only if it has exactly one lower neighbour, and it is meet irreducible if and only if it has exactly one upper neighbour.

(2) Every join-dense subset of L contains $\mathcal{J}(L)$ and every meet-dense subsets of L contains $\mathcal{M}(L)$. Conversely, $\mathcal{J}(L)$ is join-dense and $\mathcal{M}(L)$ is meet-dense in L.

Orders can easily be built from other orders and we instantiate on lattices:

- The *disjoint union* of two lattices L_1 and L_2 is another lattice $L_1 \uplus L_2$ where $x \leq y$ if and only if $x, y \in L_1$ and $x \leq y$ or $x, y \in L_2$ and $x \leq y$. This is not complete even if L_1 and L_2 are.
- The *linear or vertical sum* of two lattices L_1 and L_2 is the lattice $L_1 \oplus L_2$ defined as an order $\langle L_1 \oplus L_2, \leq \rangle$ where $x \leq y$ if and only if $x, y \in L_1$ or $x, y \in L_2$ and $x \leq y$ in either case, or $x \in P$ and $y \in Q$. For complete lattices, the top of L_1 is the single lower neighbour of the bottom of L_2. For instance $M_n = \mathbb{1} \oplus \overline{\mathbf{n}} \oplus \mathbb{1}$.
- For a family of lattices $\{L_i\}_{i=1}^n$ their *Cartesian product* $\bigtimes_{i=1}^n L_i$ will bear the componentwise order $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \Leftrightarrow x_i \leq y_i$, for all $1 \leq i \leq n$. When all the factors are the same it is customary to write the resulting expression as a power, so, for instance, the powers of $\mathbb{2}$ are the *boolean lattices*, $\mathbb{2}^n = \bigtimes_{i=1}^n \mathbb{2}$, which shows that the product lattice will be complete if the factors are complete.

We state for later use:

Lemma 2.4. *Let $L_1 \times L_2$ be the product of two complete lattices. Then:*

(1) $\mathcal{J}(L_1 \times L_2) = (\mathcal{J}(L_1) \times \{\bot_2\}) \cup (\{\bot_1\} \times \mathcal{J}(L_2))$
(2) $\mathcal{M}(L_1 \times L_2) = (\mathcal{M}(L_1) \times \{\top_2\}) \cup (\{\top_1\} \times \mathcal{M}(L_2))$

where $\bot_1$ *and* $\bot_2$ *(resp* $\top_1$ *and* $\top_2$*) are the bottom (resp. top) elements of each factor.*

The set of order ideals of a poset P is a lattice $\mathcal{O}(P)$.

Proposition 2.5. *Let* $\langle P, \leq \rangle$ *be a finite poset. Then* $\langle \mathcal{O}(P), \subseteq \rangle$ *is a lattice obtained by the embedding* $\varphi : P \to \mathcal{O}(P), \varphi(x) = \downarrow x$*, with* $\forall A_1, A_2 \in \mathcal{O}(P), A_1 \vee A_2 = A_1 \cup A_2$ *and* $A_1 \wedge A_2 = A_1 \cap A_2$*.*

Note that $x \leq y$ in $\mathcal{P}$ if and only if $\downarrow x \subseteq \downarrow y$ in $\mathcal{O}(P)$. Furthermore, $\mathcal{O}(P)$ can be obtained systematically from the ordered set in a number of cases:

Proposition 2.6. *Let* $\langle P, \leq \rangle$ *be a finite poset. Then*
(1) $\mathcal{O}(P \oplus \mathbb{1}) \cong \mathcal{O}(P) \oplus \mathbb{1}$ *and* $\mathcal{O}(\mathbb{1} \oplus P) \cong \mathbb{1} \oplus \mathcal{O}(P)$*.*
(2) $\mathcal{O}(P_1 \uplus P_2) \cong \mathcal{O}(P_1) \times \mathcal{O}(P_2)$*.*
(3) $\mathcal{O}(P^{\mathrm{d}}) \cong \mathcal{F}(P) \cong \mathcal{O}(P)^{\mathrm{d}}$*.*
(4) $\mathcal{O}(n) \cong n \oplus \mathbb{1} \cong \mathbb{1} \oplus n$*.*
(5) $\mathcal{O}(\overline{n}) \cong 2^n$*.*

2.2. *The condensation digraph of a matrix*

A *digraph (or directed graph)*, is a pair $G = (V, E)$ with V a set of *vertices* and $E \subseteq V \times V$ a set of *arcs (directed edges)*, ordered pairs of vertices, such that for every $i, j \in V$ there is at most one arc $(i, j) \in E$. If $(i, j) \in E$ then we say that *"i is a predecessor of j"* or *"j is a successor of i"*, and $E \in \mathcal{M}_n(\mathbb{B})$ is the *associated relation of* G. If there is a walk from a vertex i to a vertex j in G we say that *"i has access to j"* or *j is reachable from i*, $i \rightsquigarrow j$. Hence, *reachability* is the transitive closure of the associated relation, $\rightsquigarrow = E^+$ (Schmidt and Ströhlein 1993). However, vertices $j \in V$ with no incoming or outgoing arcs cannot be part of any cycle, hence $j \not\rightsquigarrow j$ for such nodes, so it is not reflexive, in general. $(\rightsquigarrow \cap I_V)$ is the *reflexive restriction* of $\rightsquigarrow$, that is, the biggest reflexive relation included in it.

If there is a walk from a vertex i to vertex j in G or viceversa we say that *i and j are connected*, $i \rightsquigarrow j \vee j \rightsquigarrow i$. *Connectivity* is the symmetric closure of the reachability relation: its transitive, reflexive restriction is an equivalence relation on V_G whose classes are called the *(dis)connected components of* G. Note that each of the (outwards) disconnected components is actually (inwards) connected. Let $K \geq 1$ be the number of disconnected components of G. Then V and E are partitioned into the subsets of vertices $\{V_k\}_{k=1}^K$ and arcs $\{E_k\}_{k=1}^K$ of each disconnected component $\bigcup_k V_k = V$, $V_k \cap V_l = \varnothing, k \neq l$, $\bigcup_k E_k = E$, $E_k \cap E_l = \varnothing, k \neq l$ and we may write $G = \uplus_{k=1}^K G_k$ is a disjoint union of graphs. G is called *connected* itself if $K = 1$.

On the other hand, since reachability is transitive by construction, its symmetric, reflexive restriction $i \leftrightsquigarrow j \Leftrightarrow i \rightsquigarrow j \ \wedge j \rightsquigarrow i$ is a refinement of connectivity called *strong connectivity*. Its equivalence classes are the *strongly connected components* of G. For each disconnected component G_k, let R_k be the number of its strongly connected components. If $R_k = 1$ then the k-th component is *strongly connected*, otherwise just *connected* and composed of a number of strongly connected components itself. G is *strongly connected* itself if $K = R = 1$.

Given a digraph $G = (V, E)$, the *reduced or condensation digraph,* $\overline{G} = (\overline{V}, \overline{E})$ is induced by the set $\overline{V} = V/\leftrightsquigarrow$ of strongly connected components, and $C, C' \in \overline{V}$, $(C, C') \in \overline{E}$ iff there exists one arc $(i, j) \in E$ for some $i \in C, j \in C'$ and we say that component C *has access to* C'. Clearly, "has access to" is a reflexive,

antisymmetric relation, so $\overline{G}$ is a *directed acyclic graph (dag)*. We call *accessibility* the transitive closure of this relation, which is clearly a partial order[4]. Accessibility is the reachability relation on nodes transferred to classes and completed to an order. For historical reasons to be made evident in Sections 3.1 and 3.2, we use the *downstream order* $\langle \overline{V}, \preccurlyeq \rangle$, where $C \preccurlyeq C'$ if some vertex of C has access to some vertex of C'. This the dual of the accessibility order.

Given a matrix over a semiring $A \in \mathcal{M}_n(\mathcal{K})$, its *associated digraph* $G_A = (V_A, E_A)$ can be retrieved from its weighted digraph $N_A = (V_A, E_A, w_A)$ by retaining just the set of nodes and arcs. Given a matrix A and its associated digraph $G_A = (V_A, E_A)$ the *condensation digraph* of A is the partial order of strong connectivity classes $\overline{G}_A = (\overline{V}_A, \overline{E}_A)$ as above. We will rather use $\langle \overline{V}_A, \preccurlyeq \rangle$ also in this case: Figures 1.(a), 2.(b) and 3.(b), are examples of such (duals of) condensation digraphs.

2.3. *An inductive structure for reducible matrices*

The condensation digraph of A of Section 2.2 has proven crucial to understand the structure of the spectrum and eigenspaces of A, so we next develop a representation for it in terms of an Upper Frobenius Normal Form (UFNF) (Brualdi and Ryser 1991), a block structure for matrices. Later, we will use it as a scheme for structural induction over *reducible* matrices.

In the following, for a set of indices $V_x \subseteq \overline{\mathbf{n}}$ we write $P(V_x)$ for a permutation of it, and $\mathcal{E}_{xy}$ is an empty matrix of conformal dimension most of the times represented on matrix patterns as elliptical dots.

Lemma 2.7 (Recursive Upper Frobenius Normal Form, UFNF)**.** *Let $A \in \mathcal{M}_n(S)$ be a matrix over a semiring and $\overline{G}_A$ its condensation digraph. Then,*

(1) (UFNF$_3$) If A has zero lines it can be transformed by a simultaneous row and column permutation of V_A into the following form:

$$P_3^{\mathrm{T}} \otimes A \otimes P_3 = \begin{bmatrix} \mathcal{E}_{\iota\iota} & \cdot & \cdot & \cdot \\ \cdot & \mathcal{E}_{\alpha\alpha} & A_{\alpha\beta} & A_{\alpha\omega} \\ \cdot & \cdot & A_{\beta\beta} & A_{\beta\omega} \\ \cdot & \cdot & \cdot & \mathcal{E}_{\omega\omega} \end{bmatrix} \tag{5}$$

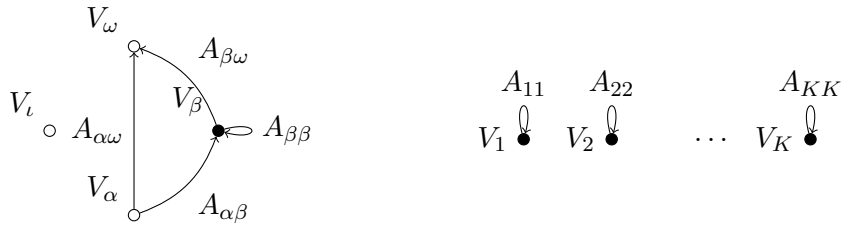
where either $A_{\alpha\beta}$ or $A_{\alpha\omega}$ or both are non-zero, and either $A_{\alpha\omega}$ or $A_{\beta\omega}$ or both are non-zero. Furthermore, P_3 is obtained concatenating permutations for the indices of simultaneously zero columns and rows V_ι, the indices of zero columns but non-zero rows V_α, the indices of zero rows but non-zero columns V_ω and the rest V_β as $P_3 = P(V_\iota)P(V_\alpha)P(V_\beta)P(V_\omega)$.

(2) (UFNF$_2$) If A has no zero lines it can be transformed by a simultaneous row and column permutation $P_2 = P(A_1) \dots P(A_k)$ into block diagonal UFNF:

$$P_2^{\mathrm{T}} \otimes A \otimes P_2 = \begin{bmatrix} A_1 & \cdot & \dots & \cdot \\ \cdot & A_2 & \dots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \dots & A_K \end{bmatrix} \tag{6}$$

where $\{A_k\}_{k=1}^K, K \geq 1$ are the matrices of connected components of $\overline{G}_A$.

[4]Recall that it can be represented as a Hasse diagram by means of its transitive-reflexive reduction.



(a) G_A for A in UFNF$_3$ (b) G_A for A in UFNF$_2$

Figure 1.: Digraphs associated to some of the specialized UFNF of Lemma 2.7

(3) (UFNF$_1$) If A is reducible with no zero lines and a single connected component it can be simultaneously row- and column-permuted by P_1 to

$$P_1^{\mathrm{T}} \otimes A \otimes P_1 = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1R} \\ \cdot & A_{22} & \cdots & A_{2R} \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & A_{RR} \end{bmatrix} \tag{7}$$

where A_{rr} are the matrices associated to each of its R strongly connected components (sorted in a topological ordering), and $P_1 = P(A_{11}) \dots P(A_{RR})$.

Proof. To prove claim 1, let $\overline{\mathrm{zc}}(A)$ and $\overline{\mathrm{zr}}(A)$ be the (possibly empty) sets of zero columns and rows, respectively, and partition $V_A = V_\iota \cup V_\alpha \cup V_\beta \cup V_\omega$, where

(1) $V_i = \overline{\mathrm{zc}}(A) \cap \overline{\mathrm{zr}}(A)$ is the set of indices of zero rows and columns,
(2) $V_\alpha = \overline{\mathrm{zc}}(A) \cap \overline{\mathrm{zr}}(A)^{\mathbf{c}}$ the set of indices of zero columns but non-zero rows,
(3) $V_\omega = \overline{\mathrm{zc}}(A)^{\mathbf{c}} \cap \overline{\mathrm{zr}}(A)$ the set of indices of zero rows but non-zero columns,
(4) $V_\beta = \overline{\mathrm{zc}}(A)^{\mathbf{c}} \cap \overline{\mathrm{zr}}(A)^{\mathbf{c}}$ the set of indices on non-zero columns and rows.

Since $V_\iota \subseteq \overline{\mathrm{zc}}(A)$ and $V_\alpha \subseteq \overline{\mathrm{zc}}(A)$, then $A_{\iota\iota} = \mathcal{E}_\iota$, and $A_{\alpha\alpha} = \mathcal{E}_\alpha$; since $V_\omega \subseteq \overline{\mathrm{zr}}(A)$, then $A_{\omega\omega} = \mathcal{E}_\omega$. If both $A_{\alpha\beta}$ and $A_{\alpha\omega}$ are null, then $V_\alpha \subseteq \overline{\mathrm{zr}}(A)$, a contradiction. Similarly, if both $A_{\alpha\omega}$ and $A_{\beta\omega}$ are null, then $V_\omega \subseteq \overline{\mathrm{zc}}(A)$, another contradiction. Hence the permutation to write A in UFNF$_3$ is $P_3 = P(V_\iota)P(V_\alpha)P(V_\beta)P(V_\omega)$.

An investigation of the associated characteristic digraphs of A—depicted in Figure 1.(a)—indicates that V_ι is the set of completely disconnected, *isolated nodes* in G_A, V_α is the set of *initial nodes* of G_A, V_ω is the set of *terminal nodes* of G_A, wherefore the only cycles in A might be those in $A_{\beta\beta}$. Note that this digraph is *not* a partial order, since if fails to be reflexive.

To prove claims 2 and 3, use Tarjan's algorithm (Tarjan 1972; Mehlhorn and Sanders 2008) to find the set of disconnected components $\{A_k\}_{k=1}^K$ on G_A. Furthermore, for component A_k the algorithm also sorts topologically its R_k strongly connected components. Let $A_{r_k r_k}$ be a block of A_k: with null non-diagonal block row this corresponds to a terminal class of $\overline{G}_A$, and with null non-diagonal block column to an initial class of $\overline{G}_A$. If both conditions apply, $A_{r_k r_k}$ is isolated and the single block in connected component k, $A_k = A_{r_k r_k}$.

Clearly, permutation $P_1(A_k) = \bigotimes_{r_k=1}^{R_k} P(A_{r_k r_k})$ renders A_k in UFNF$_1$ (Brualdi and Ryser 1991). If we gather the permutations of the disconnected blocks in whatever order then the permutation that renders A in $UFNF_2$ is $P_2 = \bigotimes_{k=1}^K P_1(A_k)$. The structure of the associated digraph, as shown in Figure 1.(b) is very simple. □

Notice that,

(1) Upper Frobenius Normal Forms (UFNF) are not unique since they rely on arbitrary and/or topological sortings of the classes in $\overline{G}_A$, which might be non-unique (Brualdi and Ryser 1991).
(2) In UFNF_3, $A_{\beta\beta}$ may still have zero columns if $A_{\alpha\beta}$ is non-zero, and/or zero rows when $A_{\beta\omega}$ is non-zero, hence it also admits a UFNF_3 . Therefore we may iterate this normal form until the innermost embedded $A_{\beta\beta}$ has no zero lines.
(3) In the UFNF_1 of a single connected block, initial classes tend to congregate in the upper left-hand corner of the submatrix while final classes tend to congregate in the lower right-hand corner.
(4) Irreducible components are the basic recursive blocks and we do not require a special form for them in this application. Sometimes we refer to them as in UFNF_0 .

Example 5 (UFNF forms of special matrices). *(1) $A = \mathcal{E}$ is in $UFNF_3$ with $V_\iota = \bar{\mathbf{n}}, V_\alpha = V_\beta = V_\omega = \varnothing$. It does not admit an $UFNF_2$ since $V_\beta = \varnothing$. In general, acyclic matrices admit a $UFNF_3$ with $V_\beta = \varnothing$. They do not admit an $UFNF_2$.*
(2) Block diagonal matrices with no zero lines are in $UFNF_3$ with $V_\beta = \bar{\mathbf{n}}, V_\iota = V_\alpha = V_\omega = \varnothing$, in $UFNF_2$ with whatever K but not in $UFNF_1$ unless $n = 1$. Diagonal matrices are a special case of this with $K = n$.
(3) Irreducible matrices are in $UFNF_3$ with $V_\beta = \bar{\mathbf{n}}, V_\alpha = V_\iota = V_\omega = \varnothing$, in $UFNF_2$ with $K = 1$, in $UFNF_1$ with $R = 1$ and in $UFNF_0$. □

Given the importance of the transitive closure of a matrix in the calculations of eigenvalues and eigenvectors highlighted by Theorem 1.1, we use the inductive structure of reducible matrices over dioids to calculate them. First we prove a simple lemma.

Lemma 2.8. *Let $A, B \in \mathcal{M}_n(\mathcal{S})$ and let P be a permutation such that $B = P^{\mathrm{T}}AP$. Then $B^+ = P^{\mathrm{T}}A^+P$ and $B^* = P^{\mathrm{T}}A^*P$.*

Proof. For the first claim $B^2 = P^{\mathrm{T}}APP^{\mathrm{T}}AP = P^{\mathrm{T}}A^2P$, since permutations cancel out by pairs. This is the basic case to induce $B^k = P^{\mathrm{T}}A^kP$. Hence

$$\begin{aligned} B^+ &= B \oplus B^2 \oplus \ldots \oplus B^k \oplus \ldots \\ &= P^{\mathrm{T}}AP \oplus P^{\mathrm{T}}A^2P \oplus \ldots \oplus P^{\mathrm{T}}A^kP \oplus \ldots \\ &= P^{\mathrm{T}}(A \oplus A^2 \oplus \ldots \oplus A^k \oplus \ldots)P \\ &= P^{\mathrm{T}}A^+P. \end{aligned}$$

As $I = P^{\mathrm{T}}IP$, and $A^* = I \oplus A^+$, the second claim follows. □

Lemma 2.9. *Let $A \in \mathcal{M}_n(A)$ be a square matrix over an idempotent semiring $\mathcal{S}$. For partition $\bar{n} = \alpha \cup \beta$ call $\textsc{Per}\,(A) = A_{\beta\alpha}A^*_{\alpha\alpha}A_{\alpha\beta} \oplus A_{\beta\beta}$. Then*

$$\begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{pmatrix}^+ = \begin{pmatrix} A^+_{\alpha\alpha} \oplus A^*_{\alpha\alpha}A_{\alpha\beta}\textsc{Per}\,(A)^*A_{\beta\alpha}A^*_{\alpha\alpha} & A^*_{\alpha\alpha}A_{\alpha\beta}\textsc{Per}\,(A)^* \\ \textsc{Per}\,(A)^*A_{\beta\alpha}A^*_{\alpha\alpha} & \textsc{Per}\,(A)^+ \end{pmatrix} \tag{8}$$

Proof. Adapted from (Golan 1999, Ch.25, p 289) □

Lemma 2.10 (Inductive structure of transitive closures). *(1) If A admits a $UFNF_1$ and the transitive closures of its strongly connected components ex-*

ist then A^+ exists, admits an UFNF$_1$ and can be iterated from

$$P^{\mathrm{T}} A^+ P = \begin{bmatrix} {A_{aa}}^+ & {A_{aa}}^* A_{ab} {A_{bb}}^* \\ \cdot & {A_{bb}}^+ \end{bmatrix}. \tag{9}$$

(2) If A admits an UFNF$_2$ and the transitive closures of its connected components exist then A^+ exists and admits an UFNF$_2$,

$$P_2^{\mathrm{T}} A^+ P_2 = \begin{bmatrix} A_1^+ & \cdot & \dots & \cdot \\ \cdot & A_2^+ & \dots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \dots & A_K^+ \end{bmatrix}. \tag{10}$$

(3) If A admits an UFNF$_3$, $V_\beta \neq \varnothing$ and the transitive closure $A^+_{\beta\beta}$ exists, then A^+ exists and admits an UFNF$_3$,

$$P_3^{\mathrm{T}} A^+ P_3 = \begin{bmatrix} \cdot\cdot & \cdot & \cdot \\ \cdot\cdot & A_{\alpha\beta} A^*_{\beta\beta} & A_{\alpha\beta} A^*_{\beta\beta} A_{\beta\omega} \oplus A_{\alpha\omega} \\ \cdot\cdot & A^+_{\beta\beta} & A^*_{\beta\beta} A_{\beta\omega} \\ \cdot\cdot & \cdot & \cdot \end{bmatrix}. \tag{11}$$

Proof. Starting with claim 1, (9) stems directly from (8) when $A_{ba} = \mathcal{E}_{ba}$. So consider A with R_k irreducible blocks. If $R_k = 1$ then $A^+ = A^+_{11}$. If $R_k = 2$, apply (9) with $V_a = V_1$ and $V_b = V_2$. For R_k, let $V_a = \cup_{r_k=1}^{R_k-1} V_{r_k}$ and $V_b = V_{R_k}$ and the (greatly involved) transitive closure of (7) follows. With the same procedure as above when further $A_{ba} = \mathcal{E}_{ba}$ and $A_{ab} = \mathcal{E}_{ba}$ we prove (10) and claim 2.

Finally, claim 3 also follows from the procedure above considering that $\mathcal{E}^+_{\iota\iota} = \mathcal{E}_{\iota\iota}$ and $\mathcal{E}^*_{\iota\iota} = I_\iota$, and the same holds for $\mathcal{E}_{\alpha\alpha}$ and $\mathcal{E}_{\omega\omega}$.

Notice that, if $V_\beta = \varnothing$ then A has no cycles and $A^+ = \mathcal{E}_A$ and $A^* = I_A$. But if $V_\beta \neq \varnothing$, when $A_{\beta\beta}$ has zero rows or columns we iterate the UFNF$_3$ on it. When $A_{\beta\beta}$ has no zero rows or columns it admits an UFNF$_2$ and the existence of a non-zero $A^+_{\beta\beta}$ and $A^*_{\beta\beta}$ can be ascertained by (10). □

The lemma above clarifies our notation: the higher the index of the UFNF the more abundant in null elements is the transitive closure, from that of the irreducible matrices—in UFNF$_0$, transitive closures with no null elements—to matrices with zero lines—in UFNF$_3$, transitive closures with many zero elements.

The particular choice of UFNF is clarified by the following Lemma, since the condensation digraph will prove important later on:

Lemma 2.11 (Scheme for structural induction over reducible matrices)**.** *Let $A \in \mathcal{M}_n(S)$ be a matrix over an entire zerosumfree semiring and $\overline{G}_A$ its condensation digraph. Then:*

(1) If A is irreducible then $\overline{G}_A \cong \mathbb{1}$.
(2) If A is in UFNF$_2$ then $\overline{G}_A = \biguplus \overline{G}_{A_k}$.
(3) If A is in UFNF$_3$ then $\overline{G}_A = \overline{G}_{A_{\beta\beta}}$.
(4) $\overline{G}_{A^{\mathrm{T}}} = (\overline{G}_A)^{\mathrm{d}}$.

Proof. It is well-known that if A is irreducible, there is a cycle between every pair i, j of nodes in V_A, hence G_A is just one strongly connected component whence claim 1. Claim 2 follows from the description of the disconnected components of G_A above. Notice from the digraph of A in UFNF$_3$ shown in Figure 1.(a) that only the

$A_{\beta\beta}$ may have the cycles needed to define the classes in $\overline{G}_A$, whence claim 3. Finally, since $G_{A^{\mathrm{T}}}$ has all its edges inverted, $\overline{G}_{A^{\mathrm{T}}} = \langle \overline{V}_{A^{\mathrm{T}}}, \overline{E}_{A^{\mathrm{T}}} \rangle = \langle \overline{V}_A, \overline{E}^{\mathrm{d}}_A \rangle = (\overline{G}_A)^{\mathrm{d}}$. □

Note that for A in UFNF_1, $\overline{G}_A$ may adopt any form as a connected ordered set. Also, by the remarks after Lemma 2.7, even $A_{\beta\beta}$ may have nodes that do not participate in any cycle and case UFNF_3 should be recurred in this component to finally find the reflexive restriction of the reachability relation in A.

3. Results

3.1. *Generic results for reducible matrices*

First we recover some definitions from (Valverde-Albacete and Peláez-Moreno 2014): call the *support of a vector* the set of indices of v whose coordinates are non-null, $\mathrm{supp}(v) = \{k \in \bar{\mathbf{n}} \mid v_k \neq \epsilon\}$. We say that v has *full support* if all of its coordinates are non-null, otherwise we say that it has *partial support.* For the case of complete semirings, call the *saturated support of an eigenvector* the set of indices of v whose coordinates are the infinite, $\text{sat-supp}\,(v) = \{k \in \bar{\mathbf{n}} \mid v_k = \top\}$. The rest of the support is the *finite support*, $\text{fin-supp}\,(v) = \{k \in \bar{\mathbf{n}} \mid \epsilon \neq v_k \neq \top\}$.

This distinction of supports is crucial since we call an eigenvalue *proper* when it has at least one eigenvector with finite coordinates, otherwise it is *improper.* The set of proper (left) eigenvalues is the *proper (left) spectrum*, $\mathrm{P}^{\mathrm{p}}(A) = \{\rho \in \mathrm{P}(A) \mid \exists v \in \mathcal{V}_\rho(A)\ \text{fin-supp}\,(v) \neq \varnothing\}$, so the *improper spectrum* is $\mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{p}}(A)$.

The following lemma clarifies the order relation between eigenvectors.

Lemma 3.1. *Let $\mathcal{X}$ be a naturally-ordered semimodule.*

(1) Vectors with incomparable supports are incomparable.
(2) If $\mathcal{X}$ is further complete, vectors with incomparable saturated supports are incomparable.

Proof. Let v and w be two vectors in $\mathcal{X}$. Comparability of supports lies in the $\subseteq$ relation: if $\mathrm{supp}(v) \parallel \mathrm{supp}(w)$ then $\mathrm{supp}(v) \not\subseteq \mathrm{supp}(w)$ and $\mathrm{supp}(w) \not\subseteq \mathrm{supp}(v)$. Therefore from $\mathrm{supp}(v) \cap \mathrm{supp}(w)^{\mathbf{c}} \neq \varnothing$ we have $v(\mathrm{supp}(v) \cap \mathrm{supp}(w)^{\mathbf{c}}) \neq \bot$ and $w(\mathrm{supp}(v) \cap \mathrm{supp}(w)^{\mathbf{c}}) = \bot$, hence $v \not\leq w$. Similarly, from $\mathrm{supp}(w) \cap \mathrm{supp}(v)^C \neq \varnothing$ we have $w \not\leq v$, therefore $v \parallel w$. Claim 2 is likewise argued on the support taking the role of $\overline{\mathbf{n}}$, and the saturated support taking the role of the original support. □

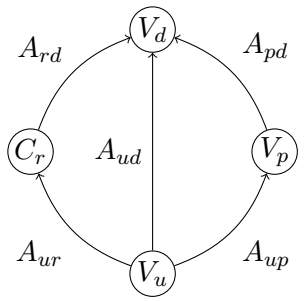
Let $A \in \mathcal{M}_n(\mathcal{S})$ be a matrix and $\langle \overline{V}_A, \preccurlyeq \rangle$ its downstream order. Consider a class $C_r \in \overline{V}_A$ and call $V_u = (\bigcup_{C' \in \downarrow C_r} C') \setminus C_r$, $V_d = (\bigcup_{C' \in \uparrow C_r} C') \setminus C_r$ and $V_p = V_A \setminus (V_u \cup C_r \cup V_d)$ the *sets of upstream, downstream and parallel vertices for C_r*, respectively. Due to permutation $P_r = P(V_u)P(C_r)P(V_p)P(V_d)$ we may suppose a blocked form of $A(C_r)$ like the one in Figure 2 without loss of generality. Notice that any of V_u, V_d or V_p may be empty. However, if V_u (resp. V_d) is not of null dimension, then A_{ur} (resp. A_{rd}) cannot be empty.

Proposition 3.2. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be a matrix over a complete commutative selective radicable semifield with $C_A^+ \neq \varnothing$. Then a scalar $\rho > \bot$ is a proper eigenvalue of A if and only if there is at least one class in its condensation digraph $C_r \in \overline{G}_A$ such that $\rho = \mu_\oplus(A_{rr})$.*

Proof. For $\rho \neq \top$, call $B = \tilde{A}^\rho = A \,/\!\!\!\!\!\cdot\, \rho = A \dot{\otimes} \rho^{-1}$ and $B_{xy} = \tilde{A}^\rho_{xy} =$

$$A(C_r) = \begin{bmatrix} A_{uu} & A_{ur} & A_{up} & A_{ud} \\ \cdot & A_{rr} & \cdot & A_{rd} \\ \cdot & \cdot & A_{pp} & A_{pd} \\ \cdot & \cdot & \cdot & A_{dd} \end{bmatrix}$$

(a) Blocked form of $A(C_r)$



(b) Dual condensation digraph of $A(C_r)$

Figure 2.: Matrix A focused on C_r, $A(C_r) = {P_r}^{\mathrm{T}} \otimes A \otimes P_r$ and associated digraph. The loops at each node, weighted by (possibly empty) A_{uu}, A_{rr}, A_{pp}, A_{dd} are not shown.

$A_{xy} \dot{\otimes} \rho^{-1}, \forall x, y \in \{u, r\}$. Use (9) in Lemma 2.10 to find the transitive reflexive closure B^* whose columns indexed by V_r are $B^*_{\cdot r} = [(B^*_{uu} B_{ur} B^*_{rr})^{\mathrm{T}} (B^*_{rr})^{\mathrm{T}} \bot_p \bot_d]^{\mathrm{T}}$, therefore

$$A \dot{\otimes} \begin{bmatrix} B^*_{uu} B_{ur} B^*_{rr} \\ B^*_{rr} \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} A_{uu} B^*_{uu} B_{ur} B^*_{rr} \dot{\oplus} A_{ur} B^*_{rr} \\ A_{rr} B^*_{rr} \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} B^*_{uu} B_{ur} B^*_{rr} \\ B^+_{rr} \\ \cdot \\ \cdot \end{bmatrix} \dot{\otimes} \rho \quad (12)$$

where the last term is obtained by factorizing $\tilde{A}^{\rho}_{rr} B^*_{rr} = B^+_{rr}$ and $\tilde{A}^{\rho}_{uu} B^*_{uu} B_{ur} B^*_{rr} \dot{\oplus} B_{ur} B^*_{rr} = B^+_{uu} B_{ur} B^*_{rr} \dot{\oplus} I_u B_{ur} B^*_{rr} = B^*_{uu} B_{ur} B^*_{rr}$. This declares as eigenvectors of A for ρ those columns of B^+ correlated to those of B^+_{rr} where $B^*_{rr} = B^+_{rr}$. Since we are looking for partially finitely supported eigenvectors and A_{rr} is irreducible, from Theorem 1.3 we know this to be the case for the critical nodes of $G_{A_{rr}}$, $V^c_r = \{i \in C_r \mid (\tilde{A}^{\rho}_{rr})^+_{ii} = e\}$, which select fully finite fundamental eigenvectors of A_{rr} for ρ, $\mathrm{FEV}_{\rho}(A_{rr}) = \{(\tilde{A}^{\rho}_{rr})^+_{\cdot i} \mid i \in V^c_r\}$. Therefore the columns of $(\tilde{A}^{\rho})^+$ selected by V^c_r are (partially) finitely supported eigenvectors of A for ρ and $\rho \in \mathrm{P}^{\mathrm{p}}(A)$.

If $\rho = \top$ and we assemble $v^{\mathrm{T}} = [{v_u}^{\mathrm{T}}\ {v_r}^{\mathrm{T}}\ \bot_d\ \bot_p]^{\mathrm{T}}$ as candidate eigenvector, we have

$$A \dot{\otimes} \begin{bmatrix} v_u \\ v_r \\ \bot_d \\ \bot_p \end{bmatrix} = \begin{bmatrix} A_{uu} \dot{\otimes} v_u \dot{\oplus} A_{ur} \dot{\otimes} v_r \\ A_{rr} \dot{\otimes} v_r \\ \bot_d \\ \bot_p \end{bmatrix} = \begin{bmatrix} v_u \\ v_r \\ \bot_d \\ \bot_p \end{bmatrix} \dot{\otimes} \top. \quad (13)$$

For $v \in \mathcal{V}_{\rho}(A)$ surely $v_r \in \mathcal{V}_{\rho}(A_{rr})$. If $v_u \in \mathcal{V}_{\top}(A_{uu})$ we must have $A_{uu} \dot{\otimes} v_u \dot{\oplus} A_{ur} \dot{\otimes} v_r = v_u \dot{\otimes} \top \dot{\oplus} A_{ur} \dot{\otimes} v_r = v_u \dot{\otimes} \top$, what entails $v_u \dot{\otimes} \top \geq A_{ur} \dot{\otimes} v_r$, whence $v_u \geq (A_{ur} \dot{\otimes} v_r) \dot{/} \top$ or $v_u \in \mathcal{V}_{\top}(A_u) \bigcap \uparrow \left[(A_{ur} \dot{\otimes} v_r) \dot{/} \top\right]$. The assembled vector is partially finitely supported if $\top \in \mathrm{P}^{\mathrm{p}}(A_{rr})$, whereas this is not warranted if only $\top \in \mathrm{P}^{\mathrm{p}}(A_{uu})$ since $(A_{ur} \dot{\otimes} v_r) \dot{/} \top$ is not (even partially) finitely supported. $\square$

A warning about notation seems necessary now. fundamental eigenvectors come in many flavors:

(1) To emphasize that they have finite components (respectively, only saturated components) we use $\mathrm{FEV}^{\mathrm{F}}(A)$ (respectively, $\mathrm{FEV}^{\top}(A)$).
(2) For a particular ρ, to emphasize that they issue from the whole matrix we use $\mathrm{FEV}_{\rho}(A)$, and if they issue from a particular UFNF form we use $\mathrm{FEV}_{\rho}^{x}(A)$ where $x \in \{0, 1, 2, 3\}$.

Lemma 3.3. *Let $A \in \mathcal{M}_n(\mathcal{S})$ be a reducible matrix over a complete radicable selective semifield. Then, there are no other finite eigenvectors in $\mathcal{V}_\rho(A)$ contributed by $\tilde{A}^\rho$ than those selected by the critical circuits in $C_r \in \overline{V}_A$ such that $\mu_\oplus(A_{rr}) = \rho$,*

$$\mathrm{FEV}^{\mathrm{F}}(A) = \cup_{C_r \in \overline{V}_A}^{\mu_\oplus(A_{rr})=\rho} \{(\tilde{A}^\rho)^+_{\cdot i} \mid i \in V_r^c\}.$$

Proof. If $\rho = \mu_\oplus(A_{rr})$, from Proposition 3.2 we see that the finite eigenvectors mentioned really belong in $\mathcal{V}_\rho(A)$. If $\rho > \mu_\oplus(A_{rr})$ then $(\tilde{A}^\rho_{rr})^+_{ii} < e = (\tilde{A}^\rho_{rr})^*_{ii}$ hence the columns selected by C_r do not generate eigenvectors. If $\rho < \mu_\oplus(A_{rr})$ then $(\tilde{A}^\rho_{rr})^+_{ij} = \top$ and whether those classes with cycle mean ρ are upstream or downstream of C_r the only value that is propagated is $\top$, hence the eigenvectors are all saturated. □

Recall from Section 2.3 that $\overline{\mathrm{zc}}(A)$ is the set of empty columns of A.

Theorem 3.4 (Spectra of generic matrices)**.** *Let $A \in \mathcal{M}_n(\overline{\mathcal{D}})$ be a reducible matrix over an entire zerosumfree semiring. Then,*
(1) If $C_A^+ = \varnothing$ then $\mathrm{P}(A) = \mathrm{P}^{\mathrm{P}}(A) = \{\epsilon\}$.
(2) If $C_A^+ \neq \varnothing$ and further $\overline{\mathcal{D}}$ is a complete selective radicable semifield,
(a) If $\overline{zc}(A) \neq \varnothing$ then $\mathrm{P}(A) = \overline{\mathcal{K}}$ and $\mathrm{P}^{\mathrm{P}}(A) = \{\perp\} \cup \{\mu_\oplus(A_{rr}) \mid C_r \in \overline{V}_A\}$.
(b) If $\overline{zc}(A) = \varnothing$ then $\mathrm{P}(A) = \overline{\mathcal{K}} \setminus \{\perp\}$ and $\mathrm{P}^{\mathrm{P}}(A) = \{\mu_\oplus(A_{rr}) \mid C_r \in \overline{V}_A\}$.

Proof. If G_A has no cycles $C_A^+ = \varnothing$, claim 1 follows from (Valverde-Albacete and Peláez-Moreno 2014, Lemma 3.6,claim 2) . But if $C_A^+ \neq \varnothing$ then by Proposition 3.2, $\mathrm{P}^{\mathrm{P}}(A) \supseteq \{\mu_\oplus(A_{rr}) \mid C_r \in \overline{V}_A\}$ and no other non-null proper eigenvalues may exist by Lemma 3.3. By (Valverde-Albacete and Peláez-Moreno 2014, Lemma 3.6) $\perp$ is only proper when $\overline{\mathrm{zc}}(A) \neq \varnothing$ hence claims 2a and 2b follow. □

Note that since $\Lambda(A) = \mathrm{P}(A^{\mathrm{T}})$ this also addresses the question of left spectra when we substitute $\overline{\mathrm{zc}}(A)$ for $\overline{\mathrm{zr}}(A)$, the set of empty rows.

Since only UFNF_3 can have empty colums, we have the following corollary.

Corollary 3.5. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be a matrix over a complete selective radicable semifield with $C_A^+ \neq \varnothing$. Then $\mathrm{P}(A) = \overline{\mathcal{K}} \setminus \{\perp\}$ and $\mathrm{P}^{\mathrm{P}}(A) = \{\mu_\oplus(A_{rr}) \mid C_r \in \overline{V}_A\}$, unless A is in UFNF$_3$ and $\overline{zc}(A) \neq \varnothing$ whence $\perp \in \mathrm{P}^{\mathrm{P}}(A) \subseteq \mathrm{P}(A)$ too.*

This solves entirely the description of the spectrum: only the description of the eigenspaces is left pending. Our aim in this respect will be to find results for other UFNFs similar to the following corollary of Theorem 1.3 for UFNF_0:

Corollary 3.6. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be an irreducible matrix over a complete commutative selective radicable semifield. For $\perp < \rho < \top$, $\mathrm{FEV}_{\rho}^{0}(A)$ is join-dense in $\mathcal{V}_\rho(A)$.*

3.2. *Eigenspaces of matrices in UFNF$_1$*

If for every parallel condensation class $V_p \subseteq V_A$ in $A(C_r)$ illustrated in Figure 2 $A_{up} \neq \mathcal{E}_{up}$ or $A_{pd} \neq \mathcal{E}_{pd}$ or both, then A is in UFNF_1 with a single connected

block. In this case, we can relate the order of the eigenvectors to the downstream order. Define the *support of a class* $\mathrm{supp}(C)$ as the support of any of the non-null eigenvectors it induces in A.

Lemma 3.7. *Let $A \in \mathcal{M}_n(\mathcal{S})$ be a matrix in UFNF$_1$ over a complete zerosumfree semiring. Then, for any class $C_r \in \overline{V}_A$, $\mathrm{supp}(C_r) = \bigcup\{C_{l_r} \mid C_{l_r} \in \downarrow C_r\}$.*

Proof. From (12) and (13), since A_{rr} is irreducible, if $\rho = \mu_\oplus(A_{rr})$ then for any $v_r \in \mathcal{V}_\rho(A_{rr})$ we have that $\mathrm{supp}(v_r) = V_r$, hence $V_r \subseteq \mathrm{supp}(C_r)$. Also, since $\mathcal{S}$ is complete and zerosumfree $(\tilde{A}^\rho)^+_{rr}$ exists and is full (Valverde-Albacete and Peláez-Moreno 2014, Proposition 2.7). Since $(\tilde{A}^\rho)^+_{uu}\tilde{A}^\rho_{ur}$ must have a full column for any $C_{l_r} \in \downarrow C_r$ signifying precisely that C_r is downstream from C_{l_r}, the product $(\tilde{A}^\rho)^+_{uu}\tilde{A}^\rho_{ur}(\tilde{A}^\rho)^+_{rr}$ for the nodes in C_{l_r} must be non-null and $V_{l_r} \subseteq \mathrm{supp}(C_r)$. □

The reason to use use the downstream order is that Lemma 3.7 establishes a bijection between downsets in $\langle \overline{V}_A, \preccurlyeq\rangle$ and supports of condensation classes which is actually an isomorphism of orders $C \preccurlyeq C' \Leftrightarrow \mathrm{supp}(C) \subseteq \mathrm{supp}(C')$. Now call $\mathrm{FEV}^{1,\top}(A) = \{v_r^\top \mid C_r \in \overline{V}_A\}$ the *set of saturated fundamental eigenvectors of A*.

Proposition 3.8. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be a matrix over a commutative complete selective radicable semifield admitting an UFNF$_1$. Then*

(1) Each class $C_r \in \overline{V}_A$ generates a distinct saturated eigenvector, $v_r^\top$.

(2) $\mathrm{FEV}^{1,\top}(A) = \{v_r^\top \mid C_r \in \overline{V}_A\} \cong \langle \overline{V}_A, \preccurlyeq\rangle \cong \overline{G}^{\mathrm{d}}_A$.

Proof. Let $v \in \mathcal{V}_\rho(A)$ where $\rho = \mu_\oplus(A_{rr})$ then by Lemma 3.7 $\mathrm{supp}(v) = \downarrow C_r$, hence $v_r^\top = \top v \in \mathcal{V}_\rho(A)$ is the unique saturated eigenvector, since $\text{sat-supp}\,(\top v) = \mathrm{supp}(\top v) = \mathrm{supp}(C)$, and the bijection follows. This is actually an order isomorphism between saturated eigenvectors and the (saturated) supports of the classes they emerge from, whence the order isomorphism in claim 2. □

Notice that $\overline{V}_{A^{\mathrm{T}}} = \overline{V}_A$ but $\overline{E}_{A^{\mathrm{T}}} = \overline{E}^{\mathrm{d}}_A$, so $\mathrm{FEV}^{1,\top}(A^{\mathrm{T}}) \cong \langle \overline{V}_A, \preccurlyeq^{\mathrm{d}}\rangle \cong \overline{G}_A$.

For every *finite* $\rho \in \mathrm{P}^{\mathrm{P}}(A)$ we define the *witness nodes* $V^c_\rho = \{i \in \overline{\mathbf{n}} \mid (\tilde{A}^\rho)^+_{ii} = e\}$ by analogy with the critical nodes of the irreducible case, and $\mathrm{FEV}^{1,\mathrm{F}}_\rho(A) = \{(\tilde{A}^\rho)^+_{\cdot i} \mid i \in V^c_\rho\}$ the *(maybe partially) finite fundamental eigenvectors of ρ*. Next, let $\delta^{-1}_\rho(\rho') = e$ if $\rho' = \rho$ and $\delta^{-1}_\rho(\rho') = \top$ otherwise. for $\rho \in \mathrm{P}(A)$ the set of *(right) fundamental eigenvectors of A in UFNF$_1$ for ρ* as

$$\mathrm{FEV}^1_\rho(A) = \cup_{\rho' \in \mathrm{P}(A)}\{\delta^{-1}_\rho(\rho') \underset{\cdot}{\otimes} \mathrm{FEV}^{1,\mathrm{F}}_{\rho'}(A)\}. \tag{14}$$

Actually, this definition absorbs two cases, explained in the lemma below.

Lemma 3.9. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be a matrix over a commutative complete selective radicable semifield admitting an UFNF$_1$. Then,*

(1) for $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$,

$$\mathrm{FEV}^1_\rho(A) = \mathrm{FEV}^{1,\top}(A)\,.$$

(2) for $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, $\rho \neq \top$,

$$\mathrm{FEV}^1_\rho(A) = \mathrm{FEV}^{1,\mathrm{F}}_\rho(A) \cup \mathrm{FEV}^{1,\top}(A) \setminus (\top \underset{\cdot}{\otimes} \mathrm{FEV}^{1,\mathrm{F}}_\rho(A)).$$

(3) for $\rho \in \mathrm{P}(A)$, $\rho \neq \top$,

$$\mathrm{FEV}^{1,\top}(A) = \top \underset{\cdot}{\otimes} \mathrm{FEV}^1_\rho(A)\,.$$

Proof. If $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$, then for all $\rho' \in \overline{\mathcal{K}}$, $\delta_\rho^{-1}(\rho') = \top$. By Proposition 3.8 claim 1 follows as we range $\rho' \in \mathrm{P}^{\mathrm{P}}(A)$. Similarly, when $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, those classes whose $\rho' \neq \rho$ supply a single saturated eigenvector. However, if $\rho' = \rho$, then $\delta_\rho^{-1}(\rho') = e$ obtains the (partially) finite fundamental eigenvectors $\mathrm{FEV}^{1,\mathrm{F}}_\rho(A)$, the saturated eigenvectors of which cannot be considered fundamental, since they can be obtained from $\mathrm{FEV}^{1,\mathrm{F}}_{\rho'}(A)$ linearly, and will not appear in $\mathrm{FEV}^1_\rho(A)$. Claim 3 is a corollary of the other two. □

Call $\mathcal{V}^\top(A) = \langle \mathrm{FEV}^{1,\top}(A)\rangle_{\overline{K}}$ the *saturated eigenspace of A* .

Corollary 3.10. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be a matrix over a commutative complete selective radicable semifield admitting an UFNF$_1$. Then,*
(1) For $\rho \in \mathrm{P}(A)$, $\mathcal{V}^\top(A) \subseteq \mathcal{V}_\rho(A)$.
(2) For $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$, furthermore, $\mathcal{V}^\top(A) = \mathcal{V}_\rho(A)$.

Proof. By (Valverde-Albacete and Peláez-Moreno 2014, Corollary 3.2), we have $\mathrm{FEV}^{1,\top}(A) \subseteq \mathcal{V}_\rho(A)$, hence $\mathcal{V}^\top(A) \subseteq \mathcal{V}_\rho(A)$. For $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$, $\mathrm{FEV}^1_\rho(A) = \mathrm{FEV}^{1,\top}(A)$ by Lemma 3.9 so $\mathcal{V}^\top(A) = \langle \mathrm{FEV}^1_\rho(A)\rangle_{\overline{\mathcal{K}}} = \langle \mathrm{FEV}^{1,\top}(A)\rangle_{\overline{\mathcal{K}}} = \mathcal{V}_\rho(A)$. □

Hence, $\mathcal{V}^\top(A)$ provides a common "scaffolding" for every eigenspace, while the peculiarities for proper eigenvalues are due to the finite eigenvectors. Also, since $\mathcal{V}^\top(A)$ is a complete lattice, $\mathrm{FEV}^{1,\top}(A) \subseteq \mathcal{V}^\top(A)$ is actually an order embedding.

Proposition 3.11. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be a matrix over a commutative complete selective radicable semifield admitting an UFNF$_1$. Then*
(1) For $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$,

$$\mathcal{U}^\top(A) = \langle \mathrm{FEV}^{1,\top}(A^{\mathrm{T}})^{\mathrm{T}}\rangle_{\mathbb{3}} \cong \mathcal{O}(\overline{G}_A) \quad \mathcal{V}^\top(A) = \langle \mathrm{FEV}^{1,\top}(A)\rangle_{\mathbb{3}} \cong \mathcal{F}(\overline{G}_A). \tag{15}$$

(2) for all $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, $\rho < \top$

$$\mathcal{U}_\lambda(A) = \langle \mathrm{FEV}^1_\lambda(A^{\mathrm{T}})^{\mathrm{T}}\rangle_{\overline{K}} \qquad \mathcal{V}_\rho(A) = \langle \mathrm{FEV}^1_\rho(A)\rangle_{\overline{K}}.$$

Proof. If $v_r^\top \in \mathrm{FEV}^{1,\top}(A)$ then $\lambda v_r^\top = \lambda(\top v_r^\top) = v_r^\top$, whence $\mathcal{V}^\top(A) = \langle \mathrm{FEV}^{1,\top}(A)\rangle_{\mathbb{3}}$. In fact, the generation process may proceed on only a subsemiring of $\overline{\mathcal{K}}$ which need not even be complete. For instance, we may use any of the isomorphic copies of $\mathbb{2}$ embedded in $\overline{\mathcal{K}}$, for instance $\{\bot, k\} \cong \mathbb{2}$, with $k \neq \bot$.

Since the number of saturated eigenvectors is finite, being identical to the number of condensation classes, we only have to worry about binary joins and meets. Recall that $v_r^\top \vee v_k^\top = v_r^\top \underset{\cdot}{\oplus} v_k^\top$ and $v_r^\top \wedge v_k^\top = v_r^\top \dot{\oplus} v_k^\top = \left((v_r^\top)^{-1} \underset{\cdot}{\oplus} (v_k^\top)^{-1}\right)^{-1}$. Then $\mathrm{supp}(v_r^\top \underset{\cdot}{\oplus} v_k^\top) = \mathrm{supp}(v_r^\top) \cup \mathrm{supp}(v_k^\top)$ and $\mathrm{supp}(v_r^\top \dot{\oplus} v_k^\top) = \left(\mathrm{supp}^{\mathbf{c}}\left(v_r^\top\right) \cup \mathrm{supp}^{\mathbf{c}}\left(v_k^\top\right)\right)^{\mathbf{c}} = \mathrm{supp}(v_r^\top) \cap \mathrm{supp}(v_k^\top)$ for $C_r, C_k \in \overline{V}_A$ and Proposition 2.5 gives $\mathcal{V}^\top(A) \cong \mathcal{O}(\langle \overline{V}_A, \preccurlyeq\rangle) \cong \mathcal{F}(\langle \overline{V}_A, \preccurlyeq^{\mathrm{d}}\rangle) \cong \mathcal{F}(\overline{G}_A)$.

For $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, $\mathrm{FEV}^1_\rho(A) \subseteq \mathcal{V}_\rho(A)$ implies that $\langle \mathrm{FEV}^1_\rho(A)\rangle_{\overline{\mathcal{K}}} \subseteq \mathcal{V}_\rho(A)$, and Lemma 3.3 ensures that no finite vectors are missing. And dually for left eigenspaces. □

This actually proves the following corollary.

Corollary 3.12. $\mathrm{FEV}^1_\rho(A)$ *is join-dense in* $\mathcal{V}_\rho(A)$.

Now, $\mathcal{V}_\rho(A)$ is a hard-to-visualize semimodule. An *eigenspace schematics* is a modified order diagram where the saturated eigenspace is represented in full but the rays generated by finite eigenvalues $\{\kappa \otimes (\tilde{A}^\rho)^+_{\cdot i} \mid i \in V^c_r, \rho = \mu_\oplus(A_{rr})\}$ are drawn with *discontinuous lines*, as in the examples below.

Apart from the eigenspace schematics, we are introducing in these examples yet another representation inspired by (15). The *(left) right eigenlattices of A for ($\lambda \in \Lambda(A)$) $\rho \in \mathrm{P}(A)$*,

$$\mathcal{L}_\lambda(A) = \langle \mathrm{FEV}^1_\rho(A^{\mathrm{T}})^{\mathrm{T}} \rangle_{\mathbb{3}} \qquad\qquad \mathcal{L}_\rho(A) = \langle \mathrm{FEV}^1_\rho(A) \rangle_{\mathbb{3}}.$$

Example 6 (Spectral lattices of irreducible matrices)**.** *Since irreducible matrices are in UFNF$_1$ with a single class,* $\mathrm{FEV}^0_{\mu_\oplus(A)}(A) = \mathrm{FEV}^1_{\mu_\oplus(A)}(A)$. *For* $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$ *we have* $\mathrm{FEV}^{0,\top}(A) = \{\top^n\}$, *whence* $\langle \overline{V}_A, \preccurlyeq \rangle \cong \mathbb{1}$ *and* $\mathcal{V}^\top(A) = \{\bot^n, \top^n\} \cong \mathbb{2} \cong \mathcal{F}(\mathbb{1})$. *For* $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, $\rho < \top$, *as proven in (Valverde-Albacete and Peláez-Moreno 2014),* $\mathcal{V}_\rho(A)$ *is finitely generable from* $\mathrm{FEV}^0_\rho(A)$, *but the form of the eigenspace and eigenlattice for* $\Lambda^{\mathrm{P}}(A) = \{\mu_\oplus(A)\} = \mathrm{P}^{\mathrm{P}}(A)$ *depends on the critical cycles and the eigenvectors they induce. However, from (Valverde-Albacete and Peláez-Moreno 2014, Example 7), if* $\mu_\oplus(A) = \top$, *then* $\mathcal{V}_\top(A)$ *may be non-finitely (join-) generable from* $\mathrm{FEV}^0_\top(A)$. □

Example 7. *Consider the matrix* $A \in \mathcal{M}_n(\overline{\mathbb{R}}_{\max,+})$ *from (Akian, Bapat, and Gaubert 2007, p. 25.7, example 2) in UFNF$_1$ depicted in Figure 3.(a). The dual condensed graph* $\overline{G}_A$ *in Figure 3.(b) has for vertex set* $\overline{V}_A = \{C_1 = \{1\}, C_2 = \{2,3,4\}, C_3 = \{5,6,7\}, C_4 = \{8\}\}$ *, so consider the strongly connected components* $G_{A_{kk}} = (C_k, E \cap C_k \times C_k), 1 \leq k \leq 4$. *Their maximal cycle means are* $\mu_k = \mu_\oplus(A_{kk})$ *:* $\mu_1 = 0$, $\mu_2 = 2$, $\mu_3 = 1$ *and* $\mu_4 = -3$ *, respectively, corresponding to critical circuits:* $C^c(G_{A_{11}}) = \{1 \circlearrowleft\}$ *,* $C^c(G_{A_{22}}) = \{2 \to 3 \to 2\}$ *,* $C^c(G_{A_{33}}) = \{5 \circlearrowleft, 6 \to 7 \to 6\}$ *,* $C^c(G_{A_{44}}) = \{8 \circlearrowleft\}$ *. Note that node 4 does not generate an eigenvector in either spectrum, since it does not belong to a critical cycle.*

Therefore $\Lambda^{\mathrm{P}}(A_3) = \mathrm{P}^{\mathrm{P}}(A_3) = \{2, 1, 0, -3\}$ *each left eigenspace is the span of the set of eigenvectors chosen from distinct critical cycles for each class of A:* $\mathcal{U}_{\mu_1}(A) = \langle (\tilde{A}_3{}^{\mu_1})^+_{1\cdot} \rangle$ *,* $\mathcal{U}_{\mu_2}(A) = \langle (\tilde{A}_3{}^{\mu_2})^+_{2\cdot} \rangle$ *,* $\mathcal{U}_{\mu_3}(A) = \langle (\tilde{A}_3{}^{\mu_3})^+_{\{5,6\}\cdot} \rangle$ *, and* $\mathcal{U}_{\mu_4}(A) = \langle (\tilde{A}_3{}^{\mu_4})^+_{8\cdot} \rangle$ *–as described by the row vectors of Figure 3.(c)–and the right eigenspaces are* $\mathcal{V}_{\mu_1}(A) = \langle (\tilde{A}_3{}^{\mu_1})^+_{\cdot 1} \rangle$ *,* $\mathcal{V}_{\mu_2}(A) = \langle (\tilde{A}_3{}^{\mu_2})^+_{\cdot 2} \rangle$ *,* $\mathcal{V}_{\mu_3}(A) = \langle (\tilde{A}_3{}^{\mu_3})^+_{\cdot\{5,6\}} \rangle$ *, and* $\mathcal{V}_{\mu_4}(A) = \langle (\tilde{A}_3{}^{\mu_4})^+_{\cdot 8} \rangle$ *–as described by the column vectors of Figure 3.(d).*

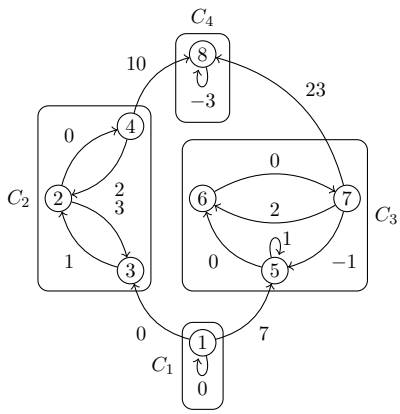
The saturated eigenspace is easily represented by means of an order diagram like that of Figure 3.(e). Note how it is embedded in that of any proper eigenvalue like $\rho = 2$ *in Figure 3.(f). Since the representation of continuous eigenspaces is problematic, we draw* schematics *of them, as in Figure 3.(f). Figure 3.(g) shows a schematic view of the union of the eigenspaces for proper eigenvalues* $\mathcal{V}(A_3) = \cup_{\rho \in \mathrm{P}^{\mathrm{P}}(A)} \mathcal{V}_\rho(A_3)$ *.* □

3.3. *Eigenspaces of matrices in UFNF$_2$*

Let the partition of V_A generating the permutation that renders A in UFNF$_2$, block diagonal form, be $V_A = \{V_k\}_{k=1}^K$, and write $A = \biguplus_{k=1}^K A_k$, $A_k = A(V_k, V_k)$.

$$A_3 = \begin{bmatrix} 0 & \cdot & 0 & \cdot & 7 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 3 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & 10 \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & 2 & \cdot & 23 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -3 \end{bmatrix}$$

(a) A reducible matrix in UFNF_1



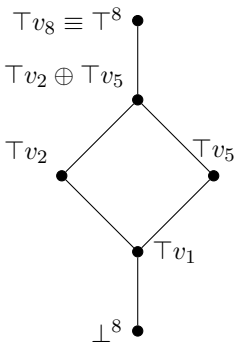
(b) Class diagram (rectangles) overlaid on $G^{\mathrm{d}}_{A_3}$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{bmatrix} 0 & \top & \top & \top & \top & \top & \top & \top \\ \cdot & 0 & 1 & -2 & \cdot & \cdot & \cdot & 6 \\ \cdot & -1 & 0 & -3 & \cdot & \cdot & \cdot & 5 \\ \cdot & \cdot & \cdot & \cdot & 0 & -1 & -2 & 20 \\ \cdot & \cdot & \cdot & \cdot & -3 & 0 & -1 & 21 \\ \cdot & \cdot & \cdot & \cdot & -2 & 1 & 0 & 22 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

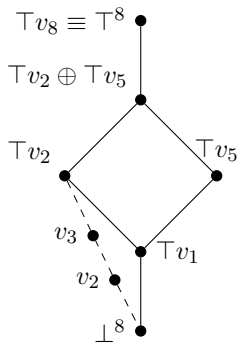
(c) Left fundamental eigenvectors

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & -3 & -2 & 6 & 5 & 4 & \top \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & \top \\ \cdot & -1 & 0 & \cdot & \cdot & \cdot & \top \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & \top \\ \cdot & \cdot & \cdot & 0 & -1 & -2 & \top \\ \cdot & \cdot & \cdot & -3 & 0 & -1 & \top \\ \cdot & \cdot & \cdot & -2 & 1 & 0 & \top \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \end{array}$$

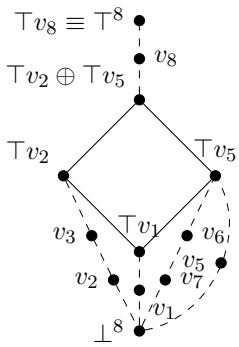
(d) Right fundamental eigenvectors



(e) $\mathcal{V}^{\top}(A_3)$



(f) Schematics of $\mathcal{V}_2(A_3)$



(g) Schematics of $\mathcal{V}(A_3)$

Figure 3.: Matrix A_3 (a), its associated digraph and class diagram (b), its left (c) and right (d) fundamental eigenvectors annotated with their eigennodes to the left and above, respectively; the eigenspace of improper eigenvectors $\mathcal{V}^{\top}(A_3)$ in (e), a schematic of the right eigenspace of proper eigenvalue $\rho = 2$, $\mathcal{V}_2(A_3)$ in (f) and the schematics of the whole right eigenspace $\mathcal{V}(A_3)$ in (g).

Lemma 3.13. *Let $A = \biguplus_{k=1}^{K} A_k \in \mathcal{M}_n(\mathcal{S})$ be a matrix in UFNF$_2$, over a semiring, and $\mathcal{V}_\rho(A_k)$ ($\mathcal{U}_\lambda(A_k)$) a right (left) eigenspace of A_k for ρ (λ). Then,*

$$\mathcal{U}_\lambda(A) \cong \bigtimes_{k=1}^{K} \mathcal{U}_\lambda(A_k) \qquad\qquad \mathcal{V}_\rho(A) \cong \bigtimes_{k=1}^{K} \mathcal{V}_\rho(A_k).$$

Proof. Let $v^k \in \mathcal{V}_\rho(A_k)$ and assemble $v = [(v^1)^{\mathrm{T}} \dots (v^K)^{\mathrm{T}}]^{\mathrm{T}}$. Then

$$A \otimes v = \begin{bmatrix} A_1 \otimes v^1 \\ \vdots \\ A_K \otimes v^K \end{bmatrix} = \begin{bmatrix} v^1 \otimes \rho \\ \vdots \\ v^K \otimes \rho \end{bmatrix} = \begin{bmatrix} v^1 \\ \vdots \\ v^K \end{bmatrix} \otimes \rho = v \otimes \rho, \tag{16}$$

and dually for left eigenvectors. Likewise, for $v \in \mathcal{V}_\rho(A)$ we have $A \otimes v = v \otimes \rho$ whence $A_k \otimes v(V_k) = v(V_k) \otimes \rho$, since A is block diagonal, with $v(V_k)$ the slice of v selected by the nodes in V_k. From this, we have $v(V_k) \in \mathcal{V}_\rho(A_k)$, completing the isomorphism. □

Note that the procedure is constructive and how the combinatorial nature of the proof makes the claim hold in any semiring. Clearly, if $\rho \in \mathrm{P}^{\mathrm{P}}(A_k)$ for any k, then $\rho \in \mathrm{P}^{\mathrm{P}}(A)$. Since $\mathrm{P}^{\mathrm{P}}(A_k) = \Lambda^{\mathrm{P}}(A_k)$ we have an alternative proof to Corollary 3.5 for matrices admitting an UFNF_2, $\mathrm{P}^{\mathrm{P}}(A) = \Lambda^{\mathrm{P}}(A) = \bigcup_{k=1}^{K} \mathrm{P}^{\mathrm{P}}(A_k)$.

In complete semirings, looking for generators for the eigenspaces and taking into consideration both (14) and Lemma 2.4, with $\delta_k(k) = e$ and $\delta_k(i) = \perp$ for $k \neq i$, we define the right fundamental eigenvectors as

$$\mathrm{FEV}^2_\rho(A) = \bigcup_{k=1}^{K} \left[\bigtimes_{i=1}^{K} \delta_k(i) \underset{\cdot}{\otimes} \mathrm{FEV}^1_\rho(A_i) \right]. \tag{17}$$

Lemma 3.13 proves that $\mathrm{FEV}^2_\rho(A) \subset \mathcal{V}_\rho(A)$, but we also have,

Lemma 3.14. *Let $A \in \mathcal{M}_n(\overline{\mathcal{D}})$ be a matrix in UFNF$_2$ over a complete idempotent semiring with $\rho \in \mathrm{P}(A)$. Then,*

(1) If $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, then $\mathrm{FEV}^{2,\mathrm{F}}_\rho(A) = \bigcup_{k|\rho\in\mathrm{P}^{\mathrm{P}}(A_k)} \left[\bigtimes_{i=1}^{K} \delta_k(i) \underset{\cdot}{\otimes} \mathrm{FEV}^{1,\mathrm{F}}_\rho(A_i) \right]$.

(2) If $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$ then $\mathrm{FEV}^2_\rho(A) = \mathrm{FEV}^{2,\top}(A)$.

(3) If $\rho \in \mathrm{P}^{\mathrm{P}}(A)$ then $\mathrm{FEV}_\rho(A) = \mathrm{FEV}^{2,\mathrm{F}}_\rho(A) \cup \mathrm{FEV}^{2,\top}(A) \setminus \top \underset{\cdot}{\otimes} \mathrm{FEV}^{2,\mathrm{F}}_\rho(A)$.

(4) $\mathrm{FEV}^{2,\top}(A) = \top \underset{\cdot}{\otimes} \mathrm{FEV}^2_\rho(A)$.

Proof. The tuple eigenvector $\bigtimes_{i=1}^{K} \delta_k(i) \underset{\cdot}{\otimes} \mathrm{FEV}^1_\rho(A_i)$ has $\perp$ in every component except the k-th which equals $v \in \mathrm{FEV}^1_\rho(A_k)$. So for $\rho \in \mathrm{P}^{\mathrm{P}}(A_k)$ then $\bigtimes_{i=1}^{K} \delta_k(i) \underset{\cdot}{\otimes} \mathrm{FEV}^{1,\mathrm{F}}_\rho(A_i) \subseteq \mathrm{FEV}^{2,\mathrm{F}}_\rho(A)$ whence claim 1. For $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$ we know that $\mathrm{FEV}^1_\rho(A_k) = \mathrm{FEV}^{1,\top}(A_k) = \top \underset{\cdot}{\otimes} \mathrm{FEV}^1_\rho(A_k)$, whence we prove claim 2 as,

$$\begin{aligned} \mathrm{FEV}^2_\rho(A) &= \bigcup_{k=1}^{K} \left[\bigtimes_{i=1}^{K} \delta_k(i) \underset{\cdot}{\otimes} \mathrm{FEV}^{1,\top}(A_i) \right] = \bigcup_{k=1}^{K} \left[\bigtimes_{i=1}^{K} \delta_k(i) \underset{\cdot}{\otimes} \top \underset{\cdot}{\otimes} \mathrm{FEV}^1_\rho(A_i) \right] \\ &= \top \underset{\cdot}{\otimes} \bigcup_{k=1}^{K} \left[\bigtimes_{i=1}^{K} \delta_k(i) \underset{\cdot}{\otimes} \mathrm{FEV}^1_\rho(A_i) \right] = \top \underset{\cdot}{\otimes} \mathrm{FEV}^2_\rho(A). \end{aligned}$$

Claim 3 follows the proof of Lemma 3.9.2, and claim 4 is a corollary of 2 and 3. □

So call $\mathrm{FEV}^{2,\top}(A)$ the *saturated fundamental eigenvectors of* A, and define the *(right) saturated eigenspace* as $\mathcal{V}^\top(A) = \langle \mathrm{FEV}^{2,\top}(A) \rangle_{\overline{\mathcal{D}}}$. The next is proven along the lines of Corollary 3.10,

Corollary 3.15. *Let $A \in \mathcal{M}_n(\mathcal{S})$ be a matrix in UFNF$_2$ over a complete selective radicable idempotent semifield. Then*
(1) For $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, $\mathcal{V}_\rho(A) \supseteq \mathcal{V}^\top(A)$.
(2) For $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$, $\mathcal{V}_\rho(A) = \mathcal{V}^\top(A)$.

Notice that the very general proposition below is for *all* complete dioids.

Proposition 3.16. *Let $A \in \mathcal{M}_n(\overline{\mathcal{D}})$ be a matrix in UFNF$_2$ over a complete dioid. Then,*
(1) For $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$,

$$\mathcal{U}^\top(A) = \langle \mathrm{FEV}^{2,\top}(A^{\mathrm{T}})\rangle_{\mathbb{3}} \cong \mathcal{O}(\overline{G}_A) \qquad \mathcal{V}^\top(A) = \langle \mathrm{FEV}^{2,\top}(A)\rangle_{\mathbb{3}} \cong \mathcal{F}(\overline{G}_A).$$

(2) For $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, $\rho < \top$,

$$\mathcal{U}_\lambda(A) = \langle \mathrm{FEV}^2_\rho(A^{\mathrm{T}})\rangle_{\overline{\mathcal{D}}} \qquad\qquad \mathcal{V}_\rho(A) = \langle \mathrm{FEV}^2_\rho(A)\rangle_{\overline{\mathcal{D}}}.$$

Proof. That the generation process ranges over $\mathbb{3}$ follows a similar proof to that of Proposition 3.11, claim 1. Since $A = \biguplus_{k=1}^K A_k$ we have $\overline{G}_A = \biguplus_{k=1}^K \overline{G}_{A_k}$, whence, by the properties of the filter and ideal completions $\mathcal{V}^\top(A) \cong \bigtimes_{k=1}^K \mathcal{V}^\top(A_k) \cong \bigtimes_{k=1}^K \mathcal{F}(\overline{G}_{A_k}) \cong \mathcal{F}(\biguplus_{k=1}^K \overline{G}_{A_k}) \cong \mathcal{F}(\overline{G}_A)$. And dually for left eigenspaces and the order filters.

By Proposition 3.11, $\mathcal{V}_\rho(A_k)$ is finitely generated, and Lemma 3.13 clarifies how this is induced on $\mathcal{V}_\rho(A)$. Looking for a set of join-dense elements, if the $\mathcal{V}_\rho(A_k)$ where finite lattices we know from Lemma 2.4 that the $\mathrm{FEV}^2_\rho(A)$ defined above are precisely the join-irreducibles obtained from the factor lattices, whence $\mathcal{V}_\rho(A) = \langle \mathrm{FEV}^2_\rho(A)\rangle_{\overline{\mathcal{D}}}$. The other claim is proven dually. □

To better represent eigenspaces, we define the *spectral lattices of A*,

$$\mathcal{L}_\lambda(A) = \langle \mathrm{FEV}^2_\rho(A^{\mathrm{T}})^{\mathrm{T}}\rangle_{\mathbb{3}} \qquad\qquad \mathcal{L}_\rho(A) = \langle \mathrm{FEV}^2_\rho(A)\rangle_{\mathbb{3}}.$$

This is clearly the product of the component spectral lattices, $\mathcal{L}_\rho(A) = \bigtimes_{k=1}^K \mathcal{L}_\rho(A_k)$.

Example 8. *Consider matrix $A_4 = A_2 \uplus A_3$ composed of matrices A_2 in Figure 4.(a), and A_3 in Figure 3.(a). By Corollary 3.5, $\mathrm{P}^{\mathrm{P}}(A_4) = \mathrm{P}^{\mathrm{P}}(A_2) \bigcup \mathrm{P}^{\mathrm{P}}(A_3) = \{\top, 2, 1, 0, -3\}$.*

For any $\rho \in \mathrm{P}(A_4) \setminus \mathrm{P}^{\mathrm{P}}(A_4)$, since $\mathcal{V}^\top(A_2) = \{\bot^4, \top^4\}$ and $\mathcal{V}^\top(A_3)$ is as in Figure 3.(e), then $\mathcal{V}^\top(A_4) = \mathcal{V}^\top(A_2) \times \mathcal{V}^\top(A_3)$ as depicted in Figure 4.(c).

When $\rho \in \mathrm{P}^{\mathrm{P}}(A_3)$ but $\rho \notin \mathrm{P}^{\mathrm{P}}(A_2)$, say $\rho = 2$, we get for $\mathcal{V}_2(A_4)$ a schematic such as in Figure 4.(d). With eigenvectors such as $v^2 = \top^4 \in \mathcal{V}_\rho(A_2) = \mathcal{V}^\top(A_2)$ and $v^3 = [-2101\bot\bot\bot\bot]^{\mathrm{T}} \in \mathcal{V}_\rho(A_3)$ we assemble $v = [(v^2)^{\mathrm{T}}(v^3)^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{V}_\rho(A_4)$. □

3.4. *Eigenspaces of matrices in UFNF$_3$*

When there are empty columns $\overline{\mathrm{zc}}(A) = V_\iota \cup V_\alpha \neq \varnothing$ the situation is clear,

Corollary 3.17. *Let $A \in \mathcal{M}_n(\mathcal{S})$ be a matrix over an entire zerosumfree semiring in UFNF$_3$. Then $\epsilon \in \mathrm{P}(A)$ and $\mathcal{V}_\epsilon(A) = \langle I_{.\overline{\mathrm{zc}}(A)}\rangle_{\mathcal{S}}$ if and only if $\overline{\mathrm{zc}}(A) \neq \varnothing$.*

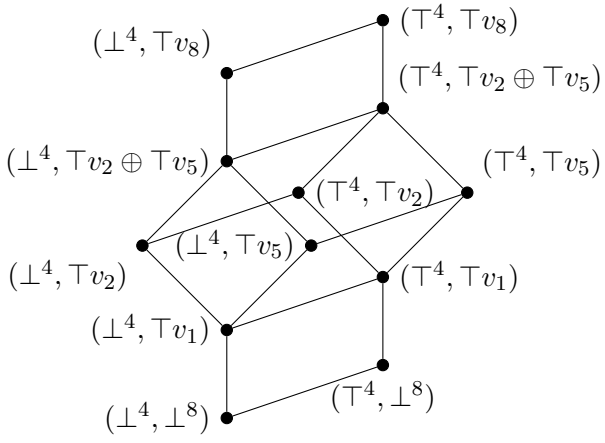
Proof. This is a corollary of (Valverde-Albacete and Peláez-Moreno 2014, Proposition 3.7 and Lemma 3.6) for A in UFNF$_3$. □

$$A_2 = \begin{bmatrix} \cdot & -2 & \top & \cdot \\ \cdot & \cdot & -3 & \top \\ 0.25 & \cdot & \cdot & -5 \\ -7 & 0.3 & \cdot & \cdot \end{bmatrix}$$

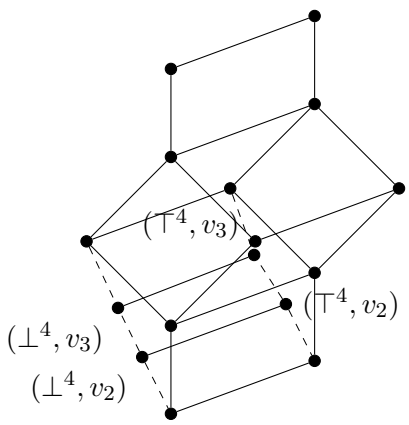
(a) Irreducible matrix A_2

$$A_4 = \begin{bmatrix} A_2 & \cdot \\ \cdot & A_3 \end{bmatrix}$$

(b) Reducible matrix A_4



(c) $\mathcal{V}^{\top}(A_4) \equiv \mathcal{V}_{\rho}(A_4)$ for $\rho \in \mathrm{P}(A_4)/\mathrm{P}^{\mathrm{P}}(A_4)$



(d) Schematics of $\mathcal{V}_{\rho=2}(A_4)$

Figure 4.: Example spectral eigenspaces for matrix $A_4 = A_2 \uplus A_3$ (eigenvector components indexed on the factor matrices they appertain to) for an improper eigenvalue (c) and for a proper eigenvalue (d).

Note that:

(1) Since $\Lambda(A) = \mathrm{P}(A^{\mathrm{T}})$, unlike for any other type of UFNF, $\mathrm{P}(A)$ and $\Lambda(A)$ may differ for UFNF_3 due to (independently) empty rows or columns.

(2) If $V_\beta = \varnothing$, the corollary describes $\mathrm{P}(A)$ completely, since the columns $A_{\cdot\omega}$ are non-void by definition and A would then have no cycles; this means that ϵ would then be the single eigenvalue of A.

(3) When the semiring is also a complete dioid $\overline{\mathcal{D}}$, if $\rho = \bot \in \mathrm{P}^{\mathrm{P}}(A)$, the eigenspace $\mathcal{V}_{\bot}(A) = \langle I_{\cdot\overline{\mathrm{zd}}(A)} \rangle_{\overline{\mathcal{D}}}$. is a complete lattice. The eigenlattice $\mathcal{L}_{\bot}(A) = \langle I_{\cdot\overline{\mathrm{zd}}(A)} \rangle_{\mathbb{3}}$ is also complete, since $\mathbb{3}$ is a complete subsemiring of $\overline{\mathcal{D}}$.

Example 9. *Consider a matrix A over any entire zerosumfree semiring in UFNF_3*

$$A = \begin{bmatrix} \cdot & x & x \\ \cdot & \cdot & x \\ \cdot & \cdot & \cdot \end{bmatrix}$$

where $x \neq \epsilon$. Since it has an empty column (row) $\epsilon \in \mathrm{P}^{\mathrm{P}}(A)$ ($\epsilon \in \Lambda^{\mathrm{P}}(A)$). And since it has no cycles $\overline{G}_A = \varnothing$, there are no other eigenvalues hence $\Lambda(A) = \{\epsilon\} = \mathrm{P}(A)$. When the semiring is a complete dioid, the schematics of the null eigenspaces and its eigenlattice are isomorphic to those in Figures 5.(b) and 5.(c). □

Proposition 3.18. *Let $A \in \mathcal{M}_n(\mathcal{D})$ be a matrix over a commutative dioid in UFNF_3 with $\overline{G}_A \neq \varnothing$. Then:*

(1) $\mathrm{P}(A)$ contains all finite eigenvalues of $A_{\beta\beta}$, $\mathrm{P}(A) \supseteq \mathrm{P}(A_{\beta\beta}) \setminus \{\epsilon\}$.

(2) Further, if $\mathcal{S}$ is a semifield, then every eigenvector of $A_{\beta\beta}$ for ρ can be uniquely extended to an eigenvector of A for ρ.

(3) *Further, if $\mathcal{S}$ is a complete (as a dioid) semifield,*
   (a) $\mathrm{P}^{\mathrm{p}}(A)$ *contains all proper finite eigenvalues of* $A_{\beta\beta}$, $\mathrm{P}^{\mathrm{p}}(A) \supseteq \mathrm{P}^{\mathrm{p}}(A_{\beta\beta}) \setminus \{\bot\}$.
   (b) *Every* $v_\beta \in \mathcal{V}_\top(A_{\beta\beta})$ *can be uniquely extended to an eigenvector of* A *for* $\top$.
   (c) *for* $\rho \in \mathrm{P}(A) \setminus \{\bot\}$, $\mathcal{V}_\rho(A) \cong \mathcal{V}_\rho(A_{\beta\beta})$.

Proof. If $\bot \in \mathrm{P}(A_{\beta\beta})$ this means there are zero columns $\overline{\mathrm{zc}}(A_{\beta\beta}) = \overline{\mathbf{m}}$ and $\{e_j \mid j \in \overline{\mathrm{zc}}(A_{\beta\beta})\} \subseteq \mathcal{V}_\epsilon(A_\beta)$. But if this is the case, since the j-th column of $A_{\alpha\beta}$ cannot be empty—or else j actually belongs in $\overline{\mathrm{zc}}(A)$ not in $\overline{\mathrm{zc}}(A_{\beta\beta})$—then $A_{\cdot j} \neq \epsilon^n$ and e_j cannot be an eigenvector of A for ϵ.

Since any null eigenvalue of $\mathrm{P}(A_{\beta\beta})$ is blocked from appearing in $\mathrm{P}(A)$ and $\overline{G}_A = \overline{G}_{A_{\beta\beta}}$, we assume $V_\beta \neq \varnothing$, $\bot \notin \mathrm{P}(A_{\beta\beta})$ and we will suppose that $A_{\beta\beta}$ is in UFNF_2.

If ρ^{-1} exists when $\mathcal{S}$ is a semifield, by (Valverde-Albacete and Peláez-Moreno 2014, Lemma 3.14) we can reduce the problem of finding its eigenspace to that of finding the eigenspace for e in $B = \tilde{A}^\rho$. Therefore we work out B^* and B^+ and compare them. Since A is in UFNF_3 its closures are given by Lemma 2.10, claim 3. Comparing them as demanded by Theorem 1.1,

$$\begin{bmatrix} I_\iota & \cdot & \cdot & \cdot \\ \cdot & I_\alpha & B_{\alpha\beta}B^*_{\beta\beta} & B_{\alpha\beta}B^*_{\beta\beta}B_{\beta\omega} \oplus B_{\alpha\omega} \\ \cdot & \cdot & B^*_{\beta\beta} & B^*_{\beta\beta}B_{\beta\omega} \\ \cdot & \cdot & \cdot & I_\omega \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & B_{\alpha\beta}B^*_{\beta\beta} & B_{\alpha\beta}B^*_{\beta\beta}B_{\beta\omega} \oplus B_{\alpha\omega} \\ \cdot & \cdot & B^+_{\beta\beta} & B^*_{\beta\beta}B_{\beta\omega} \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Clearly, no eigenvector for $\rho \neq \epsilon$ can be obtained from columns in $V_\iota \cup V_\alpha \cup V_\omega$. If $\rho \in \mathrm{P}(A_{\beta\beta})$ and $v^\beta \in \mathcal{V}_\rho(A_{\beta\beta})$ we may write,

$$B \otimes \begin{bmatrix} \cdot \\ B_{\alpha\beta} \otimes v^\beta \\ v^\beta \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ B_{\alpha\beta} \otimes v^\beta \\ v^\beta \\ \cdot \end{bmatrix} \Leftrightarrow A \otimes \begin{bmatrix} \cdot \\ \tilde{A}^\rho_{\alpha\beta} \otimes v^\beta \\ v^\beta \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \tilde{A}^\rho_{\alpha\beta} \otimes v^\beta \\ v^\beta \\ \cdot \end{bmatrix} \otimes \rho \tag{18}$$

so $v^{\mathrm{T}} = [\, \bot^{V_\iota} \ \left((A_{\alpha\beta} \otimes v^\beta) \otimes \rho^{-1}\right)^{\mathrm{T}} \ (v^\beta)^{\mathrm{T}} \ \bot^{V_\omega} \,]^{\mathrm{T}} \in \mathcal{V}_\rho(A)$, proving claims 1 and 2.

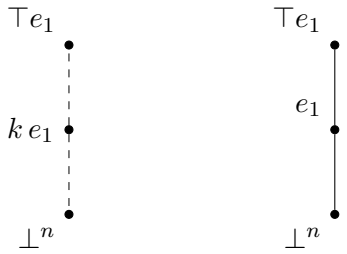
For 3a and 3b, if $\rho = \top$ in a *complete* idempotent semifield we have that $(A_{\alpha\beta} \underset{\cdot}{\otimes} v^\beta) \underset{\cdot}{/} \top = (A_{\alpha\beta} \underset{\cdot}{\otimes} v^\beta) \underset{\cdot}{\otimes} \bot$ is saturated wherever $A_{\alpha\beta} \underset{\cdot}{\otimes} v^\beta$ is and null otherwise, hence $(A_{\alpha\beta} \underset{\cdot}{\otimes} v^\beta) \underset{\cdot}{/} \top = ((A_{\alpha\beta} \underset{\cdot}{\otimes} v^\beta) \underset{\cdot}{/} \top) \underset{\cdot}{\otimes} \top$. Therefore, the reasoning in the paragraph above applies to

$$[\, \bot^{V_\iota} \quad \left((A_{\alpha\beta} \underset{\cdot}{\otimes} v^\beta) \underset{\cdot}{/} \top\right)^{\mathrm{T}} \quad (v^\beta)^{\mathrm{T}} \quad \bot^{V_\omega} \,]^{\mathrm{T}} \in \mathcal{V}_\top(A). \tag{19}$$

For the final claim, the bijection between the spaces is proven above: now consider two fundamental eigenvectors $v_1^\beta, v_2^\beta \in \mathcal{V}_\rho(A_{\beta\beta})$ in whatever relation (equality, comparable, incomparable). From (18) and (19), their extensions to the eigenspace of A, $v_1, v_2 \in \mathcal{V}_\rho(A)$ stand in the same relation. The order isomorphism is proven. □

Note that if $A_{\alpha\beta} = \bot$ any eigenvector $v^\beta \in \mathcal{V}_\rho(A_{\beta\beta})$ can be extended to $v \in \mathcal{V}_\rho(A)$ by padding it with nulls. From (18), it seems natural to define for A in UFNF_3 and

$$A_5 = \begin{bmatrix} \cdot & \mathbb{1} \\ \vdots & A_4 \end{bmatrix}$$



(a) Matrix A_5 (b) Schematics of $\mathcal{V}_\bot(A_5)$ (c) $\mathcal{L}_{\rho=\bot}(A_5)$

Figure 5.: A matrix with zero columns but no zero rows and some of its spectral eigenspaces: matrix A_5 (a), the schematics of its null eigenspace $\mathcal{V}_\bot(A_5)$ (b), and the spectral lattice of the null eigenspace $\mathcal{L}_\bot(A_5)$ (c). The other eigenspaces are isomorphic to those of A_4 in Ex. 8.

$\rho \neq \bot \in \mathrm{P}(A)$,

$$\mathrm{FEV}^3_\bot(A) = \{I_{\cdot i} \mid i \in \overline{\mathrm{zc}}(A)\} \tag{20}$$

$$\mathrm{FEV}^3_\rho(A) = \begin{cases} [\,\bot^{V_\iota}\ \left((A_{\alpha\beta} \underset{\cdot}{\otimes} v^\beta)\right) \underset{\cdot}{/} \rho\right)^{\mathrm{T}}\ (v^\beta)^{\mathrm{T}}\ \bot^{V_\omega}\,]^{\mathrm{T}} & \text{if } \overline{\mathrm{zc}}(A_{\beta\beta}) = \varnothing, v^\beta \in \mathrm{FEV}^2_\rho(A_{\beta\beta}) \\ [\,\bot^{V_\iota}\ \left((A_{\alpha\beta} \underset{\cdot}{\otimes} v^\beta)\right) \underset{\cdot}{/} \rho\right)^{\mathrm{T}}\ (v^\beta)^{\mathrm{T}}\ \bot^{V_\omega}\,]^{\mathrm{T}} & \text{if } \overline{\mathrm{zc}}(A_{\beta\beta}) \neq \varnothing, v^\beta \in \mathrm{FEV}^3_\rho(A_{\beta\beta}) \end{cases} \tag{21}$$

These definitions boil down to a $A_{\beta\beta}$ either in UFNF_2, as in Example 10, or as an empty matrix, as in Example 9. By the isomorphism, the finitely and saturatedly supported set of fundamental eigenvectors can also be defined for $\rho \in \mathrm{P}(A) \setminus \{\bot\}$ and the properties in Lemma 3.14 and Corollary 3.15 also hold, hence

Proposition 3.19. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be a matrix over a complete commutative idempotent semifield in UFNF_3. Then:*

(1) For $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$,

$$\mathcal{U}_\lambda(A) = \langle \mathrm{FEV}^{3,\top}(A^{\mathrm{T}})^{\mathrm{T}} \rangle_{\mathbb{3}} \cong \mathcal{O}(\overline{G}_A) \qquad \mathcal{V}_\rho(A) = \langle \mathrm{FEV}^{3,\top}(A) \rangle_{\mathbb{3}} \cong \mathcal{F}(\overline{G}_A).$$

(2) For $\rho \in \mathrm{P}^{\mathrm{P}}(A) \setminus \{\bot\}$,

$$\mathcal{U}_\lambda(A) = \langle \mathrm{FEV}^3_\rho(A^{\mathrm{T}})^{\mathrm{T}} \rangle_{\overline{\mathcal{K}}} \qquad \mathcal{V}_\rho(A) = \langle \mathrm{FEV}^3_\rho(A) \rangle_{\overline{\mathcal{K}}}.$$

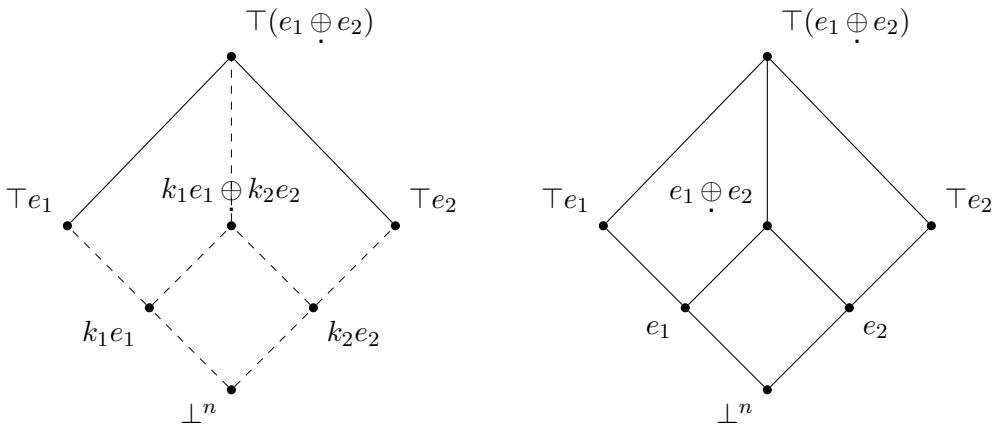
Proof. From Proposition 3.18, claim 3 as a whole, the generators for $\mathcal{V}_\rho(A_{\beta\beta})$ can be extended uniquely to generators of $\mathcal{V}_\rho(A)$, hence the claims about generation follow. Since $\overline{G}_A \cong \overline{G}_{A_{\beta\beta}}$, we have $\mathcal{V}^\top(A) \cong \mathcal{V}^\top(A_{\beta\beta}) \cong \mathcal{F}(\overline{G}_{A_{\beta\beta}}) \cong \mathcal{F}(\overline{G}_A)$. □

Define the spectral lattices as usual in the following example.

Example 10. *Retaking A_4 from Example 8, let A_5 be as in Figure 5.(a) with $V^5_\alpha = \{1\}$, $V^5_\iota = V^5_\omega = \varnothing$, $V^5_\beta = \{2-13\}$, and $V^5_{\alpha\beta} = \mathbb{0}$ is a conformant matrix with those values. We see that since $\mathrm{P}^{\mathrm{P}}(A_5) \supseteq \mathrm{P}^{\mathrm{P}}(A_4)$, and A_5 has zero columns, $\overline{zc}(A_5) = \{1\}$, $\mathrm{P}^{\mathrm{P}}(A_5) = \{\top, 2, 1, 0, -3, \bot\}$. Yet, as $\overline{zr}(A_5) = \varnothing$, $\Lambda^{\mathrm{P}}(A_5) = \{\top, 2, 1, 0, -3\}$. Therefore $\mathrm{P}(A) = \overline{\mathbb{R}}_{\max,+}$ but $\Lambda(A) = \mathbb{R}_{\max,+} \setminus \{\bot\}$, whence we have a schematics of $\mathcal{V}_\bot(A_5) = \langle I_{\cdot 1} \rangle_{\overline{\mathbb{R}}_{\max,+}}$ as in Figure 5.(b), $\mathcal{V}_\bot(A_5) = \mathcal{L}_{\rho=\bot}(A_5) \cong \mathbb{3}$ as in Figure 5.(c) but $\mathcal{U}_\bot(A_5) = \mathcal{L}_{\lambda=\bot}(A_5) = \{\bot^n\} \cong \mathbb{1}$.*

$$A_6 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \mathbb{1} & \\ \vdots & \vdots & & A_5 & \end{bmatrix}$$

(a) A matrix with zero columns and rows



(b) $\mathcal{V}_\bot(A_6)$ (c) $\mathcal{L}_{\rho=\bot}(A_6)$

Figure 6.: A matrix with zero columns and rows and some of its spectral eigenspaces: matrix A_6 (a), the schematics of its null eigenspace $\mathcal{V}_\bot(A_6)$ (b), and the spectral lattice of the null eigenspace $\mathcal{L}_\bot(A_6)$ (c). The other eigenspaces are isomorphic to those of A_4 in Ex. 8 via A_5.

For any non-null $\rho \in \mathrm{P}(A_5)$ the eigenspace is a copy of that of A_4, $\mathcal{V}_\rho(A_5) \cong \mathcal{V}_\rho(A_4)$, and likewise for the left eigenspaces. For instance, for $\rho = 2$ from Ex. 8 we have $v^4 = [(v^2)^{\mathrm{T}} \quad (v^3)^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{V}_\rho(A_4)$. Then, by (19), its isomorphic image in $\mathcal{V}_\rho(A_5)$ is $v^5 = [(\mathbb{0} \dot{\otimes} v^4)^{\mathrm{T}} \quad (v^4)^{\mathrm{T}}]^{\mathrm{T}}$.

Furthermore, let A_6 be as in Figure 6.(a) with $V^6_\iota = \{1\}$, $V^6_\alpha = \{1\}$, $V^6_\omega = \varnothing$, $V^6_\beta = \{3-15\}$, and $V^6_{\alpha\beta} = \mathbb{1}$ are conformant matrices with those values. We see that $\overline{zc}(A_6) = \{1, 2\}$ but $\overline{zr}(A_6) = \{1\}$. Analogously, $\mathrm{P}^{\mathrm{P}}(A_6) \supseteq \mathrm{P}^{\mathrm{P}}(A_5)/\{\bot\}$, but since A_6 has empty columns and rows, we have that $\Lambda^{\mathrm{P}}(A_6) = \{\top, 2, 1, 0, -3, \bot\} = \mathrm{P}^{\mathrm{P}}(A_6)$ and $\Lambda(A) = \overline{\mathbb{R}}_{\max,+} = \mathrm{P}(A)$.

The isomorphism for matrix A_5 proceeds onto the eigenspaces of A_6 whence $\mathcal{V}_\rho(A_6)$, that is $v^6 = [\bot \quad (\mathbb{1} \dot{\otimes} v^5)^{\mathrm{T}} \quad (v^5)^{\mathrm{T}}]^{\mathrm{T}}$. However, for A_6 we have $\mathcal{U}_\bot(A_6)$ isomorphic to $\mathcal{V}_\bot(A_5)$ in Figure 5.(b), and $\mathcal{L}_{\lambda=\bot}(A_6) \cong \mathbb{3}$ isomorphic to $\mathcal{L}_{\rho=\bot}(A_5)$ in Figure 5.(c), yet $\mathcal{V}_\bot(A_6) = \langle\{e_1, e_2\}\rangle_{\overline{\mathbb{R}}_{\max,+}}$ and $\mathcal{L}_{\rho=\bot}(A_6) = \langle\{e_1, e_2\}\rangle_{\mathbb{3}}$ as in Figures. 6.(b) and 6.(c). □

3.5. *Final results*

We now undertake an overarching formulation of our results. We concentrate on right eigenspaces: left eigenspaces admit dual proofs. Without loss of generality, we suppose A in an UFNF, and use structural induction on the particular form as a general technique to prove Theorem 3.20, as also illustrated in Proposition 2.6 and

Lemma 2.11. The crux of it is the definition of the fundamental eigenvectors,

$$\mathrm{FEV}_\rho(A) = \begin{cases} \mathrm{FEV}^1_\rho(A) & \text{as in (14), if } A \text{ in } \mathrm{UFNF}_1 \text{ or } \mathrm{UFNF}_0 \\ \mathrm{FEV}^2_\rho(A) & \text{as in (17), if } A \text{ in } \mathrm{UFNF}_2 \\ \mathrm{FEV}^3_\rho(A) & \text{as in (21), if } A \text{ in } \mathrm{UFNF}_3 \end{cases} \tag{22}$$

Theorem 3.20 (Eigenspaces of generic matrices)**.** *Let $A \in \mathcal{M}_n(\overline{K})$ be a matrix over a complete commutative radicable selective semifield. Then,*

(1) For any improper eigenvalue, $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$,

$$\mathcal{U}_\lambda(A) = \langle \mathrm{FEV}_\lambda(A^{\mathrm{T}})^{\mathrm{T}} \rangle_{\mathbb{3}} \cong \mathcal{O}(\overline{G}_A) \quad \mathcal{V}_\rho(A) = \langle \mathrm{FEV}_\rho(A) \rangle_{\mathbb{3}} \cong \mathcal{F}(\overline{G}_A). \tag{23}$$

(2) For proper finite eigenvalues, $\perp < \rho < \top \in \mathrm{P}^{\mathrm{P}}(A)$,

$$\mathcal{U}_\lambda(A) = \langle \mathrm{FEV}_\lambda(A^{\mathrm{T}})^{\mathrm{T}} \rangle_{\overline{\mathcal{K}}} \qquad \mathcal{V}_\rho(A) = \langle \mathrm{FEV}_\rho(A) \rangle_{\overline{\mathcal{K}}}. \tag{24}$$

Proof. (1) If A is in UFNF_0, Example 6 provide the definitions and proofs needed for (24) and (23).
(2) If A is in UFNF_1 then Proposition 3.11 proves the theorem.
(3) If A is in UFNF_2, this is Proposition 3.16
(4) If A is in UFNF_3 this is Proposition 3.19.

□

This proves the desired corollary:

Corollary 3.21. *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be a matrix over a complete commutative selective radicable semifield. For $\perp < \rho < \top \in \mathrm{P}^{\mathrm{P}}(A)$, $\mathrm{FEV}_\rho(A)$ is join-dense in $\mathcal{V}_\rho(A)$.*

We took special notice of the lattices arising from these cases and their properties– specially $\mathcal{F}(\overline{G}_A)$ and $\mathcal{O}(\overline{G}_A)$ the lattices of order filters and ideals, respectively, of the condensation digraph of A using examples to illustrate them.

Since the eigenspaces of proper eigenvalues, albeit complete lattices, are continuous, we defined more easily representable finite sublattices that show the principal order structures in the eigenspaces, our hypothesis being that these will prove effective in data mining tasks.

Proposition 3.22 (Eigenlattices of generic matrices)**.** *Let $A \in \mathcal{M}_n(\overline{K})$ be a matrix over a complete commutative radicable selective semifield. The eigenlattices*

$$\mathcal{L}_\lambda(A) = \langle \mathrm{FEV}_\lambda(A^{\mathrm{T}})^{\mathrm{T}} \rangle_{\mathbb{3}} \qquad \mathcal{L}_\rho(A) = \langle \mathrm{FEV}_\rho(A) \rangle_{\mathbb{3}},$$

are complete finite sublattices of the eigenspaces.

Proof. When $\rho \in \mathrm{P}(A) \setminus \mathrm{P}^{\mathrm{P}}(A)$, $\mathcal{L}_\rho(A) = \mathcal{V}_\rho(A) = \mathcal{V}^\top(A)$. When $\rho \in \mathrm{P}^{\mathrm{P}}(A)$, by Proposition 1.2 $\mathcal{L}_\rho(A)$ is a subsemimodule of $\mathcal{V}_\rho(A)$. But since both the generator set $\mathrm{FEV}_\rho(A)$ and the semiring guiding the generation $\mathbb{3}$ are finite, the span is finite. Since it is a complete idempotent semifield, the lattice is complete. □

4. Conclusions and discussion

This paper completes the course set out in (Valverde-Albacete and Peláez-Moreno 2014), to characterize the spectrum of matrices with entries in completed idempo-

tent semifields, as opposed to the best-known theory for matrices over incomplete idempotent semifields.

To the extent of our knowledge, this was pioneered in (Jun, Yan, and zhi Yong 2005) and both (Valverde-Albacete and Peláez-Moreno 2014) and this paper can be understood as systematic explorations to try and understand what was stated in there. For this purpose, the consideration of particular UFNF forms for the matrices has been crucial: while the description in (Jun, Yan, and zhi Yong 2005) is combinatorial, ours is constructive (see Theorem 3.20).

It is now clear that the spectral theory for *incomplete* idempotent semifields, as summarized for instance in (Bapat 1998; Butkovič, Cunninghame-Green, and Gaubert 2009), presents important differences with the new theory here. These stem fundamentally from the appearance of the top-eigenvalue either in trivial or in proper form, the possible incidence of saturated supports and the (complete) order properties of eigensemimodules.

The usual notion of spectrum as the set of eigenvectors with more than one (non-null) eigenvector appears in this context as too weak: when a matrix has at least one cycle then all the values in the semifield (except, possibly, the bottom $\perp$) belong to the spectrum. If the matrix has at least one empty column (resp. empty row) and a cycle then *all* of the semifield is the spectrum. Rather than redefine the notion of spectrum we have decided to introduce the *proper spectrum* as the set of eigenvalues with at least one vector with finite support.

For incomplete idempotent semifields both notions of spectrum coincide, a reflection of the fact that, in general, matrices over incomplete idempotent semifields admit less eigenvalues. The reason for this is easily seen in what follows: recall the condensation digraph of a matrix, $\overline{G}_A$. An *initial* component $C \in \overline{V}_A$ of this relation is not accessed by any other, except itself, $\downarrow C = \{C\}$; a component is *final* if it has only access to itself, $\uparrow C = \{C\}$; it is *isolated* if it is both initial and final, which means the class is just one strongly connected component.

Note that saturated supports will be generated for dominated classes in reducible matrices, as stated by Lemma 3.7, claim 3. Furthermore, a class $C_r \in \overline{V}_A$ is *dominated* if its maximal cycle mean is smaller than that of any of the classes in its order ideal: $\exists C_{l_r} \in \downarrow C_r,\ \mu(A_{l_r l_r}) > \mu(A_{rr})$, and we say that C_{l_r} is *dominating.*

Proposition 4.1. *In the conditions of Proposition 3.2 with $\mathcal{K}$ an incomplete idempotent semifield, C_r is spectral if and only if it is not dominated.*

Although "missing" the eigenvalues from the dominated classes, $\mathrm{P}(A)$ for $A \in \mathcal{M}_n(\mathcal{K})$ with K incomplete is never empty as some classes are never dominated: Call a class $C_r \in \overline{V}_A$ *basic* if its maximal cycle mean is that of the matrix $\mu_\oplus(A_{rr}) = \mu_\oplus(A)$.

Corollary 4.2. *In the conditions of Proposition 3.2, with $\mathcal{K}$ an incomplete idempotent semifields, basic and initial classes are always spectral.*

The situation in complete semifields is much more regular, since $(\tilde{A}^\rho)^+$ always exists.

Lemma 4.3. *Let $A \in \mathcal{M}_n(\mathcal{S})$ be a matrix over a complete radicable semiring. If C_{l_r} is dominating for any class $C_r \in \overline{V}_A$, then $V_{l_r} \subseteq \text{sat-supp}\,(C_r)$.*

Consequently, the top element in the semifield may be an eigenvalue and part of the eigenvectors, which extends spectra and eigenspaces enormously, as proven in this paper.

We next show an example of the differences between spectra in $\mathbb{R}_{\max,+}$ and in

$\overline{\mathbb{R}}_{\max,+}$.

Example 11. *Compare the following results to those of Ex. 7. Consider matrix A_3 from Figure 3, whose right spectrum is $\mathrm{P}(A)_{\mathbb{R}_{\max,+}} = \{2,1,0\}$, since C_4 is dominated by C_2 and C_3 . Then the right eigenspace for each class of A is $\mathcal{V}_{\mu_1}(A_{11}) = \langle(\tilde{A}_{11})^{+}_{\cdot 1}\rangle_{\mathbb{R}_{\max,+}}$, $\mathcal{V}_{\mu_2}(A_{22}) = \langle(\tilde{A}_{22})^{+}_{\cdot 2}\rangle_{\mathbb{R}_{\max,+}}$, and $\mathcal{V}_{\mu_3}(A_{33}) = \langle(\tilde{A}_{33})^{+}_{\cdot\{5,6\}}\rangle_{\mathbb{R}_{\max,+}}$, as described by the column vectors with no $\top$ components of Figure 3.(d).*

Likewise, the left spectrum (being the right spectrum of A^{T}) is $\Lambda(A)_{\mathbb{R}_{\max,+}} = \{2,1,-3\}$, since C_1 is dominated by C_2 and C_3—a coincidence brought about by a certain symmetry of this condensation digraph $\overline{G}_A$—and the left eigenspaces are $\mathcal{U}_{\mu_2}(A_{22}) = \langle(\tilde{A}_{22})^{+}_{2\cdot}\rangle_{\mathbb{R}_{\max,+}}$, $\mathcal{U}_{\mu_3}(A_{33}) = \langle(\tilde{A}_{33})^{+}_{\{5,6\}\cdot}\rangle_{\mathbb{R}_{\max,+}}$, and $\mathcal{U}_{\mu_4}(A_{44}) = \langle(\tilde{A}_{44})^{+}_{8\cdot}\rangle_{\mathbb{R}_{\max,+}}$, as described by the finite row vectors of Figure 3.(c). □

Our choice of definition for eigenvalues, on the other hand, results in almost identical left and right spectra. Indeed, any discrepancy between left and right spectra may only reside in the presence of the bottom eigenvalue, exclusively entailed by empty columns (resp. emtpy rows) in right (left) spectra, as collected in Theorem 3.4.

Regarding the eigenspaces, we found not only that they are complete continuous lattices for proper eigenvalues, but also that they are finite (complete) lattices for improper eigenvalues. Looking for a device to represent the information within each proper eigenspace we focus on the fundamental eigenvectors of an irreducible matrix for each eigenvalue: those with unit values in certain of their coordinates. The span of those eigenvectors by the action of the $\mathfrak{Z}$-blog generates the finite eigenlattices. Interestingly, since improper eigenvectors only have non-finite coordinates, their span by the $\mathfrak{Z}$-blog is exactly the same finite lattice as their span by the whole semifield itself.

With these building blocks it is easy to build finite lattices for reducible matrices of any UFNF description, as proven above. We believe this is a useful technique to understand and visualize the concept lattices of formal contexts with entries in an idempotent semifield (Valverde-Albacete and Peláez-Moreno 2008, 2011) which generalizes Formal Concept Analysis (Ganter and Wille 1999).

The only discrepant note in this, otherwise regular, structure is the top eigenvalue and its eigenspaces, for which we have presented an example that shows that the consideration of the fundamental eigenvectors does *not* provide a set of join-dense elements. Even more surprising fact is that the fundamental eigenvectors as described in (Valverde-Albacete and Peláez-Moreno 2014, Example 7) seem to supply a set of meet-irreducibles of $\mathcal{V}_{\top}(A)$. This is, of course, related to the issue that the meet is not join-linear nor viceversa. One can still choose fundamental eigenvectors and generate discrete–diamond–lattices therefrom but it is doubtful that they represent the eigenspace faithfully. This is a matter to be further looked into.

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