UNIVERSIDAD CARLOS III DE MADRID

WORKING PAPERS

Working Paper
Business Economic Series
WP. 16-04. September, 12nd, 2016.
ISSN 1989-8843

Instituto para el Desarrollo Empresarial
Universidad Carlos III de Madrid
C/ Madrid, 126
28903 Getafe Madrid (Spain)
FAX +34 916249607

Good deal measurement in asset pricing: Actuarial and financial implications

Alejandro Balbás[1]
Departamento de Economía e la Empresa
Universidad Carlos III de Madrid

José Garrido[2]
Department of Mathematics and Statistics.
Concordia University

Ramin Okhrati[3]
Mathematical Sciences
University of Southampton

[1] C/ Madrid, 126. 28903 Getafe (Madrid, Spain).

[2] Maisonneuve Blvd. W. Montreal, Canada H3G 1MB.

[3] Highfield Southampton SO17 1BJ (England).

Good deal measurement in asset pricing: Actuarial and financial implications

Alejandro Balbás*, **José Garrido**** **and Ramin Okhrati*****

* University Carlos III of Madrid. C/ Madrid, 126. 28903 Getafe (Madrid, Spain). alejandro.balbas@uc3m.es

** Concordia University. Department of Mathematics and Statistics. 1455 de Maisonneuve Blvd. W., Montreal, Canada H3G 1M8. jose.garrido@concordia.ca

*** University of Southampton. Mathematical Sciences. Highfield Southampton SO17 1BJ, Southampton, UK. r.okhrati@soton.ac.uk

A.M.S. 91G10, 91G20, 91B30

J.E.L. G11, G13, G22.

Key-words Risk measure; Compatibility between prices and risks; Good deal size measurement; Actuarial and financial implications.

Abstract. We will integrate in a single optimization problem a risk measure beyond the variance and either arbitrage free real market quotations or financial pricing rules generated by an arbitrage free stochastic pricing model. A sequence of investment strategies such that the couple $(risk, price)$ diverges to $(-\infty, -\infty)$ will be called good deal. We will see that good deals often exist in practice, and the paper main objective will be to measure the good deal size. The provided good deal measures will equal an optimal ratio between both risk and price, and there will exist alternative interpretations of these measures. They will also provide the minimum relative (per dollar) price modification that prevents the good deal existence. Moreover, they will be a crucial instrument to detect those securities or marketed claims which are over or under-priced. Many classical actuarial and financial optimization problems may generate wrong solutions if the used market quotations or stochastic pricing models do not prevent the good deal existence. This fact will be illustrated in the paper, and it will be pointed out how the provided good deal measurement may be useful to overcome this caveat. Numerical experiments will be yielded as well.

1 Introduction

The use of risk functions beyond the variance is becoming more and more frequent in both actuarial and financial studies. Nevertheless, when the most important arbitrage free pricing models of financial economics (binomial, trinomial, Black and Scholes, stochastic volatility, etc.) and the most important risk functions (VaR, $CVaR$, $weighted-CVaR$, $robust-CVaR$, spectral measures, etc.) are combined in a single problem, one often faces the existence of sequences of investment strategies (good deals, or GD) whose pair $(risk, price)$ diverges to $(-\infty, -\infty)$. This pathological finding has been analyzed in Balbás *et al.* (2016*a*), where explicit examples of the sequences above have been constructed and their performance empirically tested. The main conclusion was that the divergence to $(-\infty, -\infty)$ is more theoretical than real, but the performance of the constructed GD was good enough. The GD were collections of options providing much better realized Sharpe ratios than their underlying assets.

In this paper we will deal with a couple (ρ, Π) composed of the risk function ρ and the pricing rule Π. (ρ, Π) will be called non compatible if it and only if implies the GD existence, and the main objective will be the measurement of the GD size, denoted by $\tilde{N}$ or $\tilde{N}(\rho, \Pi)$. An important precedent in financial theory is the notion of arbitrage. Though the absence of arbitrage always holds in theoretical approaches, real market quotations sometimes reflect the existence of arbitrage. For this reason some years ago many authors gave several measures of the arbitrage size. This allowed them to address interesting questions such as pricing and hedging issues under transaction costs, cross-market arbitrage, integration between markets, trading systems, valuation of embedded derivatives, etc. Similarly, the existence of GD (or the lack of compatibility) must be now measured, because in some sense it is indicating a huge lack of balance between the risk that the investor is facing and the wealth that he/she is expecting. As we will see, these unbalanced situations may lead to wrong decisions in several fields. For instance, managers could pay expensive prices or compose inefficient portfolios, and insurers could buy non-optimal reinsurance contracts or receive too cheap premiums.

The arbitrage measurement was addressed from several perspectives. One of them was related to the capital profits generated by an arbitrage strategy (Balbás *et al.,* 1999). Nevertheless, if the arbitrage strategy can be repeated once and once again, the arbitrage profit will be multiplied once and once again too, and therefore it will become infinity. To prevent this caveat Balbás *et al. (*1999) measured the arbitrage level as the maximum ratio between the arbitrage income and the value of the sold assets, *i.e.*, these authors gave a relative measure of the arbitrage degree. Similarly, when $(risk, price)$ diverges to $(-\infty, -\infty)$ we will need to maximize ratios $risk/price$. Otherwise we will be facing unbounded optimization problems.

A $risk/price$ ratio will be an objective function which will not satisfy many desirable analytical properties (continuity, convexity, differentiability, etc.), and therefore its optimization will become less complex if one deals with vector optimization problems involving both risk and price. Since Harry Markowitz published his seminal results, it is known that multiobjective analyses are useful in many financial topics. In particular, in portfolio selection one has several interesting approaches such as Ballestero and Romero (1996), Ballestero *et al. (*2012), Dash and Kajiji (2014), etc. With respect to the simultaneous optimization of both risk and price, we will apply well-known results and will optimize "risk" under constraints for "price" (Sawaragi *et al.,* 1985).

The paper outline is as follows. Section 2 will be devoted to fixing notations and assumptions. The measure $\tilde{N}(\rho, \Pi)$ will be constructed in Section 3. The first approach will apply when one does not consider any theoretical pricing model, and only a finite collection of available securities and their market quotations are involved. The advantage of this approach is clear, since it is sufficient to choose a robust (or ambiguous) risk function ρ and the value of $\tilde{N}$ will become model-independent. Beyond the optimal $risk/price$ ratio, there will be a second (or dual) interpretation of $\tilde{N}$ which must be highlighted. $\tilde{N}$ coincides with the minimum relative (per dollar) price modification leading to a $GD-$free market. Moreover, the dual approach will permit the investor to identify the over-priced securities (to be sold so as to create a GD) and the under-priced ones (to be bought). After modifying the prices according to the value of $\tilde{N}$, the GD absence will be guaranteed. A numerical example will illustrate all the theoretical findings. In particular, it will show how easily the GD arises in real markets, how to implement the GD, and how the prices must be modified.

The second approach replaces real market quotations with pricing rules generated by a complete and arbitrage free stochastic pricing model. Completeness may be relaxed, as will be indicated too. Both the primal (optimal $risk/price$ ratios) and the dual (minimum relative price modifications) interpretations of $\tilde{N}$ will apply again, but important differences with respect to the model-independent approach will be also found. Indeed, if the stochastic discount factor (SDF, also called pricing kernel, Duffie, 1988) of the pricing rule Π is not essentially bounded, and the sub-gradient of the risk function ρ is composed of essentially bounded random variables, then the existence of GD is guaranteed, and the value of $\tilde{N}(\rho, \Pi)$ will be strictly higher than one. In other words, some priced-one marketed claims have a current price which should be modified more than 100%. Otherwise the lack of compatibility will remain true. This seems to be an important finding because it is reflecting that some marketed claims will be impossible to price correctly with the standard pricing methods. This could explain some empirical caveats affecting the price of several securities, including vanilla options (Bondarenko, 2014). As in the model-independent case, we will analyze some important examples. In particular, we will present a complete analysis involving the Black and Scholes model and the $CVaR$.

The presence of GD may provoke irrational solutions in many classical problems involving prices and risk functions. Section 4 will be devoted to illustrating it with some particular actuarial examples (optimal reinsurance, premium calculation, etc.) and some financial examples (asset allocation, risk management, etc.). This section is merely illustrative, so we will not fully address the solution of the presented caveats. The dimension of the paper would become enormous. Beyond the presented examples, the $\tilde{N}$ measure could apply to address the topics years ago studied by means of the arbitrage measurement, such as market integration, valuation of embedded options, trading systems, etc. Therefore, the GD size measurement may open new research lines in finance, insurance, and those fields related to prices and risks.

The last section presents the main conclusions of the paper.

2 Preliminaries and Notations

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. Denote by $\mathbb{E}(y)$

the mathematical expectation of every $\mathbb{R}-$valued random variable y defined on Ω. Denote by L^2 the Hilbert space of random variables y on Ω such that $\mathbb{E}\left(y^2\right) < \infty$, endowed with the inner product $(x,y) \rightarrow \mathbb{E}(xy)$ and norm $\|y\|_2 = \left(\mathbb{E}\left(y^2\right)\right)^{1/2}$. Let $[0,T]$ be a time interval. From an intuitive point of view, one can interpret that $y \in L^2$ represents the portfolio pay-off at T for some arbitrary investor (finance), or claims within $[0,T]$ for some arbitrary insurer (insurance). Throughout this paper y will represent the random wealth at T, although other interpretations would not modify our main conclusions. If $\rho : L^2 \longrightarrow \mathbb{R}$ is a risk measure then $\rho(y)$ may be understood as the "risk" associated with the wealth y. Let us assume that ρ satisfies a representation theorem in the line of Artzner *et al.* (1999) or Rockafellar *et al.* (2006). More precisely, consider the sub-gradient of ρ

$$\Delta_\rho = \left\{z \in L^2; -\mathbb{E}(yz) \leq \rho(y), \forall y \in L^2\right\} \subset L^2 \tag{1}$$

composed of those linear expressions lower than ρ. Δ_ρ will be convex and weakly$-$compact (Schaeffer, 1970) and ρ will be its envelope, in the sense that

$$\rho(y) = Max \ \left\{-\mathbb{E}(yz); \ z \in \Delta_\rho\right\} \tag{2}$$

will hold for every $y \in L^2$. Furthermore, we will also assume that

$$\{1\} \subset \Delta_\rho \subset \left\{z \in L^2; \ \mathbb{E}(z) = 1\right\} \tag{3}$$

and

$$\Delta_\rho \subset \left\{z \in L^2; \ \mathbb{P}(z \geq 0) = 1\right\}. \tag{4}$$

These assumptions are equivalent to the usual properties of norm-continuity, sub-additivity, homogeneity, mean dominance, translation invariance and monotonicity. To sum up, we have:

Assumption 1 $\rho : L^2 \longrightarrow \mathbb{R}$ *is norm-continuous, sub-additive* ($\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$ *if* $y_1, y_2 \in L^2$*), homogeneous* ($\rho(\alpha y) = \alpha\rho(y)$ *if* $y \in L^2$ *and* $\alpha \geq 0$*), mean dominating* ($\rho(y) \geq -\mathbb{E}(y)$ *if* $y \in L^2$*), translation invariant* ($\rho(y+k) = \rho(y) - k$ *if* $y \in L^2$ *and* $k \in \mathbb{R}$*) and decreasing* ($\rho(y_1) \leq \rho(y_2)$ *if* $y_1, y_2 \in L^2$ *and* $\mathbb{P}(y_1 - y_2 \geq 0) = 1$*).* □

Consider a closed sub-space $Y \subset L^2$ of reachable pay-offs. There are many cases included. For instance, we can consider that there exists a set $\mathcal{T} \subset [0,T]$ of trading dates, a filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ such that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$, and a $\mathbb{R}^{m+1}-$valued adapted price process $S = (S_0, S_1, ..., S_m)$ such that every $y \in Y$ is a marketed claim (or a final wealth replicated by means of a self-financing portfolio adapted to the filtration). As a second example, we can deal with a static approach such that $\mathcal{T} = \{0, T\}$ and Y is a finite-dimensional space generated by $m+1$ securities $\{S_0, S_1, ..., S_m\} \subset L^2$ available in the market. Consider also a linear and continuous pricing rule $\Pi : Y \longrightarrow \mathbb{R}$ providing us with the price $\Pi(y)$ of every $y \in Y$ at $t = 0$. Under the first framework above

$\Pi(y)$ will coincide with the initial price of the self-financing portfolio leading to the pay-off y (notice that the absence of arbitrage implies that two adapted and self-financing portfolios leading to the same pay-off will have the same initial price). Under the second framework we can consider that $\Pi(y)$ is just a trivial linear expression of the initial prices of the available assets. We will assume the existence of a riskless asset ($1 \in Y$) and a null interest rate, *i.e.*,

$$\Pi(1) = 1. \tag{5}$$

Obviously, these assumptions are not at all restrictive. In particular, (5) can be easily achieved by the usual normalization method.

Bearing in mind the properties of ρ (Assumption 1) and Π, the proof of Proposition 1 below becomes trivial.

Proposition 1 *The following statements are equivalent;*

a) There exists a sequence $(y_n)_{n=1}^{\infty} \subset Y$ such that $\Pi(y_n) \leq 0$, $n = 1, 2, ...$ and $Lim_{n\to\infty}\rho(y_n) = -\infty$.

b) For every $a \in \mathbb{R}$ there exists a sequence $(y_n)_{n=1}^{\infty} \subset Y$ such that $\Pi(y_n) \leq a$, $n = 1, 2, ...$ and $Lim_{n\to\infty}\rho(y_n) = -\infty$.

c) There exists a sequence $(y_n)_{n=1}^{\infty} \subset Y$ such that $\rho(y_n) \leq 0$, $n = 1, 2, ...$ and $Lim_{n\to\infty}\Pi(y_n) = -\infty$.

d) For every $a \in \mathbb{R}$ there exists a sequence $(y_n)_{n=1}^{\infty} \subset Y$ such that $\rho(y_n) \leq a$, $n = 1, 2, ...$ and $Lim_{n\to\infty}\Pi(y_n) = -\infty$.

e) There exists a sequence $(y_n)_{n=1}^{\infty} \subset Y$ such that $Lim_{n\to\infty}\rho(y_n) = -\infty$ and $Lim_{n\to\infty}\Pi(y_n) = -\infty$.. □

Let us introduce the notion of "compatibility" of Balbás and Balbás (2009).

Definition 2 *The couple (ρ, Π) will be said to be non-compatible if a), b), c), d) or e) above hold.* □

Remark 3 *Suppose that (ρ, Π) is non-compatible. Consider the sequence*

$$(y_n)_{n=1}^{\infty} \subset Y$$

of Proposition 1a. The price of the sequence $(y_n - \Pi(y_n) + 1)_{n=1}^{\infty}$ remains equal to one (see (5)), i.e.,

$$\Pi(y_n - \Pi(y_n) + 1) = 1, \; n = 1, 2, ... \tag{6}$$

The risk function satisfies (see Assumption 1)

$$\rho(y_n - \Pi(y_n) + 1) = \rho(y_n) + \Pi(y_n) - 1 \leq \rho(y_n) \to -\infty. \tag{7}$$

Bearing in mind that ρ is mean-dominating, the expected value of $y_n - \Pi(y_n) + 1$ satisfies

$$\mathbb{E}(y_n - \Pi(y_n) + 1) \geq -\rho(y_n - \Pi(y_n) + 1) \to +\infty. \tag{8}$$

Combining (6), (7) and (8) we have a sequence of investment strategies whose risk goes to minus infinity while its expected return goes to plus infinity. □

The "pathology" presented in Remark 3 is not at all strange in asset pricing. As illustrated by Balbás *et al.* (2016a), the most important arbitrage-free pricing models (Black and Scholes, Heston, etc.) reflect this anti-intuitive behavior when combined with the most important coherent risk measures ($CVaR$, weighted $CVaR$, etc.) or the VaR risk measure (despite the fact that the VaR does not satisfy Assumption 1). Furthermore, this caveat may also arise if one incorporates ambiguity to the pricing model (*i.e.*, $\mathbb{P}$ is not perfectly known) and deals with robust risk measures (Balbás *et al.,* 2016b). Henceforth, strategies reflecting the pathology above will be called good deals in this paper.

Definition 4 *The sequence* $(y_n)_{n=1}^{\infty} \subset Y$ *is said to be a GD if*

$$\begin{cases} \Pi(y_n) = 1, & n = 1, 2, ... \\ \mathbb{E}(y_n) \to +\infty \\ \rho(y_n) \to -\infty \end{cases} \tag{9}$$

hold. □

Remark 5 *Bearing in mind Remark* 3*, it is obvious that* (ρ, Π) *is compatible if and only if there is no* GD*.* □

3 Measuring the good deal size

A critical assumption in financial theory is the absence of arbitrage in real markets and asset pricing models. Since real market data sometimes reflect the existence of arbitrage, a major topic in finance was the measurement of the arbitrage size (Prisman, 1986, Davis *et al.,* 1993, Kamara and Miller, 1995, Chen and Knez, 1995, Kempf and Korn, 1998, etc.). This allowed the authors to address several interesting questions such as pricing and hedging issues under transaction costs, cross-market arbitrage, integration between markets, trading systems, etc. Similarly, the existence of GD (or the lack of compatibility) must be measured, because in some sense it is indicating a lack of balance between the risk that the investor is facing and the wealth that he/she is expecting. As we will see, these unbalanced situations may lead to wrong decisions in several fields. For instance, investors could pay expensive prices or compose inefficient portfolios, and insurers could buy non-optimal reinsurance contracts or receive too cheap premiums.

If we focus again on the arbitrage measurement, we will conclude that there were different approaches. Some of them were related to the fundamental theorems of asset pricing (Chen and Knez, 1995), others were justified by means of micro-structure models (Kempf and Korn, 1998), etc. The methodology of Balbás *et al.* (1999) and (2000) was related to the profits generated by the arbitrageur. We will be inspired by this approach in order to measure the GD size, since it will enable us to measure in monetary terms.

If an arbitrage strategy is available and we do not impose any constraint, then it is easy to prove that the absolute available arbitrage profit becomes unbounded. For that reason Balbás *et al.* (1999) measured in relative terms, or by mean of ratios. This caveat also applies when measuring the GD size. Indeed, Proposition 1 shows that for negative prices one can construct strategies whose risk goes to minus infinity (Proposition 1a), while for negative risks one can obtain "infinite profits" (Proposition 1c). Hence, we will give relative measures as well. More accurately, we will measure with respect to the market value of the sold assets or, equivalently, we will impose a short position lower than one dollar.

Remark 6 *Since both ρ and Π are positively homogeneous, the existence of a strategy $y \in Y$ such that $\Pi(y) \leq 0$ and $\rho(y) < 0$ will imply that $\Pi(\alpha y) \leq 0$ and $Lim_{\alpha \to +\infty} \rho(ay) = -\infty$, and the caveat of Proposition 1a will hold. Therefore, the fulfillment of the implication*

$$y \in Y, \ \Pi(y) \leq 0 \Longrightarrow -\rho(y) \leq 0 \tag{10}$$

is a necessary and sufficient condition to prevent the existence of GD. □

3.1 Market data linked measures

In the first approach we will consider a finite set of available securities

$$\{S_0, S_1, ..., S_m\} \subset L^2,$$

$S_0 = 1$ denoting the riskless asset. We will assume that $\{S_0, S_1, ..., S_m\}$ are linearly independent,[1] and their current prices $p_0 = 1, p_1, ..., p_m$ are observable in the market. In order to prevent some mathematical problems, along with Assumption 1, in Section 3.1 we will impose Assumption 2 below;

Assumption 2 $\mathbb{P}(S_j \geq 0) = 1$, $j = 1, 2, ..., m$. *Consequently, the absence of arbitrage implies that $p_j > 0$, $j = 1, 2, ..., m$.* □

The closed sub-space $Y \subset L^2$ will be the linear manifold generated by the $m + 1$ available assets, and the pricing rule Π will be the obvious one,

$$\Pi\left(\sum_{j=0}^{m} y_j S_j\right) = \sum_{j=0}^{m} y_j p_j. \tag{11}$$

Our measure $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right)$ of the GD size will be the optimal value of the optimization problem

$$Max \ -\rho\left(\sum_{j=0}^{m} (x_j - y_j) S_j\right) \quad \begin{cases} \sum_{j=0}^{m} p_j y_j \leq 1 \\ \sum_{j=0}^{m} (x_j - y_j) p_j \leq 0 \\ x_j, y_j \geq 0, \qquad j = 0, 1, ..., m \end{cases} \tag{12}$$

[1] *i.e.*, there are no non-trivial linear combinations leading to the null asset, or, equivalently, the range of the covariance matrix of $\{S_1, S_2, ..., S_m\}$ equals m.

$\left((x_j)_{j=0}^m, (y_j)_{j=0}^m\right) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ being the decision variable. The interpretation of (12) is as follows. Every portfolio $x - y = (x_j - y_j)_{j=0}^m$ is represented by the vector of purchases $x = (x_j)_{j=0}^m$ and the vector of sales $y = (y_j)_{j=0}^m$. The first constraint imposes a short position lower than one dollar (as justified above) and the second one imposes a non-positive global price. Thus, if the desired implication (10) held, then the objective function could not be positive, and the objective maximum value would be reached at $x = y = 0$ and would equal $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) = 0$. The failure of (10) would lead to a positive value of $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right)$. More accurately, we have;

Proposition 7 *Problem (12) is bounded and solvable, with an optimal value* $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) \geq 0$. *Furthermore,* (ρ, Π) *is compatible (or GD free, Remark* 5*) if and only if* $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) = 0$.

Proof. The objective function is obviously continuous (see Assumption 1) and the feasible set is obviously bounded (and therefore compact) because every p_j is positive. Hence, (12) is solvable due to the Weierstrass Theorem. Since $x = y = 0$ satisfies the problem constraints and $\rho(0) = 0$, the inequality $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) \geq 0$ becomes obvious.

Suppose that $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) > 0$. Then, the solution (x^*, y^*) of (12) satisfies

$$\rho\left(\sum_{j=0}^{m}\left(x_j^* - y_j^*\right) S_j\right) < 0$$

and $\sum_{j=0}^m \left(x_j^* - y_j^*\right) p_j \leq 0$, the implication (10) does not hold, and Remark 6 implies that there is *GD*. Conversely, suppose that $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) = 0$ and let us see that (10) will hold. If (10) failed then we could take $y \in Y$ with $\Pi(y) \leq 0$ and $-\rho(y) > 0$. y is a linear combination of $\{S_0, S_1, ..., S_m\}$ so

$$y = \sum_{j=0}^{m}(x_j - y_j) S_j$$

for some $x_j, y_j \geq 0$, $j = 0, 1, 2, ..., m$. If $\sum_{j=0}^m p_j y_j \leq 1$ then

$$\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) \geq -\rho(y) > 0$$

and we have a contradiction. If $\sum_{j=0}^m p_j y_j > 1$ then we could take $x'_j = x_j / (\sum_{j=0}^m p_j y_j)$ and $y'_j = y_j / (\sum_{j=0}^m p_j y_j)$, and we would have the same contradiction because ρ is positively homogeneous. □

Problem (12) is concave. Bearing in mind Assumption 1, (1), (2), (3) and (4), and proceeding as in Balbás and Balbás (2009) or Balbás *et al.* (2010), one can prove the existence of a linear dual problem characterizing the solutions of (12). Hence, let us present the result below whose proof will be omitted because similar ones are available in the cited reference.

Theorem 8 *Consider Problem*

$$Min\ \lambda \begin{cases} p_j \mu - \mathbb{E}\left(S_j z\right) \geq 0, & j = 0, 1, ..., m \\ p_j \left(\mu - \lambda\right) - \mathbb{E}\left(S_j z\right) \leq 0, & j = 0, 1, ..., m \\ \lambda \geq 0,\ \ \mu \geq 0,\ \ z \in \Delta_\rho \end{cases} \tag{13}$$

$(\lambda, \mu, z) \in \mathbb{R} \times \mathbb{R} \times \mathsf{L}^2$ *being the decision variable.*

a) Problem (13) is bounded and solvable, and the optimal values of (12) and (13) coincide.

b) Suppose that (x^*, y^*) *is (12)-feasible and* (λ^*, μ^*, z^*) *is (13)-feasible. Then,* (x^*, y^*) *solves (12) and* (λ^*, μ^*, z^*) *solves (13) if and only if the complementary slackness conditions below*

$$\begin{cases} \sum_{j=0}^{m} \left(x_j^* - y_j^*\right) \mathbb{E}\left(S_j z\right) \geq \sum_{j=0}^{m} \left(x_j^* - y_j^*\right) \mathbb{E}\left(S_j z^*\right), \quad \forall z \in \Delta_\rho \\ \lambda^* \left(1 - \sum_{j=0}^{m} p_j y_j^*\right) = 0 \\ \mu^* \left(\sum_{j=0}^{m} \left(x_j^* - y_j^*\right) p_j\right) = 0 \\ x_j^* \left(p_j \mu^* - \mathbb{E}\left(S_j z^*\right)\right) = 0, & j = 0, 1, ..., m \\ y_j^* \left(\mathbb{E}\left(S_j z^*\right) - \left(\mu^* - \lambda^*\right) p_j\right) = 0, & j = 0, 1, ..., m \end{cases} \tag{14}$$

hold. $\square$.

Corollary 9 *Consider Problem*

$$Min\ \lambda \begin{cases} \mu - \lambda \leq \mathbb{E}\left(\dfrac{S_j}{p_j} z\right) \leq \mu, & j = 0, 1, ..., m \\ 0 \leq \mu - \lambda \leq 1,\ \ 1 \leq \mu,\ \ z \in \Delta_\rho \end{cases} \tag{15}$$

a) Problem (15) is bounded and solvable, and the optimal values of (12) and (15) coincide.

b) Suppose that (x^*, y^*) *is (12)-feasible and* (λ^*, μ^*, z^*) *is (15)-feasible. Then,* (x^*, y^*) *solves (12) and* (λ^*, μ^*, z^*) *solves (15) if and only if the complementary* *slackness conditions below*

$$\left\{\begin{array}{ll}\sum_{j=0}^{m}\left(x_j^*-y_j^*\right)\mathbb{E}\left(S_j z\right)\geq\sum_{j=0}^{m}\left(x_j^*-y_j^*\right)\mathbb{E}\left(S_j z^*\right), & \forall z\in\Delta_\rho\\ \lambda^*\left(1-\sum_{j=0}^{m}p_j y_j^*\right)=0 & \\ \sum_{j=0}^{m}p_j x_j^*=\sum_{j=0}^{m}p_j y_j^* & \\ x_j^*\left(\mu^*-\mathbb{E}\left(\frac{S_j}{p_j}z^*\right)\right)=0, & j=0,1,...,m\\ y_j^*\left(\mathbb{E}\left(\frac{S_j}{p_j}z^*\right)-\left(\mu^*-\lambda^*\right)\right)=0, & j=0,1,...,m\end{array}\right. \qquad (16)$$

hold.

Proof. Indeed, if (λ,μ,z) is (13)-feasible and $\lambda>\mu$ then (μ,μ,z) is feasible too (see (4)) and the objective function decreases, so the constraint $\mu-\lambda\geq 0$ will not be at all restrictive. Besides, (3) along with the first constraint of (13) for $j=0$ trivially lead to $\mu\geq 1$. Moreover, (3) along with the second constraint of (13) for $j=0$ trivially lead to $\mu-\lambda\leq 1$. Lastly, $\mu\geq 1$ implies that the third condition in (14) is equivalent to the third one in (16). □

Corollary 10 (ρ,Π) *is compatible if and only if there exists* $z^*\in\Delta_\rho$ *such that* $\mathbb{E}\left(\frac{S_j}{p_j}z^*\right)=1,\ j=0,1,2,...,m.$

Proof. Indeed, If (ρ,Π) is compatible then Proposition 7 shows that the solution (λ^*,μ^*,z^*) of (15) satisfies $\lambda^*=0$. Thus, the constraints of (15) imply that $\mathbb{E}\left(\frac{S_j}{p_j}z^*\right)=\mu^*,\ j=0,1,2,...,m$. In particular, for $j=0$ we have (see (3)) $1=\mathbb{E}\left(z^*\right)=\mu^*$.

Conversely, suppose that the existence of $z^*\in\Delta_\rho$ holds. Then, take $(x^*,y^*)=(0,0)$ and $(\lambda^*,\mu^*,z^*)=(0,1,z^*)$, and it is easy to verify that they are feasible and satisfy (16), so the optimal value of (15) will become

$$\tilde{N}\left(\rho,\left(S_j\right)_{j=0}^{m},\left(p_j\right)_{j=0}^{m}\right)=\lambda^*=0,$$

and Proposition 7 shows that (ρ,Π) is compatible. □

If (ρ,Π) is not compatible one could try to modify $(p_j)_{j=0}^{m}$ so as to recover compatibility. According to the latter corollary, if (λ^*,μ^*,z^*) is the solution of (15), $p_j^*=\mathbb{E}\left(S_j z^*\right)$, $j=0,1,2,...,m$ could be a good alternative. Next, let us show that, in some sense, this is "the best alternative", since it minimizes the maximum relative (or per dollar) price modification.

Corollary 11 *Consider a dual solution* (λ^*, μ^*, z^*) *and take* $p_j^* = \mathbb{E}(S_j z^*)$, $j = 0, 1, 2, ..., m$. *Suppose that*

$$p_j^* > 0, \ j = 0, 1, ..., m.^2 \tag{17}$$

Then;

a) $p_0^* = 1$, *and the riskless rate remains the same if* $\left(p_j^*\right)_{j=0}^m$ *replaces* $(p_j)_{j=0}^m$.

b) $\tilde{N}\left(\rho, (S_j)_{j=0}^m, \left(p_j^*\right)_{j=0}^m\right) = 0$. *Moreover, if* Π^* *replaces* Π *and* $\left(p_j^*\right)_{j=0}^m$ *replaces* $(p_j)_{j=0}^m$ *in (11), we will have that* (ρ, Π^*) *is compatible.*

c)

$$\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) = Max \left\{\frac{p_j^*}{p_j} - \frac{p_i^*}{p_i}; i, j = 0, 1, ..., m\right\}. \tag{18}$$

In particular, $\left(p_j^*\right)_{j=0}^m = (p_j)_{j=0}^m$ *if and only if* (ρ, Π) *is compatible.*

d) Consider an arbitrary $\left(p_j^{**}\right)_{j=0}^m \in \mathbb{R}^{m+1}$. *If* $p_0^{**} = 1$, $p_j^{**} > 0$, $j = 0, 1, ..., m$, *and* $\tilde{N}\left(\rho, (S_j)_{j=0}^m, \left(p_j^{**}\right)_{j=0}^m\right) = 0$ *, then*

$$\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) \leq Max \left\{\frac{p_j^{**}}{p_j} - \frac{p_i^{**}}{p_i}; i, j = 0, 1, ..., m\right\}.$$

Proof. *a)* It trivially follows from (3).

b) It trivially follows from Corollary 10.

c) As in the proof of Corollary 10, if $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) = 0$ then $p_j = \mathbb{E}(S_j z^*) = p_j^*$, $j = 0, 1, 2, ..., m$, and therefore the right hand side of (18) equals zero too. Suppose that $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) > 0$. Consider the solutions (x^*, y^*) and (λ^*, μ^*, z^*) of (12) and (15). Obviously, $(x^*, y^*) \neq (0, 0)$, and the second and third conditions of (16) imply that $x^* \neq 0$ and $y^* \neq 0$. If $x_{j_1}^* > 0$ and $y_{j_2}^* > 0$ then (16) implies that

$$\mathbb{E}\left(\frac{S_{j_1}}{p_{j_1}} z^*\right) = \mu^*, \qquad \mathbb{E}\left(\frac{S_{j_2}}{p_{j_2}} z^*\right) = (\mu^* - \lambda^*).$$

Hence,

$$\frac{p_{j_1}^*}{p_{j_1}} - \frac{p_{j_2}^*}{p_{j_2}} = \mu^* - (\mu^* - \lambda^*) = \lambda^*.$$

For an arbitrary couple (i, j), and bearing in mind the constraints of (15), we have that

$$\frac{p_j^*}{p_j} - \frac{p_i^*}{p_i} \leq \mu^* - (\mu^* - \lambda^*) = \lambda^*.$$

[2] (17) will hold if $\mathbb{P}(z^* > 0) = 1$. Analogously, if Assumption 2 is replaced by the stronger property $\mathbb{P}(S_j > 0) = 1$, $j = 1, 2, ..., m$, then (17) will hold because $\mathbb{P}(z^* \geq 0) = 1$ due to (4) and $z^* \neq 0$ due to (3). Lastly, bearing in mind the constraints of (15), (17) will also hold if $\mu^* - \lambda^* > 0$.

Thus, the right hand side of (18) equals λ^*. Hence the result becomes obvious because $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) = \lambda^*$ owing to Corollary 9a.

d) Corollary 10 implies the existence of $z^{**} \in \Delta_\rho$ such that $p_j^{**} = \mathbb{E}(S_j z^{**})$, $j = 0, 1, 2, ..., m$. It is obvious that

$$\mu^{**} = Max \left\{ \mathbb{E}\left(\frac{S_j}{p_j} z^{**}\right); j = 0, 1, ..., m \right\} = Max \left\{ \frac{p_j^{**}}{p_j}; j = 0, 1, ..., m \right\}$$

$$\mu^{**} - \lambda^{**} = Min \left\{ \mathbb{E}\left(\frac{S_j}{p_j} z^{**}\right); j = 0, 1, ..., m \right\} = Min \left\{ \frac{p_j^{**}}{p_j}; j = 0, 1, ..., m \right\}$$

make $(\lambda^{**}, \mu^{**}, z^{**})$ (15)-feasible. Therefore, $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) = \lambda^* \leq \lambda^{**} = \mu^{**} - (\mu^{**} - \lambda^{**})$, *i.e.*,

$$\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) \leq$$

$$Max \left\{ \frac{p_j^{**}}{p_j}; j = 0, 1, ..., m \right\} - Min \left\{ \frac{p_j^{**}}{p_j}; j = 0, 1, ..., m \right\}$$

$$= Max \left\{ \frac{p_j^{**}}{p_j} - \frac{p_i^{**}}{p_i}; i, j = 0, 1, ..., m \right\}.$$

□

Remark 12 *Corollary* 11 *may be interpreted in terms of "fair prices". Indeed, denote by* (λ^*, μ^*, z^*) *a solution of (15). If* (ρ, Π) *is non compatible then one can build portfolios with negative risk and zero or negative price (Proposition* 1*). According to (16), this is possible if one properly buys those securities such that* $\mathbb{E}\left(\frac{S_j}{p_j} z^*\right) = \mu^*$ *and sells those ones satisfying* $\mathbb{E}\left(\frac{S_j}{p_j} z^*\right) = \mu^* - \lambda^*$*. In other words, according to the risk measure* ρ*, if* $\mathbb{E}\left(\frac{S_j}{p_j} z^*\right) = \mu^*$ *(*$\mathbb{E}\left(\frac{S_j}{p_j} z^*\right) = \mu^* - \lambda^*$*) then* S_j *is under-priced (over-priced), and, according to Corollary* 11*, the new prices* $p_j^* = \mathbb{E}(S_j z^*)$*,* $j = 0, 1, 2, ..., m$ *will provide us with the lowest relative modification leading to "fair prices" (or* $GD-$*free prices). Notice that* $p_j^* = \mathbb{E}(S_j z^*) = \mu^* p_j \geq p_j$ *if* $x_j^* > 0$ *(*$p_j^* = \mathbb{E}(S_j z^*) = (\mu^* - \lambda^*) p_j \leq p_j$ *if* $y_j^* > 0$*).* □

3.2 Numerical experiment

Let us illustrate the results of Section 3.1 with a very simple example. We will deal an arbitrage free and almost model-independent option market, and will see that some premiums must decrease more than 0.4% in order to prevent the GD existence. Furthermore, the GD will be static, which means that once it is implemented, the portfolio does not have to be rebalanced before the options maturity.[3]

As above, suppose that $S_0 = 1$ is a riskless asset and consider a security S_1 whose behavior is given by a geometric Brownian motion (GBM) with a current price, drift and volatility equaling 1, 1% and 60%, respectively. Consider also a derivative market where European calls can be traded. The unique maturity is 1/4 years (three months), and the available strikes are $\{0.82;\ 0.84;\ 0.86; ...; 1.4\}$, *i.e.*, the lowest one equals 0.82, the highest one equals 1.4, and the increment between two consecutive strikes equals 0.02. Globally, there are 32 available securities (the riskless asset, the underlying asset and 30 European calls). Suppose that the data perfectly fit the Black and Scholes model, *i.e.*, all of the market prices equal the theoretical ones given by the Black and Scholes formula. Accordingly, they become

$$\begin{pmatrix} 0.221151109; & 0.207527141; & 0.194479893; & 0.182013559; & 0.170128799 \\ 0.158822968; & 0.14809037; & 0.137922549; & 0.12830858; & 0.119235385 \\ 0.110688033; & 0.102650044; & 0.095103673; & 0.088030189; & 0.08141012 \\ 0.075223495; & 0.069450051; & 0.064069422; & 0.059061311; & 0.054405635 \\ 0.050082646; & 0.046073045; & 0.042358062; & 0.038919533; & 0.035739953 \\ 0.032802518; & 0.030091156; & 0.027590546; & 0.025286127; & 0.023164098 \end{pmatrix}$$

Obviously, since the Black and Scholes model is arbitrage free, this market is arbitrage free as well. Consider an investor who is also interested in verifying the compatibility between prices above and the $CVaR_\alpha$ risk measure, α being the level of confidence. Suppose that $\alpha = 79\%$. Despite the fact that this investor can verify that the quotations above lead to a constant implied volatility $\sigma = 0.6$, and therefore the data confirm in this case the Black and Scholes model, let us assume that he/she is still very ambiguous with respect to that. Accordingly, he/she will accept deviations between the predictions of the log-normal distribution and the realized value of S_1 in three months. He/she considers that the error between the probabilities of the log-normal distribution and the real probabilities may become 100%. In other words, for every Borel subset $B \subset \mathbb{R}$, the real probability of the event $S_1 \in B$ will be laying within the spread $[0, 2\mathbb{P}(S_1 \in B)]$, where $\mathbb{P}(S_1 \in B)$ is the theoretical probability under log-normality. In such a case, instead the $CVaR_{79\%}$ risk measure, the investor will use the robust $CVaR_{79\%}$ ($RCVaR_{79\%}$). In general,

$$RCVaR_\alpha(y) := Max \left\{ CVaR_{(Q,\alpha)}(y);\ 0 \leq \frac{dQ}{d\mathbb{P}} \leq 2 \right\}, \tag{19}$$

where Q is a $\mathbb{P}-$continuous probability measure and $CVaR_{(Q,\alpha)}(y)$ is the $CVaR_\alpha$ of y under Q. Balbás *et al.* (2016*b*) have shown that the $RCVaR_\alpha(y)$

[3] According to the empirical evidence, the available theoretical arbitrage free pricing models have many problems to match real market prices in active and liquid derivative markets (Bondarenko, 2014). Perhaps, the theoretical models should prevent the existence of GD as well.

above is well defined for every $y \in L^2$, along with the fulfillment of Assumption 1. Moreover the sub-gradient (1) is given by

$$\left\{ z \in L^2;\ 0 \leq \frac{dQ}{d\mathbb{P}} \leq 2,\ 0 \leq z \leq \frac{1}{1-\alpha}\left(\frac{dQ}{d\mathbb{P}}\right),\ \mathbb{E}\left(z\right) = 1 \right\}. \tag{20}$$

It is easy to see that the set above coincides with

$$\left\{ \begin{array}{l} \left\{ z \in L^2;\ 0 \leq z \leq \frac{2}{1-\alpha},\ \mathbb{E}\left(z\right) = 1 \right\} = \\ \left\{ z \in L^2;\ 0 \leq z \leq \frac{1}{1-(1+\alpha)/2},\ \mathbb{E}\left(z\right) = 1 \right\}. \end{array} \right. \tag{21}$$

Since this is the sub-gradient of the $CVaR_{(1+\alpha)/2}$ risk measure (Rockafellar *et al.,* 2006), $RCVaR_{\alpha} = CVaR_{(1+\alpha)/2}$ and the high ambiguity level of this example only implies that the level of confidence must properly increase. In particular, for $\alpha = 79\%$ one has $(1+\alpha)/2 = 89.5\%$, and our investor will verify the compatibility between the given market and the $CVaR_{89.5\%}$ risk measure.

Though the existence of ambiguity only implies a larger level of confidence, it is important to point out that we are dealing with an ambiguous setting. Expression (19) implies "a worst case approach", and therefore if Implication (10) fails for the given market and $RCVaR_{79\%} = CVaR_{89.5\%}$ (and therefore a GD exists, Remark 6), it will fail for every $CVaR_{(Q,79\%)}$, and Q does not have to be known. In this sense, the GD existence is model-independent, and will also hold for models beyond the Black and Scholes one.

In order to verify the existence of GD, we can solve the linear Problem (15), with Δ_ρ given by (21) for $\alpha = 79\%$ (see Anderson and Nash, 1987). The optimal value becomes

$$\tilde{N}\left(\rho, \left(S_j\right)_{j=0}^{m}, \left(p_j\right)_{j=0}^{m}\right) = 0.004203112 \approx 0.42\% \tag{22}$$

and the existence of GD (or the lack of compatibility, Remark 5) is implied by Proposition 7.

Once the lack of compatibility was confirmed, (16) will enable us to give an explicit GD and the list of under-priced (over-priced) securities. In fact, it is easy to check the fulfillment of the information below;

$$\begin{pmatrix} \textbf{Assets\_Sold\_by\_the\_GD} \\ Call\_Strike\_0.96 \\ Call\_Strike\_1.16 \\ Call\_Strike\_1.3 \\ Call\_Strike\_1.36 \\ \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} \textbf{Assets\_Bought\_by\_the\_GD} \\ Riskless\_Asset \\ Call\_Strike\_1 \\ Call\_Strike\_1.06 \\ Call\_Strike\_1.2 \\ Call\_Strike\_1.28 \\ Call\_Strike\_1.34 \\ Call\_Strike\_1.38 \\ Call\_Strike\_1.4 \end{pmatrix}$$

Accordingly, and bearing in mind that every modification of prices preventing the GD existence will conserve the same riskless rate (Corollary 11), the over-priced securities are is the European calls with strikes 0.96, 1.16, 1.3 and 1.36, while the calls of strikes 1, 1.06, 1.2, 1.28, 1.34, 1.38 and 1.4 are under-priced (Remark 12). The solution of (15) gives $\mu^* = 1$ and $\mu^* - \lambda^* = 0.995796888$, so (16), Corollary 11 and Remark 12 allow us to implement the minimum relative modification of prices preventing the GD existence. The price of the seven under-priced calls should remain the same ($\mu^* = 1$), while the price of the four over-priced calls should be multiplied by 0.9958 ($\mu^* - \lambda^* \approx 0.9958$). Thus, in this example $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right)$ gives the relative price variation of the expensive assets (see (22)). Once z^* is known, the rest of prices should decrease according to the results of Corollary 11. We will not address this straightforward modification in order to shorten the exposition.[4]

3.3 Pricing model linked measures

The approach of Section 3.1 has interesting advantages because it applies for real market data and one does not have to impose any assumption beyond the absence of arbitrage. Nevertheless, there are also some drawbacks, since it does not apply for general pricing models. In order to overcome them, we will provide a new $GD-$measure for complete pricing models, *i.e.*, cases such that $Y = L^2$. Interesting examples are, among others, the binomial model and the Black and Scholes model. If the model is incomplete we can often assume that there is an extension of Π to the whole space L^2 which still prevents the absence of arbitrage. This extension, still denoted by Π, implies that this incomplete case also fits in our general framework. The existence of the extension holds, for instance, if the set Ω only contains finitely many states (Harrison and Kreps, 1979). Therefore, cases such as the usual trinomial models are also included in or analysis. If Ω contains infinitely many states then the existence of Π is also possible. For instance, though "formally" stochastic volatility models are incomplete, in practice it is assumed the existence of volatility dependent assets making them complete. Otherwise it would be impossible to use these models

[4] The dual solution z^* is an exotic derivative of S_1, namely,

$$z^* = \begin{cases} 9.523809524, & 0.732390081 < S_1 < 0.756345122 \\ 9.523809524, & 0.771634941 < S_1 < 0.786508847 \\ 9.523809524, & 0.947638208 < S_1 < 0.962000836 \\ 6.0890897, & 1.006619358 < S_1 < 1.022152267 \\ 9.523809524, & 1.071497097 < S_1 < 1.089092088 \\ 1.228932079, & 1.089092088 < S_1 < 1.107403466 \\ 9.091953283, & 1.157082562 < S_1 < 1.178912787 \\ 5.882406772, & 1.227117704 < S_1 < 1.25412988 \\ 3.889725707, & 1.316447058 < S_1 < 1.353379522 \\ 2.694707294, & 1.373837141 < S_1 < 1.420023944 \\ 2.634172493, & 1.420023944 < S_1 < 1.446520681 \\ 2.732181867, & 1.446520681 < S_1 < 1.509427691 \\ 0.923565011, & 1.509427691 < S_1 < 1.54799097 \\ 2.472733554, & 1.54799097 < S_1 < 1.593797149 \\ 5.22380624, & 1.593797149 < S_1 < 1.650491631 \\ 5.386339925, & 1.837722647 < S_1 < 2.085596933 \\ 0, & Otherwise \end{cases}$$

so as to give a unique price of the usual derivatives. Further details about the existence of Π under general conditions for Ω may be found in Luenberger (2001).

The Riesz representation theorem (Schaeffer, 1970) implies the existence of a unique $z_\Pi \in L^2$ such that

$$\Pi(y) = \mathbb{E}(y z_\Pi) \tag{23}$$

holds for every $y \in L^2$. z_Π is usually called SDF, and it must satisfy

$$\mathbb{P}(z_\Pi > 0) = 1 \tag{24}$$

in order to prevent the arbitrage (Duffie,1988). Furthermore, (23) and (5) trivially imply that

$$\mathbb{E}(z_\Pi) = 1. \tag{25}$$

In order to prevent some mathematical problems, along with Assumption 1, in Sections 3.3 and 3.4 we will impose Assumption 3 below;

Assumption 3 *There exists* $(\mu, z) \in \mathbb{R} \times \Delta_\rho$ *such that* $\mathbb{P}(z \leq \mu z_\Pi) = 1$. □

The new measure $\tilde{N}(\rho, \Pi)$ of the GD size will be the optimal value of the optimization problem

$$Max \ -\rho(x-y) \left\{ \begin{array}{l} \Pi(y) \leq 1 \\ \Pi(x-y) \leq 0 \\ x, y \in L^2, \ x, y \geq 0 \end{array} \right. \tag{26}$$

Obviously, (26) is always feasible and $\tilde{N}(\rho, \Pi) \geq 0$ because $(x, y) = (0, 0)$ satisfies the required constraints. Next, let us give a main result whose proof is similar to those of Theorem 8 and Corollary 9.

Theorem 13 *Consider Problem*

$$Min \ \lambda \left\{ \begin{array}{l} (\mu - \lambda) z_\Pi \leq z \leq \mu z_\Pi \\ 0 \leq \mu - \lambda \leq 1, \ 1 \leq \mu, \ z \in \Delta_\rho \end{array} \right. \tag{27}$$

$(\lambda, \mu, z) \in \mathbb{R} \times \mathbb{R} \times \mathsf{L}^2$ *being the decision variable.*

a) Problem (27) is feasible, and the optimal values of (26) and (27) coincide.

b) Suppose that (x^*, y^*) *is (26)-feasible and* (λ^*, μ^*, z^*) *is (27)-feasible. Then,* (x^*, y^*) *solves (26) and* (λ^*, μ^*, z^*) *solves (27) if and only if the complementary slackness conditions below*

$$\left\{ \begin{array}{l} \mathbb{E}((x^* - y^*) z) \geq \mathbb{E}((x^* - y^*) z^*), \quad \forall z \in \Delta_\rho \\ \lambda^* (1 - \mathbb{E}(y^* z_\Pi)) = 0 \\ \mathbb{E}((x^* - y^*) z_\Pi) = 0 \\ x^* (\mu^* z_\Pi - z^*) = 0 \\ y^* (z^* - (\mu^* - \lambda^*) z_\Pi) = 0 \end{array} \right. \tag{28}$$

hold. □

Corollary 14 *The statements below are equivalent;*

a) (ρ, Π) *is compatible.*

b) $\tilde{N}(\rho, \Pi) = 0$.

c) $(x^*, y^*) = (0, 0)$ *solves (26).*

d) The solution (λ^*, μ^*, z^*) *of (27) satisfies* $\lambda^* = 0$.

e) The solution (λ^*, μ^*, z^*) *of (27) satisfies* $\mu^* = 1$.

f) The solution (λ^*, μ^*, z^*) *of (27) satisfies* $z^* = z_\Pi$.

Proof. *a)* $\Rightarrow$ *b)* If (ρ, Π) is compatible then Implication (10) holds. Thus, if (x, y) is (26)-feasible the second problem constraint implies that $-\rho(x - y) \leq 0$. Since $(0, 0)$ is (26)-feasible, *b)* becomes obvious.

b) $\Rightarrow$ *c)* $(0, 0)$ is (26)-feasible, and $-\rho(0) \geq 0$, so $(0, 0)$ solves (26) when the optimal objective equals 0.

c) $\Rightarrow$ *d)* If $(x^*, y^*) = (0, 0)$ solves (26) then the optimal value of (27) will vanish.

d) $\Rightarrow$ *e)* If the optimal value of (27) vanishes, then the first constraint will lead to $z^* = \mu^* z_\Pi$. Taking expectations, and bearing in mind (3) and (25), we have that $\mu^* = 1$.

e) $\Rightarrow$ *f)* If $\mu^* = 1$, then the first constraint of (27) implies that $z^* \leq z_\Pi$. Since both random variables have the same expectation (see (3) and (25)), we have that $z^* = z_\Pi$.

f) $\Rightarrow$ *a)* Suppose that $z^* = z_\Pi$. It is very easy to verify that $(x^*, y^*) = (0, 0)$ and $(\lambda^*, \mu^*, z^*) = (0, 1, z_\Pi)$ are feasible and satisfy (28). If $(x^*, y^*) = (0, 0)$ solves (26), then (10) must hold, and therefore (ρ, Π) must be compatible (see Remark 6). Indeed, if (10) failed because $\Pi(y) \leq 0$ and $\rho(y) < 0$ for some $y \in L^2$, then $(x, y) = (y^+, y^-)$ impede $(0, 0)$ to be a solution of (26). $\square$

Corollary 10 has a "parallel" result in the new framework.

Corollary 15 (ρ, Π) *is compatible if and only if* $z_\Pi \in \Delta_\rho$.

Proof. If (ρ, Π) is compatible then Corollary 14*f* implies that $z_\Pi \in \Delta_\rho$. Conversely, if $z_\Pi \in \Delta_\rho$ then the proof of the implication *f)* $\Rightarrow$ *a)* in Corollary 14 applies again. $\square$

Remark 16 *Corollary* 14 *shows that the lack of compatibility often holds. For instance, if* ρ *is the* $CVaR$ *then every element in* Δ_ρ *is essentially bounded (see (21)), and therefore* ρ *will not be compatible with any pricing model whose* SDF *is unbounded (Black and Scholes, stochastic volatility models in continuous time, etc.). This result was already pointed out by Balbás et al. (2016a) and others with different proofs. With a similar argument one can show that the weighted* $CVaR$ *(Rockafellar et al., 2006) and the robust* $CVaR$ *(Balbás et al., 2016b) are often non compatible with the usual continuous time pricing models of financial economics.* $\square$

Next let us show that $\tilde{N}(\rho, \Pi)$ may be understood as a "minimum relative (per dollar) price modification" preventing the existence of GD. In order words, let us give a result similar to Corollary 11.

Corollary 17 *Consider a solution* (λ^*, μ^*, z^*) *of (27), suppose that* $\mathbb{P}(z^* > 0) = 1$*, and take* $\Pi^*(y) = \mathbb{E}(yz^*)$ *for every* $y \in L^2$*. Then;*

a) $\Pi^*(1) = 1$*, and the riskless rate remains the same if* Π^* *replaces* Π*.*

b) $\tilde{N}(\rho, \Pi^*) = 0$*. Thus, if* Π^* *replaces* Π *then* (ρ, Π^*) *is compatible.*

c)

$$\tilde{N}(\rho, \Pi) \geq Sup \ \{\Pi^*(x) - \Pi^*(y);\ x, y \geq 0,\ \Pi(x) = \Pi(y) = 1\}, \tag{29}$$

and the equality holds if (26) is solvable.[5] *In particular,* $\Pi^* = \Pi$ *if and only if* (ρ, Π) *is compatible.*

d) If $z^{**} \in L^2$*,* $\mathbb{E}(z^{**}) = 1$*,* $\mathbb{P}(z^{**} > 0) = 1$*,* $\Pi^{**}(y) = \mathbb{E}(yz^{**})$ *for every* $y \in L^2$*, there are no solutions of (27) whose third component is* z^{**}*, and* $\tilde{N}(\rho, \Pi^{**}) = 0$ *, then*

$$\tilde{N}(\rho, \Pi) \leq Sup \ \{\Pi^{**}(x) - \Pi^{**}(y);\ x, y \geq 0,\ \Pi(x) = \Pi(y) = 1\}. \tag{30}$$

Proof. *a)* It trivially follows from (3).

b) If z^* replaces z_Π in (27) then it is obvious that $(\lambda^* = 0, \mu^* = 1, z^*)$ becomes (27)-feasible, and therefore $\tilde{N}(\rho, \Pi^*) = 0$.

c) As in the proof of Corollary 14, if $\tilde{N}(\rho, \Pi) = 0$ then $z^* = z_\Pi$, and therefore $\Pi^* = \Pi$ and the right hand side of (29) equals zero too. Suppose that $\tilde{N}(\rho, \Pi) > 0$. Take $x, y \geq 0$ with $\Pi(x) = \Pi(y) = 1$. The constraints of (27) imply that

$$\begin{array}{l} \mu^* - \lambda^* = (\mu^* - \lambda^*)\,\mathbb{E}(z_\Pi y) \leq \mathbb{E}(z^* y) \leq \mu^* \mathbb{E}(z_\Pi y), \\ (\mu^* - \lambda^*)\,\mathbb{E}(z_\Pi x) \leq \mathbb{E}(z^* x) \leq \mu^* \mathbb{E}(z_\Pi x) = \mu^*. \end{array}$$

Consequently,

$$\mathbb{E}(z^* x) - \mathbb{E}(z^* y) \leq \mu^* - (\mu^* - \lambda^*) = \lambda^* = \tilde{N}(\rho, \Pi).$$

Moreover, if (x^*, y^*) solves (26), the second, third, fourth and fifth equalities in (28) lead to (recall that $\lambda^* > 0$)

$$\begin{array}{l} \mu^* - \lambda^* = (\mu^* - \lambda^*)\,\mathbb{E}(z_\Pi y^*) = \mathbb{E}(z^* y^*), \\ \mathbb{E}(z^* x) = \mu^* \mathbb{E}(z_\Pi x) = \mu^*. \end{array}$$

Thus, $\mathbb{E}(z^* x) - \mathbb{E}(z^* y^*) = \mu^* - (\mu^* - \lambda^*) = \lambda^* = \tilde{N}(\rho, \Pi)$.

d) Suppose that (λ, μ, z^{**}) is never (27)-feasible for $0 \leq \mu - \lambda \leq 1$ and $\mu \geq 1$. Then, for every $\mu \geq 1$ the inequality $z^{**} \leq \mu z_\Pi$ will not hold, because if it held then $\lambda = \mu$ would make $(\lambda = \mu, \mu, z^{**})$ (27)-feasible. Thus, for every $\mu \geq 1$ there

[5] We will see that (26) is not necessarily solvable, *i.e.*, it does not necessarily attain its optimal value. This is a difference between Problems (12) and (26) (Proposition 7 and Theorem 19).

exists $x_\mu \geq 0$ in L^2 such that $\mathbb{E}\left(z^{**}x_\mu\right) > \mu\mathbb{E}\left(z_\Pi x_\mu\right)$. Moreover, $\mathbb{E}\left(z_\Pi x_\mu\right) > 0$ due to (24). Replacing x_μ with $x_\mu / \mathbb{E}\left(z_\Pi x_\mu\right)$ if necessary, and still denoting x_μ, one can suppose that $\mathbb{E}\left(z_\Pi x_\mu\right) = 1$ and $\mathbb{E}\left(z^{**}x_\mu\right) > \mu$. Taking $y_\mu = 1$ (riskless security) we have

$$\mathbb{E}\left(z^{**}x_\mu\right) - \mathbb{E}\left(z^{**}y_\mu\right) \geq \mu - 1,$$

which tends to $+\infty$ as so does μ. Hence, the right hand side of (30) is unbounded and (30) becomes obvious.

Suppose that (λ, μ, z^{**}) is (27)-feasible for some $0 \leq \mu - \lambda \leq 1$ and $\mu \geq 1$. Take

$$\mu^{**} = Inf \ \left\{\mu \geq 1;\ z^{**} \leq \mu z_\Pi\right\}, \tag{31}$$

and it is obvious that

$$z^{**} \leq \mu^{**} z_\Pi. \tag{32}$$

If $\mu^{**} = 1$ then (32) implies that $z^{**} \leq z_\Pi$, and (3) and (25) will imply that $z^{**} = z_\Pi$. Whence, $(\lambda^{**} = 0, \mu^{**} = 1, z^{**} = z_\Pi)$ will solve (27), against the assumptions. Thus,

$$\mu^{**} > 1. \tag{33}$$

Take

$$\lambda^{**} = Inf \ \left\{\lambda;\ 0 \leq \mu^{**} - \lambda \leq 1,\ \left(\mu^{**} - \lambda\right) z_\Pi \leq z^{**}\right\}. \tag{34}$$

The set above is non void because it obviously contains $\lambda = \mu^{**}$. Furthermore,

$$0 \leq \mu^{**} - \lambda^{**} \leq 1$$

and

$$\left(\mu^{**} - \lambda^{**}\right) z_\Pi \leq z^{**} \tag{35}$$

obviously hold. Suppose that $\mu^{**} - \lambda^{**} = 1$. Then, (35) implies that $z_\Pi \leq z^{**}$, and (3) and (25) imply that $z^{**} = z_\Pi$. Once again we get a contradiction because $(\lambda^{**} = 0, \mu^{**} = 1, z^{**} = z_\Pi)$ will solve (27), and therefore

$$0 \leq \mu^{**} - \lambda^{**} < 1. \tag{36}$$

Bearing in mind (33) and (36), we can take $\varepsilon > 0$ such that

$$0 \leq \mu^{**} - \left(\lambda^{**} - \varepsilon\right) < 1, \qquad \mu^{**} - \varepsilon > 1.$$

(31) and (34) lead to the existence of $x_\varepsilon, y_\varepsilon \geq 0$ in L^2 such that $\mathbb{E}\left(z^{**}x_\varepsilon\right) > \left(\mu^{**} - \varepsilon\right)\mathbb{E}\left(z_\Pi x_\varepsilon\right)$ and $\mathbb{E}\left(z^{**}y_\varepsilon\right) < \left(\mu^{**} - \left(\lambda^{**} - \varepsilon\right)\right)\mathbb{E}\left(z_\Pi y_\varepsilon\right)$. Therefore, normalizing so that $\Pi\left(x_\varepsilon\right) = \Pi\left(y_\varepsilon\right) = 1$, and still denoting x_ε and y_ε,

$$\begin{aligned}&\mathbb{E}\left(z^{**}x_\varepsilon\right) - \mathbb{E}\left(z^{**}y_\varepsilon\right) > \left(\mu^{**} - \varepsilon\right)\mathbb{E}\left(z_\Pi x_\varepsilon\right) - \left(\mu^{**} - \left(\lambda^{**} - \varepsilon\right)\right)\mathbb{E}\left(z_\Pi y_\varepsilon\right)\\ &= \left(\mu^{**} - \varepsilon\right) - \left(\mu^{**} - \left(\lambda^{**} - \varepsilon\right)\right) = \lambda^{**} - 2\varepsilon.\end{aligned}$$

Moreover, since (32), (33), (35) and (36) make $(\lambda^{**}, \mu^{**}, z^{**})$ (27)-feasible, $\lambda^{**} \geq \lambda^*$ must hold, and therefore $\mathbb{E}\left(z^{**}x_\varepsilon\right) - \mathbb{E}\left(z^{**}y_\varepsilon\right) > \lambda^* - 2\varepsilon = \tilde{N}\left(\rho, \Pi\right) - 2\varepsilon$. If ε converges to zero we will have (30). □

Remark 18 *As in Remark* 12*, one can use the latter corollary so as to recover "fair prices". Indeed, if* Π^* *replaces* Π *then compatibility will hold, the over-priced marketed claims, characterized by*

$$\Pi^*(y) = \mathbb{E}(yz^*) = (\mu^* - \lambda^*)\,\mathbb{E}(yz_\Pi) = (\mu^* - \lambda^*)\,\Pi(y)$$

(see (28)) will recover a "fair price" once the initial one $\Pi(y)$ *is multiplied by* $\mu^* - \lambda^*$*, and the under-priced marketed claims, characterized by*

$$\Pi^*(x) = \mathbb{E}(xz^*) = \mu^*\mathbb{E}(xz_\Pi) = \mu^*\Pi(x)$$

will recover a "fair price" once the initial one $\Pi(x)$ *is multiplied by* μ^**.*

3.4 Lack of compatibility between the CVaR and the Black and Scholes model or other continuous time pricing processes

Bearing in mind Remark 16, it may be interesting to give the value of $\tilde{N}(\rho, \Pi)$ for some important risk measures and pricing models. This is the purpose of this section. Along with Assumptions 1 and 3, in this section we will also impose Assumption 4 below;

Assumption 4 There does not exists any $(\beta, z) \in \mathbb{R} \times \Delta_\rho$ such that $\beta > 0$ and $\mathbb{P}(\beta z_\Pi \leq z) = 1$. $\square$

Assumption 4 frequently holds in practice. For instance, it holds if z_Π is not essentially bounded (Black and Scholes, stochastic volatility, etc.) and Δ_ρ is composed of essentially bounded random variables ($CVaR$, and very often the $RCVaR$ and the weighted $CVaR$, see (20) and (21)). Besides, Assumption 4 enables us to simplify Problem (27).

Theorem 19 *a)* (ρ, Π) *is not compatible.*

b) Consider Problem

$$Min\ \mu \begin{cases} z \leq \mu z_\Pi \\ 1 \leq \mu,\ z \in \Delta_\rho \end{cases} \tag{37}$$

$(\mu, z) \in \mathbb{R} \times \mathsf{L}^2$ *being the decision variable. Then,* (λ^*, μ^*, z^*) *solves (27) if and only if* $\lambda^* = \mu^*$ *and* (μ^*, z^*) *solves (37). Consequently, (37) is bounded and solvable, and its optimal value equals* $\tilde{N}(\rho, \Pi)$*.*

c) If (μ^*, z^*) *solves (37) and* $\mathbb{P}(z^* > 0) = 1$*, then Problem (26) is not solvable, although it is bounded and its optimal value is* $\tilde{N}(\rho, \Pi) > 0$*.*

d) Suppose that $\alpha \in (0,1)$*,* (μ^*, z^*) *is (37)-feasible and* $\rho = CVaR_\alpha$*. Then,* (μ^*, z^*) *solves (37) if and only if*

$$z^*(\omega) = Min\ \left\{\mu^* z_\Pi(\omega),\ \frac{1}{1-\alpha}\right\} \tag{38}$$

out of a $\mathbb{P}-$null set. Furthermore, $\mathbb{P}(z^* > 0) = 1$ and therefore Problem (26) is not solvable.

e) Suppose that $\alpha \in (0,1)$, $(\mu^*, z^*) \in (1,\infty) \times L^2$ and $\rho = CVaR_\alpha$. Then, (μ^*, z^*) solves (37) if and only if

$$\mathbb{E}\left(Min\left\{\mu^* z_\Pi, 1/(1-\alpha)\right\}\right) = 1 \tag{39}$$

and (38) holds.

Proof. *a*) If (ρ, Π) were compatible then Corollary 14 shows that $(\lambda^*, \mu^*, z^*) = (0, 1, z_\Pi)$ would solve (27), and therefore it would be (27)-feasible. Thus, $z_\Pi \in \Delta_\rho$ should hold and $\beta = 1$ would contradict Assumption 4.

b) If (λ, μ, z) is (27)-feasible, then Assumption 4 trivially implies that $\lambda = \mu$. Then, the equivalence between Problems (27) and (37) becomes straightforward and *b*) trivially follows from Theorem 13.

c) Take the solution (λ^*, μ^*, z^*) of (27). Statement *a*) and Corollary 14 imply that $\mu^* > 1$, and Assumption 4 and the constraints of (27) imply that $\mu^* - \lambda^* = 0$. If (x^*, y^*) solved (26) then (28) would imply $y^* z^* = 0$, and $\mathbb{P}(z^* > 0) = 1$ would imply $\mathbb{P}(y^* = 0) = 1$. Notice that $\lambda^* = \mu^* > 1$ and $\mathbb{P}(y^* = 0) = 1$ contradict the second condition of (28), so (26) cannot be solvable. The rest of the proof trivially follows form Theorem 13.

d) Suppose that (μ^*, z^*) solves (37). $z^* \leq \mu^* z_\Pi$ obviously must hold, and $z^* \leq 1/(1-\alpha)$ holds because $z^* \in \Delta_\rho$ (see (20) with $\frac{dQ}{d\mathbb{P}} = 1$). Hence,

$$z^* \leq Min\left\{\mu^* z_\Pi, 1/(1-\alpha)\right\}. \tag{40}$$

Suppose that (40) is not a equality. Then,

$$1 = \mathbb{E}(z^*) < \mathbb{E}\left(Min\left\{\mu^* z_\Pi, 1/(1-\alpha)\right\}\right).$$

Consider $\varepsilon > 0$ with $\mu^* - \varepsilon > 1$ (recall that $\mu^* > 1$ due to *a*) and Corollary 14*e*) and $\mathbb{E}\left(Min\left\{\mu^* z_\Pi, 1/(1-\alpha)\right\}\right) - \varepsilon > 1$. Obviously,

$$\begin{aligned} &Min\left\{\mu^* z_\Pi, 1/(1-\alpha)\right\} - Min\left\{(\mu^* - \varepsilon) z_\Pi, 1/(1-\alpha)\right\} \\ &\leq \mu^* z_\Pi - (\mu^* - \varepsilon) z_\Pi = \varepsilon z_\Pi. \end{aligned}$$

Thus, bearing in mind (25),

$$\mathbb{E}\left(Min\left\{(\mu^* - \varepsilon) z_\Pi, 1/(1-\alpha)\right\}\right) \geq \mathbb{E}\left(Min\left\{\mu^* z_\Pi, 1/(1-\alpha)\right\}\right) - \varepsilon > 1. \tag{41}$$

Since $\mathbb{P}((\mu^* - \varepsilon) z_\Pi > 0) = 1$ due to (24),

$$\mathbb{P}\left(Min\left\{(\mu^* - \varepsilon) z_\Pi, 1/(1-\alpha)\right\} > 0\right) = 1$$

becomes obvious, and $\mathbb{P}\left(Min\left\{(\mu^* - \varepsilon) z_\Pi, 1/(1-\alpha)\right\} \leq 1/(1-\alpha)\right) = 1$ is obvious too. Thus, (41) leads to

$$\frac{Min\left\{(\mu^* - \varepsilon) z_\Pi, 1/(1-\alpha)\right\}}{\mathbb{E}\left(Min\left\{(\mu^* - \varepsilon) z_\Pi, 1/(1-\alpha)\right\}\right)} \in \Delta_\rho,$$

and

$$\frac{Min\left\{\left(\mu^*-\varepsilon\right)z_\Pi,1/\left(1-\alpha\right)\right\}}{\mathbb{E}\left(Min\left\{\left(\mu^*-\varepsilon\right)z_\Pi,1/\left(1-\alpha\right)\right\}\right)}\leq\left(\mu^*-\varepsilon\right)z_\Pi$$

implies that

$$\left(\mu^*-\varepsilon,\frac{Min\left\{\left(\mu^*-\varepsilon\right)z_\Pi,1/\left(1-\alpha\right)\right\}}{\mathbb{E}\left(Min\left\{\left(\mu^*-\varepsilon\right)z_\Pi,1/\left(1-\alpha\right)\right\}\right)}\right)$$

is (37)-feasible. We have a contradiction because (μ^*, z^*) solves (37). Hence (40) is an equality, and (38) holds.

Conversely, if (38) holds and (μ^*, z^*) does not solve (37) then the solution (μ, z) of (37) satisfies $\mu < \mu^*$, and the proved implication leads to

$$z = Min\left\{\mu z_\Pi, 1/\left(1-\alpha\right)\right\}. \tag{42}$$

Bearing in mind (3), we have the chain

$$\begin{aligned}1 = \mathbb{E}\left(z^*\right) = \mathbb{E}\left(Min\left\{\mu^* z_\Pi, 1/\left(1-\alpha\right)\right\}\right)\\ \geq \mathbb{E}\left(Min\left\{\mu z_\Pi, 1/\left(1-\alpha\right)\right\}\right) \geq \mathbb{E}\left(z\right) = 1\end{aligned}$$

and therefore

$$Min\left\{\mu^* z_\Pi, 1/\left(1-\alpha\right)\right\} = Min\left\{\mu z_\Pi, 1/\left(1-\alpha\right)\right\}.$$

Hence, $\mu < \mu^*$ and (24) imply that $\mu z_\Pi \geq 1/\left(1-\alpha\right)$, and (42) leads to $z = 1/\left(1-\alpha\right)$. Therefore, $\mathbb{E}\left(z\right) = 1/\left(1-\alpha\right) > 1$, and we have a contradiction with (3).

e) If (38) and (39) hold then (μ^*, z^*) is (37)-feasible. Therefore, (μ^*, z^*) solves (37) due to *d*).

Conversely, suppose that (μ^*, z^*) solves (37). Then, (38) follows from *d*). Besides, $z^* \in \Delta_\rho$ implies that $\mathbb{E}\left(z^*\right) = 1$, so (38) leads to (39). □

Remark 20 *Theorem* 19 *shows how different are going to be the given measures* $\tilde{N}\left(\rho, \left(S_j\right)_{j=0}^m, \left(p_j\right)_{j=0}^m\right)$ *of Section* 3.1 *and* $\tilde{N}\left(\rho, \Pi\right)$ *of Section* 3.3. *While* $\tilde{N}\left(\rho, \left(S_j\right)_{j=0}^m, \left(p_j\right)_{j=0}^m\right)$ *can take every non negative value, and the null value will sometimes hold,* $\tilde{N}\left(\rho, \Pi\right)$ *will be almost always strictly positive (Theorem* 19*a) and strictly higher than* $1 = 100\%$ *(Theorem* 19*b). In particular, for* $\rho = CVaR_\alpha$*, and bearing in mind Corollary* 17 *and Theorem* 19*d, if the pricing rule is modified so as to prevent the existence of GD, the relative (per dollar) price modification might be larger than* 100% *for some marketed claims. Otherwise the existence of GD could remain true, though it is important to point out that Corollary* 17 *just provide an upper bound, rather that the exact price relative variation. Anyway, (30) justifies that every substitution of* z_Π *must be implemented with a solution of (37) (see also Remark* 22 *below).* □

Remark 21 *Expressions (38) and (39) significantly facilitate the practical computation of* $\left(\mu^* = \tilde{N}\left(\rho, \Pi\right), z^*\right)$ *if* $\rho = CVaR_\alpha$*. In real examples, and according to* 18*d) and* 18*e), the key condition to estimate* μ^* *is the equality*

$$\mathbb{E}\left(Min\left\{\mu^* z_\Pi, 1/\left(1-\alpha\right)\right\}\right) = 1. \tag{43}$$

It seems to be clear that Monte Carlo simulation methods may be useful so as to match (43), though we will not address any numerical experiment in order to shorten the exposition. □

Remark 22 *Let us focus on the Black and Scholes model. Without loss of generality, if one looks for a GD only composed of European style derivatives,*[6] *then one can simplify the structure of the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Indeed, assume that* $\Omega = (0,1)$ *and* $\mathbb{P}$ *is the Lebesgue measure on the Borel* $\sigma-$*algebra of this set. The value at* T *of the underlying asset will have a log-normal distribution which can be given by*

$$S(\omega) = W Exp\ \left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T}\Phi^{-1}(\omega)\right) \tag{44}$$

for $\omega \in (0,1)$*,* $W > 0$ *denoting the current price, and* r *and* σ *denoting drift and volatility. Obviously,* $\Phi : \mathbb{R} \longmapsto (0,1)$ *is the cumulative distribution function of the standard normal distribution.*

This simplification cannot be implemented when pricing path dependent or American style derivatives. In both situations the dynamic evolution of the GBM plays a critical role. Thus, when we choose the simple probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *above we know that we are missing information. However, our simplification is interesting because the exposition is shortened, it becomes much easier, and it provides closed formulas for* z^**. We will still obtain solutions of (26) and (27) that will allow the investor to create the sequences of Proposition 1 or satisfying (9). The only restriction is that our sequences will be composed of European style derivatives and might become sub-optimal if more complex securities were involved.*

It is known that z_π *is also log-normal and it is the first derivative of the one to one strictly increasing function (Wang, 2000)*

$$(0,1) \ni \omega \leftrightarrow g(\omega) = \Phi\left(\gamma + \Phi^{-1}(\omega)\right) \in (0,1), \tag{45}$$

where

$$\gamma = \frac{r}{\sigma}\sqrt{T}. \tag{46}$$

Computing the derivative in (45) we have that

$$z_\Pi(\omega) = Exp\ \left(-\frac{\gamma^2}{2} - \gamma\Phi^{-1}(\omega)\right) \tag{47}$$

$\omega \in (0,1)$*, which easily allows us to verify that* $(0,1) \ni \omega \leftrightarrow z_\Pi(\omega) \in \mathbb{R}$ *is continuous and strictly decreasing. Since is strictly decreasing and* $\mu^* > 1$*, the computation of* $\left(\mu^* = \tilde{N}(\rho, \Pi), z^*\right)$ *simplifies to the estimation of* $p \in (0,1)$ *such that (see (38), (39) and (43))*

$$\frac{p}{1-\alpha} + \frac{1}{(1-\alpha) z_\Pi(p)} \int_p^1 z_\Pi(\omega)\, d\omega = 1. \tag{48}$$

[6] Remark 20 applies for more complex derivatives.

In fact, if one solves (48) then

$$\mu^* = \frac{1}{(1-\alpha) z_\Pi(p)}, \qquad z^*(\omega) = \begin{cases} \dfrac{1}{1-\alpha}, & \omega \le p \\ \\ \mu^* z_\Pi(\omega), & \omega \ge p \end{cases}$$

In order to solve (48) one can change the variable $\omega = \Phi(u-\gamma)$ *in the integral, and straightforward manipulations lead to the new equation*

$$\frac{p}{1-\alpha} + \frac{1}{(1-\alpha) z_\Pi(p)} \Phi\left(-\gamma - \Phi^{-1}(p)\right) = 1, \tag{49}$$

which may be solved with numerical methods. If one solved (49) for the parameters used in Section 3.2*, i.e.,* $\alpha = 89.5\%$*,* $r = 1\%$*,* $\sigma = 60\%$*,* $T = 1/4$ *and (see (46))* $\gamma = 0.007900634$*, the result would satisfy* $\mu^* > 1$*. In Section* 3.2 *we obtained* $\tilde{N}\left(\rho, (S_j)_{j=0}^m, (p_j)_{j=0}^m\right) \approx 0.42\%$*. With the same drift, volatility, expiration date, pricing model and risk measure we can obtain* $\tilde{N}(\rho, \Pi) = \mu^* > 100\% > 0.42\%$*. Obviously, the GD size increases because now we are considering every* $y \in L^2$ *as a reachable pay-off, and in Section* 3.2 *we only dealt with finitely many options. The GD size increases as so does the set of available securities. Nevertheless, the difference*

$$100\% - 0.42\% = 99.58\%$$

is really relevant. According to Corollaries 11 *and* 17*, the minimum relative price modification preventing the GD existence might significantly increase as the number of available options tends to infinity, though it is important to point out that Corollary* 17 *just provide an upper bound, rather that the exact price relative variation. Anyway, (30) and* $\mu^* > 100\%$ *justify that every substitution of* z_Π *must be implemented with a solution of (37) (see Remark* 20 *above).* □

Remark 23 *(Beyond the log-normal distribution) Though Remark* 21 *yields a general enough estimation method, the simplification of Remark* 22 *may be interesting when dealing with European style derivatives. In such a case (44) and (47) are no strictly necessary, and the methodology may be extended beyond the Black and Scholes model. Instead of (44), suppose that* S_1 *is the random value at* T *of a stochastic pricing process. Suppose that the model has been calibrated and we have chosen a unique SDF* z_π *such that (23) applies. Suppose finally that the cumulative distribution function* $F : (U,V) \overset{\prime}{\longrightarrow} (0,1)$ *of the random variable* S_1 *is a one to one continuous bijection for some* $-\infty \le U < V \le \infty$*.*[7] *Then, the simplification (44) may be adapted to this new framework, in the sense that one can take*

$$S_1(\omega) = F^{-1}(\omega),$$

ω *being a uniform distribution on* $(0,1)$*. Moreover,* z_π *may be also understood as a function* $(0,1) \ni \omega \overset{\prime}{\longrightarrow} z_\pi(\omega) \in (0,\infty)$*. This setting allows us to easily extend the methodology of Remark* 22*.* □

[7] This assumption is not at all restrictive. It holds for many continuous distributions (exponential, normal, log-normal, Gamma, Pareto, etc) used in Financial Economics.

4 Some actuarial and financial implications

Many classical problems in finance and insurance deal with risk optimization. This section will be devoted to illustrating how classical problems may become unbounded if one faces lack of compatibility. As a consequence, one must recover compatibility before making decisions. Otherwise the problem solution will not make economic sense or lead to wrong decisions.

We will select a few actuarial and financial problems. This is not at all an exhaustive collection of potential applications, but the purpose of this section is just illustrative; One must prevent the existence of GD. Furthermore, for the same reason, we will not present global solutions of the proposed problems, which would require a significantly larger paper.

4.1 Actuarial examples

Let us focus on a couple of actuarial classical topics. The first one is the optimal reinsurance problem. Since Borch (1960) and Arrow (1963) proved that, under adequate assumptions, the stop-loss contract minimizes the standard deviation of the ceding company final wealth, this problem has been once and once again revisited by many authors. The most recent approaches deal with general risk measures rather than the standard deviation (see, amongst many others, Zhuang *et al.*, 2016, Weng and Zhuang, 2016, etc.), and sometimes also incorporate the effect of the financial market (Guan and Liang, 2014, Peng and Wang, 2016, etc.). Let us point out how the incorporation of the financial market effect may lead to non well-posed optimization problems.

Under the notations of Sections 2 and 3 let us consider that the random variable $u_0 \in L^2$ represents the indemnifications to be paid by a insurer within the time period $[0, T]$. Decompose $u_0 = u_r + u_c$, $u_r, u_c \in L^2$ denoting the retained and ceded risk after a reinsurance contract. If $y \in Y$ is the pay-off provided by the financial market, the insurer final wealth will equal $y - u_r$. Thus, if $C > 0$ represents the capital to diversify between the financial market and the reinsurance contract, the optimal reinsurance problem may become

$$Min\ \rho\left(y-u_r\right) \quad \begin{cases} \Pi\left(y\right)+\left(1+K\right)\mathbb{E}\left(u_0-u_r\right)\leq C \\ 0\leq u_r\leq u \\ y\in Y,\ u_r\in\mathcal{M} \end{cases} \tag{50}$$

(y, u_r) being the decision variable, $K > 0$ denoting the loading rate and $\mathcal{M}$ denoting the set of risks $z \in L^2$ such that z and $u_0 - z$ are co-monotone with u_0.[8] The objective function of (50) implies that the reinsurer prices according to the "expected value premium principle". This assumption may be significantly relaxed and the rest of the example will remain true, but we will present an example as simple as possible because we only have illustrative purposes.

[8] Recall that u_0 and u_1 are co-monotone if

$$\mathbb{P}\left(\left(u_0\left(\omega_1\right)-u_0\left(\omega_2\right)\right)\left(\left(u_1\left(\omega_1\right)-u_2\left(\omega_2\right)\right)\right)\geq 0\right)=1.$$

Proposition 24 *If there is a GD then Problem (50) is unbounded, i.e., there are sequences of feasible decisions whose risk diverges to* $-\infty$.

Proof. Consider the sequence $(y_n)_{n=1}^{\infty} \subset Y$ satisfying (9), and take $(y_n - 1)_{n=1}^{\infty} \subset Y$ and $u_r = u_0 \in \mathcal{M}$. Then $(y_n - 1, u_r)$ is feasible because

$$\Pi(y_n - 1) + (1 + K)\, \mathbb{E}(u_0 - u_r) = 0 \leq C.$$

Moreover, Assumption 1 implies that

$$Lim_{n\to\infty}\rho(y_n - 1 - u_r) \leq \rho(-1 - u_r) + Lim_{n\to\infty}\rho(y_n) = -\infty,$$

and Problem (50) is unbounded. □

Proposition 24 above shows that very important classical actuarial problems do not make sense under the presence of GD. Actually, the provided proof only indicates that the insurer must retain the global actuarial risk because it will be compensated in the financial market.[9]

From a theoretical viewpoint, optimal reinsurance approaches involving both actuarial risk and asset pricing models must deal with GD free models. Otherwise the solution will not exist or will not make sense. A potential solution overcoming this caveat could be to recover the GD absence by modifying the pricing rule according to the lines of Corollary 17.

Approaches involving both actuarial risk and static financial strategies created with real market quotations should also verify the fulfillment of the equality $\tilde{N}\left(\rho, (S_j)_{j=0}^{m}, (p_j)_{j=0}^{m}\right) = 0$. If it does not hold, then some additional restriction should be incorporated to the y variable of (50) in order to prevent unbounded solutions. For instance, one could consider an upper bound to the short position value, in the line of the constraint imposed in (12). Once these additional restrictions have been incorporated, the existence of GD will have positive effects on the reached risk level, in the sense that the minimum value of the objective $\rho(y - u_r)$ will decrease. The global fall of $\rho(y - u_r)$ will be closely related to the value of $\tilde{N}\left(\rho, (S_j)_{j=0}^{m}, (p_j)_{j=0}^{m}\right)$.

As a second actuarial example, suppose that a insurer is computing the loading rate K that a set of policies must pay. Denoting by $(u_i)_{i=1}^{N} \subset L^2$ the random indemnifications associated to the involved policies, the global portfolio premium will become $(1 + K)\sum_{i=1}^{N} \mathbb{E}(u_i)$. If this money in invested in the financial market, and $y \in Y$ is the random reached pay-off, the final insurer wealth will become $y - \sum_{i=1}^{N} u_i$. Thus, the insurer problem may become

$$Min\ K \begin{cases} \Pi(y) - (1 + K)\sum_{i=1}^{N} \mathbb{E}(u_i) \leq 0 \\ \rho\left(y - \sum_{i=1}^{N} u_i\right) \leq 0 \\ y \in Y,\ K \in \mathbb{R} \end{cases} \tag{51}$$

[9] Notice that the independence between the financial market and the global indemnification did not have to be imposed.

(K, y) being the decision variable. As in Proposition 24, and bearing in mind Proposition 1, if (ρ, Π) is non compatible it is easy to see that (51) is unbounded, *i.e.*, there are sequences of feasible solutions provoking that K diverges to $-\infty$. Hence, the comments about the optimal reinsurance problem apply again. The insurer may modify the pricing rule if a pricing model is used, or he/she must incorporate bounds in the short position value when dealing with real market quotations.

4.2 Financial Examples

Since Rockafellar and Uryasev (2000) gave a simple procedure to minimize the $CVaR$, many studies in portfolio choice and asset allocation have extended the classical approach of Markowitz once the standard deviation was replaced by an alternative risk measure or robust risk measure (VaR, $CVaR$, $RCVaR$, etc.). Amongst many others, interesting examples are Stoyanov *et al.* (2007), Haugh and Lo (2001), Dupacová and Kopa (2014), Balbás *et al.* (2016*b*), Zhao and Xiao (2016), etc. Expression (9) shows that the GD existence implies the availability of sequences of investment strategies whose expected return diverges to $+\infty$ while their risk diverges to $-\infty$. Once again we will be facing unbounded problems and theoretical results without economic sense. In Balbás *et al.* (2016*b*) the authors propose to enlarge the ambiguity level of the investor. Alternatively, for non ambiguous agents one could deal with the ideas of this paper. Corollaries 11 and 17 propose ways to modify the pricing rules, while Problems (12) and (26) propose solutions making the problems bounded. Anyway, according to the empirical analysis yielded by Balbás *et al.* (2016*a*), the intuition is that high values of $\tilde{N}\left(\rho, (S_j)_{j=0}^{m}, (p_j)_{j=0}^{m}\right)$ or $\tilde{N}(\rho, \Pi)$ will imply that the investor might be able to create strategies with a very attractive $return/risk$ ratio.

Many more classical financial problems may be treated with risk measures. For instance, pricing and hedging issues (Goovaerts and Laeven, 2008, Balbás *et al.,* 2010), risk management (Ahn *et al.*, 1999, Constantinides *et al.*, 2011), regulatory capital etc. All of them will often lead to unrealistic solutions in the presence of GD, which implies that the pricing rules will have to be changed. If there are no pricing processes involved, and only market quotations are being considered, appropriate bounds must be imposed. Both, constraints in the line of Problems (12) and (12), and constraints related to the limit order book will have to be considered. It may be important to point out that the additional constraints of Problems (12) and (12) will be quite similar to those related to the restrictions of the limit order book.

5 Conclusion

The existence of GD is anti-intuitive and should not make any economic sense, but it often holds in practice. The GD size has been measured for both real market quotations and theoretical pricing models. In both cases the provided measure has optimized the strategy risk with respect to the value of the sold assets, which means that we are measuring in monetary and relative terms.

If only real market quotations are involved and the risk measure is robust then the approach is also model-independent. In this case the GD measure has a dual interpretation in terms of the minimum relative (per dollar) price modification preventing the GD existence. Moreover, it yields information about which are the over-priced and the under-priced securities. Numerical examples have been studied.

If a pricing model is involved then the dual interpretation above still applies, as well as the comments about over or under priced pay-offs, but there are also very important differences with respect to the model-independent case. Firstly, if the SDF is not essentially bounded then GD existence will always hold if the risk measure sub-gradient is composed of essentially bounded random variables. Secondly, the GD size will be much higher. Actually, it will be higher than 100%, while this value is quite difficult to reach with a finite collection of real market data. Explicit expressions of the GD size have been given for the Black and Scholes model.

Lastly, it is important to remark that the GD existence may provoke pathologies in many classical actuarial and financial problems. Concrete examples have been provided. The developed methodology allows the agent to overcome these pathologies and prevent wrong decisions.

Acknowledgments

Research partially supported by "*Ministerio de Economía*" (grant $ECO2012-39031-C02-01$, Spain) and the *Natural Sciences and Engineering Research Council (NSERC)* of Canada grant 36860-2012. The usual caveat applies.

References

Ahn, D., J. Boudoukh, M. Richardson and R.F. Whitelaw (1999). Optimal risk management using options. *The Journal of Finance*, 54, 359 - 375.

Anderson, E.J. and P. Nash (1987). *Linear programming in infinite-dimensional spaces*. John Wiley & Sons.

Arrow, K.J. (1963). Uncertainty and the welfare of medical care. *American. Economic. Review*, 53, 941–973.

Artzner, P., F. Delbaen, J.M. Eber and D. Heath (1999). Coherent measures of risk. *Mathematical Finance*, 9, 203 - 228.

Balbás, A., B. Balbás and R. Balbás (2016*a*). Outperforming benchmarks with their derivatives: Theory and empirical evidence. *The Journal of Risk*, 18, 4, 25 - 52.

Balbás, A., B. Balbás and R. Balbás (2016*b*). Good deals and benchmarks in robust portfolio selection. *European Journal of Operational Research*, 250, 666 - 678.

Balbás, A. and R. Balbás (2009). Compatibility between pricing rules and risk measures: The $CCVaR$. *Revista de la Real Academia de Ciencias, RACSAM*, 103, 251 - 264.

Balbás, A., R. Balbás and J. Garrido (2010). Extending pricing rules with general risk functions. *European Journal of Operational Research*, 201, 23 - 33.

Balbás A., I.R. Longarela and J. Lucia (1999). How financial theory applies to catastrophe-linked derivatives. An empirical test of several pricing models. *Journal of Risk and Insurance*, 66, 4, 551 - 582.

Balbás A., I.R. Longarela and A. Pardo (2000). Integration and arbitrage in the Spanish financial markets: An empirical approach. *Journal of Futures Markets*, 20, 4, 321 - 344.

Ballestero, E., M. Bravo, B. Perez-Gladish, M. Arenas-Parra and D. Pla-Santamaria (2012). Socially Responsible Investment: A multicriteria approach to portfolio selection combining ethical and financial objectives. *European Journal of Operational Research*, 216, 487 - 494.

Ballestero, E., and C. Romero (1996). Portfolio Selection: A Compromise Programming Solution. *Journal of the Operational Research Society*, 47, 1377 - 1386.

Bondarenko, O. (2014). Why are put options so expensive? *Quarterly Journal of Finance*, 4, 3, 1450015.

Borch, K. (1960). An attempt to determine the optimum amount of stop loss reinsurance. *In: Transactions of the 16th International Congress of Actuaries I*, 597–610.

Chen, Z., and P.J. Knez (1995). Measurement of market integration and arbitrage. *The Review of Financial Studies*, 8, 2, 287 - 325.

Constantinides, G.M., M. Czerwonko, J.C. Jackwerth and S. Perrakis (2011). Are options on index futures profitable for risk-averse investors? Empirical evidence. *The Journal of Finance*, 66, 1407-1437.

Dash, G.H., and N. Kajiji (2014). On multiobjective combinatorial optimization and dynamic interim hedging of efficient portfolios. *International Transactions in Operational Research*, 21, 899 - 918.

Davis M.H.A., V.G. Panas and T. Zariphopoulou (1993). European option pricing with transaction costs. *Siam Journal of Control and Optimization*, 31, 470 - 493.

Duffie, D. (1988). *Security markets: Stochastic models*. Academic Press.

Dupacová, J. and M. Kopa (2014). Robustness of optimal portfolios under risk and stochastic dominance constraints. *European Journal of Operational Research*, 234, 434 - 441.

Goovaerts, M.J. and R. Laeven (2008). Actuarial risk measures for financial derivative pricing. *Insurance: Mathematics and Economics*, 42, 540-547.

Guan, G. and Z. Liang (2014). Optimal reinsurance and investment strategies for insurer under interest rate and inflation risks. *Insurance: Mathematics and Economics*, 55, 105 – 115.

Harrison, J. and D. Kreps (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20, 381 - 408.

Haugh, M.B. and A.W. Lo (2001). Asset allocation and derivatives. *Quantitative Finance*, 1, 45–72.

Kamara, A., and T.W. Miller (1995). Daily and intradaily tests of European put-call parity. *Journal of Financial and Quantitative Analysis*, 30, 4, 519 - 541.

Kempf, A., and O. Korn (1998). Trading system and market integration. *Journal of Financial Intermediation*, 7, 220 - 239.

Luenberger, D.G. (2001). Projection pricing. *Journal of Optimization Theory and Applications*, 109, 1 - 25.

Peng, X. and W. Wang (2016). Optimal investment and risk control for an insurer under inside information. *Insurance: Mathematics and Economics*, 69, 104 - 116.

Prisman, E.Z. (1986). Valuation of risky assets in arbitrage free economies with frictions. *The Journal of Finance*, 41, 3, 545 - 556.

Rockafellar, R. T. and S. Uryasev (2000). Optimization of conditional-value-at-risk. *The Journal of Risk*, 2, 21 - 42.

Rockafellar, R. T., S. Uryasev and M. Zabarankin (2006). Generalized deviations in risk analysis. *Finance and Stochastics*, 10, 51 - 74.

Sawaragi, Y., H. Nakayama and T. Tanino (1985). *Theory of Multiobjective Optimization*. Elsevier,

Schaeffer, H.H. (1970). *Topological vector spaces*. Springer.

Stoyanov, S.V., S.T. Rachev and F.J. Fabozzi (2007). Optimal financial portfolios. *Applied Mathematical Finance*, 14, 401-436.

Wang, S.S. (2000). A class of distortion operators for pricing financial and insurance risks. *Journal of Risk and Insurance*, 67, 15-36.

Weng, C. and S.C. Zhuang (2016). CDF formulation for solving an optimal reinsurance problem. *Scandinavian Actuarial Journal*, forthcoming, DOI: 10.1080/03461238.2016.1167114.

Zhao, P. and Q. Xiao (2016). Portfolio selection problem with Value-at-Risk constraints under non-extensive statistical mechanics. *Journal of Computational and Applied Mathematics*, 298, 74-91.

Zhuang, S.C., C. Weng, K.S. Tan and H. Assa (2016). Marginal indemnification function formulation for optimal reinsurance. *Insurance: Mathematics and Economics*, 67, 65-76.