VARIANTS OF P-FRAMES AND ASSOCIATED RINGS

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Declaration

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I declare that *Variants of P-Frames and Associated Rings* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

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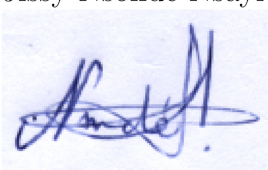

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Abstract

We study variants of P-frames and associated rings, which can be viewed as natural generalizations of the classical variants of P-spaces and associated rings. To be more precise, we define quasi m-rings to be those rings in which every prime d-ideal is either maximal or minimal. For a completely regular frame L, if the ring $\mathcal{R}L$ of real-valued continuous functions of L is a quasi m-ring, we say L is a quasi cozero complemented frame. These frames are less restricted than the cozero complemented frames. Using these frames we study some properties of what are called quasi m-spaces, and observe that the property of being a quasi m-space is inherited by cozero subspaces, dense z-embedded subspaces, and regular-closed subspaces among normal quasi m-space.

M. Henriksen, J. Martínez and R. G. Woods have defined a Tychonoff space X to be a quasi P-space in case every prime z-ideal of $C(X)$ is either minimal or maximal. We call a point I of βL a quasi P-point if every prime z-ideal of $\mathcal{R}L$ contained in the maximal ideal associated with I is either maximal or minimal. If all points of βL are quasi P-points, we say L is a quasi P-frame. This is a conservative definition in the sense that X is a quasi P-space if and only if the frame $\mathfrak{O}X$ is a quasi P-frame. We characterize these frames in terms of cozero elements, and, among cozero complemented frames, give a sufficient condition for a frame to be a quasi P-frame.

A Tychonoff space X is called a weak almost P-space if for every two zero-sets E and F of X with $\mathrm{Int}E \subseteq \mathrm{Int}F$, there is a nowhere dense zero-set H of X such that $E \subseteq F \cup H$. We present the pointfree version of weakly almost P-spaces. We define weakly regular rings by a condition characterizing the rings $C(X)$ for weak almost P-spaces X. We show that a reduced f-ring is weakly regular if and only if every prime z-ideal in it which contains only zero-divisors is a d-ideal. We characterize the frames L for which the ring $\mathcal{R}L$ of real-valued continuous functions on L is weakly regular.

We introduce the notions of boundary frames and boundary rings, and use them to give another ring-theoretic characterization of boundary spaces. We show that X is a boundary space if and only if $C(X)$ is a boundary ring.

A Tychonoff space whose Stone-Čech compactification is a finite union of closed subspaces each of which is an F-space is said to be finitely an F-space. Among normal spaces, S. Larson gave a characterization of these spaces in terms of properties of function rings $C(X)$. By extending this notion to frames, we show that the normality restriction can actually be dropped, even in spaces, and thus we sharpen Larson's result.

keywords: P-frame, quasi P-frame, quasi cozero complemented frame, quasi m-space, weak almost P-frame, weakly regular ring, boundary frame, boundary ring, finitely an F-frame.

Contents

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Dedication

To my mother, Elisabeth Niangui Nsayi,

and my father, Theophile Nsonde.

Chapter 1

Introduction and preliminaries

1.1 A brief history on P-spaces and P-frames

A P-space is a topological space in which every countable intersection of open sets is open. An equivalent condition is that countable unions of closed sets are closed. In other words, G_δ sets are open and F_δ sets are closed. These spaces were introduced by L. Gilman and M. Henriksen [38], and their various characterizations are documented in the classical text *Rings of Continuous Functions* by Gillman and Jerison.

An extension of the study of P-spaces to pointfree topology was initiated by Ball and Walter-Wayland in [4], who called a frame L a P-frame in case every cozero element in L is complemented. It has recently been shown by Ball, Walters-Wayland and Zenk [5] that, in stark contrast with P-spaces, there are P-frames with quotients which are not P-frames. P-frames have been characterized in terms of ring-theoretic properties of the ring of continuous real-valued functions on a frame L by Dube in [22], and also by Ball, Walters-Wayland and Zenk in the aforementioned article. Dube defined the m-topology on the ring $\mathcal{R}L$ of continuous real functions on a frame L and showed that if the frame L belongs to a certain class of frames properly containing the spatial ones, then L is a P-frame if and only if every ideal of $\mathcal{R}L$ is m-closed.

1.2 Synopsis of the thesis

The thesis consists of six chapters, the first of which is a brief overview of the theory of frames. It is the chapter in which we fix notation and provide the requisite background needed to read the thesis. All spaces in the thesis are Tychonoff, all frames are completely regular, and all rings are commutative with identity.

In Chapter 2 we look at some properties of quasi m-spaces from a ring-theoretic perspective. These spaces were defined by Azarpanah and Karavan [1] as those X for which every prime d-ideal of $C(X)$ is either a maximal ideal or a minimal prime ideal. We prove, for instance, that among subspaces that inherit the property of being a quasi m-space are cozero subspaces, dense z-embedded subspaces, and regular-closed subspaces among the normal quasi m-spaces. The ring-theoretic approach that we take actually yields the above results within the broader context of frames.

In Chapter 3 we study quasi P-frames. We define these frames by generalizing the condition employed by Henriksen, J. Martínez and Woods [40] to define quasi P-spaces. The definition is "conservative", which is to say a space is a quasi P-space if and only if the frame of its open sets is a quasi P-frame. We give a localic characterization of quasi P-frames. Among cozero complemented frames, we give a sufficient condition for a frame to be a quasi P-frame. We show that a perfectly normal frame is a quasi P-frame precisely when every nowhere dense quotient of it is closed.

A space X is called a weak almost P-space if for every two zero-sets E and F of X with $\operatorname{int} E \subseteq \operatorname{int} F$, there is a nowhere dense zero-set H of X such that $E \subseteq F \cup H$. In Chapter 4 we present the pointfree version of weakly almost P-spaces. We define weakly regular rings by a condition characterizing the rings $C(X)$ for weak almost P-spaces X. We show that a reduced f-ring is weakly regular if and only if every prime z-ideal in it which contains only zero-divisors is a d-ideal. We characterize the frames L for which the ring $\mathcal{R}L$ of real-valued continuous functions on L is weakly regular. We show that if the coproduct of two Lindelöf frames is of this kind, then so is each summand. Also, a continuous Lindelöf frame is of this kind if and only if its Stone-Čech compactification is of this kind.

A space X is called a boundary space if the boundary of every zero-set of X is contained in a zero-set with empty interior. These spaces were characterized by Azarpanah and Karavan [1] as precisely those X for which every prime ideal of $C(X)$ that consists entirely of zero-divisors is a d-ideal. In Chapter 5 we introduce the notions of boundary frames and boundary rings. This is with the view to giving another ring-theoretic characterization of boundary spaces. We show that X is a boundary space if and only if $C(X)$ is a boundary ring. We also show that if $X \times Y$ is z-embedded in $\beta X \times \beta Y$, then X and Y are boundary spaces if $X \times Y$ is a boundary space. We also provide a frame version of this result.

A space whose Stone-Čech compactification is a finite union of closed subspaces each of which is an F-space is said to be finitely an F-space. Larson [45] has shown that for normal spaces X, the property of being finitely an F-space can be characterized in terms of algebraic properties of the ring $C(X)$. By extending this concept to frames in Chapter 6, we show that the normality restriction can actually be dropped, even in spaces, and thus sharpen Larson's result.

1.3 Frames and their homomorphisms

In this section we recall some definitions and results concerning frames that are needed in the sequel. For general information on frames, we refer to [44] and [57].

A *frame* is a complete lattice L in which binary meets distribute over arbitrary joins, that is,

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$

for every $a \in L$ and $\{b_i \,|\, i \in I\} \subseteq L$. We shall denote the *bottom* element of L by $\bot$ or 0, and the *top* element by $\top$ or 1.

A σ-*frame* is a complete lattice L with top and bottom, which is countably satisfying the distributive law

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$

for every $a \in L$ and any countable $\{b_i \,|\, i \in I\} \subseteq L$.

A *frame homomorphism* is a map $h : L \to M$ between frames L and M preserving finite meets and arbitrary joins, including the top element and the bottom element. The frames and frame homomorphisms form a category **Frm**.

An important class of frames arises from topology. For any topological space X, the lattice $\mathfrak{O}X$ of open subsets of X is a frame, furthermore any continuous map $f : X \to Y$ between topological spaces gives rise to a frame homomorphism $\mathfrak{O}f : \mathfrak{O}Y \to \mathfrak{O}X$. The resulting correspondence from topological spaces to frames, and continuous maps to frame homomorphisms, constitutes a *contravariant functor* between the category **Top** of topological spaces and continuous maps and the category **Frm**.

Associated with any frame homomorphism $h : L \to M$ is a map $h_* : M \to L$, known as the *right adjoint* of h, which is not necessarily a frame homomorphism, but preserves arbitrary meets, and defined by

$$h_*(y) = \bigvee\{x \in L \mid h(x) \leq y\}.$$

The following property holds for every $x \in L$, and every $y \in M$:

$$h(x) \leq y \quad \Longleftrightarrow \quad x \leq h_*(y)$$

A frame homomorphism $h : L \to M$ is *dense* if for every $a \in L$, $h(a) = 0$ implies $a = 0$. This holds if and only if $h_*(0) = 0$. A frame homomorphism $h : L \to M$ is *codense* if for every $a \in L$, $h(a) = 1$ implies $a = 1$. A frame homomorphism $h : L \to M$ is *onto* if and only if $hh_* = id_M$.

By a *quotient* M of a frame L we mean a homomorphic image of L, which we shall frequently write as $h : L \to M$ in the category of frames where h is an onto homomorphism. In such a case we shall refer to h as a *quotient map*. When we say a quotient $h : L \to M$ has a property of frames we shall mean that M has that property. Likewise, to say a quotient $h : L \to M$ has a property of homomorphisms means that h has that property.

Given a frame L. We call $D \subseteq L$ a *downset* if $x \in D$ and $y \leq x$ implies $y \in D$, and we call $U \subseteq L$ an *upset* if $u \in U$ and $u \leq v$ implies $v \in U$. For any $a \in L$, we write

$$\downarrow a = \{x \in L \mid x \leq a\} \quad \text{which is a downset,}$$

and

$$\uparrow a = \{x \in L \mid a \leq x\} \quad \text{which is an upset.}$$

We note hat $\downarrow a$ is a frame whose bottom element is $0 \in L$ and top element is a, and $\uparrow a$ is a frame whose bottom element is a an top element $1 \in L$. In fact these frames are quotients of L via the maps $L \to \downarrow a$ and $L \to \uparrow a$ given, respectively, by $x \mapsto x \wedge a$ and $x \mapsto x \vee a$. These quotients are known as the *open quotients* and *closed quotients*.

The *pseudocomplement* of an element x of L is the element

$$x^* = \bigvee\{y \in L \mid x \wedge y = 0\}.$$

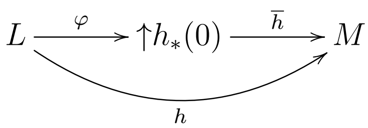
We note that $x \wedge x^* = 0$. However $x \vee x^* = 1$ does not hold in general.

(i) In the case where $x \vee x^* = 1$, we say x is *complemented*.

(ii) An element $x \in L$ is *dense* if $x^* = 0$; and it is *regular* if $x = x^{**}$. For every $x \in L$, $x \leq x^{**}$ always holds.

The *Booleanization* of a frame L is the Boolean algebra $\mathfrak{B}L = \{x^{**} \mid x \in L\}$ of its regular elements with meet as in L and join $\bigvee_{\mathfrak{B}L} S = (\bigvee S)^{**}$ for each $S \subseteq \mathfrak{B}L$. The map $L \to \mathfrak{B}L$ which sends each $x \in L$ to x^{**} is a dense onto frame homomorphism.

A result often used in frame theory is that every frame homomorphism $h : L \to M$ has a *dense-onto factorization*



where φ is the onto homomorphism $x \longmapsto x \vee h_*(0)$ and $\overline{h}$ the dense homomorphism mapping as h.

We say that x is *rather below* y or x is *well inside* y, written $x \prec y$, if there is a separating element $u \in L$ such that $x \wedge u = 0$ and $y \vee u = 1$. We say a frame L is *regular* if every $x \in L$ is expressible as

$$x = \bigvee\{y \in L \mid y \prec x\}.$$

By a *scale* in frame we mean a sequence $\{c_q \mid q \in \mathbb{Q} \cap [0,1]\} = (c_q)$ of elements in L indexed by the rational numbers in $[0,1]$, such that whenever $p < q$, then $c_p \prec c_q$. We defined the *completely below* relation $\prec\!\!\prec$ on L by: $x \prec\!\!\prec y$ if there is a scale (c_p) such that $x \leq c_0$ and $c_1 \leq y$. We say L is *completely regular* if every $x \in L$ is expressible as

$$x = \bigvee\{y \in L \mid y \prec\!\!\prec x\}.$$

A frame L is *normal* if for any elements $a, b \in L$, such that $a \vee b = 1$, there are elements $c, d \in L$ such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$.

By a *cover* A of a frame L we mean a subset of L such that $\bigvee A = 1$. We write $\mathrm{Cov}(L)$ for the set of all covers of the frame. A frame L is *compact* if for any $A \in \mathrm{Cov}(L)$, there is a finite $F \subseteq A$ in $\mathrm{Cov}(L)$. A frame L is *Lindelöf* if every cover has a countable subcover.

By a *compactification* of a completely regular frame L we mean a dense onto frame homomorphism $h : M \to L$ with M compact regular. The realization of βL we shall use is the following. An ideal I of L is called *completely regular* if for any $a \in I$ there exists $b \in L$ such that $a \prec\!\!\prec b$. The frame βL is the frame of all completely regular ideals of L. We write $j_L : \beta L \to L$ for the dense onto frame homomorphism $j_L(I) = \bigvee I$. Its right adjoint will be denoted by r_L. It is given by $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$. The Stone-Čech compactification of L is denoted by $\beta L \to L$ or simply βL.

An element c of a frame L is said to be *compact* if for any $S \subseteq L$, $c \leq \bigvee S$ implies $c \leq \bigvee T$, for some finite $T \subseteq S$.

By a *point* of a frame L we mean a prime element, that is, an element $p < 1$ such that for any a and b in L, $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We denote by $\mathrm{Pt}(L)$ the set of all points of L. We remark that, subject to appropriate choice principles (which we assume throughout), a compact regular frame has *enough points*, which means that every element is a meet of points. Also, for regular frames, "point" and "maximal element" are synonyms, where the latter is understood to mean maximal strictly below the top.

An ideal of a lattice is said to be a *σ-ideal* if it is closed under countable joins. A *nucleus* on a frame L is a closure operator $\ell : L \to L$ such that $\ell(a \wedge b) = \ell(a) \wedge \ell(b)$ for all $a, b \in L$. The set

$$\mathrm{Fix}(\ell) = \{a \in L \mid \ell(a) = a\}$$

is a frame with meet as in L and join given by

$$\bigvee_{\mathrm{Fix}(\ell)} S = \ell(\bigvee S)$$

for every $S \subseteq \mathrm{Fix}(\ell)$.

1.4 Rings and f-rings

Throughout this thesis all rings considered are commutative with identity 1 and the term "space" means a Tychonoff space. Let A be a ring. The *annihilator* of $S \subseteq A$ is the ideal

$$\mathrm{Ann}(S) = \{a \in A \mid as = 0 \quad \text{for every } s \in S\}.$$

If S is a singleton, say $S = \{a\}$, we shall abbreviate $\mathrm{Ann}(\{a\})$ as $\mathrm{Ann}(a)$. The double annihilator will be written as $\mathrm{Ann}^2(S)$ or $\mathrm{Ann}^2(a)$.

A ring is said to be *reduced* if it has no non-zero nilpotent elements. We shall write $\mathrm{Max}(A)$ for the set of all maximal ideals of A. For an ideal I of A we write

$$\mathfrak{M}(I) = \{M \in \mathrm{Max}(A) \mid M \supseteq I\},$$

and abbreviate $\mathfrak{M}(\{a\})$ as $\mathfrak{M}(a)$.

An *f-ring* is a lattice-ordered ring A in which the identity

$$(a \wedge b)c = (ac) \wedge (bc)$$

holds for all $a, b, c \in A$ with $c \geq 0$.

An f-ring A is said to have *bounded inversion* if any $a \geq 1$ is a unit in A. The *bounded part* of an f-ring A, denoted A^*, is the subring

$$A^* = \{a \in A \mid |a| \leq n \cdot 1 \text{ for some } n \in \mathbb{N}\}.$$

It is not hard to show that, for any $a \in A$, $\frac{a}{1+|a|} \in A^*$. A prime ideal P in a reduced ring is minimal prime if and only if for every $a \in P$ there is an $a' \notin P$ such that $aa' = 0$ (see [39]). If the sum of positive elements in an f-ring is zero, then each summand is zero.

1.5 Function rings

Our approach to pointfree function rings is that of [8]. We give a brief description. The *frame of reals*, $\mathfrak{L}(\mathbb{R})$, is defined by generators which are pairs (p, q) of rationals, and the relations (R1) through (R4) below:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$, whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $1_{\mathfrak{L}(\mathbb{R})} = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

A *continuous real-valued function* on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$. The ring $\mathcal{R}L$ has as its elements continuous real-valued functions on L, with operations determined by the operations of $\mathbb{Q}$ viewed as a lattice-ordered ring as follows:

For $\diamond \in \{+, \cdot, \wedge, \vee\}$ and $\alpha, \beta \in \mathcal{R}L$,

$$\alpha \diamond \beta(p, q) = \bigvee\{\alpha(r, s) \wedge \beta(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\},$$

where $\langle \cdot, \cdot\rangle$ denotes the open interval in $\mathbb{Q}$, and the given condition means that $x \diamond y \in \langle p, q\rangle$ for any $x \in \langle r, s\rangle$ and $y \in \langle t, u\rangle$.

For any $\alpha \in \mathcal{R}L$ and $p, q \in \mathbb{Q}$,

$$(-\alpha)(p, q) = \alpha(-q, -p).$$

The ring $\mathcal{R}L$ is a reduced f-ring with identity, and for any Tychonoff space X, $C(X)$ is isomorphic to $\mathcal{R}(\mathfrak{O}X)$. Furthermore, $\mathcal{R}L$ has bounded inversion. The correspondence $L \mapsto \mathcal{R}L$ is functorial, where, for any frame homomorphism $h\colon L \to M$, the ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ is given by $\mathcal{R}h(\alpha) = h \cdot \alpha$ – the centre dot designating composition.

An element a of L is a *cozero element* if there is a sequence (a_n) in L such that $a_n \prec\!\prec a$ for each n and $a = \bigvee a_n$. The *cozero part* of L, denoted by $\operatorname{Coz} L$, is the *regular sub-σ-frame* consisting of all the cozero elements of L.

The *cozero map*, $\operatorname{coz}: \mathcal{R}L \to L$, is given by

$$\operatorname{coz}\varphi = \bigvee\{\varphi(p,0) \vee \varphi(0,q) \mid p,q \in Q\}.$$

The association $L \mapsto \mathcal{R}L$ is functorial, with $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ taking δ to $h \cdot \delta$, for any $h : L \to M$. Furthermore, $\operatorname{coz}(h \cdot \delta) = h(\operatorname{coz}\delta)$.

The following presents some of the properties of coz needed here, see [10].

Lemma 1.5.1. *For any $\gamma, \delta \in \mathcal{R}L$,*

(1) $\operatorname{coz}\gamma = \operatorname{coz}|\gamma|$,

(2) $\operatorname{coz}(\gamma\delta) = \operatorname{coz}\gamma \wedge \operatorname{coz}\delta$,

(3) $\operatorname{coz}(\gamma + \delta) \leq \operatorname{coz}\gamma \vee \operatorname{coz}\delta$,

(4) $\varphi \in \mathcal{R}L$ *is invertible if and only if* $\operatorname{coz}\varphi = 1$,

(5) $\operatorname{coz}\varphi = 0$ *if and only if* $\varphi = \boldsymbol{0}$,

(6) $\operatorname{coz}(\gamma + \delta) = \operatorname{coz}\gamma \vee \operatorname{coz}\delta$ *if* $\gamma, \delta \geq 0$.

The maximal ideals of $\mathcal{R}L$ are described in [24] as follows: For any $I \in \beta L$, the ideals $\boldsymbol{M}^I$ of $\mathcal{R}L$ are defined by

$$\boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\}.$$

For any $a \in L$, we abbreviate $\boldsymbol{M}^{r_L(a)}$ as $\boldsymbol{M}_a$, and remark that

$$\boldsymbol{M}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\}.$$

The maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^I$, for $I \in \operatorname{Pt}(\beta L)$. Annihilator ideals in $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}_{a^*}$, for $a \in L$ [26, Lemma 3.1]. In particular, for any $\alpha \in \mathcal{R}L$, $\operatorname{Ann}(\alpha) = \boldsymbol{M}_{(\operatorname{coz}\alpha)^*}$. Recall that an element of a frame is *dense* if it has nonzero meet with every nonzero element. Any $\alpha \in \mathcal{R}L$ is a non-divisor of zero if and only if $\operatorname{coz}\alpha$ is dense [24, Corollary 4.2].

An ideal I of a ring A is called a *d-ideal* if $\mathrm{Ann}^2(a) \subseteq I$, for every $a \in I$. On the other hand, I is called a *z-ideal* if whenever $a \in I$ and b is an element of A contained in every maximal ideal containing a, then $b \in I$.

In $\mathcal{R}L$, z-ideals and d-ideals are characterized in terms of the cozero map as follows (see [43]).

Lemma 1.5.2. *The following are equivalent for an ideal Q of $\mathcal{R}L$.*

(a) *Q is a z-ideal.*

(b) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $\operatorname{coz} \beta \leq \operatorname{coz} \alpha$, then $\beta \in Q$.*

(c) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $\operatorname{coz} \beta = \operatorname{coz} \alpha$, then $\beta \in Q$.*

Lemma 1.5.3. *The following are equivalent for an ideal Q of $\mathcal{R}L$.*

(a) *Q is a d-ideal.*

(b) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $(\operatorname{coz} \alpha)^* = (\operatorname{coz} \beta)^*$, then $\beta \in Q$.*

(c) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $(\operatorname{coz} \alpha)^* \leq (\operatorname{coz} \beta)^*$, then $\beta \in Q$.*

(d) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $\operatorname{coz} \beta \leq (\operatorname{coz} \alpha)^{**}$, then $\beta \in Q$.*

1.6 The coreflections λL and υL

The regular Lindelöf coreflection of L, denoted λL, is the frame of σ-ideals of $\operatorname{Coz} L$ (see [48]). The join map $\lambda_L \colon \lambda L \to L$ is a dense onto frame homomorphism, and is the coreflection map to L from Lindelöf frames.

The map $\eta L : \beta L \to \lambda L$ given by $\eta_L(I) = \langle I \rangle_\sigma$, where $\langle . \rangle_\sigma$ signifies σ-ideal generation in $\operatorname{Coz} L$, is a dense onto frame homomorphism. In fact, $\eta_L : \beta L \to \lambda L$ realizes the Stone-Čech compactification of λL. To see this, recall that a frame homomorphism is called *coz-codense* (or *coz-faithful*) if it is one-one on cozero elements. Now suppose that $j_L : \beta L \to L$ factorizes as $\beta L \xrightarrow{g} M \xrightarrow{h} L$ with g onto and h coz-codense. Then

$g : \beta L \to M$ is the Stone-Čech compactification of M. We use [4, Corollary 8.2.7] to prove the following assertion. Let $c_1 \vee c_2 = 1$ in $\mathrm{Coz}\, M$. Then $h(c_1) \vee h(c_2) = 1$ in $\mathrm{Coz}\, L$. By the result cited from [4], there are cozero elements d_1, d_2 of βL such that $d_1 \vee d_2 = 1_{\beta L}$ and $j_L(d_i) = h(c_i)$ for $i = 1, 2$. Thus, $h(g(d_i)) = h(c_i)$, and hence $g(d_i) = c_i$ because h is coz-codense. Thus, by [4, Corollary 8.2.7] again, $h : \beta L \to M$ is the Stone-Čech compactification of M. Since $\lambda L \to L$ is coz-codense and $j_L : \beta L \to L$ factorizes as $\beta L \xrightarrow{\eta_L} \lambda L \xrightarrow{\lambda_M} L$, the claim is established.

Realcompact frames are coreflective in completely regular frames (**CRFrm**) (see, for instance, [11] and [49] for details). The realcompact coreflection of L, denoted υL, is constructed in the following manner. For any $t \in L$, let $[\![t]\!] = \{x \in \mathrm{Coz}\, L \mid x \le t\}$; so that if $c \in \mathrm{Coz}\, L$, then $[\![c]\!]$ is the principal ideal of $\mathrm{Coz}\, L$ generated by c. The map $\ell \colon \lambda L \to \lambda L$ given by

$$\ell(J) = \Big[\!\!\Big[\bigvee J \Big]\!\!\Big] \wedge \bigwedge \{P \in \mathrm{Pt}(\lambda L) \mid J \le P\}$$

is a nucleus. The frame υL is defined to be $\mathrm{Fix}(\ell)$. The join map $\upsilon_L \colon \upsilon L \to L$ is a dense onto frame homomorphism whose right adjoint is given by $a \mapsto [\![a]\!]$. It is the coreflection map to L from realcompact frames.

1.7 Binary coproducts of frames

The coproduct $L \oplus M$ of two frames may be constructed in the following simple way. First take the Cartesian product $L \times M$ with the usual partial order and consider

$$\mathfrak{D}(L \times M) = \{U \subseteq L \times M \mid {\downarrow} U = U \ne \emptyset\}.$$

Call a $U \in \mathfrak{D}(L \times M)$ *saturated* if

(1) for any subset $A \subseteq L$ and any $b \in M$, if $A \times \{b\} \subseteq U$ then $(\bigvee A, b) \in U$, and

(2) for any $a \in L$ and any subset $B \subseteq M$, if $\{a\} \times B \subseteq U$ then $(a, \bigvee B) \in U$.

The set A (resp. B) can be void; hence, in particular, each saturated set contains the set

$\mathbb{O} = \{(0,b),(a,0) \mid a \in L, b \in M\}$. It is easy to see that for each $(a,b) \in L \times M$,

$$a \oplus b = {\downarrow}(a,b) \cup \mathbb{O} \quad \text{is saturated.}$$

To finish the construction take

$$L \oplus M = \{U \in \mathfrak{D}(L \times M) \mid U \text{ is saturated}\}$$

with the coproduct injections

$$i_L = (a \mapsto a \oplus 1) : L \to L \oplus M, \qquad i_M = (b \mapsto 1 \oplus b) : M \to L \oplus M.$$

Note that one has for each saturated U,

$$\bigvee\{a \oplus b \mid (a,b) \in U\} = \bigcup\{a \oplus b \mid (a,b) \in U\},$$

and if $a \oplus b \leq c \oplus d$ and $b \neq 0$, then $a \leq c$.

The results in the following lemma appear in [19].

Lemma 1.7.1. (1) $0_{L\oplus M} = \downarrow(1,0) \cup \downarrow(0,1)$.

(2) $a \oplus b = 0 \iff a = 0$ *or* $b = 0$, *consequently,* $(a \oplus b)^* = (a^* \oplus 1) \vee (1 \oplus b^*)$.

(3) $a \oplus (\bigvee_i b_i) = \bigvee_i (a \oplus b_i)$ *and* $(\bigvee_i b_i) \oplus a = \bigvee_i (b_i \oplus a)$.

(4) $a \leq c$ *and* $b \leq d \implies a \oplus b \leq c \oplus d$.

(5) $0 \neq a \oplus b \leq c \oplus d \implies a \leq c$ *and* $b \leq d$.

Recall that if, for $i = 1,2$, $h_i \colon M_i \to L_i$ are frame homomorphisms, then the map $h_1 \oplus h_2 \colon M_1 \oplus M_2 \to L_1 \oplus L_2$ given by

$$(h_1 \oplus h_2)\Big(\bigvee_\alpha (x_\alpha \oplus y_\alpha)\Big) = \bigvee_\alpha \big(h_1(x_\alpha) \oplus h_2(y_\alpha)\big)$$

is a frame homomorphism.

Chapter 2

Quasi m-spaces and quasi cozero complemented frames

2.1 Introduction

We denote the annihilator of a set S by $\mathrm{Ann}(S)$, and abbreviate $\mathrm{Ann}(\{a\})$ as $\mathrm{Ann}(a)$. Double annihilators are written as $\mathrm{Ann}^2(S)$. An ideal I of a ring A is called a *d-ideal* if $\mathrm{Ann}^2(a) \subseteq I$, for every $a \in I$. On the other hand, I is called a *z-ideal* if whenever $a \in I$ and b is an element of A contained in every maximal ideal containing a, then $b \in I$. The symbols $\mathrm{Spec}(A)$, $\mathrm{Max}(A)$ and $\mathrm{Min}(A)$ have their usual meaning; namely, the set of prime, maximal and minimal prime ideals of A, respectively. We write $\mathrm{Spec}_d(A)$ and $\mathrm{Spec}_z(A)$ for the set of prime d-ideals and prime z-ideals of A, respectively.

Consider the following conditions on a ring A:

(dMin) $\quad \mathrm{Spec}_d(A) \subseteq \mathrm{Min}(A)$

(dMax) $\quad \mathrm{Spec}_d(A) \subseteq \mathrm{Max}(A)$

(dMM) $\quad \mathrm{Spec}_d(A) \subseteq \mathrm{Min}(A) \cup \mathrm{Max}(A)$

(zMin) $\quad \mathrm{Spec}_z(A) \subseteq \mathrm{Min}(A)$

(zMax) $\quad \mathrm{Spec}_z(A) \subseteq \mathrm{Max}(A)$

(zMM) $\quad \operatorname{Spec}_z(A) \subseteq \operatorname{Min}(A) \cup \operatorname{Max}(A).$

One of our goals is to characterize frames L such that, for each of these conditions, the ring $\mathcal{R}L$ satisfies that condition. In the class of reduced rings, three of these are all equivalent, and are equivalent to the property of being von Neumann regular (vNR). Precisely,

$$\text{zMin} \iff \text{zMax} \iff \text{dMax} \iff \text{vNR}.$$

Indeed, (zMin) is equivalent to (vNR) because maximal ideals are z-ideals, and a reduced ring is von Neumann regular if and only if every maximal ideal is minimal prime. Next, (zMax) implies (dMax) because every d-ideal is a z-ideal([55, Proposition 2.12]); (dMax) implies (vNR) because every minimal prime ideal is a d-ideal; and, finally, (vNR) implies (zMax) because every prime ideal in a von Neumann regular ring is a maximal ideal. Thus, $\mathcal{R}L$ satisfies any (and hence all) of these three if and only if L is a P-frame because L is a P-frame if and only if $\mathcal{R}L$ is von Neumann regular [12].

We shall see that $\mathcal{R}L$ satisfies (dMin) precisely if L is cozero complemented. The more substantive results concern those L for which $\mathcal{R}L$ satisfies (dMM).

2.2 Characterizations of quasi cozero complemented frames

In this section we proceed to characterize frames L for which every prime d-ideal of $\mathcal{R}L$ is either a maximal ideal or a minimal prime ideal. But first we start by justifying the claim made in the introduction that every prime d-ideal of $\mathcal{R}L$ is minimal prime if and only if L is cozero complemented. Let us recall the definition. A frame L is *cozero complemented* if for every $c \in \operatorname{Coz} L$ there is a $d \in \operatorname{Coz} L$ such that $c \vee d$ is dense and $c \wedge d = 0$.

We recall that a ring R is said to have *Property A* if every finitely generated ideal of R which consists entirely of zero-divisors has nonzero annihilator (see, for instance, [42]). In [2, Proposition 1.26] the authors show, among other things, that if a ring R has Property A, then every prime d-ideal of R is minimal prime if and only if for every

$a \in R$ there exists $b \in R$ such that $\operatorname{Ann}(a) = \operatorname{Ann}^2(b)$. Proposition 1.1 in [32] shows that L is cozero complemented if and only if for every $\alpha \in \mathcal{R}L$ there is a $\beta \in \mathcal{R}L$ such that $\operatorname{Ann}(\alpha) = \operatorname{Ann}^2(\beta)$. Now let us show that $\mathcal{R}L$ has Property A.

Lemma 2.2.1. *The ring $\mathcal{R}L$ has property A.*

Proof. In fact, let $Q = \langle \alpha_1, \alpha_2 \cdots \alpha_m \rangle$ be a finitely generated ideal of $\mathcal{R}L$ consisting of zero-divisors. Then $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_m^2 \in Q$ and therefore a zero-divisor. Note that

$$\bigvee \{\operatorname{coz} \alpha \mid \alpha \in Q\} = \operatorname{coz}(\alpha_1^2) + \operatorname{coz}(\alpha_2^2) + \cdots + \operatorname{coz}(\alpha_m^2),$$

which is not dense because $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_m^2$ is a zero divisor. Thus by [24, Lemma 4.3], $\operatorname{Ann}(Q) \neq 0$. □

We therefore have the following proposition.

Proposition 2.2.2. *Every prime d-ideal of $\mathcal{R}L$ is minimal prime if and only if L is cozero complemented.*

Remark 2.2.3. There is an alternative affirmation of this result. It is shown in [29, Proposition 3.1] that the lattice $\operatorname{Did}(\mathcal{R}L)$ of d-ideals of $\mathcal{R}L$ is a coherent frame, and, is in fact, the frame of d-elements of $\operatorname{Rad}(\mathcal{R}L)$, where the latter denotes the frame of radical ideals of $\mathcal{R}L$. Now, by [52], every prime d-ideal of $\mathcal{R}L$ is minimal prime if and only if $\operatorname{Did}(\mathcal{R}L)$ is regular. This in turn is equivalent to L being cozero complemented in light of [29, proposition 5.5]. In fact an algebraic frame is regular if and only if every compact element is complemented, and the compact elements of $\operatorname{Did}(\mathcal{R}L)$ are precisely the ideals $\boldsymbol{M}_{c^{**}}$, for $c \in \operatorname{Coz} L$ ([29, Proposition 4.1]).

Now we investigate when every prime d-ideal in $\mathcal{R}L$ is either a maximal or a minimal prime ideal. We first obtain a characterization for reduced f-rings. It will generalize the equivalence of conditions (i) and (ii) in [1, Theorem 3.2]. We start with a lemma which is itself an f-ring version of [1, Lemma 3.1]. Observe that a directed union of d-ideals is a d-ideal.

Lemma 2.2.4. *Let A be a reduced f-ring. Then, for any $a \in A$, the set*

$$\bigcup \{\mathrm{Ann}^2(a^2 + t^2) \mid ta = 0\}$$

is a d-ideal of A.

Proof. It suffices to show that the collection $\{\mathrm{Ann}^2(a^2+t^2) \mid t \in \mathrm{Ann}(a)\}$ is directed. Let u and v be in $\mathrm{Ann}(a)$. Then $u^2 + v^2 \in \mathrm{Ann}(a)$. We claim that

$$\mathrm{Ann}^2(a^2 + u^2) \subseteq \mathrm{Ann}^2(a^2 + (u^2 + v^2)^2).$$

We prove the claim by showing that $\mathrm{Ann}(a^2 + u^2) \supseteq \mathrm{Ann}(a^2 + (u^2 + v^2)^2)$. Let x be in the set on the right. Then

$$x^2(a^2 + u^2 + p) = 0 \quad \text{where} \quad p = u^2 + 2u^2v^2 + v^4 \geq 0.$$

Since squares are positive in any f-ring, and since whenever the sum of positive elements is zero then each summand is zero, it follows that $x^2(a^2+u^2) = 0$, implying $x^2(a^2+u^2)^2 = 0$, whence $x(a^2 + u^2) = 0$ because A is reduced. Thus, $x \in \mathrm{Ann}(a^2 + u^2)$, and the claim follows. Similarly, $\mathrm{Ann}^2(a^2 + v^2) \subseteq \mathrm{Ann}^2(a^2 + (u^2 + v^2)^2)$, and hence the collection

$$\{\mathrm{Ann}^2(a^2 + t^2) \mid t \in \mathrm{Ann}(a)\}$$

is directed. Therefore its union is a d-ideal. □

In the upcoming proof we are going to use the fact that a prime ideal minimal over a d-ideal is itself a d-ideal [55, Theorem 2.5]. We shall also have to keep in mind that a prime ideal P in a reduced ring is minimal prime if and only if, for every $a \in P$, there exists $b \notin P$ such that $ab = 0$ [39].

Proposition 2.2.5. *The following are equivalent for a reduced f-ring A.*

(1) *Every prime d-ideal of A is either a maximal ideal or a minimal prime ideal.*

(2) *For every maximal ideal M of A and every pair a, b of elements in M, there exists $u \in \mathrm{Ann}(a)$ and $v \notin M$ such that $\mathrm{Ann}(a^2 + u^2) \subseteq \mathrm{Ann}^2(bv)$.*

Proof. (1) ⇒ (2): Suppose (2) fails. Then there is a maximal ideal M of A and elements $a, b \in M$ such that for every $u \in \operatorname{Ann}(a)$ and $v \notin M$, $\operatorname{Ann}(a^2 + u^2) \nsubseteq \operatorname{Ann}(bv)$. Define subsets I and S of A by

$$I = \bigcup\{\operatorname{Ann}^2(a^2 + t^2) \mid ta = 0\} \quad \text{and} \quad S = \{b^n r \mid r \notin M \text{ and } n = 0, 1, \ldots\}.$$

It is easy to check that S is multiplicatively closed because M is a prime ideal. We claim that $S \cap I = \emptyset$. If not, then $b^n r \in I$ for some n and $r \notin M$, and hence $b^n r \in \operatorname{Ann}^2(a^2 + t^2)$ for some $t \in \operatorname{Ann}(a)$. Thus, $\langle b^n r \rangle \subseteq \operatorname{Ann}^2(a^2 + t^2)$, which implies

$$\operatorname{Ann}(a^2 + t^2) = \operatorname{Ann}(\operatorname{Ann}^2(a^2 + t^2)) \subseteq \operatorname{Ann}(b^n r) = \operatorname{Ann}(br),$$

because A is reduced. But this violates the supposition. There is therefore a prime ideal P which contains I and misses S. Without loss of generality, we may assume P is minimal with this property. Then P is a d-ideal ([55, Theorem 2.5]). Clearly, $A \smallsetminus M \subseteq S$, so that, in light of $P \cap S = \emptyset$, we have $P \subseteq M$. Observe that $a \in P$ because $a \in \operatorname{Ann}^2(a) \subseteq I \subseteq P$. Also, $\operatorname{Ann}(a) \subseteq P$ because of the following. If $ta = 0$, then $\operatorname{Ann}^2(a^2 + t^2) \subseteq I \subseteq P$. Consider any $z \in A$ with $z(a^2 + t^2) = 0$. Then $z^2 a^2 = z^2 t^2 = 0$, whence $tz = 0$. Thus, $t \in \operatorname{Ann}^2(a^2 + t^2) \subseteq P$. It follows therefore that P is not a minimal prime ideal. Then P is a maximal ideal by (1), and hence $P = M$. But $b \in M \cap S = P \cap S = \emptyset$, a contradiction.

(2) ⇒ (1): Let P be a prime d-ideal and M a maximal ideal with $P \subseteq M$. Suppose, for contradiction, that $P \neq M$ and P is not a minimal prime ideal. Since P is properly contained in M, there exists $b \in M \smallsetminus P$. Because P is not a minimal prime ideal, there is an $a \in A$ such that $a \in P$ and $\operatorname{Ann}(a) \subseteq P$. By (2), there exist $u \in \operatorname{Ann}(a)$ and $v \notin M$ such that $\operatorname{Ann}(a^2 + u^2) \subseteq \operatorname{Ann}(bv)$. Since $a^2 + u^2 \in P$ and P is a d-ideal, it follows that $bv \in P$. Since $b \notin P$, this implies $v \in P \subseteq M$, which is a contradiction. □

Definition 2.2.6. A ring is a *quasi m-ring* if every prime d-ideal in it is either maximal or minimal prime. If $\mathcal{R}L$ is a quasi m-ring, we shall say L is a *quasi cozero complemented frame.*

Remark 2.2.7. A remark concerning this terminology is in order. In [1] the authors call a Tychonoff space X an *m-space* if for every zero-set Z of X there is a zero-set Z' of X

such that $Z \cap Z'$ is nowhere dense and $Z \cup Z' = X$. They then say a space X is a *quasi m-space* if $C(X)$ is a quasi m-ring as we have defined it above. However, spaces with the formerly stated property have come to be known as cozero complemented – a moniker that has come to be commonly used.

In light of Proposition 2.2.2, every cozero complemented frame is a quasi cozero complemented frame. We shall shortly give a frame-theoretic characterization of quasi cozero complemented frames. Let us recall some facts from [26]. For any $\alpha \in \mathcal{R}L$,

$$\operatorname{Ann}(\alpha) = \boldsymbol{M}_{(\operatorname{coz}\alpha)^*}.$$

Thus,

$$\begin{aligned}\operatorname{Ann}(\alpha) \subseteq \operatorname{Ann}(\beta) &\iff \boldsymbol{M}_{(\operatorname{coz}\alpha)^*} \subseteq \boldsymbol{M}_{(\operatorname{coz}\beta)^*} \\ &\iff (\operatorname{coz}\alpha)^* \leq (\operatorname{coz}\beta)^*.\end{aligned}$$

Proposition 2.2.8. *The following are equivalent for a completely regular frame L.*

(1) *L is a quasi cozero complemented frame.*

(2) *λL is a quasi cozero complemented frame.*

(3) *υL is a quasi cozero complemented frame.*

(4) *For every $I \in \operatorname{Pt}(\beta L)$ and $\alpha, \beta \in \boldsymbol{M}^I$, there exists $\gamma \in \operatorname{Ann}(\alpha)$ and $\delta \notin \boldsymbol{M}^I$ such that $\operatorname{Ann}(\alpha^2 + \gamma^2) \subseteq \operatorname{Ann}(\beta\delta)$.*

(5) *For every $I \in \operatorname{Pt}(\beta L)$ and $c, d \in \operatorname{Coz} L$ with $r_L(c) \vee r_L(d) \leq I$, there exist $u, v \in \operatorname{Coz} L$ with $c \wedge v = 0, r_L(u) \vee I = 1_{\beta L}$ and $(c \vee u)^* \leq (d \wedge v)^*$.*

Proof. The equivalence of (1), (2) and (3) follows from the fact that the rings $\mathcal{R}L, \mathcal{R}(\lambda L)$ and $\mathcal{R}(\upsilon L)$ are all isomorphic. The equivalence of (1) and (4) follows from Proposition 2.2.5 because $\mathcal{R}L$ is a reduced f-ring. The discussion preceding the statement of the proposition shows that (4) and (5) are equivalent. □

This proposition generalizes [1, Theorem 3.2], and includes a characterization which does not exist for spaces. In [1], Azarpanah and Karavan mentioned that a Tychonoff space is a quasi m-space if and only if its Stone-Čech compactification is a quasi m-space. We do not know if this extends to frames.

2.3 Subspaces of quasi m-spaces

In this section, the results about certain subspaces inheriting the property of being a quasi m-space will be corollaries of a ring-theoretic result which we now prepare to present. We require some definitions concerning ring homomorphisms. We will show that these are motivated by certain types of frame homomorphisms.

Definition 2.3.1. A ring homomorphism $\phi\colon A \to B$ is *weakly skeletal* if, for any pair of elements $a_1, a_2 \in A$, the containment $\operatorname{Ann}(a_1) \subseteq \operatorname{Ann}(a_2)$ implies $\operatorname{Ann}(\phi(a_1)) \subseteq \operatorname{Ann}(\phi(a_2))$.

Let us show how this is motivated by certain types of frame homomorphisms. Recall that a frame homomorphism $h\colon M \to L$ is said to be *skeletal* if it maps dense elements to dense elements. As observe in [54], that h is skeletal if and only if, for any $a, b \in M$,

$$a^* = b^* \implies h(a)^* = h(b)^*.$$

Now we can weaken this by requiring that the elements a, b be restricted to cozero elements. Let us therefore agree to say a frame homomorphism $h : M \to L$ is *weakly skeletal* if, for any $c, d \in \operatorname{Coz} M$,

$$c^* = d^* \quad \implies \quad h(c)^* = h(d)^*.$$

In the proof that follows we shall use the facts that, for any $\alpha, \beta \in \mathcal{R}M$, $\operatorname{Ann}(\alpha) = \boldsymbol{M}_{(\operatorname{coz}\alpha)^*}$ and $\operatorname{Ann}(\alpha) = \operatorname{Ann}(\beta)$ if and only if $(\operatorname{coz}\alpha)^* = (\operatorname{coz}\beta)^*$.

Lemma 2.3.2. *A frame homomorphism $h\colon M \to L$ is weakly skeletal if and only if the ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ is weakly skeletal.*

Proof. (⇒) Let $\alpha, \beta \in \mathcal{R}M$ be such that $\mathrm{Ann}(\alpha) \subseteq \mathrm{Ann}(\beta)$. For brevity, write $a = \mathrm{coz}\,\alpha$ and $b = \mathrm{coz}\,\beta$. Then $\boldsymbol{M}_{a^*} \subseteq \boldsymbol{M}_{b^*}$, which implies $a^* \leq b^*$. Consequently,

$$a^* = a^* \wedge b^* = (a \vee b)^*.$$

Since $a \vee b$ is a cozero element, the weak skeletality of h implies

$$h(a)^* = \big(h(a \vee b)\big)^* = \big(h(a) \vee h(b)\big)^* = h(a)^* \wedge h(b)^*,$$

whence $h(a)^* \leq h(b)^*$. Since $h(\mathrm{coz}\,\alpha) = \mathrm{coz}\,\big(\mathcal{R}h(\alpha)\big)$, we conclude that $\mathrm{Ann}(\mathcal{R}h(\alpha)) \subseteq \mathrm{Ann}(\mathcal{R}h(\beta))$. Therefore $\mathcal{R}h$ is weakly skeletal.

(⇐) Let $a, b \in \mathrm{Coz}\,M$ be such that $a^* = b^*$. For brevity, write $a = \mathrm{coz}\,\alpha$ and $b = \mathrm{coz}\,\beta$. Then $(\mathrm{coz}\,\alpha)^* = (\mathrm{coz}\,\beta)^*$, which implies $\mathrm{Ann}(\alpha) = \mathrm{Ann}(\beta)$. The weak skeletality of $\mathcal{R}h$ implies

$$\mathrm{Ann}(\mathcal{R}h(\alpha)) = \mathrm{Ann}(\mathcal{R}h(\beta)),$$

whence $\big(\mathrm{coz}(\mathcal{R}h(\alpha))\big)^* = \big(\mathrm{coz}(\mathcal{R}h(\beta))\big)^*$. Since $h(\mathrm{coz}\,\alpha) = \mathrm{coz}\,\big(\mathcal{R}h(\alpha)\big)$, we conclude that $\big(h(\mathrm{coz}\,\alpha)\big)^* = \big(h(\mathrm{coz}\,\beta)\big)^*$, whence $h(a)^* = h(b)^*$ Therefore h is weakly skeletal. □

Remark 2.3.3. In light of the way we have defined weak skeletality for frame homomorphisms, one may wonder if we should rather not have defined a ring homomorphism $\phi : A \to B$ to be weakly skeletal if, for any $u, v \in A, \mathrm{Ann}(u) = \mathrm{Ann}(v)$ implies $\mathrm{Ann}(\phi(u)) = \mathrm{Ann}(\phi(v))$. This latter definition would be formally weaker, as one readily checks. But it does not seem to imply the first. However, if A and B are reduced f-rings, then the two are equivalent. To see the nontrivial implication, let $\mathrm{Ann}(u) \subseteq \mathrm{Ann}(v)$ for $u, v \in A$. Then $\mathrm{Ann}(u) = \mathrm{Ann}(u) \cap \mathrm{Ann}(v) \subseteq \mathrm{Ann}(u^2 + v^2)$. Let $r \in \mathrm{Ann}(u^2 + v^2)$. Then $r^2u^2 + r^2v^2 = 0$, which implies $ru = rv = 0$, showing that $\mathrm{Ann}(u^2 + v^2) \subseteq \mathrm{Ann}(u)$, and hence $\mathrm{Ann}(u) = \mathrm{Ann}(u^2 + v^2)$. Then, as above,

$$\mathrm{Ann}(\phi(u)) = \mathrm{Ann}(\phi(u)^2 + \phi(v)^2) = \mathrm{Ann}(\phi(u)) \cap \mathrm{Ann}(\phi(v)),$$

so that $\mathrm{Ann}(\phi(u)) \subseteq \mathrm{Ann}(\phi(v))$ as required.

Next, we give an algebraic characterization of coz-onto frame homomorphisms, which we will then use as basis for defining "coz-onto" ring homomorphisms. We will of course give them a different name because calling them coz-onto would be stretching nomenclature too far. For a ring A and an element $a \in A$, denote by $\mathfrak{M}(a)$ the set of all maximal ideals of A which contain a.

Lemma 2.3.4. *A frame homomorphism $h : M \to L$ is coz-onto if and only if for every $\alpha \in \mathcal{R}L$ there is a $\gamma \in \mathcal{R}M$ such that $\mathfrak{M}(\alpha) = \mathfrak{M}(\mathcal{R}h(\gamma))$.*

Proof. We need only observe that, in any frame N, for any $\tau, \rho \in \mathcal{R}N$, $\mathfrak{M}(\tau) = \mathfrak{M}(\rho)$ if and only if $\operatorname{coz}\alpha = \operatorname{coz}\beta$; and $\boldsymbol{M}_{\operatorname{coz}\tau} = \bigcap \mathfrak{M}(\tau)$, by the same argument as in the proof of [31, Lemma 3.2]. □

Definition 2.3.5. We say a ring homomorphism $\phi : A \to B$ is $\mathfrak{M}$-*full* if, for every $b \in B$, there is an $a \in A$ such that $\mathfrak{M}(b) = \mathfrak{M}(\phi(a))$.

An example of a $\mathfrak{M}$-full homomorphism is the inclusion $A^* \to A$, where A is an f-ring with bounded inversion. For, if $a \in A$, then $\frac{a}{1+|a|}$ is an element of A^* with $\mathfrak{M}(a) = \mathfrak{M}(\frac{a}{1+|a|})$.

Proposition 2.3.6. *Let $\phi\colon A \to B$ be a $\mathfrak{M}$-full weakly skeletal homomorphism. If A is a quasi m-ring, then B is also a quasi m-ring.*

Proof. Let M be a maximal ideal of B and b_1, b_2 be elements of M. For each $i = 1, 2$, we can find, by heaviness of ϕ, an element $a_i \in A$ such that $\mathfrak{M}(b_i) = \mathfrak{M}(\phi(a_i))$. Let N be a maximal ideal of A with $\phi^{-1}[M] \subseteq N$. Then $\phi(a_i) \in M$, which implies $a_1, a_2 \in \phi^{-1}[M] \subseteq N$. Since A is a quasi m-ring, there are elements $u, v \in A$ such that

$$u \in \operatorname{Ann}(a_1), \qquad v \notin N, \qquad \operatorname{Ann}(a_1^2 + u^2) \subseteq \operatorname{Ann}(a_2 v).$$

We show that the elements $\phi(u)$ and $\phi(v)$ of B satisfy the requirements of Proposition 2.2.5 for a_1 and a_2. We have that $\phi(u) \in \operatorname{Ann}(b_1)$ because $b_1 = \phi(a_1)$. Also, $\phi(v) \notin M$, for otherwise $v \in \phi^{-1}[M] \subseteq N$. Since ϕ is weakly skeletal and $\operatorname{Ann}(a_1^2 + u^2) \subseteq \operatorname{Ann}(a_2 v)$, it follows that

$$\operatorname{Ann}\big(\phi(a_1^2 + u^2)\big) = \operatorname{Ann}\big(b_1^2 + \phi(u)^2\big) \subseteq \operatorname{Ann}\big(\phi(a_2)\phi(v)\big) = \operatorname{Ann}\big(b_2\phi(v)\big).$$

Therefore $\phi(u)$ and $\phi(v)$ do indeed satisfy the desired requirements. □

We observed above that, for a reduced f-ring A with bounded inversion, the inclusion map $A^* \to A$ is $\mathfrak{M}$-full. We now note that it is also weakly skeletal. Denote annihilation in A^* by Ann_*. Let a and b be elements of A^* with $\mathrm{Ann}_*(a) \subseteq \mathrm{Ann}_*(b)$. Let $r \in \mathrm{Ann}(a)$. Then $\frac{ar}{1+|a|} = 0$, and therefore $\frac{r}{1+|r|} \in \mathrm{Ann}_*(a) \subseteq \mathrm{Ann}_*(b)$. Thus $rb = 0$, which yields the result. We therefore have the following corollary. Recall that $\mathcal{R}(\beta L)$ is isomorphic to $\mathcal{R}^*L$ [12].

Corollary 2.3.7. *Let A be a reduced f-ring with bounded inversion. If A^* is a quasi m-ring, then so is A. Consequently, if βL is a quasi cozero complemented frame, then L is a quasi cozero complemented frame.*

Since, as observed above, a frame homomorphism $h\colon M \to L$ is weakly skeletal (resp., coz-onto) precisely when the ring homomorphism $\mathcal{R}M \to \mathcal{R}L$ is weakly skeletal (resp., $\mathfrak{M}$-full), the following corollary is apparent.

Corollary 2.3.8. *Let $h\colon M \to L$ be a coz-onto weakly skeletal frame homomorphism. If M is a quasi cozero complemented frame, then L is also a quasi cozero complemented frame.*

We now apply this to identify certain subspaces of quasi m-spaces which inherit the property of being a quasi m-space. Recall that a frame homomorphism $h\colon M \to L$ is called *nearly open* [13] if $h(a^*) = h(a)^*$ for every $a \in M$. Nearly open homomorphisms include dense onto homomorphisms (see [13, Lemma 2.1]), and they are skeletal. Other skeletal homomorphisms are the following. An element $r \in M$ is called *regular* if $r = r^{**}$. For a regular $r \in M$, denote the pseudocomplement in $\uparrow r$ by $(\,)^{\#}$. It is shown in [34, Lemma 4.5] that, for any $t \in \uparrow r$, $t^{\#} = (t \wedge r^*)^*$. In the proof that follows we shall use the fact that, in any frame $(u \wedge v)^{**} = u^{**} \wedge v^{**}$ because the mapping $x \mapsto x^{**}$ is a nucleus.

Lemma 2.3.9. *For any regular $r \in M$, the homomorphism $\kappa_r\colon M \to \uparrow r$ given by $\kappa_r(a) = r \vee a$ is skeletal.*

Proof. Let a and b be elements of M with $a^* = b^*$. Then

$$\begin{aligned}(r \vee a)^{\#} = \big((r \vee a) \wedge r^*\big)^* &= (a \wedge r^*)^* \\ &= (a \wedge r^*)^{***} \\ &= (a^{**} \wedge r^*)^* = (b^{**} \wedge r^*)^* = (r \vee b)^{\#},\end{aligned}$$

which proves the claim. □

For any frame L and $c \in \operatorname{Coz} L$, the open quotient map $L \to {\downarrow}c$ is coz-onto [4, Corollary 3.2.11], and for any normal L and $a \in L$, the closed quotient map $L \to {\uparrow}a$ is coz-onto [4, Theorem 8.3.3]. Therefore the following corollary is apparent.

Corollary 2.3.10. *The following statements hold for completely regular frames.*

(1) *If L is a quasi cozero complemented frame and $c \in \operatorname{Coz} L$, then ${\downarrow}c$ is a quasi cozero complemented frame.*

(2) *If L is a normal quasi cozero complemented frame and r is a regular element of L, then ${\uparrow}r$ is a quasi cozero complemented frame.*

(3) *A nearly open coz-quotient of a quasi cozero complemented frame is a quasi cozero complemented frame. Hence, a dense coz-quotient of a quasi cozero complemented frame is a quasi cozero complemented frame.*

That L is a quasi cozero complemented frame whenever βL is could also be deduced from the last statement in this corollary. Applied to spaces, this corollary tells us the following. Recall that a subspace X of a space Y is *nearly open* in Y if every open set in X is dense in some open set in Y. Dense subspaces are nearly open.

Corollary 2.3.11. *For Tychonoff spaces we have the following results.*

(1) *A cozero subspace of a quasi m-space is a quasi m-space.*

(2) *A regular-closed subspace of a normal quasi m-space is an m-space.*

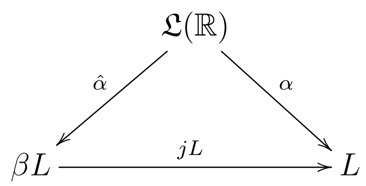
(3) *A nearly open z-embedded subspace of a quasi m-space is a quasi m-space. Hence, a dense z-embedded subspace of a quasi m-space is a quasi m-space.*

We now say something about Oz-spaces. Recall from [15] that a subset S of a topological space X is *z-embedded* in X in case each zero-set of S is the restriction to S of a zero-set of X, (A zero-set is the set of zeros of a real-valued continuous function), a Tychonoff space X is called an *Oz-space* if every open subspace of X is z-embedded. This notion was extended to frames in [9].

Proposition 2.3.12. *An Oz-space is a quasi m-space if and only if every open subspace is a quasi m-space. An Oz-frame L is a quasi cozero complemented frame if and only if $\downarrow a$ is a quasi cozero complemented frame for every $a \in L$.*

In conclusion, we mentioned earlier that we have not been able to determine if the Stone-Čech compactification of every quasi cozero complemented frame is a quasi cozero complemented frame. We do however have a class of frames (not necessarily spatial) for which this can be assert. Following [32], we say a point I of βL is *sharp* if, for any $c \in \operatorname{Coz} L$, $r_L(c) \leq L$ implies $c \in I$. We say it is *almost sharp* if, for any $c \in \operatorname{Coz} L$, $r_L(c) \leq I$ implies c is not dense. It is shown in [32] that L is a P-frame (resp. almost P-frame) precisely when every point of βL is sharp (resp. almost sharp).

We aim to show that if βL has no almost sharp point, or if the join of every almost sharp point is not the top, then βL is a quasi cozero complemented frame exactly when L is a quasi cozero complemented frame. We need some background. Since $j_L \colon \beta L \to L$ is a C^*-quotient map, for every $\alpha \in \mathcal{R}^*L$ there is an element $\hat{\alpha} \in \mathcal{R}L$ such that the triangle



commutes. As shown in [28, Lemma 3.8], maximal ideals of $\mathcal{R}^*L$, are exactly the ideals

$$\boldsymbol{M}^{*I} \quad = \quad \{\alpha \in \mathcal{R}^*L \mid \operatorname{coz}(\hat{\alpha}) \leq I\}$$

for $I \in Pt(\beta L)$. Below we shall use the fact, proved in [31, Lemma 3.8], that if A is a reduced f-ring with bounded inversion, then the extension, I^e, of any d-ideal in A^* is a d-ideal in A, and the contraction, J^c, of any d-ideal J of A is a d-ideal of A^*. Furthermore, $I^{ec} = I$ and $J^{ce} = J$. It is easy to show that we have a similar situation with "d-ideal" replaced by " minimal prime ideal".

Proposition 2.3.13. *Let L be a frame such that βL has no almost sharp point, or join of every sharp point is not the top. Then βL is a quasi cozero complemented frame if and only if L is a quasi cozero complemented frame.*

Proof. Only the right-to-left implication needs verification, and for this it suffices to show that $\mathcal{R}^*L$ is a quasi m-ring under the pertinent hypotheses. So let P be a prime d-ideal of $\mathcal{R}^*L$. Then P^e is a prime d-ideal in $\mathcal{R}L$, and so, in view of $\mathcal{R}L$ being a quasi m-ring, $P^e \in \mathrm{Min}(\mathcal{R}L)$ or $P^e \in \mathrm{Max}(\mathcal{R}L)$.

Case (i): Suppose βL has no almost sharp point. Then P^e cannot be a maximal ideal, for if it were, then there would be a point I of βL such that $P^e = \boldsymbol{M}^{\boldsymbol{I}}$. But then since P^e is a d-ideal, every element of $\boldsymbol{M}^{\boldsymbol{I}}$ is a zero-divisor, implying that, for any $\gamma \in \mathcal{R}L$, if $\gamma \in \boldsymbol{M}^{\boldsymbol{I}}$, then $\operatorname{coz}\gamma$ is not dense. That is, if $c \in \operatorname{Coz} L$ and $r_L(c) \leq I$ then c is not dense. This makes I an almost sharp point. Therefore P^e is a minimal prime ideal, and hence P^{ec} is a minimal prime ideal, that is, P is a minimal prime ideal. Therefore $\mathcal{R}^*L$ (and hence $\mathcal{R}(\beta L)$) is a quasi m-ring.

Case (ii): Suppose βL has almost sharp points, and each has join unequal to the top. Let P be a prime d-ideal in $\mathcal{R}^*L$. As above, if $P^e \in \mathrm{Min}(\mathcal{R}L)$ we are done. So suppose $P^e \in \mathrm{Max}(\mathcal{R}L)$. Take a point I in βL such that $P^e = \boldsymbol{M}^{\boldsymbol{I}}$. Then, as observed above, I is almost sharp, and hence $\bigvee I < 1$, by hypothesis. We claim that $P = \boldsymbol{M}^{*\boldsymbol{I}}$. Proving this will complete the proof. To show this, it suffices to show that $(\boldsymbol{M}^{\boldsymbol{I}})^c = \boldsymbol{M}^{*\boldsymbol{I}}$. By [28, Proposition 4.2], we need only show that $\boldsymbol{M}^{*\boldsymbol{I}}$ contains no unit of $\mathcal{R}L$. Suppose, for contradiction, that some $\alpha \in \boldsymbol{M}^{*\boldsymbol{I}}$ is a unit of $\mathcal{R}L$. Since $\alpha \in \boldsymbol{M}^{*\boldsymbol{I}}$, $\operatorname{coz}(\hat{\alpha}) \leq I$. Since α is invertible in $\mathcal{R}L$, we have

$$1 = \operatorname{coz}\alpha = \operatorname{coz}(j_L \cdot \alpha) = \bigvee \operatorname{coz}(\hat{\alpha}) \leq \bigvee I,$$

which yields the contradiction we seek. □

Chapter 3

Quasi P-frames

We will introduce quasi P-frames, and the definition we use is motivated by the definition of quasi P-spaces [40]. Although P-frames generalize P-spaces in the sense that a Tychonoff space is a P-space precisely when the frame of its open sets is a P-frame, it has recently been shown by Ball, Walters-Wayland and Zenk [5] that, in stark contrast with P-spaces, there are P-frames with quotients which are not P-frames. In this chapter we examine how far the theory of quasi P-frames parallels that of quasi P-spaces as defined by Henriksen, Martínez and Woods [40].

3.1 Characterizations of quasi P-frames

In this section we consider frames L for which $\mathcal{R}L$ satisfies condition (zMM). That is, frames L such that every prime z-ideal of $\mathcal{R}L$ is minimal or maximal. It will turn out that these are frame versions of what are called quasi P-spaces in [40]. That is why in the definition that follows we use the term quasi P-point. Although in spaces quasi P-points are points of X, we find it appropriate to use the same term for points of βL in frames.

Definition 3.1.1. A point I of βL is a quasi *P-point* if whenever Q is a prime z-ideal of $\mathcal{R}L$ with $Q \subseteq \boldsymbol{M}^I$, then $Q = \boldsymbol{M}^I$ or Q is a minimal prime ideal. We say L is *quasi P-frame* if every point of βL is a quasi P-point.

Since maximal ideals of $\mathcal{R}L$ are in one-one correspondence with the points of βL, it is evident that L is a quasi P-frame precisely if $\mathcal{R}L$ satisfies the condition (zMM). In [50], Martínez defines the dimension of L, $\dim(L)$, to be the maximum of the lengths n of chains of primes $p_0 < p_1 < \cdots < p_n$, if such a maximum exists, and infinity, otherwise. In [53, Theorem 2.8] Martínez and Zenk give a criterion, in terms of compact elements, for determining an algebraic frame L and a nonnegative integer k when $\dim(L) \leq k$. Applied to a coherent frame (so that the top is compact), and considering the case $k = 1$, this yield the following characterization:

For a coherent frame L, $\dim(L) \leq 1$ if and only if for every pair $a, b \in \mathfrak{k}(L)$, there is a pair $c, d \in \mathfrak{k}(L)$ such that

$$a \wedge c = 0, \qquad b \vee d = 1_L, \qquad b \wedge d \leq a \vee c.$$

Now, as shown in [31, Proposition 3.5], the lattice $\mathrm{Zid}(\mathcal{R}L)$ of z-ideals of $\mathcal{R}L$ is a coherent frame whose lattice of compact elements is

$$\mathfrak{k}\big(\mathrm{Zid}(\mathcal{R}L)\big) = \{\boldsymbol{M}_c \mid c \in \mathrm{Coz}\, L\}.$$

Since maximal ideals are z-ideals, and the minimal prime elements of $\mathrm{Zid}(\mathcal{R}L)$ are precisely the minimal prime ideals of $\mathcal{R}L$, it follows that L is a quasi P-frame if and only if, $\dim(\mathrm{Zid}(\mathcal{R}L)) \leq 1$. Thus, exactly as in the case of quasi P-spaces ([53, Remark 4.6(b)]), we have the following characterization.

Proposition 3.1.2. *L is a quasi P-frame if an only if for every pair a, b of cozero elements of L, there is a pair c, d of cozero elements of L such that*

$$a \wedge c = 0, \qquad b \vee d = 1, \qquad b \wedge d \leq a \vee c.$$

This immediately shows that a Tychonoff space X is a quasi P-space if and only if $\mathfrak{O}X$ is a quasi P-frame. Of course this can also be shown directly from the definition. In [40, Remark 2.9] it is demonstrated that any infinite P-space is a quasi P-space whose Stone-Čech compactification is not a quasi P-space. Using the proposition above we show that if $L \to M$ is a coz-onto homomorphism and L is a quasi P-frame, then M is also a quasi P-frame. Recall that a frame homomorphism $h : L \to M$ is *coz-onto* if for every $d \in \operatorname{Coz} M$, there exists $c \in \operatorname{Coz} L$ such that $h(c) = d$.

Lemma 3.1.3. *Let $h : L \to M$ is a coz-onto homomorphism and L is a quasi P-frame, then M is a quasi P-frame.*

Proof. Let $a, b \in \operatorname{Coz} M$. Since h is coz-onto, there exist $u, v \in \operatorname{Coz} L$ such that $h(u) = a$ and $h(v) = b$. Since L is quasi P-frame, there exist $u', v' \in \operatorname{Coz} L$ such that $u \wedge u' = 0$, $\quad v \vee v' = 1$ and $v \wedge v' \leq u \vee u'$. Now $h(u) \wedge h(u') = h(0)$, $\quad h(v) \vee h(v') = h(1)$ and $h(v) \wedge h(v') \leq h(u) \vee h(u')$ since frame homomorphisms preserve cozero elements. Further

$$a \wedge h(u') = 0, \quad b \vee h(v') = 1$$

and

$$b \wedge h(v') \leq a \vee h(u').$$

Therefore $h(u')$ and $h(v')$ are cozero elements in M, and hence M is quasi P-frame. □

In [34] it is shown that for a frame surjection $h : L \to M$, if M is Lindelöf and L is completely regular, then h is coz-onto. Then we have the following.

Proposition 3.1.4. *Let L be a quasi P-frame and $h : L \to M$ be a quotient of L with M Lindelöf frame. Then M is quasi P-frame.*

Corollary 3.1.5. *Let L be a quasi P-frame, then $\downarrow c$ is a quasi P-frame for every $c \in \operatorname{Coz} L$.*

Thus in particular, from Lemma 3.1.3, we have

> *if βL is a quasi P-frame, then L is a quasi P-frame.*

Since a frame L is pseudocompact exactly when βL is isomorphic to υL (see [11]), and since $\mathcal{R}L$ is isomorphic to $\mathcal{R}(\upsilon L)$, it is easy to see that

> *a pseudocompact frame is quasi P-frame if and only if its Stone-Čech compactification is a quasi P-frame.*

3.2 Some special quasi P-frames

Recall that a completely regular frame L is *cozero complemented* if for each $u \in \operatorname{Coz} L$, there is a $v \in \operatorname{Coz} L$ such that $u \wedge v = 0$ and $u \vee v$ is dense in L. In preparation for the following result (which generalizes [40, Theorem 5.5]), we observe the following about cozero complemented frames.

Lemma 3.2.1. *If L is cozero complemented and Q is a prime ideal of $\mathcal{R}L$ which is not minimal prime, then Q contains a non-divisor of zero.*

Proof. For, there is a $\gamma \in Q$ such that $\operatorname{Ann}(\gamma) \subseteq Q$. Since L is cozero complemented, there is a $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\gamma \wedge \operatorname{coz}\delta = 0$ and $\operatorname{coz}\gamma \vee \operatorname{coz}\delta$ is dense, and therefore $\operatorname{coz}(\gamma^2 + \delta^2)$ is dense. Then $\gamma^2 + \delta^2$ is a non-divisor of zero, and since $\gamma\delta = \mathbf{0}, \delta \in Q$, whence $\gamma^2 + \delta^2 \in Q$. □

Proposition 3.2.2. *Let L be a completely regular frame.*

(*a*) *If L is a quasi P-frame and $I \subseteq \operatorname{Coz} L$ is a prime ideal in $\operatorname{Coz} L$ containing a dense cozero element, then I is a maximal ideal in $\operatorname{Coz} L$.*

(*b*) *If L is cozero complemented, and any prime ideal $I \subseteq \operatorname{Coz} L$ containing a dense cozero element is maximal, then L is a quasi P-frame.*

Proof. (a) Let $Q = \{\gamma \in \mathcal{R}L \mid \operatorname{coz}\gamma \in I\}$. We claim that Q is a prime z-ideal. We show first that Q is an ideal of $\mathcal{R}L$. Consider any $\alpha, \beta \in Q$, and any $\gamma \in \mathcal{R}L$. Then $\operatorname{coz}\alpha$ and $\operatorname{coz}\beta$ are elements of I, which implies $\operatorname{coz}\alpha \vee \operatorname{coz}\beta \in I$ since I is an ideal in $\operatorname{Coz} L$. Since $\operatorname{coz}(\alpha+\beta) \leq \operatorname{coz}\alpha \vee \operatorname{coz}\beta$, it follows that $\operatorname{coz}(\alpha+\beta) \in I$ since I is a downset. Thus, $\alpha + \beta \in Q$. Similarly, from the relations $\operatorname{coz}(\alpha\gamma) = \operatorname{coz}\alpha \wedge \operatorname{coz}\gamma \leq \operatorname{coz}\alpha$, we deduce that $\alpha\gamma \in Q$. Therefore Q is an ideal in $\mathcal{R}L$.

It is easy to see that Q is a z-ideal since $\operatorname{coz}\alpha = \operatorname{coz}\beta$ and $\beta \in Q$ implies $\operatorname{coz}\alpha \in I$, whence $\alpha \in Q$. To see that Q is prime, consider any $\alpha, \beta \in \mathcal{R}L$ such that $\alpha\beta \in Q$. Then $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \operatorname{coz}(\alpha\beta) \in I$, implying $\operatorname{coz}\alpha \in I$ or $\operatorname{coz}\beta \in I$ since I is a prime ideal in $\operatorname{Coz} L$. It follows therefore that $\alpha \in Q$ or $\beta \in Q$. Hence Q is prime.

Now, being a prime ideal in $\mathcal{R}L$, Q is contained in a maximal ideal M, say, and contains a minimal prime ideal P, say. By hypothesis, there is a dense cozero element c in L which is contained in I. Take $\gamma \in \mathcal{R}L$ such that $\operatorname{coz}\gamma = c$. Then $\gamma \in Q$, and γ is a non-divisor of zero since $\operatorname{coz}\gamma$ is dense. Thus, $\gamma \notin P$ because minimal prime ideals consist entirely of zero-divisors. So $Q \neq P$, and therefore $Q = M$ since L is a quasi P-frame.

We show from this that I is a maximal ideal in $\operatorname{Coz} L$. Consider any $d \in \operatorname{Coz} L$ with $d \notin I$. We must show that $\langle I, d\rangle$, the ideal generated by I and d, is the whole of $\operatorname{Coz} L$. Take $\gamma \geq 0$ in $\mathcal{R}L$ with $d = \operatorname{coz}\gamma$. Then $\gamma \notin Q$. Since Q is a maximal ideal in $\mathcal{R}L$, the ideal $\langle Q, \gamma\rangle = \mathcal{R}L$, so $\mathbf{1} = q + \delta\gamma$, for some $q \in Q$ and $\delta \in \mathcal{R}L$. Thus, $1 = \operatorname{coz}(q + \delta\gamma) \leq \operatorname{coz} q \vee \operatorname{coz}\gamma \in \langle I, d\rangle$. Therefore $\langle I, d\rangle = \operatorname{Coz} L$, showing that I is a maximal ideal in $\operatorname{Coz} L$.

(b) Let Q be a prime z-ideal which is not minimal prime. Since L is cozero complemented, Lemma 3.2.1 shows that Q contains some non-divisor of zero, say γ. Put $I = \{\operatorname{coz}\alpha \mid \alpha \in Q\}$. It is easy to check that I is a prime ideal of $\operatorname{Coz} L$. Since γ is a non-divisor of zero, $\operatorname{coz}\gamma$ is dense. But $\operatorname{coz}\gamma \in I$; therefore I is a maximal in $\operatorname{Coz} L$, by hypothesis. Arguing as above, we have that Q is a maximal ideal in $\mathcal{R}L$. It follows therefore that L is a quasi P-frame. □

Recall that if u and v are cozero elements in L, then $r_L(u \vee v) = r_L(u) \vee r_L(v)$. For the proof of the next proposition, we shall use the following fact: a frame L is a *P-frame* if and only if for every $a \in \operatorname{Coz} L$, there exists $b \in \operatorname{Coz} L$ such that $a \wedge b = 0$ and $a \vee b = 1$.

Proposition 3.2.3. *Let L be a cozero complemented frame such that, for every dense $c \in \operatorname{Coz} L$, $\uparrow c$ is a P-frame and $\kappa_c \colon L \to \uparrow c$ is coz-onto. Then L is a quasi P-frame.*

Proof. Let Q be a prime z-ideal of $\mathcal{R}L$ which is not minimal prime. Then Q contains some non-divisor of zero, say γ. For brevity, write $c = \operatorname{coz}\gamma$. Since c is dense, as γ is a non-divisor of zero, the map $\kappa_c \colon L \to \uparrow c$ is coz-onto, and $\uparrow c$ is a P-frame. Now, Q is contained in some (unique) maximal ideal, say $\boldsymbol{M^I}$, for some $I \in Pt(\beta L)$. We shall be done if we can show that $Q = \boldsymbol{M^I}$. Let $\alpha \in \boldsymbol{M^I}$, and write $a = \operatorname{coz}\alpha$. Then $r_L(a) \leq I$. Since $c \vee a \in \operatorname{Coz}(\uparrow c)$ and $\uparrow c$ is a P-frame, by hypothesis, $(c \vee a)^{\#} \vee (c \vee a) = 1$, where

$(c \vee a)^{\#}$ denotes the pseudocomplement of $c \vee a$ in ${\uparrow}c$. Since $(c \vee a)^{\#}$ is complemented in ${\uparrow}c$ it is a cozero element in this frame, and hence, in light of κ_c being coz-onto, there is an $s \in \operatorname{Coz} L$ such that $(c \vee a)^{\#} = c \vee s$. Thus,

$$(c \vee a) \wedge (c \vee s) = c \quad \text{and} \quad (c \vee a) \vee (c \vee s) = 1.$$

Since both α and γ are in $\boldsymbol{M}^I$, we have

$$r_L(a \vee c) = r_L(a) \vee r_L(c) \leq I,$$

which then quickly shows that $r_L(c \vee s) \nleq I$, lest I be the top. Pick $\tau \in \mathcal{R}L$ with $\operatorname{coz} \tau = c \vee s$. Then $\tau \notin \boldsymbol{M}^I$, and hence $\tau \notin Q$. Since $\operatorname{coz}((\alpha^2 + \gamma^2)\tau) = \operatorname{coz} \gamma$, and Q is a z-ideal, it follows that $(\alpha^2 + \gamma^2)\tau \in Q$. Thus $\alpha^2 + \gamma^2 \in Q$ because Q is a prime ideal. Consequently $\alpha \in Q$ because Q is a z-ideal. Therefore $\boldsymbol{M}^I \subseteq Q$, and hence equality. □

The next result extends [40, Theorem 5.6]. The proof we give though is completely different from that which could be modeled on the one given in [40]. We use Proposition 3.1.2.

Proposition 3.2.4. *Let L be a quasi P-frame and c be a dense cozero element of L such that $\kappa_c \colon L \to {\uparrow}c$ is coz-onto. Then ${\uparrow}c$ is a P-frame.*

Proof. Let $z \in \operatorname{Coz}({\uparrow}c)$. Since κ_c is coz-onto, there is an $a \in \operatorname{Coz} L$ such that $z = c \vee a$. Now c and a are cozero elements of the quasi P-frame L, so, by Proposition 3.1.2, there are cozero elements u and v of L such that

$$c \wedge u = 0, \qquad a \vee v = 1, \qquad a \wedge v \leq c \vee u.$$

Since c is dense, $u = 0$, and hence $a \wedge v \leq c$. Now, $c \vee v$ is a cozero element of ${\uparrow}c$ with the property that

$$(c \vee a) \vee (c \vee v) = 1, \quad \text{and} \quad (c \vee a) \wedge (c \vee v) = c \vee (a \wedge v) = c = 0_{\uparrow c}.$$

This shows that z is complemented in $\operatorname{Coz}({\uparrow}c)$. Therefore ${\uparrow}c$ is a P-frame. □

From the previous two propositions we immediately deduce the following.

Corollary 3.2.5. *If L is a cozero complemented frame such that, for every dense $c \in \operatorname{Coz} L$, the map $\kappa_c \colon L \to \uparrow c$ is coz-onto, then L is a quasi P-frame if and only if $\uparrow c$ is a P-frame.*

We have seen above that if $h : M \to L$ is coz-onto, then L is a quasi P-frame if M is quasi P-frame. We shall now give an instance where the property of being quasi-P is transferred by a homomorphism from the codomain to the domain. Recall that for a frame homomorphism $h \colon M \to L$ is called *closed* if, for every $a \in M$ and $b \in L$, $h_*(h(a) \vee b) = a \vee h_*(b)$. For regular frames, $h \colon M \to L$ is *closed* if and only if $h(a) \vee b = 1$ implies $a \vee h_*(b) = 1$. In the previous chapter we discussed what we called weakly skeletal maps. Here we need a somewhat weaker condition.

Motivated by the fact that a frame homomorphism $h : M \to L$ is skeletal if it sends dense elements to dense elements, we shall say $h : M \to L$ is *coz-skeletal* if it sends dense cozero elements to dense cozero elements. We suggest that the reader see [40] for comparison with the spatial case. Weak skeletality implies coz-skeletality because if $c \in \operatorname{Coz} M$ is dense, then $c^* = 1^*$, and $1 \in \operatorname{Coz} M$.

Proposition 3.2.6. *Let $h \colon M \to L$ be an injective coz-skeletal closed map. Suppose M and L are cozero complemented, and that, further, L is normal and quasi P-frame. Then M is a quasi P-frame.*

Proof. We show first that M is normal, whence closed quotients will be coz-quotients. Let $a \vee b = 1$ in M. Then $h(a) \vee h(b) = 1$. Since L is normal, there are elements $u, v \in L$ such that

$$u \wedge v = 0 \quad \text{and} \quad h(a) \vee u = 1 = h(b) \vee v.$$

We show that $h_*(u)$ and $h_*(v)$ satisfy the normality requirements for a and b. To start, $h_*(u) \wedge h_*(v) = h_*(u \wedge v) = 0$. Since h is a closed map,

$$a \vee h_*(u) = 1 = b \vee h_*(v).$$

Therefore M is normal. We now apply Corollary 3.2.5. Let c be a dense cozero element of M. Then $h(c)$ is a dense cozero element of L by the coz-skeletality of h. Thus, by

Corollary 3.2.5, $\uparrow h(c)$ is a P-frame. Let $w \in \mathrm{Coz}(\uparrow c)$. Then $h(w) \in \mathrm{Coz}(\uparrow h(c))$, as one checks easily. Note that, since $h(c) \in \mathrm{Coz}\, L$ and $\kappa_{h(c)} \colon L \to \uparrow h(c)$ is coz-onto, the cozero elements of $\uparrow h(c)$ are precisely the cozero elements of L which are above $h(c)$. Since $\uparrow h(c)$ is P-frame, there is a $t \in \mathrm{Coz}\, L$ above $h(c)$ such that

$$h(w) \vee t = 1 \quad \text{and} \quad h(w) \wedge t = h(c).$$

The equality on the left implies $w \vee h_*(t) = 1$ since h is a closed map, and the one on the right implies $h\big(w \wedge h_*(t)\big) \leq h(c)$, so that $w \wedge h_*(t) \leq c$ because h is one-one. By normality, there is a $d \in \mathrm{Coz}\, M$ such that $d \leq h_*(t)$ and $w \vee d = 1$. Therefore $c \vee d$ is a cozero element of $\uparrow c$ such that $w \vee (c \vee d) = 1$ and, in light of $c \leq h_*(t)$,

$$c \leq w \wedge (c \vee d) \leq w \wedge \big(c \vee h_*(t)\big) = w \wedge h_*(t) \leq c.$$

Therefore $c \vee d$ misses w in the frame $\uparrow c$ and joins it at the top. This shows that $\uparrow c$ is a P-frame, and hence M is a quasi P-frame. □

We end with a result which tells us, among other things, that in the class of metrizable frames the quasi-P ones are precisely those whose nowhere dense quotients are closed. In [20] a frame homomorphism $h : M \to M$ is called *nowhere dense* if for each nonzero $x \in M$, there is a nonzero $y \leq x$ such that $h(y) = 0$. This is a conservative extension of the topological concept of nowhere density because, as shown in [20, Proposition 3.9], a subspace S of a topological space X is nowhere dense if and only if the frame homomorphism $\mathfrak{O}X \to \mathfrak{O}S$, induced by the subspace inclusion $S \hookrightarrow X$, is nowhere dense. It is shown in [23, Lemma 3.2] that h is nowhere dense if and only if $h_*(0)$ is a dense element. Further, [20, Proposition 3.9] shows that $h : M \to L$ is nowhere dense precisely if, viewed as locales, $\mathrm{Fix}(h_* h)$ has zero meet (in the co-frame of sublocales) with the smallest dense sublocale of L. Thus, this notion of nowhere density agrees with that of Plewe [56].

Recall that a topological space is said to be a *nodec space* [18] if every nowhere dense subspace is closed. We extend this to frames.

Definition 3.2.7. A frame L is *nodec* if, for every nowhere dense quotient map $g \colon L \to N$, the homomorphism $\uparrow g_*(0) \to N$, mapping as g, is an isomorphism.

In localic terms this definition says a locale is nodec if every nowhere dense sublocale is closed. We invite the reader to compare this with the notion of strongly submaximal locales defined in [20] by decreeing that every complemented dense sublocale be open. As remarked in [20], a locale is strongly submaximal if and only if its complemented nowhere dense sublocales are closed. So, nodec frames are more restricted than the strongly submaximal ones.

We remind the reader that a frame is called *perfectly normal* if it is normal and every element is a cozero element. Within the category of completely regular frames it suffices to define L to be perfectly normal if $\operatorname{Coz} L = L$ because $\operatorname{Coz} L$ is a normal lattice for completely regular L. Observe that perfectly normal frames are cozero complemented because $c \vee c^*$ is always dense.

Proposition 3.2.8. *A perfectly normal frame is quasi P-frame if and only if it is nodec.*

Proof. ($\Rightarrow$) Let M be a perfectly normal quasi P-frame, and let $M \xrightarrow{q} N$ be a nowhere dense quotient of M. Then $q_*(0)$ is a dense cozero element of M and the quotient map $\kappa_{q_*(0)} \colon M \to {\uparrow} q_*(0)$ is coz-onto. Therefore, by Corollary 3.2.5, ${\uparrow} q_*(0)$ is a P-frame. We show that the homomorphism ${\uparrow} q_*(0) \to M$ which maps as q is codense, which will imply it is one-one and hence an isomorphism. Consider any $a \in {\uparrow} q_*(0)$ with $q(a) = 1$. Since $a \in \operatorname{Coz}({\uparrow} q_*(0))$ there is a $d \in \operatorname{Coz}({\uparrow} q_*(0))$ such that

$$a \wedge d = q_*(0) \quad \text{ and } \quad a \vee d = 1.$$

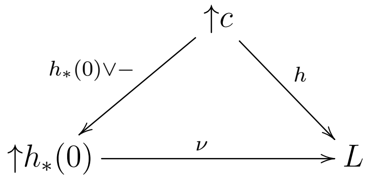
This implies $q(a) \wedge q(d) = 0$, so that $q(d) = 0$, hence $d \leq q_*(0)$, whence $d = q_*(0)$. As a consequence,

$$1 = a \vee d = a \vee q_*(0) = a.$$

Therefore M is a nodec frame.

($\Leftarrow$) Suppose M is a perfectly normal nodec frame. Let c be a dense cozero element of M. We aim to show that ${\uparrow} c$ is a P-frame. We will actually prove that it is Boolean; and for this it suffices to show that each of its quotients is closed. So let ${\uparrow} c \xrightarrow{h} L$ be a quotient of ${\uparrow} c$, and consider the composite $M \xrightarrow{\kappa_c} {\uparrow} c \xrightarrow{h} L$, which is a quotient map out of a nodec frame. We show that it is nowhere dense. Take any $x \in L$ such that

$(h\kappa_c)_*(0) \wedge x = 0$. Since $(\kappa_c)_*$ is the inclusion map $\uparrow c \to M$, this implies $h_*(0) \wedge x = 0$, whence $x = 0$ because $h_*(0)$ is a dense element of M as it is above the dense element c. Therefore the homomorphism $\nu\colon \uparrow h_*(0) \to L$, mapping as $h\kappa_c$, is an isomorphism. We therefore have the following commutative diagram.



This shows that L is (isomorphic to) a closed quotient of $\uparrow c$, and hence $\uparrow c$ is Boolean, and is therefore a P-frame. Thus, M is a quasi P-frame. □

Remark 3.2.9. A localic proof that a sublocale S of a nowhere dense sublocale N of a locale A is nowhere dense in A is immediate. Indeed, recall that a sublocale N of a locale A is nowhere dense if $N \wedge \mathrm{d}(A) = \mathsf{O}$, where $\mathrm{d}(A)$ denotes the smallest dense sublocale of A. So if $S \leq N$, then $S \wedge \mathrm{d}(A) \leq N \wedge \mathrm{d}(A) = \mathsf{O}$, showing that S is nowhere dense. If we had not sought to keep the algebraic flavour throughout, we would rather have used that.

Chapter 4

Weak almost P-frames

4.1 Introduction

It is well known that, for any Tychonoff space X, the ring $C(X)$ is regular in the sense of Von Neumann precisely if X is a P-space (see [37]). This result also holds in the broader context of frames [12]. Call a ring *almost regular* if each of its elements is either a zero-divisor or a unit. Every regular ring is almost regular, but not conversely. The Tychonoff spaces X for which $C(X)$ is almost regular are exactly the almost P-spaces that were introduced by Veksler in [61].

Less restricted than almost P-spaces are what Azarpanah and Karavan [1] call *weak almost P-spaces*. These are spaces X such that for every two zero-sets E and F of X with $\operatorname{Int} E \subseteq \operatorname{Int} F$, there is a nowhere dense zero-set H of X such that $E \subseteq F \cup H$. In [1] the authors characterize these spaces as precisely those X for which every singular (i.e. consisting entirely of zero-divisors) prime z-ideal of $C(X)$ is a d-ideal. We will provide another ring-theoretic characterization of these spaces.

In fact, defining a frame L to be a weak almost P-frame if it satisfies a property which is a frame-theoretic enunciation of the definition of weak almost P-spaces, we will obtain a ring-theoretic characterization of these frames which bears immediate resemblance to the frame-theoretic definition. That characterization will then be the basis for our definition of weakly regular rings. It will turn out that an f-ring is weakly regular if and only if it satisfies the prime z-ideal condition mentioned above which characterizes the rings $C(X)$ for X a weak almost P-space.

On the frame-theoretic side of things, we show that if βL is a weak almost P-frame then so is L; and conversely if L is a continuous Lindelöf frame. Another result with the Lindelöf condition concerns coproducts. It says if the coproduct $L \oplus M$ of Lindelöf frames is a weak almost P-frame then so is each summand. Applied to spaces, we deduce from it that if $X \times Y$ is a weak almost P-space where X and Y are Lindelöf with at least one of them locally compact, then X and Y are weak almost P-spaces. The result about coproducts hinges on the fact (established in the course of the proof of the proposition) that every cozero element of the coproduct $L \oplus M$ of Lindelöf frames is a countable join of "cozero rectangles" $a \oplus b$, for a and b cozero elements in L and M, respectively. It thus seemed appropriate that we end this section with characterizations of when every cozero element of a binary coproduct of frames is a countable join of cozero rectangles.

Every P-space is an almost P-space, and every almost P-space is a weak almost P-space. For rings, regularity implies almost regularity quite easily. Less obvious is that every almost regular f-ring is weakly regular. This we show in our last result which also points out the position of weak regularity in relation to other variants of regularity.

4.2 Characterizations of weak almost P-frames

Azarpanah and Karavan [1] define a Tychonoff space X to be a *weak almost P-space* if whenever E and F are zero-sets with $\mathrm{Int}E \subseteq \mathrm{Int}F$, then $E \subseteq F \cup W$ for some nowhere dense zero-set W of X. For any $U \in \mathfrak{O}X$, $\mathrm{Int}(X \smallsetminus U) = X \smallsetminus \mathrm{cl}\, U = U^*$. Consequently, the condition defining weak almost P-spaces is equivalent to saying whenever U and V are cozero-sets in X with $U^* \subseteq V^*$, then $V \cap W \subseteq U$ for some dense cozero-set W of X. We thus formulate the following definition.

Definition 4.2.1. A completely regular frame L is a *weak almost P-frame* if whenever a and b are cozero elements of L with $a^* \leq b^*$, then there is a dense cozero element c such that $b \wedge c \leq a$.

It is immediate that a Tychonoff space X is a weak almost P-space if and only if $\mathfrak{O}X$ is a weak almost P-frame. Here are some examples.

Example 4.2.2. Recall that a frame L is called an *almost P-frame* if $c = c^{**}$ for every $c \in \operatorname{Coz} L$. Every almost P-frame is a weak almost P-frame because $a^* \leq b^*$ for $a, b \in \operatorname{Coz} L$ implies $b = b^{**} \leq a^{**} = a$, so that we can take the top element of L as a witnessing dense cozero element.

Example 4.2.3. A frame L is called *cozero complemented* if for every $c \in \operatorname{Coz} L$ there is a $d \in \operatorname{Coz} L$ such that $c \wedge d = 0$ and $c \vee d$ is dense. Let L be cozero complemented and $a^* \leq b^*$ for some $a, b \in \operatorname{Coz} L$. Pick $c \in \operatorname{Coz} L$ such that $a \wedge c = 0$ and $a \vee c$ is dense. Since $c \leq a^* \leq b^*$, so that $b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = b \wedge a \leq a$, it follows that every cozero complemented frame is a weak almost P-frame.

We acknowledge one of the examiners for bringing to our attention a counterexample which is needed in 4.2.2 and 4.2.3, i.e an example of the weak almost P-frame which is neither almost P-frame nor cozero complemented. In the context of topology, $X = \{0, 1, 2, \cdots, \frac{1}{n}, \cdots\}$ as a subspace of reals is a weak almost P-space which is neither an almost P-space nor cozero complemented.

We now seek a ring-theoretic characterization of these frames. Let us formulate the following definition, which, as the calculations that follow will show, is motivated by the frame-theoretic definition above.

Definition 4.2.4. A ring A is *weakly regular* if for any $a, b \in A$ with $\operatorname{Ann}(a) \subseteq \operatorname{Ann}(b)$, there is a non-divisor of zero $c \in A$ such that $bc \in M(a)$.

The terminology suggests a weakening of regularity. That is indeed the case for reduced rings. To see this, recall that a reduced ring is regular if and only if $\operatorname{Min}(A) = \operatorname{Max}(A)$. Thus, if $\operatorname{Ann}(a) \subseteq \operatorname{Ann}(b)$, then

$$b \in \operatorname{Ann}^2(b) \subseteq \operatorname{Ann}^2(a) = \bigcap\{P \in \operatorname{Min}(A) \mid a \in P\} = \bigcap \mathfrak{M}(a) = M(a),$$

so that $c = 1$ is a non-divisor of zero with $bc \in M(a)$.

Proposition 4.2.5. *The following are equivalent for a completely regular frame L.*

(1) *L is a weak almost P-frame.*

(2) *For any $a, b \in \operatorname{Coz} L$ with $a^* = b^*$, there is a dense $c \in \operatorname{Coz} L$ such that $a \wedge c = b \wedge c$.*

(3) *$\mathcal{R}L$ is a weakly regular ring.*

Proof. (1) $\Rightarrow$ (2): Suppose a and b are cozero elements with $a^* = b^*$. Then there are dense cozero elements u and v such that $b \wedge u \leq a$ and $a \wedge v \leq b$. Now $c = u \wedge v$ is a dense cozero element with $a \wedge c = b \wedge c$ because $b \wedge u \leq a$ implies $b \wedge u \wedge v \leq a \wedge u \wedge v$, and similarly for the other inequality.

(2) $\Rightarrow$ (3): Let $\alpha, \beta \in \mathcal{R}L$ be such that $\operatorname{Ann}(\alpha) \subseteq \operatorname{Ann}(\beta)$. Then $\boldsymbol{M}_{(\operatorname{coz}\alpha)^*} \subseteq \boldsymbol{M}_{(\operatorname{coz}\beta)^*}$, which implies $(\operatorname{coz}\alpha)^* \leq (\operatorname{coz}\beta)^*$, and hence

$$(\operatorname{coz}(\alpha^2 + \beta^2))^* = (\operatorname{coz}\alpha \vee \operatorname{coz}\beta)^* = (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = (\operatorname{coz}\alpha)^*.$$

It therefore follows from (2) that there is a positive $\gamma \in \mathcal{R}L$ such that $\operatorname{coz}\gamma$ is dense and $\operatorname{coz}\alpha \wedge \operatorname{coz}\gamma = \operatorname{coz}\gamma \wedge \operatorname{coz}(\alpha^2 + \beta^2)$. Consequently,

$$\operatorname{coz}(\gamma\beta) = \operatorname{coz}(\gamma\beta^2) \leq \operatorname{coz}(\gamma\alpha^2 + \gamma\beta^2) = \operatorname{coz}\alpha \wedge \operatorname{coz}\gamma \leq \operatorname{coz}\alpha.$$

Let Q be a maximal ideal of $\mathcal{R}L$ containing α. Pick $I \in \operatorname{Pt}(\beta L)$ such that $Q = \boldsymbol{M}^{I}$. Then $r_L(\operatorname{coz}\alpha) \leq I$, which implies $r_L(\operatorname{coz}(\beta\gamma)) \leq I$, so that $\beta\gamma \in \boldsymbol{M}^{I}$. Consequently, $\beta\gamma \in M(\alpha)$. Now, γ is a non-divisor of zero because $\operatorname{coz}\gamma$ is dense, therefore $\mathcal{R}L$ is a weakly regular ring.

(3) $\Rightarrow$ (1): Suppose $a, b \in \operatorname{Coz} L$ are such that $a^* \leq b^*$. Pick $\alpha, \beta \in \mathcal{R}L$ with $a = \operatorname{coz}\alpha$ and $b = \operatorname{coz}\beta$. Now $a^* \leq b^*$ implies $\operatorname{Ann}(\alpha) \subseteq \operatorname{Ann}(\beta)$, and hence, by (3), there is a non-divisor of zero γ such that $\beta\gamma \in M(\alpha) = \boldsymbol{M}_{\operatorname{coz}\alpha}$. Thus, $c = \operatorname{coz}\gamma$ is a dense cozero element of L such that $b \wedge c \leq a$. Therefore L is a weak almost P-frame. □

Corollary 4.2.6. *The following are equivalent for a completely regular frame L.*

(1) *L is a weak almost P-frame.*

(2) *υL is a weak almost P-frame.*

(3) *λL is a weak almost P-frame.*

Since a completely regular frame L is pseudocompact precisely when υL is isomorphic to βL – which also is the case for Tychonoff spaces – we deduce from the above that:

Corollary 4.2.7. *A pseudocompact frame L is a weak almost P-frame if and only if βL is a weak almost P-frame. Similarly, a pseudocompact space X is a weak almost P-space if and only if βX is a weak almost P-space.*

We recall from [7, Lemma 2] that if $h\colon M \to L$ is dense, then the ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ is one-one. Also, recall from [4] that a quotient map $h\colon M \to L$ is a C-quotient map precisely when $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ is onto. This is however not the definition used in [4].

Corollary 4.2.8. *Let $h\colon M \to L$ be a dense C-quotient map. Then M is a weak almost P-frame if and only if L is weak almost-P.*

Interpreting this result for Tychonoff spaces we obtain the following.

Corollary 4.2.9. *A dense C-embedded subspace of a Tychonoff space is a weak almost P-space if and only if the containing space is a weak almost P-space.*

In the less restricted case we have the following corollary. Let us recall that if $h\colon M \to L$ is dense, then $h(a) = h(b)$ implies $a^* = b^*$ for very $a, b \in L$. This is so because

$$h(a^* \wedge b) = h(a^*) \wedge h(b) \leq h(a)^* \wedge h(b) = h(b)^* \wedge h(b) = 0,$$

so that $a^* \wedge b = 0$ by density, and hence $a^* \leq b^*$, whence equality follows by symmetry. Recall also that a dense onto frame homomorphism preserves pseudocomplements, hence it preserves (and reflects) dense elements.

Corollary 4.2.10. *Let $h\colon M \to L$ be a dense coz-onto frame homomorphism. If M is a weak almost P-frame, then L is a weak almost P-frame. Hence, if βL is a weak almost P-frame, then so is L.*

Proof. Suppose $a^* = b^*$ for some $a, b \in \operatorname{Coz} L$. Since h is coz-onto, there exist $u, v \in \operatorname{Coz} M$ such that $h(u) = a$ and $h(v) = b$. Then $h(u^*) = h(v^*)$, from which we can deduce by what we have observed above that $u^* = v^*$. Since M is a weak almost-P, there is a dense $w \in \operatorname{Coz} M$ such that $v \wedge w = u \wedge w$. Thus, $h(w)$ is a dense cozero element of L such that $b \wedge h(w) = a \wedge h(w)$. Therefore L is a weak almost-P. □

In spaces this result yields the following.

Corollary 4.2.11. *A dense z-embedded subspace of a weak almost P-space is a weak almost P-space. Hence, X is a weak almost P-space if βX is a weak almost P-space.*

We have not been able to determine if βL is always a weak almost P-frame whenever L is. We do however have a case when this happens. Recall that the "well below" relation $\ll$ in a frame L is defined by

$$a \ll b \iff b \leq \bigvee S \text{ for some } S \subseteq L \text{ implies } a \leq \bigvee T \text{ for some finite } T \subseteq S.$$

The frame L is then called *continuous* if $a = \bigvee\{x \in L \mid x \ll a\}$ for every $a \in L$. In a regular continuous frame,

$$a \ll b \iff a \prec b \text{ and } \uparrow a^* \text{ is compact,}$$

a consequence of which is that, in a continuous regular frame, $a_i \ll b_i$ for $i = 1, 2$ implies $a_1 \wedge a_2 \ll b_1 \wedge b_2$. This in general is not the case, and the frames for which it holds are called stably continuous.

In the proof that follows we shall use the fact that if $I \prec\!\!\prec J$ in βL, then $\bigvee I \in J$. For verification see for instance the paragraph preceding Example 4 in [24].

Proposition 4.2.12. *Let L be a continuous Lindelöf frame. Then βL is a weak almost P-frame if and only if L is a weak almost P-frame.*

Proof. Only the right-to-left implication needs proof. So assume L is a weak almost P-frame, and let $U, V \in \operatorname{Coz}(\beta L)$ be such that $U^* \leq V^*$. Pick cozero elements U_n in βL such that $U_n \prec\!\!\prec U_{n+1}$ and $U = \bigvee_n U_n$. For each n, put $u_n = \bigvee U_n$ and observe that

$u_n \prec\!\!\prec u_{n+1}$. Since $U_n \leq r_L(u_n) \leq U_{n+1}$, it follows that $U = \bigvee_n r_L(u_n)$. Similarly, there are cozero elements v_n in L such that $V = \bigvee_n r_L(v_n)$. Now let u and v be the cozero elements of L given by $u = \bigvee u_n$ and $v = \bigvee v_n$. Since $u = j_L(U_n)$ and $v = j_L(V_n)$, it follows from $U^* \leq V^*$ that $u^* \leq v^*$. Since L is a weak almost P-frame, there is a dense $d \in \operatorname{Coz} L$ such that $v \wedge d \leq u$. Since L is a continuous frame and $d \in \operatorname{Coz} L$, so that, by [10, Corollary 4], d is a Lindelöf element in L, there are elements d_n in L such that $d_n \ll d$ for every n and $d = \bigvee d_n$. We may assume that each $d_n \in \operatorname{Coz} L$ because the well below relation interpolates in a continuous frame. Now, in view of the fact that $j_L \colon \beta L \to L$ is coz-onto, there exists, for each n, $D_n \in \operatorname{Coz}(\beta L)$ such that $j_L(D_n) = d_n$. The element $D = \bigvee_n D_n$ is a cozero element in βL. Since

$$j_L(D) = j_L\Big(\bigvee_n D_n\Big) = \bigvee_n d_n = d,$$

it follows that D is dense because d is dense. We claim that $D \wedge V \leq U$. To see this, observe first that $D_n \leq r_L(d_n)$ since $j_L(D_n) = d_n$, and hence

$$D \wedge V \leq \bigvee_n r_L(d_n) \wedge \bigvee_m r_L(v_m) = \bigvee_{n,m} r_L(d_n \wedge v_m).$$

Now, for any pair of indices (n, m),

$$d_n \wedge v_m \ll d \wedge v \leq u = \bigvee u_n,$$

which, in view of the fact that the sequence (u_n) increases, implies there is an index k such that $d_n \wedge v_m \leq u_k \prec\!\!\prec u_{k+1}$, so that $d_n \wedge v_m \in r_L(u_{k+1}) \subseteq U$. Therefore $r_L(d_n \wedge v_m) \leq U$, and hence $D \wedge V \leq U$. Thus, βL is a weak almost P-frame. □

Corollary 4.2.13. *A locally compact Lindelöf space is a weak almost P-space if and only if its Stone-Čech compactification is a weak almost P-space.*

In [24] it is shown that if the coproduct of two frames is an almost P-frame, then each summand is an almost P-frame. We prove a similar result for Lindelöf weak almost P-frames. Recall that if $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$, then $c \oplus d \in \operatorname{Coz}(L \oplus M)$ because $c \oplus d = i_L(c) \wedge i_M(d)$ for the coproduct injections $L \xrightarrow{i_L} L \oplus M \xleftarrow{i_M} M$. It is shown in [13] that, for any $x \in L$ and $y \in M$, $(x \oplus y)^{**} = x^{**} \oplus y^{**}$. Thus, if $x \oplus y$ is dense, then both x and y are dense.

Proposition 4.2.14. *If the coproduct of two Lindelöf frames is a weak almost P-frame, then each summand is a weak almost P-frame.*

Proof. Let L and M be such frames. Suppose $a^* \leq b^*$ for some $a, b \in \operatorname{Coz} L$. Then $a \oplus 1$ and $b \oplus 1$ are cozero elements of $L \oplus M$. Since $b^{**} \leq a^{**}$, we have

$$(b \oplus 1)^{**} = b^{**} \oplus 1 \leq a^{**} \oplus 1 = (a \oplus 1)^{**},$$

which implies $(a \oplus 1)^* \leq (b \oplus 1)^*$. So, by hypothesis, there is a dense $U \in \operatorname{Coz}(L \oplus M)$ such that

$$(b \oplus 1) \wedge U \leq a \oplus 1. \tag{\#}$$

We claim that there are sequences (c_n) and (d_n) in $\operatorname{Coz} L$ and $\operatorname{Coz} M$ respectively such that

$$U = \bigvee_{n=1}^{\infty} (c_n \oplus d_n).$$

To show this, we write U as a join of basic elements, say $U = \bigvee_{\alpha} (a_\alpha \oplus b_\alpha)$. By complete regularity, for each α there are cozero elements $\{c_i^{(\alpha)}\}$ in L and cozero elements $\{d_j^{(\alpha)}\}$ in M such that

$$a_\alpha = \bigvee_i c_i^{(\alpha)} \quad \text{and} \quad b_\alpha = \bigvee_j d_j^{(\alpha)}.$$

Consequently,

$$a_\alpha \oplus b_\alpha = \bigvee_i c_i^{(\alpha)} \oplus \bigvee_j d_j^{(\alpha)} = \bigvee_{i,j} (c_i^{(\alpha)} \oplus d_j^{(\alpha)}),$$

so that

$$U = \bigvee_{\alpha,i,j} (c_i^{(\alpha)} \oplus d_j^{(\alpha)}).$$

Since U is a cozero element in a Lindelöf frame, it is a Lindelöf element, and hence we can find countably many $c_n \in \operatorname{Coz} L$ and countably many $d_n \in \operatorname{Coz} M$ such that $U = \bigvee_n (c_n \oplus d_n)$. Now

$$\begin{aligned}(b \oplus 1) \wedge \bigvee_{n=1}^{\infty} (c_n \oplus d_n) &= \bigvee_{n=1}^{\infty} \big((b \oplus 1) \wedge (c_n \oplus d_n)\big) \\ &= \bigvee_{n=1}^{\infty} \big((b \wedge c_n) \oplus d_n\big).\end{aligned}$$

Thus, the inequality in (#) implies $\bigvee_{n=1}^{\infty}\big((b\wedge c_n)\oplus d_n\big)\le a\oplus 1$, whence $(b\wedge c_n)\oplus d_n\le a\oplus 1$ for every n, and hence $b\wedge c_n\le a$ for every n, which implies $b\wedge\bigvee_n c_n\le a$. To finish the proof we show that the cozero element $\bigvee_n c_n$ of L is dense. Since U is dense and

$$U=\bigvee_n(c_n\oplus d_n)\le\Big(\bigvee_n c_n\Big)\oplus\Big(\bigvee_n d_n\Big),$$

it follows that $\Big(\bigvee_n c_n\Big)\oplus\Big(\bigvee_n d_n\Big)$ is dense, whence $\bigvee_n c_n$ is dense. Therefore L is a weak almost P-frame. Similarly, M is a weak almost P-frame. □

Corollary 4.2.15. *Let X and Y be Lindelöf spaces with one of them locally compact. If $X\times Y$ is a weak almost P-space, then both X and Y are weak almost P-spaces.*

Proof. By [44, Proposition II 13], $\mathfrak{O}(X\times Y)$ is isomorphic to $\mathfrak{O}X\oplus\mathfrak{O}Y$. Therefore $\mathfrak{O}X\oplus\mathfrak{O}Y$ is a weak almost P-frame, and so $\mathfrak{O}X$ and $\mathfrak{O}Y$ are weak almost P-frames, which implies X and Y are weak almost P-spaces. □

Remark 4.2.16. The fact that $L\oplus M$ is Lindelöf was used only to enable us to write the cozero element U as a join of countable many "cozero rectangles" $c_n\oplus b_n$. The result therefore is true for any pair (L,M) of frames for which every cozero element of $L\oplus M$ is a join of countably many cozero rectangles. We end the section with a digression from our main train of thought, and give characterizations of such pairs.

To start, we mention that the term "cozero rectangle" is borrowed from [16], and the pointed analogues of the characterizations that follow are in that paper, excluding, of course, the one about the Lindelöf coreflections.

Proposition 4.2.17. *The following are equivalent for frames K,N,L and M.*

(1) *Every cozero element of $L\oplus M$ is a countable join of cozero rectangles.*

(2) *For any coz-onto homomorphisms $h\colon K\to L$ and $g\colon N\to M$, the homomorphism $h\oplus g\colon K\oplus N\to L\oplus M$ is coz-onto.*

(3) *$j_L\oplus j_M\colon \beta L\oplus\beta M\to L\oplus M$ is coz-onto.*

(4) $\lambda_L \oplus \lambda_M \colon \lambda L \oplus \lambda M \to L \oplus M$ *is coz-onto.*

Proof. (1) ⇒ (2): Given a cozero element $U = \bigvee_n (a_n \oplus b_n)$ in $L \oplus M$, take, for each n, cozero elements u_n in K and cozero elements v_n in N such that $h(u_n) = a_n$ and $g(v_n) = b_n$. Then $\bigvee_n (u_n \oplus v_n)$ is a cozero element of $K \oplus N$ mapped to U by $h \oplus g$.

(2) ⇒ (3): This is trivial because the Stone-Čech maps are coz-onto.

(3) ⇒ (1): Let $U \in \mathrm{Coz}(L \oplus M)$. By (3), there is a $V \in \mathrm{Coz}(\beta L \oplus \beta M)$ such that

$(j_L \oplus j_M)(V) = U$. As we observed in the proof of Proposition 4.2.14, there are cozero elements $c_n \in \mathrm{Coz}(\beta L)$ and $d_n \in \mathrm{Coz}(\beta M)$ such that $V = \bigvee_n (c_n \oplus d_n)$ because $\beta L \oplus \beta M$ is Lindelöf. Thus,

$$U = (j_L \oplus j_M)\Big(\bigvee_n (c_n \oplus d_n)\Big) = \bigvee_n \big(j_L(c_n) \oplus j_M(d_n)\big),$$

which is a countable join of cozero rectangles.

(1) ⇔ (4): The same line of argument as the foregoing one shows this since $\lambda L \oplus \lambda M$ is Lindelöf. □

4.3 More on weakly regular f-rings

The characterization in statement (2) of Proposition 4.2.5 suggests an analogous characterization for weakly regular rings. For f-rings we indeed do have such. Recall that a *radical ideal* is one which whenever it contains a power of an element, then it already contains the element. For any a in a ring, $M(a)$ is a radical ideal. Observe that if $\mathfrak{M}(x) \subseteq \mathfrak{M}(y)$, then $\mathfrak{M}(xy) = \mathfrak{M}(y)$. The last implication in the following proof is modeled on that of [1, Theorem 4.2 (ii)].

Proposition 4.3.1. *The following properties of a reduced f-ring A are equivalent.*

(1) *A is weakly regular.*

(2) *For any $a, b \in A$, $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ implies there is a non-divisor of zero c such that $ac \in M(b)$ and $bc \in M(a)$.*

(3) *Every singular prime z-ideal in A is a d-ideal.*

Proof. (1) $\Rightarrow$ (2): Assume (1) and suppose that $\text{Ann}(a) = \text{Ann}(b)$ for some $a, b \in A$. Then there are non-divisor of zero u and v such that $au \in M(b)$ and $bv \in M(a)$. Hence $c = uv$ is a non-divisors of zero with $ac \in M(b)$ and $bc \in M(a)$.

(2) $\Rightarrow$ (3): Let P be a singular prime z-ideal in A. Suppose that, for some $a, b \in A$, $\text{Ann}(a) = \text{Ann}(b)$ and $a \in P$. We must show that $b \in P$. By (2), there is a non-divisor of zero c such that $bc \in M(a)$. Thus, $\mathfrak{M}(a) \subseteq \mathfrak{M}(bc)$, which implies $\mathfrak{M}(bc) = \mathfrak{M}(abc)$. Since $abc \in P$ and P is a z-ideal, it follows that $bc \in P$, and hence $b \in P$ because P is prime and $c \notin P$.

(3) $\Rightarrow$ (1): Let $\text{Ann}(a) \subseteq \text{Ann}(b)$ and suppose, by way of contradiction, that for any non-divisor of zero c, $bc \notin M(a)$. Define the set $S \subseteq A$ by

$$S = \{b^n c \mid c \text{ is a non-divisor of zero and } n = 0, 1, 2 \ldots\},$$

and note that S is multiplicatively closed. Also, $M(a) \cap S = \emptyset$ because if $b^n c \in M(a)$ for some n and some non-divisor of zero c, then $(bc)^n \in M(a)$, so that $bc \in M(a)$ since $M(a)$ is a radical ideal. Let P be a prime ideal minimal over $M(a)$ and disjoint from S. Since $M(a)$ is a z-ideal, it follows from [55, Theorem 1.1] that P is a z-ideal. Since S contains all non-divisor of zero, P is singular, and hence, by hypothesis, P is a d-ideal, and therefore $\text{Ann}^2(a) \subseteq P$ as $a \in P$. Consequently, the relations $b \in \text{Ann}^2(b) \subseteq \text{Ann}^2(a) \subseteq P$ imply $b \in P$, which is a contradiction because $b \in S$. Therefore A is weakly regular. $\square$

Below we use the fact that if A is a reduced f-ring with bounded inversion and

$$S = \{a \in A^* \mid a \text{ is invertible in A}\},$$

then $A = A^*[S^{-1}]$ (see [29, Lemma 3.4]). That is, A is the ring of fractions of its bounded part relative to the set S. A consequence of this is that ideals of A are exactly the ideal

$$I^e = \{s^{-1}u \mid s \in S \text{ and } u \in I\}$$

for I an ideal of A^*.

Corollary 4.3.2. *Let A be a reduced f-ring with bounded inversion.*

(a) *If A is weakly regular and every prime singular z-ideal of A^* extends to a z-ideal in A, then A^* is weakly regular.*

(b) *If A^* is weakly regular and every prime singular z-ideal of A contracts to a z-ideal in A^*, then A is weakly regular.*

Proof. (a) Let I be a singular prime z-ideal in A^*. Then of course I^e is a prime ideal in A, and it consists entirely of zero-divisors. By hypothesis, I^e is a z-ideal, and hence it is a d-ideal since A is weakly regular. By [29, Lemma 3.8], I^{ec} is a d-ideal in A^*, and by [29, Proposition 3.0], $I = I^{ec}$. Therefore A^* is weakly regular.

(b) The proof goes along the lines of that of (a). □

Remark 4.3.3. In [43, Corollary 2.6.1], Ighedo shows that every z-ideal of any ring $\mathcal{R}L$ contracts to a z-ideal of $\mathcal{R}^*L$. Thus, the (b) part of the foregoing corollary gives another reaffirmation of the fact that if βL is a weak almost P-frame, then so is L since $\mathcal{R}(\beta L)$ is isomorphic to $\mathcal{R}^*L$.

We conclude by examining the position of weak regularity for f-rings vis-à-vis other weaker variants of regularity. First let us recall some terminology. Endo [35] calls a ring *quasi-regular* if its classical ring of quotients is regular. In [36, Theorem 2.2], Evans characterizes quasi-regular rings internally. He shows that A is quasi-regular if and only if for every $a \in A$ there exists $b \in A$ such that $\mathrm{Ann}^2(a) = \mathrm{Ann}(b)$ if and only if for every $a \in A$ there exists a non-divisor of zero $d \in A$ such that $a^2 = ad$. At the beginning of the paper we agreed to say a ring is almost regular if every non-divisor of zero in it is invertible.

The proposition that we shall prove shortly is motivated by what happens in function rings $\mathcal{R}L$. Recall that in frames we have the following irreversible implications. We use the abbreviations AP, WAP and CC for "almost P", "weak almost P", and "cozero complemented", respectively.

$$\text{P} \implies \text{AP} \implies \text{WAP} \qquad \text{and} \qquad \text{P} \implies \text{CC} \implies \text{WAP}$$

Furthermore,

$$\text{CC} + \text{AP} \implies \text{P}.$$

Frames L with any of these properties have ring-theoretic characterizations. We list them below, giving reference where each characterization first appeared.

1. L is a P-frame if and only if $\mathcal{R}L$ is a regular ring [12].

2. L is an almost P-frame if and only if $\mathcal{R}L$ is an almost regular ring [24].

3. L is cozero complemented if and only if $\mathcal{R}L$ is a quasi-regular ring [32].

We now show that the ring analogues of the implications above hold for reduced f-rings.

Proposition 4.3.4. *For reduced f-rings the following implications hold.*

(1) *Regularity $\implies$ almost regularity $\implies$ weak regularity.*

(2) *Regularity $\implies$ quasi-regularity $\implies$ weak regularity.*

(3) *Quasi-regularity + almost regularity $\implies$ regularity.*

Proof. (1) To show the first implication, suppose A is a regular ring, and let $a \in A$ be a non-divisor of zero. Pick $b \in A$ such that $a = a^2 b$. Then $a(1 - ab) = 0$, and hence $ab = 1$ since a is not a divisor of zero. Therefore a is invertible, and hence A is almost regular.

For the second implication, assume A is almost regular, and suppose $\text{Ann}(a) \subseteq \text{Ann}(b)$ for some $a, b \in A$. Let M be a maximal ideal of A containing a. Since M consists entirely of zero-divisors, $\text{Ann}^2(u)$ is a proper ideal of A for every $u \in M$. Indeed, if 1 were in $\text{Ann}^2(u)$ we would have $\text{Ann}(u) = 0$, whence u would be a non-divisor of zero. Let $u, v \in M$, and suppose $w \in \text{Ann}(u^2 + v^2)$. Then $(wu)^2 + (wv)^2 = 0$, and hence $(wu)^2 = 0$, which implies $wu = 0$ since A is reduced. Thus, $\text{Ann}(u^2 + v^2) \subseteq \text{Ann}(u)$, which implies $\text{Ann}^2(u) \subseteq \text{Ann}^2(u^2 + v^2)$. Therefore the set

$$K = \bigcup \{\text{Ann}^2(x) \mid x \in M\}$$

is a directed union of proper d-ideals of A, and is therefore a proper d-ideal with $M \subseteq K$, and therefore $M = K$. Thus, M is a d-ideal. Since $\text{Ann}(a) \subseteq \text{Ann}(b)$, we have $\text{Ann}(a) = \text{Ann}(a^2 + b^2)$, which implies $a^2 + b^2 \in M$ because $a \in M$ and M is a d-ideal. This implies $b^2 \in M$, and hence $b \in M$. Because M is an arbitrary maximal ideal containing a, it follows that $b \in M(a)$. So choosing $c = 1$, we have that c is a non-divisor of zero with $bc \in M(a)$. Therefore A is weakly regular.

(2) Only the second implication need be shown. Assume that A is quasi-regular, and suppose $\text{Ann}(a) \subseteq \text{Ann}(b)$ for some $a, b \in A$. By [36, Theorem 2.2], there is a non-divisor of zero c such that $a^2 = ac$. Then $a(a - c) = 0$, so that $b(a - c) = 0$ since $a - c \in \text{Ann}(a) \subseteq \text{Ann}(b)$. So, $bc = ba$, and hence bc is in every ideal that contains a. Therefore $bc \in M(a)$, showing that A is weakly regular.

(3) Let $a \in A$. Again by Evans' result, $a^2 = ac$ for some non-divisor of zero c, which is then invertible since A is almost regular. Thus $a = a^2c^{-1}$, and therefore A is regular. □

We end with a result on direct products, and for that we need the following lemma.

Lemma 4.3.5. *Let $(A_i)_{i \in I}$ be a family of weakly regular rings. For any $a = (a_i) \in \prod A_i$ we have the following:*

(a) $\prod M(a_i) = M(a)$.

(b) $\prod \text{Ann}(a_i) = \text{Ann}(a)$.

Proof. (a) Let $(z_i) \in \prod M(a_i)$, and take any maximal ideal $\mathcal{M}$ of A containing a. There exists $i_0 \in I$ such that $\mathcal{M} = \prod J_i$, where $J_{i_0} \in \text{Max}(A_{i_0})$ and $J_i = A_i$ for $i \neq i_0$. Then $a_{i_0} \in J_{i_0}$, which implies $z_{i_0} \in J_{i_0}$ because $z_{i_0} \in M(a_{i_0})$, and consequently $z_i \in M(a_i)$ since $\mathcal{M}$ is an arbitrary maximal ideal of A containing a. This establishes the containment $\prod M(a_i) \subseteq M(a)$. For the opposite inclusion, let $x = (x_i) \in M(a)$. We must show that for any fixed index k, $x_k \in M(a_k)$. Consider any maximal ideal N of A_k containing a_k. Let $\mathcal{M}$ be the maximal ideal of $\prod A_i$ defined by $\mathcal{M} = \prod J_i$ with $J_k = N$ and $J_i = A_i$ for $i \neq k$. Then $a \in \mathcal{M}$, which implies $x \in \mathcal{M}$, and hence $x_k \in N$. Thus, $x_k \in M(a_k)$, and it follows therefore that $x \in \prod M(a_i)$, establishing the desired inclusion. Consequently, $\prod M(a_i) = M(a)$.

(b) Let $x = (x_i) \in \prod A_i$. Then,

$$\begin{aligned} x \in \prod \operatorname{Ann}(a_i) \quad &\iff \quad x_i \in \operatorname{Ann}(a_i) \quad \text{for each } i \\ &\iff \quad x_i a_i = 0 \quad \text{for each } i \\ &\iff \quad xa = 0 \\ &\iff \quad x \in \operatorname{Ann}(a). \end{aligned}$$

Therefore $\prod \operatorname{Ann}(a_i) = \operatorname{Ann}(a)$. □

For use in the proof that follows, we recall that if (X_i) and (Y_i) are families of nonempty sets of some set S such that $\prod X_i \subseteq \prod Y_i$, then $X_i \subseteq Y_i$ for each index i.

Proposition 4.3.6. *The direct product of any family of weakly regular rings is a weakly regular ring if and only if each factor is a weakly regular ring.*

Proof. ($\Leftarrow$) Let $(A_i)_{i \in I}$ be a family of weakly regular rings, and consider two elements $a = (a_i)$ and $b = (b_i)$ of $\prod A_i$ such that $\operatorname{Ann}(a) \subseteq \operatorname{Ann}(b)$. By the second part in the foregoing lemma, $\prod \operatorname{Ann}(a_i) \subseteq \prod \operatorname{Ann}(b_i)$, which implies each $\operatorname{Ann}(a_i) \subseteq \operatorname{Ann}(b_i)$ because annihilator ideals are nonempty. For each i there is a non-divisor of zero, $c_i \in A_i$, such that $b_i c_i \in M(a_i)$. The element $c = (c_i)$ is a non-divisor of zero in $\prod A_i$. Now, by the first part in the lemma above,

$$bc = (b_i)(c_i) \in \prod M(a_i) \subseteq M(a).$$

Therefore $\prod A_i$ is a weakly regular ring.

($\Rightarrow$) Suppose $\prod A_i$ is a weakly regular ring. Fix an index k, and let $x, y \in A_k$ be such that $\operatorname{Ann}(x) \subseteq \operatorname{Ann}(y)$. Let $a = (a_i)$ and $b = (b_i)$ be the elements of $\prod A_i$ such that $a_k = x$, $b_k = y$, and for $i \neq k$, $a_i = 1$ and $b_i = 1$. Observe that, by construction of the elements a and b and the fact that $\operatorname{Ann}(x) \subseteq \operatorname{Ann}(y)$, we have $\operatorname{Ann}(a) \subseteq \operatorname{Ann}(b)$. By the present hypothesis, there is a non-divisor of zero, say $d = (d_i)$, in $\prod A_i$ such that $bd \in M(a)$. Note that d_k is a non-divisor of zero in A_k, and $b_k d_k \in M(a_k)$ since $M(a) = \prod M(a_i)$. Thus, $d_k y \in M(x)$, which proves that A_k is a weakly regular ring. □

Chapter 5

Boundary frames

5.1 Introduction

In [1], the authors define a space X to be a boundary space (they write "∂-space") if the boundary of every zero-set is contained in a zero-set with empty interior. They then prove that X is a boundary space if and only if every prime ideal of $C(X)$ that consists entirely of zero-divisors is a d-ideal (see the definition of a d-ideal below).

Coming from the pointfree topology vantage point, we are able to give another ring-theoretic characterization of these spaces; one that bears close resemblance to the topological definition. Let us expatiate. For any a in a ring A, we write $M(a)$ to designate the intersection of all maximal ideals of A containing a. As usual, $\mathrm{Ann}(a)$ denotes the annihilator of a. We say an ideal of A is a frontier ideal if it is of the form $M(a)+\mathrm{Ann}(a)$, for some $a \in A$. We then call A a boundary ring if every frontier ideal of A contains a non-divisor of zero. The characterization is then that a space X is a boundary space if and only if $C(X)$ is a boundary ring. When we work with frames and their rings of real-valued continuous functions, it becomes apparent what motivates the definition of boundary ring we have just mentioned.

As the reader has likely surmised from the previous paragraph, our approach will be frame-theoretic in certain instances. We shall thus start by adapting the definition of a boundary space to frames in a conservative way, and call a frame with the defining property a boundary frame. Then we will show that a frame L is a boundary frame if and only if λL, the Lindelöf coreflection of L, is a boundary frame, if and only if υL, the realcompact coreflection of L, is a boundary frame. Incidentally, in the process of doing so we will bring to the fore what appears hitherto not to have been noticed regarding the right adjoint of the coreflection map $\lambda L \to L$. The main result is also proved using frame-theoretic language. Here the main reason for such an approach is clarity and transparency of the proof. There is not much we are able to say about products of boundary spaces. We do though prove that if X and Y have the property that $X \times Y$ is z-embedded in $\beta X \times \beta Y$, then a necessary condition that $X \times Y$ be a boundary space is that both X and Y be boundary spaces.

We end with some purely ring-theoretic results about boundary rings. We show that every reduced quasi-regular f-ring is a boundary ring. Following that, we show that for any bounded ring A, the ideals of A which inherit the property of being a boundary ring (when viewed as rings in their own right) are exactly those which contain "internal" non-divisors of zero. We end by showing that the direct product of rings is a boundary ring if and only if each factor is a boundary ring.

5.2 A ring theoretic characterization of boundary frames

We start by extending the notion of being a boundary space to frames. Let us write $\operatorname{bd}(S)$ for the boundary of a subset S of a Tychonoff space X. If Z is a zero-set of X, then $X \smallsetminus \operatorname{bd}(Z) = (X \smallsetminus Z) \cup (X \smallsetminus Z)^*$, where $(^*)$ denotes pseudocomplement in the frame $\mathfrak{O}X$. Observe that a zero-set (in fact, any closed set) W has empty interior precisely when its complement $X \smallsetminus W$ is dense. Thus, X is a boundary space if and only if for every cozero-set C of X, there exists a dense cozero-set D of X such that $D \subseteq C \cup (X \smallsetminus \overline{C})$. Since C^* in the frame $\mathfrak{O}X$ is $X \smallsetminus \overline{C}$, this motivates the following definition.

Definition 5.2.1. A completely regular frame L is a *boundary frame* if for every $c \in \operatorname{Coz} L$ there exists a dense $d \in \operatorname{Coz} L$ such that $d \leq c \vee c^*$.

It is evident that a Tychonoff space X is a boundary space if and only if the frame $\mathfrak{O}X$ is a boundary frame. Before we proceed, let us compare these frames with others of a similar kind.

Example 5.2.2. Here are comparisons of boundary frames with the ones mentioned above.

(a) Every cozero complemented frame (and hence every P-frame) is a boundary frame. Indeed, if $c \in \operatorname{Coz} L$, where L is cozero complemented, and $d \in \operatorname{Coz} L$ is such that $c \wedge d = 0$ and $c \vee d$ is dense, then $d \leq c^*$, so that $c \vee d$ is a dense cozero element below $c \vee c^*$.

(b) Every boundary frame is a weak almost P-frame. Indeed, suppose $a^* \leq b^*$ for some $a, b \in \operatorname{Coz} L$. Take a dense $c \in \operatorname{Coz} L$ with $c \leq a \vee a^*$. Then $b \wedge c \leq a$. Finally, it is immediate that a frame is a P-frame if and only if it is both an almost P-frame and a boundary frame.

We shall now prove frame-theoretically that L is a boundary frame if and only if λL is a boundary frame if and only if υL is a boundary frame. A little later we will have a ring-theoretic reaffirmation of this result, coming about as a corollary of our main result. The merit in frame-theoretic proof is that not only is the result deducible directly from the definition, it also brings to the fore what appears hitherto not to have been noticed regarding the right adjoint of the map $\lambda L \to L$. We start with the following lemma.

Lemma 5.2.3. *If $h : M \to L$ is a dense coz-onto frame homomorphism, and M is a boundary frame, then L is a boundary frame.*

Proof. Let $c \in \operatorname{Coz} L$. Since h is coz-onto, there exists $a \in \operatorname{Coz} M$ such that $h(a) = c$. Since M is boundary frame, there exists a dense $b \in \operatorname{Coz} M$ such that $b \leq a \vee a^*$. Then $h(b)$ is a dense cozero element in L with $h(b) \leq h(a) \vee h(a^*) = h(a) \vee h(a)^* = c \vee c^*$. Therefore L is a boundary frame. □

Applied to spaces, this yields the following. Recall that a subspace S of a Tychonoff space X is *z-embedded* in case every zero-set of S is a trace on S of some zero-set of X.

Corollary 5.2.4. *A dense z-embedded subspace of a boundary space is a boundary space.*

Let us recall from [3, Corollary to Lemma 1.9] that if $a \wedge b = 0$ in a frame L, then $r_L(a \vee b) = r_L(a) \vee r_L(b)$. We will use this in the proof of the lemma that follows. Recall also that for any normal frame M, $r_M(a \vee b) = r_M(a) \vee r_M(b)$ for all $a, b \in M$. Note that the composite $\beta L \xrightarrow{\eta_L} \lambda L \xrightarrow{\lambda_M} L$ is the map $\beta L \xrightarrow{j_L} L$, so that $(\eta_L)_* \cdot (\lambda_L)_* = r_L$. That is, $(\eta_L)_*(\llbracket a \rrbracket) = r_L(a)$ for every $a \in L$.

Lemma 5.2.5. *If $a \wedge b = 0$ in a completely regular frame L, then $\llbracket a \vee b \rrbracket = \llbracket a \rrbracket \vee \llbracket b \rrbracket$.*

Proof. Since λL is a normal frame and $\eta_L : \beta L \to \lambda L$ is (isomorphic to) the Stone-Čech compactification of λL,

$$(\eta_L)_*(\llbracket a \rrbracket \vee \llbracket b \rrbracket) = (\eta_L)_*(\llbracket a \rrbracket) \vee (\eta_L)_*(\llbracket b \rrbracket) = r_L(a) \vee r_L(b).$$

On the other hand,

$$(\eta_L)_*(\llbracket a \rrbracket \vee \llbracket b \rrbracket) = r_L(a \vee b) = r_L(a) \vee r_L(b),$$

since $a \wedge b = 0$. Consequently, $(\eta_L)_*(\llbracket a \rrbracket \vee \llbracket b \rrbracket) = (\eta_L)_*(\llbracket a \rrbracket) \vee (\eta_L)_*(\llbracket b \rrbracket)$, which, on applying the map η_L, yields $\llbracket a \vee b \rrbracket = \llbracket a \rrbracket \vee \llbracket b \rrbracket$. $\square$

Let us note the following regarding dense frame homomorphisms. If $h : M \to L$ is a dense frame homomorphism, then $h_* h(a^*) = a^*$ for every $a \in M$. Indeed, the equality $h\left(h_* h(a^*) \wedge a\right) = 0$ implies $h_* h(a^*) \wedge a = 0$ by density of h, whence $h_* h(a^*) \leq a^*$ and hence we claimed equality. In particular, let $I \in \lambda L$, and put $a = \bigvee I$. Then in view of the fact that the right adjoint of a dense onto frame homomorphism preserves pseudocomplements, $I^* = \llbracket a^* \rrbracket = \llbracket a \rrbracket^*$. Recall also that if $h : M \to L$ is dense onto, then $h_*(z)$ is dense in M whenever z is dense in L.

Proposition 5.2.6. *The following are equivalent for a completely regular frame L.*

(1) *L is a boundary frame.*

(2) *λL is a boundary frame.*

(3) *υL is a boundary frame.*

Proof. (1) $\Rightarrow$ (2) Assume $I \in \mathrm{Coz}(\lambda L)$. There exists $c \in \mathrm{Coz}\, L$ such that $I = [\![c]\!]$. Since L is a boundary frame, there exists a dense cozero element d such that $d \leq c \vee c^*$. The cozero element $[\![d]\!]$ of λL is dense and satisfies

$$[\![d]\!] \leq [\![c \vee c^*]\!] = [\![c]\!] \vee [\![c^*]\!] = I \vee I^*,$$

in light of the foregoing lemma since $c \wedge c^* = 0$. Therefore λL is a boundary frame.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1). These implications follow from Lemma 5.2.3 since each of these maps $l_L : \lambda L \to \upsilon L$ and $\upsilon_L : \upsilon L \to L$ is dense coz-onto. □

Remark 5.2.7. We can deduce from this result that a Tychonoff space X is a boundary space if and only if υX is a boundary space. Of course this also follows from [1, Theorem 4.4] because the rings $C(X)$ and $C(\upsilon X)$ are isomorphic.

We shall now give a characterization of boundary frames L in terms of properties of the ring $\mathcal{R}L$. The characterization of boundary space X in terms of $C(X)$ will then follows as a corollary in view of the fact that X is a boundary space if and only if $\mathfrak{O}X$ is a boundary frame, and $C(X)$ is isomorphic to $\mathcal{R}(\mathfrak{O}X)$. We start with a definition.

Definition 5.2.8. An ideal of a ring A is a *frontier ideal* if it is of the form $M(a)+\mathrm{Ann}(a)$ for some $a \in A$. We say A is a *boundary ring* if every frontier ideal in A contains a non-divisor of zero.

Example 5.2.9. Here are some easy examples.

(a) Every von Neumann regular ring with $1 \neq 0$ is a boundary ring. To see this, let $a \in A$, and take $b \in A$ such that $a = a^2b$. Then $a(1-ab) = 0$, so that $1-ab \in \mathrm{Ann}(a)$. Since $ab \in M(a)$, it follows that $1 \in M(a) + \mathrm{Ann}(a)$. Since 1 is a non-divisor of zero, A is a boundary ring.

(b) Every integral domain is a boundary ring.

In order to prove the main result, we shall need the following lemma which was proved in [30]. We include the proof for the sake of completeness.

Lemma 5.2.10. *For any $\alpha \in \mathcal{R}L, M(\alpha) = \boldsymbol{M}_{\operatorname{coz}\alpha}$.*

Proof. It is shown in the proof of [29, Lemma 3.2] that, for any $I \in \beta L$,

$$\boldsymbol{M}^I = \bigcap\{\boldsymbol{M}^J \mid J \in \mathrm{Pt}(\beta L) \text{ and } I \subseteq J\}.$$

Since for any $J \in \mathrm{Pt}(\beta L)$ we have $\alpha \in \boldsymbol{M}^J$ if and only if $r_L(\operatorname{coz}\alpha) \subseteq J$, it follows that

$$\begin{aligned} \boldsymbol{M}_{\operatorname{coz}\alpha} = \boldsymbol{M}^{r_L(\operatorname{coz}\alpha)} &= \bigcap\{\boldsymbol{M}^J \mid J \in \mathrm{Pt}(\beta L) \text{ and } r_L(\operatorname{coz}\alpha) \subseteq J\} \\ &= \bigcap\{\boldsymbol{M}^J \mid J \in \mathrm{Pt}(\beta L) \text{ and } \alpha \in \boldsymbol{M}^J\} \\ &= M(\alpha), \end{aligned}$$

which proves the result. □

Recall from [37, Lemma 14.8] that the sum of any two z-ideals in $C(X)$ is a z-ideal. The proof in [37] uses propierties of βX. Rudd [58] gives an elementary proof which uses no properties of βX.

Theorem 5.2.11. *A Tychonoff space X is a boundary space if and only if $C(X)$ is a boundary ring.*

Proof. (⇒) Assume X is a boundary space, and let $L = \mathfrak{O}X$. Then L is a boundary frame. Let I be a frontier ideal of $\mathcal{R}L$. Pick α such that

$$I = M(\alpha) + \mathrm{Ann}(\alpha) = \boldsymbol{M}_{\operatorname{coz}\alpha} + \boldsymbol{M}_{(\operatorname{coz}\alpha)^*}.$$

For brevity, write $a = \operatorname{coz}\alpha$. We claim that $\boldsymbol{M}_a + \boldsymbol{M}_{a^*} = \boldsymbol{M}_{a\vee a^*}$. The containment $\boldsymbol{M}_a + \boldsymbol{M}_{a^*} \subseteq \boldsymbol{M}_{a\vee a^*}$ is immediate. For the other inclusion, let $\gamma \in \boldsymbol{M}_{a\vee a^*}$, and write $c = \operatorname{coz}\gamma$. Then $c \leq a \vee a^*$. Find a sequence (c_n) in $\operatorname{Coz} L$ such that $c_n \prec\!\!\prec c$ for each n, and $c = \bigvee c_n$. Then, for each n, $c_n \prec\!\!\prec a \vee a^*$. By [3, Lemma 1], $c_n \wedge a^* \prec\!\!\prec a^*$. Pick $d_n \in \operatorname{Coz} L$ with $c_n \wedge c^* \prec\!\!\prec d_n \prec\!\!\prec a^*$. Put $d = \bigvee d_n$, and pick a positive $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\delta = d$. Since $d \leq a^*$, $\delta \in \boldsymbol{M}_{a^*}$. Observe that $c \wedge a^* \leq d$ and consequently,

$$\begin{aligned} \operatorname{coz}\gamma = c &= (c \wedge a) \vee (c \wedge a^*) \\ &\leq (c \wedge a) \vee d \\ &= \operatorname{coz}(\gamma^2\alpha^2) \vee \operatorname{coz}\delta \\ &= \operatorname{coz}(\gamma^2\alpha^2 + \delta). \end{aligned}$$

Since $\gamma^2\alpha^2 \in \boldsymbol{M}_a$ and $\delta \in \boldsymbol{M}_{a^*}$, $\tau = \gamma^2\alpha^2 + \delta$ is an element of the z-ideal $\boldsymbol{M}_a + \boldsymbol{M}_{a^*}$ with $\operatorname{coz}\gamma \leq \operatorname{coz}\tau$, it follows that $\tau \in \boldsymbol{M}_a + \boldsymbol{M}_{a^*}$. Thus $\boldsymbol{M}_{a\vee a^*} \subseteq \boldsymbol{M}_a + \boldsymbol{M}_{a^*}$, and hence the claimed equality. Now, since L is a boundary frame, there exists a dense cozero element u with $u \leq a \vee a^*$. For any $\mu \in \mathcal{R}L$ with $\operatorname{coz}\mu = u$, we have that μ is non-divisor of zero belonging to I. Therefore $\mathcal{R}L$ is a boundary ring. Since $\mathcal{R}L = \mathcal{R}(\mathfrak{O}X)$ is isomorphic to $C(X)$, it follows that $C(X)$ is a boundary ring.

($\Leftarrow$) Suppose $C(X)$ is a boundary ring, so $\mathcal{R}L$ is a boundary ring as well, where, as before, $L = \mathfrak{O}X$. Let $c \in \operatorname{Coz} L$, and take $\gamma \in \mathcal{R}L$ with $\operatorname{coz}\gamma = c$. Then there is a non-divisor of zero $\delta \in M(\gamma) + \operatorname{Ann}(\gamma) = \boldsymbol{M}_{c\vee c^*}$. Therefore $\operatorname{coz}\delta$ is a dense cozero element of L such that $\operatorname{coz}\delta \leq c \vee c^*$, which shows that L is a boundary frame. Thus, X is a boundary space. This completes the proof. □

Remark 5.2.12. The foregoing proof hinges on the fact that the sum of two z-ideals in $C(X)$ is a z-ideal. The statement of the theorem can thus be broadened as follows: *If L is a completely regular frame such that the sum of two z-ideals in $\mathcal{R}L$ is a z-ideal, then L is a boundary frame if and only if $\mathcal{R}L$ is a boundary ring.* Now we do not know if the sum of two z-ideals of $\mathcal{R}L$ is a z-ideal for every completely regular frame L. This question was actually asked by Ighedo [43] in her thesis. There are however non-spatial frames in whose rings of real-valued continuous functions the sum of two z-ideals is a z-ideal. Indeed, as observed in [27], if $\mathfrak{m}$ is an uncountable cardinal, and $L = \mathfrak{O}(\mathbb{R}^m)$, then λL is an example of non-spatial frame whose ring of real-valued continuous functions is isomorphic to a $C(X)$.

We end this section with an alternative proof of 5.2.11 which has been pointed out to us by one of the examiners. It is a purely $C(X)$ proof.

Proof. First we let $C(X)$ be a boundary ring. Let $f \in C(X)$. By definition, there is a non-divisor of zero $r \in C(X)$ such that $r \in M(f) + \operatorname{Ann(f)}$. Hence $f = g + h$, where $g \in M(f)$ and $h \in \operatorname{Ann(f)}$, i.e, $Z(f) \subseteq Z(g)$ and $\operatorname{cl}_X(X \smallsetminus Z(f)) \subseteq Z(h)$. Thus $\operatorname{bd}\big(Z(f)\big) = Z(f) \cap \operatorname{cl}_X(X \smallsetminus Z(f)) \subseteq Z(g) \cap Z(h) \subseteq Z(g+h) = Z(r)$. Next, suppose that X is a boundary space and $f \in C(X)$. Hence $\operatorname{bd}\big(Z(f)\big) = Z(f) \cap \operatorname{cl}_X(X \smallsetminus Z(f)) \subseteq Z(r)$, for some non-divisor of zero $r \in C(X)$. Now consider the following functions:

$$h(t) = \begin{cases} r(t) & \text{if} \quad t \in \mathrm{cl}_X(X \setminus Z(f)) \\ 0 & \text{if} \quad t \in Z(f) \end{cases} \qquad k(t) = \begin{cases} 0 & \text{if} \quad t \in \mathrm{cl}_X(X \setminus Z(f)) \\ r(t) & \text{if} \quad t \in Z(f). \end{cases}$$

Clearly h, k are well defined for $\mathrm{bd}\big(Z(f)\big) = Z(f) \cap \mathrm{cl}_X(X \setminus Z(f)) \subseteq Z(r)$, and $h, k \in C(X)$, by pasting lemma, $h \in M(f)$ for, $Z(f) \subseteq Z(h)$ and $k \in \mathrm{Ann}(f)$ since $\mathrm{cl}_X(X \setminus Z(f)) \subseteq Z(k)$. But $r = h + k$, so $r \in M + \mathrm{Ann}(f)$. □

5.3 On product of boundary spaces

The result we present is a necessary condition for certain products of spaces to be a boundary space. It is reminiscent of Curtis' [17] result that if a product $X \times Y$ is an F-space, then both X and Y are F-spaces, and more.

In [16], Blair and Hager prove that, for a pair X and Y of Tychonoff spaces, the product $X \times Y$ is z-embedded in $\beta X \times \beta Y$ if and only if every cozero-set of $X \times Y$ is of the form $\bigcup_{n=1}^{\infty} (C_n \times D_n)$, for some sequences (C_n) and (D_n) of cozero-sets of X and Y, respectively.

In what follows we shall at times not denote the closure of a set U in a space S by $\overline{U}$, but rather by $\mathrm{cl}_S(U)$. Where we use the over-line it will be clear from the context where the closure is contemplated.

Theorem 5.3.1. *If $X \times Y$ is a boundary space that is z-embedded in $\beta X \times \beta Y$, then both X and Y are boundary spaces.*

Proof. We prove that X is a boundary space. Let C be a cozero-set of X. Then $C \times Y$ is a cozero-set of $X \times Y$ because $C \times Y = \pi_X^{-1}[C]$, for the projection map $\pi_X : X \times Y \to X$. Since $X \times Y$ is a boundary space, there is a dense cozero-set $\mathcal{C}$ of $X \times Y$ such that $\mathcal{C} \subseteq (C \times Y) \cup \big((X \times Y) \setminus \overline{C \times Y}\big)$. Since $X \times Y$ is z-embedded in $\beta X \times \beta Y$, by hypothesis, there are sequences (U_n) and (V_n) of cozero-sets of X and Y, respectively, such that $\mathcal{C} = \bigcup_{n=1}^{\infty} (U_n \times V_n)$. We may assume, without loss of generality, that each U_n and each V_n is non-empty. Define $U = \bigcup_{n=1}^{\infty} U_n$, and note that U is a cozero-set of X. We show that it is dense in X. Indeed,

$$\begin{aligned} X \times Y = \mathrm{cl}_{X\times Y}(\mathcal{C}) &= \mathrm{cl}_{X\times Y}\left(\bigcup_{n=1}^{\infty}(U_n \times V_n)\right) \\ &\subseteq \mathrm{cl}_{X\times Y}\left(\bigcup_{n=1}^{\infty} U_n \times \bigcup_{n=1}^{\infty} V_n\right) \\ &= \mathrm{cl}_X\left(\bigcup_{n=1}^{\infty} U_n\right) \times \mathrm{cl}_Y\left(\bigcup_{n=1}^{\infty} V_n\right). \end{aligned}$$

Thus, $\overline{U} = X$, showing that U is dense in X. Now observe that

$$\begin{aligned} \mathcal{C} \subseteq (C \times Y) \cup \big((X \times Y) \smallsetminus \overline{C \times Y}\big) &= (C \times Y) \cup \big((X \times Y) \smallsetminus (\overline{C} \times \overline{Y})\big) \\ &= (C \times Y) \cup \big((X \smallsetminus \overline{C}) \times Y\big) \\ &= \big(C \cup (X \smallsetminus \overline{C})\big) \times Y. \end{aligned}$$

We claim that $U \subseteq C \cup (X \smallsetminus \overline{C})$. Let $x \in U$. Pick an index k with $x \in U_k$. Since $V_k \neq \emptyset$, take any $y \in V_k$. Then

$$(x, y) \in U_k \times V_k \subseteq \mathcal{C} \subseteq \big(C \cup (X \smallsetminus \overline{C})\big) \times Y,$$

which implies $x \in \big(C \cup (X \smallsetminus \overline{C})\big)$, thus proving that $U \subseteq \big(C \cup (X \smallsetminus \overline{C})\big)$. Therefore X is a boundary space. A similar argument shows that Y is a boundary space. □

Remark 5.3.2. We can also prove that if L and M are frames such that every cozero element of $L \oplus M$ is of the form $\bigvee_{n=1}^{\infty}(a_n \oplus b_n)$, for some sequences (a_n) and (b_n) in $\mathrm{Coz}\, L$ and $\mathrm{Coz}\, M$, respectively, then L and M are boundary frames if $L \oplus M$ is boundary frame. Such a proof however would not yield the topological result above because, in general, the frames $\mathfrak{O}(X \times Y)$ and $\mathfrak{O}X \oplus \mathfrak{O}Y$ are not isomorphic.

Proposition 5.3.3. *Let L and M be frames such that every cozero element of $L \oplus M$ is of the form $\bigvee_{n=1}^{\infty}(a_n \oplus b_n)$, for some sequences (a_n) and (b_n) in $\mathrm{Coz}\, L$ and $\mathrm{Coz}\, M$, respectively. If $L \oplus M$ is boundary frame, then L and M are boundary frames.*

Proof. Let $x \in \mathrm{Coz}\, L$. Then $x \oplus 1 \in \mathrm{Coz}(L \oplus M)$. By hypothesis, there is a dense $U = \bigvee_{n=1}^{\infty}(a_n \oplus b_n) \in \mathrm{Coz}(L \oplus M)$, for some sequences (a_n) and (b_n) of non-zero cozero elements of L and M, respectively such that $U \leq (x \oplus 1) \vee (x \oplus 1)^*$. Now

$$\begin{aligned}\bigvee_{n=1}^{\infty}(a_n \oplus b_n) &\leq (x \oplus 1) \vee (x \oplus 1)^* \\ &\leq (x \oplus 1) \vee ((x^* \oplus 1) \vee (1 \oplus 1^*)) \\ &\leq (x \oplus 1) \vee (x^* \oplus 1) \\ &\leq (x \vee x^*) \oplus 1.\end{aligned}$$

Hence $(a_n \oplus b_n) \leq (x \vee x^*) \oplus 1$ for every n, whence $a_n \leq x \vee x^*$ for every n, which implies $\bigvee_n a_n \leq x \vee x^*$. We claim that the cozero element $\bigvee_n a_n$ of L is dense. Since U is dense and

$$U = \bigvee_n (a_n \oplus b_n) \leq \Big(\bigvee_n a_n\Big) \oplus \Big(\bigvee_n b_n\Big),$$

it follows that $\Big(\bigvee_n a_n\Big) \oplus \Big(\bigvee_n b_n\Big)$ is dense, whence $\bigvee_n a_n$ is dense. Therefore L is a boundary frame. Similarly, M is a boundary frame. □

5.4 Some comments on boundary rings

We observed in Example 5.2.2 that every cozero complemented space (there we used frames) is a boundary space. Now, X is cozero complemented if and only if for every $f \in C(X)$ there exists $g \in C(X)$ such that $\mathrm{Ann}(f) = \mathrm{Ann}^2(g)$. Rings with this property are called *quasi-regular*. See [36] for some other properties of quasi-regular rings. Thus, if $C(X)$ is quasi-regular, then it is a boundary ring. We show that, in fact, this is the case for all reduced f-rings.

Proposition 5.4.1. *Every reduced quasi-regular f-ring is a boundary ring.*

Proof. Let I be a frontier ideal in a reduced quasi-regular f-ring A. Pick $a \in A$ such that $I = M(a) + \mathrm{Ann}(a)$. Since A is quasi-regular, there exists $b \in A$ such that $\mathrm{Ann}(a) = \mathrm{Ann}^2(b)$. Thus, $I = M(a) + \mathrm{Ann}^2(b)$. Observe that the element $d = a^2 + b^2$ belongs to I. We claim that d is a non-divisor of zero. Consider any $r \in A$ with $r(a^2 + b^2) = 0$. Since squares are positive in f-rings, this implies $r^2a^2 = 0 = r^2b^2$, whence $ra = 0 = rb$ because A is reduced. But now $ra = 0$ implies $r \in \mathrm{Ann}(a) = \mathrm{Ann}^2(b)$. Consequently,

$r^2 = 0$, implying $r = 0$. Therefore d is a non-divisor of zero belonging to I, and so A is a boundary ring. $\square$

We show next that some ideals of a boundary ring, when viewed as rings in their own right, are themselves boundary rings. Not every ideal of a boundary ring inherits this property, as the following example bears testimony.

Example 5.4.2. The ring $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ is a boundary ring. One can verify this directly by brute force, or one can peek ahead to Theorem 5.4.4 where it is shown that direct products of boundary rings are boundary rings; so that the isomorphism $\mathbb{Z}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ proves the claim since $\mathbb{Z}_2$ is a field, and hence a boundary ring. Now the ideal $J = \{\overline{0}, \overline{2}\}$ is not a boundary ring because there is no non-divisor of zero in J.

Recall from [59, Corollary 3.6] that if A is a ring that is a $\mathbb{Q}$-algebra (for instance an f-ring) and J is an ideal in A, then the set of maximal ideals of J is

$$\operatorname{Max}(J) = \{M \cap J \mid M \in \operatorname{Max}(A) \,\text{and}\, M \not\supseteq J\}.$$

In what follows we shall write $M_J(a)$ and $\operatorname{Ann}_J(a)$ to indicate that the stated ideal is contemplated in the ring J. Without a subscript, the ideals $M(a)$ and $\operatorname{Ann}(a)$ will be taken as ideals in A. It will also be helpful to write $\operatorname{Ndz}(A)$ and $\operatorname{Ndz}(J)$ for the sets of non-divisors of zero of A and J, respectively, with J viewed as a ring.

Theorem 5.4.3. *Let A be a boundary ring that is a $\mathbb{Q}$-algebra. A necessary and sufficient condition that an ideal J of A be a boundary ring is that* $\operatorname{Ndz}(J)$ *be nonempty.*

Proof. The condition is clearly necessary. To prove sufficiency, let $a \in J$. Since A is a boundary ring, the frontier ideal $M(a) + \operatorname{Ann}(a)$ contains an element d which belongs to $\operatorname{Ndz}(A)$. So there exist $u \in M(a)$ and $v \in \operatorname{Ann}(a)$ such that $d = u + v$. Take any $c \in \operatorname{Ndz}(J)$, and observe that $cd \in \operatorname{Ndz}(J)$. Now, we shall be done if we can show that $cu \in M_J(a)$ and $cv \in \operatorname{Ann}_J(a)$. The latter is immediate. To prove the former, we consider two cases. First, if there is no maximal ideal in J containing a (for instance, if A is a local ring and J its unique maximal ideal), then $M_J(a) = J$. Next, suppose I is a maximal ideal of J containing a. Pick $N \in \mathrm{Max}(A)$ with $I = N \cap J$. Then $a \in N$, and therefore $u \in N$ because u belongs to every maximal ideal of A containing a. Since J is an ideal in A, and $c \in J$, it follows that $cu \in N \cap J = I$. In consequence, $cu \in M_J(a)$, and the result follows. □

We end with a purely ring-theoretic result. Recall that an ideal $\mathcal{I}$ of a direct product $\prod A_i$ of family $(A_i)_{i\in I}$ of rings is a maximal ideal if and only if it is of the form $\mathcal{I} = \prod J_i$, where there exists $i_0 \in I$ such that $J_{i_0} \in \mathrm{Max}(A_{i_0})$ and $J_i = A_i$ for all $i \in I \setminus \{i_0\}$. We shall not decorate the notations $M(a)$ and $\mathrm{Ann}(a)$ in what follows; the element a will make it clear where the ideal in question resides.

Theorem 5.4.4. *The direct product of any family of boundary rings is a boundary ring if and only if each factor is a boundary ring.*

Proof. ($\Leftarrow$) Let $(A_i)_{i\in I}$ be a family of boundary rings. Let $(a_i) \in \prod A_i$. For brevity, we write $a = (a_i)$ and $A = \prod A_i$. Consider the boundary ideal $M(a) + \mathrm{Ann}(a)$ of A. For each $i \in I$ there exists a non-divisor of zero $d_i \in M(a_i) + \mathrm{Ann}(a_i)$. Pick $u_i \in M(a_i)$ and $v_i \in \mathrm{Ann}(a_i)$ such that $d_i = u_i + v_i$. Let $d = (d_i)$. One checks routinely that d is a non-divisor of zero in A. We claim that

$$\prod_{i\in I} M(a_i) \subseteq M(a) \quad \text{and} \quad \prod_{i\in I} \mathrm{Ann}(a_i) \subseteq \mathrm{Ann}(a).$$

The latter is easy to check, and, in fact, the containment is equality. To prove the former, let $(z_i) \in \prod M(a_i)$, and take any maximal ideal $\mathcal{M}$ of A containing a. There exists $i_0 \in I$ such that $\mathcal{M} = \prod J_i$, where $J_{i_0} \in \mathrm{Max}(A_{i_0})$ and $J_i = A_i$ for $i \neq i_0$. Then $a_{i_0} \in J_{i_0}$, which implies $z_{i_0} \in J_{i_0}$ because $z_{i_0} \in M(a_{i_0})$, and consequently $z_i \in M(a_i)$ since $\mathcal{M}$ is an arbitrary maximal ideal of A containing a. This establishes the claimed containment. Now

$$d = (u_i) + (v_i) \in \prod_{i\in I} M(a_i) + \prod_{i\in I} \mathrm{Ann}(a_i) \subseteq M(a) + \mathrm{Ann}(a),$$

which shows that A is a boundary ring.

($\Rightarrow$) Suppose $\prod A_i$ is a boundary ring. Fix any index k. Let $x \in A_k$, and let $a = (a_i)$ be the element of $\prod A_i$ such that $a_i = 1$ if $i \neq k$, and $a_k = x$. We claim that $M(a) = \prod J_i$, where

$$J_i = \begin{cases} A_i & \text{if} \quad i \neq k \\ M(x) & \text{if} \quad i = k. \end{cases}$$

To see this, let $\mathcal{I}$ be a maximal ideal of $\prod A_i$ containing a. Then $\mathcal{I}$ must be of the form $\prod J_i$, with each $J_i = A_i$ for $i \neq k$, and $J_k \in \mathrm{Max}(A_k)$. The claim follows easily from this observation. By hypothesis, there is a non-divisor of zero $d = (d_i) \in A$ such that

$$d \in M(a) + \mathrm{Ann}(a) = M(a) + \prod_{i \in I} \mathrm{Ann}(a_i).$$

Now, d_k is a non-divisor of zero in A_k, for if z were a non-zero element of A_k with $zd_k = 0$, then the element $b = (b_i)$ of $\prod A_i$ for which

$$b_i = \begin{cases} 0 & \text{if} \quad i \neq k \\ z & \text{if} \quad i = k \end{cases}$$

would be a non-zero element of $\prod A_i$ with $bd = 0$. Pick element $u = (u_i) \in M(a)$ and $v = (v_i) \in \prod \mathrm{Ann}(a_i)$ such that $d = u + v$. Then $u_k \in M(x)$ and $v_k \in \mathrm{Ann}(x)$, which shows that the non-divisor of zero $d_k \in M(x) + \mathrm{Ann}(x)$. Therefore A_k is a boundary ring. $\square$

Chapter 6

Frames that are finitely an F-frame

6.1 Introduction

There are various topological properties of a space X which can be characterized in terms of algebraic properties of the ring $C(X)$ of continuous real-valued functions on X. For instance, X is a P-space (meaning that every G_δ-set is open) precisely when $C(X)$ is von Neumann regular, and X is an F-space (meaning that every cozero-set is C^*-embedded) if and only if every finitely generated ideal in $C(X)$ is principal. See [37] for other such properties.

A space X is *finitely an F-space* if βX can be written as a union $K_1 \cup \cdots \cup K_n$, where each K_i is a closed set in βX and is an F-space in the subspace topology. These spaces were first considered in [41], and have since been studied by Larson in a series of papers, including [46] and [47]. In the former paper she gives a characterization, among normal spaces, in terms of an algebraic condition on the ring $C(X)$. In fact, she shows that the condition is sufficient for X to be finitely an F-space with no normality assumed, and necessary if X is normal. Thus, for normal spaces there is an algebraic characterization.

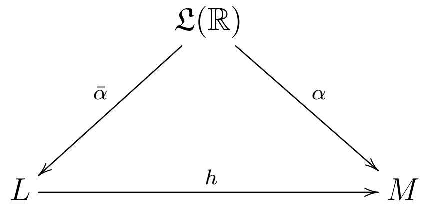
Our goal in this chapter is to unshackle the characterization from normality. This we achieve by working with frames instead of spaces. We thus have to define, in a conservative way, what it means to say a frame is finitely an F-frame. By "conservative" we mean that a space must be finitely an F-space if and only if the frame of its open sets is finitely an F-frame.

We shall then exploit the presence of the Lindelöf coreflection, λL, of a frame L, the normality of the frame λL, and the fact that the rings $\mathcal{R}L$ and $\mathcal{R}(\lambda L)$ are isomorphic. Loosely speaking, one can say the normality that one needs to assume in spaces is already present (albeit at a higher level) if we work frames.

To some extent our pattern of proofs will be modeled on that of Larson. Indeed, in one instance we will piggyback on her proof of the corresponding implication in spaces.

C^*-quotients

Our approach to the ring $\mathcal{R}L$ follows that [8], so that $\mathcal{R}L$ is the f-ring whose members are the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, where $\mathfrak{L}(\mathbb{R})$ is the frame of reals. Recall from [4] that a quotient map $h\colon L \to M$ is called a *C^*-quotient map*, and we then say M is a *C^*-quotient* of L, in case for every bounded $\alpha \in \mathcal{R}M$ there exists some $\bar{\alpha} \in \mathcal{R}L$ such that the triangle below commutes.



A *C-quotient map* is defined similarly but without the restriction that α be bounded. There are several characterizations of C^*-quotient maps in [4], and we shall have occasion to use one of the characterizations in [4, Theorem 7.1.1].

f-Rings that are finitely 1-convex

We refer the reader to [60] for concepts regarding f-rings. Our f-rings are commutative with identity element. An f-ring A is 1-*convex* if whenever $0 \leq a \leq b$ in A, then $a = bc$ for some $c \in A$. A ring A is called a *Bézout ring* if every finitely generated ideal of A is principal. In [51, Theorem 1] it is shown that a reduced f-ring with bounded inversion is Bézout if and only if it is 1-convex.

We follow [46] in defining fibre products and f-rings that are finitely 1-convex. Let A_1, A_2, B be f-rings and $\phi_i \colon A_i \to B$, for $i = 1, 2$, be surjective ℓ-ring homomorphisms. The *fibre product* of A_1 and A_2 relative to the pair (ϕ_1, ϕ_2) is the ring $A_1 \times_B A_2$ that equalizes $\phi_1\pi_1$ and $\phi_2\pi_2$ in the (pullback) square below.

$$\begin{array}{ccc} A_1 \times_B A_2 & \xrightarrow{\pi_2} & A_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ A_1 & \xrightarrow{\phi_1} & B \end{array}$$

Explicitly,

$$A_1 \times_B A_2 = \{(a_1, a_2) \in A_1 \times A_2 \mid \phi_1(a_1) = \phi_2(a_2)\}.$$

An f-ring is a *finite fibre product* of the f-rings $A_1, A_2, \ldots, A_n$ if it can be constructed in a finite number of steps, where every step consists of taking the fibre product of two f-rings, each satisfying either the property that it is one of the A_i not used in a previous step, or it is a fibre product obtained in an earlier step of the construction. A *finitely* 1-*convex* f-ring is one that is either 1-convex, or can be written as a finite fibre product of 1-convex f-rings.

6.2 Ring theoretic characterization

Let us recall from [4] that a completely regular frame L is called an *F-frame* if, for each $c \in \operatorname{Coz} L$, the quotient map $L \mapsto {\downarrow} c$ is a C^*-quotient map. There are several characterizations which we shall use freely. In particular, L is an F-frame if and only if $\mathcal{R}L$ is a Bézout ring ([25, Proposition 3.2]).

Proposition 6.2.1. *The following are equivalent for a completely regular frame L.*

(1) *L is an F-frame.*

(2) *$\mathcal{R}L$ is 1-convex.*

(3) *For any $a, b \in \operatorname{Coz} L$ with $a \wedge b = 0$, there exist $c, d \in \operatorname{Coz} L$ such that $c \vee d = 1$ and $a \wedge c = 0 = b \wedge d$.*

We shall need to know that the homomorphic image of an F-frame under a coz-onto homomorphism is an F-frame. For the proof we shall use [34, Proposition 3.3] which, paraphrased, states the following:

> *A frame homomorphism $h\colon L \to M$ is coz-onto if and only if for all $a, b \in \operatorname{Coz} M$ with $a \wedge b = 0$, there exist $c, d \in \operatorname{Coz} L$ such that $c \wedge d = 0$, $h(c) = a$, and $h(d) = b$.*

Lemma 6.2.2. *If $h\colon L \to M$ is a coz-onto frame homomorphism and L is an F-frame, then M is an F-frame.*

Proof. Let $a \wedge b = 0$ in $\operatorname{Coz} M$. By the result cited above, there exist $c, d \in \operatorname{Coz} L$ with $c \wedge d = 0$ and $h(c) = a$, $h(d) = b$. Since L is an F-frame, the second characterization in Proposition 6.2.1 yields $u, v \in \operatorname{Coz} L$ such that $u \vee v = 1$ and $u \wedge c = v \wedge d = 0$. Then $h(u)$ and $h(v)$ are cozero elements of M with the desired property. □

A special case that we shall apply in the proof of the main theorem is the following corollary.

Corollary 6.2.3. *A closed quotient of a normal F-frame is an F-frame.*

We want to define the property of being finitely an F-frame in a conservative way. It is convenient to use localic language. Consider the following property that a topological space X, or a locale L can have:

(fin-F) *The space* X (resp. the locale L) is a union (resp. join) of finitely many closed subspaces (resp. sublocales) each of which is an F-space (resp. F-frame).

In frame language, L has property fin-F if and only if there are finitely many elements $a_1, \ldots, a_n$ in L such that $a_1 \wedge \cdots \wedge a_n = 0$, and each $\uparrow a_i$ is an F-frame. If K is a closed subspace of X, then $U = X \smallsetminus K$ is an element of the frame $\mathfrak{O}X$ such that $\uparrow U$ is isomorphic to $\mathfrak{O}K$. Thus, X has property fin-F if and only if $\mathfrak{O}X$ has property fin-F. Also, if $h\colon L \to M$ is a frame isomorphism, then, for any $a \in L$, the mapping $\uparrow a \to \uparrow h(a)$, effected by h, is easily checked to be a frame isomorphism. Consequently, L has property fin-F if and only if M has the property.

Definition 6.2.4. A frame L is *finitely an F-frame* if βL has property fin-F.

It is clear that an F-frame is finitely an F-frame. Equally clear is that L is finitely an F-frame if and only if βL is finitely an F-frame. Since $\beta(\lambda L)$ is isomorphic to βL, and $\beta(\lambda L)$ is isomorphic to $\beta(\upsilon L)$ it follows that:

Proposition 6.2.5. *The following are equivalent for L.*

(1) *L is finitely an F-frame.*

(2) *βL is finitely an F-frame.*

(3) *λL is finitely an F-frame.*

(4) *υL is finitely an F-frame.*

Proposition 6.2.6. *A Tychonoff space X is finitely an F-space if and only if $\mathfrak{O}X$ is finitely an F-frame.*

Proof. Recall that the frames $\beta(\mathfrak{O}X)$ and $\mathfrak{O}(\beta X)$ are isomorphic. Now,

$$\begin{aligned} X \text{ is finitely an } F\text{-space} &\iff \beta X \text{ has property fin-F} \\ &\iff \mathfrak{O}(\beta X) \text{ has property fin-F} \\ &\iff \beta(\mathfrak{O}X) \text{ has property fin-F} \\ &\iff \mathfrak{O}X \text{ is finitely an } F\text{-frame,} \end{aligned}$$

which proves the proposition. □

In [46] Larson shows that if $C(X)$ is finitely 1-convex, then X is finitely an F-space, and conversely if X is normal. We sharpen this result by showing that normality is actually not needed. This will be a consequence of a result in frames. Recall from [46] that if an f-ring A is finitely 1-convex, then so is its bounded part A^*.

We remind the reader that a ring A is a *subdirect product* of the rings $\{B_i \mid i \in I\}$ if there is an injective ring homomorphism $\phi\colon A \to \prod_i B_i$ such that the composite

$$A \xrightarrow{\quad\phi\quad} \prod_i B_i \xrightarrow{\quad\pi_j\quad} B_j$$

is surjective for each $j \in I$, where π_j is the j^{th} canonical projection. For use in the upcoming proof we recite the following result from [47]. Here we paraphrase it somewhat.

Lemma 6.2.7. *If A is a subdirect product of the rings A_1 and A_2, then A is isomorphic to some fibre product of A_1 and A_2*

Let L be a frame, $a \in L$, and $f \in \mathcal{R}L$. In the lemma that follows we shall write $f_{|\mathfrak{c}(a)}$ for the element of $\mathcal{R}(\uparrow a)$ given by the composite $\mathfrak{L}(\mathbb{R}) \xrightarrow{f} L \xrightarrow{\kappa_a} \uparrow a$. The notation is chosen to reflect that we think of $f_{|\mathfrak{c}(a)}$ as the restriction of f to the closed sublocale $\mathfrak{c}(a)$.

Lemma 6.2.8. *Let L be a normal frame and let a and b be elements of L with $a \wedge b = 0$. Then $\mathcal{R}L$ is isomorphic to some fibre product $\mathcal{R}(\uparrow a) \times_B \mathcal{R}(\uparrow b)$.*

Proof. We shall show that $\mathcal{R}L$ is a subdirect product of $\mathcal{R}(\uparrow a)$ and $\mathcal{R}(\uparrow b)$, whence the result will follow from Lemma 6.2.7. Denote by π_a and π_b, respectively, the canonical projection maps

$$\mathcal{R}(\uparrow a) \xleftarrow{\quad\pi_a\quad} \mathcal{R}(\uparrow a) \times \mathcal{R}(\uparrow b) \xrightarrow{\quad\pi_b\quad} \mathcal{R}(\uparrow b).$$

We need to produce an injective ℓ-ring homomorphism $\phi\colon \mathcal{R}L \to \mathcal{R}(\uparrow a) \times \mathcal{R}(\uparrow b)$ such that the composites $\pi_a\phi$ and $\pi_b\phi$ are surjective. Define ϕ by setting

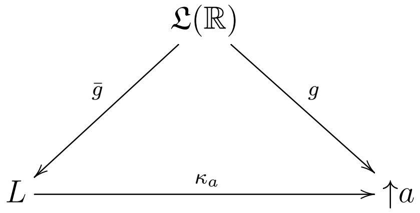
$$\phi(f) = (f_{|\mathfrak{c}(a)}, f_{|\mathfrak{c}(b)}).$$

Since each of the mappings $f \mapsto f_{|\mathfrak{c}(a)}$ and $f \mapsto f_{|\mathfrak{c}(b)}$ is an ℓ-ring homomorphism, it is clear that ϕ is an ℓ-ring homomorphism. To show that it is injective, consider any $f, g \in \mathcal{R}L$

with $(f_{|\mathfrak{c}(a)}, f_{|\mathfrak{c}(b)}) = (g_{|\mathfrak{c}(a)}, g_{|\mathfrak{c}(b)})$. For any $p \in \mathbb{Q}$ we have

$$\begin{aligned} f(p,-) &= f(p,-) \vee (a \wedge b) \\ &= \Big(f(p,-) \vee a\Big) \wedge \Big(f(p,-) \vee b\Big) \\ &= f_{|\mathfrak{c}(a)}(p,-) \wedge f_{|\mathfrak{c}(b)}(p,-) \\ &= g_{|\mathfrak{c}(a)}(p,-) \wedge g_{|\mathfrak{c}(b)}(p,-) \\ &= \Big(g(p,-) \vee a\Big) \wedge \Big(g(p,-) \vee b\Big) \\ &= g(p,-) \vee (a \wedge b) \\ &= g(p,-) \end{aligned}$$

which shows that $f = g$. Therefore ϕ is injective. To see that the composite $\pi_a\phi$ is surjective, let $g \in \mathcal{R}(\uparrow a)$. Since $\kappa_a \colon L \to \uparrow a$ is a C-quotient map, there exists a $\bar{g} \in \mathcal{R}L$ such that the triangle



commutes. Therefore

$$\pi_a\phi(\bar{g}) = \pi_a(\bar{g}_{|\mathfrak{c}(a)}, \bar{g}_{|\mathfrak{c}(b)}) = \pi_a(\kappa_a\bar{g}, \kappa_b\bar{g}) = \kappa_a\bar{g} = g,$$

which proves that $\pi_a\phi$ is surjective. Similarly, $\pi_b\phi$ is surjective. Thus, $\mathcal{R}L$ is a subdirect product of $\mathcal{R}(\uparrow a)$ and $\mathcal{R}(\uparrow a)$, as desired. □

Let us recall from [21, Proposition 2.1] the following result, which we paraphrase and state less generally than in [21].

Lemma 6.2.9. *If $h \colon L \to M$ is a C^*-quotient map, then $\uparrow r_L h_*(0) \to M$ is the Stone-Čech compactification of M.*

We are now sufficiently equipped to prove the main localic result from which the spatial one we desire will follow as a corollary.

Theorem 6.2.10. *A completely regular frame L is finitely an F-frame if and only if $\mathcal{R}L$ is finitely* 1*-convex.*

Proof. ($\Leftarrow$) Suppose that $\mathcal{R}L$ is finitely 1-convex. Then $\mathcal{R}^*L$ is finitely 1-convex, and hence $\mathcal{R}(\beta L)$ is finitely 1-convex since $\mathcal{R}(\beta L)$ is isomorphic to $\mathcal{R}^*L$. Since βL is spatial, there is a space X such that βL is isomorphic to $\mathfrak{O}X$. Hence $C(X)$ is isomorphic to $\mathcal{R}(\beta L)$, so that $C(X)$ is finitely 1-convex, which means X is finitely an F-space, and therefore $\mathfrak{O}X$ is finitely an F-frame. Consequently, βL is finitely an F-frame, and thus L is finitely an F-frame.

($\Rightarrow$) Suppose that L is finitely an F-frame. We shall first look at the case when L is normal, and from there deduce the general case. So assume, for a moment, that L is normal. Consider any $I \in \beta L$ for which ${\uparrow}I$ is an F-frame. Put $a = \bigvee I$. We claim that $\mathcal{R}({\uparrow}a)$ is 1-convex. To prove the claim it suffices to show that ${\uparrow}a$ is an F-frame. Observe that $I \leq r_L(a)$ since r_L is the right adjoint of the join map $\beta L \to L$, hence $r_L(a)$ is a closed quotient of the normal F-frame ${\uparrow}I$. Of course ${\uparrow}I$ is normal by Corollary 6.2.3 since it is a closed quotient of the normal frame βL. So ${\uparrow}r_L(a)$ is a closed quotient of the normal F-frame ${\uparrow}I$, which implies that ${\uparrow}r_L(a)$ is an F-frame by Corollary 6.2.3. But now the right adjoint of the map $\kappa_a\colon L \to {\uparrow}a$ is the inclusion map, and the zero of ${\uparrow}a$ is a, hence ${\uparrow}r_L(a)$ is isomorphic to $\beta({\uparrow}a)$, by Lemma 6.2.9, as $\kappa_a\colon L \to {\uparrow}a$ is a C^*-quotient map because L is normal. It follows therefore that ${\uparrow}a$ is an F-frame, whence $\mathcal{R}({\uparrow}a)$ is 1-convex, as claimed.

Now, since L is finitely an F-frame, there are elements $I_1, \ldots, I_n$ in βL such that $I_1 \wedge \cdots \wedge I_n = 0$ and each frame ${\uparrow}I_k$ is an F-frame. For each $k = 1, \ldots, n$ put $a_k = \bigvee I_k$, and observe that $a_1 \wedge \cdots \wedge a_n = 0$. As just observed, each $\mathcal{R}({\uparrow}a_k)$ is a 1-convex f-ring. A simple induction using Lemma 6.2.8 shows that $\mathcal{R}L$ is (isomorphic to) a finite fibre product of the 1-convex f-rings $\mathcal{R}({\uparrow}a_1), \ldots, \mathcal{R}({\uparrow}a_n)$. That is, $\mathcal{R}L$ is finitely 1-convex.

We now relax the normality condition. So assume L is finitely an F-frame. Then λL is finitely an F-frame, and since λL is normal, we have that $\mathcal{R}(\lambda L)$ is finitely 1-convex. But the ring $\mathcal{R}(\lambda L)$ is isomorphic to $\mathcal{R}L$, so $\mathcal{R}L$ is also finitely 1-convex. □

Since the rings $C(X)$ and $\mathcal{R}(\mathfrak{O}X)$ are isomorphic for any Tychonoff space X, the following characterization follows from Proposition 6.2.6 and Theorem 6.2.10.

Corollary 6.2.11. *A Tychonoff space X is finitely an F-space if and only if $C(X)$ is finitely 1-convex.*

6.3 Inheritance by quotients

It is not always the case that if a topological property is inherited by subspaces, then the corresponding frame property is inherited by quotients. A famous (or should that be "infamous") example is that, whereas subspaces of P-spaces are P-spaces, not every quotient of a P-frame is a P-frame [5]. In [45], Larson shows that cozero-sets inherit the property of being finitely an F-space, as do C^*-embedded subspaces [46]. We show that similar inheritances occur for frames. We start with the easier of the two.

Theorem 6.3.1. *A C^*-quotient of a frame that is finitely an F-frame is itself finitely an F-frame.*

Proof. Let $h\colon L \to M$ be a C^*-quotient map with L finitely an F-frame. By the preceding lemma, ${\uparrow} r_L h_*(0)$ is (isomorphic to) βM. Since L is finitely an F-frame, there exist $I_1, \ldots, I_n$ in βL such that $I_1 \wedge \cdots \wedge I_n = 0$, and ${\uparrow} I_k$ is an F-frame for each $k = 1, \ldots, n$. Then

$$\big(r_L(h_*(0)) \vee I_1\big) \wedge \cdots \wedge \big(r_L(h_*(0)) \vee I_n\big) = 0_{{\uparrow} r_L(h_*(0))}.$$

For each k, ${\uparrow} I_k$ is a normal frame, being a closed quotient of the normal frame βL. But ${\uparrow}(r_L(h_*(0)) \vee I_k)$ is a closed quotient of ${\uparrow} I_k$, so ${\uparrow}(r_L(h_*(0)) \vee I_k)$ is a C^*-quotient of the normal F-frame ${\uparrow} I_k$, and is therefore itself an F-frame by Corollary 6.2.3. Thus βM has the fin-F property, which says M is finitely an F-frame. □

Remark 6.3.2. We could also have argued as follows. We claim that a Lindelöf quotient of a frame with property fin-F has property fin-F. To see this, let $\phi\colon A \to B$ be a quotient map with B Lindelöf and A having property fin-F. Pick $a_1, \ldots, a_n$ in A with $a_1 \wedge \cdots \wedge a_n = 0$ and such that each ${\uparrow} a_i$ is an F-frame. For any $i \in \{1, \ldots, n\}$, the map ${\uparrow} a_i \to {\uparrow} \phi(a_i)$, mapping as ϕ, is a surjective frame homomorphism. It is therefore coz-onto because $\uparrow\phi(a_i)$ is Lindelöf as it is a closed quotient of a Lindelöf frame, and any surjective homomorphism with a Lindelöf codomain is coz-onto by [34, Proposition 3.2]. Thus, $\uparrow\phi(a_i)$ is an F-frame by Lemma 6.2.2. The theorem would then follow by [21, Proposition 2.1] which ensures that if $h\colon L \to M$ is a C^*-quotient map, then βM is a quotient of βL.

We now move to cozero quotients. That is, we aim to show that if L is finitely an F-frame and $c \in \operatorname{Coz} L$, then $\downarrow c$ is finitely an F-frame. We remind the reader that if L is a Lindelöf frame, then an element $a \in L$ is a cozero element if and only if $\downarrow a$ is Lindelöf [10, Corollary 4]. We need a preliminary result.

Let $h\colon M \to L$ be a surjective frame homomorphism, $a \in L$, and $b \in M$ such that $h(b) = a$. The map $\downarrow b \to \downarrow a$ effected by h is a surjective frame homomorphism. We shall say it is induced by h, and write $h\colon \downarrow b \to \downarrow a$.

Lemma 6.3.3. *Let L be a completely regular frame and $c \in \operatorname{Coz} L$. Then there exists $C \in \operatorname{Coz}(\beta L)$ such that $\bigvee C = c$, and the induced map $j_L\colon \downarrow C \to \downarrow c$ is a C^*-quotient map.*

Proof. Since $c \in \operatorname{Coz} L$, there is a sequence (c_n) in L such that $c_n \prec\!\!\prec c_{n+1}$ for every n, and $c = \bigvee c_n$. Put $C = \bigvee\{r_L(c_n) \mid n \in \mathbb{N}\}$ in βL, and observe that $C \in \operatorname{Coz}(\beta L)$ since $r_L(c_n) \prec\!\!\prec r_L(c_{n+1})$ for each n. Also,

$$\begin{aligned} \bigvee C = j_L(C) &= j_L\Big(\bigvee\{r_L(c_n) \mid n \in \mathbb{N}\}\Big) \\ &= \bigvee\{j_L r_L(c_n) \mid n \in \mathbb{N}\} \\ &= \bigvee_n c_n = c. \end{aligned}$$

To show that the induced map $\downarrow C \to \downarrow c$ is a C^*-quotient map we use [4, Theorem 7.2.7]. So let $u, v \in \operatorname{Coz}(\downarrow c)$ with $u \vee v = 1_{\downarrow c} = c$. We must produce $U, V \in \operatorname{Coz}(\downarrow C)$ with $U \vee V = C$, $\bigvee U \leq u$, and $\bigvee V \leq v$. By [4, Proposition 3.2.10], $u, v \in \operatorname{Coz} L$, hence $r_L(u \vee v) = r_L(u) \vee r_L(v)$. Since $c_1 \prec\!\!\prec c = u \vee v$, we have $c_1 \in r_L(u \vee v) = r_L(u) \vee r_L(v)$. This enables us to find $s_1 \prec\!\!\prec u$ and $t_1 \prec\!\!\prec v$ such that $c_1 = s_1 \vee t_1$. Similarly, we can find $w_2 \prec\!\!\prec u$ and $z_2 \prec\!\!\prec v$ such that $c_2 = w_2 \vee z_2$. It will soon be apparent why we write w_2 and z_2 instead of s_2 and t_2. Observe that $s_1 \vee w_2 \prec\!\!\prec u$ and $t_1 \vee z_2 \prec\!\!\prec v$. Thus we can pick s_2 and t_2 with $s_1 \vee w_2 \lll s_2 \lll u$ and $t_1 \vee z_2 \lll t_2 \lll v$, so that $s_1 \lll s_2$, $t_1 \lll t_2$, and $c_2 \leq s_2 \vee t_2$. Continuing this way we can construct sequences (s_n) and (t_n) in L such that, for each n,

$$s_n \lll s_{n+1}, \quad t_n \lll t_{n+1}, \quad c_n \leq s_n \vee t_n.$$

Let S and T be the cozero elements of βL given by $S = \bigvee r_L(s_n)$ and $T = \bigvee r_L(t_n)$. Then $U = C \wedge S$ and $V = C \wedge T$ are cozero elements of $\downarrow C$, with $S \leq r_L(u)$ and $T \leq r_L(v)$. Therefore

$$\bigvee U = j_L(C \wedge S) = \bigvee C \wedge \bigvee S \leq c \wedge u = u,$$

and similarly $\bigvee V \leq v$. It thus remains to show that $U \vee V = C$. Since for any n we have $c_n \leq s_n \vee t_n \in S \vee T$, it follows that $C \leq S \vee T$, whence $C = C \wedge (S \vee T) = U \vee V$. This completes the proof. □

We need to do some ground-clearing in preparation for the proof of the following result. We shall use both the concepts of nuclei and sublocales. The background is in our references [44] and [57]. Recall that nuclei are compared pointwise. That is, if j and k are nuclei on a frame L, then $j \leq k$ if and only if $j(a) \leq k(a)$ for every $a \in L$. Further, if $j \leq k$, then the map $\mathrm{Fix}(j) \to \mathrm{Fix}(k)$ given by $x \mapsto k(x)$ is a surjective frame homomorphism whose right adjoint is the inclusion.

For a frame L and $a \in L$, we let $\nu_a \colon L \to L$ be the nucleus defined by

$$\nu_a(x) = a \to x = \bigvee \{z \in L \mid a \wedge z \leq x\}$$

where the arrow signifies the Heyting implication. Recall that $a \wedge (a \to x) = a \wedge x$. The map $\nu_a \colon \downarrow a \to \mathrm{Fix}(\nu_a)$ is an isomorphism of frames whose inverse (and therefore right adjoint) is the map $\hat{a} \colon \mathrm{Fix}(\nu_a) \to \downarrow a$ given by $\hat{a}(x) = a \wedge x$. Recall that when viewed as a map $\kappa_a \colon L \to L$, κ_a is a nucleus with $\mathrm{Fix}(\kappa_a) = \uparrow a$. It is shown in [14] that if j is a nucleus on L and $a \in L$, then the join $j \vee \kappa_a$ is the composite $j\kappa_a$, so that we have a frame homomorphism $j\kappa_a \colon \mathrm{Fix}(j) \to \mathrm{Fix}(j\kappa_a)$ the bottom of which is $j(a)$.

By a *cozero-sublocale* of a frame L we mean any open sublocale of the form $\mathfrak{o}(c)$, for some $c \in \mathrm{Coz}\, L$. Observe that if L is a frame, C a cozero sublocale of L, and S any sublocale of L, then $S \cap C$ is a cozero-sublocale of S.

Theorem 6.3.4. *If L is finitely an F-frame and $c \in \operatorname{Coz} L$, then $\downarrow c$ is also finitely an F-frame.*

Proof. By Lemma 6.3.3 there is a $C \in \operatorname{Coz}(\beta L)$ such that the map $\downarrow C \to \downarrow c$ given by join is a C^*-quotient map. Since this map is clearly dense, it follows from [21, Corollary 2.2] that $\beta(\downarrow c)$ is isomorphic to $\beta(\downarrow C)$. We shall therefore be done if we can show that $\beta(\downarrow C)$ has the fin-F property. Since L is finitely an F-frame, there are elements $I_1, \ldots, I_n$ in βL such that each $\uparrow I_i$ is an F-frame and $I_1 \wedge \cdots \wedge I_n = 0$. Then in the frame $\downarrow C$ we have that $(C \wedge I_1) \wedge \cdots \wedge (C \wedge I_n) = 0$. The right adjoint $r_{\downarrow C}$ preserves meets, so

$$r_{\downarrow C}(C \wedge I_1) \wedge \cdots \wedge r_{\downarrow C}(C \wedge I_n) = 0.$$

We aim to show that each of the closed quotients $\uparrow r_{\downarrow C}(C \wedge I_i)$ of $\beta(\downarrow C)$ is an F-frame. This we do by showing that each is the Stone-Čech compactification of some F-frame. Let $I \in \{I_1, \ldots, I_n\}$, and consider the composite

$$\beta(\downarrow C) \xrightarrow{\widehat{C}} \downarrow C \xrightarrow{\nu_C} \operatorname{Fix}(\nu_C) \xrightarrow{\nu_C \kappa_I} \operatorname{Fix}(\nu_C \kappa_I).$$

Since $\nu_C \kappa_I$ is the join in the assembly of βL of the nuclei ν_C and κ_I, the frame $\operatorname{Fix}(\nu_C \kappa_I)$ is the meet $\mathfrak{o}(C) \cap \mathfrak{c}(I)$ in the coframe $\mathcal{S}\ell(\beta L)$ of the sublocales of βL, and is therefore a closed sublocale of $\mathfrak{o}(C)$. Since βL is Lindelöf and $C \in \operatorname{Coz}(\beta L)$, we have that $\mathfrak{o}(C)$ is Lindelöf, and therefore normal. Consequently, the homomorphism $\nu_C \kappa_I \colon \operatorname{Fix}(\nu_C) \to \operatorname{Fix}(\nu_C \kappa_I)$ is a C^*-quotient map because a closed quotient of a normal frame is a C^*-quotient. For brevity, write $\ell \colon \downarrow C \to \operatorname{Fix}(\nu_C \kappa_I)$ for the composite

$$\downarrow C \xrightarrow{\nu_C} \operatorname{Fix}(\nu_C) \xrightarrow{\nu_C \kappa_I} \operatorname{Fix}(\nu_C \kappa_I).$$

Since $\nu_C \colon \downarrow C \to \operatorname{Fix}(\nu_C)$ is an isomorphism, it follows that ℓ is a C^*-quotient map, and hence, by Lemma 6.2.9,

$$\beta(\operatorname{Fix}(\nu_C \kappa_I)) = \uparrow r_{\downarrow C}(\ell_*(\nu_C(I))) = \uparrow r_{\downarrow C}(\ell_*(C \to I))$$

since the bottom of $\mathrm{Fix}(\nu_C\kappa_I)$ is $\nu_C(I) = C \to I$. Now

$$\begin{aligned}\ell_*(C \to I) &= (\nu_C)_*(\nu_C\kappa_I)_*(C \to I)\\ &= (\nu_C)_*(C \to I) \quad \text{since } (\nu_C\kappa_I)_* \text{ is inclusion}\\ &= C \wedge (C \to I)\\ &= C \wedge I.\end{aligned}$$

Consequently, $\beta(\mathrm{Fix}(\nu_C\kappa_I)) = {\uparrow} r_{\downarrow C}(C \wedge I)$. We show that $\mathrm{Fix}(\nu_C\kappa_I)$ is an F-frame. Since $\mathrm{Fix}(\kappa_I)$ is an F-frame and $\mathrm{Fix}(\nu_C\kappa_I) = \mathrm{Fix}(\kappa_I) \cap \mathrm{Fix}(\nu_C)$, it follows that $\mathrm{Fix}(\nu_C\kappa_I)$ is a cozero sublocale of $\mathrm{Fix}(\kappa_I)$, and hence, by Lemma 6.2.2, $\mathrm{Fix}(\nu_C\kappa_I)$ is an F-frame. We are done. □

Corollary 6.3.5. *If L is normal and finitely an F-frame, then ${\uparrow}a$ is finitely an F-frame, for any $a \in L$.*

Proof. We first show that the Stone-Čech compactification of ${\uparrow}a$ can be realized as a some closed quotient of βL. Consider the composite $\beta L \xrightarrow{j_L} L \xrightarrow{\kappa_a} {\uparrow}a$, and the closure $g\colon {\uparrow}r_L(a) \to {\uparrow}a$ of ${\uparrow}a$ in βL, where g maps as $\kappa_a j_L$. We claim that $g\colon {\uparrow}r_L(a) \to {\uparrow}a$ is (isomorphic to) the Stone-Čech compactification of ${\uparrow}a$. We apply Theorem 7.1.1 and Corollary 8.2.7 of [4]. So let $u \vee v = 1$ in $\mathrm{Coz}({\uparrow}a)$. Since L is normal, the map $\kappa_a\colon L \to {\uparrow}a$ is a C^*-quotient map, by [4, Theorem 8.3.3], there exist $c, d \in \mathrm{Coz}\, L$ with $c \vee d = 1$ such that $u = a \vee c$ and $v = a \vee d$. Since $j_L\colon \beta L \to L$ is a C^*-quotient map, there exist $s, t \in \mathrm{Coz}(\beta L)$ with $s \vee t = 1$ such that $j_L(s) = c$ and $j_L(t) = d$. Thus, $g(s) = u$ and $g(t) = v$, showing that $g\colon {\uparrow}r_L(a) \to {\uparrow}a$ is the Stone-Čech compactification of ${\uparrow}a$.

Now, since L is finitely an F-frame, there exist $I_1, \dots, I_n$ in βL such that $I_1 \wedge \cdots \wedge I_n = \bot$, and ${\uparrow}I_k$ is an F-frame for each $k = 1, \dots, n$. Then

$$\big(r_L(a) \vee I_1\big) \wedge \cdots \wedge \big(r_L(a) \vee I_n\big) = 0_{{\uparrow}r_L(a)}.$$

For each k, ${\uparrow}I_k$ is a normal frame, being a closed quotient of the normal frame βL. But ${\uparrow}(r_L(a) \vee I_k)$ is a closed quotient of ${\uparrow}I_k$, so ${\uparrow}(r_L(a) \vee I_k)$ is a C^*-quotient of the normal F-frame ${\uparrow}I_k$, and is therefore itself an F-frame by what we observed earlier. Thus $\beta({\uparrow}a)$ has the fin-F property, that is ${\uparrow}a$ is finitely an F-frame. □

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