Essays in Search Theory

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Declaration

I, Eeva Mauring, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

This thesis consists of three papers on search theory. Chapter 2 studies stationary cutoff-strategy equilibria of a dynamic market model where buyers sample sellers sequentially from an unknown distribution. Buyers learn about the distribution from the sampled sellers and a private "trade signal". The trade signal reveals whether a randomly chosen seller traded yesterday. The signal's precision and the market distribution of options are determined in equilibrium. Observing a trade (as opposed to no trade) is good news about the distribution. Buyers who observe a trade use a higher cutoff than buyers who observe no trade, despite buyers' learning from sampled sellers that puts a countervailing pressure on the cutoffs. The trade signal may reduce market efficiency, while an appropriate exogenous signal increases efficiency.

Chapter 3 extends the standard sequential search model by allowing the agent who inspects items sequentially (the "searcher") to differ from the agent who chooses from the set of inspected items (the "chooser"). I show for a general joint distribution of the agents' preferences that the searcher's optimal policy is a cutoff rule. The cutoff is weakly decreasing in time, i.e., exhibits the "discouragement effect". I characterise the cutoff and discuss some testable implications of the discouragement effect.

Chapter 4 relaxes the standard sequential search model's assumption that the searching agent makes no choice mistakes. In my model, once the agent stops the search process, she chooses the best inspected item with probability $1-\varepsilon$ and uniformly among the remaining inspected items with probability ε. I show that her optimal policy is a stochastic cutoff rule and that she may both experience regret and search longer than an agent who makes no mistakes.

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Chapter 1

Introduction

This thesis consists of three chapters, connected by a common theme: decision-making of agents who face informational imperfections. In the models of all three chapters agents on one side of the market (e.g., buyers or job seekers) do not know exactly which potential partner on the other side of the market (e.g., seller or employer) offers which option (e.g., price or remuneration package). A buyer samples sellers sequentially in order to discover which options the sellers offer and chooses when to stop searching for further alternatives. Such search models are a good description of the way people make many economically relevant decisions, including job and house search, finding business opportunities, and purchases of consumer goods. In particular, the models capture the idea that it takes time and resources to find an option that the person likes. In the standard sequential search model (see McCall (1970) for the seminal contribution) the agent who searches knows the distribution of options that sellers offer, makes his decisions independently of other agents, and never errs.

In the three chapters that comprise this thesis, I relax various assumptions of the standard search model. This is important for several reasons. First, making small changes to the assumptions allows us to evaluate the robustness of the standard model's results. The importance of the standard model in both applied and theoretical work, especially in macroeconomics, labour economics and industrial organisation, cannot be overstated. Determining robust predictions of search models thus helps us both to build better models and give robust policy advice. Second, relaxing the standard assumptions allows us to analyse novel questions that are relevant in search environments. For example, Chapter 2 of this thesis analyses the effect of different types of information on a market by assuming that people do not know the exact distribution of offers they face. Imperfect knowledge of the offer distribution is a reasonable assumption of at least some people's knowledge of some markets, but is ruled out in the standard model. Finally, relaxing the standard assumptions allows us to match the reality better. The standard model makes multiple assumptions about the search environment and process, which are often unrealistic. In Chapter 3 I show that some of the standard model's results no longer hold if we assume that multiple agents conduct the search process, which is a realistic assumption in many organisational settings. I now explain in more detail which assumptions I relax in the three chapters of this thesis.

In Chapter 2, *Learning from Trades*, I analyse the equilibria of a dynamic market model with pairwise meetings where buyers face an unknown distribution of options. In many real markets a buyer is unsure about the distribution of options that he faces, e.g., a house-hunter is unsure about the joint distribution of houses' attributes and prices. Often, buyers in these markets learn about the distribution from information that reflects the frequency of trading in the market, e.g., the house-hunter learns if a house is still advertised.

I characterise and analyse the efficiency of cutoff-strategy equilibria of the model when buyers learn about the unknown distribution of options from a "trade signal". Cutoff strategy means that a buyer stops the search process if he finds an option that yields him higher utility than a cutoff utility. The trade signal reveals to a buyer whether a randomly chosen seller traded yesterday.

The signal's precision and the market distribution of options are determined in equilibrium.

In equilibrium, observing a trade (as opposed to no trade) is good news about the distribution. Buyers who observe a trade use a higher cutoff than buyers who observe no trade, despite buyers' learning from options that puts a countervailing pressure on the cutoffs. In particular, the level of a cutoff depends on the informativeness of both the trade signal and observing an option equal to the cutoff. In equilibrium, observing an option just equal to the cutoff used by buyers who observe no trade is better news than observing an option equal to the cutoff used by buyers who observe a trade.

I show that the efficiency of a market where buyers learn from the trade signal can be lower than of a market where buyers do not observe this information. The extra information makes buyers, on average, too optimistic about the unknown distribution because the buyers who observe no trade are relatively too optimistic. Optimistic buyers search longer, which is inefficient in this model.

In contrast, the efficiency of a market where buyers observe the realisation of a signal with an appropriate exogenously given precision is higher than both of a market where buyers do not observe this information and of a market where buyers know the true distribution. The paper highlights that the type of information that buyers access is a crucial determinant of whether more information improves market efficiency. This is an important consideration for policy-makers and especially relevant in a computerised world as the cost of emitting information is low.

In Chapter 3, *A Two-Agent Model of Sequential Search and Choice*, I study a natural extension of the standard single-agent sequential search problem: a two-agent (or, equivalently, a multi-selves) search problem with misaligned preferences. One of the agents, the searcher, inspects items sequentially and the other, the chooser, makes the final choice among the inspected items. Their preferences differ. Many real-world situations can be viewed as two-party search problems: a political leader chooses a policy option from a pool collected by his advisers, a boss hires a new worker from the application stack compiled by the HR manager, and a person chooses from the investment options shortlisted by an earlier "self".

I characterise the optimal stopping rule of the searcher who knows that the final choice is made according to the chooser's preferences. The optimal stopping rule is a cutoff rule. The cutoff decreases weakly in time, that is, exhibits the "discouragement effect": the searcher is discouraged from searching longer and for better items as time goes on. The cutoff decreases weakly in time because of the following. At any point in time, the cutoff depends on the items that the searcher has inspected. In particular, the searcher's expected value from continuing with the search process, thus, the cutoff, decreases in the chooser's utility from the chooser-preferred inspected item, i.e., the item that would be chosen were the searcher to stop immediately. The result is intuitive: an item that yields a high utility for the chooser acts as a restriction on the searcher's problem as he has to find an item that is even better for the chooser in order to induce the chooser to change her choice. Since the chooser's utility from the chooser-preferred item weakly increases in time, the searcher's cutoff weakly decreases in time.

I discuss the testable implications of the discouragement effect and explain how the characteristics of my model differ from two single-agent search models that feature a time-varying cutoff (convex search costs or deadline). In particular, in my model the cutoff decreases endogenously over time and the finally chosen item is never an item found earlier, in contrast to the other models.

In Chapter 4, *Search with Mistakes*, I analyse a single-agent sequential search problem, where the searcher is boundedly rational in that she makes mistakes when choosing. With some probability she chooses not the item that she intended to, but any other of the inspected items. A person may choose an unintended item because she trembles, is inattentive, or cannot determine the utilities of items precisely

I show, first, that the agent's optimal cutoff is history-dependent. This is because the agent's expected value from continuing with the search process is affected by each new item that she inspects as each is chosen with positive probability. The new item's utility positively affects the continuation value: a low-utility item (weakly) lowers and a high-utility item (weakly) increases the continuation value. Thus, the cutoff can both decrease and increase in time.

Second, I show that, for some parameter values, the erring agent may search longer than an unerring agent and the erring agent's behaviour exhibits regret. The erring agent searches longer to insure herself against trembles: she accumulates a set of items most of which have an acceptable quality. Her behaviour exhibits regret if she stops when her expected value from stopping is lower than it was in the past when she chose to continue. Regret is possible after the agent receives low-utility items that lower both her optimal cutoff and stopping value, but in a way that makes stopping optimal.

Finally, I explain how the characteristics of the agent's optimal behaviour in my model differ from an unerring agent's optimal behaviour in some other extensions to the standard sequential search model that generate a history-dependent cutoff. For example, neither convex search costs nor a deadline alone generate regret. A deadline and either uncertain or costly recall together can generate regret, but for a different reason than my model.

Chapter 2

Learning from Trades

2.1 Introduction

In many markets, a buyer discovers what option a seller sells (e.g., a product's characteristics and price) only when he visits the seller's store or website. If the market distribution of options is unknown to the buyer, he learns about the distribution from the options that he observes during the visits. In many instances, a buyer learns about the unknown distribution also from other sources of information, for example, aggregate statistics, sellers' sales talk, hearsay, and observing if others trade.

In this paper I study how a signal about trading frequency affects the equilibrium of a sequential search model where the market distribution of options is unknown to buyers. A signal about trading frequency is both practically relevant because it is often present in real markets and theoretically interesting because its precision is determined in equilibrium. I characterise the model's equilibria and show that a market with the signal on trading frequency may be less efficient than a market without the signal. This is among the first papers to study a search model where buyers' actions have both informational and payoff externalities on other buyers. Buyers' actions have informational externalities because they determine the informativeness of the signals other buyers get and payoff externalities because the market distribution of options (or, payoffs) is determined in equilibrium. Payoff externalities exist in reality, are a novel feature of my model, and matter for my results.

The model that I propose describes various dynamic markets where a person who searches for a good option faces an unknown distribution of options. A person who searches for a house is often unsure about the distribution of the characteristics of houses within his budget. A job-seeker may not know the distribution of pay packages across firms. In these markets, first, one person's decisions affect the distribution of options that the others face: if a person bought a certain house or accepted a job, it is not available to the others. Second, people often learn about the unknown distribution by observing a signal about trading frequency. A job-seeker learns if a vacancy is still available and a house-hunter hears if a colleague has moved. In these markets, both the distribution of the available options and the precision of the signal are determined by the actions of all agents in equilibrium.

In my model, an equal amount of buyers and non-strategic sellers enter the market in each period. The distribution of options ("qualities", for concreteness) among the entering sellers (the "state") is fixed for all periods. The state is unknown to buyers and can be either "good" or "bad": good means more high-quality sellers. A buyer meets a seller and decides whether to accept the seller's quality. If he accepts, both he and the seller exit the market. Otherwise, the buyer continues to search for another period. A buyer dies after two periods, hence, he accepts any quality when "old". A "young" buyer decides whether to accept or continue based on the quality of the seller he met and on the realisation of a private "trade signal".

The trade signal reveals to a buyer whether a randomly drawn seller traded yesterday: if the seller traded, the buyer observes a "trade", and if the seller did not trade, the buyer observes "no trade". The signal's precision is determined in equilibrium. This is the trade-signal regime. I also analyse two benchmark regimes. Under the no-signals' regime, buyers receive no signal and under the known-state regime, buyers know the true state.

I study the model's symmetric stationary equilibria in cutoff strategies: a young buyer accepts a quality if it exceeds a cutoff quality ("cutoff" for short). I focus on inefficiency due to delay in trading. Delay is the only source of inefficiency in my model because buyers are homogeneous and receive positive utility from all qualities, the amounts of buyers and sellers are equal, matching is frictionless, and the quality distribution of entrants is fixed. Delay is positively related to equilibrium cutoff(s).

The three main results of the paper are as follows. First, I characterise the cutoff-strategy equilibrium under the trade-signal regime. The equilibrium cutoff that a buyer uses depends on the signal realisation that he observes and is higher if the realisation indicates the good rather than the bad state. A trade indicates the good state because a trade is more likely in the good state. A trade is more likely in the good state because buyers use cutoff strategies: more qualities exceed the cutoffs in the good state.

In equilibrium, buyers who observe a trade use a higher cutoff than buyers who observe no trade, despite buyers' learning from sampled options that puts a countervailing pressure on the cutoffs. In particular, the level of a cutoff depends on the informativeness of both the trade signal and a quality equal to the cutoff. In equilibrium, observing a quality equal to the lower cutoff is better news about the state than observing a quality equal to the higher cutoff.

Second, for a restricted set of parameter values I show that the trade-signal regime is less efficient than the no-signals' regime.[1] Intuitively, buyers use too high cutoffs on average under the trade-signal regime because the lower cutoff,

[1] My numerical results suggest that the result holds throughout the parameter space.

used by buyers who observe no trade, is relatively "too high". The cutoff is "too high" because observing a quality equal to the lower cutoff is better news about the state than a quality equal to the higher cutoff. Thus, buyers are on average "too optimistic" under the trade-signal regime and delay trading, which is inefficient.

Finally, I show that an exogenous-signal regime that reveals the bad state with one signal realisation and shrouds the state with the other realisation is more efficient than both the no-signals' and the known-state regimes. The efficiency-improving exogenous signal imitates the more efficient of the known-state and no-signals' regimes state by state: the market is more efficient under the known-state regime than the no-signals' regime in the bad state and vice versa in the good state.

Related literature. Wolinsky (1990) established the literature on learning in dynamic markets with pairwise meetings. In Wolinsky (1990), an uninformed agent learns about the state of the market from the bargaining position of his partner (which is equivalent to the option that a buyer observes in my model). At the time of making a purchase decision all buyers of the same age have the same belief about the unknown distribution in Wolinsky (1990), but they can have different beliefs in my model. In Section 2.6 I argue that the heterogeneity in beliefs is an important driver of my result on the trade-signal regime. In Wolinsky (1990) and several related models[2] buyers learn only from sampled options, but in my model learn from sampled options and an additional signal.

The idea that a buyer in a search market learns from others' actions is present in adverse-selection search models such as Kircher and Postlewaite (2008), Hendricks et al. (2012), and Garcia and Shelegia (2015). In these models a buyer learns about a single seller's option from others' behaviour, while in my model he learns about the aggregate state of the market.

[2]See, e.g., Serrano and Yosha (1993), Isaac (2010), and Isaac (2011).

Sequentially arriving buyers learn about the quality of a single seller from their own experience and from additional information in other dynamic adverse selection models such as Hörner and Vieille (2009), Kim (2014), Kaya and Kim (2015), and Lauermann and Wolinsky (2015). In these models, there are no payoff externalities (i.e., a buyer's actions do not affect the distribution of payoffs that later-arriving buyers face), while payoff externalities are present in my model. Hörner and Vieille (2009) and Kim (2014) show that more information may delay trade or hurt efficiency respectively. I show a similar result in a novel setting, a model with informational and payoff externalities.

Models of observational learning in financial markets are also related. Multiple papers consider the effect of exogenous signals and demonstrate that greater transparency can lead to less efficiency.[3] The closest among these is Asriyan et al. (2015). In Asriyan et al. (2015) the distribution of payoffs that a single buyer faces essentially does not depend on the actions of other buyers, whereas it does in my model.[4]

Papers on social learning where an agent's action/outcome is observed by a subset of the other agents, for example Ellison and Fudenberg (1995), Bala and Goyal (1998), Araujo and Camargo (2006), and Camargo (2014), are also related.[5] The most related among these is Camargo (2014). In Camargo (2014), like in my paper, an agent learns from the action of one other randomly chosen agent in the economy. The crucial difference between these social learning models and mine is that in the former there are no payoff externalities: an agent's actions do not affect the distribution of payoffs that other agents face.

[3]See, e.g., Daley and Green (2012), Cespa and Vives (2015), and Duffie et al. (2015).

[4]In Asriyan et al. (2015) payoff externalities are not present in equilibrium because, by assumption, a single seller always meets multiple buyers who bid so that the winning buyer always receives zero expected utility (as do losing buyers).

[5]Ellison and Fudenberg (1995) study the long-run behaviour of agents who choose between alternatives based on a simple decision rule that depends on others' choices. Araujo and Camargo (2006) study the stability of fiat money in a model where an agent learns from the experience of his "parent". Bala and Goyal (1998) study social learning in a network.

The rest of the paper is structured as follows. In Section 2.2 I introduce the model. In Section 2.3, I analyse a single buyer's decision problem who does not know the true state. I then analyse the equilibria of the model with many buyers. I introduce the equilibrium and efficiency concepts and prove the existence of an equilibrium in Section 2.4. The known-state and no-signals' regimes are analysed in Section 2.5. The trade-signal and exogenous-signal regimes are analysed in Sections 2.6 and 2.7 respectively. In each section, I first characterise the equilibrium under that regime and then compare efficiency across regimes. Section 2.8 summarises and discusses alternative modelling choices.

2.2 Model

Time is discrete and runs from $-\infty$ to ∞. The market is characterised by state $s \in \{\gamma, \beta\}$, which is fixed for all periods. The amount of agents in the market is measured at the start of a period, after entry.

Sellers. In each period a mass one of infinitesimal, non-strategic, and infinitely-lived sellers enter the market. Each seller has one unit of an indivisible good of quality q for sale for a given price, normalised to zero.[6] If the state is "bad", $s = \beta$, the quality distribution among the entering sellers, $F_\beta(q)$, is $U[0,1]$. If the state is "good", $s = \gamma$, the entry distribution, $F_\gamma(q)$, is $U[0,a]$ with $a \in (1,2]$.[7]

Buyers. In each period a mass one of infinitesimal, homogeneous, risk-neutral, and short-lived buyers enter the market. A buyer has a unit demand and his utility from quality q is $u(q) = q$. He discounts future payoffs at rate

[6] More generally, q stands for the indirect utility that a buyer obtains from the seller denoted by q.

[7] I make the assumption that $a \leq 2$ to ensure equilibrium existence as explained in more detail when I introduce the equilibrium concept, on p. 29.

$\delta \in (0, 1)$.[8] A buyer dies after two periods. In state s, the amount of buyers who entered today, called "young" buyers, is one. In state s, the amount of buyers who entered yesterday and did not exit ("old" buyers) is defined to be O_s.[9] The total amount of buyers is $1 + O_s$ in state s.

Timing. In each period, first, new entrants enter. Second, each buyer is matched to a seller. Third, buyers update their beliefs. Fourth, buyers who want to purchase do so and the agents who trade exit the market. Finally, the period ends, old buyers die, and the agents who neither traded nor died are carried over to the next period.

Matching. A buyer is randomly matched to a seller in each period and meets each seller with equal probability. Matching is frictionless: each seller is matched to exactly one buyer in a period. The quality of the seller that a young buyer meets is denoted q_1 and that an old buyer meets, q_2.

Information. Buyers do not know the true state s. A buyer assigns prior probability $\pi \in (0, 1)$ to the event $s = \gamma$ and updates his belief according to Bayes' rule. I denote the prior odds by $\omega := \frac{\pi}{1-\pi}$.

A young buyer updates his belief based on two pieces of information. He observes the quality of the seller he meets, q_1, and the realisation of a private signal with binary outcome $i \in \{G, B\}$.[10] Conditional on the true state, the signal realisations are i.i.d. across buyers and periods. The precision of the signal is $P(G|\gamma) =: p_G$ and $P(B|\beta) =: p_B$. Without loss of generality, I assume that $p_G \geq 1 - p_B$ so that outcome G is (weakly) good news, i.e., (weakly) indicative of state γ. A young buyer's belief after observing quality q_1 and signal outcome i is $\pi(q_1, i)$ and his private history is (q_1, i). His posterior odds are $\frac{\pi(q_1,i)}{1-\pi(q_1,i)}$.

[8] The assumption $\delta < 1$ is used for efficiency comparisons, as explained on p. 30.

[9] I study stationary equilibria so I omit time indicators on stock and flow variables.

[10] The quality q_1 is a signal about the state, too, if it informs the buyer about which state is more likely. However, I restrict the term "signal" to refer to only the second piece of information that a young buyer receives.

An old buyer learns from the quality of the seller he meets, q_2.[11] An old buyer's private history is (q_1, i, q_2).

Information regimes. The precision of the signal that a young buyer receives differs across the four information regimes that I consider:

1. *Known state.* Buyers know the state. Equivalently, they receive a perfectly informative signal with precision $p_G = p_B = 1$.

2. *No signals.* Buyers receive no signals. Equivalently, they receive a perfectly uninformative signal with precision $p_B = 1 - p_G$.

3. *Trade signal.* Buyers receive a "trade signal" with outcome $i \in \{T, N\}$: a buyer b observes whether a randomly drawn seller (equivalently, a buyer) traded yesterday, without observing the quality of the seller.[12] If b observes that the seller traded, I say b observes a "trade" (outcome T) and if he observes that the seller did not trade, I say b observes "no trade" (outcome N). The signal's precision is determined in equilibrium: $P(T|s) =: t_s$ is the equilibrium probability that a randomly drawn seller trades in state s. A trade is informative if the equilibrium probability of a trade differs across states. If a trade is good news (as I will prove in Section 2.6), then outcome T corresponds to G (and N to B) and the signal's precision is $p_G = t_\gamma$ and $p_B = 1 - t_\beta$.

4. *Exogenous signal.* Buyers receive a private signal with an exogenously given precision $p_G = 1$ and $p_B \in (0, 1)$.[13]

[11]I do not specify an old buyer's belief because it is irrelevant for his optimal decision, as I argue shortly.

[12]Note that it is irrelevant whether the buyer observes the trading activity of the previous or an earlier period because I focus on the model's stationary equilibria, as defined in Section 2.4.

[13]I restrict the set of values that the precision of the exogenous signal can take because the set $(p_G = 1, p_B)$ is sufficiently rich to demonstrate that the exogenous-signal regime can be more efficient than the known-state and no-signals' regimes.

Table 2.1 summarises the four regimes. When comparing the information regimes, I use the calligraphic superscripts $\mathcal{K}$, $\mathcal{N}$, $\mathcal{T}$, and $\mathcal{E}$ for the four regimes respectively. I suppress the superscripts when there is no ambiguity about which regime is considered.

Table 2.1: Summary of the information regimes

Regime	Signal outcomes		Precision	
	i	j	$P(i\|s=\gamma)$	$P(j\|s=\beta)$
Known-state ($\mathcal{K}$)	G	B	1	1
No-signals' ($\mathcal{N}$)	G	B	p_G	$1-p_G$
Trade-signal ($\mathcal{T}$)	T	N	t_γ	$1-t_\beta$
Exogenous-signal ($\mathcal{E}$)	G	B	p_G	p_B

Strategies. A young buyer's strategy specifies for each possible private history whether to accept the quality he observes, q_1, or to continue to search. Search is without recall: an old buyer cannot accept the quality he was offered when young. An old buyer's strategy is whether to accept or reject the quality q_2 and he optimally accepts any q_2. Thus, a relevant strategy only specifies the conditions under which a young buyer accepts q_1 rather than continues.

Formally, a (relevant) strategy σ is a mapping from the space of a young buyer's private histories to the space of all probability distributions over his actions "accept" and "continue", $\sigma : [0, a]\times\{G, B\} \rightarrow \Omega(\{A, C\})$, where Ω is the set of all probability distributions over accepting q_1 (A) and continuing (C). A strategy σ is a cutoff strategy if there exists a unique number $\bar{q}_i$ ("cutoff") for $i = G, B$ such that a young buyer who has observed signal outcome i accepts all $q_1 \geq \bar{q}_i$ and continues after $q_1 < \bar{q}_i$. I call this strategy $\sigma = (\bar{q}_G, \bar{q}_B)$.

I describe the equilibrium concept and provide the existence result after analysing the problem of a single buyer who faces an unknown distribution of options.

2.3 Single-buyer problem

In this section I present the solution to the optimisation problem of a single buyer who does not know the market distribution of options. I also describe how the optimal policy and the expected quality that the buyer accepts depend on his prior beliefs. Proposition 1 summarises the results.

Proposition 1. *A single buyer's optimal policy is a cutoff rule. The optimal cutoff* $\bar{\bar{q}}$ *is*

$$\bar{\bar{q}} = \frac{\delta a(1+\omega)}{2(a+\omega)}.$$

The cutoff increases in the buyer's prior odds, ω*.*

The expected quality that the buyer accepts is

$$\frac{1}{2a(1+\omega)}[a(a\omega+1)+\omega\bar{\bar{q}}(a\delta-\bar{\bar{q}})+a\bar{\bar{q}}(\delta-\bar{\bar{q}})],$$

which also increases in ω*.*

Proof. In Appendix A. □

Consider the market described in Section 2.2 with the difference that in total only one buyer enters the market. The buyer does not know whether the true state s is good or bad. Since in each period the distribution of entering sellers is F_s and there is no other buyer to buy from these sellers, the market distribution of qualities that the single buyer faces is identical to the entry distribution. I derive the buyer's optimal policy.

Consider the single buyer when old. An old single buyer accepts all qualities since he has no further opportunities to search. Thus, I only consider the buyer's optimal decision when young in what follows. A young buyer has to decide whether to accept the quality of the seller he meets, q_1, or to continue to search for another period. The young buyer's optimal policy is to accept all qualities that exceed his continuation value, i.e., his discounted expected utility from continuing with the search process for another period and then accepting any offer. Thus, the optimal policy is a cutoff rule and the cutoff equal to the continuation value.

The continuation value of a young buyer who meets a seller with quality q_1 is

$$\delta[\pi(q_1)\mathbb{E}_{F_\gamma}(q) + (1 - \pi(q_1))\mathbb{E}_{F_\beta}(q)].$$

The first term in the squared brackets accounts for the possibility that the true state is good, in which case the buyer expects to receive the mean quality $\mathbb{E}_{F_\gamma}(q)$ when old. The buyer's belief depends on q_1 because all qualities are offered with different probabilities under the two possible distributions: $f_\gamma(q)$ and $f_\beta(q)$ differ for all $q \in [0, a]$. The second term accounts for the possibility that the true state is bad and is interpreted analogously. The buyer discounts the future at rate δ. The only unknown quantity in the above expression is the buyer's posterior belief $\pi(q_1)$.

The buyer's posterior odds after any $q_1 \leq 1$ are

$$\frac{\pi(q_1)}{1 - \pi(q_1)} = \omega \frac{f_\gamma(q_1)}{f_\beta(q_1)} = \frac{\omega}{a}, \tag{2.1}$$

because random matching means that the probability of observing any quality q_1 in state s is equal to its density. Note that the posterior belief is the same for all $q_1 \leq 1$. The buyer becomes more pessimistic about the state after any $q_1 \leq 1$ because these qualities are more likely in the bad rather than the good state. The buyer knows that the state is good after any $q_1 > 1$, i.e., $\pi(q_1) = 1$ for all $q_1 > 1$, because sellers with such qualities do not enter the market if the state is bad.

The buyer is indifferent between continuing to search for another period and accepting quality $q_1 = \bar{\bar{q}}$ that solves

$$\bar{\bar{q}} = \delta[\pi(\bar{\bar{q}})\mathbb{E}_{F_\gamma}(q) + (1 - \pi(\bar{\bar{q}}))\mathbb{E}_{F_\beta}(q)].$$

The RHS of this equation is constant for all $\bar{\bar{q}} \leq 1$ and the LHS increases in $\bar{\bar{q}}$ so the equation has at most one solution. The equation has exactly one solution because after any $q_1 \leq 1$ the buyer's continuation value is strictly below δ even if he knows the state to be good: the mean quality in the good state is at most one. Thus, also the optimal cutoff $\bar{\bar{q}}$ is below δ.

The optimal cutoff $\bar{\bar{q}}$ is solved for explicitly by using equation (2.1) for beliefs and is presented in Proposition 1. It is straightforward to check that the optimal cutoff increases in the buyer's prior odds that the state is good, ω (or, equivalently, in the prior belief π). A more optimistic buyer is willing to forgo some medium qualities that a less optimistic buyer accepts because the former is more hopeful that he draws a high quality when old.

The expected quality that the buyer accepts is

$$\sum_s P(s)[(1 - F_s(\bar{\bar{q}}))\mathbb{E}_{F_s}(q|q > \bar{\bar{q}}) + F_s(\bar{\bar{q}})\delta\mathbb{E}_{F_s}(q)], \tag{2.2}$$

The first term in the squared brackets accounts for the possibility that in state s the young buyer's offer q_1 exceeds $\bar{\bar{q}}$, in which case the buyer accepts the offer. The second term accounts for the possibility that q_1 falls below $\bar{\bar{q}}$, in which case the buyer accepts any offer when old. Equation (2.2) simplifies to the expression in Proposition 1 after some manipulation. The expected accepted quality increases in the buyer's prior odds that the state is good, ω. To see that, consider an increase in ω, i.e., suppose that the buyer becomes more optimistic about the state. If the buyer did not change his cutoff in response, the effect on the expected accepted quality would be positive because the buyer expects to get the higher, good-state, mean quality with a larger probability. The buyer, however, optimally also increases the cutoff he uses, which has an ambiguous effect on the expected accepted quality. But the total effect on the expected accepted quality is positive because the buyer could also leave the cutoff unchanged if it were optimal.

I turn to the equilibrium analysis of the model in the subsequent sections. The most important change that many buyers introduce is that the market distribution of qualities that a buyer faces is different from the entry distribution, because the market distribution is partly determined by the equilibrium behaviour of all buyers.

2.4 Equilibrium concept, existence, and efficiency

In this section, I introduce the equilibrium and efficiency concepts and prove the existence of an equilibrium in cutoff strategies.

2.4.1 Equilibrium concept

I study the model's symmetric stationary equilibria in cutoff strategies. I focus on cutoff strategies because, first, the unique equilibrium under the known-state regime (for any a) is in cutoff strategies. Second, an equilibrium in cutoff strategies exists for all parameter values under the trade-signal regime.[14] Third, cutoff strategies are simple, hence, a plausible description of reality.

A strategy profile $\sigma^* = (\bar{q}_G, \bar{q}_B)$ is an equilibrium if for all (q_1, i), σ^* is

[14]The assumption $a \leq 2$ is sufficient to guarantee that an equilibrium in cutoff strategies exists under the no-signals' regime (and is also sufficient for the trade-signal regime). It is known already from Rothschild (1974) that the optimal stopping rule of a buyer who faces an unknown distribution of options may not be a cutoff rule and depends on the distributions the buyer considers possible.

(a) consistent: $H_s(q) =: H_s$, $s = \gamma, \beta$, is the equilibrium distribution of qualities in state s induced by σ^*;

(b) optimal: a buyer prefers accepting q_1 to continuing for all $q_1 \geq \bar{q}_i$ and vice versa for all $q_1 < \bar{q}_i$, i.e., for $i = B, G$,

$$q_1 \geq \delta \sum_{s=\gamma,\beta} P(s|q_1, i)\mathbb{E}_{H_s}(q_2) \quad \text{for all } q_1 \geq \bar{q}_i,$$

$$q_1 < \delta \sum_{s=\gamma,\beta} P(s|q_1, i)\mathbb{E}_{H_s}(q_2) \quad \text{for all } q_1 < \bar{q}_i;$$

(c) uses Bayes updating: $\frac{\pi(q_1,i)}{1-\pi(q_1,i)} = \omega \frac{P(q_1|\gamma)}{P(q_1|\beta)} \frac{P(i|\gamma)}{P(i|\beta)}$, where $P(x|s)$ is the equilibrium probability of observing event x in state s;

(d) stationary: the equilibrium strategy σ^* and distribution H_s, $s = \gamma, \beta$, are independent of the time period.

To distinguish the equilibrium cutoffs under the four regimes, I use different denotation for the cutoffs: $\bar{q}_\beta$ and $\bar{q}_\gamma$ under the known-state, $\bar{q}$ under the no-signals', $\bar{q}_N$ and $\bar{q}_T$ under the trade-signal, and $\bar{q}_B$ and $\bar{q}_G$ under the exogenous-signal regime.

2.4.2 Efficiency

Delay in trading is the only possible source of inefficiency in the model because buyers receive positive utility from any quality, the quality distribution of entrants is fixed, matching is frictionless, and the amounts of buyers and sellers equal.[15] If buyers discount future payoffs, i.e., $\delta < 1$, an equilibrium is the less efficient the longer the delay.[16] The (expected) delay as measured across the

[15] Welfare is equal to a young buyer's expected utility of participating in the market, which is a standard efficiency measure in this literature (see Lauermann (2012)).

[16] If buyers did not discount, i.e., $\delta = 1$, delay would not matter.

possible states is

$$D := \pi O_\gamma + (1 - \pi) O_\beta, \tag{2.3}$$

as a buyer delays trade only if he becomes old.

2.4.3 Equilibrium existence

I prove that an equilibrium in cutoff strategies exists under the four regimes.[17]

Proposition 2. *An equilibrium in cutoff strategies exists under the known-state, no-signals', trade-signal, and exogenous-signal regimes. Under the known-state regime, the unique equilibrium is in cutoff strategies.*

Proof. In Appendix A. □

The idea of the proof is standard. I first suppose that all buyers but one use a cutoff strategy and show that the best response of the single buyer is to use a cutoff strategy, too. Then I show that the equation that an equilibrium cutoff must satisfy has a solution. Finally, I prove that under the known-state regime the unique equilibrium is in cutoff strategies.

I have to prove existence and cannot rely on results present in the literature because the distribution of options is determined in equilibrium in my model. I explain the complication in more detail in Section 2.5.

2.5 Known-state and no-signals' regimes

In this section I first present the equilibrium cutoffs and distribution of qualities under both regimes (Proposition 3). I argue informally why a cutoff-strategy

[17] I cannot show that the cutoff-strategy equilibrium is unique under the trade-signal and exogenous-signal regimes. However, as all the results of the paper (most importantly, comparisons across regimes) hold for all of the equilibria, should there be many, I use the singular "equilibrium" throughout the paper.

equilibrium exists by describing a young buyer's decision. I then characterise the equilibria. Finally, I compare the delay under the two regimes (Proposition 4).

Under the known-state regime, buyers know the true state s. Under the no-signals' regime, buyers do not know the state and a young buyer learns about the state only from the offer of the seller he meets, q_1. I present the equilibrium cutoffs and distribution of qualities in Proposition 3.

Proposition 3. *1. Under the known-state regime, the equilibrium cutoffs $\bar{q}_\beta$ and $\bar{q}_\gamma$ are*

$$\bar{q}_\beta = \delta \frac{4-\delta-\sqrt{\delta(8-3\delta)}}{2(2-\delta)^2}, \qquad \textit{and} \qquad \bar{q}_\gamma = a\bar{q}_\beta.$$

The equilibrium density of qualities in state $s = \gamma, \beta$ is

$$h_s^{\mathcal{K}}(q) = \begin{cases} \frac{f_s(q)}{\sqrt{F_s(\bar{q}_s)}} & \textit{if } q < \bar{q}_s, \\ \frac{f_s(q)}{1+\sqrt{F_s(\bar{q}_s)}} & \textit{if } q \geq \bar{q}_s. \end{cases}$$

2. Under the no-signals' regime, the equilibrium cutoff $\bar{q}$ solves

$$\bar{q} = \frac{\delta}{2-\delta} \frac{\omega(1+\sqrt{\bar{q}})(a-\sqrt{a\bar{q}}) + (a+\sqrt{a\bar{q}})(1-\sqrt{\bar{q}})}{\omega(1+\sqrt{\bar{q}}) + a + \sqrt{a\bar{q}}}.$$

The equilibrium density of qualities in state $s = \gamma, \beta$ is

$$h_s^{\mathcal{N}}(q) = \begin{cases} \frac{f_s(q)}{\sqrt{F_s(\bar{q})}} & \textit{if } q < \bar{q}, \\ \frac{f_s(q)}{1+\sqrt{F_s(\bar{q})}} & \textit{if } q \geq \bar{q}. \end{cases}$$

Proof. In Appendix A. □

In the proof, I derive the equations that the cutoffs solve and the equilibrium distribution of qualities by using the equilibrium description provided in the proof of Proposition 2. I now describe a young buyer's decision and explain why I need to prove existence of an equilibrium in my model.

2.5.1 A young buyer's decision

First consider the known-state regime. A young buyer knows that the true state is s and must decide whether to accept the quality he observes, q_1, or to continue to search for another period. An old buyer accepts any quality. Hence, a young buyer accepts q_1 only if it exceeds his continuation value: the discounted mean quality in state s, where the expectation is taken under the equilibrium distribution h_s. For a fixed behaviour of all other buyers, the mean quality in a given state is constant. Hence, a buyer's optimal rule is a cutoff rule and the cutoff $\bar{q}_s$ equals his continuation value in state s. The cutoff is a solution to a fixed-point equation because the equilibrium distribution of options h_s depends on the cutoff.

Now consider the no-signals' regime. A young buyer b's choice is the same as under the known-state regime, except that b's continuation value is a discounted average of the mean qualities in the good and bad states because he does not know the true state s if $q_1 \leq 1$.[18] Buyer b optimally accepts q_1 iff it exceeds his continuation value. I show that b's continuation value is discontinuous in q_1 if other buyers use a cutoff strategy so that it is not evident that b's best response is a cutoff rule.

Suppose that all young buyers but b use a cutoff $\hat{q}$. In the continuation value, the weight that b puts on the good-state mean quality is his posterior

[18] A young buyer who observes quality $q_1 \geq 1$ optimally accepts q_1 under all information regimes as $q_1 \geq 1$ exceeds the discounted expected value of quality q_2, i.e., the discounted mean quality. In equilibrium, the mean quality is less than one because a cutoff strategy means that high qualities are accepted more often than low, so that the mean quality under the equilibrium distribution is smaller than under the entry distribution, $\mathbb{E}_{H_s}(q) < \mathbb{E}_{F_s}(q)$, and $\mathbb{E}_{F_s}(q) \leq 1$ for $s = \gamma, \beta$.

belief that the true state is good. His posterior differs from the prior as the probability of observing quality $q_1 \leq 1$ differs in the two possible states. If all other buyers use cutoff $\hat{q}$, then observing any quality below $\hat{q}$ is equally informative about the state (because if all buyers but one use a single cutoff, the market distribution in state $s = \gamma, \beta$ is piece-wise constant with a jump at the cutoff; see Proposition 3). Likewise, observing any quality below one and above the cutoff $\hat{q}$ is equally informative about the state. Thus, as a function of q_1, a young buyer's belief about the true state (and hence his continuation value) has a discontinuity at $\hat{q}$ and it is no longer evident that the fixed-point equation that determines the equilibrium cutoff has a solution. In Appendix A I show that the equation has a solution.

2.5.2 Equilibrium characterisation

I describe the equilibrium distribution of qualities and establish the relative size of the cutoffs under the two regimes. The equilibrium distribution of qualities H_s differs from the entry distribution, F_s. In particular, since high qualities are accepted by all buyers while low qualities are accepted by only old buyers, high qualities are under-represented in the equilibrium distribution as compared to the entry distribution. The difference between the mean quality under the equilibrium and entry distributions captures the payoff externality that other buyers impose on a single buyer.

The ratio of equilibrium densities, $\frac{h_\gamma(q_1)}{h_\beta(q_1)}$, captures the change in the buyer's belief that the state is good after observing a certain quality q_1. Since this ratio varies with q_1, young buyers have heterogeneous beliefs about the state under the no-signals' regime. An analogous heterogeneity in beliefs drives some of the results under the trade-signal regime, in Section 2.6. The ratio of equilibrium densities is not monotone increasing in q_1, despite the ratio of entry densities being monotone.

I compare the sizes of the cutoffs under the two regimes in Lemma 1.

Lemma 1. *The buyers' equilibrium cutoff under the no-signals' regime is between the cutoffs used under the known-state regime:* $\bar{q} \in (\bar{q}_\beta, \bar{q}_\gamma)$.

Proof. In Appendix A. □

The result is intuitive. Suppose that buyers know the state at first. The optimal cutoff in state s equals the discounted mean quality in state s. Now consider making the state unknown (as under the no-signals' regime). Then the optimal cutoff is equal to an average between the discounted mean qualities in the good and bad states. Thus, if we ignore the effect that buyers' actions have on the equilibrium distributions, a buyer's optimal cutoff under the no-signals' regime is between the cutoffs of the known-state regime. Buyers' actions affect the equilibrium distributions via the equilibrium cutoff, but not enough to overturn the intuitive effect of making the state unknown.

I show in Appendix A that in a given state, the increasing function that maps the cutoff into the probability of becoming old is the same under the known-state and no-signals' regimes.[19] Hence, Lemma 1 implies Corollary 1, which I use to provide intuition for the efficiency result in Section 2.7.

Corollary 1. *In the bad state the expected delay is longer under the no-signals' than the known-state regime, and vice versa in the good state.*

2.5.3 Efficiency comparison

Proposition 4 compares efficiency under the no-signals' and known-state regimes.

[19] See equations (A.3) and (A.12), on p. 114 and p. 121 respectively.

Proposition 4. *The expected delay under the known-state regime is*

$$D^{\mathcal{K}} = \sqrt{\bar{q}_\beta},$$

and under the no-signals' regime

$$D^{\mathcal{N}} = \pi\sqrt{\frac{\bar{q}}{a}} + (1-\pi)\sqrt{\bar{q}}.$$

The known-state regime is more efficient than the no-signals' regime (i.e., $D^{\mathcal{K}} < D^{\mathcal{N}}$) if $\delta > \frac{2}{3}$ and vice versa if $\delta < \frac{2}{3}$.

Proof. In Appendix A. □

In the proof, I show that the expected delays under the no-signals' and known-state regimes are equal if $a = 1$ and that $D^{\mathcal{N}}$ increases in a if $\delta > \frac{2}{3}$ and decreases if $\delta < \frac{2}{3}$, while $D^{\mathcal{K}}$ is independent of a. Note that even the known-state regime is not fully efficient because a buyer does not internalise the externality that he imposes on others. A low quality that a young buyer b rejects remains in the market until it is accepted by some (old) buyer. The rejection by b delays the realisation of gains from trade, creating an inefficiency. In Lauermann (2012), a similar reason causes the symmetric information benchmark to be inefficient (Proposition 2). The comparative efficiency result in Proposition 4 is driven by the specific assumption that the entry distributions of qualities are uniform and does not extend to more general entry distributions.

2.6 Trade-signal regime

In this section, I first present the buyers' equilibrium cutoffs and distribution of qualities under the trade-signal regime (Proposition 5). I then characterise the equilibrium in steps: I show that a trade is good news (Lemma 3), that a low quality is better news than a higher quality (Lemma 4), and the relative size of the equilibrium cutoffs (Lemma 5).[20] The characterisation is summarised in Proposition 6. Finally, I show for certain parameter values that this regime is less efficient than the no-signals' regime (Proposition 7).[21]

Under the trade-signal regime, a young buyer learns about the true state s from the quality he observes, q_1, and a trade signal. The trade signal reveals to a buyer b whether a randomly chosen seller traded in the previous period, without revealing the seller's quality. I say that b observes a "trade" ($i = T$) if the seller traded and observes "no trade" ($i = N$) if the seller did not trade. Conditional on the true state s, the signal's realisations are i.i.d. across buyers and periods. The precision of the signal is determined in equilibrium: the per-period probability of observing a trade in state s, t_s, is the equilibrium probability that a randomly selected seller (equivalently, buyer) trades in state s. A buyer who observes a trade uses the cutoff $\bar{q}_T$ and a buyer who observes no trade, the cutoff $\bar{q}_N$. The trade signal is a natural way to model a type of information that is often observed in markets: information about the trading frequency of other agents.

Proposition 5. *Under the trade-signal regime, the equilibrium cutoffs* $\bar{q}_T$ *and*

[20]Lemmas 2, 3, 4, and 5 prove that the characterisation is consistent and part one of the proof of Proposition 5 that it is unique.

[21]My numerical results across the grid of (δ, a, π) suggest that the result holds for all parameter values.

$\bar{q}_N$ solve the following system of equations:

$$\begin{cases} \bar{q}_T = \dfrac{\delta}{2} \dfrac{\omega(1+O_\beta)^2 \frac{1}{aO_\gamma}\left[a^2 - \frac{a^2-\bar{q}_T^2}{1+O_\gamma} - \frac{\bar{q}_T^2-\bar{q}_N^2}{2+O_\gamma}\right] + a(1+O_\gamma)^2 \frac{1}{O_\beta}\left[1 - \frac{1-\bar{q}_T^2}{1+O_\beta} - \frac{\bar{q}_T^2-\bar{q}_N^2}{2+O_\beta}\right]}{\omega(1+O_\beta)^2 + a(1+O_\gamma)^2}, \\ \bar{q}_N = \dfrac{\delta}{2} \dfrac{\omega(2+O_\beta) \frac{1}{aO_\gamma}\left[a^2 - \frac{a^2-\bar{q}_T^2}{1+O_\gamma} - \frac{\bar{q}_T^2-\bar{q}_N^2}{2+O_\gamma}\right] + a(2+O_\gamma) \frac{1}{O_\beta}\left[1 - \frac{1-\bar{q}_T^2}{1+O_\beta} - \frac{\bar{q}_T^2-\bar{q}_N^2}{2+O_\beta}\right]}{\omega(2+O_\beta) + a(2+O_\gamma)}, \\ aO_\gamma^2 = \dfrac{\bar{q}_T + (1+O_\gamma)\bar{q}_N}{2+O_\gamma}, \\ O_\beta^2 = \dfrac{\bar{q}_T + (1+O_\beta)\bar{q}_N}{2+O_\beta}. \end{cases}$$

The equilibrium density of qualities in state $s = \gamma, \beta$ is

$$h_s(q) = \begin{cases} \frac{f_s(q)}{O_s} & \text{if } q < \bar{q}_N, \\ \frac{f_s(q)(1+O_s)}{O_s(2+O_s)} & \text{if } q \in [\bar{q}_N, \bar{q}_T), \\ \frac{f_s(q)}{1+O_s} & \text{if } q \geq \bar{q}_T, \end{cases}$$

where O_γ and O_β are defined by the above system of equations.

Proof. In Appendix A. □

In the proof of Proposition 2 I showed that an equilibrium with characteristics as summarised in Proposition 6 exists. The first part of the proof of Proposition 5 (together with Lemma 4) shows that this characterisation is unique. The second part is similar to the proof of Proposition 3.

Since trades and no trades take place in both states if cutoff strategies are used, both cutoffs $\bar{q}_T$ and $\bar{q}_N$ are used under the trade-signal regime in a given state, while a single cutoff is used under the benchmark regimes. Accordingly, under the trade-signal regime, the equilibrium density of qualities, h_s, is piecewise constant and has two downward jumps, at $\bar{q}_N$ and $\bar{q}_T$.

2.6.1 Trade is good news

A trade is good news if the probability of observing a trade is higher in the good state than in the bad. Lemma 3 establishes that a trade is good news. I first show that the amount of old buyers is smaller in the good state than in the bad.

Lemma 2. *The amount of old buyers is smaller in the good state than in the bad state, i.e.,* $O_\gamma < O_\beta$*, in an equilibrium where* $\bar{q}_T > \bar{q}_N$ *and* $t_\gamma > t_\beta$*.*

Proof. In Appendix A. □

Intuitively, the amount of old buyers is positively related to the entry amount of sellers with qualities below $\bar{q}_N$ and below $\bar{q}_T$ because buyers use cutoff strategies. Fewer low-quality sellers enter the market in the good state so a young buyer is less likely to observe a below-cutoff quality and become old in the good state. In the context of a real-estate market, Lemma 2 suggests that fewer houses remain unsold from one month to the next if many high-quality houses become available each month.

Lemma 2 implies that a trade is good news.

Lemma 3. *The probability of observing a trade is higher in the good state than in the bad, i.e.,* $t_\gamma > t_\beta$*, in an equilibrium where* $O_\gamma < O_\beta$*.*

Proof. In Appendix A. □

Intuitively, trades happen more frequently in the good state than in the bad because buyers use cutoff strategies and more sellers whose qualities exceed the cutoffs enter in the good state. In a real-estate market context, learning that a colleague has moved is good news about the qualities of houses on the market because a person knows that his colleague has certain standards for the house that he wants to live in (i.e., uses a cutoff strategy).

2.6.2 Low quality is better news than high

If a young buyer learned only from the trade signal, then a trade being good news would imply that a buyer who observes a trade uses a higher cutoff than a buyer who observes no trade, i.e., that $\bar{q}_T > \bar{q}_N$. But a young buyer learns from both the trade signal and the quality of the seller he meets, q_1. In fact, by the definition of a cutoff, a young buyer who has observed quality $q_1 = \bar{q}_i$ and signal realisation i must be just indifferent between accepting the offer $q_1 = \bar{q}_i$ and continuing. The buyer's information ($q_1 = \bar{q}_i$ and i) affects his continuation value via his belief about the state. Thus, the good-news content of the quality $q_1 = \bar{q}_i$ matters for the level of the cutoff $\bar{q}_i$, $i = N, T$.

A trade being good news (Lemma 3) would translate directly into a higher $\bar{q}_T$ than $\bar{q}_N$ if $q_1 = \bar{q}_T$ was (weakly) better news than $q_1 = \bar{q}_N$. But the next Lemma shows the opposite: $q_1 = \bar{q}_T$ is strictly worse news than $q_1 = \bar{q}_N$ because the equilibrium odds of observing $q_1 = \bar{q}_T$ are lower than of observing $q_1 = \bar{q}_N$. The equilibrium odds of observing a quality q_1 are equal to the equilibrium densities' ratio of q_1 because of random matching.

Lemma 4. *In any equilibrium with a binary signal that has outcomes B and G and a precision such that $P(B|\beta) > P(B|\gamma)$, if the equilibrium cutoffs satisfy $\bar{q}_B < \bar{q}_G$ and $O_\gamma < O_\beta$, the equilibrium densities' ratio of $q = \bar{q}_B$ is higher than of $q = \bar{q}_G$, i.e., $\frac{h_\gamma(\bar{q}_B)}{h_\beta(\bar{q}_B)} > \frac{h_\gamma(\bar{q}_G)}{h_\beta(\bar{q}_G)}$.*

Proof. In Appendix A. □

Lemma 4 shows that the equilibrium densities' ratio is not monotone increasing in quality (despite the ratio of entry densities being monotone increasing). The intuition behind the result in Lemma 4 is that a buyer knows that other young buyers reject the offer $q_1 = \bar{q}_N$ more often in the good state (because good news, i.e., trades, are observed more often in the good state), while no buyer rejects offer $q_1 = \bar{q}_T$. That is, a seller of quality $q = \bar{q}_N$ is more likely to stay on the market in the good state than in the bad, while a seller of quality $q = \bar{q}_T$ stays on the market in neither of the states. Thus, observing $q_1 = \bar{q}_N$ is relatively better news than observing $q_1 = \bar{q}_T$. In Wolinsky (1990) such an asymmetry in the inference from options does not happen: all buyers of the same age must have observed the same sequence of options in the past.

2.6.3 Relative size of equilibrium cutoffs

Despite $q_1 = \bar{q}_N$ being better news than $q_1 = \bar{q}_T$, observing $q_1 = \bar{q}_N$ and no trade together is worse news than observing $q_1 = \bar{q}_T$ and a trade together.

Lemma 5. *The equilibrium cutoff used by a buyer who has observed a trade, $\bar{q}_T$, is higher than the cutoff used by a buyer who has observed no trade, $\bar{q}_N$, in an equilibrium where $O_\gamma < O_\beta$, $t_\gamma > t_\beta$, and $\frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)} > \frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)}$.*

Proof. In Appendix A. □

Intuitively, the effect of the trade signal outweighs the effect of the quality draw q_1 because q_1 is less informative than the trade signal. By definition, q_1 is a single quality draw. In contrast, the trade signal aggregates the experience of many buyers and their respective quality draws. Since most quality draws are informative, the aggregate is more informative than the single draw. As a result, the equilibrium cutoff used after a trade is higher than the cutoff used after no trade.

The equilibrium characterisation result is summarised in Proposition 6.

Proposition 6. *Under the trade-signal regime, in equilibrium*

(i) the cutoff used after a trade is higher than the cutoff used after no trade, i.e., $\bar{q}_T > \bar{q}_N$,

(ii) the lower quality $q = \bar{q}_N$ is better news than the higher quality $q = \bar{q}_T$, i.e., $\frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)} > \frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)}$, and

(iii) a trade is good news, i.e., $t_\gamma > t_\beta$.

Proof. Lemmas 2, 3, 4, and 5 prove that the characterisation is consistent. Its uniqueness is proven in the proof of Proposition 5.[22] □

2.6.4 Trade signal can reduce efficiency

Now I show that (for certain parameter values) the trade-signal regime is less efficient than the no-signals' regime.[23]

Proposition 7. *The expected delay under the trade-signal regime is*

$$D^{\mathcal{T}} = \pi O_\gamma + (1-\pi)O_\beta,$$

where O_γ and O_β are defined by the system of equations in Proposition 5.

A sufficient condition for the no-signals' regime to be more efficient than the trade-signal regime (i.e., $D^{\mathcal{N}} < D^{\mathcal{T}}$) is that $\omega = 1$ and $a \le \bar{a}(\delta)$, where

$$\bar{a}(\delta) := 2\delta \frac{4 - \delta - \sqrt{\delta(8-3\delta)}}{(2-\delta)^2}.$$

Proof. In Appendix A. □

In the proof I show by a series of approximations that a lower bound on the expected delay under the trade-signal regime is greater than the expected delay

[22]The only proof that relies on the assumption that f_γ and f_β are uniform is Lemma 4. In the proof of Proposition 5 I show that the only alternative characterisation of an equilibrium in a model where $\frac{f_\gamma(q)}{f_\beta(q)}$ increases in q is that a trade is good news and the cutoff used after a trade is higher than the cutoff used after no trade, but a lower quality $q = \bar{q}_N$ is worse news than a higher quality $q = \bar{q}_T$.

[23]My numerical results suggest that the result holds throughout the parameter space.

under the no-signals' regime. Note that a necessary condition for $a \leq \bar{a}(\delta)$ to hold is that $\delta > \frac{2}{3}$.

Intuitively, under the trade-signal regime buyers use "too high" cutoffs, in particular $\bar{q}_N$, on average as compared to the cutoff used under the no-signals' regime. As explained above, in equilibrium, observing the quality $q_1 = \bar{q}_N$ makes a buyer more optimistic than the quality $q_1 = \bar{q}_T$. Accordingly, under the trade-signal regime the equilibrium cutoff used after no trade is relatively "too high" as compared to the cutoff used after a trade, which leads to a longer delay under the trade-signal than the no-signals' regime. Proposition 7 suggests that trades could happen faster in a real estate market if people would not share their experiences on the market with each other.

Allowing buyers to observe options from a continuum, or, more precisely, from a set with more than two elements, is important for the efficiency result. Recall that in Wolinsky (1990), at the time of making a purchase decision, all buyers of the same age hold the same beliefs about the state. I show in Appendix A (Claim 1, p. 144) that in a version of my model that corresponds most closely to Wolinsky (1990), i.e., with two possible qualities, where buyers of the same age have the same beliefs, the trade signal can improve efficiency.

Finally, I comment on the importance of payoff externalities for the efficiency result. Payoff externalities arise because the market distribution of options is determined in equilibrium. The equilibrium distribution of options determines the informativeness and good-news content of a certain option. The asymmetric good-news content of different options leads to "too optimistic" buyers on average under the trade-signal regime. In fact, if payoff externalities are ignored (i.e., if we assume that the distribution of options that buyers face is identical to the entry distribution) then the trade signal does not change market efficiency as compared to the no-signals' regime (the proof is in Appendix A, Claim 2, p. 148).

2.7 An exogenous signal that improves efficiency

The aim of this section is to show that the trade signal is special: there exist precisions of a private signal that improve upon market efficiency as compared to both the no-signals' and known-state regimes (Proposition 9). Before presenting the result, I derive the buyers' equilibrium cutoffs and distribution of qualities under the exogenous-signal regime (Proposition 8).

Under the exogenous-signal regime, young buyers learn about the true state s from q_1 and a private signal with realisation $i \in \{G, B\}$. The precision of the private signal is exogenously given: $p_G := P(G|\gamma) = 1$ and $p_B := P(B|\beta) \in (0, 1)$. Realisation B reveals the bad state and realisation G is good news as $p_G > 1 - p_B$.

Proposition 8. *Under the exogenous-signal regime with* $p_G = 1$ *and* $p_B \in (0, 1)$*, the equilibrium cutoffs* $\bar{q}_G$ *and* $\bar{q}_B$ *solve the following system of equations:*

$$\begin{cases} \bar{q}_G = \dfrac{\frac{\delta}{2}\omega(1+O_\beta)(a-\sqrt{a\bar{q}_G}+\bar{q}_G)+(a+\sqrt{a\bar{q}_G})(1-p_B)\bar{q}_B}{\omega(1+O_\beta)+(a+\sqrt{a\bar{q}_G})(1-p_B)}, \\ \bar{q}_B = \dfrac{\delta}{2O_\beta}\left[1-\dfrac{1-\bar{q}_G^2}{1+O_\beta}-\dfrac{p_B(\bar{q}_G^2-\bar{q}_B^2)}{p_B+O_\beta}\right], \\ O_\beta^2 = \dfrac{(1-p_B)O_\beta\bar{q}_G+p_B(1+O_\beta)\bar{q}_B}{p_B+O_\beta}. \end{cases}$$

The equilibrium density of qualities in state $s = \gamma, \beta$ *is*

$$h_s(q) = \begin{cases} f_s(q)O_s^{-1} & \text{if } q < \bar{q}_B, \\ f_s(q)(O_s+P(B|s))^{-1} & \text{if } q \in [\bar{q}_B, \bar{q}_G), \\ f_s(q)(1+O_s)^{-1} & \text{if } q \geq \bar{q}_G. \end{cases}$$

where O_β *is defined by the above system of equations and* $O_\gamma = \sqrt{\frac{\bar{q}_G}{a}}$.

Proof. In Appendix A. □

The proof of Proposition 8 is similar to the proof of Proposition 5.

The equilibrium densities have similar features under the exogenous-signal and trade-signal regimes: both feature downward jumps at the two relevant cutoffs in each state. Under the exogenous-signal regime, the cutoff used after signal realisation G is higher than the cutoff used after realisation B, i.e., $\bar{q}_G > \bar{q}_B$. A buyer is maximally pessimistic after B because he knows that the state is bad, while G leaves a buyer uncertain about the state.

I now prove that precisions for the exogenous signal exist such that the exogenous-signal regime is more efficient than both the known-state and the no-signals' regime.

Proposition 9. *The expected delay under the exogenous-signal regime is*

$$D^{\mathcal{E}} = \pi O_\gamma + (1 - \pi)O_\beta,$$

where $O_\gamma = \sqrt{\frac{\bar{q}_G}{a}}$ *and* O_β *is defined by the system of equations in Proposition 8.*

Precisions of the exogenous signal exist such that the exogenous-signal regime is more efficient than the more efficient of the no-signals' and known-state regimes.

Proof. In Appendix A. □

The proof shows that if p_B is close to one, the exogenous-signal regime is more efficient than the known-state regime for all δ. I show separately that if $\delta < \frac{2}{3}$ and p_B is close to zero, the exogenous-signal regime is more efficient than the no-signals' regime (recall Proposition 4: the no-signals' regime is more efficient than the known-state regime if $\delta < \frac{2}{3}$).

The intuition behind the result in Proposition 9 is the following. Recall Corollary 1: in the bad state, delay is shorter under the known-state regime than under the no-signals' regime and vice versa in the good state. The efficiency-improving exogenous-signal regime tries to mimic the more efficient of the known-state and no-signals' regimes state by state. The exogenous signal reveals the state in which buyers trade quickly if they know the state (i.e., the bad state) and shrouds the state where buyers trade slowly if they know the state (i.e., the good state).

The trade signal cannot be more efficient than the more efficient of the known-state and no-signals' regimes because it cannot mimic the more efficient of the two extreme information regimes state by state. In particular, the trade signal can never reveal one state. Any optimal strategy prescribes that a young buyer rejects some qualities (e.g., $q_1 = 0$) and accepts others (e.g., $q_1 = 1$) so trades and no trades take place in both states. Hence, rather than inducing the buyers to shift towards using the more efficient cutoff state by state, a trade signal moves the buyers' cutoffs in the same direction in the two states: towards the no-signals' cutoff as compared to the known-state cutoffs and towards the known-state cutoffs as compared to the no-signals' cutoff. Not surprisingly, this cannot be better than the more efficient of the known-state and no-signals' regimes.

In fact, the precision of the trade signal is such that the regime is less efficient than the no-signals' regime even if the no-signals' regime is less efficient than the known-state regime (i.e., for $\delta > \frac{2}{3}$). Recall the intuition: the cutoff used after no trade, $\bar{q}_N$, is too high as compared to the other cutoff, $\bar{q}_T$, because observing the low quality $q_1 = \bar{q}_N$ is better news than observing the higher quality $q_1 = \bar{q}_T$. In contrast, the similar discrepancy between the good-news content of $q_1 = \bar{q}_B$ versus $q_1 = \bar{q}_G$ has no effect on the cutoff $\bar{q}_B$ under the exogenous-signal regime because $p_G = 1$: the buyer knows that the state is bad after B. Hence, $\bar{q}_B$ is not too high relative to $\bar{q}_G$.

I argued in Section 2.6 that if payoff externalities are ignored (i.e., if we assume that the distribution of options that buyers face is identical to the entry distribution) then the trade signal does not change market efficiency as compared to the no-signals' regime. In fact, if the payoff externalities are ignored, neither the trade signal nor the exogenous signal change market efficiency as compared to the no-signals' regime (see Claim 2 in Appendix A, p. 148).

2.8 Conclusion

I summarise the paper and then discuss some alternative modelling choices.

In many markets, buyers do not know exactly the distribution of options they face. In a model of such a market, I analyse the effect a signal about trading frequency ("trade signal") on the model's cutoff-strategy equilibrium. In equilibrium, observing that another market participant traded is good news. Buyers who observe a trade use a higher cutoff than buyers who observe no trade, despite buyers' learning from relevant sampled options that puts a countervailing pressure on the cutoffs. Contrary to the intuition that more information increases market efficiency, the trade signal can reduce market efficiency as compared to a market without this signal. That is, a real estate market could be more efficient if people would not talk to each other about their experiences on the market. The reason is that buyers are "too optimistic" on average in a market with the trade signal and delay trade, which is inefficient. Conversely, a private signal with an appropriately chosen precision increases market efficiency as compared to both a market without this signal and a market where buyers know the distribution of options.

This exogenous signal is less likely to arise endogenously in a market than a trade signal because the informational requirements for generating the exogenous signal are much larger. Generating the exogenous signal requires the knowledge of the true distribution of options while a trade signal requires sellers to report their own past trading behaviour. The exogenous signal with the efficiency-improving precision can be implemented, for example, by a consumer-protection agency. If the true distribution is bad, the agency sometimes (but not always) produces a condemning report about the qualities in the market. No report is produced if the distribution is good. Hence, a condemning report reveals the bad distribution, but no report leaves the buyers in darkness about the true distribution.

Strategic information transmission. I argue that strategic sellers would emit and strategic buyers acquire the information provided by a trade signal, leading to an outcome that is worse for all impatient agents on the aggregate.

First, impatient sellers would emit information on trades. High-quality sellers trade in their entry period and prefer (weakly) not to reveal that they traded. But low-quality sellers may not trade in their entry period and prefer strictly to reveal not trading. Buyers understand that signal realisation "trade" or no information can come from a high-quality seller and "no trade" comes from a low-quality seller (just like in my model). Hence, "no trade" makes a buyer more willing to accept a low quality, which is in the interests of a low-quality seller.

Second, buyers would acquire information on trades. Since a trade signal informs the buyer about the true state and improves his assessment of his continuation value, he would optimally acquire it.

General entry distributions of sellers. Suppose that the quality distributions of the entering sellers, F_γ and F_β, are some continuous distributions with a common positive support and F_γ first-order stochastically dominates F_β. The equations that describe an equilibrium in cutoff strategies are straightforward to derive, but the cutoffs do not have closed-form solutions.

However, I expect the three main results of the paper (the equilibrium characterisation, that a trade signal can reduce welfare as compared to no signals, and that an appropriate exogenous signal improves welfare as compared to both no signals and known state) to continue to hold. I showed in Section 2.6 that under the assumption that F_γ first-order stochastically dominates F_β, the only alternative cutoff-strategy equilibrium characterisation to the one provided in Proposition 6 is that a trade is good news, the cutoff used after a trade is (weakly) higher than after no trade, and the lower quality $q = \bar{q}_N$ is worse news than the higher quality $q = \bar{q}_T$. Two crucial asymmetries in the model drive the two other results. The first asymmetry, between updating after $q_1 = \bar{q}_N$ and $q_1 = \bar{q}_T$, drives the result that the trade signal can reduce efficiency. This asymmetry continues to exist (unless the ratio $\frac{f_\gamma}{f_\beta}$ grows too quickly) because a cutoff strategy affects the equilibrium distribution asymmetrically under a private signal regime. The second asymmetry, that delay is shorter in the bad state under the known-state than the no-signals' regime and vice versa in the good state, drives the result that an exogenous signal with precision $p_G = 1$ and $p_B \in (0, 1)$ improves efficiency. I expect this asymmetry to continue to exist because I expect the cutoff used under the no-signals' regime to be between the two cutoffs used under the known-state regime.

More possible states. Many possible quality distributions means that a buyer has a distribution of beliefs over all the possible states and updates this distribution when new information arrives. I expect the three main results to continue to hold because there is no argument above that relies on the assumption that only two distributions are possible.

Long-lived buyers. If buyers live infinitely, then the existence of a stationary equilibrium in cutoff strategies fails, but can be restored with an additive search cost. If buyers live for $\infty > L > 2$ periods, the analysis of the model does not change substantially. In each life period, a buyer uses some cutoff that depends on his belief. The three main results of the model should continue to hold.

Chapter 3

A Two-Agent Model of Sequential Search and Choice[1]

3.1 Introduction

Standard sequential search models with recall build on the assumption that the search and choice stages comprise an undivided whole: the person who searches can stop and choose an item from the accumulated choice set at any time during the search process. This is an innocuous assumption if the preferences of the person are stable over time. In this paper, I extend the standard search model by allowing the preferences according to which the final choice is made to differ from the preferences according to which search is conducted. The set-up has two natural interpretations. First, the preferences belong to different parties: a "searcher" compiles a choice set via sequential search and a "chooser" chooses from the collected choice set. Second, the preferences belong to one individual, but change between the search and choice stages. I show that the searcher's optimal policy is a cutoff rule and characterise the cutoff.

[1]A version of this chapter was published in *Journal of Economic Behavior & Organization*, 2016, vol. 123, pp. 122–137, doi:10.1016/j.jebo.2015.12.010.

Examples of such search problems are an HR manager collecting applications for a boss who wants to hire a new worker and a real estate agent collecting offers for a client interested in buying a flat. An example involving a person and a set of individuals is a spouse looking for a job that determines the living place of the couple. A person who is interested in the return while searching for an investment opportunity, but later tempted to invest in an option that involves the least paperwork is a "multi-selves" example. More generally, many household decisions, organisational decisions involving different phases and multiple agents, choice processes partially outsourced to external partners, and political decisions involving advisers feature one party compiling a choice set for another party via search.

In this paper I analyse the optimal policy of a searcher who compiles a choice set for a chooser. I describe the model as a two-agent search problem. The searcher (he) and the chooser (she) have preferences over all items in some grand set of alternatives and the preferences are distributed according to a general full-support distribution function. The searcher has access to an arrival process. In each period, one item arrives and the searcher discovers how much utility both he and the chooser receive from the item if it is chosen. The searcher decides in each period whether to stop or continue the search process. If he stops the process, all the items that have arrived are presented to the chooser. The chooser then chooses the best item in the choice set according to her preferences, unless all the items in the set yield her less utility than her exogenous outside option. Utilities are realised when an item or the outside option is chosen. The process ends after the chooser moves. The searcher's problem is to choose an optimal policy, knowing the chooser's choice rule.

First, I derive and characterise the searcher's optimal policy. His optimal policy is a cutoff rule and the cutoff depends on the item that would be chosen by the chooser were the searcher to stop immediately, x_m. The searcher's cutoff is the lower the higher the chooser values this item because it acts as a restriction for the searcher: if he is unsatisfied with the utility he would receive from x_m, a new item is chosen only if the chooser's utility from it exceeds her utility from x_m. This has two implications. First, if the searcher finds an item that has a very high value for the chooser, he optimally stops searching regardless of the value that the item yields him. Second, the observed cutoff that the searcher uses weakly decreases in time, although the search horizon is infinite and the search environment stationary. I call this the "discouragement effect". The discouragement effect is present because the restriction that x_m poses on the searcher's problem becomes more stringent over time: as time passes, the searcher is less and less likely to find an item that the chooser values higher than the chooser-preferred item in the choice set. The searcher is thus optimally willing to accept a lower-value item. The time-decreasing cutoff is in contrast with the standard single-agent search model where a stationary environment translates into a stationary cutoff. The searcher's cutoff in my model is defined implicitly by a differential equation. I use a specific joint distribution where the utilities' correlation is captured by a single parameter to numerically show that an increase in the correlation parameter unambiguously increases the searcher's cutoff, in line with intuition.

Second, I compare the optimal cutoff of the searcher in the main model with imperfectly correlated preferences to the benchmark where the agents' preferences are perfectly aligned. I first show that in the main model the searcher's cutoff is always lower than in the benchmark: the searcher is "less picky". The reason is that the chooser chooses according to her preferences not the searcher's, which lowers the latter's continuation value, thus, his cutoff. I then provide an example where, as a result of a mean-preserving spread, the searcher's cutoff decreases in the main model, while it always increases in the benchmark: the searcher is “more conservative” in the main model.[2] A mean-preserving spread increases the probability that an item arrives that yields very high utility to the chooser, which restricts the searcher and lowers his continuation value.

Third, I explain how the model’s characteristics differ from those of two single-agent search models that feature a cutoff that varies in time. The first model has convex search costs and the second, a deadline. Both models result in an optimal cutoff that decreases over time (for a fixed searcher-preferred item in the choice set) because they assume non-stationarity of the environment. My model features a time-decreasing cutoff in a stationary environment. Also, in those models returning to an item found earlier is possible, while search always stops with the item found last in my model. The searcher returns to an item found earlier in the models with convex search costs or a deadline because his cutoff decreases exogenously over time. In my model, the decrease is endogenous: it happens only if there is a change in the item that would be chosen if the searcher stopped. I suggest three tests on data that allow us to reject one or more of the three models.

Finally, I extend the model to allow the searcher to hide items. The searcher can hide an item only upon its arrival and succeeds with some given probability that is strictly less than one. He observes whether he succeeded before making the decision whether to stop or continue. As in the main model, I find that the searcher’s cutoff is unambiguously decreasing in the value that the chooser receives from the chooser-preferred item in the choice set. For independent uniform utilities, I show that the cutoff is strictly increasing in the probability that the searcher succeeds in hiding. The constraint of having to account for the chooser’s preferences becomes the less restrictive the likelier that the searcher can ignore those preferences.

[2] The terms “less picky” and “more conservative” are borrowed from Albrecht et al. (2010).

Related literature. This paper is closely related in spirit to other papers on multi-agent search. In "committee" search problems the committee has a common arrival process and must agree on when to stop.[3] In "couple" search problems each person has his own arrival process, but they pool income.[4] In these papers some part of the entire search process is joint, while distinct parties are engaged in distinct stages of the process in my paper. I borrow the terms "less picky" and "more conservative" in their specific meaning from Albrecht et al. (2010) (AAV henceforth). AAV analyse a committee search problem, where M members of an N-member committee must agree in order to stop the search process. They find that a committee is both less picky and more conservative than a single searching agent. I find that the searcher is both less picky and more conservative if his and the chooser's preferences are misaligned as opposed to when they are perfectly aligned. These results echo those of AAV, but the reasons behind the results are somewhat different, as I explain in Section 3.5.

The paper is also related to the literature on delegated choice in a principal-agent set-up.[5] In these papers an agent makes the final choice and the principal either restricts the set of items that the agent can choose from or designs a contract. The available set of items is assumed to be given so the problem is static, unlike in this paper.

Finally, the paper is related to a recent literature on delegated search, where an agent conducts search on behalf of the principal in a principal-agent set-up, by Postl (2004), Armstrong and Vickers (2008), Lewis (2012), Kováč et al. (2014), and Ulbricht (2016). These models study how the principal can

[3]For example, see Albrecht et al. (2010), Compte and Jehiel (2010), Bergemann and Välimäki (2011), Kamada and Muto (2015), and Moldovanu and Shi (2013).

[4]For example, see Dey and Flinn (2008), Ek and Holmlund (2010), Flabbi and Mabli (2012), and Guler et al. (2012).

[5]For example, see Holmström (1977), Armstrong (1995), Alonso and Matouschek (2008), Armstrong and Vickers (2010), Amador and Bagwell (2013), and Kováč and Krähmer (2015).

direct the agent to conduct search in the best possible manner for her, whereas here I focus on the "agent's" optimal policy in the case when the "principal" cannot affect the search process directly. In Postl (2004), Lewis (2012), and Ulbricht (2016) utility is transferable, the agent does not receive direct utility from a chosen item, and the focus is on the principal's optimal contracts. In the other two papers, which are the most closely related to mine, utility is non-transferable. I point out how the modelling choices in Armstrong and Vickers (2008) (AV henceforth) and Kováč et al. (2014) (KKT henceforth) result in a simpler optimal policy for the agent in AV and KKT than in my paper. In AV, the working paper version of Armstrong and Vickers (2010), the agent makes the final choice and the principal permits the choice among a subset of all items. In an extension to the main model of AV, the agent collects the items via costly search. The most important aspect of my paper that sets it apart from AV is that the final choice is made by the "principal", whereas in AV it is made by the agent. In AV the agent's cutoff is constant in time. In KKT the agent's preferences differ cardinally, but not ordinally, from the principal's and the final choice is made by the principal. They solve for the principal's optimal mechanism. The most important aspect of my paper that sets it apart from KKT is that the agent and principal do not necessarily agree on which item is the best, whereas they do in KKT. The latter assumption together with assuming that the agent wants to stop with any item trivialises the unrestricted agent's optimal stopping rule in KKT, unlike in this paper.

The next section contains the details of the model. Section 3.3 introduces the benchmarks: the searcher's optimal policy when the chooser's preferences are either perfectly aligned or opposed to his. Section 3.4 solves for the searcher's optimal policy when the agents' preferences are arbitrarily imperfectly correlated. Section 3.5 characterises the solution. The final section concludes. All omitted proofs are in Appendix B.

3.2 Model

A "chooser" ("she") has to choose an item. The chooser makes the final choice from a choice set, but the choice set is compiled by someone else, a "searcher" ("he"). The final choice determines the agents' payoffs. The searcher compiles the choice set over time via sequential search. He chooses when to stop and take the accumulated choice set to the chooser.

The arrival process. Time, t, is discrete and $t = 1, 2, ..., \infty$. The searcher uncovers one new item in each period. He cannot affect how frequently or which items arrive. Search costs $c > 0$ per period for the searcher.[6]

Preferences. The t'th item that arrives, $x_t = (u_t, v_t)$, is worth u_t to the searcher and v_t to the chooser. The x_t are independent draws from a time-invariant distribution $H(u, v)$ with support $[0, 1]^2$. The marginal distributions of u and v are $F_u(\cdot)$ and $F_v(\cdot)$ respectively. The conditional distribution of u is $G(u|v)$. The associated pdfs are h, f_u, f_v and g, with $f_u(u) > 0$ for all $u \in [0, 1]$. I assume that the cost of search is relatively low, $c < \mathbb{E}[u]$, so that at least one period of search is desirable for the searcher. In the main part of the paper, Section 3.4, the joint distribution H has full support. In the benchmarks of Section 3.3, the preferences are either perfectly aligned (i.e., $P(v = u) = 1$ and $u \sim F_u$) or opposed (i.e., $P(v = 1 - u) = 1$ and $u \sim F_u$). The chooser has an outside option $\bar{v} \in [0, 1]$.

Actions. In period t the searcher chooses an action $a_t \in \{S, C\}$. If the searcher continues ($a_t = C$), the chooser does not get to act in period t. If the searcher stops ($a_t = S$), he takes the accumulated choice set to the chooser. He cannot hide or lie about items.[7] Let the v-maximal item in a choice set be denoted $x_m = (u_m, v_m)$. From any non-empty choice set brought to the

[6]The main results of the paper are not sensitive to the assumption of an additive search cost as opposed to discounting. I will comment on the result that does change in Section 3.4.

[7]The assumption of no hiding is relaxed in Section 3.6.

chooser, she chooses x_m if $v_m \geq \bar{v}$, which yields utility v_m to the chooser and u_m to the searcher. If $v_m < \bar{v}$, the chooser chooses her outside option, which yields utility $\bar{v}$ to her and zero utility to the searcher. The entire process ends after the choice of an item or the outside option.

Timing. In each period, first, the searcher uncovers an item and pays the search cost. Second, he chooses whether to stop or continue to search. If the searcher stops searching, the chooser chooses an item from the choice set or her outside option, after which utilities are realised and the entire process ends. If the searcher continues search, the process moves to the next period.

Problem. The searcher's problem is to maximise his expected utility from the search process, taking as given the chooser's outside option and her choice rule. The searcher refuses to start the search process if his expected utility is negative.

3.3 Two benchmarks

In this section, I derive the searcher's optimal policy in two benchmark versions of the model: in the first benchmark, the chooser's preferences are perfectly aligned with the searcher's, and in the second, her preferences are perfectly opposed to the searcher's.

3.3.1 Perfectly aligned preferences

Suppose that the chooser's preferences are perfectly aligned with the searcher's, i.e., $v = u$ with probability one and $u \sim F_u$. If the chooser's outside option is zero, the searcher's problem is equivalent to the standard single-agent sequential search problem (see McCall (1970) for the seminal contribution). In the standard search model, the searcher's optimal policy is a cutoff rule and the optimal cutoff is equal to the searcher's expected value from starting the search process (if the value is positive).

The presence of the outside option changes the analysis, but not considerably. The searcher's optimal policy is a cutoff rule because of the standard argument: if it is optimal for the searcher to stop with some item which yields him u, then it is optimal for him to stop with any item that yields him $u' > u$. If the u-value of an item that he finds exceeds the cutoff, he stops and the chooser chooses this item; the searcher continues otherwise. The cutoff does not depend on time as the problem is stationary. Since the chooser has an outside option worth $\bar{v}$, the searcher must wait for an item that exceeds the outside option to receive non-zero utility from the search process. The searcher uses one of two possible cutoffs depending on the size of the chooser's outside option. First, if the outside option is low, the cutoff equals $\tilde{u}$, the optimal cutoff in the standard single-agent sequential search problem. This cutoff is optimal if $\bar{v} \leq \tilde{u}$. McCall (1970) shows that $\tilde{u}$ solves

$$\int_{\tilde{u}}^{1} u - \tilde{u} \, \mathrm{d}F_u(u) = c.$$

The cutoff decreases in the cost of search, in line with intuition.

Second, if the chooser's outside option $\bar{v}$ is higher than $\tilde{u}$, the searcher is restricted by $\bar{v}$ in the sense that in the absence of it he would stop with items with $u \in [\tilde{u}, \bar{v})$, but has to continue in the presence of $\bar{v}$. Then the searcher optimally searches until he finds the first item that exceeds the chooser's outside option (if his expected payoff from search is positive). His expected payoff from the search process, $U_a(\cdot)$, is

$$U_a(\bar{v} > \tilde{u}) = P(u \geq \bar{v})\mathbb{E}[u|u \geq \bar{v}] + P(u < \bar{v})U_a(\bar{v} > \tilde{u}) - c. \tag{3.1}$$

The first term on the right-hand side (RHS) accounts for the possibility that the first item's value exceeds the outside option, in which case the searcher stops and receives u. If the value of the first item is lower than the outside option, he continues and his expected continuation value is the same as at the start of today. The searcher also pays the search cost c. The expected payoff decreases in the cost of search and the chooser's outside option. If the outside option is very high, i.e., $\bar{v} > v_a^*$ where v_a^* solves $U_a(v_a^* > \tilde{u}) = 0$, the searcher refuses to search because his expected payoff from searching is negative.

3.3.2 Perfectly opposed preferences

The chooser's preferences are perfectly opposed to the searcher's if $v = 1 - u$ with probability one and $u \sim F_u$. I argue that the searcher's optimal cutoff is zero: he stops with any first item acceptable to the chooser, if he searches at all.

The searcher knows that the chooser chooses according to v. As a result, he optimally stops after uncovering the first item which satisfies $v \geq \bar{v}$ if he searches at all. The reason is as follows. Without loss of generality, suppose that $v_1 \geq \bar{v}$. Then for any realisation of u_1, x_2 satisfies one of the following. Either $v_2 \geq v_1$ (so that $u_2 \leq u_1$) or $v_2 < v_1$ (so that $u_2 > u_1$). If the searcher stopped after observing the second item, in the former case the chooser would choose x_2 and in the latter case x_1. Since $u_2 \leq u_1$ in the former case and the searcher would end up getting x_1 at an extra search cost in the latter case, he prefers stopping after x_1 for any realisation of x_2.

The searcher's expected payoff from stopping with the first item which satisfies $v \geq \bar{v}$, $U_o(\cdot)$, is

$$U_o(\bar{v}) = P(v \geq \bar{v})\mathbb{E}[1 - v | v \geq \bar{v}] + P(v < \bar{v})U_o(\bar{v}) - c.$$

The equation is interpreted analogously to equation (3.1), except that here the searcher receives $u = 1 - v$ when he stops. The searcher's expected payoff is decreasing in the cost of search and the chooser's outside option. The expected payoff from searching is positive if $\bar{v} < v_o^*$ where v_o^* solves $U_o(v_o^*) = 0$. If the expected payoff from searching is negative, the searcher refuses to search and receives zero.

The results of the first two sections are summarised in the following:

Lemma 6. *(i) If the agents' preferences are perfectly aligned, the searcher's optimal policy is a cutoff rule and the cutoff is* $\max\{\tilde{u}, \bar{v}\}$*, where* $\tilde{u}$ *solves*

$$\int_{\tilde{u}}^{1} u - \tilde{u} \, \mathrm{d}F_u(u) = c.$$

(ii) If the agents' preferences are perfectly opposed, the searcher's optimal policy is a cutoff rule and the cutoff is zero.

3.4 Imperfectly correlated preferences

In this section the searcher's preferences, u, are imperfectly correlated with the chooser's preferences, v. The only assumption I make is that their joint distribution $H(u, v)$ has full support. I derive the first main result of this paper: the searcher's optimal policy is a cutoff rule, the cutoff is $\bar{u} = \max\{0, \bar{u}(v_m)\}$, and decreases in v_m.

The searcher's optimal policy is a cutoff rule because of the standard argument: if it is optimal for the searcher to stop with some item $x = (u, v)$ (which yields him u), then it is optimal for him to stop with any $x' = (u', v)$ with $u' > u$. The searcher stops if the utility that he receives from the item that would be chosen by the chooser if the searcher was to stop immediately (i.e., from the v-maximal item in the choice set) exceeds the cutoff, and continues otherwise. The optimal cutoff $\bar{u}(\cdot)$ is equal to the searcher's value from continuing, $U(\cdot)$, if the value is positive as in the standard search problem (and zero if the value is negative).

The cutoff differs from the standard search problem's in several ways. Recall that $x_m = (u_m, v_m)$ is the v-maximal item in the choice set. First, the cutoff is defined only if the choice set contains at least one item with $v \geq \bar{v}$, i.e., if $v_m \geq \bar{v}$. If no such item exists in the choice set, the searcher optimally continues for all u_m (if it was optimal for him to start the search process). Second, the positive part of the cutoff $\bar{u}(\cdot)$, or, equivalently, the searcher's continuation value $U(\cdot)$, depends on v_m if $v_m > \bar{v}$, but not on u_m. Suppose that the v-maximal item in the choice set is $x_m = (u', v')$ with $v' = v_m > \bar{v}$. If the searcher stops, x_m is chosen and he receives u'. If he is not satisfied with u', i.e., if $u' < \bar{u}(\cdot)$, he continues. I argue that his continuation value depends on v', but not on u'. The first part is simple: the continuation value depends on v' because a new item $x'' = (u'', v'')$ is chosen only if $v'' > v'$. The second part requires considering two scenarios. Suppose a new item (u'', v'') arrives. If $v'' > v'$, then v_m changes to v'': the new item would be chosen if the searcher stopped. Hence, u' is irrelevant. If $v'' \leq v'$, then v_m stays v'. But then nothing has changed as compared to the previous period (when the searcher continued) so that $u' < \bar{u}(\cdot)$ must still hold. In neither of the cases does the searcher's continuation value depend on u'. Hence, the expected value of continuing depends only on v_m so I write $U(v_m)$ and $\bar{u}(v_m)$. Third, the cutoff is weakly positive for $v_m \in [\bar{v}, v^*]$. The lower bound $\bar{v}$ is explained above. The upper bound v^* is present because v_m acts as a restriction: if v_m is very high, finding an item with $v > v_m$ becomes so unlikely that the searcher's continuation value becomes negative and he is better off accepting any positive u_m. The upper bound is defined as $\bar{u}(v_m = v^*) = 0$ and $v^* \in [0, 1]$ is guaranteed to exist for all $c > 0$ because of the full support assumption.

I solve for the searcher's continuation value for various values that v_m can take. Suppose first that $v_m \leq \bar{v}$. Then no item in the choice set is acceptable to the chooser. The searcher's expected value from continuing (equivalently, starting) the search process is

$$U(\bar{v}) = \int_{\bar{v}}^{1} \int_{0}^{1} \max\{u, U(v)\} h(u, v) \text{ d}u \text{ d}v + \int_{0}^{\bar{v}} U(\bar{v}) f_v(v) \text{ d}v - c, \quad (3.2)$$

if the next item that he finds is (u, v). The first term on the RHS accounts for the possibility that v exceeds the chooser's outside option: the item would be chosen if the searcher stopped. The searcher chooses optimally whether to stop or continue, where the continuation value is now a function of v. The second term on the RHS accounts for the possibility that v is below the chooser's outside option: the searcher continues and the continuation value is the same as at the start of today. The last term accounts for the search cost. As $U(\bar{v})$ is also the searcher's expected payoff from the entire search process, he optimally starts search if and only if $U(\bar{v}) \geq 0$.[8]

Suppose now that $v_m > \bar{v}$. The searcher's expected value from continuing is

$$U(v_m) = \int_{v_m}^{1} \int_{0}^{1} \max\{u, U(v)\} h(u, v) \text{ d}u \text{ d}v + \int_{0}^{v_m} U(v_m) f_v(v) \text{ d}v - c, \quad (3.3)$$

which is interpreted analogously to (3.2), except that $\bar{v}$ in (3.2) is replaced by v_m in (3.3).

As argued above, the searcher's optimal cutoff $\bar{u}(\cdot)$ is equal to his continuation value $U(\cdot)$ if the latter is positive, which is true for $v_m \in [\bar{v}, v^*]$ for some $v^* \in [0, 1]$ that satisfies $U(v^*) = 0$. The optimal cutoff is zero if the continuation value is negative, i.e., if $v_m > v^*$. Thus, by substituting $U(v_m) = \bar{u}(v_m)$

[8]If I assume discounting instead of an additive search cost, then the searcher optimally starts search for all $\bar{v} < 1$ as $U(\bar{v}) > 0$ for all positive discount factors. This is the only result that qualitatively changes in a model with discounting.

into equation (3.3) and using the fact that the agent continues for $x = (u, v)$ s.t. $v > v_m$ and $u < \bar{u}(v)$, I can rewrite (3.3) for $v_m \in [\bar{v}, v^*]$ as

$$\bar{u}(v_m) = [1 - F_v(v_m)]^{-1} \int_{v_m}^{v^*} \left[\int_0^{\bar{u}(v)} \bar{u}(v) h(u, v) \, \mathrm{d}u + \int_{\bar{u}(v)}^1 u h(u, v) \, \mathrm{d}u \right] \mathrm{d}v, \tag{3.4}$$

where v^* satisfies $\int_0^1 \int_{v^*}^1 uh(u, v) \, \mathrm{d}v \, \mathrm{d}u = c$. As search becomes more costly, i.e., c increases, v^* decreases. Intuitively, if search becomes more costly, it becomes unprofitable for the searcher to continue searching at lower values of v_m. Note that the function $\bar{u}(v_m)$ does not depend on $\bar{v}$: the chooser's outside option affects the searcher's expected payoff from the entire search process, but does not affect his optimal policy for $v_m > \bar{v}$. The above integral equation can be converted into an ODE by differentiation:

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{f_v(v_m)}{1 - F_v(v_m)} \int_{\bar{u}(v_m)}^1 u - \bar{u}(v_m) \, \mathrm{d}G(u|v_m). \tag{3.5}$$

On the RHS of the equation, the first term is the hazard rate, i.e., the probability that an item with $v = v_m$ arrives, given that such an item has not arrived earlier. The integral term is the utility that the searcher expects to get from the new item, u, in excess of the continuation value $\bar{u}(v_m)$, which is realised only if u exceeds the continuation value. The slope of the cutoff depends on the search cost c indirectly via the level of the cutoff $\bar{u}(v_m)$, because the level depends on the search cost via v^* (see equation (3.4)).

Equation (3.5) does not, in general, have a closed form solution (I provide a closed form solution for an example below; further examples can be provided under the assumption that u and v are independent). However, equation (3.5) together with the initial condition $\bar{u}(v^*) = 0$ fully pin down the function $\bar{u}(v_m)$, which proves the first main result of this paper.

Proposition 10. *If the agents' preferences are imperfectly correlated, the* *searcher's optimal policy is a cutoff rule. The cutoff is* $\bar{u} = \max\{0, \bar{u}(v_m)\}$, *where* $\bar{u}(v_m)$ *solves*

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{f_v(v_m)}{1 - F_v(v_m)} \int_{\bar{u}(v_m)}^{1} u - \bar{u}(v_m) \ \mathrm{d}G(u|v_m),$$

for $v_m \in [\bar{v}, v^*]$, *where* v^* *satisfies* $\bar{u}(v^*) = 0$.

The cutoff $\bar{u}(v_m)$ is clearly decreasing in v_m for any joint distribution of u and v with full support. The chooser's value from the v-maximal item in the choice set, v_m, acts as a restriction on the searcher's problem because a new item $x' = (u', v')$ is chosen only if $v' > v_m$. The stricter the restriction, the lower the searcher's expected payoff from the process. I provide sufficient conditions for the cutoff $\bar{u}(v_m)$ to be concave in the Appendix.

The fact that the cutoff $\bar{u}(v_m)$ decreases in v_m means that the searcher does not start searching for high enough outside options for the chooser, $\bar{v}$. The chooser's outside option acts as a similar restriction on the searcher's problem as v_m. If the outside option is very high, i.e., $\bar{v} > v^*$, the searcher prefers to receive payoff zero to starting the process and making a loss in expectation. The critical outside option above which the searcher prefers not to start searching is lower if the agents' preferences are misaligned as opposed when they are perfectly aligned ($v^* < v_a^*$), in line with intuition.

Example 1 (Analytic solution)**.** *Suppose the utilities are independent and h uniform on* $[0,1]^2$. *The searcher's expected value from continuing, equation* (3.4), *simplifies to*

$$\bar{u}(v_m) = \frac{1}{2}(1 - v_m)^{-1} \int_{v_m}^{v^*} 1 + \bar{u}(v)^2 \ \mathrm{d}v,$$

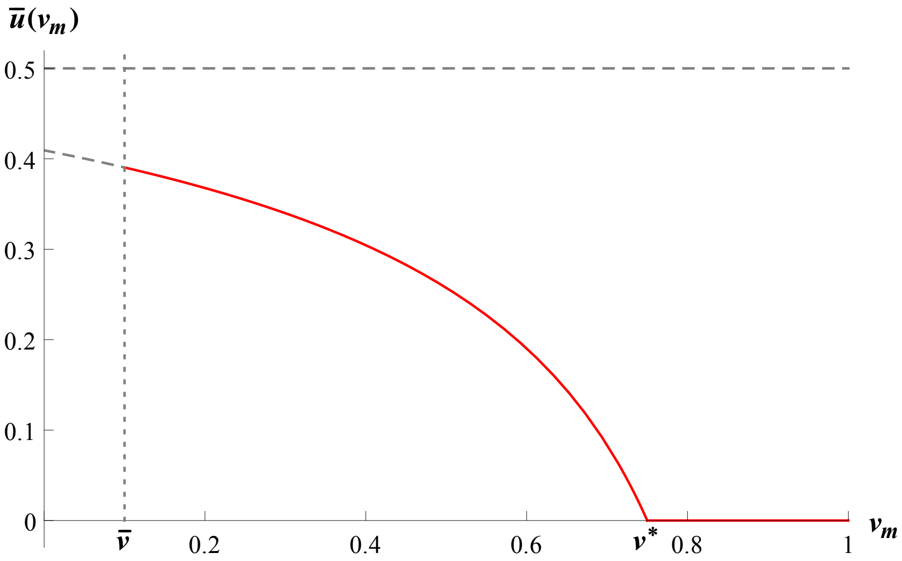

Figure 3.1: Searcher's cutoff as a function of v_m if utilities are independent and uniform for $\bar{v} = \frac{1}{10}$ and $c = \frac{1}{8}$. Dashed line: $\tilde{u}$.

and the associated ODE (3.5) *to*

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{1}{2}(1 - v_m)^{-1}(1 - \bar{u}(v_m))^2. \tag{3.6}$$

In Appendix B I solve equation (3.6) *in detail using a standard method and the initial condition* $\bar{u}(v^*) = 0$. *The explicit solution is*

$$\bar{u}(v_m) = 1 - \left[\frac{1}{2}\ln\left(\frac{1 - v_m}{1 - v^*}\right) + 1\right]^{-1}, \tag{3.7}$$

where the value v^* *has a simple closed form:* $v^* = 1 - 2c$. *The conditions that guarantee that* $\bar{u}(v_m)$ *is concave in* v_m *are satisfied. An example of the cutoff as a function of* v_m *is depicted in Figure 3.1.*

3.5 Characterisation

In this section, I first prove that the cutoff policy exhibits the "discouragement effect" and derive testable implications of my model (Sections 3.5.1 and 3.5.2).

I then show that, as compared to when the agents' preferences are perfectly aligned, if the preferences are misaligned, the searcher is "less picky" and "more conservative" (Sections 3.5.3 and 3.5.4).[9] These two results echo the results of AAV, but the mechanisms are slightly different as I explain below. I then provide a numerical example where a higher correlation of preferences results in an unambiguously higher cutoff. Finally, I show how the implications of my model differ from two models that extend the standard single-agent search model and generate a cutoff that varies over time. Omitted details are in Appendix B.

3.5.1 The cutoff exhibits the discouragement effect

I present the second main result of the paper, that the searcher's cutoff exhibits the discouragement effect.

Definition 1 (Discouragement effect)**.** *A cutoff policy exhibits the* discouragement effect *if the cutoff weakly decreases in time.*

I call a time-decreasing cutoff the discouragement effect because the longer the searcher searches, the lower-utility items he is willing to accept and the more likely he is to stop: he is discouraged from searching longer and for better items as time goes on.

Proposition 11. *The cutoff policy $\bar{u}(v_m)$ exhibits the discouragement effect.*

Proof. The cutoff $\bar{u}(v_m)$ is weakly decreasing in time if and only if v_m is weakly increasing in time because $\frac{\partial \bar{u}(v_m)}{\partial v_m} < 0$ (by Proposition 10). Since v_m is the maximum utility that the chooser gets from a choice set collected up to some time t, v_m is formally the tth (or largest) order statistic of the choice set at time t: $v_m = \max\{v_1, v_2, ..., v_t\}$. But the tth order statistic must weakly increase in

[9] The terms "less picky" and "more conservative" are borrowed from AAV.

t: for any v_m at date t, either $v_{t+1} \leq v_m$, in which case v_m is left unchanged, or $v_{t+1} > v_m$, in which case v_m takes on the new, higher value v_{t+1}. □

Proposition 2 states that he searcher's cutoff stochastically weakly decreases over time, despite the search environment being stationary. This is in contrast to the standard sequential search model where a stationary environment translates into a stationary cutoff.

The source of the discouragement effect in my model is an endogenous restriction on the searcher's problem that becomes more stringent over time. In particular, the v-value of the v-maximal item in the choice set, v_m, acts as a restriction on the searcher's problem. The value v_m restricts the searcher because he has to uncover a new item that yields higher utility to the chooser than v_m in order for her to choose the new item. The higher v_m, the less likely is the searcher to uncover an item with a v-value that exceeds v_m. But v_m weakly increases in time because it is the largest order statistic of the choice set according to the chooser's utility. If search has gone on for long enough, v_m is high and the searcher optimally stops after uncovering any new item (u, v) with $v > v_m$ regardless of how low u is. Suppose that the HR manager in the example in the Introduction cares about the amiability of a future colleague and the boss about the new worker's qualifications. The discouragement effect means that if the HR manager receives the application of a highly qualified worker who does not seem like an amiable colleague, he stops looking for other applicants since he anticipates that the boss hires the highly qualified applicant.

3.5.2 Testable implications of the discouragement effect

I describe the testable implications of my model and especially the discouragement effect. I assume that we have data on multiple search instances involving the same (or a group of representative) searcher(s) and the same chooser.

Each search instance is identified with an observation i. I assume that each observation contains information on the duration of search (denoted D_i), the identity of item that is the final choice (denoted M_i), and on the searcher's and chooser's utility from the finally chosen item $x_{M_i} = (u_{M_i}, v_{M_i})$. Data on utilities is unlikely to occur in field settings, but can be generated in a laboratory experiment. The model has three testable implications.

1. u_{Mi} and v_{Mi} are negatively correlated across i. This is a direct implication of the negatively-sloped cutoff. If at any point during the search process v_m is low, the searcher only stops if u_m is high. Conversely, if v_m is high, then a low u_m is sufficient for the searcher to stop. Across many instances, the utilities from the finally chosen item should, thus, be negatively correlated.

2. D_i and u_{Mi} (v_{Mi}) are negatively (positively) correlated across i. The longer the searcher searches, the more likely is v_m to be high. But when v_m is high, the searcher accepts items with lower u-values as compared to when v_m is low. Across many instances, the search duration and the searcher's utility from the finally chosen item should, thus, be negatively correlated.

3. $D_i = 1$ for some observations i. The searcher stops after any first item x_1 if $v_1 > v^*$. In this case, the utility that he gets from the item, u_1, is irrelevant for his stopping problem and he stops immediately.

I explain in Section 3.5.6 how my model's implications differ from the implications of two other models with a time-varying cutoff.

3.5.3 The searcher is less picky if the preferences are misaligned

I borrow the terminology from AAV and say that a searcher with a lower cutoff is "less picky" than a searcher with a higher cutoff.

Proposition 12. *The searcher is less picky if the agents' preferences are misaligned as compared to when they are perfectly aligned, i.e.,* $\max\{\tilde{u}, \bar{v}\} > \bar{u}(v_m)$ *for all* v_m.

Proof. In Appendix B. □

The reason why the searcher whose preferences differ from the chooser's is willing to accept a lower utility item is that his search process is restricted by the chooser's preferences: an item that the searcher "likes" is chosen only if the chooser "likes" it, too. If their preferences are not perfectly aligned, the agents "like" the same item with a probability less than one. This reduces the searcher's value from continuing with the search process, hence, his optimal stopping cutoff. In terms of the hiring example, if amiable people have higher qualifications, then the HR manager is optimally satisfied with a more amiable new colleague as opposed to when amiability and qualifications are not perfectly correlated, despite the HR manager's preferences not changing.

In AAV, a committee is less picky than a single searcher because the committee members need to compromise. In my model, the searcher is less picky if his and the chooser's preferences are misaligned because choice is made according to the chooser's not the searcher's preferences.

3.5.4 The searcher is more conservative if the preferences are misaligned

I say that the searcher is "more conservative" if his cutoff may decrease as a result of a mean-preserving spread (MPS) to the distribution of utilities, following AAV. In order to make the comparison between the benchmark and full model and give an unambiguous meaning to a MPS, I assume in this section that the marginal distributions of u and v equal: $F_u = F_v =: F$. I show by example that the searcher is more conservative if his and the chooser's preferences are misaligned as compared to when they are perfectly aligned.

Proposition 13. *If $F_u = F_v$ and the joint distribution of the utilities is subjected to a mean-preserving spread, the searcher's cutoff always increases when the preferences are perfectly aligned, but for a certain set of parameter values the cutoff decreases when the agents' preferences are misaligned.*

Proof. (a) If the joint distribution of the utilities is subjected to a mean-preserving spread, the searcher's cutoff $\tilde{u}$ always increases when the preferences are perfectly aligned. For proof, see e.g., AAV.

(b) If the joint distribution of the utilities is subjected to a mean-preserving spread, the searcher's cutoff may decrease when the agents' preferences are misaligned.

By example: I provide an example where a MPS leads to a decrease in $\bar{u}(v_m)$. Assume that u and v are independent and have the same uniform marginal $\mathcal{U}[a, b]$ for $0 < a < b$. An example of a MPS for the distribution is $\mathcal{U}[a', b']$ such that $0 < a' < a$, $b' > b$ and $a + b = a' + b'$.

The cutoff of the searcher satisfies the ODE

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{(b - \bar{u}(v_m))^2}{2(b-a)(b-v_m)}.$$

The searcher prefers continuing to accepting any item if his value from continuing is higher, i.e., if $U(v_m) \geq a$. Define v^* by $U(v^*) = a$, or $\bar{u}(v^*) = a$. The closed form is $v^* = b - 2c$. Then I can solve the ODE by using similar methods as in the proof of Claim 3 (in Appendix B). The result is

$$\bar{u}(v_m) = \frac{2(b+a) - 2b - b\ln[\frac{b-v^*}{b-v_m}]}{2 - \ln[\frac{b-v^*}{b-v_m}]}.$$

Differentiating the expression with respect to b while keeping $a+b$ constant gives

$$\frac{\partial \bar{u}(v_m)}{\partial b} = 4v_m - 2(a+b) + (b - v_m)\left(\ln\left[\frac{b-v^*}{b-v_m}\right]\right)^2.$$

This is negative, for example, if $b = 3$, $a = 1$, $v_m = 1.5$ and $c = \frac{1}{4}$.

□

The reason why a MPS leads to an unambiguous increase in the searcher's optimal cutoff in a single-agent search problem is that a MPS increases the option value of searching by making really high (and really low) draws possible. The gain from the really high draws outweighs the loss from the really low draws because the latter are not accepted.

But if the searcher's and chooser's preferences are misaligned, the searcher's optimal cutoff may decrease under a MPS: the searcher behaves more conservatively under more risk by accepting lower-utility items. Two counteracting effects lie behind the result. On the one hand, a MPS of the u-value distribution benefits the searcher through the same mechanism as in the single-agent setup: the option value of searching increases and the cutoff rises. On the other, a MPS increases the probability of items with high v-value occurring. A high v-value acts as a stricter constraint for the searcher, thus decreasing his value of continuing and his cutoff. In the example in the proof of Proposition

13, the negative effect outweighs the positive.

In AAV, a committee is more conservative than a single searcher. The result emanates from one committee member exerting a negative externality on another, e.g., in a unanimity committee a member can veto stopping in a case where everyone but the vetoing member would receive a high utility from the last item. The externality can become more severe under a MPS. Here, the result emanates from the fact that under a MPS, the searcher is more likely to find an item with a v-value that restricts him more.

3.5.5 The searcher prefers more aligned preferences

I use a simple parametric family of joint distributions to demonstrate numerically that the searcher is unambiguously better off as the agents' preferences become more aligned. Analytic results are not available even for this simple family. Let the marginals of u and v be uniform on $[0, 1]$ throughout. The family of distributions deals with positive and negative correlation separately. For positive correlation, let the correlation be governed by parameter $q \in [0, 1]$. The conditional distribution of u given v is

$$u|v = \begin{cases} v & \text{with probability} \quad q, \\ \sim \mathcal{U}[0,1], u \perp\!\!\!\perp v & \quad -''- \quad 1-q. \end{cases}$$

The searcher's optimal cutoff is the solution to a system of two differential equation that have the form

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{1}{2}(1-v_m)^{-1}[A(1-\bar{u}(v_m))^2 + B(v_m - \bar{u}(v_m))],$$

where A and B depend only on q. I provide the exact differential equations in Appendix B, but omit them here as they do not possess closed form solutions that could be interpreted.

For negative correlation, let the correlation be governed by parameter $r \in [0, 1]$. The conditional distribution of u given v is

$$u|v = \begin{cases} 1 - v & \text{with probability} \quad r, \\ \sim \mathcal{U}[0,1], u \perp\!\!\!\perp v & \quad -''- \quad\quad 1 - r. \end{cases}$$

The searcher's optimal cutoff is the solution to a system of at most two differential equations, with a general form given by

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{1}{2}(1 - v_m)^{-1}[A(1 - \bar{u}(v_m))^2 + B(1 - v_m - \bar{u}(v_m))],$$

where A and B depend only on r. Again, I provide the exact differential equations in Appendix B.

The positive and negative correlation parameters q and r can be comprised in a single parameter ρ: $\rho = q$ for $q \geq 0$ and $r = 0$, and $\rho = -r$ for $r \geq 0$ and $q = 0$. I show numerically that $\bar{u}(v_m)$ increases in ρ. The result is illustrated in Figure 3.2 (a lower curve corresponds to a lower level of correlation). The dashed grey line corresponds to $\tilde{u} = 1 - \sqrt{2c}$. The intuition behind the result is simple: if the utilities become more correlated, then the searcher is better off as his continuation value, thus, his cutoff, increases. An increase in ρ affects the cutoff via several channels. First, a higher ρ directly increases the searcher's value from stopping because a high v is more likely to be accompanied by a high u. Second, a higher ρ indirectly increases the searcher's continuation value both through the decreased likelihood that a high v is restrictive and through the increased future value of stopping.

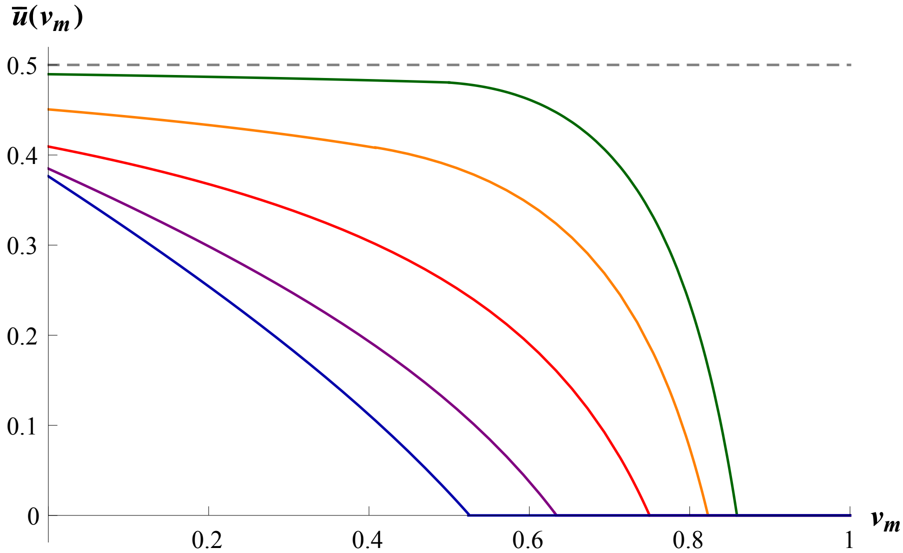

Figure 3.2: Searcher's cutoff $\bar{u}(v_m)$ at correlation levels $\rho = -0.9, -0.5, 0, 0.5, 0.9$ (from lowest to highest curve respectively; dashed line: $\tilde{u}$), for $\bar{v} = 0$, $c = \frac{1}{8}$.

3.5.6 Differences with other models with time-varying cutoffs

Here, I first explain how the characteristics of my model differ from two single-agent search models that feature a time-varying cutoff. I then explain how data allows us to test between the three models. Recall that in my model the searcher's cutoff decreases (weakly) over time because v_m increases (weakly) over time. Extensions to the standard single-agent search model that produce a time-varying cutoff are, for example, a finite horizon, i.e., a deadline, (see Gronau (1971) and Lippman and McCall (1976)) or convex search costs (see Stiglitz (1987)). For a fixed u-value of the u-maximal item in a choice set (denoted u_w), the cutoff in these models is decreasing over time.

First, I compare the reasons behind the decreasing cutoffs. The reason behind a decreasing cutoff (for a fixed u_w) in all the models is that the searcher's value of continuing decreases over time. However, the deeper reasons differ. In the case of a deadline, the decrease is due to the exogenous end of search possibilities beyond the deadline. In the case of convex search costs, the decrease is due to search becoming exogenously more expensive over time. In fact, in these models the environment is non-stationary to start with (so that a non-stationary optimal policy is expected), contrary to my model.

Second, for a fixed u_w, the time path of the cutoff in a model with a deadline or convex search costs is strictly concave in time (for infinitesimally short time periods) whereas it is not concave in time in my model. In a model with a deadline, the cutoff is concave in time because for a fixed time increment, the loss of future search opportunities for the searcher becomes larger the closer the deadline. In a model with convex search costs, the cutoff is concave in the number of items (equivalently, in the number of time periods). In my model, the cutoff is not concave in time because with positive probability, a new item's v-value does not change v_m, in which case the cutoff is constant in time, thus, not concave. The time path of the cutoff cannot be constant in time in a model with either a deadline or convex search costs.

Third, the models generate different outcomes. In my model, the item that is finally chosen is always the one uncovered last. This is because the cutoff decreases only if the v-maximal item changes. If a new item becomes the v-maximal item in the choice set, it changes the cutoff $\bar{u}(v_m)$, and the new item may be attractive for the searcher to stop with (if $u_m \geq \bar{u}(v_m)$). Alternatively, the new item does not become the v-maximal item in the choice set and the u-value of the v-maximal item still falls below the cutoff. In contrast, there may be return to an item uncovered earlier in a model with either a deadline or convex search costs. This is because in those models the cutoff can decrease independently of the changes in the choice set. An item that yielded too little utility to warrant stopping in the past may exceed the decreased cutoff. In sum, in my model the final choice is always the item uncovered last despite the searcher's decreasing cutoff.

Finally, I explain how data allows to test between the three models. For this exercise, let us first assume that each observation i in the data corresponds to a search instance and contains information on the duration of the search process, D_i, and the identity of the item that is the final choice, M_i. Suppose we have data on $i \geq 1$ of such search instances. Then the following predictions are made.

1. If in any observation the finally chosen item is not the last item, i.e., $M_i \neq D_i$ for an i, we can reject my model. This prediction is a straightforward implication of the last of the three differences between the three models that I discussed above.

2. If the deadline in the deadline model is some known number T_i for each observation and the duration of search exceeds the deadline in any observation, i.e., $D_i > T_i$ for some i, we can reject the deadline model.

If each observation would additionally contain information about the utilities of all the items that the searcher has observed, a further prediction is made.

3. If the finally chosen item does not have the highest utility for the searcher among all the observed items, i.e., if for some i, $u_{M_i} \neq \text{argmax}_{x \in C_{D_i}} u(x)$, where C_{D_i} is the choice set collected until the process ends, we can reject the deadline and the convex search cost models. In my model, the final choice is made according the chooser's preferences, v, but is made according to the searcher's preferences, u, in the two other models. Thus, in the two models the final choice must be the u-maximal item, but not in my model.

3.6 An extension: hiding

Suppose that the searcher can hide the items that he wants to at the arrival of the items, but is not always successful.[10] The searcher takes two actions in any period: $a_{1t} \in \{H, D\}$ and $a_{2t} \in \{S, C\}$, where H stands for for trying to hide, D for not trying to hide, S for stopping, and C for continuing. In particular, after an item x_t arrives and the searcher has found out its utilities (u_t, v_t), he can attempt to hide the item x_t. He succeeds with probability $p \in (0, 1)$. After taking the hiding action and observing its outcome, he chooses whether to stop or continue.

I derive the searcher's optimal hiding and stopping policy. In sum, the searcher (weakly) prefers hiding all items that he does not want to stop with. He prefers stopping to continuing if his value from stopping exceeds the value from continuing. Let us consider all the possible cases that the searcher may encounter. Suppose that the v-maximal item found until today is $x_m = (u_m, v_m)$, the searcher decided to continue yesterday (so that $u_m < U(v_m)$), and the item found today is $x = (u, v)$. I describe the searcher's optimal policy for all possible values of x_m and x. If $v < v_m$, the searcher continues (with or without attempting to hide x) because x_m is chosen if the searcher stopped (so that his continuation value is definitely $U(v_m)$) and $u_m < U(v_m)$ still holds. If $v \geq v_m$, it must be that $U(v) \leq U(v_m)$ because a higher v_m is a greater restriction for the searcher (I verify later that this property holds). Then the maximum continuation value that the searcher can achieve is, after successfully hiding x, $U(v_m)$. Hence, if $u > U(v_m)$ it is optimal for the searcher to not to hide the item and stop. Suppose that $u < U(v_m)$: the searcher would like to continue if he could guarantee himself the continuation value $U(v_m)$. Hence, the searcher attempts hiding, $a_{1t} = H$, and if he succeeds, continues as $u_m < U(v_m)$. If

[10] I thank Ludo Visschers for proposing this particular hiding technology.

he fails, he compares u to his continuation value $U(v)$: if $u \geq U(v)$, he stops, and continues otherwise. In sum, the optimal sequence of actions for the agent after receiving $x = (u, v)$ with $v \geq v_m$ is

$$(a_{1t}, a_{2t}) = \begin{cases} (D, S) & \text{if } u \geq U(v_m), \\ (H, C) & \text{if } u < U(v_m) \text{ and hiding succeeds}, \\ (H, S) & \text{if } u \in [U(v), U(v_m)) \text{ and hiding fails}, \\ (H, C) & \text{if } u < U(v) \text{ and hiding fails}. \end{cases} \tag{3.8}$$

The optimal sequence of actions for the agent after receiving $x = (u, v)$ with $v < v_m$ is

$$(a_{1t}, a_{2t}) = (\{H, D\}, C). \tag{3.9}$$

Thus, the optimal policy of the searcher can no longer be characterised by a cutoff only, but the buyer's continuation value $U(v_m)$ is sufficient to describe his optimal policy.

Note that if v_m is very high, then the searcher's continuation value is negative for the same reason as in the main model. For any $p < 1$, if v_m is very high, the probability that an item arrives with $v > v_m$ (and $u > u_m$) is very low, so that the searcher's benefit from continuing is less than the cost c. As before, let the smallest v_m s.t. $U(v_m) \leq 0$ be v^*.

Formally, the searcher's continuation value when the v-maximal item is (u_m, v_m) and the newly arrived item is (u, v) satisfies

$$\begin{aligned} U(v_m) = & \int_0^{v_m} U(v_m) f_v(v) \text{ d}v + \int_{v_m}^{v^*} \int_0^{U(v)} [pU(v_m) + (1-p)U(v)] h(u, v) \text{ d}u \\ & + \int_{U(v)}^{U(v_m)} [pU(v_m) + (1-p)u] h(u, v) \text{ d}u + \int_{U(v_m)}^{1} u h(u, v) \text{ d}u \text{ d}v \end{aligned} \tag{3.10}$$

$$+\int_{v^*}^{1}\int_{0}^{U(v_m)}[pU(v_m)+(1-p)u]h(u,v)\ \mathrm{d}u+\int_{U(v_m)}^{1}uh(u,v)\ \mathrm{d}u\ \mathrm{d}v-c,$$

where v^* satisfies $U(v^*)=0$. Each of the terms corresponds to the optimal sequence of actions for the searcher as described in equations (3.8) and (3.9). For example, the first terms reads that if the new item's v-value falls below the v-value of the v-maximal item found so far, then the searcher optimally continues to search (as described in equation (3.9)). The last double integral reads that if $v > v^*$ (so we know that the searcher's continuation value is negative if he fails to hide x), the searcher stops for sure if u exceeds his best possible continuation value $U(v_m)$ (last term). If u falls short of this continuation value, the searchers tries to hide x. If he succeeds, he continues. If he fails, he stops.

In a similar fashion as in the main model, I derive the differential equation that the continuation value satisfies by differentiating (3.10) with respect to v_m and obtain the ODE

$$\frac{\partial U(v_m)}{\partial v_m}\int_{v_m}^{1}\int_{0}^{1}h(u,v)\ \mathrm{d}u-p\int_{0}^{U(v_m)}h(u,v)\ \mathrm{d}u\ \mathrm{d}v$$

$$=-\int_{U(v_m)}^{1}h(u,v_m)(u-U(v_m))\ \mathrm{d}u,$$

with terminal condition $U(v^*)=0$, or $\int_{v^*}^{1}\int_{0}^{1}uh(u,v)\ \mathrm{d}u\ \mathrm{d}v=c$. Since the RHS of the ODE is negative and the multiplier after $\frac{\partial U(v_m)}{\partial v_m}$ on the LHS positive, the continuation value is unambiguously decreasing in v_m. I explain why v^* is independent of p. The value v^* is defined as the smallest v_m such that $U(v_m=v^*)\leq 0$ so that the searcher wants to stop with any u_m when $v_m=v^*$. Suppose that $v_m=v^*$ and the searcher continues (so that $u_m=0$). Then the best continuation value that he can hope for is $U(v^*)=0$. If an item (u,v) arrives with $v>v^*$, the searcher optimally accepts any u because $U(v)<0$

for all $v > v^*$. But the expected value of u (given that $v > v^*$) is independent of p, hence, v^* is independent of p.

The comparative static derivative of $U(v_m)$ with respect to p cannot be determined without solving for the function $U(v_m)$. I show for an example below that, in line with intuition, the searcher's continuation value is strictly increasing in p for all $v_m < v^*$.

Example 2 (Analytic solution, continued)**.** *Suppose that the utilities are independent and h uniform on $[0,1]^2$. The searcher's expected value from continuing simplifies to*

$$U(v_m)(1-v_m) = \int_{v^*}^{1}\int_{0}^{U(v_m)} [pU(v_m)+(1-p)u]\ \mathrm{d}u + \int_{U(v_m)}^{1} u\ \mathrm{d}u\ \mathrm{d}v - c$$

$$+\int_{v_m}^{v^*}\int_{0}^{U(v)} [pU(v_m)+(1-p)U(v)]\ \mathrm{d}u + \int_{U(v)}^{U(v_m)} [pU(v_m)+(1-p)u]\ \mathrm{d}u + \int_{U(v_m)}^{1} u\ \mathrm{d}u\ \mathrm{d}v.$$

The derivative of the above is

$$\frac{\partial U(v_m)}{\partial v_m} = -\frac{1}{2}(1-v_m)^{-1}(1-pU(v_m))^{-1}(1-U(v_m))^2,$$

with terminal condition $U(v^*) = 0$, which yields $v^* = 1-2c$. Note that the equation collapses to equation (3.6) if $p = 0$.

I use the same method to solve the ODE as before, using the initial condition $U(1-2c) = 0$. The implicit solution for $U(v_m)$ is

$$2(1-p)\frac{U(v_m)}{1-U(v_m)} - 2p\ln(1-U(v_m)) = \ln\left(\frac{1-v_m}{1-v^*}\right).$$

The RHS is constant in $U(\cdot)$ and p so the derivative $\frac{\mathrm{d}U(\cdot)}{\mathrm{d}p} = -\frac{LHS_p}{LHS_{U(\cdot)}} > 0$ as $LHS_{U(\cdot)} > 0$ and $LHS_p < 0$ (straightforward to verify from above). In line with intuition, the searcher becomes unambiguously better off as his probability of successfully hiding the items that he wants to hide increases: he is less likely

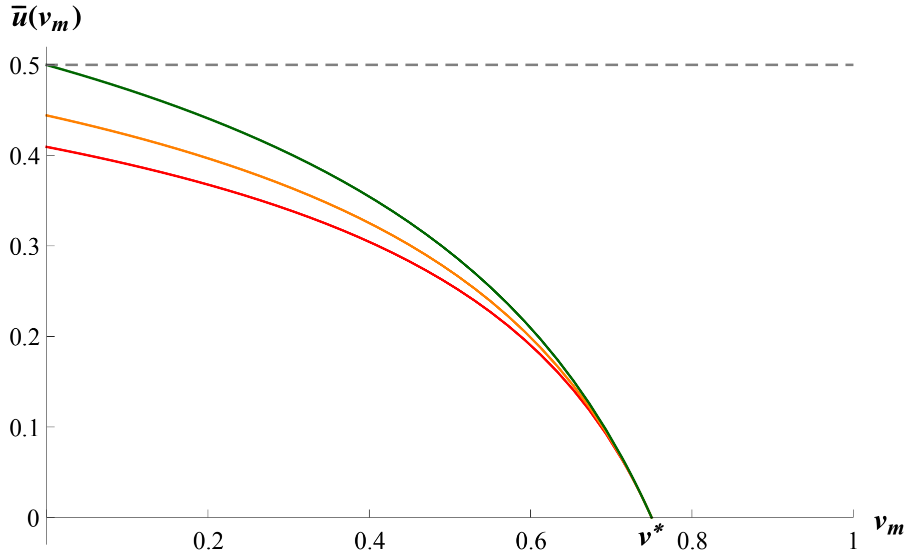

Figure 3.3: Searcher's cutoff $\bar{u}(v_m)$ at hiding probabilities $p = 0, 0.5, 0.999$ (from lowest to highest curve respectively), for $\bar{v} = 0$, $c = \frac{1}{8}$. Dashed line: $\tilde{u}$.

to have to account for the chooser's preferences as p increases. The result is illustrated on Figure 3.3: higher curves stand for higher p.

Allowing the searcher to hide items acts in a similar manner as making the agents' preferences more aligned in the sense that the searcher's expected continuation value increases. However, the parametric examples illustrate that there are differences (see Figure 3.2 and Figure 3.3). For a given correlation structure, an increase in p shifts the searcher's continuation value up, without changing the value of v_m *above which the continuation value is zero* (v^*)*. Conversely, for a given p, an increase in the correlation parameter shifts both the searcher's continuation value and* v^* *up. The reason behind the difference is as follows. If the utilities become more positively correlated, then a high* v_m *is not as great a restriction on the searcher's problem: the probability that an item arrives tomorrow with a v-value that exceeds* v_m *does not change, but it is more likely that the high v-value is accompanied by a high u-value. Thus, the searcher's continuation value at any given* v_m *goes up, including at* $v_m = v^*$*. If, instead, the probability that the searcher successfully hides an item increases,* *then if the searcher fails to hide an item with a high v_m, the high v_m is exactly as great a restriction on the searcher's continuation problem as in the original model (i.e., where $p = 0$).*

3.7 Conclusion

I study a sequential search problem where the preferences according to which the final item is chosen differ from the preferences according to which search is conducted. A natural interpretation of this set-up is that the preferences belong to separate parties: a searcher and a chooser. Alternatively, the preferences of an individual change between the search and choice stages. I show that the optimal policy of the searcher is a cutoff rule and that the cutoff depends on the items that the searcher has found so far. Due to this dependence, the search behaviour features the discouragement effect: the cutoff decreases weakly in time. The cutoff is characterised in detail in Section 3.5. The characteristics of my model differ from two single-agent search models that feature a time-varying cutoff (convex search costs or deadline). In particular, my model features a cutoff that decreases endogenously over time and never generates return to an item rejected earlier, in contrast to the other models.

I interpret some of the results in the context of the multi-selves example presented in Introduction. The cutoff decreasing in v_m means that a person who has an investment opportunity readily available that involves minimal paperwork (i.e., if the outside option for the chooser is high) optimally does not even attempt to look for an opportunity with a higher return as he knows he will choose the minimal-paperwork option when the time to invest arrives. A person who is intimidated by paperwork optimally stops searching at an opportunity that offers a lower return (but little paperwork) than a person who likes paperwork: the former is less picky. He is less picky not because he cares less about returns, but because he anticipates that he chooses a minimal paperwork option when investing.

The model can be extended in several ways. First, if the model is limited to the principal-agent set-up it is reasonable to think that the principal (the chooser) has a direct influence on the agent's (the searcher's) search process. The possibilities of extending the model are rich due to the many possible assumptions that can be made on the action space and commitment power of the principal.

For example, the principal's optimal restrictions on the length of search are interesting to study because these are prevalent in real life. I have derived partial results on the principal's optimal minimal search duration restriction, $\bar{t}$ (equivalently in this model, on the principal's optimal restriction on the minimal number of options that she requires the agent to inspect). The agent's optimal policy is not affected by the minimal search duration restriction because after $\bar{t}$ periods have passed, the agent's problem looks exactly like for the searcher who does not face time restrictions (as in the main part of this Chapter).

The principal's optimal minimal search duration restriction is more complicated to derive. In a two-period model, I have shown for the same class of joint distributions as in Section 3.5.5 that the principal's optimal minimal search duration restriction is shorter if the agent's and principal's utilities are more positively correlated. I have not been able to derive analytic results for the infinite-horizon model because the functional form of the searcher's optimal cutoff is complicated even for this simple class of joint distributions. However, I expect the result of the two-period model to generalise because of the following. The minimal search duration restriction helps the principal to prevent the agent to stop searching too early. The agent is more likely to stop too early from the principal's viewpoint if he inspects an item that has a low value for the principal and a high value for the agent. Such items are more likely to occur if the agent's and principal's utilities are more negatively correlated, thus lengthening the minimal search duration restriction that is optimal for the principal.

Results on more general time restrictions are complex to derive because a maximal search duration restriction (a deadline) means that the agent's cutoff depends directly on the amount of time available before the deadline. In fact, it is unclear that the optimal restriction in the time dimension is a connected interval.

Second, the model in this paper is very general with respect to the joint distribution of utilities that is considered. If the model is restricted to some application, an application-motivated simplifying restriction on the joint distribution would allow for a more detailed analysis of various aspects of the optimal cutoff. Third, my model can be enriched by deriving the chooser's outside option from the model by considering many searchers competing for the chooser. I conjecture that a robust equilibrium is such where each searcher reports to the chooser as soon as he finds the first item. Fourth, my model provides a natural framework in which to think about issues related to naivete and sophistication in a search framework. This forms part of my planned future work.

Chapter 4

Search with Mistakes

4.1 Introduction

The standard sequential search model with recall assumes that the person who searches always succeeds in choosing the item that she intends to choose from the items she has inspected, never erring. I modify the standard model by allowing the person to make mistakes: she mostly chooses the item she intends to choose, but sometimes makes a mistake and chooses another item. A person may choose an unintended item for one or more of several reasons. The person may choose an item she knows to be inferior because she trembles or because she is inattentive. She may choose an inferior item because she cannot determine the items' utilities exactly or because her preferences change between the time she assessed the items' utilities and the time the utility of the final choice is realised. I derive and characterise the optimal choice and stopping rules of the person who makes choice mistakes.

The model that I use is the following. A decision-maker (DM) inspects items over discrete time. An item is either a low- or a high-utility item. In each period, the DM pays a constant search cost and decides whether to stop or continue to search. If she continues, the next period arrives and she inspects another item. If she stops, she attempts to choose an item from amongst all the inspected items. She chooses the intended item with probability $1 - \varepsilon$ and chooses uniformly among the other inspected items, i.e., makes a mistake, with probability ε.[1] By assumption, the probability of making a mistake is less than a half, so the DM optimally attempts to choose the utility-maximising item. Given the optimal choice rule, the DM's optimal policy is fully characterised by her optimal stopping rule.

I show, first, that the DM's optimal stopping rule is a cutoff rule and the cutoff is history-dependent. As usual, the DM's optimal cutoff equals her expected value from continuing to search. I argue that the continuation value can both increase and decrease in time. The DM's continuation value is positively related to her stopping value because she receives positive utility only when she stops the search process. When she stops, she optimally intends to choose the utility-maximising inspected item, but may make a mistake and choose any other inspected item. The utility from the utility-maximising inspected item weakly increases in time, but the utility from making a mistake can both increase and decrease. Thus, the DM's stopping value can both increase and decrease in time and the optimal cutoff inherits these traits.

Second, I show that under some parameter values the erring DM stops searching later than an unerring DM, i.e., the erring DM prefers larger choice sets than the unerring DM. This result is in contrast to the intuition that an erring DM prefers to simplify her choice as compared to the unerring DM by having fewer items to choose from. The erring DM prefers larger choice sets because she wants to insure herself against her mistakes.

Third, I show that the DM's behaviour can exhibit regret. The DM's

[1] An alternative interpretation of the choice process is akin to the two-agent or multi-selves interpretation as in Chapter 3: the DM chooses from the choice set with probability $1 - \varepsilon$ and someone else, whose preferences the DM does not know, with probability ε. I maintain the erring single-agent interpretation throughout the paper and in Section 4.5 explain how the results in my model differ from those in Chapter 3.

behaviour exhibits regret if she stops when her expected value from stopping is lower than it was in the past when she chose to continue. Regret occurs if the DM receives low-utility items that lower both her optimal cutoff and stopping value, but in a way that makes stopping optimal.

Finally, I explain how the characteristics of the DM's optimal behaviour in my model differ from an unerring DM's optimal behaviour in some other extensions to the sequential search model that generate a history-dependent cutoff. Such extensions are, for example, convex search costs, a deadline, and a deadline together with either uncertain or costly recall (see Stiglitz (1987), Gronau (1971) and Lippman and McCall (1976), Akın and Platt (2014), and Janssen and Parakhonyak (2014) respectively). The unique feature of my model as compared to these is that, for a fixed utility-maximising inspected item, the optimal cutoff can increase over time in my model, while the cutoff weakly decreases over time in the other models. The increase is possible in my model because a new item with a relatively high utility increases the DM's expected stopping value via the mistake utility. Regret cannot be generated by extensions of the standard sequential search model where the DM can always pick the item that she intends to from amongst the inspected items. Thus, neither convex search costs nor a deadline alone generate regret. A deadline and either uncertain or costly recall together can generate regret: regret occurs if an item that was unattractive at the start of search becomes attractive if the deadline is close, but is no longer available or is costly to retrieve.

Literature. My model is related to the well-established literature on stochastic choice. I classify the best-known models into three groups following Loomes et al. (2002). The first group contains a model where the mistake occurs in the choice stage. In Harless and Camerer (1994) a DM makes the utility-maximising choice with probability $1 - \varepsilon$ and makes a mistakes with probability ε. As Harless and Camerer (1994) study choices across pairs of items, they do not specify how a mistake is made among three or more items. I take the stance that the mistake is a true tremble: if the DM does not pick the best item for her, she is equally likely to choose among all other items. The second group contains models where the mistake occurs in the stage of calculating items' utilities, such as Fechner (1860/1948), Marschak (1960), and Block and Marschak (1960).[2] The third group contains models where where the mistake occurs in the stage of specifying the utility function, for example, Becker et al. (1963), Loomes and Sugden (1995), and Gul and Pesendorfer (2006). In these models, the DM's utility function is either one of multiple functions or contains a random element.

In another branch of related literature, stochastic choice is the result of a constrained or non-standard optimisation problem. For example, in Smith and Walker (1993), Mattsson and Weibull (2002), Caplin and Dean (2015), Matějka and McKay (2015), and Cheremukhin et al. (2015) stochastic choice arises because information acquisition is costly. In Masatlioglu et al. (2012) and Manzini and Mariotti (2014) the DM does not consider each alternative in the choice set. In Swait and Marley (2013) the DM maximises two separate goals. In Payró and Ülkü (2015) the DM sometimes chooses an item that is similar to the utility-maximising item (where similarity can be defined in terms of utility). In Koida (2015) the DM's choice depends on her previously chosen mood. These models study static choice behaviour, whereas I analyse how stochastic choice affects the optimal dynamic search behaviour of a DM. In Chapter 3 of this thesis the DM is not the one who makes the final choice from the choice set, but knows the preferences of the agent who makes the final choice. An interpretation of my model is that the DM is the one who makes

[2]Fechner (1860/1948) proposes that a separable error term enters the DM's estimate of utility differences between items, whereas Marschak (1960) and Block and Marschak (1960) propose that the error term enters the estimate of an individual item's utility (the latter is known as the random utility model). The well-known Luce (1959) model is equivalent to the random utility model.

the final choice with probability $1 - \varepsilon$ and does not know the preferences of the agent who makes the final choice with the rest of the probability.

Most search models assume that the searching DM makes no mistakes. An exception is Caplin et al. (2011), where the DM observes her utility from an item with an error term (as in Fechner (1860/1948)). The DM's optimal policy is to search as in the standard model (see, e.g., McCall (1970)), except that she applies the optimal cutoff to the observed (not true) utility of an item. In my model, the DM can calculate an item's utility, but trembles when choosing (as in Harless and Camerer (1994)), which results in an optimal policy for the DM that differs from the standard model.

The connection between regret and stochastic choice has been studied in the past. Regret is experienced if the actual choice is inferior to a potential choice.[3] Loomes and Sugden (1982) and Bell (1985) study the static choice of a DM who experiences regret, Irons and Hepburn (2007) study the effect of regret preferences on optimal search behaviour, and Strack and Viefers (2015) on dynamic decisions more generally. In these models, regret is the primitive of the models and choice mistakes a result of regret. In my model, regret is a result and choice mistakes the primitive of the model.

The rest of the paper is structured as follows. The model is described in Section 4.2. A reminder of the standard sequential search model is in Section 4.3. I derive the optimal stopping rule of a DM who makes mistakes in Section 4.4 and characterise it in Section 4.5. Section 4.6 concludes.

[3] I introduce the definitions of regret used by different authors in more detail in Section 4.5.3.

4.2 Model

A decision-maker searches for a good item over time. In each period of time $t = 1, 2, ...$ she inspects one item x_t. The DM gets utility $x_t \in \{L, H\}$, $\infty > H > L \geq 0$, if she chooses this item.[4] The probability that an item is of high utility is $P(x_t = H) = p \in (0, 1)$. Search costs $c > 0$ per period. Search is with recall: if the DM stops the process at time t, she can (try to) choose any of the items that she inspected at t or earlier. Utility is realised after the choice of an item. The choice set at t, X_t, consists of all the inspected items up to period t: $X_t = \{x_1, x_2, ..., x_t\}$. The DM's problem is to decide when to stop the search process and which item from the choice set to (try to) choose.

The DM makes mistakes when choosing. If she stops the search process at t and intends to choose some item from the choice set, she succeeds with probability $1-\varepsilon \geq \frac{1}{2}$. With probability ε the DM makes a mistake and chooses any of the other items in the choice set with an equal probability. I assume that the DM makes no mistake if she only has one item in the choice set. At any time t, let m_t denote (the utility of) the utility-maximising item in the choice set and n_t the expected utility of making the mistake. The optimal choice rule for the DM is to try to choose m_t because she is more likely to succeed than fail ($1 - \varepsilon \geq \varepsilon$). I characterise her optimal stopping rule in the next two sections.

4.3 Benchmark: no mistakes

This section contains a brief reminder of the DM's optimal policy in the standard sequential search model with recall, as in McCall (1970). In each period

[4] In what follows, I let H to stand for an item x_t with utility H for brevity. The model's results continue to hold if x_t can take more than two values. I make the binary support assumption to simplify some proofs.

t, the DM decides whether to stop the search process and choose the best item she has inspected or to continue to search. She stops if the best inspected item's utility m_t exceeds the optimal cutoff level of utility, $\tilde{x}$, which is equal to her expected value from continuing, $\tilde{V}$. The cutoff $\tilde{x}$ is stationary because the environment is stationary. I derive the optimal cutoff $\tilde{x}$.

Suppose that the DM's optimal policy is to accept the first H and continue if she uncovers only Ls.[5] Then her expected continuation value is

$$\tilde{V} = pH + (1-p)\tilde{V} - c.$$

With probability p, the item inspected next is H, in which case the DM stops and gets utility H. With probability $1-p$, the next item is L, in which case she continues and gets the same expected continuation value as at the start of today, $\tilde{V}$. She always pays the search cost c. As the optimal cutoff equals the continuation value, the closed form for the optimal cutoff is

$$\tilde{x} = H - \frac{c}{p},$$

which decreases in the cost of search and increases both in the utility value of H and in the probability that an H arrives.

The assumption that guarantees that accepting the first H and continuing after Ls is optimal is

Assumption 1. *The parameter values satisfy $H - L > \frac{c}{p}$ so that the DM who makes no mistakes optimally continues if her value from stopping is L.*

I consider parameter values that satisfy Assumption 1 throughout the paper.

I summarise the DM's optimal policy in Lemma 1.

[5] I make an assumption on the parameter values below that guarantees that this is the optimal policy because other parameter values trivialise the searcher's problem.

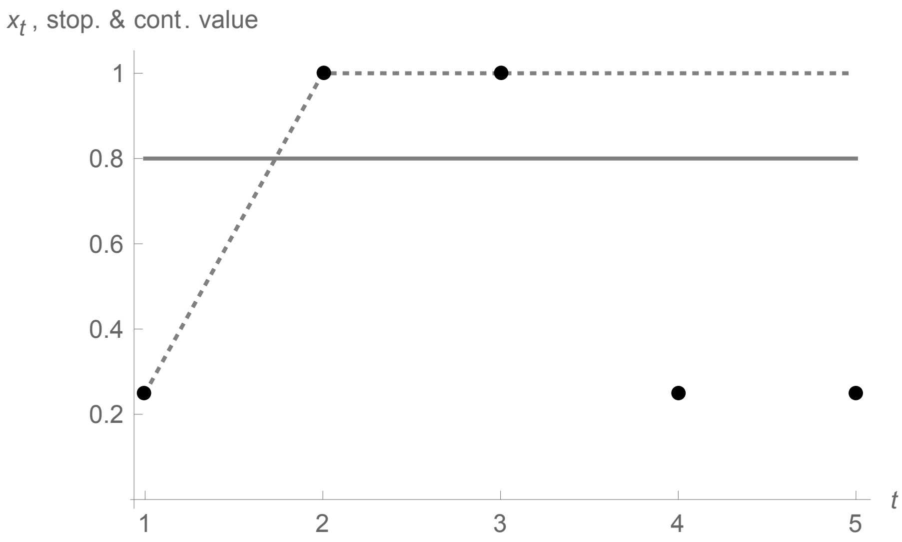

Figure 4.1: Stopping value and optimal cutoff on an unerring DM as a function of time for $c = \frac{1}{10}$, $p = \frac{1}{2}$, $H = 1$, $L = \frac{1}{4}$, and $(x_1, x_2, x_3, x_4, x_5) = (L, H, H, L, L)$. Solid line: cutoff $\tilde{x}$; dotted line: stopping value; dots: x_t.

Lemma 7. *The optimal policy of a DM who makes no mistakes is to stop searching at the earliest time T when her stopping value m_T exceeds the cutoff $\tilde{x} = H - \frac{c}{p}$.*

A representative sequence of events is depicted in Figure 4.1. The DM's optimal cutoff is $\tilde{x} = \frac{8}{10}$, she receives an $L = \frac{1}{4}$ in the first period and an $H = 1$ in the second period. She stops in the second period because her stopping value exceeds her continuation value. If she continued instead, her stopping value would remain unchanged at H in all the following periods because she is always able to choose an H from the choice set. I call a DM who makes no mistakes an *unerring DM.*

4.4 Optimal policy

A decision-maker who makes choice mistakes must in each period decide whether to stop the search process and try to pick the best item she has found so far or continue to search. I present the optimal policy of the erring DM in the below Proposition and prove its optimality thereafter. The optimal policy is characterised in Section 4.5.

Proposition 14. *The optimal policy of the DM who makes mistakes is to stop searching at the earliest time T when her stopping value $(1-\varepsilon)m_T + \varepsilon n_T$ exceeds the cutoff $\bar{x}(m_T, n_T, T)$.*

As in the standard search model, in each period t the DM must compare her expected value from stopping the search process to her expected value from continuing with the process. Unlike in the standard model, the DM's expected value from stopping is not the utility of the best item found so far, m_t, but is an average between the best item's and mistake utilities:

$$(1-\varepsilon)m_t + \varepsilon n_t. \tag{4.1}$$

The mistake utility at t, n_t, is an average of the utilities of all the items in the choice set, except the best one:

$$n_t = \frac{\sum_{x \in X_t \setminus m_t} x}{t-1}. \tag{4.2}$$

The erring DM's stopping value can develop differently from the unerring DM's, as illustrated in Figure 4.2. In particular, the stopping value of an unerring DM never decreases in time, whereas the the value of an erring DM can decrease. Also, the the stopping value of an unerring DM depends on the best item's utility only, whereas the value of an erring DM depends on all the items in the choice set.

Let the DM's expected continuation value at time t as a function of the best item's utility, mistake utility, and time be denoted by $V(m_t, n_t, t)$.[6] I argue

[6] I could equivalently express the continuation value as a function of the number of inspected Hs and Ls, but choose m_t, n_t, and t as state variables to show that the results

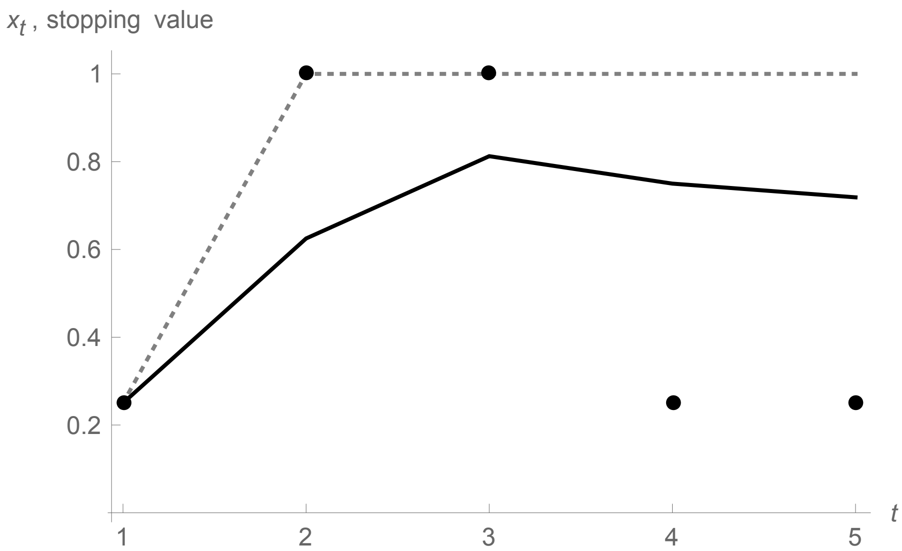

Figure 4.2: Stopping value as a function of time for $\varepsilon = 0.5$, $H = 1$, $L = \frac{1}{4}$, and $(x_1, x_2, x_3, x_4, x_5) = (L, H, H, L, L)$. Dotted line: unerring agent; solid line: erring agent; dots: x_t.

that the continuation value depends on the choice set only via the best item's utility m_t, mistake utility n_t, and date t. The continuation value depends on m_t, n_t, and t because the DM's stopping value tomorrow depends on all three. The stopping value tomorrow is fully determined by m_{t+1} and n_{t+1} (see equations (4.1) and (4.2)). The best item's utility tomorrow, m_{t+1}, is fully determined by m_t and the new item's utility x_{t+1} as $m_{t+1} = \max\{m_t, x_{t+1}\}$. The mistake utility tomorrow, n_{t+1}, is fully determined by the mistake utility today, the best item's utility today, the new item's utility, and the time period as $n_{t+1} = \frac{(t-1)n_t + x_{t+1}}{t}$ if $x_{t+1} < m_t$ and $n_{t+1} = \frac{(t-1)n_t + m_t}{t}$ if $x_{t+1} \geq m_t$. The continuation value depends on no other information, first, because the search cost and probability of inspecting an H are constant and, second, because only the DM's stopping value tomorrow (and at future dates) matters as she gets positive utility only when she stops. Altogether, at any date t the DM compares her stopping value $(1 - \varepsilon)m_t + \varepsilon n_t$ to the continuation value $V(m_t, n_t, t)$.

extend to settings where items' utilities can take more than two values.

The continuation value depends on the entire choice set via m_t, n_t, and t, whereas in Chapter 3 the continuation value depends on a single item in the choice set.

The continuation value can in principle be derived from the following Bellman equation. At any date t, for any utility of the best item $m_t = m \in \{H, L\}$ and any mistake utility $n_t = n \in [L, H]$, the Bellman equation is

$$V(m,n,t) = p \max\left\{V(H, \frac{(t-1)n+m}{t}, t+1), (1-\varepsilon)H + \varepsilon\frac{(t-1)n+m}{t}\right\}$$

$$+(1-p)\max\left\{V(m, \frac{(t-1)n+L}{t}, t+1), (1-\varepsilon)m + \varepsilon\frac{(t-1)n+L}{t}\right\} - c. \tag{4.3}$$

If an H arrives at $t+1$, the best item in the choice set at $t+1$ is H. If an L arrives at $t+1$, the best item in the choice set at $t+1$ remains m. The new item x changes the mistake value to $\frac{(t-1)n+m}{t}$ if $x = H$ and to $\frac{(t-1)n+L}{t}$ if $x = L$. Both the stopping and continuation values change accordingly. Search cost c has to be paid per period.

The DM optimally stops at t if her expected stopping value $(1-\varepsilon)m_t + \varepsilon n_t$ exceeds her continuation value $V(m_t, n_t, t)$. Thus, the optimal policy is a cutoff rule with a cutoff $\bar{x}(m_t, n_t, t)$ that is equal to the continuation value. The optimal cutoff decreases in the cost of search c and in the mistake probability ε because the stopping value decreases in both. An explicit form for the optimal cutoff $\bar{x}(m_t, n_t, t)$ does not exist because equation (4.3) does not have a closed form solution: at each date the arguments of the value function take on new values. However, aspects of the optimal behaviour can be characterised, which I do in the next section.

The DM who makes mistakes never wants to stop when she is sure to get utility L if

$$V(L, L, t) = p\max\{V(H, L, t+1), (1-\varepsilon)H + \varepsilon L\}$$

$$+(1-p)\max\{V(L, L, t+1), L\} - c > L,$$

holds. The condition is guaranteed to hold if $(1-\varepsilon)(H-L) > \frac{c}{p}$ as the continuation value after only Ls can be approximated down to

$$V(L, L, t) \geq p[(1-\varepsilon)H + \varepsilon L] + (1-p)L - c.$$

I formalise the assumption below.

Assumption 2. *The parameter values satisfy* $(1-\varepsilon)(H-L) > \frac{c}{p}$ *so that the DM who makes mistakes optimally continues if her expected value from stopping is* L.

Assumption 1 is guaranteed to hold if Assumption 2 holds. A DM who makes mistakes prefers continuing to stopping and receiving utility L if her expected gain from waiting for an H, conditional on being able to pick that item, exceeds the expected cost. A DM who does not make mistakes knows that she is able to pick an H once it arrives. Thus, if the stopping value is L, it is easier to provide incentives to continue to an unerring than an erring DM.

4.5 Characterisation

In this section I show that the erring DM's optimal cutoff is history-dependent, her optimal behaviour may exhibit regret, and that for some parameter values she searches longer than an unerring DM.

4.5.1 History-dependent cutoff

I prove in Proposition 15 that as a function of time, the erring DM's continuation value, hence, her optimal cutoff can both decrease and increase. The history-dependent cutoff used by an erring DM is in contrast to the stationary cutoff used by an unerring DM. In order to prove Proposition 15, I first establish that for a fixed time period t, the DM's continuation value increases in both the utility of the best item, m_t, and in the mistake utility, n_t.

Lemma 8. *For fixed m (n) and t, the continuation value $V(m,n,t)$ is increasing in n (m).*

Proof. Consider period t with $m_t = m$ and $n_t = n$. The DM's expected stopping value $(1-\varepsilon)m+\varepsilon n$ unambiguously increases in m and n. Recall that the DM's continuation value $V(m,n,t)$ is given by equation (4.3). I argue that the continuation value increases in n for a fixed m and t, and increases in m for a fixed n and t. The DM receives positive utility from the search process only after stopping at some date. Thus, the DM's continuation value is positively related to future stopping values. But future stopping values unambiguously increase in m (for a fixed n and t) and n (for a fixed m and t). Hence, the continuation value increases in n for a fixed m and t, and increases in m for a fixed n and t. □

For any given t, the continuation value of the DM must increase in both of its other arguments, the best item's utility and the mistake utility. The reason is that the DM cares only about the utility that she gets from eventually stopping, which increases in both. We are ready to prove the main result of this subsection.

Proposition 15. *As a function of time, the DM's cutoff $\bar{x}(m,n,t)$ can both increase and decrease.*

Proof. (a) A decrease is possible: I argue that for some t large enough, if $m = H$ and $n = L$, the continuation value decreases in t.

First, I show that if $m = H$ and $n = L$, then for t' large enough, the DM prefers to stop, i.e., that for t' large enough, $V(H,L,t') < (1-\varepsilon)H + \varepsilon L$. The continuation value at a large $t = t'$ is

$$V(H, L, t') = p \max\{V(H, \frac{(t'-1)L + H}{t'}, t'+1), (1-\varepsilon)H + \varepsilon\frac{(t'-1)L + H}{t'}\}$$

$$+(1-p)\max\{V(H, L, t'+1), (1-\varepsilon)H + \varepsilon L\} - c$$

$$\approx p\max\{V(H, L, t'+1), (1-\varepsilon)H+\varepsilon L\}+(1-p)\max\{V(H, L, t'+1), (1-\varepsilon)H+\varepsilon L\}-c$$

$$\approx \max\{V(H, L, t'), (1-\varepsilon)H + \varepsilon L\} - c,$$

where the first approximation is true for large enough t' because an additional H does not change the mistake utility much if t' is large, which in turn implies the second approximation: the date does not influence $V(H, L, t')$ for large enough t'. But then it must be optimal for the DM to stop at t' because the stopping value $(1-\varepsilon)H + \varepsilon L$ exceeds the continuation value $V(H, L, t')$. In the preceding period, $t'-1$, the continuation value is

$$V(H, L, t'-1) = p\max\{V(H, \frac{(t'-2)L + H}{t'-1}, t'), (1-\varepsilon)H + \varepsilon\frac{(t'-2)L + H}{t'-1}\}$$

$$+(1-p)[(1-\varepsilon)H + \varepsilon L] - c > (1-\varepsilon)H + \varepsilon L - c \approx V(H, L, t').$$

That is, for $m = H$ and $n = L$ the continuation value decreases in t at large enough t.

(b) An increase is possible: I prove by example that it is possible that the continuation value increases in t.

Assume that the DM wants to continue after the sequence $(x_1 = L, x_2 = H)$, or that $V(H, L, 2) > (1-\varepsilon)H + \varepsilon L$. A sufficient condition for the assumption to hold is that $\frac{\varepsilon}{2}(H - L) > \frac{c}{p}$, because

$$V(H, L, 2) \geq p[(1-\varepsilon)H + \varepsilon\frac{L + H}{2}] + (1-p)[(1-\varepsilon)H + \varepsilon L] - c.$$

I show that under this assumption, the DM's continuation value can increase at $t = 2$ after $x_1 = L$, i.e., that $V(H, L, 2) > V(L, L, 1)$. The continuation value after $x_1 = L$ is

$$V(L, L, 1) = p\max\{V(H, L, 2), (1-\varepsilon)H + \varepsilon L\} + (1-p)\max\{V(L, L, 2), L\} - c.$$

I approximate the continuation value up:

$$V(L, L, 1) < p\max\{V(H, L, 2), (1-\varepsilon)H + \varepsilon L\} + (1-p)\max\{V(H, L, 2), L\} - c$$
$$= V(H, L, 2) - c,$$

where the inequality follows from Lemma 8 and the equality from the assumption that $V(H, L, 2) > (1 - \varepsilon)H + \varepsilon L$. Thus, the continuation value increases in t for some realisations of the items' values. □

Lemma 8 shows that at any given time, the DM's continuation value increases in both the best item's and the mistake utilities. The utility of the best item in the choice set, m_t, can only weakly increase in time because it is the tth statistic in the choice set at t: for any m_t, a new item has either weakly lower value than m_t, in which case $m_{t+1} = m_t$, or higher value than m_t, in which case $m_{t+1} > m_t$. But the utility from making a mistake, n_t, can both increase and decrease in time: it increases after the arrival of an item with value $x \in (n_t, m_t)$ and decreases after the arrival of an item with value $x < n_t$. Because of these changes, the stopping value can either increase (after an increase in m_t or n_t) or decrease (after a decrease in n_t). Thus, the DM's continuation value and optimal cutoff can both increase and decrease in time.[7] Note that an increase in the cutoff is also possible for a fixed best item in the choice set, m_t: the increase happens due to an increase in the mistake utility

[7] In Chapter 3, the optimal cutoff can only decrease in time.

n_t. I show in Section 4.5.4 that this feature distinguishes my model from some other search models that feature a history-dependent cutoff.

As a result, it is conceivable that for a given (m_t, n_t), the DM stops at $t+1$ both if her cutoff increases and decreases. That is a direct implication of the result that the DM's stopping and continuation values move in the same direction, for all values of the new item x_{t+1}. That is, the DM's behaviour can potentially exhibit endogenous search fatigue.

4.5.2 Longer search

I prove that an unerring DM sometimes searches longer, i.e., prefers larger choice sets than an unerring DM, in contrast to the intuition that the erring DM prefers simpler choice situations.

Proposition 16. *Under some parameter values, histories exist such that the erring DM stops searching later than an unerring DM.*

Proof. I first argue that the erring DM can stop strictly later than an unerring DM. Suppose that both the erring and an unerring agents receive the sequence $(x_1 = L, x_2 = H)$. At $t = 1$, both agents continue under Assumption 2 on the parameter values. At $t = 2$, the unerring DM stops, whereas the erring DM continues if $\frac{\varepsilon}{2}(H - L) > \frac{c}{p}$ (see the proof of Proposition 15). Thus, the unerring DM stops earlier than the erring DM.[8] □

Under Assumption 2, the erring DM prefers (weakly) larger choice sets than an unerring DM because she wants to insure herself against her mistakes. If the mistake probability is high, then it is very risky for the DM to stop and try to pick an H if there are only a few Hs and many Ls in the choice

[8]Note that if Assumption 2 does not hold, but Assumption 1 does, i.e., if $H - L \in (\frac{c}{p}, \frac{c}{p(1-\varepsilon)})$, then the erring DM stops at $t = 1$ with probability one, while the unerring DM stops at $t = 1$ only if she uncovers an H.

set. If the probability of an H arriving is large enough, the DM prefers to continue searching in order to improve her mistake utility and insure herself against trembles.[9] Note that the intuition behind this result is similar to that in the directed search model of Galenianos and Kircher (2009), where searching agents can visit multiple firms. In particular, in Galenianos and Kircher (2009) an agent visits both a low- and high-priced firm that in equilibrium serve the agent with a low and a high probability respectively. She visits the low-priced firm to try to get a good deal and insures herself against the possibility of not being able to buy at the low-priced firm by also visiting a high-priced firm.

4.5.3 Regret

I show that an erring DM may regret her earlier decision to continue. I define regret analogously to the definition of regret over past decisions used in Strack and Viefers (2015): the DM experiences regret when her utility from stopping is lower than it was any time in the past.

Definition 2 (Regret)**.** *The DM's behaviour exhibits regret when stopping in period T if her expected stopping value was higher any time in the past, i.e., if* $\max_{t<T}(1-\varepsilon)m_t + \varepsilon n_t > (1-\varepsilon)m_T + \varepsilon n_T$.

In Loomes and Sugden (1982) and Bell (1985) the DM experiences regret if the realised payoff from an (action) choice is lower than from a potential (action) choice. The erring DM clearly experiences regret in this sense because she makes mistakes. In Irons and Hepburn (2007) the DM experiences regret if the maximal expected utility from searching the entire set of items is higher than when stopping. The erring DM clearly may experience regret in this sense (if the items' utilities can take more than two values or if Assumption 2 fails)

[9] In Chapter 3, the searching agent's cutoff is below the standard model's cutoff so he never stops searching later than in the standard model.

because she uses a cutoff that is below the maximal potential utility. I thus adopt the above notion of regret which is analogous to that used in Strack and Viefers (2015).

Proposition 17. *The erring DM's optimal behaviour can exhibit regret.*

Proof. Suppose that the erring DM started with the sequence (L, H, H), optimally continued, and thereafter received only Ls. Then at $t = 3$ her stopping value was $(1-\varepsilon)H + \varepsilon\frac{L+H}{2}$ but she optimally chose to continue. But at some large $t = t'$, her continuation value is $(1-\varepsilon)H + \varepsilon\frac{(t'-2)L+H}{t'-1}$, which for t' large enough is approximately equal to $(1-\varepsilon)H + \varepsilon L$. Recall from the proof of Proposition 15 that there exists a time t' large enough such that a DM with $m = H$ and $n = L$ wants to stop at t'. Hence, there also exists a time $t'' > t'$ large enough such that a DM with $m = H$ and $n \approx L$ wants to stop at t''. But then the optimal policy of the DM who sees the sequence $(L, H, H, L, L, L, ...)$ and stops at t'' exhibits regret. □

Suppose that the erring DM inspects some Hs, but not early enough to warrant stopping. If she then receives multiple Ls in a row, her stopping value decreases, but so does her continuation value. If the DM receives enough Ls, she stops, despite her stopping value being lower than before she inspected the sequence of Ls. The erring DM can experience regret because her stopping and continuation values move in the same direction: a bad item makes both stopping and continuing less valuable. For an unerring DM, the continuation value is constant so the DM never regrets having continued in the past.[10]

[10]In Chapter 3, the searching agent can experience regret because he is not in charge of the final choice.

4.5.4 Other models with a history-dependent cutoff

I explain how the characteristics of the DM's optimal behaviour in my model differ from an unerring DM's optimal behaviour in some other search models that generate a history-dependent cutoff. Such models are, for example, a model with a deadline (see Gronau (1971) and Lippman and McCall (1976); I call it the pure deadline model below), convex search costs (see Stiglitz (1987)), and a combination of a deadline and costly or uncertain recall (see Janssen and Parakhonyak (2014) and Akın and Platt (2014) respectively). I show that the unique feature of my model is that the DM's cutoff can increase in time when the utility from the best item in the choice set remains unchanged.

I first explain how the stopping value changes in time in these models. In the pure deadline and convex search costs' models, the DM's stopping value is the best item's utility found so far (m_t) because returning to an item found earlier is always possible and costless. In the costly recall model, the stopping value is the maximum of the value of the item found today and the best item found earlier minus the recall cost ($\max\{m_{t-1} - b, x_t\}$, where b is the cost of recall). In the uncertain recall model, the stopping value is either the value of the item found today or the best item found so far (x_t or m_t).

These differences in the stopping values produce different outcomes in terms of the optimal cutoff's development in time and regret. In all of the models, the optimal cutoff can decrease in time. In case of a deadline, the DM's continuation value, thus, optimal cutoff, decreases because her opportunities to search further are restricted from some period onwards. In case of convex search costs, the continuation value decreases because observing another item becomes more costly over time. However, in only my model can the cutoff increase in time for a fixed utility-maximising item in the choice set (m_t). For a fixed m_t, an increase in the cutoff happens in my model if the mistake value n_t increases because n_t affects future stopping values, thus, the expected continuation value. But in the other models, the future stopping value depends on the past only via m_t, which increases in time.

Regret can be generated in the models with uncertain or costly return, but not in the pure deadline and convex search costs' models. The pure deadline and convex search costs' models do not generate regret because the DM's stopping value weakly increases in time so whenever she stops, her past stopping value cannot have been higher and thus cannot trigger regret. The uncertain and costly return models can generate regret. If recall is uncertain, regret is triggered when an item that was unacceptable in terms of utility in the past becomes both acceptable (because the cutoff has decreased) and unavailable. If recall is costly, regret is triggered when an item that was unacceptable in terms of utility in the past becomes acceptable (because the cutoff has decreased) since now it costs to retrieve the item. In my model, regret is triggered when the stopping value decreases due to the arrival of low-utility items.

4.6 Conclusion

Economic agents sometimes choose items that they did not intend to choose, either because they are not able to determine which item has the highest utility, their utility assessment changes between the choice and utility realisation phases, they tremble or are inattentive. This paper investigates what happens if a decision-maker (DM) who searches for a good item makes mistakes when choosing. The erring DM's optimal policy is a cutoff rule, as in the standard sequential search model, but unlike in the standard model, the cutoff is history-dependent. The erring DM's cutoff can both increase and decrease in time. The value of the best item in the choice set weakly increases in time, but the value from making a mistake can both increase and decrease. Thus, the DM's stopping value can both increase and decrease in time and the optimal cutoff inherits these traits. The DM may experience regret: regret occurs if she stops the process at some period t, despite the stopping value having been higher in the past, when she continued. A DM who makes mistakes can search longer, or prefer larger choice sets, than a DM who makes no mistakes.

I conclude by discussing the robustness of my model's results to alternative modelling choices.

More possible utilities. If the items' utilities can take on more than two possible values (e.g., $x_t \sim F$ with support $[\underline{x}, \bar{x}]$, $0 < \underline{x} < \bar{x} < \infty$ and $F'(x) > 0$ for all $x \in [\underline{x}, \bar{x}]$), all of the model's results continue to hold.

Other mistake processes. Other plausible mistake processes are such that the probability that the DM chooses a particular item is positively related to its utility. A candidate is, for example, a version of Luce (1959) model where the probability that an item is chosen from a choice set X is proportional to its utility, i.e., $P(x \in X \text{ is chosen}) = \frac{x}{\sum_{y \in X} y}$. This choice process means that the stopping and continuation values, thus the optimal cutoff of the DM, depend on the entire choice set that she has collected. The results of my model are robust to the change.

Mistake processes such that the DM's continuation value does not depend on the entire choice set she has collected are, for example, the DM choosing the second-best item or, more generally, among the n items with utility just below the best, if she fails to choose the best item. These processes would represent an almost-rational DM. However, both of these processes are arbitrary and the analysis thus omitted.

Appendix A

Appendix for Chapter 2

This section contains the proofs and some formal calculations to accompany the informal arguments made in the body of the Chapter.

A.1 Single-buyer problem

Proof of Proposition 1. The explicit form of the cutoff is easily obtained by using equation (2.1) for the buyer's beliefs and the mean qualities in the expression for his continuation value where $q_1 = \bar{\bar{q}}$. The expected quality that the buyer accepts, Eq, is

$$Eq := \sum_s P(s)[(1 - F_s(\bar{\bar{q}}))\mathbb{E}_{F_s}(q|q > \bar{\bar{q}}) + F_s(\bar{\bar{q}})\delta\mathbb{E}_{F_s}(q)]$$

$$= \pi\left[\left(1 - \frac{\bar{\bar{q}}}{a}\right)\frac{\bar{\bar{q}} + a}{2} + \frac{\bar{\bar{q}}}{a}\delta\frac{a}{2}\right] + (1 - \pi)\left[(1 - \bar{\bar{q}})\frac{\bar{\bar{q}} + 1}{2} + \bar{\bar{q}}\delta\frac{1}{2}\right],$$

which, after some manipulation, simplifies to the expression in the body of the Chapter.

To see the comparative statistics' result, take the derivative of the optimal cutoff and the expected accepted quality with respect to the prior odds, ω.

The derivatives are

$$\frac{\partial \bar{\bar{q}}}{\partial \omega} = \frac{\delta a(a-1)}{2(a+\omega)^2},$$

and

$$\frac{\partial Eq}{\partial \omega} = \frac{1}{2a(1+\omega)^2}\{\frac{(a-1)}{2(a+\omega)}[2(a+\omega)(a+\bar{\bar{q}}^2) - \delta a(\omega+1)\bar{\bar{q}}]$$

$$+(\omega+1)[\omega(a\delta - \bar{\bar{q}})\bar{\bar{q}}' + a(\delta - \bar{\bar{q}})\bar{\bar{q}}']\},$$

where $\bar{\bar{q}}' := \frac{\partial \bar{\bar{q}}}{\partial \omega}$. The first derivative is clearly positive because $a > 1$. The second derivative is positive because in the first squared brackets the first term is larger than the second, factor by factor, and in the second squared brackets, the optimal cutoff is less than the discount factor. □

A.2 Equilibrium existence

Proof of Proposition 2. In order to show that an equilibrium in cutoff strategies exists, I need to show that the best response of a buyer to others' using a cutoff strategy $(\bar{q}_B, \bar{q}_G)$ is to use a cutoff strategy (Part a), and that this cutoff strategy is $(\bar{q}_B, \bar{q}_G)$ (Part b). Part c shows uniqueness for the known-state regime.

Part a. A cutoff strategy is a best response to others' using a cutoff strategy $(\bar{q}_B, \bar{q}_G)$ with $\bar{q}_G \geq \bar{q}_B$.

First, I derive the parts of equilibrium that are common for the four regimes. When the common parts end, I show separately for each regime that either a sufficient condition (denoted (SC)) or a necessary and sufficient condition (denoted (NC)) for the best response of a buyer to be a cutoff strategy is satisfied.

Suppose that all young buyers but one, b, use the cutoffs $(\bar{q}_G, \bar{q}_B)$ with $\bar{q}_G \geq \bar{q}_B$ when young. Buyer b who observes quality $q_1 \in [0,1]$ and signal realisation $i \in \{G, B\}$ decides whether to accept q_1 or continue, taking other buyers' actions as given.[1] His best response is to accept q_1 if q_1 exceeds his continuation value, $V(q_1, i)$, and to continue otherwise.[2] A sufficient condition for b's best response to be a cutoff rule is

(SC): $V(q_1, i)$ decreases weakly in q_1 for $i = G, B$.

A necessary and sufficient condition for b's best response to be a cutoff rule is

(NC): for $i = G, B$, there exists a unique $\hat{q}_i$ such that for all $q_1 < \hat{q}_i$, the buyer's continuation value is higher than q_1, or $q_1 < V(q_1, i)$, and for all $q_1' \geq \hat{q}_i$, the value is lower than q_1', or $V(q_1', i) \leq q_1'$.

I show that the sufficient condition (SC) is satisfied for the known-state, no-signals' and trade-signal regimes ((i) to (iii) below) and that the necessary and sufficient condition (NC) is satisfied for the exogenous-signal regime (iv).

The continuation value of b equals the discounted expected value of q_2 (i.e., mean quality) because b accepts any quality when old. After q_1 and signal realisation i, given that others use cutoff strategy $(\bar{q}_G, \bar{q}_B)$, the continuation value is

$$V(q_1, i) = \delta[\pi(q_1, i)\mathbb{E}_{H_\gamma}(q) + (1 - \pi(q_1, i))\mathbb{E}_{H_\beta}(q)],$$

where H_s is the distribution of qualities induced by strategy $(\bar{q}_G, \bar{q}_B)$ in state s (derived below, see (A.2)) and $\pi(q_1, i)$ is b's posterior belief that the state is good after quality q_1 and signal realisation i. As $\mathbb{E}_{H_\gamma}(q) > \mathbb{E}_{H_\beta}(q)$ must hold when all buyers (but one) use a cutoff strategy, the continuation value decreases in q_1 if $\pi(q_1, i)$ decreases in q_1.

[1]Recall that b optimally accepts $q_1 \geq 1$ under all information regimes as $q_1 \geq 1$ exceeds the discounted mean qualities. If others use cutoff strategies, then the mean quality that b faces is smaller than under the entry distribution, $\mathbb{E}_{H_s}(q) < \mathbb{E}_{F_s}(q)$, and $\mathbb{E}_{F_s}(q) \leq 1$ for $s = \gamma, \beta$.

[2]The continuation value could equivalently be expressed as a function of posterior beliefs, but since this would increase the notational burden, I express the value as a function of q_1 and i.

I derive b's beliefs. I focus on the posterior odds instead of the posterior beliefs because the two are equivalent (especially, the posterior increases iff the posterior odds increase), but the odds easier to interpret. A young buyer updates his belief that the true state is good based on the signal realisation i and q_1. After any $q_1 \leq 1$ and realisation i, the buyers' posterior odds are

$$\frac{\pi(q_1, i)}{1 - \pi(q_1, i)} = \omega \frac{h_\gamma(q_1)}{h_\beta(q_1)} \frac{P(i|\gamma)}{P(i|\beta)}. \tag{A.1}$$

Random matching means that the buyer observes q_1 in state s with a probability equal to the equilibrium density of q_1, $h_s(q_1)$. Hence, the odds of observing quality q_1 are $\frac{h_\gamma(q_1)}{h_\beta(q_1)}$. The final term in (A.1) is the odds of observing signal outcome i. For all the regimes except the trade-signal regime, this term is clearly a constant because the precision of the signal is exogenously given. I argue that the term is constant also for the trade-signal regime. Under the trade-signal regime, the probabilities of observing outcomes G and B are determined by the probability that a randomly chosen seller trades, t_s. But the behaviour of all agents but b is fixed in this step and b has measure zero so the odds of observing outcome i are constant in q_1 (t_s is derived formally below). Hence, in order to show that $\pi(q_1, i)$ decreases weakly in q_1, I need to show that $\frac{h_\gamma(q_1)}{h_\beta(q_1)}$ decreases weakly in q_1 (or that the odds of q_1 do not affect the posterior).

I derive the equilibrium distribution of sellers to determine whether the odds of q_1, $\frac{h_\gamma(q_1)}{h_\beta(q_1)}$, weakly decrease in q_1. An intuitive way to derive $h_s(q)$ is to impose the stationarity condition: the distribution of sellers is independent of the time period only if the inflow of sellers of a certain quality equals the outflow of sellers of that quality. For any q, an amount $f_s(q)$ of sellers with quality q enter. If buyers use the cutoff strategy $(\bar{q}_B, \bar{q}_G)$ with $\bar{q}_B \leq \bar{q}_G$, a seller of quality $q < \bar{q}_B$ trades with a buyer only if he meets an old buyer. In equilibrium, there is a fraction $h_s(q)$ of these sellers in the market and because of random matching, each is equally likely to meet any buyer. The probability of meeting an old buyer is $\frac{O_s}{1+O_s}$ (where O_s is the amount of old buyers in state s) as the total number of buyers (and sellers) in the market is $1+O_s$. The total number of sellers with quality q in the market is $h_s(q)(1+O_s)$. Altogether, an amount $h_s(q)O_s$ sellers with quality $q < \bar{q}_B$ trade in each period. Hence, the equilibrium density for sellers with quality $q < \bar{q}_B$ is determined by the equality $f_s(q) = h_s(q)O_s$. A seller of $q \in [\bar{q}_B, \bar{q}_G)$ trades with an old buyer and a young buyer who has observed the bad signal outcome. By a similar argument as in the previous case, for these sellers the equilibrium density is determined by the equality $f_s(q) = h_s(q)(O_s + P(B|s))$. A seller of quality $q \geq \bar{q}_G$ trades with any buyer so the equilibrium density is determined by the equality $f_s(q) = h_s(q)(O_s + 1)$. In sum, the equilibrium density of qualities in state s, $h_s(q)$, is

$$h_s(q) = \begin{cases} f_s(q)O_s^{-1} & \text{if } q < \bar{q}_B, \\ f_s(q)(O_s + P(B|s))^{-1} & \text{if } q \in [\bar{q}_B, \bar{q}_G), \\ f_s(q)(O_s + 1)^{-1} & \text{if } q \geq \bar{q}_G. \end{cases} \tag{A.2}$$

The equilibrium density is a piecewise constant weakly decreasing function in each state as $O_s^{-1} \geq (O_s + P(B|s))^{-1} \geq (O_s + 1)^{-1}$. In equilibrium, qualities lower than a cutoff are overrepresented as compared to the entry distribution: buyers impose a negative payoff externality on others.

Now I show separately for the known-state, no-signals', and trade-signal regimes that the odds of quality q_1 weakly decrease in q_1 or that they do not matter for the posterior odds.

(i) The posterior odds of b are constant under the known-state regime, regardless of the odds of q_1: in (A.1), the final term is equal to either zero or plus infinity, hence, the posterior odds are constant in q_1. The sufficient condition (SC) is satisfied for the known-state regime: b's continuation value is weakly decreasing in q_1 as it is independent of q_1. Let $\bar{q}_\gamma$ denote the cutoff used in the good state and $\bar{q}_\beta$ the cutoff used in the bad state under the known-state regime.

(ii) Under the no-signals' regime, a single cutoff is used ($\bar{q}_B = \bar{q}_G =: \bar{q}$) because the signal is perfectly uninformative. Hence, in state s the amount of old buyers, $O_s^{\mathcal{N}}$, is derived by integrating $h_s(q)$ from zero to $\bar{q}$ and solving for $O_s^{\mathcal{N}}$:

$$O_s^{\mathcal{N}} = \int_0^{\bar{q}} h_s^{\mathcal{N}}(q) \, \mathrm{d}q = F_s(\bar{q}) \frac{1}{O_s^{\mathcal{N}}},$$

which gives,

$$O_s^{\mathcal{N}} = \sqrt{F_s(\bar{q})}. \tag{A.3}$$

Note that $O_\gamma^{\mathcal{N}} < O_\beta^{\mathcal{N}}$ as $a > 1$.

Using (A.2), we get the odds of q_1 under the no-signals' regime:

$$\frac{h_\gamma^{\mathcal{N}}(q_1)}{h_\beta^{\mathcal{N}}(q_1)} = \begin{cases} \frac{O_\beta^{\mathcal{N}}}{a O_\gamma^{\mathcal{N}}} & \text{if } q_1 < \bar{q}, \\ \frac{1+O_\beta^{\mathcal{N}}}{a(1+O_\gamma^{\mathcal{N}})} & \text{if } q_1 \geq \bar{q}. \end{cases}$$

In order to show that the odds of q_1 weakly decrease in q_1, it is sufficient to check that the odds of $q_1 = 0$ are at least as high as the odds of $q_1 = \bar{q}_N$, or equivalently, that $\frac{O_\beta^{\mathcal{N}}}{a O_\gamma^{\mathcal{N}}} \geq \frac{1+O_\beta^{\mathcal{N}}}{a(1+O_\gamma^{\mathcal{N}})}$. But the inequality holds as $O_\gamma^{\mathcal{N}} < O_\beta^{\mathcal{N}}$. Altogether, b's continuation value decreases (weakly) in q_1 under the no-signals' regime so the sufficient condition (SC) is satisfied.

(iii) Under the trade-signal regime, assume that a trade is good news ($t_\gamma > t_\beta$) and that the cutoff used by buyers who have observed a trade, $\bar{q}_T$, is higher than the cutoff used by buyers who have observed no trade, $\bar{q}_N$ (in terms of the above notation, $\bar{q}_G = \bar{q}_T$ and $\bar{q}_B = \bar{q}_N$). Then the missing probability in the equilibrium distribution of qualities, (A.2), is $P(B|s) = 1 - t_s$, where t_s is the probability of observing a trade. The odds of quality q_1 are, explicitly

$$\frac{h_\gamma^{\mathcal{T}}(q_1)}{h_\beta^{\mathcal{T}}(q_1)} = \begin{cases} \frac{O_\beta^{\mathcal{T}}}{aO_\gamma^{\mathcal{T}}} & \text{if } q_1 < \bar{q}_N, \\ \frac{(O_\beta^{\mathcal{T}}+1-t_\beta)}{a(O_\gamma^{\mathcal{T}}+1-t_\gamma)} & \text{if } q_1 \in [\bar{q}_N, \bar{q}_T), \\ \frac{1+O_\beta^{\mathcal{T}}}{a(1+O_\gamma^{\mathcal{T}})} & \text{if } q_1 \geq \bar{q}_T. \end{cases} \tag{A.4}$$

To get an explicit form of the distribution, I derive the probability of observing a trade and the amount of old buyers.

The probability of observing a trade is the equilibrium probability that a randomly chosen seller trades in a period because a buyer does not observe the quality of the seller whose trade/no trade he observes. The total amount of sellers that trade and exit in a period is one because the market is stationary and the entry amount of sellers is equal to one. The total amount of sellers is equal to the amount of buyers, $1 + O_s$, in state s. Hence, the probability of observing a trade in state s is

$$t_s = \frac{1}{1 + O_s^{\mathcal{T}}}. \tag{A.5}$$

Note that $O_\gamma^{\mathcal{T}} < O_\beta^{\mathcal{T}}$ implies that $t_\gamma > t_\beta$, as assumed above.

The amount of old buyers in the market equals the amount of sellers that are carried from one period to the next. This amount is made up, first, of all the sellers with quality $q < \bar{q}_N$ that have entered the market and have always met young buyers, and, second, of all the sellers with quality $q \in [\bar{q}_N, \bar{q}_T)$ that have entered and have always met young buyers who have observed trades. The probability that a seller who entered j periods ago has always met young buyers is $(1 + O_s)^{-j}$ and that he has always met young buyers who have observed trades is $t_s^j (1 + O_s)^{-j}$. In sum, the amount of sellers that are carried from one period to the next is

$$O_s^{\mathcal{T}} = \sum_{j=1}^{\infty} F_s(\bar{q}_N) \left(\frac{1}{1 + O_s^{\mathcal{T}}} \right)^j + (F_s(\bar{q}_T) - F_s(\bar{q}_N)) \left(\frac{t_s}{1 + O_s^{\mathcal{T}}} \right)^j$$

$$= \frac{1}{O_s^{\mathcal{T}}} \frac{F_s(\bar{q}_T) + (1 + O_s^{\mathcal{T}})F_s(\bar{q}_N)}{2 + O_s^{\mathcal{T}}}. \tag{A.6}$$

The last fraction decreases in $O_s^{\mathcal{T}}$ and as fewer sellers both with qualities below $\bar{q}_N$ and below $\bar{q}_T$ enter the market in the good state, it must be that $O_\gamma^{\mathcal{T}} < O_\beta^{\mathcal{T}}$.

I now show that the odds of q_1, (A.4), decrease (weakly) in q_1. The first inequality that has to be satisfied is $\frac{O_\beta^{\mathcal{T}}}{aO_\gamma^{\mathcal{T}}} > \frac{(O_\beta^{\mathcal{T}}+1-t_\beta)}{a(O_\gamma^{\mathcal{T}}+1-t_\gamma)}$. After cross-multiplying, inserting the equation for t_s, and collecting terms, the inequality simplifies to

$$O_\beta^{\mathcal{T}} \frac{O_\gamma^{\mathcal{T}}}{1 + O_\gamma^{\mathcal{T}}} > O_\gamma^{\mathcal{T}} \frac{O_\beta^{\mathcal{T}}}{1 + O_\beta^{\mathcal{T}}},$$

which holds as $O_\beta^{\mathcal{T}} > O_\gamma^{\mathcal{T}}$. The second inequality that has to be satisfied is $\frac{(O_\beta^{\mathcal{T}}+1-t_\beta)}{a(O_\gamma^{\mathcal{T}}+1-t_\gamma)} > \frac{1+O_\beta^{\mathcal{T}}}{a(1+O_\gamma^{\mathcal{T}})}$. After cross-multiplying and collecting terms, the inequality simplifies to

$$t_\gamma(1 + O_\beta^{\mathcal{T}}) > t_\beta(1 + O_\gamma^{\mathcal{T}}),$$

which holds as $O_\beta^{\mathcal{T}} > O_\gamma^{\mathcal{T}}$ and $t_\gamma > t_\beta$. In sum, the odds $\frac{h_\gamma^{\mathcal{T}}(q_1)}{h_\beta^{\mathcal{T}}(q_1)}$ decrease (weakly) in q_1. Altogether, under the trade-signal regime b's continuation value decreases (weakly) in q_1 so that the sufficient condition (SC) is satisfied and b's best response is a cutoff rule if others use strategy $(\bar{q}_T, \bar{q}_N)$ with $\bar{q}_T > \bar{q}_N$.

(iv) Under the exogenous-signal regime, assume that all buyers but b optimally use strategy $(\bar{q}_G, \bar{q}_B)$ with $\bar{q}_\gamma > \bar{q}_G > \bar{q}_B > \bar{q}_\beta$. I show that the necessary and sufficient condition for the best response for b to be a cutoff strategy (NC) holds under the exogenous-signal regime with $p_G = 1$ and $p_B \in (0, 1)$. Recall

(NC): for signal realisation $i = B, G$ there exists a unique $\hat{q}_i$ such that for all $q_1 < \hat{q}_i$, b's continuation value is higher than q_1, or $q_1 < V(q_1, i)$, and for all $q_1' \geq \hat{q}_i$, b's continuation value is lower than q_1', or $V(q_1', i) \leq q_1'$.

As before, a sufficient condition for the existence of such $\hat{q}_i$ is that $\pi^{\mathcal{E}}(q_1, i)$, weakly decreases in q_1. The sufficient condition holds for $i = B$: if $p_G = 1$, then b's belief after B and any q_1 is zero, $\pi^{\mathcal{E}}(q_1, B) = 0$, as he knows that the state is bad (in (A.1), $P(B|\gamma) = 1 - p_G = 0$).

Suppose b observed $i = G$. I show that the sufficient condition that the odds of q decrease weakly in q does not hold, but the necessary and sufficient condition (NC) holds. To see that, write down the buyer's odds of observing q_1 explicitly:

$$\frac{h^{\mathcal{E}}_{\gamma}(q_1)}{h^{\mathcal{E}}_{\beta}(q_1)} = \begin{cases} \frac{O^{\mathcal{E}}_{\beta}}{aO^{\mathcal{E}}_{\gamma}} & \text{if } q_1 < \bar{q}_B, \\ \frac{O^{\mathcal{E}}_{\beta}+p_B}{aO^{\mathcal{E}}_{\gamma}} & \text{if } q_1 \in [\bar{q}_B, \bar{q}_G), \\ \frac{1+O^{\mathcal{E}}_{\beta}}{a(1+O^{\mathcal{E}}_{\gamma})} & \text{if } q_1 \geq \bar{q}_G. \end{cases}$$

Comparing the odds of $q_1 = 0$ and $q_1 = \bar{q}_B$, we see that the odds of q_1 do not weakly decrease in q_1. I show that a unique $\hat{q}_i$ as required by (NC) exists nevertheless.

In order to do so, I establish that the amount of old buyers is smaller in the good than in the bad state state so that the odds of q_1 are lowest for $q_1 \geq \bar{q}_G$. Similarly to the trade-signal regime, the amount of old buyers, $O^{\mathcal{E}}_s$, is equal to the amount of sellers that are carried from one period to the next. This amount is made up of all the sellers with quality $q < \bar{q}_B$ that have always met young buyers, and of all the sellers with quality $q \in [\bar{q}_B, \bar{q}_G)$ that have always met young buyers who have observed signal realisations G. That is,

$$O^{\mathcal{E}}_s = \sum_{j=1}^{\infty} F_s(\bar{q}_B) \left(\frac{1}{1 + O^{\mathcal{E}}_s} \right)^j + (F_s(\bar{q}_G) - F_s(\bar{q}_B)) \left(\frac{P(G|s)}{1 + O^{\mathcal{E}}_s} \right)^j$$

$$= \frac{1}{O_s^{\mathcal{E}}} \frac{P(G|s) O_s^{\mathcal{E}} F_s(\bar{q}_G) + (1 + O_s^{\mathcal{E}})(1 - P(G|s)) F_s(\bar{q}_B)}{1 + O_s^{\mathcal{E}} - P(G|s)}. \tag{A.7}$$

In the good state, $P(G|\gamma) = 1$ so the last fraction simplifies to $F_\gamma(\bar{q}_G)$. In the bad state, $P(G|\beta) = 1 - p_B$ and I show that, as a function of $O_s^{\mathcal{E}}$, the last fraction in (A.7) exceeds $F_\gamma(\bar{q}_G)$ for all $O_s^{\mathcal{E}}$. To see that, note that if $s = \beta$, the fraction increases in $O_s^{\mathcal{E}}$ and at $O_s^{\mathcal{E}} = 0$ the fraction equals $F_\beta(\bar{q}_B)$. But $F_\beta(\bar{q}_B)$ exceeds $F_\gamma(\bar{q}_G)$ because the thresholds are ordered $\bar{q}_\gamma > \bar{q}_G > \bar{q}_B > \bar{q}_\beta$ and under the known-state regime, $F_\gamma(\bar{q}_\gamma) = F_\beta(\bar{q}_\beta)$.[3] Hence, $O_\gamma^{\mathcal{E}} < O_\beta^{\mathcal{E}}$ must hold. But this implies that the odds of $q_1 = \bar{q}_G$ are lower than of $q_1 = 0$.

I now show that the necessary and sufficient condition (NC) holds after $i = G$. Since $\bar{q}_G$ is an optimal cutoff used by other buyers, it must be that after observing $(q_1, i) = (\bar{q}_G, G)$ the buyers are just indifferent between accepting the offer $q_1 = \bar{q}_G$ and continuing, or $V^{\mathcal{E}}(\bar{q}_G, G) = \bar{q}_G$. Buyer b's continuation value after G is piecewise constant because the beliefs are piecewise constant (see (A.1) and (A.2) in conjunction). Hence, it is clear that for all $q' \geq \bar{q}_G$, the constant continuation value $V^{\mathcal{E}}(q', G)$ is less than the increasing q'. I need to show that for all $q < \bar{q}_G$, the continuation value exceeds q, i.e., that $V^{\mathcal{E}}(q, G) > q$. To see that this holds, note that the odds of q_1, hence, the continuation value, are lowest for $q' \geq \bar{q}_G$. Hence, for all $q < \bar{q}_G$, it must be that the continuation value $V^{\mathcal{E}}(q, G)$ is higher than $V^{\mathcal{E}}(\bar{q}_G, G)$. But the continuation value $V^{\mathcal{E}}(\bar{q}_G, G)$ is equal to $\bar{q}_G$ so is higher than q. In sum, the unique $\hat{q}_i$ that satisfies the necessary and sufficient condition is $\hat{q}_i = \bar{q}_G$. This completes the proof that under the exogenous-signal regime with $p_G = 1$ and $p_B \in (0, 1)$, the necessary and sufficient condition (NC) holds: a buyer's best response is a cutoff rule if others use strategy $(\bar{q}_G, \bar{q}_B)$ with $\bar{q}_\gamma > \bar{q}_G > \bar{q}_B > \bar{q}_\beta$. This completes Part a of the proof.

[3] I show that $F_\gamma(\bar{q}_\gamma) = F_\beta(\bar{q}_\beta)$ under the known-state regime in the proof of Proposition 3 and do not repeat the argument here to save space.

Part b. Equilibrium cutoff qualities.

Let us treat the four regimes together again. An equilibrium cutoff $\bar{q}_i$ equals the buyer's continuation value after observing $q_1 = \bar{q}_i$ and realisation i for $i = G, B$:

$$\bar{q}_i = \delta[\pi(\bar{q}_i, i)\mathbb{E}_{H_\gamma}(q) + (1 - \pi(\bar{q}_i, i))\mathbb{E}_{H_\beta}(q)]. \tag{A.8}$$

I show that this equation has a solution for $i = G, B$. At $\bar{q}_i = 0$, the RHS of equation (A.8) is positive because either $\pi(\bar{q}_i, i)$ or $1 - \pi(\bar{q}_i, i)$ is positive. At both $\bar{q}_i = 0$ and $\bar{q}_i = 1$, the RHS is smaller than one because $\mathbb{E}_{H_s}(q) < \mathbb{E}_{F_s}(q) \leq 1$ and $\delta < 1$. Hence, the LHS and RHS of equation (A.8) cross at least once so that an equilibrium in cutoff strategies exists under all four regimes.

I need to verify that the equilibrium cutoffs satisfy the assumptions that I made above: that $\bar{q}_T > \bar{q}_N$ under the trade-signal regime and that $\bar{q}_\gamma > \bar{q}_G > \bar{q}_B > \bar{q}_\beta$ under the exogenous-signal regime. First, under the trade-signal regime, it is sufficient to show that $\pi^{\mathcal{T}}(\bar{q}_T, T) > \pi^{\mathcal{T}}(\bar{q}_N, N)$ as $\mathbb{E}_{H_\gamma}(q) > \mathbb{E}_{H_\beta}(q)$ (see (A.8)). Inserting the definition of t_s, (A.5), and $h^{\mathcal{T}}(q)$, (A.4), into (A.1), yields

$$\frac{\pi^{\mathcal{T}}(\bar{q}_T, T)}{1 - \pi^{\mathcal{T}}(\bar{q}_T, T)} = \omega \frac{h^{\mathcal{T}}_\gamma(\bar{q}_T)}{h^{\mathcal{T}}_\beta(\bar{q}_T)} \frac{t_\gamma}{t_\beta} = \frac{\omega}{a} \left(\frac{1 + O^{\mathcal{T}}_\beta}{1 + O^{\mathcal{T}}_\gamma} \right)^2, \tag{A.9}$$

and

$$\frac{\pi^{\mathcal{T}}(\bar{q}_N, N)}{1 - \pi^{\mathcal{T}}(\bar{q}_N, N)} = \omega \frac{h^{\mathcal{T}}_\gamma(\bar{q}_N)}{h^{\mathcal{T}}_\beta(\bar{q}_N)} \frac{1 - t_\gamma}{1 - t_\beta} = \frac{\omega}{a} \left(\frac{2 + O^{\mathcal{T}}_\beta}{2 + O^{\mathcal{T}}_\gamma} \right). \tag{A.10}$$

But cross-multiplying and simplifying $\left(\frac{1+O^{\mathcal{T}}_\beta}{1+O^{\mathcal{T}}_\gamma}\right)^2 > \frac{2+O^{\mathcal{T}}_\beta}{2+O^{\mathcal{T}}_\gamma}$ shows that it holds if $O^{\mathcal{T}}_\gamma < O^{\mathcal{T}}_\beta$ (which was established in Part a) so indeed $\pi^{\mathcal{T}}(\bar{q}_T, T) > \pi^{\mathcal{T}}(\bar{q}_N, N)$.

Second, consider the exogenous-signal regime. I show that $\bar{q}_\gamma > \bar{q}_G$ and $\bar{q}_B > \bar{q}_\beta$ in the proof of Proposition 8 (after presenting the explicit equation that the equilibrium cutoffs satisfy under the two regimes). For $\bar{q}_G > \bar{q}_B$, it is sufficient to show that $\pi^{\mathcal{E}}(q,G) > \pi^{\mathcal{E}}(q,B)$ for any $q < 1$. But this clearly holds as $\pi^{\mathcal{E}}(q,G) > 0$ (see (A.1) for $P(G|\gamma) = p_G = 1$) and $\pi^{\mathcal{E}}(q,B) = 0$.

This completes the proof of the existence of an equilibrium in cutoff strategies for all four regimes.[4]

Part c. Equilibrium uniqueness under the known-state regime.

A simple argument shows that the unique equilibrium under the known-state regime is in cutoff strategies. For any fixed behaviour of other buyers, b's continuation value, i.e., the discounted mean quality is constant as he knows the state: hence, b's optimal strategy is a cutoff strategy. □

A.3 Known state and no signals

Proof of Proposition 3. I derive the explicit forms of the cutoff qualities for the known-state regime (Part 1) and the equation that the cutoff solves for the no-signals' regime (Part 2). To do that, I simplify equation (A.8) in the proof of Proposition 2 by deriving the mean quality in state s and buyers' beliefs under both regimes. The equilibrium distribution of qualities is derived as a side-product.

Part 1. Known-state regime.

Under the known-state regime, all buyers know the state after the signal realisation: their posterior belief is $\pi^{\mathcal{K}}(q_1,G) = 1$ if $s = \gamma$ and $\pi^{\mathcal{K}}(q_1,B) = 0$ if $s = \beta$. A buyer uses cutoff $\bar{q}_\gamma$ after realisation G and $\bar{q}_\beta$ after realisation B, and each satisfies equation (A.8). The equilibrium density (A.2) in state s is

[4] Note that the existence proof for the known-state regime does not depend on the entry distributions F_γ and F_β.

explicitly,

$$h_s^{\mathcal{K}}(q) = \begin{cases} f_s(q)(O_s^{\mathcal{K}})^{-1} & \text{if } q < \bar{q}_s, \\ f_s(q)(O_s^{\mathcal{K}}+1)^{-1} & \text{if } q \geq \bar{q}_s. \end{cases} \tag{A.11}$$

The density is piecewise constant so that the mean quality under the equilibrium distribution (A.11) is

$$\mathbb{E}_{H_s^{\mathcal{K}}}(q) = H_s^{\mathcal{K}}(\bar{q}_s)\frac{\bar{q}_s}{2} + (1 - H_s^{\mathcal{K}}(\bar{q}_s))\frac{x + \bar{q}_s}{2},$$

where $x = 1$ if $s = \beta$ and $x = a$ if $s = \gamma$. The probability that a young buyer becomes old, i.e., observes an unacceptable offer $q_1 < \bar{q}_s$, $H_s^{\mathcal{K}}(\bar{q}_s)$, is equal to the equilibrium amount of old buyers, $O_s^{\mathcal{K}}$, because the measure of young buyers is one. The probability is derived by integrating $h_s^{\mathcal{K}}(q)$ from zero to $\bar{q}_s$ and solving for $O_s^{\mathcal{K}}$:

$$H_s^{\mathcal{K}}(\bar{q}_s) = O_s^{\mathcal{K}} = \int_0^{\bar{q}_s} h_s^{\mathcal{K}}(q)\ \mathrm{d}q = F_s(\bar{q}_s)\frac{1}{O_s^{\mathcal{K}}},$$

which gives,

$$O_s^{\mathcal{K}} = \sqrt{F_s(\bar{q}_s)}. \tag{A.12}$$

Hence, the mean quality in state s simplifies to

$$\mathbb{E}_{H_s^{\mathcal{K}}}(q) = xO_s^{\mathcal{K}}\frac{(O_s^{\mathcal{K}})^2}{2} + x(1 - O_s^{\mathcal{K}})\frac{1 + (O_s^{\mathcal{K}})^2}{2}$$

$$= \frac{x + \bar{q}_s - \sqrt{x\bar{q}_s}}{2}.$$

Plugging this (and the correct constant belief) into (A.8) and simplifying gives that the equilibrium cutoff in state γ, $\bar{q}_\gamma$, solves

$$\bar{q}_\gamma = \frac{\delta}{2-\delta}(a - \sqrt{a\bar{q}_\gamma}),$$

and in state β, $\bar{q}_\beta$, solves

$$\bar{q}_\beta = \frac{\delta}{2-\delta}(1-\sqrt{\bar{q}_\beta}).$$

The explicit solutions to these are given in the statement of the Proposition. Note that $\bar{q}_\gamma = a\bar{q}_\beta$ means that $F_\gamma(\bar{q}_\gamma) = F_\beta(\bar{q}_\beta)$.

Part 2. No-signals' regime.

Under the no-signals' regime, the signal is perfectly uninformative. A buyer's posterior belief is $\pi^{\mathcal{N}}(q_1, G) = \pi^{\mathcal{N}}(q_1, B)$ and $\bar{q}_G = \bar{q}_B =: \bar{q}$. The equilibrium cutoff solves equation (A.8) with belief $\pi^{\mathcal{N}}(\bar{q})$ that satisfies

$$\frac{\pi^{\mathcal{N}}(\bar{q})}{1-\pi^{\mathcal{N}}(\bar{q})} = \omega\frac{1+\sqrt{\bar{q}}}{a+\sqrt{\bar{q}a}}. \tag{A.13}$$

The mean qualities are derived in the same way as under the known-state regime as a single cutoff is used in one state under both regimes. Thus, $\bar{q}$ solves

$$\bar{q} = \frac{\delta}{2-\delta}[\pi(\bar{q})(a-\sqrt{a\bar{q}}) + (1-\pi(\bar{q}))(1-\sqrt{\bar{q}})], \tag{A.14}$$

The exact expression in the Proposition is obtained by plugging $\pi(\bar{q})$, (A.13), into (A.14). The equilibrium distribution is obtained by substituting $\mathcal{K}$ with $\mathcal{N}$ and $\bar{q}_s$ with $\bar{q}$ in (A.11). $\square$

Proof of Lemma 1. Recall that the cutoff $\bar{q}$ can be written as in equation (A.14). Note that $a - \sqrt{aq} > 1 - \sqrt{q}$ for all $q \in (0,1)$ and both sides of this inequality decrease in q. Hence, $\bar{q} > \frac{\delta}{2-\delta}(1-\sqrt{\bar{q}})$ and $\bar{q} < \frac{\delta}{2-\delta}(a-\sqrt{a\bar{q}})$. Recall that the known-state cutoffs were defined by $\bar{q}_\beta = \frac{\delta}{2-\delta}(1-\sqrt{\bar{q}_\beta})$ and $\bar{q}_\gamma = \frac{\delta}{2-\delta}(a-\sqrt{a\bar{q}_\gamma})$. Hence, the inequality $\bar{q} > \frac{\delta}{2-\delta}(1-\sqrt{\bar{q}})$ implies that $\bar{q} > \bar{q}_\beta$ and the inequality $\bar{q} < \frac{\delta}{2-\delta}(a-\sqrt{a\bar{q}})$ implies that $\bar{q} < \bar{q}_\gamma$. $\square$

Proof of Proposition 4. The expected delay in the two regimes is derived by plugging equation (A.12) or (A.3) into equation (2.3). To show that $D^{\mathcal{N}} > D^{\mathcal{K}}$, I show that $D^{\mathcal{N}}|_{a=1} = D^{\mathcal{K}}$ and $\frac{\partial D^{\mathcal{N}}}{\partial a} > 0$ for $\delta > \frac{2}{3}$ and $\frac{\partial D^{\mathcal{N}}}{\partial a} < 0$ for $\delta < \frac{2}{3}$.

First, note that if $a = 1$, then the good distribution F_γ is the same as the bad distribution F_β, which translates directly into $H_\gamma(q) = H_\beta(q)$. Hence, $a - \sqrt{aq} = 1 - \sqrt{q}$ and $\bar{q} = \bar{q}_\beta$ so that $D^{\mathcal{N}}|_{a=1} = D^{\mathcal{K}}$.

Second, I show that the derivative $\frac{\partial D^{\mathcal{N}}}{\partial a}$ is positive for $\delta > \frac{2}{3}$ and negative for for $\delta < \frac{2}{3}$. Under the no-signals' regime, the derivative of the expected delay is

$$\frac{\partial D^{\mathcal{N}}}{\partial a} = \frac{\pi}{2}\frac{1}{a\sqrt{a\bar{q}}}\left(a\frac{\partial \bar{q}}{\partial a} - \bar{q}\right) + \frac{1-\pi}{2}\frac{1}{\sqrt{\bar{q}}}\frac{\partial \bar{q}}{\partial a}.$$

I derive $\frac{\partial \bar{q}}{\partial a}$. By inserting $\pi(\bar{q})$ and multiplying by a positive amount, equation (A.14) can be rewritten as

$$C := \pi(1+y)[\delta(a-\sqrt{a}y)-(2-\delta)y^2]+(1-\pi)(a+\sqrt{a}y)[\delta(1-y)-(2-\delta)y^2] = 0, \tag{A.15}$$

where $y := \sqrt{\bar{q}}$. The two terms in the squared brackets are positive and negative respectively.

Totally differentiating C and using equation (A.15) to substitute terms that have multiplier $1-\pi$ with terms that have multiplier π, I get

$$\frac{\partial \bar{q}}{\partial a} = \frac{y^2(1+y)[2\delta a + 2\sqrt{a}y(2-\delta) + (2-\delta)y^2][(2-\delta)y^2 - \delta(1-y)]}{aC_2} > 0,$$

where $y := \sqrt{\bar{q}}$ and

$$C_2 := (1+y)(\sqrt{a}+y)[(2-\delta)y^2 - \delta(1-y)][\delta\sqrt{a} + 2(2-\delta)y]$$

$$+[\delta(a-\sqrt{a}y)-(2-\delta)y^2]\{(\sqrt{a}+y)[2\delta+(2-\delta)y(2+y)]+(1+y)[(2-\delta)y^2-\delta(1-y)]\}.$$

Showing that $\frac{\partial D^{\mathcal{N}}}{\partial a} > 0$ is equivalent to showing that

$$C_3 := a\frac{\partial \bar{q}}{\partial a}\{(\sqrt{a}+y)[(2-\delta)y^2 - \delta(1-y)] + (1+y)[\delta(a-\sqrt{a}y) - (2-\delta)y^2]\}$$

$$-y^2(\sqrt{a}+y)[(2-\delta)y^2 - \delta(1-y)] > 0,$$

where I have again used equation (A.15) to substitute terms that have multiplier $1-\pi$ with terms that have multiplier π and then multiplied by a positive term. Plugging in the expression for $\frac{\partial \bar{q}}{\partial a}$, collecting terms and dividing with positive amounts $(2-\delta)y^2 - \delta(1-y)$, and $\delta(a-\sqrt{a}y) - (2-\delta)y^2$, I get that $C_3 > 0$ is equivalent to

$$(3\delta - 2)[2\sqrt{a}y(\sqrt{a}-1) + y^2(a-1)] > 0,$$

which holds iff $\delta > \frac{2}{3}$ because $a > 1$. This completes the proof. □

A.4 Trade signal

Proof of Proposition 5. I first show that in every equilibrium under the trade-signal regime, a trade is good news, $q = \bar{q}_N$ is better news than $q = \bar{q}_T$, and $\bar{q}_T > \bar{q}_N$ holds (Part 1). I then derive the exact forms of the equations that the equilibrium cutoffs and distribution of qualities satisfy (Part 2).

Part 1. I show that under the trade-signal regime, in all equilibria in cutoff strategies, trade is good news, $q = \bar{q}_N$ is better news than $q = \bar{q}_T$, and $\bar{q}_T > \bar{q}_N$. The existence of such an equilibrium was proven in Proposition 2.

In order to do so, I consider all the possible characterisations and show that the only possible combination is the one above. First recall that the relative size of the equilibrium cutoff $\bar{q}_i$ is determined by how optimistic a young buyer is after observing quality $q_1 = \bar{q}_i$ and signal outcome i as opposed to $q_1 = \bar{q}_j$ and signal outcome j. That is, if $q_1 = \bar{q}_i$ is better news than $q_1 = \bar{q}_j$ and signal outcome i better news that j, then cutoff $\bar{q}_i$ is definitely higher than cutoff $\bar{q}_j$. In the other cases, the relative magnitude of the cutoffs is ambiguous. Therefore, an exhaustive list of the cases that I need to consider is

(i) T is better news than N and $q_1 = \bar{q}_T$ is better news than $q_1 = \bar{q}_N$ (together implying that $\bar{q}_T > \bar{q}_N$).

(ii) T is better news than N and $q_1 = \bar{q}_T$ is worse news than $q_1 = \bar{q}_N$ and

  (a) $\bar{q}_T > \bar{q}_N$, or

  (b) $\bar{q}_T < \bar{q}_N$.

(iii) T is worse news than N and $q_1 = \bar{q}_T$ is worse news than $q_1 = \bar{q}_N$ (together implying that $\bar{q}_T < \bar{q}_N$).

(iv) T is worse news than N and $q_1 = \bar{q}_T$ is better news than $q_1 = \bar{q}_N$ and

  (a) $\bar{q}_T > \bar{q}_N$, or

  (b) $\bar{q}_T < \bar{q}_N$.

I show in turn that all of the cases but (ii)(a) lead to a contradiction.

(i) T is better news than N and $q_1 = \bar{q}_T$ is better news than $q_1 = \bar{q}_N$ (together implying that $\bar{q}_T > \bar{q}_N$), i.e., $t_\gamma > t_\beta$ and $\frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)} > \frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)}$.

We know from Lemma 4 that in any equilibrium under a regime with a binary signal such that $P(B|\beta) > P(B|\gamma)$, if $O_\gamma < O_\beta$ (so that $t_\gamma > t_\beta$) and the equilibrium cutoffs satisfy $\bar{q}_B < \bar{q}_G$, the odds of $q = \bar{q}_B$ are higher than of $q = \bar{q}_G$, i.e., $q = \bar{q}_B$ is better news than $q = \bar{q}_G$. Substituting N for B means that if $P(N|\beta) = 1 - t_\beta > P(N|\gamma) = 1 - t_\gamma$ and $\bar{q}_N < \bar{q}_T$, then $\frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)} < \frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)}$, which contradicts the initial assumption about the relative size of the odds.[5]

(ii) T is better news than N and $q_1 = \bar{q}_T$ is worse news than $q_1 = \bar{q}_N$, i.e., $t_\gamma > t_\beta$ and $\frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)} < \frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)}$.

(a) $\bar{q}_T > \bar{q}_N$: in the proofs of Lemmas 2, 3, 4, and 5 together I showed that T is better news than N and $q_1 = \bar{q}_T$ is worse news than $q_1 = \bar{q}_N$ are consistent with $\bar{q}_T > \bar{q}_N$.

(b) $\bar{q}_T < \bar{q}_N$: I show that assuming $\bar{q}_T < \bar{q}_N$ leads to a contradiction. Consider a young buyer who observes $q_1 = \bar{q}_T$. We know that he is just indifferent between accepting this quality and continuing if he has observed signal outcome T, i.e., $\bar{q}_T = V^{\mathcal{T}}(q_1 = \bar{q}_T, T)$. But trade being good news means that for any given $q \in [0, 1]$, a buyer who observes q and T is more optimistic than a buyer who observes q and N, i.e., $V^{\mathcal{T}}(q, T) > V^{\mathcal{T}}(q, N)$ for all $q \in [0, 1]$. But then a buyer who observes $q_1 = \bar{q}_T$ and signal outcome N must strictly prefer accepting this quality to continuing as $\bar{q}_T = V^{\mathcal{T}}(q_1 = \bar{q}_T, T) > V^{\mathcal{T}}(q_1 = \bar{q}_T, N)$. This is a contradiction because $\bar{q}_T < \bar{q}_N$ means that the buyer should instead prefer to continue after $q_1 = \bar{q}_T$ and N.

(iii) T is worse news than N and $q_1 = \bar{q}_T$ is worse news than $q_1 = \bar{q}_N$ (together implying that $\bar{q}_T < \bar{q}_N$), i.e., $t_\gamma < t_\beta$ and $\frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)} < \frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)}$.

An analogous argument to that provided for (i) (where now T is substituted for B) proves that this case is impossible.[6]

[5]Note that if $\frac{f_\gamma(q)}{f_\beta(q)}$ increases in q, Lemma 4 is no longer guaranteed to hold.

[6]This proof relies on the assumption that $\frac{f_\gamma(q)}{f_\beta(q)}$ is constant in q, but for general distributions F_γ and F_β such that F_γ first-order stochastically dominates F_β, (iii) is ruled out in (iv)(b).

(iv) T is worse news than N and $q_1 = \bar{q}_T$ is better news than $q_1 = \bar{q}_N$, i.e., $t_\gamma < t_\beta$ and $\frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)} > \frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)}$.

(a) $\bar{q}_T > \bar{q}_N$: By an analogous argument as in (ii)(b), $\bar{q}_T > \bar{q}_N$ leads to a contradiction.

(b) $\bar{q}_T < \bar{q}_N$: I argue that $\bar{q}_T < \bar{q}_N$ and $t_\gamma < t_\beta$ together lead to a contradiction. I use an argument analogous to that in the proof of Lemma 2: I derive the equation that the amount of old sellers in state s must satisfy and show that it leads to $O_\gamma < O_\beta$. But here, $O_\gamma < O_\beta$ contradicts the assumption that $t_\gamma < t_\beta$ as $t_s = (1+O_s)^{-1}$.

The amount of old buyers in the market is equal to the amount of sellers that are carried from one period to the next. This amount is made up of all the sellers with quality $q < \bar{q}_T$ that have entered and always met young buyers and of all the sellers with quality $q \in [\bar{q}_T, \bar{q}_N)$ that have entered and always met young buyers who have observed no trades. The probability that a seller who entered j periods ago has always met young buyers is $(1 + O_s)^{-j}$ and that he has always met young buyers who have observed no trades is $(1 - t_s)^j (1 + O_s)^{-j}$. In sum, the amount of sellers that are carried from one period to the next is

$$O_s = \sum_{j=1}^{\infty} F_s(\bar{q}_T) \left(\frac{1}{1 + O_s} \right)^j + (F_s(\bar{q}_N) - F_s(\bar{q}_T)) \left(\frac{1 - t_s}{1 + O_s} \right)^j$$

$$= \frac{1}{O_s} \frac{(1 + O_s)F_s(\bar{q}_T) + O_s^2 F_s(\bar{q}_N)}{1 + O_s + O_s^2}.$$

Equivalently, the amount of old sellers in state s satisfies the equation

$$O_s^2 = \frac{(1 + O_s)F_s(\bar{q}_T) + O_s^2 F_s(\bar{q}_N)}{1 + O_s + O_s^2}. \tag{A.16}$$

I show that equation (A.16) implies that $O_\gamma < O_\beta$, which contradicts the assumption that $t_\gamma < t_\beta$. The derivative of the RHS of (A.16) with respect to O_s is

$$\frac{\partial RHS}{\partial O_s} = \frac{O_s(2+O_s)(F_s(\bar{q}_N) - F_s(\bar{q}_T))}{(1+O_s+O_s^2)^2},$$

which is positive for all O_s. As a function of O_s, the RHS of (A.16) evaluated at $s = \gamma$ is lower than the RHS evaluated at $s = \beta$ because fewer sellers both with qualities below $\bar{q}_N$ and below $\bar{q}_T$ enter the market in the good state than in the bad state (for all F_γ and F_β such that F_γ first–order stochastically dominates F_β). But the LHS of (A.16) increases in O_s so that the solutions to equation (A.16) in the two states must satisfy $O_\gamma < O_\beta$. But this contradicts the assumption that $t_\gamma < t_\beta$ because $t_s = (1+O_s)^{-1}$.[7]

Altogether, only case (ii)(a), i.e., T is better news than N, $q_1 = \bar{q}_T$ is worse news than $q_1 = \bar{q}_N$, and $\bar{q}_T > \bar{q}_N$, is possible in equilibrium.

Part 2. I derive the system of equations that the equilibrium cutoffs $\bar{q}_T$ and $\bar{q}_N$ satisfy. I write out the mean quality in state s and the beliefs relevant for the cutoffs, and then plug them into equation (A.8). Most of the detail that I need for the derivation is presented in the proof of Proposition 2.

First, I derive the mean quality in state s if the equilibrium distribution is described by two cutoffs as in (A.2) (the derivation is valid both for the

[7]Note that this proof does not rely on the assumption about the informativeness of different qualities so that it also rules out (iii) for general distributions F_γ and F_β such that F_γ first-order stochastically dominates F_β.

trade-signal and the exogenous-signal regimes). I rewrite the density, (A.2), as

$$h_s(q) = \begin{cases} f_s(q)O_s^{-1} & \text{if } q < \bar{q}_B, \\ f_s(q)O_s^{-1}\left(1 - \frac{P(B|s)}{P(B|s)+O_s}\right) & \text{if } q \in [\bar{q}_B, \bar{q}_G), \\ f_s(q)O_s^{-1}\left(1 - \frac{1}{O_s+1}\right) & \text{if } q \geq \bar{q}_G. \end{cases}$$

Denoting $f_s(q) = \frac{1}{x}$ where $x = 1$ if $s = \beta$ and $x = a$ if $s = \gamma$ and using the above equation for $h_s(q)$, the mean quality in state s can be written as

$$\mathbb{E}_{H_s}(q) = O_s^{-1}\left[\mathbb{E}_{F_s}(q) - \frac{P(B|s)}{P(B|s)+O_s}\int_{\bar{q}_B}^{\bar{q}_G} \frac{q}{x}\,\mathrm{d}q - \frac{1}{1+O_s}\int_{\bar{q}_G}^{x} \frac{q}{x}\,\mathrm{d}q\right]$$

$$= \frac{1}{2xO_s}\left[x^2 - \frac{P(B|s)}{P(B|s)+O_s}(\bar{q}_G^2 - \bar{q}_B^2) - \frac{1}{1+O_s}(x^2 - \bar{q}_G^2)\right]. \qquad \text{(A.17)}$$

Under the trade-signal regime, no trade is bad news so the term $\frac{P(B|s)}{P(B|s)+O_s}$ in the mean is $\frac{1-t_s}{1-t_s+O_s} = \frac{1}{2+O_s}$, where the last step follows from plugging in the equation for t_s, (A.5). Plugging the explicit term into (A.17) gives the exact form of the expected value of $\mathbb{E}_{H_s}(q)$ used in the statement of the Proposition.

The beliefs of the buyers that are relevant for the equilibrium cutoffs, i.e., after observing $(q_1, i) = (\bar{q}_T, T)$ and $(\bar{q}_N, N)$ were derived earlier and the result was equations (A.9) and (A.10). Using equations (A.17), (A.9), and (A.10) in (A.8), gives the two first equations in the statement of the Proposition.

I derive the last two equations in the statement of the Proposition. The amount of old buyers is given by equation (A.6), which can be simplified to

$$O_s = \frac{1}{x}\frac{\bar{q}_T + (1+O_s)\bar{q}_N}{O_s(2+O_s)},$$

by plugging in $F_s(\bar{q}_i) = \frac{\bar{q}_i}{x}$, where $x = 1$ if $s = \beta$ and $x = a$ if $s = \gamma$, and collecting terms. Rearranging the equation for $s = \gamma$ completes the system of equations in the Proposition. The exact form of the equilibrium density of qualities is obtained by plugging $P(B|s) = 1 - t_s = \frac{O_s}{1+O_s}$ into equation (A.2) and rearranging the density for $q \in [\bar{q}_N, \bar{q}_T)$. □

The proofs of Lemmas 2, 3, and 5 (and a specific version of Lemma 4) appear in the proof of Proposition 2, but I restate them here for the completeness of the argument.

Proof of Lemma 2. Suppose that $\bar{q}_T > \bar{q}_N$ and $t_\gamma > t_\beta$. The amount of old buyers in the market is equal to the amount of sellers that are carried from one period to the next. These sellers consist of those with quality $q < \bar{q}_N$ that have ever entered and always met young buyers, and those with quality $q \in [\bar{q}_N, \bar{q}_T)$ that have ever entered and always met young buyers who observed good news: trades.

In state s the per-period probability that a seller meets a young buyer is $(1 + O_s)^{-1}$ and that he meets a young buyer who has observed a trade is $t_s (1 + O_s)^{-1}$ because matching is random, the amount of young buyers is one, and the total amount of buyers is $1+O_s$. Since meetings are independent across periods, the probability that a seller who entered j periods ago has always met young buyers is $(1 + O_s)^{-j}$ and that he has always met young buyers who observed trades is $t_s^j (1 + O_s)^{-j}$. Thus, the amount of sellers that are carried from one period to the next is

$$O_s = \sum_{j=1}^{\infty} F_s(\bar{q}_N) \left(\frac{1}{1+O_s} \right)^j + (F_s(\bar{q}_T) - F_s(\bar{q}_N)) \left(\frac{t_s}{1+O_s} \right)^j$$

$$= \frac{1}{O_s} \frac{F_s(\bar{q}_T) + (1+O_s)F_s(\bar{q}_N)}{2+O_s}, \tag{A.18}$$

where I have inserted the equilibrium probability of a trade, (A.19) in the proof of Lemma 3, in the last step. The last fraction in (A.18) decreases in O_s and as fewer sellers both with qualities below $\bar{q}_N$ and below $\bar{q}_T$ enter the market in the good state, it must be that $O_\gamma < O_\beta$.[8] □

Proof of Lemma 3. Suppose that $O_\gamma < O_\beta$. The probability of observing a trade is the equilibrium probability that a randomly chosen seller trades in a period. Mass one of sellers that trade (and exit) in a period because the market is stationary and mass one of sellers is enter. The total amount of sellers is equal to the amount of buyers, $1 + O_s$, in state s, because equal amount of buyers and sellers enter and, in equilibrium, a buyer and a seller leave the market together. Hence, the probability of a trade in state s is

$$t_s = (1 + O_s)^{-1}. \tag{A.19}$$

A trade is, thus, more probable in the good state because $O_\gamma < O_\beta$.[9] □

Proof of Lemma 4. Suppose that $\bar{q}_B < \bar{q}_G$ and $O_\gamma < O_\beta$. The odds of observing quality q are obtained by inserting the equilibrium density, equation (A.2) in the Appendix, into $\frac{h_\gamma(q)}{h_\beta(q)}$:

$$\frac{h_\gamma(q)}{h_\beta(q)} = \begin{cases} \frac{f_\gamma(q)}{f_\beta(q)} \frac{O_\beta}{O_\gamma} & \text{if } q < \bar{q}_B, \\ \frac{f_\gamma(q)}{f_\beta(q)} \frac{O_\beta + P(B|\beta)}{O_\gamma + P(B|\gamma)} & \text{if } q \in [\bar{q}_B, \bar{q}_G), \\ \frac{f_\gamma(q)}{f_\beta(q)} \frac{O_\beta + 1}{O_\gamma + 1} & \text{if } q \geq \bar{q}_G. \end{cases}$$

I show that $\frac{h_\gamma(\bar{q}_B)}{h_\beta(\bar{q}_B)} > \frac{h_\gamma(\bar{q}_G)}{h_\beta(\bar{q}_G)}$ holds. The inequality is explicitly $\frac{O_\beta + P(B|\beta)}{a(O_\gamma + P(B|\gamma))} > \frac{O_\beta + 1}{a(O_\gamma + 1)}$. Now the LHS of this inequality is larger than $\frac{O_\beta + P(B|\gamma)}{a(O_\gamma + P(B|\gamma))}$ as $P(B|\beta) > P(B|\gamma)$. But then the initial inequality must hold because $\frac{\partial}{\partial z} \frac{O_\beta + z}{O_\gamma + z} \propto -(O_\beta - O_\gamma) < 0$ as $O_\beta > O_\gamma$.[10] □

[8] The proof uses only the assumption that F_γ first-order stochastically dominates F_β.

[9] The proof uses only the assumption that F_γ first-order stochastically dominates F_β.

[10] The proof relies on the assumption that $\frac{f_\gamma(q)}{f_\beta(q)} = \frac{1}{a}$ is constant in q. The proof still holds

Proof of Lemma 5. Suppose that $O_\gamma < O_\beta$, $t_\gamma > t_\beta$, and $\frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)} > \frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)}$. The relative size of the cutoffs depends only on the relative size of the buyers' beliefs after observing $q_1 = \bar{q}_T$ and a trade as opposed to $q_1 = \bar{q}_N$ and no trade. A young buyer's posterior odds that are relevant for the equilibrium cutoffs are (see equations (A.9) and (A.10) in the Appendix)

$$\frac{\pi(\bar{q}_T, T)}{1-\pi(\bar{q}_T, T)} = \omega \frac{h_\gamma(\bar{q}_T)}{h_\beta(\bar{q}_T)} \frac{t_\gamma}{t_\beta} = \frac{\omega}{a}\left(\frac{1+O_\beta}{1+O_\gamma}\right)^2,$$

and

$$\frac{\pi(\bar{q}_N, N)}{1-\pi(\bar{q}_N, N)} = \omega \frac{h_\gamma(\bar{q}_N)}{h_\beta(\bar{q}_N)} \frac{1-t_\gamma}{1-t_\beta} = \frac{\omega}{a}\left(\frac{2+O_\beta}{2+O_\gamma}\right).$$

It is straightforward to verify that $\frac{\pi(\bar{q}_T,T)}{1-\pi(\bar{q}_T,T)} > \frac{\pi(\bar{q}_N,N)}{1-\pi(\bar{q}_N,N)}$ as $O_\gamma < O_\beta$ so that the optimal cutoffs satisfy $\bar{q}_N < \bar{q}_T$. □

Proof of Proposition 7. The expected delay is obtained by plugging equation (A.6) into (2.3).

The proof of $D^{\mathcal{T}} > D^{\mathcal{N}}$ holds for $\pi = \frac{1}{2}$ and all a, δ such that $O_\gamma^{\mathcal{T}} > \frac{1}{2}$, for which the condition $\bar{q}_\beta > \frac{a}{4}$ is sufficient (I show below that $a(O_\gamma^{\mathcal{T}})^2 > \bar{q}_N > \bar{q}_\beta$). Note that a necessary condition for $\bar{q}_\beta > \frac{a}{4}$ to hold is that $\delta > \frac{2}{3}$ as $\bar{q}_\beta|_{\delta=\frac{2}{3}} = \frac{1}{4}$.

The proof is in four steps. I first show that a sufficient condition for $D^{\mathcal{T}} > D^{\mathcal{N}}$ is that $O_\beta^{\mathcal{T}} > O_\beta^{\mathcal{N}}$ (Step 1). Then I approximate $O_\beta^{\mathcal{T}}$ down to $\tilde{O}_\beta^{\mathcal{T}}$ (Step 2). The rest of the proof shows that $\tilde{O}_\beta^{\mathcal{T}} > O_\beta^{\mathcal{N}}$ (Steps 3 and 4). Hence, $O_\beta^{\mathcal{T}} > O_\beta^{\mathcal{N}}$ must hold.

Step 1. It is sufficient to show that $O_\beta^{\mathcal{T}} > O_\beta^{\mathcal{N}}$.

Recall that the expected delay is $D = \pi O_\gamma + (1-\pi) O_\beta$. Under the no-signals' regime, $a(O_\gamma^{\mathcal{N}})^2 = (O_\beta^{\mathcal{N}})^2 = \bar{q}$. Under the trade-signal regime, the

if $\frac{f_\gamma(\bar{q}_B)}{f_\beta(\bar{q}_B)}$ is not "too much lower" than $\frac{f_\gamma(\bar{q}_G)}{f_\beta(\bar{q}_G)}$, i.e., holds for general entry distributions F_γ and F_β if the entry densities' ratio $\frac{f_\gamma(q)}{f_\beta(q)}$ does not grow "too quickly" in q.

equations for $O_s^{\mathcal{T}}$ are, explicitly:

$$(O_\beta^{\mathcal{T}})^2 = \frac{\bar{q}_T + (1+O_\beta^{\mathcal{T}})\bar{q}_N}{2+O_\beta^{\mathcal{T}}}, \tag{A.20}$$

and

$$a(O_\gamma^{\mathcal{T}})^2 = \frac{\bar{q}_T + (1+O_\gamma^{\mathcal{T}})\bar{q}_N}{2+O_\gamma^{\mathcal{T}}}, \tag{A.21}$$

so that $a(O_\gamma^{\mathcal{T}})^2 > (O_\beta^{\mathcal{T}})^2$ as $O_\gamma^{\mathcal{T}} < O_\beta^{\mathcal{T}}$ and $\bar{q}_T > \bar{q}_N$. Note that equation (A.21) proves that $a(O_\gamma^{\mathcal{T}})^2 > \bar{q}_N$ as claimed at the start of the proof. Hence, in order to show that $D^{\mathcal{T}} > D^{\mathcal{N}}$, it is sufficient to show that $O_\beta^{\mathcal{T}} > O_\beta^{\mathcal{N}}$.

Step 2. I approximate $O_\beta^{\mathcal{T}}$ down to $\tilde{O}_\beta^{\mathcal{T}}$ by approximating $\bar{q}_N$ and $\bar{q}_T$ down. As I deal almost solely with the trade-signal regime in this step, I suppress the superscript $\mathcal{T}$.

Consider the trade-signal regime. Let $\tilde{q}_i$ equal

$$\tilde{q}_i = \delta\left[\frac{\pi(\tilde{q}_i,i)}{2}a(1-\tilde{O}_\gamma+\tilde{O}_\gamma^2) + \frac{1-\pi(\tilde{q}_i,i)}{2}(1-\tilde{O}_\beta+\tilde{O}_\beta^2)\right], \tag{A.22}$$

for $i = T, N$, where $\tilde{O}_s$ on the RHS is the amount of old buyers in state s if the cutoffs are $\tilde{q}_i$ instead of $\bar{q}_i$. I show that $\tilde{q}_i$ is less than the equilibrium cutoff $\bar{q}_i$.

The equilibrium cutoff $\bar{q}_i$ satisfies

$$\bar{q}_i = \delta\left[\pi(\bar{q}_i,i)\mathbb{E}_{H_\gamma}(q) + (1-\pi(\bar{q}_i,i))\mathbb{E}_{H_\beta}(q)\right]. \tag{A.23}$$

I show that $\mathbb{E}_{H_\gamma}(q) > \frac{a}{2}(1-O_\gamma+O_\gamma^2)$ and $\mathbb{E}_{H_\beta}(q) > \frac{1}{2}(1-O_\beta+O_\beta^2)$. The expected value of q in state s can be written as

$$\mathbb{E}_{H_s}(q) = \frac{1}{2xO_s}\left[x^2 - \frac{x^2-\bar{q}_T^2}{1+O_s} - \frac{\bar{q}_T^2-\bar{q}_N^2}{2+O_s}\right],$$

where $x = a$ if $s = \gamma$ and $x = 1$ if $s = \beta$, so that $\mathbb{E}_{H_s}(q) > \frac{x}{2}(1-O_s+O_s^2)$ is equivalent to

$$x^2 - \frac{x^2 - \bar{q}_T^2}{1+O_s} - \frac{\bar{q}_T^2 - \bar{q}_N^2}{2+O_s} > x^2 O_s(1 - O_s + O_s^2).$$

Rearranging the inequality and using equations (A.20) and (A.21), I can instead show that

$$\frac{\bar{q}_T}{\bar{q}_T + (1+O_s)\bar{q}_N} > \frac{1}{2+O_s},$$

which holds as $\bar{q}_T > \bar{q}_N$. Hence, the RHS of equation (A.23) is approximated down by substituting $\mathbb{E}_{H_s}(q)$ with $\frac{x}{2}(1 - O_s + O_s^2)$ for $s = \gamma, \beta$:

$$\bar{q}_i > \delta\left[\frac{\pi(\bar{q}_i, i)}{2} a(1 - O_\gamma + O_\gamma^2) + \frac{1 - \pi(\bar{q}_i, i)}{2}(1 - O_\beta + O_\beta^2)\right] := Y(\bar{q}_i). \tag{A.24}$$

A sufficient condition for $Y(\bar{q}_i)$ to be monotonically increasing in the equilibrium cutoff $\bar{q}_i$ is that $2O_\gamma > 1$ as

$$\frac{\partial Y(\bar{q}_i)}{\partial \bar{q}_i} = \frac{\delta}{2}\left\{\frac{\partial \pi(\bar{q}_i, i)}{\partial \bar{q}_i}[a(1 - O_\gamma + O_\gamma^2) - (1 - O_\beta + O_\beta^2)]\right.$$

$$\left. + \frac{\partial O_\gamma}{\partial \bar{q}_i}\pi(\bar{q}_i, i)a(2O_\gamma - 1) + \frac{\partial O_\beta}{\partial \bar{q}_i}(1 - \pi(\bar{q}_i, i))(2O_\beta - 1)\right\},$$

and $\frac{\partial O_s}{\partial \bar{q}_i} > 0$, $\frac{\partial \pi(\bar{q}_i, i)}{\partial \bar{q}_i} > 0$, and $2O_\beta > 1$. First, $\frac{\partial O_s}{\partial \bar{q}_i} > 0$, as

$$\frac{\partial O_s}{\partial \bar{q}_i} = \frac{k}{2xO_s(2+O_s) + xO_s^2 - \bar{q}_N} > 0,$$

where $k = 1$ for $i = T$ and $k = 1 + O_s$ for $i = N$, and $x = a$ for $s = \gamma$ and $x = 1$ for $s = \beta$.

Second, $\frac{\partial \pi(\bar{q}_i, i)}{\partial \bar{q}_i} > 0$ as

$$\frac{\partial \pi(\bar{q}_T, T)}{\partial \bar{q}_T} \propto (1 + O_\gamma)\frac{\partial O_\beta}{\partial \bar{q}_T} - (1 + O_\beta)\frac{\partial O_\gamma}{\partial \bar{q}_T} > 0,$$

if

$$4(aO_\gamma - O_\beta) + 7(aO_\gamma^2 - O_\beta^2) + 3(aO_\gamma^3 - O_\beta^3) + \bar{q}_N(O_\beta - O_\gamma) > 0.$$

Since $aO_\gamma > \sqrt{a}O_\beta$, $aO_\gamma^2 > O_\beta^2$, and $aO_\gamma^3 > \frac{O_\beta^3}{\sqrt{a}}$, the LHS of the above is greater than

$$4O_\beta(\sqrt{a}-1) + 3O_\beta^3\frac{1}{\sqrt{a}}(1-\sqrt{a}) \propto 4\sqrt{a} - 3O_\beta^2,$$

which is positive. Hence, $\frac{\partial \pi(\bar{q}_T, T)}{\partial \bar{q}_T} > 0$.

Third, $\frac{\partial \pi(\bar{q}_N, N)}{\partial \bar{q}_N} > 0$. A sufficient condition for the derivative

$$\frac{\partial \pi(\bar{q}_N, N)}{\partial \bar{q}_N} \propto (2+O_\gamma)\frac{\partial O_\beta}{\partial \bar{q}_N} - (2+O_\beta)\frac{\partial O_\gamma}{\partial \bar{q}_N},$$

to be positive is that $\frac{\partial \pi(\bar{q}_T, T)}{\partial \bar{q}_T} > 0$, which is satisfied.

Finally, $2O_\beta > 1$ as $O_\beta^2 > \bar{q}_N > \bar{q}_\beta$ and $\bar{q}_\beta > \frac{1}{4}$ for all $\delta > \frac{2}{3}$. Altogether, $\frac{\partial Y(\bar{q}_i)}{\partial \bar{q}_i} > 0$ for sure if $2O_\gamma > 1$.

The RHS of expression (A.24), $Y(\bar{q}_i)$, evaluated at $\bar{q}_i = 0$ is greater than zero and at $\bar{q}_i = 1$, less than one. Hence, the cutoff that solves equation (A.22), $\tilde{q}_i$, is less than $\bar{q}_i$. Note that $\bar{q}_\gamma > \tilde{q}_T > \tilde{q}_N > \bar{q}_\beta$ holds (because $\tilde{q}_T$ can be approximated up to $\bar{q}_\gamma$ by setting $\pi(\tilde{q}_T, T) = 1$ and $\tilde{q}_N$ down to $\bar{q}_\beta$ by setting $\pi(\tilde{q}_N, N) = 0$). Hence, $\bar{q}_N > \bar{q}_\beta$ holds, which was claimed at the start of this proof.

The amount of old buyers in state s if the cutoffs are $\tilde{q}_i$ instead of $\bar{q}_i$, $\tilde{O}_\beta^{\mathcal{T}}$, is, explicitly,

$$(\tilde{O}_\beta^{\mathcal{T}})^2 = \frac{\tilde{q}_T + (1+\tilde{O}_\beta^{\mathcal{T}})\tilde{q}_N}{2+\tilde{O}_\beta^{\mathcal{T}}}, \tag{A.25}$$

with $\tilde{O}_\beta^{\mathcal{T}} < O_\beta^{\mathcal{T}}$ because $O_\beta(\bar{q}_i)$ increases in the cutoff $\bar{q}_i$. Hence, to prove the Proposition, it suffices to show that $\tilde{O}_\beta^{\mathcal{T}} > O_\beta^{\mathcal{N}}$. Recall that $\bar{q}_\beta \geq \frac{1}{4}$ for all $\delta \in [2/3, 1]$ so that $\tilde{O}_\beta^{\mathcal{T}}, \sqrt{a}\tilde{O}_\gamma^{\mathcal{T}} > \frac{1}{2}$.

Step 3. I write comparable equations for $O_\beta^{\mathcal{N}}$ and $\tilde{O}_\beta^{\mathcal{T}}$.

Consider the no-signals' regime. Suppressing the superscript $\mathcal{N}$, I can rewrite the equation for $\bar{q}$, (A.14), as

$$2O_\beta^2 = \delta[1 - O_\beta + O_\beta^2 + \pi(\bar{q})(a - 1 + O_\beta - \sqrt{a}O_\beta)] := Z_1(O_\beta), \qquad \text{(A.26)}$$

by multiplying the equation through by $2 - \delta$, substituting O_γ with $\frac{O_\beta}{\sqrt{a}}$, and collecting terms.

Consider the trade-signal regime and suppress $\mathcal{T}$. I insert $\tilde{q}_i$ as defined by equation (A.22) into the equation for $(\tilde{O}_\beta^{\mathcal{T}})^2$, (A.25), and rearrange to get an expression comparable to equation (A.26):

$$2O_\beta^2 = \delta[1 - O_\beta + O_\beta^2 + \frac{\pi(\tilde{q}_T, T) + (1 + O_\beta)\pi(\tilde{q}_N, N)}{2 + O_\beta}(a - 1 + aO_\gamma^2 - O_\beta^2 + O_\beta - aO_\gamma)], \qquad \text{(A.27)}$$

where $O_\beta := \tilde{O}_\beta^{\mathcal{T}}$ and $O_\gamma := \tilde{O}_\gamma^{\mathcal{T}}$.

Step 4. In a sequence of algebra-heavy steps, I show that, as a function of O_β, the RHS of equation (A.27) is higher than the RHS of equation (A.26) for all O_β. This implies that $\tilde{O}_\beta^{\mathcal{T}} > O_\beta^{\mathcal{N}}$.

Consider equations (A.26) and (A.27) as functions of O_β. Since the last term on the RHS of equation (A.27) increases in O_γ if $O_\gamma > \frac{1}{2}$ and $O_\gamma > \frac{O_\beta}{\sqrt{a}}$ under the trade-signal regime, I approximate the RHS of equation (A.27) down by substituting O_γ with $\frac{O_\beta}{\sqrt{a}}$ in the last term:

$$1 - O_\beta + \frac{\pi(\tilde{q}_T, T) + (1 + O_\beta)\pi(\tilde{q}_N, N)}{2 + O_\beta}(a - 1 + aO_\gamma^2 - O_\beta^2 + O_\beta - aO_\gamma)$$

$$> 1 - O_\beta + \frac{\pi(\tilde{q}_T, T) + (1 + O_\beta)\pi(\tilde{q}_N, N)}{2 + O_\beta}(a - 1 + O_\beta^2 - O_\beta^2 + O_\beta - \sqrt{a}O_\beta) := Z_2(O_\beta).$$

To show that $(\tilde{O}_\beta^{\mathcal{T}})^2 > (O_\beta^{\mathcal{N}})^2$, it is sufficient to show that $Z_2(O_\beta) > Z_1(O_\beta)$

for all O_β. Equivalently, it is sufficient to show that

$$\frac{\pi(\tilde{q}_T, T) + (1+O_\beta)\pi(\tilde{q}_N, N)}{2+O_\beta} > \pi(\bar{q}). \tag{A.28}$$

Inequality (A.28) can be rewritten by using equations (A.13), (A.9), (A.10) (and because under the no-signals' regime $O_\gamma = \frac{O_\beta}{\sqrt{a}}$) as

$$\frac{(1+O_\beta)}{(1+O_\beta)^2 + a(1+O_\gamma)^2} + \frac{(2+O_\beta)}{(2+O_\beta)+a(2+O_\gamma)} > \frac{(2+O_\beta)}{(1+O_\beta)+a+\sqrt{a}O_\beta}, \tag{A.29}$$

for all O_β and $O_\gamma := \tilde{O}_\gamma^{\mathcal{T}}$ as defined by

$$a(\tilde{O}_\gamma^{\mathcal{T}})^2 = \frac{\tilde{q}_T + (1+\tilde{O}_\gamma^{\mathcal{T}})\tilde{q}_N}{2+\tilde{O}_\gamma^{\mathcal{T}}}. \tag{A.30}$$

Note that $a(\tilde{O}_\gamma^{\mathcal{T}})^2 > (\tilde{O}_\beta^{\mathcal{T}})^2$. After cross-multiplying inequality (A.29) and collecting terms, it becomes

$$\underbrace{a^2\left[(1+O_\beta)\left(1+\frac{O_\beta}{\sqrt{a}}\right)(2+O_\gamma) - (1+O_\gamma)^2(2+O_\beta)\right]}_{A_1}$$

$$+\underbrace{a(O_\beta - O_\gamma)(1-O_\beta-O_\beta^2) + \sqrt{a}(2+O_\beta)(O_\beta^2 - \sqrt{a}O_\gamma^2)}_{A_2}$$

$$\underbrace{-\sqrt{a}(\sqrt{a}O_\gamma - O_\beta)(2+O_\beta)[1+(1+O_\beta)^2 + a(1+O_\gamma)^2]}_{A_3} > 0.$$

I show separately that $A_1 > 0$ and $A_2 + A_3 > 0$.

Step 4a. $A_1 > 0$.

Showing that $A_1 > 0$ is equivalent to showing that $\frac{A_1}{a\sqrt{a}} > 0$ or, after expanding the LHS of $\frac{A_1}{a\sqrt{a}} > 0$ and collecting terms, that

$$\underbrace{2(O_\beta^2 - \sqrt{a}O_\gamma^2) - (\sqrt{a}-1)O_\gamma O_\beta}_{B_1} + \underbrace{\sqrt{a}(O_\beta - O_\gamma) - (2+O_\gamma O_\beta)(\sqrt{a}O_\gamma - O_\beta)}_{B_2} > 0.$$

The first part, B_1, is greater than

$$B_1' := 2(O_\beta^2 - \sqrt{a}O_\gamma^2) - (\sqrt{a}-1)O_\beta^2 = \frac{1}{\sqrt{a}}[(3\sqrt{a}-a)O_\beta^2 - 2aO_\gamma^2],$$

as $O_\gamma < O_\beta$. Inserting the equations for O_β^2 and aO_γ^2, (A.25) and (A.30) respectively, B_1' can be rewritten as

$$B_1' = \frac{(3\sqrt{a}-a)(2+O_\gamma)[\tilde{q}_T + (1+O_\beta)\tilde{q}_N] - 2(2+O_\beta)[\tilde{q}_T + (1+O_\gamma)\tilde{q}_N]}{\sqrt{a}(2+O_\beta)(2+O_\gamma)}$$

$$> -\frac{2(O_\beta - O_\gamma)(\tilde{q}_T - \tilde{q}_N)}{\sqrt{a}(2+O_\beta)(2+O_\gamma)} =: B_1'',$$

where the last step follows from $3\sqrt{a}-a \geq 2$ for all $a \in (1,2]$. Hence, $B_1 > B_1''$.

The other part, B_2, can be written as

$$B_2 = \sqrt{a}(O_\beta - O_\gamma) - (2 + O_\gamma O_\beta)\frac{(O_\beta - O_\gamma)(\tilde{q}_T - \tilde{q}_N)}{(\sqrt{a}O_\gamma + O_\beta)(2+O_\beta)(2+O_\gamma)},$$

where the fraction is a rewritten version of the difference of equations (A.30) and (A.25).

Now $A_1 = a\sqrt{a}(B_1 + B_2) > a\sqrt{a}(B_1'' + B_2) > 0$ and showing the last inequality is equivalent to showing that $B_1'' + B_2 > 0$, or

$$\sqrt{a}(O_\beta - O_\gamma) - \frac{(2+O_\gamma O_\beta)(O_\beta - O_\gamma)(\tilde{q}_T - \tilde{q}_N)}{(\sqrt{a}O_\gamma + O_\beta)(2+O_\beta)(2+O_\gamma)} - \frac{2(O_\beta - O_\gamma)(\tilde{q}_T - \tilde{q}_N)}{\sqrt{a}(2+O_\beta)(2+O_\gamma)} > 0.$$

I approximate the LHS down in several steps and show that the resulting expression is positive.

First, I show that $(\sqrt{a}-1)\tilde{q}_N > \tilde{q}_T - \tilde{q}_N$ or, $\sqrt{a}\tilde{q}_N > \tilde{q}_T$. Showing the latter is equivalent to showing that $\sqrt{a}\pi(\tilde{q}_N, N) > \pi(\tilde{q}_T, T)$, or

$$\sqrt{a}\frac{2+O_\beta}{2+O_\beta + a(2+O_\gamma)} > \frac{(1+O_\beta)^2}{(1+O_\beta)^2 + a(1+O_\gamma)^2}.$$

Cross-multiplying the inequality and collecting terms gives

$$(\sqrt{a}-1)(2+O_\beta)(1+O_\beta)^2 + a[2(\sqrt{a}-1)(1+O_\gamma O_\beta) - 2(O_\beta^2 - \sqrt{a}O_\gamma^2)$$

$$+(4+O_\gamma O_\beta)(\sqrt{a}O_\gamma - O_\beta) + \sqrt{a}O_\beta - O_\gamma] > 0,$$

which holds as $\sqrt{a}-1 > O_\beta^2 - \sqrt{a}O_\gamma^2$. Hence, I can substitute $(\tilde{q}_T - \tilde{q}_N)$ with $(\sqrt{a}-1)\tilde{q}_N$ and, instead of $B_1'' + B_2 > 0$, show that the following inequality holds:

$$B_2' := a - \frac{(\sqrt{a}-1)\tilde{q}_N[\sqrt{a}(2+O_\gamma O_\beta) + 2(\sqrt{a}O_\gamma + O_\beta)]}{(\sqrt{a}O_\gamma + O_\beta)(2+O_\beta)(2+O_\gamma)} > 0.$$

The LHS of this inequality can be approximated down by making the numerator of the negative fraction larger and the denominator smaller:

$$B_2' > a - \frac{(\sqrt{a}-1)O_\beta^2[\sqrt{a}\cdot 3 + 2(\sqrt{a}+1)]}{(O_\beta + O_\beta)\cdot 2\cdot 2} \propto 8a - (\sqrt{a}-1)O_\beta(5\sqrt{a}+2),$$

where the substitutions work because $\tilde{q}_N < O_\beta^2$ (follows from equation (A.25)), $\sqrt{a}O_\gamma > O_\beta$, and $O_\gamma, O_\beta \in (0,1)$. The last expression can in turn be approximated down by making the negative part larger:

$$8a - (\sqrt{a}-1)O_\beta(5\sqrt{a}+2) > 8a - 5a - 2,$$

as $\sqrt{a}-1 < 1$, $O_\beta < 1$, and $\sqrt{a} < a$. But $3a-2$ is positive, which concludes the proof that $A_1 > 0$.

Step 4b. $A_2 + A_3 > 0$. Recall that

$$A_2 + A_3 = a(O_\beta - O_\gamma)(1 - O_\beta - O_\beta^2) + \sqrt{a}(2+O_\beta)(O_\beta^2 - \sqrt{a}O_\gamma^2)$$

$$-\sqrt{a}(\sqrt{a}O_\gamma - O_\beta)(2+O_\beta)[1+(1+O_\beta)^2 + a(1+O_\gamma)^2].$$

I will show that a more stringent condition than $A_2 + A_3 > 0$ holds by approximating $\sqrt{a}O_\gamma - O_\beta$ up.

The term $\sqrt{a}O_\gamma - O_\beta$ is proportional to the difference between equations (A.30) and (A.25):

$$\sqrt{a}O_\gamma - O_\beta = \frac{(O_\beta - O_\gamma)(\tilde{q}_T - \tilde{q}_N)}{(\sqrt{a}O_\gamma + O_\beta)(2+O_\beta)(2+O_\gamma)}$$

$$= \frac{(O_\beta - O_\gamma)(\pi(\tilde{q}_T, T) - \pi(\tilde{q}_N, N))}{(\sqrt{a}O_\gamma + O_\beta)(2+O_\beta)(2+O_\gamma)} \frac{\delta}{2}[a(1-O_\gamma+O_\gamma^2) - (1-O_\beta+O_\beta^2)] =: B_3$$

where the last step follows from plugging in the definitions of $\tilde{q}_T$ and $\tilde{q}_N$, equation (A.22). I approximate B_3 up by showing that $\pi(\tilde{q}_T, T) - \pi(\tilde{q}_N, N) < \frac{O_\beta - O_\gamma}{3}$. To approximate the difference $\pi(\tilde{q}_T, T) - \pi(\tilde{q}_N, N)$ up, I insert equations (A.9) and (A.10) and collect terms:

$$\pi(\tilde{q}_T, T) - \pi(\tilde{q}_N, N) = \frac{a(O_\beta - O_\gamma)[3 + 2(O_\beta + O_\gamma) + O_\gamma O_\beta]}{[(1+O_\beta)^2 + a(1+O_\gamma)^2][2 + O_\beta + a(2+O_\gamma)]}.$$

Hence, after collecting terms, $\pi(\tilde{q}_T, T) - \pi(\tilde{q}_N, N) < \frac{O_\beta - O_\gamma}{3}$ is equivalent to

$$a - 2 + aO_\gamma(1-a) + 2a(1-a) + a - 4O_\beta + aO_\beta(1-2O_\beta) + a(1-2aO_\gamma)$$

$$< 2O_\beta^2 + O_\beta(1+O_\beta)^2 + aO_\gamma^2(2+O_\beta) + aO_\gamma O_\beta(1+O_\beta) + a^2 O_\gamma[(1+2O_\gamma) + (1+O_\gamma)^2].$$

The LHS of the last inequality is negative because $2 \geq a > 1$, $2O_\beta > 1$ and $2aO_\gamma > 1$ as $2\sqrt{a}O_\gamma > 1$. Hence, the inequality holds, or $\pi(\tilde{q}_T, T) - \pi(\tilde{q}_N, N) < \frac{O_\beta - O_\gamma}{3}$.

Return to B_3. By the last approximation, we know that

$$B_3 < \frac{(O_\beta - O_\gamma)^2}{3(\sqrt{a}O_\gamma + O_\beta)(2+O_\beta)(2+O_\gamma)} \frac{\delta}{2}[a(1-O_\gamma+O_\gamma^2) - (1-O_\beta+O_\beta^2)],$$

and by approximating the last term on the RHS up, that

$$B_3 < \frac{\delta(O_\beta - O_\gamma)^2}{6(\sqrt{a}O_\gamma + O_\beta)(2+O_\beta)(2+O_\gamma)}(a-1),$$

where I have used the fact that $-(aO_\gamma - O_\beta) + aO_\gamma^2 - O_\beta^2 < 0$ (as $O_\gamma < O_\beta$). I approximate the RHS up further by reducing the denominator so that

$$B_3 < \frac{\delta(O_\beta - O_\gamma)^2}{6 \cdot 2O_\beta \cdot (2+O_\beta) \cdot 2}(a-1) = \frac{\delta(O_\beta - O_\gamma)^2(a-1)}{24O_\beta(2+O_\beta)},$$

as $O_\gamma \geq 0$ and $\sqrt{a}O_\gamma > O_\beta$. Altogether, I have shown so far that

$$\sqrt{a}O_\gamma - O_\beta < \frac{\delta(O_\beta - O_\gamma)^2(a-1)}{24O_\beta(2+O_\beta)}.$$

Return to $A_2 + A_3$. We now know that

$$A_2 + A_3 > a(O_\beta - O_\gamma)(1 - O_\beta - O_\beta^2) + \sqrt{a}(2+O_\beta)(O_\beta^2 - \sqrt{a}O_\gamma^2)$$

$$-\sqrt{a}\frac{\delta(O_\beta - O_\gamma)^2(a-1)}{24O_\beta}[1 + (1+O_\beta)^2 + a(1+O_\gamma)^2].$$

Instead of showing that $A_2 + A_3 > 0$, I show that the RHS is positive, or,

$$\underbrace{(2+O_\beta)(O_\beta^2 - \sqrt{a}O_\gamma^2) - \sqrt{a}O_\beta(O_\beta - O_\gamma)}_{B_4}$$

$$\underbrace{+\sqrt{a}(O_\beta - O_\gamma)(1 - O_\beta^2) - \frac{\delta(O_\beta - O_\gamma)^2(a-1)}{24O_\beta}[1 + (1+O_\beta)^2 + a(1+O_\gamma)^2]}_{B_5} > 0.$$

I show in turn that $B_4 > 0$ and $B_5 > 0$.

I rewrite $\sqrt{a}B_4 > 0$ and insert the definitions of O_β^2 and aO_γ^2 to get

$$\sqrt{a}B_4 = (2+O_\beta)\sqrt{a}\frac{\tilde{q}_T + (1+O_\beta)\tilde{q}_N}{2+O_\beta} - (2+O_\beta)\frac{\tilde{q}_T + (1+O_\gamma)\tilde{q}_N}{2+O_\gamma}$$

$$-\frac{O_\beta}{O_\beta+O_\gamma}\left(a\frac{\tilde{q}_T+(1+O_\beta)\tilde{q}_N}{2+O_\beta}-\frac{\tilde{q}_T+(1+O_\gamma)\tilde{q}_N}{2+O_\gamma}\right)>0.$$

Multiplying the inequality through with $(2+O_\gamma)(2+O_\beta)(O_\beta+O_\gamma)$ and collecting terms yields

$$B_4\propto\sqrt{a}(2+O_\gamma)[\tilde{q}_T+(1+O_\beta)\tilde{q}_N][(2+O_\beta)(O_\beta+O_\gamma)-\sqrt{a}O_\beta]$$

$$-(2+O_\beta)[\tilde{q}_T+(1+O_\gamma)\tilde{q}_N][(2+O_\beta)(O_\beta+O_\gamma)-O_\beta]>0.$$

As $O_\beta>O_\gamma$, the above inequality holds for sure if I replace $(1+O_\gamma)\tilde{q}_N$ with $(1+O_\beta)\tilde{q}_N$ in the second row, or (after collecting terms), equivalently, if

$$[O_\gamma(2+O_\beta)+(1+O_\beta)O_\beta](\sqrt{a}O_\gamma-O_\beta)$$

$$+(\sqrt{a}-1)\left[\frac{1}{\sqrt{a}}(4\sqrt{a}O_\gamma-aO_\beta)+O_\gamma O_\beta(2-\sqrt{a})+O_\beta(2-\sqrt{a})+2O_\beta^2\right]>0.$$

The inequality holds as $4\sqrt{a}O_\gamma>2>aO_\beta$. Hence, $B_4>0$.

Finally, I need to show that $B_5>0$. I approximate B_5 down by approximating its second, the negative, term up:

$$B_5>\sqrt{a}(O_\beta-O_\gamma)(1-O_\beta^2)-\frac{\delta(O_\beta-O_\gamma)^2(a-1)}{24\frac{1}{2}}\cdot 12,$$

where I used $O_\beta\in[\frac{1}{2},1)$ and $1+(1+O_\beta)^2+a(1+O_\gamma)^2<12$ because $aO_\gamma^2<\tilde{q}_T<1$. Hence, it is sufficient to show that

$$B_5':=\sqrt{a}(1-O_\beta^2)-\delta(O_\beta-O_\gamma)(a-1)>0.$$

But $O_\gamma>\frac{O_\beta}{\sqrt{a}}$ so B_5' is greater than

$$\sqrt{a}(1-O_\beta^2)-\delta\left(O_\beta-\frac{O_\beta}{\sqrt{a}}\right)(a-1)$$

$$\propto a(1-O_\beta^2) - \delta\left(\sqrt{a}-1\right)O_\beta(a-1) > a(1-O_\beta^2) - \delta\frac{1}{2}O_\beta,$$

where the last follows from noting that $\sqrt{a}-1 < \frac{1}{2}$ and $a-1 \leq 1$. Since $\tilde{q}_T > O_\beta^2$ and $\bar{q}_\gamma > \tilde{q}_T$, it is sufficient to show that

$$2a(1-\bar{q}_\gamma) - \delta\sqrt{\bar{q}_\gamma} > 0,$$

or that

$$2a > 2a\bar{q}_\gamma + \delta\sqrt{\bar{q}_\gamma}. \tag{A.31}$$

The LHS is constant in δ, but the RHS increases in δ as $\bar{q}_\gamma$ increases in δ. The derivative of $\bar{q}_\gamma$ is proportional to

$$\frac{\partial\bar{q}_\gamma}{\partial\delta} \propto (2-\delta)\left[4-2\delta-\sqrt{\delta(8-3\delta)}-\frac{\delta(4-3\delta)}{\sqrt{\delta(8-3\delta)}}\right]+2\delta[4-\delta-\sqrt{\delta(8-3\delta)}]. \tag{A.32}$$

Since $\delta(8-3\delta)$ increases in δ, I know that $\sqrt{\delta(8-3\delta)} \in (2,\sqrt{5})$ for all $\delta \in [2/3,1]$ and $\sqrt{5} < 2.25$. Hence, the RHS of (A.32) is greater than

$$\frac{1}{2}\left[(2-\delta)(8-8\delta-4.5+3\delta^2)+\delta(7-4\delta)\right]$$

$$\propto 7+10\delta^2-12.5\delta-3\delta^3 > 7(1-\delta)^2+3\delta^2(1-\delta) > 0,$$

so that $\frac{\partial\bar{q}_\gamma}{\partial\delta} > 0$.

Hence, it is sufficient to show that inequality (A.31) holds for $\delta = 1$, or, equivalently, that

$$4 > 2a(3-\sqrt{5}) + \sqrt{\frac{2(3-\sqrt{5})}{a}}. \tag{A.33}$$

The derivative of the RHS of (A.33) with respect to a is proportional to $4a^{\frac{3}{2}}(3-$

$\sqrt{5}) - \sqrt{2(3-\sqrt{5})}$. But

$$a^{\frac{3}{2}}(3-\sqrt{5}) - \sqrt{2(3-\sqrt{5})} > 2(3-\sqrt{5}) - \sqrt{2(3-\sqrt{5})} > 0,$$

where the first inequality follows from $a > 1$ and the second from $2(3-\sqrt{5}) > 1$. Hence, the RHS of (A.33) increases in a so that it is sufficient to show that inequality (A.33) holds for $a = 2$, which is true.

Altogether, this shows that $B_5 > 0$, which in turn proves that $A_2 + A_3 > 0$, as required. This completes the proof of the Proposition. □

I now prove that the trade-signal regime can be more efficient than the no-signals' regime if the quality distribution has binary support as claimed on p. 43.

Claim 1. *In a model where the quality distribution has binary support, the trade-signal regime can be more efficient than the no-signals' regime.*

Proof. Consider the same model as in the main part of the paper, except that now qualities can take two values, $q \in \{q_L, q_H\}$. In the bad state, all entering sellers offer quality q_L and in the good state, a fraction x_γ offer the high quality q_H and the rest offer quality q_L. Let $x_\beta := 0$ so that I can treat the two states simultaneously below. Since the aim of this claim is to demonstrate a possibility result, i.e., that the trade-signal regime can be more efficient than the no-signals' regime, I assume (for simplicity) that a buyer's utility from the qualities are $u(q_L) = \frac{1}{2}$ and $u(q_H) = 1$ and that the prior odds ω equal to one. Note that, as in the main model, old buyers optimally accept all qualities and a young buyer optimally accepts q_H. I show that in a region of the parameter space where the unique equilibrium under the no-signals' regime is such that a young buyer rejects a low quality with probability one, an equilibrium exists under the trade-signal regime such that a young buyer rejects a low quality with probability less than one. That is, a buyer is less likely to delay under the trade-signal regime.

First, consider the no-signals' regime and recall that we look for an equilibrium where a young buyer rejects q_L with probability one. Let p be the probability with which the young buyer accepts q_L. A buyer's utility from accepting $q_1 = q_L$ is $\frac{1}{2}$. His continuation value after receiving $q_1 = q_L$ is

$$W^{\mathcal{N}}(p) = \delta\left[\pi(q_L)\left(1 - \lambda_\gamma + \lambda_\gamma\frac{1}{2}\right) + (1 - \pi(q_L))\frac{1}{2}\right],$$

where the posterior belief $\pi(q_L)$ is the probability that the state is good after $q_1 = q_L$ and λ_γ is the equilibrium proportion of low-quality sellers. Both depend on p. A buyer gets utility one only if the state is good and he meets a high-quality seller tomorrow. Otherwise, he gets utility of a half.

The optimal probability for a young buyer to reject $q_1 = q_L$ is $p = 0$ if $W^{\mathcal{N}}(p = 0) \leq \frac{1}{2}$. But if all buyers accept low qualities, the equilibrium distribution is the same as the entry distribution and $\lambda_\gamma(p = 0) = 1 - x_\gamma$.

The optimal probability is $p = 1$ if $W^{\mathcal{N}}(p = 1) \geq \frac{1}{2}$. If all young buyers reject the low quality, then the equilibrium amount of old buyers is

$$O_s = \lambda_s = \frac{1 + O_s - x_s}{1 + O_s},$$

where the last equality follows from noting that the only high-quality seller in the market are those who just entered, in the amount x_s. The solution is $\lambda_\gamma(p = 1) = \sqrt{1 - x_\gamma}$. The posterior odds of a buyer after observing $q_1 = q_L$ are

$$\frac{\pi(q_L)}{1 - \pi(q_L)} = \lambda_\gamma,$$

because the buyer receives a low-quality offer with probability λ_γ in the good state and probability one in the bad state.

Thus, the equilibrium probability that a young buyer rejects q_L is one if the young buyer does not want to deviate, i.e., if $W^{\mathcal{N}}(p=1) \geq \frac{1}{2}$ which can be rearranged to give

$$\delta > \frac{1}{1 + \frac{\sqrt{1-x_\gamma}}{\sqrt{1-x_\gamma}+1}(1 - \sqrt{1-x_\gamma})} := \bar{\delta}_2^{\mathcal{N}}.$$

Likewise, we get that the equilibrium probability that a young buyer rejects q_L is zero if

$$\delta < \frac{1}{1 + \frac{1-x_\gamma}{1-x_\gamma+1} x_\gamma} := \bar{\delta}_1^{\mathcal{N}}.$$

Hence, the equilibrium where a young buyer delays with certainty exists if $\delta > \bar{\delta}_2^{\mathcal{N}}$ and is unique if $\bar{\delta}_1^{\mathcal{N}} < \bar{\delta}_2^{\mathcal{N}}$ (the latter holds for all $x_\gamma \leq \bar{x}_2$). I now show that for some $x_\gamma \leq \bar{x}_2$ and $\delta > \bar{\delta}_2^{\mathcal{N}}$, an equilibrium under the trade-signal regime exists such that a young buyer delays with probability less than one.

Consider the trade-signal regime. I derive the conditions under which in equilibrium a young buyer continues after $q_1 = q_L$ if he observes a trade and accepts $q_1 = q_L$ if he observes no trade. His continuation value after observing $q_1 = q_L$ and signal outcome i is

$$W^{\mathcal{T}}(i) = \delta \left[\pi(q_L, i) \left(1 - \lambda_\gamma + \lambda_\gamma \frac{1}{2} \right) + (1 - \pi(q_L, i)) \frac{1}{2} \right],$$

where the posterior belief $\pi(q_L, i)$ is the probability that the state is good after $q_1 = q_L$ and signal outcome i, and λ_γ is the equilibrium proportion of low-quality sellers. I will derive them now.

A young buyer only becomes old if he observes $q_1 = q_L$ and trade so the amount of old buyers is

$$O_s = \lambda_s t_s,$$

where t_s is the probability of observing a trade in state s. The probability of observing a trade is $t_s = \frac{1}{1+O_s}$ because amount one of buyers trade and exit in each period and the total amount of buyers is $1 + O_s$. Thus, the amount of old buyers can be solved from the equation $O_s(1 + O_s)^2 = 1 + O_s - x_s$, which gives a unique solution O_s that decreases in x_s. Since $x_\gamma > x_\beta$, $O_\gamma < O_\beta$ so that trade is indeed good news as I assumed earlier.

A buyer's posterior odds are

$$\frac{\pi(q_L, T)}{1 - \pi(q_L, T)} = \lambda_\gamma \frac{t_\gamma}{t_\beta} = \frac{(1 + O_\gamma - x_\gamma)(1 + O_\beta)}{(1 + O_\gamma)^2},$$

after a trade and

$$\frac{\pi(q_L, N)}{1 - \pi(q_L, N)} = \lambda_\gamma \frac{1 - t_\gamma}{1 - t_\beta} = \frac{(1 + O_\gamma - x_\gamma)(1 + O_\beta)O_\gamma}{(1 + O_\gamma)^2 O_\beta},$$

after no trade. A buyer is more optimistic after a trade, or $\pi(q_L, T) > \pi(q_L, N)$ because $O_\gamma < O_\beta$.

Altogether, it is optimal for a young buyer to continue after observing $q_1 = q_L$ and a trade if $W^{\mathcal{T}}(T) \geq \frac{1}{2}$ and to accept $q_1 = q_L$ after observing $q_1 = q_L$ and no trade if $W^{\mathcal{T}}(N) \leq \frac{1}{2}$. The conditions can be rearranged to give

$$\delta \geq \frac{1}{1 + \frac{(1+O_\gamma - x_\gamma)(1+O_\beta)}{(1+O_\gamma)^2 + (1+O_\gamma - x_\gamma)(1+O_\beta)} \frac{x_\gamma}{1+O_\gamma}} =: \bar{\delta}_1^{\mathcal{T}},$$

and

$$\delta \leq \frac{1}{1 + \frac{(1+O_\gamma - x_\gamma)(1+O_\beta)O_\gamma}{(1+O_\gamma)^2 O_\beta + (1+O_\gamma - x_\gamma)(1+O_\beta)O_\gamma} \frac{x_\gamma}{1+O_\gamma}} =: \bar{\delta}_2^{\mathcal{T}}.$$

The fact that there are more old buyers in the bad state guarantees that $\bar{\delta}_1^{\mathcal{T}} < \bar{\delta}_2^{\mathcal{T}}$ always holds.

Finally, to prove that the trade-signal regime can be more efficient than the no-signals' regime, I need to show that within the region of the parameter space where in the unique equilibrium under the no-signals' regime young

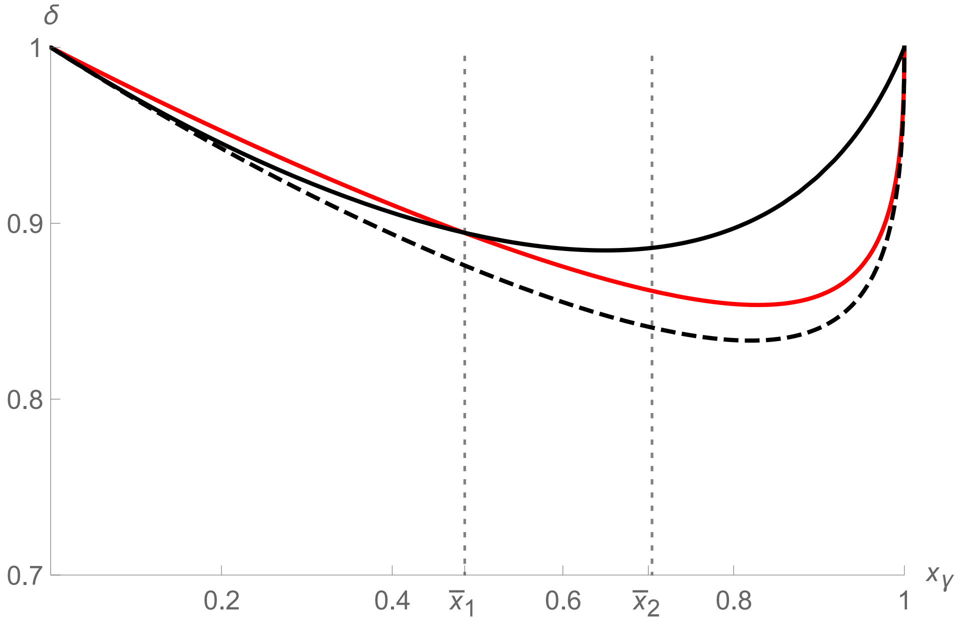

Figure A.1: Trade-signal regime is more efficient than no-signals' regime if $x_\gamma \in (\bar{x}_1, \bar{x}_2)$ and $\delta \in (\bar{\delta}_2^{\mathcal{N}}, \bar{\delta}_2^{\mathcal{T}})$: $\bar{\delta}_2^{\mathcal{N}}$ (red), $\bar{\delta}_1^{\mathcal{T}}$ (dashed) and $\bar{\delta}_2^{\mathcal{T}}$ (black).

buyers delay with certainty, an equilibrium of the above type exists under the trade-signal regime. That is, I need to show that $\bar{\delta}_2^{\mathcal{N}} < \bar{\delta}_2^{\mathcal{T}}$ is possible for some $x_\gamma \leq \bar{x}_2$. Figure A.1 illustrates that there exists a unique $\bar{x}_1 < \bar{x}_2$ such that if $x_\gamma = \bar{x}_1$, then $\bar{\delta}_2^{\mathcal{N}} = \bar{\delta}_2^{\mathcal{T}}$ and for all $x_\gamma > \bar{x}_1$, $\bar{\delta}_2^{\mathcal{N}} < \bar{\delta}_2^{\mathcal{T}}$, as required. □

Here, I formally prove that payoff externalities matter for the efficiency results as claimed on pp. 43 and 47.

Claim 2. *For any private signal precisions $p_G \in [0,1]$ and $p_B \in [0,1]$ with $p_G \geq 1 - p_B$, if we assume that the equilibrium distribution is the same as the entry distribution, the delay is independent of p_G and p_B.*

Proof. Consider the exogenous signal regime with some given $P(G|\gamma) = p_G \in [0,1]$ and $P(B|\beta) = p_B \in [0,1]$ with $p_G \geq 1 - p_B$ (so that G is at least weakly better news than B) and assume that the equilibrium distribution in state s is given by F_s. I aim to show that the delay, $D^{\mathcal{E}}$, is independent of both p_G and p_B.

The buyers' equilibrium beliefs are derived analogously to equation (A.1), except that now the equilibrium density of any $q \in [0,1]$ is one in the bad state and $\frac{1}{a}$ in the good state. Hence, the belief after $(\bar{q}_G, G)$, or equation (A.35), becomes

$$\frac{\pi(\bar{q}_G, G)}{1-\pi(\bar{q}_G, G)} = \omega \frac{f_\gamma(\bar{q}_G)}{f_\beta(\bar{q}_G)} \frac{p_G}{1-p_B} = \omega \frac{p_G}{a(1-p_B)},$$

and the belief after $(\bar{q}_B, B)$ is

$$\frac{\pi(\bar{q}_B, B)}{1-\pi(\bar{q}_B, B)} = \omega \frac{f_\gamma(\bar{q}_B)}{f_\beta(\bar{q}_B)} \frac{1-p_B}{p_G} = \omega \frac{1-p_B}{ap_G}.$$

The buyer's expected value of continuing in state γ is $\frac{a}{2}$ and in state β is $\frac{1}{2}$. An equilibrium cutoff must again equal a buyer's discounted expected continuation value, which, after collecting terms, gives

$$\bar{q}_G = \frac{\delta a}{2} \frac{\omega p_G + (1-p_B)}{\omega p_G + a(1-p_B)},$$

and

$$\bar{q}_B = \frac{\delta a}{2} \frac{\omega(1-p_G) + p_B}{\omega(1-p_G) + ap_B}.$$

Since a buyer becomes old only if he observes an unacceptable offer when young, the probability of becoming old in state s is

$$O_s = P(G|s)F_s(\bar{q}_G) + P(B|s)F_s(\bar{q}_B).$$

As a result, the expected delay is explicitly

$$D = \pi \left[p_G \frac{\bar{q}_G}{a} + (1-p_G)\frac{\bar{q}_B}{a} \right] + (1-\pi)\left[(1-p_B)\bar{q}_G + p_B \bar{q}_B\right]$$

which can be rearranged to give

$$D = 1 + (1-\pi)\frac{1}{a}\left\{\bar{q}_B\left[\omega(1-p_G) + ap_B\right] + \bar{q}_G[\omega p_G + a(1-p_B)]\right\} = 1 + \frac{\delta}{2},$$

where the last step follows from inserting the explicit solutions for $\bar{q}_G$ and $\bar{q}_B$ and collecting terms. □

A.5 Exogenous signal

Proof of Proposition 8. I derive the system of equations that the equilibrium cutoffs $\bar{q}_G$ and $\bar{q}_B$ satisfy. I write out the mean quality in state s and the beliefs relevant for the cutoffs, and then plug them into equation (A.8). Most of the detail that I need for the derivation is presented in the proofs of Proposition 2 and of Proposition 5.

The derivation of the mean quality in the proof of Proposition 5, (A.17), works for the exogenous-signal regime, too, hence, I can plug in the exogenous signal's precision into (A.17) to get

$$\mathbb{E}_{H_\gamma}(q) = \frac{a^2 O_\gamma + \bar{q}_G^2}{2aO_\gamma(1+O_\gamma)}, \tag{A.34}$$

and

$$\mathbb{E}_{H_\beta}(q) = \frac{1}{2O_\beta}\left[1 - \frac{p_B}{p_B + O_\beta}(\bar{q}_G^2 - \bar{q}_B^2) - \frac{1}{1+O_\beta}(1-\bar{q}_G^2)\right].$$

Since the posterior of a buyer is zero after B, the explicit form for $\mathbb{E}_{H_\beta}(q)$ plugged into (A.8) gives the second equation in the statement of the Proposition.

After $(q_1, i) = (\bar{q}_G, G)$ the buyers' beliefs are, explicitly,

$$\frac{\pi(\bar{q}_G, G)}{1-\pi(\bar{q}_G, G)} = \omega\frac{h_\gamma(\bar{q}_G)}{h_\beta(\bar{q}_G)}\frac{p_G}{1-p_B} = \frac{\omega}{a}\frac{1+O_\beta}{1+O_\gamma}\frac{1}{1-p_B}, \tag{A.35}$$

which are used in the first equation in the statement of the Proposition. To obtain the exact version of the first equation, I need to derive the explicit form of the probability of becoming old in the good state. As only signal realisation G occurs in the good state, only cutoff $\bar{q}_G$ is used in the good state so the derivation of O_γ works exactly as under the known-state and no-signals' regimes. Accordingly, the functional form of O_γ is the same under the three regimes:

$$O_\gamma = \sqrt{F_\gamma(\bar{q}_G)}. \tag{A.36}$$

I simplify equation (A.7) for $s = \beta$:

$$O_\beta^2 = \frac{(1-p_B)O_\beta \bar{q}_G + p_B(1+O_\beta)\bar{q}_B}{O_\beta + p_B}. \tag{A.37}$$

Equation (A.37) is the third equation in the statement of the Proposition. To get the exact form of the first equation, I use (A.36) in (A.34) and (A.35) that, together with the second equation in the statement of the Proposition, in turn are plugged into (A.8). This completes the proof of the Proposition.

In order to complete the proof of Proposition 2, I need to show that $\bar{q}_\gamma > \bar{q}_G$ and $\bar{q}_B > \bar{q}_\beta$.

I first show that show that $\bar{q}_\gamma > \bar{q}_G$. Under the exogenous-signal regime, the cutoff $\bar{q}_G$ satisfies

$$\bar{q}_G = \delta[\alpha \mathbb{E}_{H_\gamma^\varepsilon}(q) + (1-\alpha)\mathbb{E}_{H_\beta^\varepsilon}(q)],$$

for an appropriate α so that the cutoff is less than the discounted mean quality in the good state: $\bar{q}_G < \delta \mathbb{E}_{H_\gamma^\varepsilon}(q)$. Using, (A.34) and (A.36), the discounted mean quality in the good state can be written explicitly as

$$\delta \mathbb{E}_{H_\gamma^\varepsilon}(q) = \frac{\delta}{2}(a - \sqrt{a\bar{q}_G} + \bar{q}_G),$$

so that

$$\bar{q}_G < \frac{\delta}{2-\delta}(a - \sqrt{a\bar{q}_G}).$$

But recall that under the known-state regime, cutoff $\bar{q}_\gamma$ satisfies equation

$$\bar{q}_\gamma = \frac{\delta}{2-\delta}(a - \sqrt{a\bar{q}_\gamma}).$$

As $\frac{\delta}{2}(a - \sqrt{aq})$ decreases in q, it must be that $\bar{q}_G < \bar{q}_\gamma$.

Finally, I show that $\bar{q}_B > \bar{q}_\beta$. Under the exogenous-signal regime, the cutoff $\bar{q}_B$ satisfies

$$\bar{q}_B = \delta \mathbb{E}_{H_\beta^{\mathcal{E}}}(q),$$

and under the known-state regime, the cutoff $\bar{q}_\beta$ satisfies

$$\bar{q}_\beta = \delta \mathbb{E}_{H_\beta^{\mathcal{K}}}(q).$$

I compare the mean quality in the bad state in the case where the buyers who know that the state is bad use the optimal cutoff $\tilde{q}$ and others use $\bar{q}_G > \tilde{q}$ (as is the case under the exogenous-signal regime) to the case where all buyers use cutoff $\tilde{q}$ (as is the case under the known-state regime). In case both cutoffs $\tilde{q}$ and $\bar{q}_G$ are used, the qualities $q > \tilde{q}$ that are left on the market by some buyers increase the discounted mean quality in the market (because $\tilde{q}$ equals the discounted mean quality) . Thus, the mean quality must be higher in the market where some buyers use cutoff $\tilde{q}$ and others $\bar{q}_G > \tilde{q}$ as compared to a market where all buyers use cutoff $\tilde{q}$. Hence, it must be that $\bar{q}_B > \bar{q}_\beta$. □

Proof of Proposition 9. The expected delay is obtained by plugging equation (A.7) into (2.3). I show that the exogenous-signal regime with $p_G = 1$ and $p_B = 1 - \eta$ for $\eta > 0$ small is more efficient than the known-state regime, i.e., $D^{\mathcal{E}} < D^{\mathcal{K}}$, (Part 1). I then prove that the exogenous-signal regime with $p_G = 1$

and $p_B = 1-\eta$ for $\eta > 0$ small is more efficient than the the no-signals' regime if $\delta > \frac{2}{3}$ (Part 2a) and with $p_G = 1$ and $p_B = 1 - \eta$ for $\eta \approx 1$ if $\delta < \frac{2}{3}$ (Part 2b).

Consider the exogenous-signal regime with $p_G = 1$ and $p_B = 1 - \eta$, $\eta \in (0, 1)$. Recall that $p_G = 1$ means that in the good state only cutoff $\bar{q}_G$ is used and $O_\gamma = \sqrt{F_\gamma(\bar{q}_G)}$ (see (A.36)). Hence, for any $\eta \in (0, 1)$, if the precision of the exogenous signal is $p_G = 1$ and $p_B = 1 - \eta$, the expected delay under the exogenous-signal regime is

$$D^{\mathcal{E}} = \pi\sqrt{\frac{\bar{q}_G}{a}} + (1-\pi)O_\beta,$$

where $O_\beta = (1-\eta)\frac{\bar{q}_B}{O_\beta} + \eta\left(\frac{\bar{q}_B}{O_\beta} + \frac{\bar{q}_G - \bar{q}_B}{O_\beta + 1 - \eta}\right)$ (which is obtained by plugging $p_B = 1 - \eta$ into (A.37)).

Part 1. $D^{\mathcal{E}} < D^{\mathcal{K}}$ if the precision of the exogenous signal is $p_G = 1$ and $p_B = 1 - \eta$ for $\eta > 0$ small.

If $p_G = p_B = 1$, then $D^{\mathcal{E}} = D^{\mathcal{K}}$. I show that $D^{\mathcal{E}} < D^{\mathcal{K}}$ for $p_G = 1$ and $p_B = 1 - \eta$ for $\eta > 0$ small by proving that the derivative of $D^{\mathcal{E}}$ with respect to η is negative as $\eta \to 0$.

The derivative of the expected delay under the exogenous-signal regime with respect to η is

$$\frac{\partial D^{\mathcal{E}}}{\partial \eta} = \frac{\pi}{2}\frac{\bar{q}'_G}{\sqrt{a\bar{q}_G}} + (1-\pi)O'_\beta \propto \frac{\omega}{2}\frac{\bar{q}'_G}{\sqrt{a\bar{q}_G}} + O'_\beta, \tag{A.38}$$

where $\bar{q}'_G := \frac{\partial \bar{q}_G}{\partial \eta}$ and $O'_\beta := \frac{\partial O_\beta}{\partial \eta}$. I use the rewritten versions of the three equations in the statement of Proposition 8 to derive the missing derivatives in equation (A.38). The equations give

$$Y_1 := O_\beta^2(O_\beta + 1 - \eta) - \bar{q}_B(O_\beta + 1)(1 - \eta) - \bar{q}_G\eta O_\beta = 0,$$

$$Y_2 := 2\bar{q}_B - \delta\bar{q}_B^2 O_\beta^{-1} - \delta(\bar{q}_G^2 - \bar{q}_B^2)(O_\beta + 1 - \eta)^{-1} - \delta(1 - \bar{q}_G^2)(O_\beta + 1)^{-1} = 0,$$

$$Y_3 := \omega(1 + O_\beta)[\delta(a + \bar{q}_G - \sqrt{a\bar{q}_G}) - 2\bar{q}_G] + 2\eta(\bar{q}_B - \bar{q}_G)(a + \sqrt{a\bar{q}_G}) = 0.$$

The derivatives required for $\frac{\partial D^{\mathcal{E}}}{\partial \eta}$ are obtained by totally differentiating Y_1 through Y_3. In the limit as $\eta \to 0$, $\bar{q}_B \to \bar{q}_\beta$ (so that $O_\beta^2 \to \bar{q}_B$) and $\bar{q}_G \to \bar{q}_\gamma$, so that the derivatives become

$$\lim_{\eta\to 0} O'_\beta = \frac{\bar{q}'_B(O_\beta + 1) + (\bar{q}_G - \bar{q}_B)O_\beta}{2O_\beta(O_\beta + 1)},$$

$$\lim_{\eta\to 0} \bar{q}'_B = -\frac{\delta}{2(O_\beta + 1)} \frac{\bar{q}_B O'_\beta(O_\beta + 1)^2 + (\bar{q}_G^2 - \bar{q}_B^2)(O'_\beta - 1) + (1 - \bar{q}_G^2)O'_\beta}{O_\beta + 1 - \delta O_\beta},$$

and

$$\lim_{\eta\to 0} \bar{q}'_G = \frac{4(\bar{q}_G - \bar{q}_B)(a + \sqrt{a\bar{q}_G})}{\omega(1 + O_\beta)\left[\delta\left(2 - \sqrt{\frac{a}{\bar{q}_G}}\right) - 4\right]}.$$

Inserting the second into the first, we get

$$\lim_{\eta\to 0} O'_\beta = \frac{(\bar{q}_G - \bar{q}_B)(2\bar{q}_B - 2\delta\bar{q}_B + 2O_\beta + \delta\bar{q}_G)}{\delta + 8\bar{q}_B - 3\delta\bar{q}_B + 4\bar{q}_B O_\beta - \delta\bar{q}_B O_\beta + 4O_\beta}.$$

Thus, in the limit, the derivative $\frac{\partial D^{\mathcal{E}}}{\partial \eta}$ is proportional to

$$\lim_{\eta\to 0} \frac{\partial D^{\mathcal{E}}}{\partial \eta} \propto -\frac{2(a + \sqrt{a\bar{q}_G})}{(4\sqrt{a\bar{q}_G} - 2\delta\sqrt{a\bar{q}_G} + \delta a)} + \frac{(1 + O_\beta)(2\bar{q}_B - \delta\bar{q}_B + 2O_\beta + \delta\bar{q}_G)}{\delta + 8\bar{q}_B - 3\delta\bar{q}_B + 4\bar{q}_B O_\beta - 2\delta\bar{q}_B O_\beta + 4O_\beta}.$$

I show that the cross-multiplied version of the RHS is negative. Using the identities $2\bar{q}_B = \delta - \delta O_\beta + \delta\bar{q}_B$ and $a\bar{q}_B = \bar{q}_G$ that hold in the limit $\eta \to 0$, I rewrite the RHS as

$$-\sqrt{a\bar{q}_G}[9\delta(1 - O_\beta) + 8\bar{q}_B O_\beta(1 - \delta) + 4\bar{q}_B(1 - \delta)] - \delta\bar{q}_G O_\beta[2(1 - \bar{q}_G) + 2 - \delta a]$$

$$-4\bar{q}_G(1-O_\beta) - \delta\bar{q}_G[2(1-\bar{q}_G) + \frac{2}{\delta} - \delta a] - \bar{q}_G(2-\delta) - \delta\sqrt{a\bar{q}_G}(1-\delta)$$

$$-\delta^2\bar{q}_G^2(O_\beta + 1) - 2\delta^2\bar{q}_B\bar{q}_G,$$

which is negative as $a \leq 2$ and $\delta < 1$. Hence, the delay is shorter for precision $p_G = 1$ and $p_B = 1-\eta$ with $\eta > 0$ small enough than for precision $p_G = 1$ and $p_B = 1$, or, $D^{\mathcal{E}} < D^{\mathcal{K}}$ if $p_G = 1$ and $p_B = 1-\eta$ for $\eta > 0$ small.

Part 2a. For $\delta > \frac{2}{3}$, $D^{\mathcal{E}} < D^{\mathcal{N}}$ if the precision of the exogenous signal is $p_G = 1$ and $p_B = 1-\eta$ for $\eta > 0$ small.

Recall that under the exogenous-signal regime the expected delay is

$$D^{\mathcal{E}} = \pi\sqrt{\frac{\bar{q}_G}{a}} + (1-\pi)O_\beta.$$

Now $\lim_{\eta\to 0} D^{\mathcal{E}} = D^{\mathcal{K}}$ and we know from Proposition 4 that $D^{\mathcal{K}} < D^{\mathcal{N}}$ if $\delta > \frac{2}{3}$. Since everything is continuous, if $p_G = 1$ there exists a small neighbourhood of $p_B = 1$ s.t. $D^{\mathcal{E}} < D^{\mathcal{N}}$.

Part 2b. For $\delta < \frac{2}{3}$, $D^{\mathcal{E}} < D^{\mathcal{N}}$ if the precision of the exogenous signal is $p_G = 1$ and $p_B = 1-\eta$ for $\eta \approx 1$.

I first take the limits of the exogenous-signal regime's cutoffs as $\eta \to 1$. By plugging $\eta = 1$, or $p_B = 0$, into the system of equations that determines the cutoffs of the exogenous-signal regime (see Proposition 8), I get

$$\begin{cases} \bar{q}_G = \dfrac{\frac{\delta}{2}\omega(1+O_\beta)(a - \sqrt{a\bar{q}_G} + \bar{q}_G) + (a + \sqrt{a\bar{q}_G})\bar{q}_B}{\omega(1+O_\beta) + (a+\sqrt{a\bar{q}_G})}, \\ \bar{q}_B = \dfrac{\delta}{2O_\beta}\left[1 - \dfrac{1-\bar{q}_G^2}{1+O_\beta}\right], \\ O_\beta^2 = \bar{q}_G. \end{cases}$$

Combining the three equations, I get that in the limit as $\eta \to 1$, $\bar{q}_G$ is defined by the following equation:

$$\lim_{\eta\to 1}\bar{q}_G = \frac{\delta}{2}\frac{\omega(1+\sqrt{\bar{q}_G})(a-\sqrt{a\bar{q}_G}+\bar{q}_G)+(a+\sqrt{a\bar{q}_G})(1-\sqrt{\bar{q}_G}+\bar{q}_G)}{\omega(1+\sqrt{\bar{q}_G})+(a+\sqrt{a\bar{q}_G})},$$

which is identical to the equation that defines the equilibrium cutoff under the no-signals' regime, $\bar{q}$ (see Proposition 3). Hence, $\lim_{\eta\to 1}\bar{q}_G = \bar{q}$. However, note that the limit of $\bar{q}_B$ is much smaller than $\bar{q}$:

$$\lim_{\eta\to 1}\bar{q}_B = \frac{\delta}{2}(1-\sqrt{\bar{q}}+\bar{q}),$$

which is much smaller than $\bar{q}$ because $a-\sqrt{a\bar{q}}+\bar{q} > 1-\sqrt{\bar{q}}+\bar{q}$.

Since everything is continuous, we know that there exists a neighbourhood of $\eta = 1$ s.t. for $p_G = 1$ and $p_B = 1-\eta$, $\bar{q}_G \approx \bar{q}$, but $\bar{q}_B < \bar{q}$, so that $O_\beta^2 < \bar{q}_G \approx \bar{q}$. Accordingly, the expected delay under the exogenous-signal regime with this precision is

$$D^{\mathcal{E}}|_{\eta\approx 1} = \pi\sqrt{\frac{\bar{q}_G}{a}} + (1-\pi)O_\beta \approx 1+\pi\sqrt{\frac{\bar{q}}{a}} + (1-\pi)O_\beta,$$

which is smaller than $D^{\mathcal{N}}$ because $O_\beta < \sqrt{\bar{q}}$. □

Appendix B

Appendix for Chapter 3

This section contains the proofs that were omitted in the body of the Chapter.

B.1 Imperfectly correlated preferences

The following provides sufficient conditions for the searcher's cutoff $\bar{u}(v_m)$ to be concave and is a corollary to Proposition 10.

Corollary 2. *Sufficient conditions for the cutoff $\bar{u}(v_m)$ to be concave are: (a) monotone increasing hazard rate for f_v: $\frac{\partial}{\partial v_m}\frac{f_v(v_m)}{1-F_v(v_m)} \geq 0$, and (b) $\frac{\partial g(u|v_m)}{\partial v_m} \geq 0$ for all $u \in [\bar{u}(v_m), 1]$.*

Proof. I derive the sufficient conditions for $\bar{u}(v_m)$ to have a negative second derivative. Differentiating equation (3.5) gives

$$\frac{\partial^2 \bar{u}(v_m)}{\partial v_m^2} = -\left(\frac{\partial}{\partial v_m}\frac{f_v(v_m)}{1-F_v(v_m)}\right)\int_{\bar{u}(v_m)}^{1} g(u|v_m)(u-\bar{u}(v_m))\,\mathrm{d}u$$

$$-\frac{f_v(v_m)}{1-F_v(v_m)}\left[\int_{\bar{u}(v_m)}^{1}\frac{\partial g(u|v_m)}{\partial v_m}(u-\bar{u}(v_m)) - g(u|v_m)\frac{\partial \bar{u}(v_m)}{\partial v_m}\,\mathrm{d}u\right],$$

which is negative for sure if (a) $\frac{\partial}{\partial v_m}\frac{f_v(v_m)}{1-F_v(v_m)} \geq 0$ and (b) $\frac{\partial g(u|v_m)}{\partial v_m} \geq 0$ for $u \in [\bar{u}(v_m), 1]$ as $\frac{\partial \bar{u}(v_m)}{\partial v_m} < 0$. For example, log-concave distributions satisfy (a) and independent u and v satisfy (b). □

Claim 3. *For u independent of v and h uniform on $[0,1]^2$, the searcher's cutoff is given by equation* (3.7)*.*

Proof. I solve the differential equation for $\bar{u}(v_m)$ as follows. Rewrite (3.6) as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}(1-x)(1-y)^2,$$

and rearrange to separate the variables:

$$(1-y)^{-2}\ \mathrm{d}y = -\frac{1}{2}(1-x)\ \mathrm{d}x.$$

I use the standard method, integrating and rearranging, to solve the above equation. Integrating both sides of the equation yields

$$(1-y)^{-1} = \frac{1}{2}\ln(1-x) + k,$$

where k is a constant and which can be rearranged to yield

$$y = 1 - \left\{\frac{1}{2}\ln(1-x) + k\right\}^{-1}. \tag{B.1}$$

The constant k is pinned down by the initial condition $\bar{u}(v^*) = \bar{u}(1-2c) = 0$:

$$k = 1 - \frac{1}{2}\ln(1-v^*).$$

Inserting k to (B.1) and reverting to the original notation yields the result. □

B.2 Characterisation

This section contains the omitted details of Section 3.5.

B.2.1 The searcher is less picky if the preferences are misaligned

Proposition 12. *The searcher is less picky if the agents' preferences are misaligned as compared to when they are perfectly aligned, i.e.,* $\max\{\tilde{u}, \bar{v}\} > \bar{u}(v_m)$ *for all* v_m*.*

Proof. I show that $\tilde{u} > \bar{u}(v_m)$ for all v_m. Recall that $\tilde{u}$ is the optimal cutoff if the searcher could choose himself and equals the searcher's value from the search process. Recall that $\bar{u}(v_m)$ is the optimal cutoff if the chooser chooses instead of the searcher and equals the searcher's value from the search process. The proof is by noting that the searcher can do as least as well in the absence of a chooser as in the presence of her.

Let us call a searcher who can choose himself A and call a searcher who searches for a chooser B. Consider a sequence of items drawn from the distribution H with misaligned preferences (u_1, v_1), (u_2, v_2), (u_3, v_3), etc., that is relevant for B and the equivalent sequence (u_1, u_1), (u_2, u_2), (u_3, u_3), etc., relevant for A. A can mimic the behaviour of B by using the cutoff $\bar{u}(v_1)$ in the first period, cutoff $\bar{u}(\max\{v_1, v_2\})$ in the second, and so on. However, A can do strictly better in expectation as he can ignore the v-value of the items. For example, if $v_1 > v^*$, then $\bar{u}(v_1) = 0$ so that B would accept (u_1, v_1) even if $u_1 \leq \epsilon$ for $\epsilon > 0$ very small. A can do better by continuing to search. The probability of an item with $u_1 < \epsilon$ and $v_1 \geq v^*$ occurring is positive under H because of the full support assumption. Since A's value from searching equals $\tilde{u}$, B's value equals $\bar{u}(\bar{v})$, and $\bar{u}(v_m)$ decreases in v_m, it must be that $\tilde{u} \geq \bar{u}(v_m)$ for all v_m. □

B.2.2 The searcher prefers more aligned preferences

For positive correlation parametrised by q, the differential equations that pin down the solution are, for $v_m < \hat{v}$,

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{1}{2}(1 - v_m)^{-1}(1 - q)(1 - \bar{u}(v_m))^2,$$

with initial condition $\bar{u}(\hat{v}) = \hat{v}$, and for $v_m \in (\hat{v}, v^*)$,

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{1}{2}(1 - v_m)^{-1}[(1 - q)(1 - \bar{u}(v_m))^2 + 2q(v_m - \bar{u}(v_m))],$$

with initial condition $\bar{u}(v^*) = 0$. The differential equations do not have manageable closed form solutions (i.e., the solutions are nonlinear functions involving Bessel and gamma functions) and are thus omitted.

For negative correlation parametrised by r, the relevant system of equations depends on the size of c (or, more precisely, on whether the curve $\bar{u}(v_m)$ intersects with the line $1 - v_m$). For c large enough, the curve and line do not intersect and the differential equation that pins down the solution is for any $v_m < v^*$,

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{1}{2}(1 - v_m)^{-1}[(1 - r)(1 - \bar{u}(v_m))^2 + 2r(1 - v_m - \bar{u}(v_m))],$$

with initial condition $\bar{u}(v^*) = 0$. Again, the differential equation does not have a manageable closed form solution and is omitted.

For small search cost, the curve $\bar{u}(v_m)$ and line $1 - v_m$ intersect. Let $\bar{u}(\hat{v}_i) = 1 - \hat{v}_i$ for $i = 1, 2$ such that $\hat{v}_1 < \hat{v}_2$. The differential equation that pins down the solution is for $v_m < \hat{v}_1$ and $v_m \in (\hat{v}_2, v^*)$

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{1}{2}(1 - v_m)^{-1}[(1 - r)(1 - \bar{u}(v_m))^2 + 2r(1 - v_m - \bar{u}(v_m))],$$

with initial conditions $\bar{u}(\hat{v}_1) = 1 - \hat{v}_1$ and $\bar{u}(v^*) = 0$ for $v_m < \hat{v}_1$ and $v_m \in (\hat{v}_2, v^*)$ respectively, and for $v_m \in (\hat{v}_1, \hat{v}_2)$

$$\frac{\partial \bar{u}(v_m)}{\partial v_m} = -\frac{1}{2}(1 - v_m)^{-1}(1 - r)(1 - \bar{u}(v_m))^2,$$

with initial condition $\bar{u}(\hat{v}_2) = 1 - \hat{v}_2$. The solutions to these equations are not informative and thus omitted.

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