VOL. 11, NO. 7, APRIL 2016 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

SOLVING NON-LINEAR DAMPED DRIVEN SIMPLE PENDULUM WITH SMALL AMPLITUDE USING A SEMI ANALYTICAL METHOD

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ABSTRACT

In this paper, we present a semi analytical solution for a damped driven pendulum with small amplitude, by using the differential transformation method. We begin by showing how the differential transformation method applies to the non-linear dynamical system. The method transformed the differential equation governing the motion of the pendulum into its algebraic form. The results obtained are in good agreement with the solution in the literature. The results show that the technique introduced is easy to apply to such dynamical system.

Keywords: semi analytical method, small amplitude, non-linear damped driven simple pendulum.

1. INTRODUCTION

The purpose of this paper is to employ the differential transformation method to the second order linear ordinary differential equation associated with pendulum dynamics. Most of the other methods available to solve the governing equations of motion of pendulum are trial and error in nature, thereby making them computationally intensive [1]. The differential transform method is a numerical method which requires no linearization or discretization [3]. In this work, we considered a damped driven pendulum as dynamical system whose governing equation is a second order linear ordinary differential equation. We solve this governing equation using DTM. The dynamic effect of the angular displacement and angular driven force are estimated from the solution of the governing equation [7].

2. BASIC DEFINITION

With reference to the articles [1, 2, 3] the differential transformation of a function r(x) can be written as:

$$R(t) = \frac{1}{K!}\left[\frac{d^k r(x)}{dx}\right] x = 0 \qquad (1)$$

where r(x) is referred to as the original function and R(t) is the transformed function.

The Taylor series expansion of r(x) about x=0 is the differential inverse transform of R(t). This can be defined as follows:

$$r(x) = \sum_{k=0}^{\infty} R(t)x^k \qquad (2)$$

Equation (2) can be written using (1), as:

$$r(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k r(x)}{dx}\right] \qquad (3)$$

2.1 Some operation properties of differential transformation

Combining equations (1) and (2), we derive the following mathematical operations whose proofs can be found in [4] and [5]:

a) $If\ r(x) = p(x) \pm g(x), then\ R(t) = P(t) \pm R(t)$
b) $If\ r(x) = cp(x), then\ R(t) = cP(t), where\ c\ is\ a\ constant$
c) $If\ r(x) = \frac{d^n p(x)}{dx^n}, then\ R(t) = \frac{(k+n)!}{k} P(k+n)$
d) $If\ r(x) = \sin(ax+b),\ then\ R(t) = \frac{a^k}{k!} sin\left(\frac{k\pi}{2} + b\right)$
e) $If\ r(x) = e^{\lambda x}, then\ R(t) = \frac{\lambda^k}{k!}, where\ \lambda\ is\ a\ constant$

2.2 DTM on second order ODE

DTM can be used for second order linear ordinary differential equations as follows [2, 11]:
Given

$$r(x) = \frac{1}{a}[f(x) - cy(x) - by^1 x] \qquad (4)$$

The differential transformation of (4), by applying the above properties, is as follows:

$$(k+1)(k+2)R(k+2) = \frac{1}{a}[F(x) - b(k+1)R(k+1) - cR(k)] \qquad (5)$$

$$R(k+2) = \frac{1}{a(k+1)(k+2)}[F(x) - b(k+1)R(k+1) - cR(k)] \qquad (6)$$

Equation (6) is subject to initial condition $R(0) = R_0 = a_1$

$$\text{and } R^1(0) = R(1) = a_2 \qquad (7)$$

Equation (7) is referred to as a recursive formula.

ARPN Journal of Engineering and Applied Sciences

The appropriate solution

$$r(x) = \sum_{k=0}^{\infty} R(t)x^k \quad (8)$$

is obtained by using equation (2) and (7).

2.3 The governing equation

The equation of motion for damped driven pendulum mass m and length l is given as [7]:

$$ml^2 \frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + mgl\, sin\theta = C\cos(w_\Delta t) \quad (9)$$

With its dimensionless form as:

$$\frac{d^2\theta}{dt^2} + \frac{1}{q}\frac{d\theta}{dt} + sin\theta = a\cos(w_\Delta t) \quad (10)$$

where m is the mass of the pendulum bob, l is the length of the string, θ is the angular displacement, γ is the dissipation coefficient, g is the acceleration due to gravity, t represents time, w_Δ is the angular driven force, while C is the amplitude of the driving force. Q is the damping parameter. The right hand side terms represent acceleration, damping and gravitation respectively. The term on the right hand side is a sinusoidal driving torque [7, 14].

For small angle of displacement, θ;

$$\text{Sin}\theta = \theta \quad (11)$$

So equation (2) can be written as

$$\frac{d^2\theta}{dt^2} + \frac{1}{q}\frac{d\theta}{dt} + \theta = a\cos(w_\Delta t) \quad (12)$$

Let $\emptyset = W_D t$, the phase of the driving force term (7)

Then equation (3) becomes

$$\theta''(t) + \frac{1}{q}\theta'(t) + \theta(t) = a\cos\emptyset \quad (13)$$

At equilibrium,

$$\theta(0) = 0 \;\; i.e.(t = 0) \quad (14)$$

Equation (4) becomes

$$0 = a\cos\emptyset \quad (15)$$

That is, $0 = \cos\emptyset$ (16)

$\Rightarrow \emptyset = 90^0 \; or \; 360^0 + 90$ (17)

In general $\phi = 2n\pi + 90$, where n = 1, 2, 3,… (18)

Now when the pendulum is not at equilibrium (i.e.$\theta \neq 0$), the value of ϕ will be outside the above range.

For this paper, the three dimensions for this system represented by equation (9), are $\emptyset, \theta \; and \; \theta'$. Also both $\emptyset \; and \; \theta$ have been restricted to reside within $-\pi \; and \; \pi, and \; 0 \; and \; 2\pi$ respectively, in order to simplify the result of the system (13). We also let q = $^1/_2$ and a = 0.

The dimensionless governing equation to be transformed, using DTM, becomes

$$\theta''(t) = -2\theta'(t) - \theta(t) \quad (19)$$

Subject to

$$\theta(0) = 0, \;\; \theta'(0) = -1 \quad (20)$$

3. APPLICATION OF DTM

In this section, we will apply the discussed DTM to solve equations (5) and (6). We start by re-writing equation (5) and (6) in a standard form and take the differential transform (DT) as follows:

$$DT[\theta''(t) = -2\theta'(t) - \theta(t)\,] \quad (21)$$

$$\Rightarrow (k+1)(k+2)Y(k+2) = -2(k+1)Y(k+1) - Y(k) \quad (22)$$

$$\therefore Y(k+2) = \frac{1}{(k+1)(k+2)}[-2(k+)Y(k+1) - Y(k)] \quad (23)$$

With the initial conditions Y(0) = 0, Y(0) = -1 (24)

By using the recursive relation in (9) with $k \geq 0$, we obtain values for Y(2), Y(3), Y(4), …, as follows: [2,10].

For k=0; we have

$$Y(2) = \frac{1}{2}(2) = 1 \quad (25)$$

For k=1, we have,

$$Y(3) = \frac{1}{6}[(-2)(2)(1) - (-1)] = -\,^1/_2 \quad (26)$$

For k=2, we have

$$Y(4) = \frac{1}{12}[(-2)(3)(-\,^1/_2) - 1] = -\,^1/_6 \quad (27)$$

For k=3, we have

$$Y(5) = \frac{1}{20}[-2(4)(^1/_6) - (-\,^1/_2)] = -\,^1/_{24} \quad (28)$$

And so on.

Generally, for $k + 2 = n, (k = 0,1,2,3, \dots)$ we have

ARPN Journal of Engineering and Applied Sciences

$$Y(n) = 1, -\tfrac{1}{2}, \tfrac{1}{6}, -\tfrac{1}{24}, \ldots, \quad (29)$$

The terms are given as:

Y(n) = Y(n-1) x $-^1/_n$,

Where

Y(0) = -1 and n = 1, 2, 3, … (30)

That is,

$$\left.\begin{aligned} Y(1) &= -1 \times -1 = 1 \\ Y(2) &= 1 \times -\tfrac{1}{2} = -\tfrac{1}{2} \\ Y(3) &= -\tfrac{1}{2} \times -\tfrac{1}{3} = \tfrac{1}{6} \\ Y(4) &= \tfrac{1}{6} \times -\tfrac{1}{4} = -\tfrac{1}{24} \\ Y(5) &= -\tfrac{1}{24} \times -\tfrac{1}{5} = \tfrac{1}{220} \end{aligned}\right\} \quad (31)$$

From equation (8),

$$r(x) = \sum_{k=0}^{\infty} R(t)x^k$$

So let

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \quad (32)$$

$$\therefore y(x) = Y(0) + Y(1)x + y(2)x^2 + y(3)x^3 + y(4)x^4 + \ldots \quad (33)$$

$$= 0 - x + 1x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 + \cdots \quad (34)$$

$$= -x + x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 - \tfrac{1}{24}x^5 + \cdots \quad (35)$$

Equation (34) can now be written as,

$$i.e.\ \theta(t) = -t + t^2 - \tfrac{1}{2}t^3 + \tfrac{1}{6}t^4 + \cdots \quad (36)$$

For t = 1, 2, 3, 4, 5,…, we have

$$\theta(0) = 0 \quad (37)$$

$$\theta(1) = -1 + 1 - \tfrac{1}{2} + \tfrac{1}{6} + \cdots = -\tfrac{1}{3} \quad (38)$$

$$\theta(2) = -2 + 4 - 4 + \tfrac{8}{3} + \cdots = \tfrac{1}{3} \quad (39)$$

$$\theta(3) = -3 + 9 - \tfrac{27}{2} + \tfrac{27}{2} + \cdots = 3 \quad (40)$$

$$\theta(4) = -4 + 16 - 32 + \tfrac{128}{3} + \cdots = \tfrac{68}{3} = 22\tfrac{2}{3} \quad (41)$$

$$\theta(5) = -5 + 5^2 - \tfrac{1}{2}(5^3) + \tfrac{1}{6}(5^4) + \cdots = 1632\frac{1}{2} \quad (42)$$

and so on.

Now for the velocity;

Differentiating equation (36), with respect to t, we have

$$\theta'(t) = -1 + 2t - \frac{3}{2}t^2 + \frac{2}{3}t^3 + \ldots \quad (43)$$

$$\therefore \theta'(0) = -1 + 0 - 0 + 0 + \ldots \quad (44)$$

$$\theta'(1) = -1 + 2 - \frac{3}{2} + \frac{2}{3} + \ldots \quad (45)$$

$$\theta'(2) = -1 + 4 - 6 + \frac{16}{3} + \ldots \quad (46)$$

$$\theta'(3) = -1 + 6 - \frac{27}{2} + 18 + \ldots \quad (47)$$

$$\theta'(4) = -1 + 8 - 24 + \frac{128}{3} + \ldots \quad (48)$$

$$\theta'(5) = -1 + 10 - \frac{75}{2} + \frac{250}{3} + \ldots \quad (49)$$

and so on.

For the acceleration, we differentiate equation (44) with respect to t, we have;

$$\theta''(t) = 0 + 2 - 3t + 2t^2 + \ldots \quad (50)$$

$$\therefore$$

$$\theta''(0) = 0 + 2 - 0 + 0 + \ldots \quad (51)$$

$$\theta''(1) = 0 + 2 - 3 + 2 + \ldots \quad (52)$$

$$\theta''(2) = 0 + 2 - 6 + 8 + \ldots \quad (53)$$

$$\theta''(3) = 0 + 2 - 9 + 18 + \ldots \quad (54)$$

$$\theta''(4) = 0 + 2 - 12 + 32 + \ldots \quad (55)$$

$$\theta''(5) = 0 + 2 - 15 + 50 + \ldots \quad (56)$$

and so on.

Truncation the values of the angular displacement, velocity and acceleration after the fourth terms, we have respectively;

$$\theta(1) = -\tfrac{1}{3} \quad (57)$$

$$\theta(2) = \tfrac{1}{3} \quad (58)$$

$$\theta(3) = 3 \quad (59)$$

$$\theta(4) = 22\tfrac{2}{3} \quad (60)$$

$$\theta(5) = 1632\frac{1}{2} \quad (61)$$

VOL. 11, NO. 7, APRIL 2016 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

$$\theta^{/}(1)=1 \tag{62}$$
$$\theta^{/}(2)=2\frac{1}{3} \tag{63}$$
$$\theta^{/}(3)=9\frac{1}{2} \tag{64}$$
$$\theta^{/}(4)=25\frac{2}{3} \tag{65}$$
$$\theta^{/}(5)=54\frac{5}{6} \tag{66}$$

$$\theta^{//}(1)=1 \tag{67}$$
$$\theta^{//}(2)=4 \tag{68}$$
$$\theta^{//}(3)=11 \tag{69}$$
$$\theta^{//}(4)=22 \tag{70}$$
$$\theta^{//}(5)=37 \tag{71}$$

4. RESULT DISCUSSIONS

With the assumptions in place, the results obtained show that the angular displacement increases as time increases, within the range of time considered (t = 1, 2, 3, 4, 5). From Figure-1, it can be seen that the increase at the beginning was not very pronounced, but after t=4 the increase in angular displacement becomes so pronounced and sharp. Also both angular velocity and angular acceleration increase as time increases, as we can see from Figure-2 and Figure-3. However Figure-4 compares the increase in angular velocity with the increase in angular acceleration: The Figure shows that, at any given time, the increase in angular acceleration is more than the increase in angular velocity. Similarly in Figure-5, the increases in angular acceleration, angular velocity and angular acceleration are compared. We notice that increase in angular displacement was at a point less than increase in both angular velocity and angular acceleration, but from time t = 5, the increase in angular displacement becomes higher than that of both increase in angular velocity and angular acceleration. From equation (18), the phase of the driven force can be written generally as $\phi = 2n\pi + 90$. But for the interval being considered in this paper, the phase of the driven force $\emptyset\ is\ 90^0$.

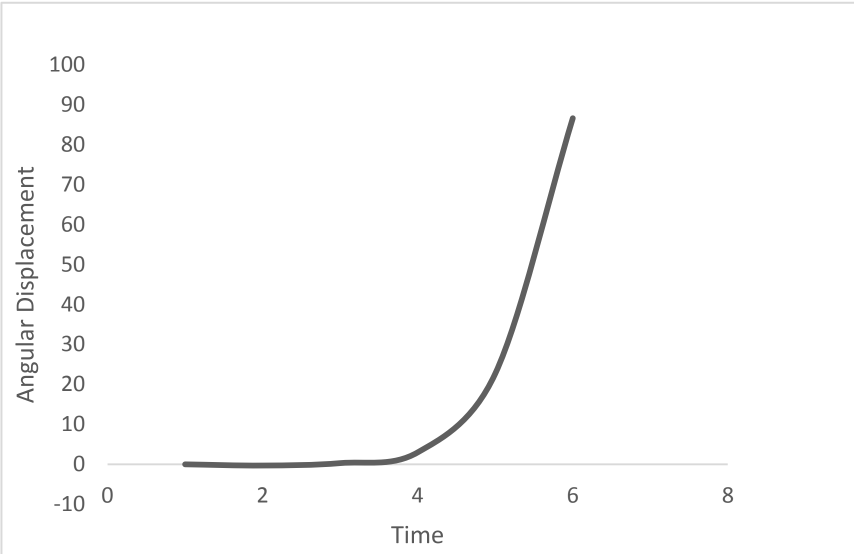

Figure-1. Angular displacement of the pendulum at various times.

ARPN Journal of Engineering and Applied Sciences

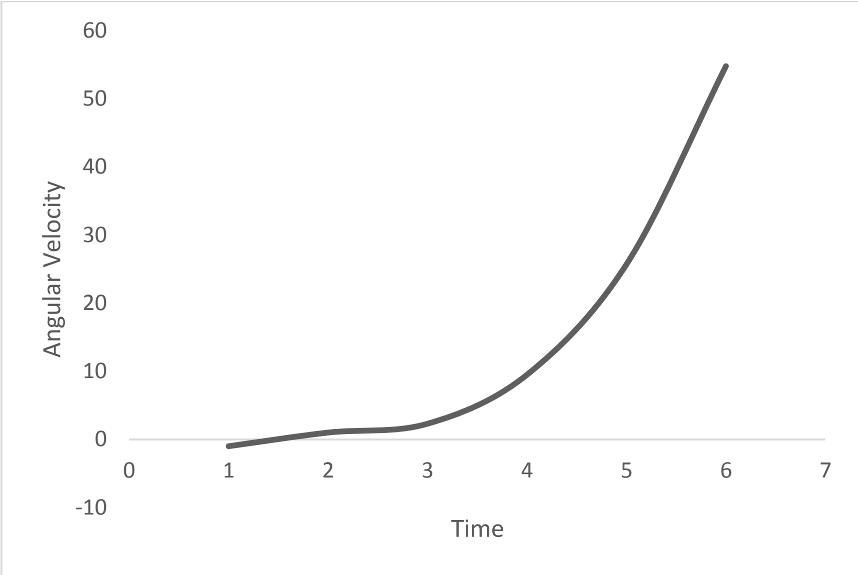

Figure-2. Angular velocity of the pendulum at various times.

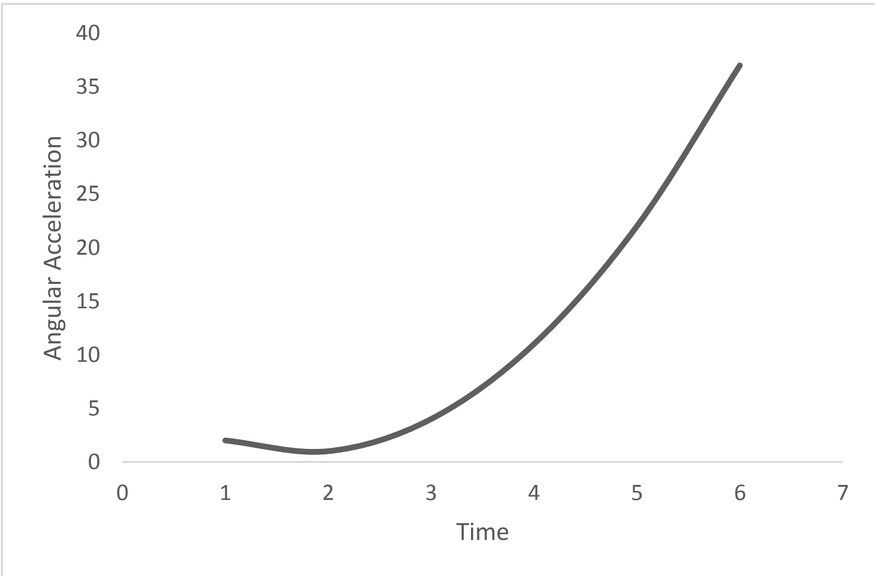

Figure-3. Angular acceleration of the pendulum at various times.

VOL. 11, NO. 7, APRIL 2016 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

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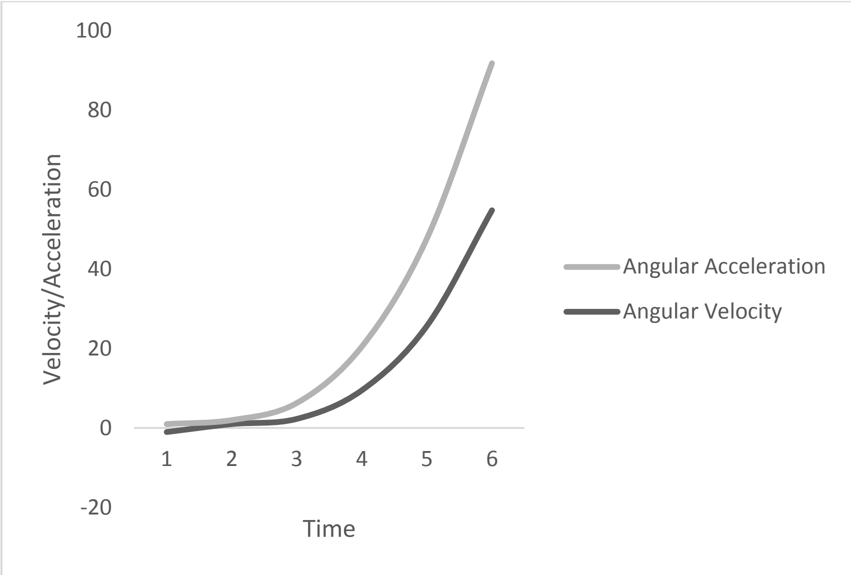

Figure-4. Comparing the angular acceleration with the Angular velocity of the pendulum at various times.

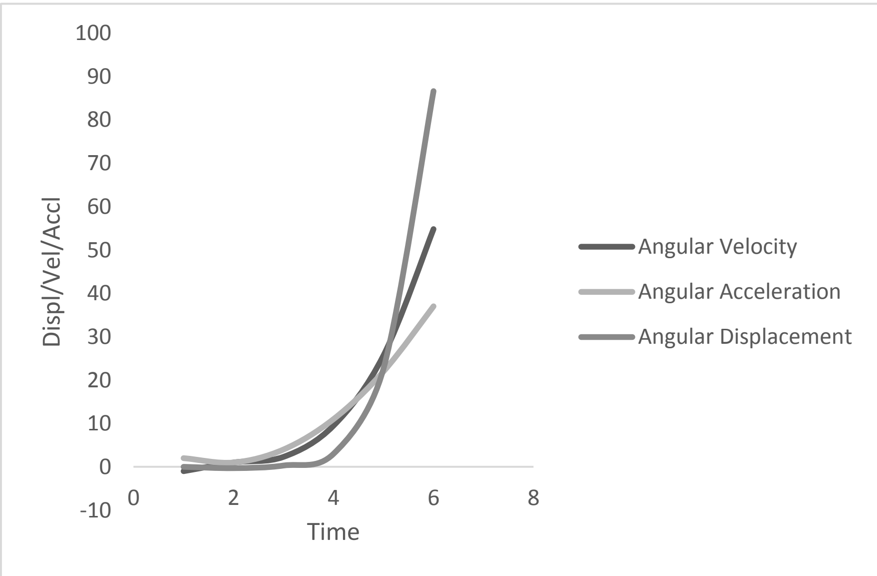

Figure-5. Comparing the angular acceleration with the Angular velocity angular displacement of the pendulum at various times.

5. CONCLUSIONS

A non-linear damped driven simple pendulum with small amplitude problem was solved using differential transformation method (DTM). The obtained results are plotted on graphs and are in agreement with the ones in literature. Deductions are made from both the graphs and equations. The differential transformation method is an effective method for solving the type of problem being considered in this paper. It is also easy to apply.

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