DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

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ABSTRACT OF THE INVESTIGATIONS PRESENTED TO OBTAIN THE ACADEMIC DEGREE OF DOCTOR OF MATHEMATICS

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TARTU 1992

The present research work was carried out at the P.P.Shirshov Institute of Oceanology, USSR Acad. Sci. (1980-83); the Institute of Thermophysics and Electrophysics, Estonian Acad. Sci. (1983-90) and the Institute of Ecology and Marine Research, Estonian Acad. Sci. (1990-92).

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The open defence of the thesis will be held on November 2, 1992 at 16.30 at the Council Hall of Tartu University (Ülikooli 18, Tartu, Estonia EE2400)

The thesis is available at Tartu University Library.

The abstracts were mailed to libraries and experts on September 28, 1992.

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1. Introduction

1.1. Statistical description of weakly nonlinear stochastic wave systems and the kinetic equation

In nature the fields of hydrodynamic motions are, as a rule, extremely complex systems consisting, formally, of a vast number of wavelike harmonics. They can be treated as random wave fields with continuous spectral density (spectrum) of energy. In that case the behaviour of the particular wave components is usually of no interest, and the wave field evolution is determined by the temporal behaviour of its statistical moments or cumulants. Their evolution is described by an infinite system of coupled equations similar to the BBGKY system. As a general rule, this system cannot be truncated because of the incessant generation of the higher moments by the nonlinear coupling (Monin and Yaglom, 1967). Truncation is possible only in a few particular cases; for example, in the case of the Gaussian systems the third and higher order cumulants are identically zero.

Of particular interest are weakly nonlinear waves - a wave class "intermediate" between linear and nonlinear waves. They are usually defined heuristically as waves for which nonlinear effects are negligible in the time scale comparable with their characteristic period, but for longer time scales nonlinearity may have significant impact. Mathematically, such waves are described by an equation in which the nonlinear terms are multiplied by a small parameter, ε, called the measure of nonlinearity. Physically, this means that the particle's velocity in a particular wave is expected to be small as compared to the latter's phase velocity.

Hasselmann (1962), studying weakly nonlinear surface waves, pointed out that the initial correlations between the harmonics are broken down relatively fast by dispersive effects in comparison with the rate of growth of the correlations by nonlinear effects. Consequently, he argued, a weakly nonlinear dispersive wave field can be assumed to be approximately Gaussian in the sense that the fourth and higher order cumulants can be discarded. This assumption, called the quasi-Gaussian approximation, makes it possible to obtain a closed equation describing the evolution of the energy spectrum (the second-order cumulant). This equation is

called the kinetic equation and describes the slow (as compared to the characteristic period of the waves) spectral evolution of a wave system due to the lowest-order resonant interactions between the wave harmonics (triad interactions for capillary waves, Rossby waves, internal waves etc., four-wave interactions for surface waves). Below we shall deal only with the class of waves allowing triad interactions.

The kinetic equation is a hydrodynamical analogue of the Boltzmann equation for an idealized gas. In order to establish this analogy it is convenient to consider a wave field as a superposition of a huge number of localized wave packets, each associated with its mean wave vector and frequency. As distinct from the classical kinetic theory, interaction ("collision") between packets may occur only if at least three of them are simultaneously present within a small area of space. Although at any given point there may coexist a continuum of different packets, actual energy exchange occurs only if the wave vectors and frequencies of three packets satisfy certain geometrical conditions called resonance conditions (see below). These conditions "preselect" the "colliding" waves, limiting the "number" of the interacting waves and ensuring that the triad interactions occur relatively rarely and the time of "free" propagation of the packets is much longer than the duration of the "collision".

Travelling through each other during a short time interval (as compared to the time scale of spectral evolution), the interacting waves execute an "elementary" interaction, and propagate into various directions afterwards. The "elementary" interactions are assumed to be independent and separated in time and space. These interactions are integrated by the collision integral of the kinetic equation in the same way as the collision integral of the Boltzmann equation integrates the collisions of particles.

1.2. Rossby waves

The Rossby wave is an example of low frequency oscillation found in rotating systems and which owes its existence to the variation of the vertical component of the background rotation (or, equivalently, the Coriolis parameter). Rossby waves represent the so-called synoptic-scale motions in the oceans and the atmosphere. For Earth's conditions these motions normally have the horizontal length scale $L \geq 100$ kilometers, the vertical length scale of the order of the full depth of the ocean (atmosphere) and the time scale of several dozens of days. As the synoptic motions contain

a notable part of the whole wave energy in the Earth's oceans and atmosphere, a clearer understanding of their main properties will be unquestionably useful for the promotion of the study of the dynamics of the atmosphere and the oceans.

Below we shall limit ourselves to a simplified treatment of the synoptic motions, in which only the main factors of those forming the Rossby wave field are included. Namely, the variation of the Coriolis force (β-effect), the vertical structure of the motions (baroclinicity), the nonlinear effects and the presence of a great number of wave harmonics are taken into account. No forcing, and usually no dissipation, is taken into account. The propagation of Rossby waves on the Earth may be described by treating the Earth's surface as an infinite flat plane in which the Coriolis parameter varies in the North-South direction; such a rotating plane is known as a β-plane (Kamenkovich, 1972). This approximation, although it excludes several interesting features associated with a countable set of wave harmonics on a sphere and the effect of the shores, greatly simplifies the analytical investigation into the equations of the motion. The equation for the weakly nonlinear Rossby waves in a barotropic (vertically homogeneous) ocean (atmosphere) of constant depth within the bounds of the described model has the following form:

$$\partial(\Delta\psi - a^2\psi)/\partial t + \partial\psi/\partial x = \varepsilon J(\Delta\psi,\psi), \qquad (1)$$

where ψ is the stream function, (x,y) - the Cartesian coordinates on the β-plane; the x-axis directed to the North and the y-axis to the East; t-time, $\varepsilon = U/(\beta L^2) \ll 1$ - the measure of nonlinearity; β - the North-South derivative of the Coriolis parameter, U - characteristic velocity scale, $J(f,g) = f_x g_y - f_y g_x$, a^{-1} - Rossby deformation radius, depending on the background physics. The equation for the Rossby waves in a baroclinic ocean is qualitatively similar to Eq. (1). Also, Eq. (1) turns out to be identical (to within the physical meaning of parameters and coordinates) to the equation for drift waves in plasma (Hasegawa, Maclennan and Kodama, 1979).

The main mathematical advantages of basing this study on Rossby waves are the following:

(1) Rossby waves are two-dimensional (2D) waves;

(2) Three-wave resonance takes place in this wave system (unlike, for example, the surface gravity waves);

(3) Unlike most wave classes of physical interest, Rossby waves have an explicit algebraic (polynomial fractional) dispersion relation;

(4) Rossby waves are anisotropic: their frequency depends on the direction of wave propagation.

The first two features reduce the number of the geometrical conditions for the interacting waves as well as the multiplicity of the collision integral as compared to the case of 3D waves. The third property greatly simplifies the analytical investigation of the solutions of the kinetic equation. Also, it makes possible complete analysis of the resonance and double resonance conditions. The last feature gives rise to anisotropic thermodynamically equilibrium solutions, which do not exist, for example, within the bounds of the general theory of isotropic 2D and 3D turbulence.

2. Kinetic equation for Rossby waves

2.1. The kinetic equation for barotropic waves

The kinetic theory of Rossby waves goes back to a report by Kenyon (1964), who derived the kinetic equation for barotropic waves within the bounds of the described model:

$$\frac{\partial F}{\partial \tau} = 8\pi \int C_{\kappa_1\kappa_2}(C_{\kappa_1\kappa_2}F^1F^2 + C_{\kappa\kappa_1}FF^1 + C_{\kappa\kappa_2}FF^2)N^{-1}\delta(\omega^{012})\delta(\vec{\kappa}_{012})d\vec{\kappa}_{12}. \quad (2)$$

Here $F=F(\vec{\kappa},\tau)$ is the energy spectrum, $\vec{\kappa}=(k,l)=(\kappa\cos\varphi,\kappa\sin\varphi)$ - the wave vector, $C_{\kappa_i\kappa_j}=(\vec{\kappa}_i\times\vec{\kappa}_j)(\kappa_j^2-\kappa_i^2)/2$ - interaction coefficients; $\tau=\varepsilon^2 t$ - slow time, $F^i=F(\vec{\kappa}_i,\tau)$, $i=1,2$; $N=(\kappa^2+a^2)(\kappa_1^2+a^2)(\kappa_2^2+a^2)$; $\delta(x)$ - Dirac delta function, $\omega^{012}=\omega(\vec{\kappa})+\omega(\vec{\kappa}_1)+\omega(\vec{\kappa}_2)$; $\omega(\vec{\kappa})=-k/(\kappa^2+a^2)$ - dispersion relation (frequency) of the linear barotropic Rossby waves; $\vec{\kappa}_{012}=\vec{\kappa}+\vec{\kappa}_1+\vec{\kappa}_2$; $d\vec{\kappa}_{12}=dk_1dl_1dk_2dl_2$; $(\vec{\kappa}_1,\vec{\kappa}_2)\in R^2(\vec{\kappa}_1)\times R^2(\vec{\kappa}_2)$.

Eq. (2) serves as an integro-differential equation containing the simplest partial differential operator and the kernel with quadratic nonlinearity. The unknown function, F, depends on three variables: slow time and two wave vector components. Additionally, the kernel contains three Dirac delta functions, which means that the right-hand side of Eq. (2) (called collision integral) may be reduced to a one-dimensional integral over a specified curve. These delta functions reflect the well-known fact that the most intensive energy exchange occurs between Rossby wave harmonics, the wave vectors and frequencies of which satisfy the resonance conditions $\omega^{012}=0$; $\vec{\kappa}_{012}=\vec{0}$.

2.2. Kinetic equation for two-layer model

Synoptic motions in nature are, as a rule, essentially depth-dependant (baroclinic). Mathematically, the vertical structure of the motions in a stratified fluid may be described in terms of a linear combination of a barotropic and a certain (maybe infinite) number of baroclinic modes (Phillips, 1966). This decomposition is equivalent to a model consisting of several non-mixing vertically homogeneous layers. Since a substantial amount of the energy of the synoptic motions of the ocean is usually contained in the barotropic and first baroclinic modes, we shall mostly use the traditional two-layer model in which the motion is decomposed into the barotropic and the (first) baroclinic vertical mode.

The slow evolution of the energy spectra of baroclinic and barotropic modes within the bounds of the two-layer model is described by the following system of integro-differential equations (Kozlov, Reznik and Soomere, 1987):

$$\frac{\partial F_p}{\partial \tau} = 8\pi \sum_{m,n=0}^{1} I_{pmn}; \; I_{pmn} = \int C^{pmn}_{\vec{\kappa}_1 \vec{\kappa}_2} K_{pmn} N^{-1}_{pmn} \delta(\omega^{012}_{pmn}) \delta(\vec{\kappa}_{012}) d\vec{\kappa}_{12}, \qquad (3)$$

where $F_p = F_p(\vec{\kappa}, \tau)$ is the spectrum of the barotropic (p=0) or the baroclinic (p=1) mode, I_{pmn}, p,m,n=0,1 - interaction (collision) integrals describing the energy alteration of the wave with $\vec{\kappa}$ of mode p due to the interaction with waves belonging to modes m, n; $K_{pmn} = C^{pmn}_{\vec{\kappa}_1 \vec{\kappa}_2} F^1 F^2_n + C^{pmn}_{\vec{\kappa}\vec{\kappa}_1} F_p F^1_m + C^{pmn}_{\vec{\kappa}\vec{\kappa}_2} F_p F^2_n$; $N_{pmn} = (\kappa^2 + a_p^2)(\kappa_1^2 + a_m^2)(\kappa_2^2 + a_n^2)$; $C^{pmn}_{\vec{\kappa}_i \vec{\kappa}_j} = \gamma^p_{mn} (\vec{\kappa}_i \times \vec{\kappa}_j)(\kappa_j^2 + a_n^2 - \kappa_i^2 - a_m^2)/2$ - interaction coefficients, the coefficients γ^p_{mn} depending on the particular vertical structure of the motion, $\omega^{012}_{pmn} = \omega_p(\vec{\kappa}) + \omega_m(\vec{\kappa}_1) + \omega_n(\vec{\kappa}_2)$; $\omega_p(\vec{\kappa}) = -k/(\kappa^2 + a_p^2)$ - dispersion relation of the Rossby waves of the p-th mode. The quantities a_p^{-1} are called the barotropic (p=0) and baroclinic (p=1) Rossby radii and depend on the background physics. Further on we regard the waves with vectors $\vec{\kappa}, \vec{\kappa}_1, \vec{\kappa}_2$ as always representing the modes with the numbers p,m,n, respectively. For simplicity, we shall call the system (3) a baroclinic kinetic equation.

3. Double resonance

3.1. Correctness of the kinetic equation

The correctness of the derivation of the kinetic equation as well as

its applicability has been discussed for a long time. Hasselmann (1962), who first derived this equation for hydrodynamical systems, used the quasi-Gaussian approximation. Although in the course of time even an initial Gaussian state will be distorted by nonlinear effects, he argues that probably this distortion has no significant impact on the evolution of the second cumulant if the nonlinearity is weak. Benney and Newell (1969) showed that higher cumulants, formally omitted from the derivation of the kinetic equation, but incessantly regenerated by the nonlinear coupling, indeed grow up fast and soon develop a cusp-like structure in Fourier space. Fortunately, this effect does not distort the kinetic equation.

Reznik (1984b) elaborated the derivation of the kinetic equation, taking into account the full behaviour of the fourth-order cumulants and ignoring the fifth and higher order moments. He reached exactly the same shape of the equation as in the earlier derivations. He argues that the kinetic equation does not depend on the particular truncation of the hierarchy of the cumulant equations of weakly nonlinear wave systems.

The mathematical question of the validity of the kinetic equation in wave vector space was first discussed by Hasselmann (1962), This equation works only for limited time. intervals $\tau \ll \varepsilon^{-2}$ and for limited wave vector areas $\kappa \ll \varepsilon^{-2}$. If τ is of greater value. higher order interactions will become significant, and spectral evolution will be described by another equation (Benney and Newell, 1969). For interactions of very short waves the initial assumption of weak nonlinearity is not valid (Reznik, 1986).

3.2. Double resonance

In addition to the above-mentioned general limitations, the kinetic equation may fail if the function ω^{012} has multiple zeros on the hypersurface $\vec{\kappa}_{012}=0$. For several two- and higher dimensional waves (sound waves, shallow water waves etc., called semi-dispersive waves) Newell and Aucoin (1971) obtained a closure of the hierarchy of the moment equations despite the double zeros of the function ω^{012}. As distinct from the fully dispersive case, the linear dispersion, breaking down the initial correlations and approaching the wave system towards the Gaussian state, works only along specified rays in the 2D semi-dispersive case.

Reznik (1984b; 1986) proposed a general condition ensuring the correctness of the kinetic equation. This equation is valid for a specified wave vector $\vec{\kappa}$ only on condition that the quantity $\Delta = | \vec{\nabla}\omega(\vec{\kappa}_1)-\vec{\nabla}\omega(\vec{\kappa}_2) |$

($\vec{\nabla}_{\vec{\kappa}} = (\partial/\partial k, \partial/\partial l)$ is the group velocity operator), has a nonzero lower boundary for all vectors[1] $\vec{\kappa}_1, \vec{\kappa}_2$ resonantly interacting with $\vec{\kappa}$. The case $\Delta = 0$ was mentioned first by Benney and Saffman (1966) in their one-dimensional analysis and was called double resonance. We shall call the corresponding waves (wave vectors $\vec{\kappa}$, points in wave vector space or triads $\vec{\kappa}, \vec{\kappa}_1, \vec{\kappa}_2$) double resonance waves (wave vectors, points or triads).

The equality $\Delta = 0$ means that interacting waves with $\vec{\kappa}_1, \vec{\kappa}_2$ have equal group velocities. In that case the breakdown of the kinetic equation is anticipated: since these waves (or wave packets) travel together for a long time, their initial correlations will not break down, and an approximately Gaussian state will not be achieved. Mathematically, the collision integral may have nonintegrable singularities, as distinct from the quite usual situation when it has integrable singularities (Valenzuela and Laing, 1972; Hasselmann and Hasselmann, 1985).

3.3. The set of double resonance points for Rossby waves

The set of double resonance points forms a proper subset of the (k,l) plane. The system of equations with respect to six coordinates of the double resonance wave vectors $\vec{\kappa}, \vec{\kappa}_1, \vec{\kappa}_2$ associated with the mode numbers p,m,n, consists of the resonance conditions $\omega^{012}_{pmn} = 0$; $\vec{\kappa}_{012} = \vec{0}$ and $\Delta = 0$ (Soomere, 1990; 1992a). Generally, these conditions form a system of algebraic equations of the 180-th degree. Also, this system contains three parameters a_p, a_m, a_n. As a rule, the described equations have a one-dimensional set of solutions $S^*_{2R}(p,m,n)$, called the double resonance curve. This curve is symmetrical to both coordinate axes and always contains the l-axis. For brevity, we shall sometimes speak of the double resonance curve, having in mind the set $S_{2R}(p,m,n) = S^*_{2R}(p,m,n) \setminus \{\vec{\kappa} = (0,l)\}$. The complete set of the double resonance points for a certain mode p is described by

$$S_{2R}(p) = \bigcup_{m,n} S^*_{2R}(p,m,n). \qquad (4)$$

Analyzing the double resonance conditions if $k \neq 0$, it is convenient to use the polar coordinates $\kappa, \varphi, \kappa_0, \varphi'$, where $\vec{\kappa}_0 = 2\vec{\kappa}_1 + \vec{\kappa}$, $\kappa_0 = |\vec{\kappa}_0|$, $\varphi' = \angle(\vec{\kappa}, \vec{\kappa}_0)$ (Longuet-Higgins and Gill, 1967). Applying these unknowns, the condition

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[1] For brevity, we shall speak of interactions of the wave vectors $\vec{\kappa}$, meaning wave interactions with these wave vectors.

$\vec{\kappa}_{012}=\vec{0}$ is satisfied identically. The set $S_{2R}(p,m,n)$ consists of solutions of the system

$$\omega^{012}_{pmn} = \partial\omega^{012}_{pmn}/\partial\kappa_0 = \partial\omega^{012}_{pmn}/\partial\varphi' = 0;\ \kappa\geq 0, \kappa_0 > 0 \qquad (5)$$

and of double resonance points with $\kappa_0=0$.

The case $\kappa_0=0$ means that $\vec{\kappa}_1=\vec{\kappa}_2=-\vec{\kappa}/2$. This type of double resonance generally occurs only when an arbitrary wave interacts with the waves representing the fixed mode ($a_m=a_n$). The set of such double resonance points consists either of the l-axis and the circle $3\kappa^2=4(a_m^2-a_p^2)$ (case $a_m^2>a_p^2$) or only of the l-axis (case $a_m^2\leq a_p^2$).

Eqs. (5) can be reduced to a certain algebraic equation $Q_2=0$. The latter's left-hand side serves as a polynomial of the fifth degree with respect to κ_0^2, in which the quantities $\kappa_0,\kappa,a_p,a_m,a_n$ occur jointly to an even degree. The polynomial Q_2 generally has no nontrivial polynomial factors with rational coefficients. In the case of synoptic motions in the oceans the values of the Rossby radii for dissimilar modes normally differ greatly, which means that usually we have either $a_m=a_n$ or $a_m^2>\!<a_n^2$. In these two cases of significant practical interest decomposition of the polynomial Q_2 is possible.

In the first case the complete set of double resonance triplets consists either of the l-axis only ($a_m^2\leq a_p^2$), or of the l-axis, the circle $S_R=\{\vec{\kappa}\mid 3\kappa^2=4(a_m^2-a_p^2)\}$ and a certain curve S_κ, linking the origin with a specified point of the circle mentioned. The described set contains seven singularity points: the origin, two crosspoints of the circle S_R with the l-axis and four coinciding points of this circle and the curve S_κ (Figure 1). The singularity points of the double resonance curve correspond to the special case of the double resonance triplets, in which all the group velocities are the same.

In the second case one can apparently take $a_n=0$ (or $a_m=0$) expecting that this approximation will not distort the double resonance curve significantly. The resulting equation exactly describes the double resonance curve for interactions with a barotropic wave in the case of the infinite Rossby radius. Here the polynomial Q_2 has a decomposition $Q_2=Q_3Q_4$, where $Q_3>0$, provided $\kappa>0$, and Q_4 is a cubic polynomial with respect to κ_0^2.

The equation $Q_4=0$ describes a unique real branch of the double resonance curve in each quadrant. This branch is placed in a bounded relatively close vicinity of the l-axis, touches it at the origin and links the origin with a specified point of the l-axis, forming the image of a flower-

leaf. In each quadrant, it has the shape of a fishhook, containing one broken point and having continuous derivatives elsewhere (Figure 2). The set $S^{*}_{2R}(p,m,n)$ contains three singularity points: the origin and two crosspoints of the described branch with the l-axis.

In the most general case $a_m \neq a_n$; $a_m a_n > 0$, the equation $Q_2 = 0$ always describes a fishhook-like branch of the curve $S_{2R}(p,m,n)$, similar to the one just described. In case $a_p < a_m$; $a_p \leq a_n$, relatively close to the k-axis there exists an oval branch of the double resonance curve, twice crossing this axis, and in the case $a_p = 0$ going through the origin (Figure 2). Usually the set $S^{*}_{2R}(p,m,n)$ contains three singularity points lying on the l-axis. Two additional singularity points on the k-axis exist iff $a_m > a_p = a_n > 0$.

It is of interest to consider the behaviour of the branches of the double resonance curve if $a_n \Rightarrow a_m$; a_p fixed (Figure 2). There exists only the fishhook-like branch on condition that $a_p > a_n$. The oval branch arises in case $a_p = a_n$ as a separated singularity point on the k-axis and increases

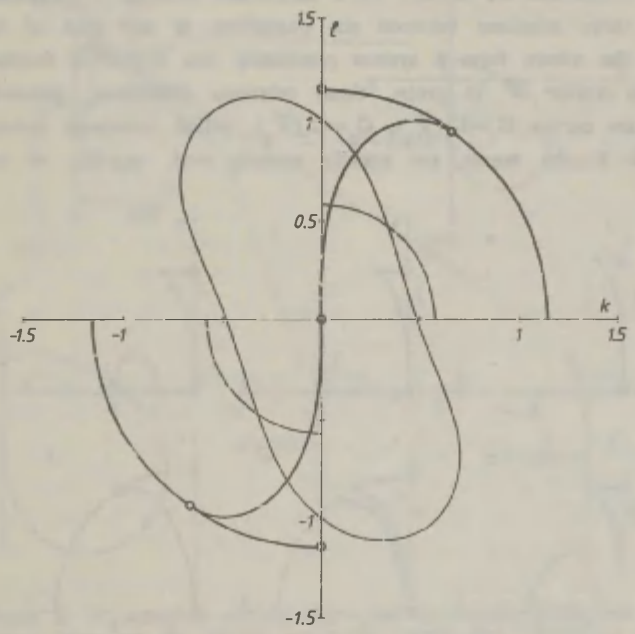

Figure 1. The double resonance curve (*solid curve*) and the locus of the other members $\vec{\kappa}_1, \vec{\kappa}_2$ of the double resonance triplets for the case $a_p = 1$, $a_m = a_n = 1$; $kl \geq 0$. The *small circles* mark the singularity points of the double resonance curve.

as the difference between a_m and a_n decreases. If $a_n \Rightarrow a_m$, one part of the oval combines with the upper part of the fishhook and with it forms the circle S_R. Another part merges with the lower part of the fishhook and in case $a_n = a_m$ forms the "double" curve S_x, linking the origin with a specified point of the circle S_R.

The question concerning all real branches of the double resonance curve is not yet completely solved, but probably there exist only the two branches described above.

A significant feature of double resonance triplets is that these triads usually contain one double resonance wave and two "normal" waves. For these "normal" waves all interactions are simple resonant interactions and the kinetic equation is valid for them.

3.4. Topology of the resonance curves

The conditions for Rossby wave harmonics forming a resonance triplet represent three relations between six quantities; if any two of these are specified, the others form a system possessing one degree of freedom. Supposing the vector $\vec{\kappa}$ is given, these relations determine, generally speaking, certain curves $G_1 = G_1(\vec{\kappa}_1)$; $G_2 = G_2(\vec{\kappa}_2)$, called resonance curves. These curves for Rossby waves are usually smooth oval, egg-like or hourglass-

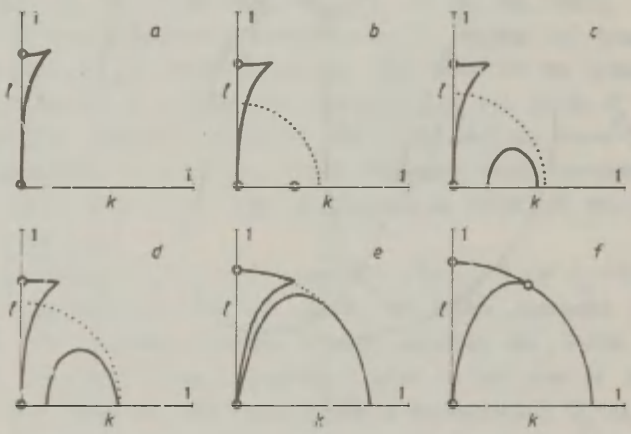

Figure. 2. The double resonance curve (*solid curve*) in the domain $k,l \geq 0$ and $a_p^2 = 0.5$, $a_m^2 = 1$, for the cases: $a_n^2 = 0$ (*a*); $a_n^2 = 0.5$ (*b*); $a_n^2 = 0.55$ (*c*); $a_n^2 = 0.7$ (*d*); $a_n^2 = 0.9$ (*e*); $a_n^2 = 1$ (*f*). The *dotted line* shows the circle $\kappa_0 = 0$. The *small circles* mark the singularity points of the double resonance curve.

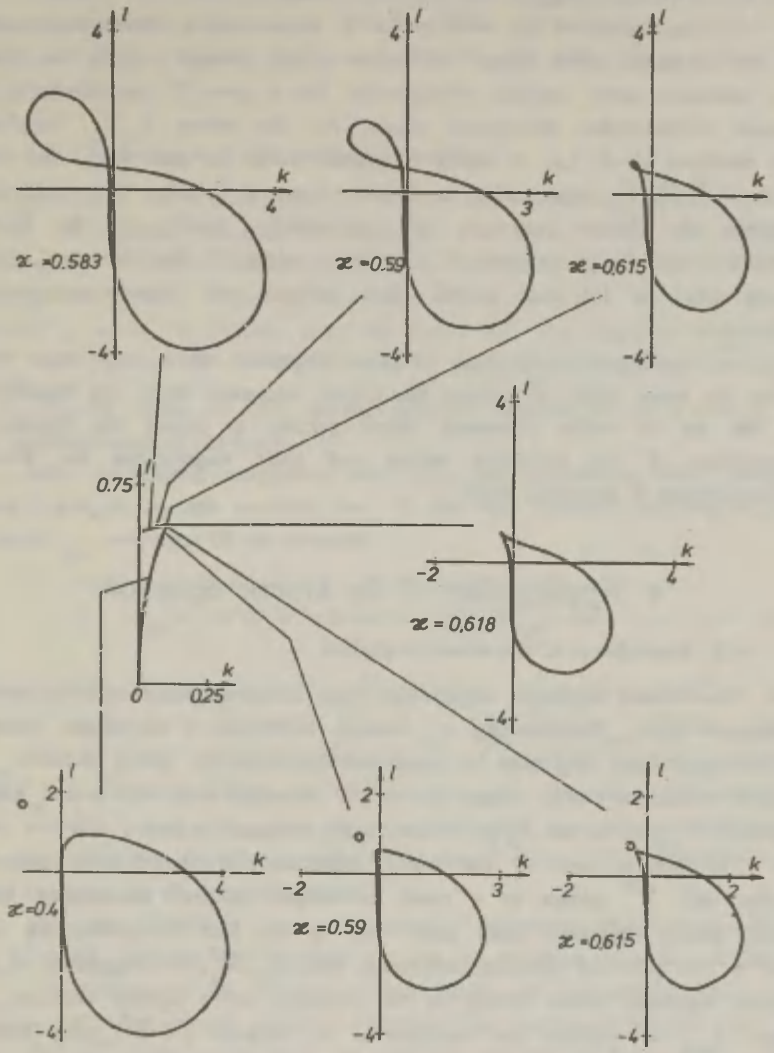

Figure 3. A selection of resonance curves, corresponding to wave vectors lying on the double resonance curve and containing singularity point in the case $a_p = a_m = 1$, $a_n = 0$. The position of the wave vector on the double resonance curve (*solid line* on the central picture) is shown using the *fine straight lines*. The *small circles* mark the separated singularity points of the double resonance curve.

like curves (Longuet-Higgins and Gill, 1967, Soomere, 1990).

A displacement of the wave vector $\vec{\kappa}$ causes only a smooth deformation of the resonance curve unless this vector passes through a point for which the resonance curve contains a singularity. For a given $\vec{\kappa}$ the curves G_1, G_2 remain differentiable everywhere except for the points $\vec{\kappa}_1, \vec{\kappa}_2$, satisfying the condition $\Delta = 0$; i.e. if double resonance occurs for this wave, the wave vectors $\vec{\kappa}, \vec{\kappa}_1, \vec{\kappa}_2$ representing a double resonance triplet. The connnexion between the double resonance (or, equivalently, validity of the kinetic equation) and the occurrence of singularity points of the resonance curves is in force in the case of all wave systems with three-wave resonant interactions.

As topological interactions of the resonance curve may occur only when the wave vector $\vec{\kappa}$ crosses the double resonance curve, the description of the set of double resonance points permits to sketch the topological description of the resonance curves and their singularities for Rossby waves (Figure 3, Soomere, 1990).

4. Applicability of the kinetic equation

4.1. Breakdown of the kinetic equation

The kinetic equation apparently fails at the points of the double resonance curve. The quantity Δ_{mn}, which is usually a continuous function of its arguments, may also be infinitesimally small for waves close to the double resonance waves. Thus, the kinetic equation may fail in the neighbourhood $V_{2R\varepsilon}$ of the set $S_{2R}(p)$ of the double resonance points.

The greater part of the energy alteration of the double resonance waves with $\vec{\kappa}^*$ occurs as a result of simple resonant interactions. Generally double resonance takes place only in the case of interactions with one or two pairs of specified harmonics with $\vec{\kappa}^*_1, \vec{\kappa}^*_2$. A breakdown of the kinetic equation occurs owing to the inability of a certain collision integral I_{pqrn} to describe the interactions of vectors $\vec{\kappa} \approx \vec{\kappa}^*$ with vectors $\vec{\kappa}_1 \approx \vec{\kappa}^*_1$; $\vec{\kappa}_2 \approx \vec{\kappa}^*_2$, the other integrals I_{pqrn}: $|q-m| + |r-n| > 0$ having generally nothing extraordinary. Thus, only a rather limited set of interactions cannot be described by the kinetic equation.

All collision integrals turn into zero provided $k=0$ because then the interactions coefficient $C^{pmn}_{\kappa_1 \kappa_2}$ vanishes. Thus, double resonance of waves with $k=0$ has no impact on the total energy exchange and the kinetic equa-

tion remains valid on the l-axis. This conclusion serves as a generalization of the result obtained by Reznik (1986) and reflects a well-known feature of Rossby waves: triad interactions do not alter the amplitude of the zonal flow.

4.2. Convergence of the collision integrals

Obviously, the kinetic equation is valid at a specified point $\vec{\kappa}$ only if all its collision integrals converge in this point. Any wave system governed by a kinetic equation conserves the enstrophy of the motion. Considering the Rossby wave systems with finite enstrophy only, we have $F_p = o(\kappa^{-4})$, $\kappa \to \infty$. It follows from the latter that the functions K_{pmn} and $C^{pmn}_{\kappa_i \kappa_j}$ are bounded for the arbitrary set of parameters p,m,n and the finite $\vec{\kappa}, \vec{\kappa}_1, \vec{\kappa}_2$. By virtue of this relation collision integrals always converge in the neighbourhood of the l-axis.

After performing integration over k_2, l_2 and introducing polar coordinates $\kappa, \varphi, \kappa_0, \varphi'$ we can conclude that, in the case arbitrary spectrum F, an integral I_{pmn} converges iff the integrals

$$J_{pmn} = \int_0^{2\pi} \kappa_o^3 \sin^2\varphi' (\kappa_0 \kappa \cos\varphi' + a_n^2 - a_m^2) \mid R^{-1}_{pmn} \mid d\varphi' \qquad (6)$$

converge for all the branches $\kappa_o(\varphi') \geq 0$. Here $\kappa_o(\varphi')$ is defined as the positive solution of the equation $P_{pmn} = -16k^{-1} N_{pmn} \omega^{012}_{pmn} = 0$ and $R_{pmn} = 0.25 \partial P_{pmn} / \partial \kappa_0$. Obviously, these integrals converge if the quantities κ_o, R^{-1}_{pmn} are bounded. Thus, all collision integrals describing the interaction between waves with $a_p \leq a_m = a_n$ (in particular, the interaction integral of the barotropic kinetic equation) converge. Also, all collision integrals converge for all "simple" points.

For the particular case of double resonance points we have $P_{pmn}(\hat{\varphi}') = \partial R_{pmn}(\hat{\varphi}') / \partial \kappa_0 = \partial P_{pmn}(\hat{\varphi}') / \partial \varphi' = 0$ for a specified $\hat{\varphi}'$. The function R_{pmn} (if defined in the vicinity of the point $\hat{\varphi}'$) generally satisfies the relation $R_{pmn} \alpha \varphi' - \hat{\varphi}'$ if $\varphi' \Rightarrow \hat{\varphi}'$, therefore the collision integral (if it exists as an improper integral) can diverge for double resonance points with $\mid \vec{k} \mid \geq \delta > 0$. Also, this integral may not exist for certain double resonance points. For example, there are several cases when the resonance curve is defined for the double resonance value $\hat{\varphi}'$ only, but the integrand in (6) is infinite in this point, and the actual value of the integral remains undefined.

4.3. Energy alteration of the double resonance waves

Reznik (1986) argued that the barotropic kinetic equation remains correct in the vicinity of the l-axis. Estimating the energy balance within "almost" double resonance triplets (i.e. the vector $\vec{\kappa}$ lying close to the l-axis, $\Delta \ll 1$), it was shown that the resonant interactions mostly lead to energy exchange between waves with $\vec{\kappa}_1, \vec{\kappa}_2$ and have practically no impact on the energy of the wave with $\vec{\kappa}$. Thus, the amplitude of an "almost" double resonance wave alters mainly owing to simple resonant interactions. On the other hand, the contribution to the energy exchange rate of an "almost" double resonance wave due to the interactions within the "almost" double resonance triplets, computed by formally applying the kinetic equation, is infinitesimally small. Thus, the kinetic equation remains correct in the vicinity of the l-axis in the sense that the error of computing the energy exchange rate for waves with $|k| \ll l$ is apparently negligible as compared to the total energy exchange rate of these waves. Below we shall treat the correctness of the kinetic equation and the collision integrals just in this sense.

An analogous speculation holds true in the case of interactions between harmonics representing arbitrary modes, and the kinetic equation for baroclinic Rossby waves is also valid in the vicinity of the l-axis.

Mathematically, these arguments reflect the fact that the collision integral converges in the vicinity of the l-axis and the energy exchange rate for "almost" double resonance waves computed by formally applying the kinetic equation is of the same order (i.e. negligible) as the actual value.

The integral (6) converges for double resonance points lying on either of the coordinate axes, or for double resonance vectors $\vec{\kappa}^*_1 = \vec{\kappa}^*_2$. The former case corresponds to the interaction of collinear wave vectors. The latter occurs in the case of the branch of the double resonance curve described by $3\kappa^2 = 4(a_m^2 - a_p^2)$, provided $a_p > a_m = a_n$. The kinetic equation remains valid in the vicinities of both the above-mentioned sets of double resonance points (in particular, in the vicinities of all the singularity points of the double resonance curve) and apparently fails in certain vicinity of the rest of the set $S_{2R}(p)$.

4.4. Double resonance and total energy alteration

Let us apply formally the kinetic equation to describe the temporal

evolution of the whole Rossby wave field. A certain collision integral fails in the vicinity $V_\delta = V_{2R\delta} \setminus V_{\delta 1} \setminus V_{\delta 2}$ of the curve $S^*_{2R}(p,m,n)$, where $\delta \ll 1$ is the width of the vicinity, $V_{\delta 1}$ denotes the vicinity of the coordinate axes, $V_{\delta 2}$ - the vicinity of the circle $3\kappa^2 = 4(a_m^2 - a_p^2)$ if $a_m = a_n$ or $V_{\delta 2} = \varnothing$ if $a_m \neq a_n$. This integral generally overestimates the absolute value of the energy exchange rate for "almost" double resonance triplets. The maximum error due to its inability to describe the energy exchange rate of the waves with $\vec{\kappa} = (\hat{\kappa}, \hat{\varphi}) \in V_\delta$ can be estimated roughly as

$$I_\delta^{pmn} = |k^{-1}| \int_{V_\delta} \int_0^{2\pi} \kappa \kappa_0 \, |K_{pmn} R^{-1}_{pmn}| \, d\varphi' d\kappa d\varphi \leq M \int_{V_\delta} \int_0^{2\pi} |R^{-1}_{pmn}| \, d\varphi' d\kappa d\varphi, \quad (7)$$

where $M < \infty$. It can be shown that $R_{pmn} \propto (\varphi - \hat{\varphi})^{3/2}$ or $R_{pmn} \propto (\varphi - \hat{\varphi})^{3/3}$ if $\varphi \to \hat{\varphi}$ on condition that the point $(\hat{\kappa}, \hat{\varphi})$ is not a singularity point of the double resonance curve or does not lie on the k-axis. After performing the integration in (7) over φ first, it is easy to verify that $I_\delta^{pmn} < \infty$ and $\lim I_\delta^{pmn} = 0$; $\delta \to 0$. This result essentially depends on correctness of the kinetic equation in the vicinity of the singularity points of the double resonance curve.

The latter relations mean that the error caused by the inability of the kinetic equation to describe the energy exchange of "almost" double resonance Rossby waves is in a certain sense small in comparison with the integral intensity of nonlinear interactions. Accordingly, it can be supposed that during the numerical solution of the kinetic equation for Rossby waves we may disregard the evolution of the spectrum in the vicinity of S_{2R}, expecting that the arising inaccuracy will not essentially distort the whole spectrum.

Thorough understanding of the geometry of the set of the double resonance points and triplets is necessary to establish the applicability of the kinetic equation in the case of other wave systems. This equation is apparently applicable in the case of arbitrary waves if (1) the length of the double resonance curve (more precisely, of that branch of the double resonance curve in the vicinity of which the kinetic equation fails) is finite, (2) the number of the double resonance triplets corresponding to each double resonance point is limited, (3) there exist no singularity points of the double resonance curve, or the kinetic equation remains correct in the vicinity of these points.

5. Stationary and equilibrium solutions

5.1. Conservation laws and irreversibility

The wave systems governed by a kinetic equation conserve their energy, enstrophy and wave momentum. The idea of proving that within the bounds of the kinetic equation, used widely in proving the other assertions given below, is as follows (Soomere, 1983). The interaction coefficients of the barotropic kinetic equation satisfy the Jacobi identity $C_{\kappa_1\kappa_2}/\omega(\vec{\kappa})=C_{\kappa\kappa_1}/\omega(\vec{\kappa}_2)=C_{\kappa\kappa_2}/\omega(\vec{\kappa}_1)=h$ on the hypersurface $\omega^{012}=0$; $\vec{\kappa}_{012}=\vec{0}$ (Kenyon, 1964). Multiplying (2) by an arbitrary function $S(\vec{\kappa},\tau)$, integrating it over $\vec{\kappa}$ and adding the resultant equation to the equations obtained from the latter by means of substitutions $\vec{\kappa}\Leftrightarrow\vec{\kappa}_1$; $\vec{\kappa}\Leftrightarrow\vec{\kappa}_2$, we obtain:

$$\int S(\vec{\kappa},\tau)\frac{\partial F}{\partial \tau} = 8\pi\int\left[S(\vec{\kappa},\tau)\omega(\vec{\kappa})+S(\vec{\kappa}_1,\tau)\omega(\vec{\kappa}_1)+S(\vec{\kappa}_2,\tau)\omega(\vec{\kappa}_2)\right]\times$$

$$\times\left[\frac{\omega(\vec{\kappa})}{F}+\frac{\omega(\vec{\kappa}_1)}{F_1}+\frac{\omega(\vec{\kappa}_2)}{F_2}\right]\frac{b^2FF_1F_2}{3N}\,\delta(\omega^{012})\delta(\vec{\kappa}_{012})\mathrm{d}\vec{\kappa}\,\mathrm{d}\vec{\kappa}_{12} \qquad (8)$$

The latter integral vanishes if the expression in the small square brackets equals the argument of a delta function. The choices $S=\mathrm{const}$; $S=\mathrm{const}(\kappa^2+a^2)$ and $S=\mathrm{const}1/(\kappa^2+a^2)/k$ yield $E=\int F\mathrm{d}\vec{\kappa}=\mathrm{const}$; $Y=\int(\kappa^2+a^2)F\mathrm{d}\vec{\kappa}=\mathrm{const}$ or $L=\int 1k^{-1}(\kappa^2+a^2)F\mathrm{d}\vec{\kappa}=\mathrm{const}$, respectively.

Furthermore, by taking $S=F^{-1}$ we get $\int F^{-1}(\partial F/\partial\tau)\mathrm{d}\vec{\kappa}=\partial(\int \ln F\mathrm{d}\vec{\kappa})/\mathrm{d}\tau\geq 0$. The latter inequality serves as an analogy to Boltzmann's H-theorem and proves the irreversibility of the spectrum changes; the quantity $H=\int \ln F\mathrm{d}\vec{\kappa}$ playing the role of entropy. Analogous results hold true also for the baroclinic kinetic equation (Kozlov, Reznik and Soomere, 1987).

5.2. Differeutiable thermodynamically equilibrium solutions

Thermodynamically equilibrium solutions of the kinetic equation are defined as a subclass of its stationary solutions, satisfying the condition $\partial H/\partial T=0$. In principle, Eq. (2) has at least two classes of stationary solutions. The first consists of positively defined spectra satisfying the system of equations $\omega(\vec{\kappa})F^{-1}+\omega(\vec{\kappa}_1)F_1^{-1}+\omega(\vec{\kappa}_2)F_2^{-1}=0$; $\omega^{012}=0$; $\vec{\kappa}_{012}=0$. The second class consists of spectra in which the product F_iF_j is nonzero only at points $\vec{\kappa}_i,\vec{\kappa}_j$, whose coordinates satisfy the condition $C_{\kappa_i\kappa_j}=0$.

There exists only one family of differentiable solutions $F_{eq}=(p+q\kappa^2)^{-1}$ in the first class, with $p>0$; $q\geq 0$ as arbitrary constants (Reznik, 1984a). These solutions are linearly stable with respect to small disturbances. It is of importance that these spectra are isotropic and coincide with the well-known equilibrium spectrum of 2D isotropic turbulence.

As for the two-layer model, the only family of differentiable equilibrium solutions has the following form (Kozlov, Reznik and Soomere, 1987): $F_{eq}=[p+q(\kappa^2+a_1^2)]^{-1}$; $G_{eq}=[p+q(\kappa^2+a_1^2)]^{-1}$. These solutions are linearly stable with respect to small disturbances. The arbitrary constants $p>0$; $q\geq 0$ are the same for both spectrum components. Thus, the final state of the spectral evolution of baroclinic Rossby wave systems must be essentially baroclinic, and no decaying of the higher mode(s) should be expected. Although in case $a_0=0$ there exists a purely barotropic solution ($G\equiv 0$) of the baroclinic kinetic equation, this solution is apparently unstable.

5.3. Generalized equilibrium solutions

One of the most distinctive features of the evolution of Rossby waves is a relatively intensive energy transfer to the zonal component of motion, i.e. mainly to the relatively close vicinity of the l-axis. As will be seen below, this transfer takes place irrespective of the initial form of the spectrum and gives rise to essentially anisotropic spectra. This feature contradicts the fact that the only possible equilibrium-state solution of the kinetic equation in the space of functions differentiable in the usual sense is isotropic. Reznik and Soomere (1983b) have introduced solutions from a wider class of functions, including Dirac delta functions, which they called generalized spectra.

The simplest generalized solution of the kinetic equation is $F_z=f(l)\delta(k)$, where $f(l)\geq 0$ is an arbitrary continuous function. This spectrum corresponds to a stationary rectilinear zonal motion and is stable with respect to arbitrary perturbations.

The function $F_{zd}=f(l)\delta(k)+F_d$, equal to the sum of an arbitrary differentiable symmetric $[F_d(-k,l)=F_d(k,l)]$ solution F_d of Eq. (2) and of the spectrum of a zonal current F_z, also satisfies the kinetic equation. The law of the increasing of the entropy $H_d=\int \ln F_d \, d\vec{\kappa}$ of the non-zonal part of the motion holds true, law guaranteeing that the spectral evolution of this wave system is also irreversible, and will tend to an equilibrium

state. The latter is described by the anisotropic spectrum $F_{zeq}=F_{zd}+F_{eq}=f(l)\delta(k)+(p+q\kappa^2)^{-1}$; $p>0$; $q\geq 0$, consisting of the sum of an arbitrary zonal flow and of the differentiable equilibrium spectrum. This spectrum is also linearly stable with respect to small perturbations. Therefore, an intensive energy transfer to the vicinity of the vertical axis by no means contradicts the fact that the system tends towards the thermodynamical equilibrium. However, the distribution of energy between the zonal and nonzonal components as well as the shape of the function f(l) depends on the initial state.

All the results presented in this section have been generalized for the case of the baroclinic kinetic equation (3) (Kozlov, Reznik and Soomere, 1987). The anisotropic generalized thermodynamically equilibrium spectrum, linearly stable with respect to small perturbations, is expressed by $F_{zeq}=F_z+F_{eq}=f(l)\delta(k)+[p+q(\kappa^2+a_1^2)^{-1}]$; $G_{zeq}=G_{zd}+G_{eq}=f(l)\delta(k)+[p+q(\kappa^2+a_0^2)^{-1}]$; $p>0$; $q\geq 0$. The continuous functions f(l), g(l) representing the barotropic and baroclinic parts of the zonal flow are independent here, whereas the smooth parts of the spectra are related.

5.4. Generalized stationary solutions

There exists a much wider class of generalized solutions of the kinetic equation, consisting of spectra in which the product F_iF_j is nonzero only at points $\vec{\kappa}_i, \vec{\kappa}_j$, whose coordinates satisfy the condition $C_{\kappa_i\kappa_j}=(\vec{\kappa}_i\times\vec{\kappa}_j)(\kappa_j^2-\kappa_i^2)/2=0$ (Soomere, 1987). All such spectra are concentrated either on an arbitrary line passing through the origin or on an arbitrary circle centered at the origin, i.e. spectra of the forms $F^{(1)}=f(\kappa)\delta(pk+ql)$ or $F^{(2)}=f(\varphi)\delta(\kappa-r)$, $f,r\geq 0$. The spectrum $F^{(1)}$ represents a straight line flow on the β-plane; the spectrum $F^{(2)}$ - a system of waves of equal length. The existence of these solutions to the kinetic equation results from a well-known property of Rossby wave triad interactions, namely, that neither the waves with collinear wave vectors nor the waves of equal length exchange energy between each other. But in physical terms, the existence of the solutions $F^{(1)}$ with $q\neq 0$ is not entirely comprehensible, because nonzonal straight flow does not satisfy the initial equation (1) for Rossby waves (Kamenkovich and Reznik, 1978).

Analysis of the spectra $F_d^{(1)}=F^{(1)}+F_d$; $F_d^{(2)}=F^{(2)}+F_d$, where F_d is an arbitrary continuous function, shows that the evolution of the spectrum F_d is irreversible. The spectra $F_d^{(2)}$ with $f\neq 0$ cannot be equilibrium solutions of the kinetic equation. The spectrum $F_d^{(1)}$ represents an equilibrium state

only when $F_d^{(1)}=F_{zeq}$.

Physically, the existence of a unique equilibrium spectrum F_{zeq} among the solutions of the kinetic equation $F_d^{(1)},F_d^{(2)}$ is a fine expression of the "equilibrium" between the impact of the β-effect and weak nonlinearity. The presence of nonlinearity results in an expansion of the initially narrow spectral peaks, causing the spectra to tend towards the isotropic equilibrium state. But as a result of the β-effect, the zonal component of motion is simultaneously intensified to a certain degree. The "interaction" of these two factors makes possible the existence of the anisotropic equilibrium spectra F_{zeq}.

The results described in this section have also been generalized for the case of the baroclinic kinetic equation (3).

6. Numerical method for solving the Cauchy problem for the kinetic equation

6.1. Principles of the numerical method: barotropic case

The problem consists in solving the Cauchy problem for the nonlinear integro-differential equation (2) in the infinite domain (Soomere, 1983; Reznik and Soomere, 1983a; 1984a,b). It is convenient to reduce the delta functions of the collision integral I_{pmn}, by first performing the integration over k_2,l_2 and then introducing the polar coordinates $\kappa,\varphi,\kappa_0,\varphi'$ (Section 3.2). After performing the integration over κ_0, the collision integrals are reduced to certain single integrals over a bounded interval.

The function F is calculated at the nodes of a rectangular grid covering the bounded region Ω of the k,l plane. The values of $F(\vec{\kappa}_1),F(\vec{\kappa}_2)$ at points $\vec{\kappa}_1,\vec{\kappa}_2$, not coinciding with the grid nodes, are interpolated between the nodes adjacent to these points. Collision integrals are approximated by using the Gauss' quadrature formula.

The resulting system of ordinary differential equations is not closed because in any finite region Ω there exist vectors $\vec{\kappa}_1$, which interact with vectors lying outside Ω. Two methods were used in order to handle such interactions. According to the first method, we assumed $F(\vec{\kappa})\equiv 0$ outside the region Ω (scheme A). Physically, this is equivalent to the introduction of infinite viscosity outside of Ω. Since the Rossby wave spectrum $F=o(\kappa^{-4})$; $\kappa\Rightarrow\infty$, in the event of a sufficiently large region Ω, the error introduced by this truncation is very small. Another truncation scheme was also used,

consisting in completely ignoring all interactions involving vectors lying outside Ω (scheme B). This method is equivalent to replacing the infinite region of integration in Eq. (2) by a finite region $\Omega(\vec{\kappa}_1)\times\Omega(\vec{\kappa}_2)$. It is of importance that all conservation laws of the kinetic equation (2) remain in force in this approximation.

The greatest difficulty in the development of the numerical scheme was the proper choice of the grid size, the interpolation method, the quadrature and the finite-difference representation of the time derivative, all of which together determine the operational effectiveness of the scheme. The listed parameters were optimized according to the demand that the rms. error of the calculations must be less than 1% per unit of time (Soomere, 1983). Most of the calculations were carried out for the nodes of a 97×114 rectangular grid covering the region $\Omega=\{\kappa\le4\}$. The grid has a finer resolution in the vicinity of the l-axis. The collision integral was replaced by the Gauss' quadrature formula with 24 nodes in the segment $[0,\pi]$. A double linear interpolation method was used. The resulting system of ordinary differential equations was solved by using an implicit second-order Adams' scheme.

A check was made on the proper operation of the scheme from the behaviour of the conservation integrals. Still another property that permits judging the operational correctness of the scheme is the average modulus $<\kappa>=\int\kappa F d\vec{\kappa}/E$ of the wave vectors. Analogically with 2D turbulence, the quantity $<\kappa>$ should decrease with time, as noted in all cases observed. Every time step of this numerical scheme takes about 15 minutes of the CPU time of EC-1022 (IBM 360). The evolution of more than ten initial spectra was investigated during maximally 1000 time steps.

6.2. Specificity of the numerical scheme in the baroclinic case

The numerical scheme for solving the Cauchy problem for the baroclinic kinetic equation (3) is rather similar to the analogous scheme for the barotropic case (Soomere and Rannat, 1990; 1991) and we shall describe only its specific features.

Since the density of water masses in the Earth's oceans varies insignificantly, for practical purposes in Eqs. (3) it is allowed to disregard the quantities of the order of the relative density alteration. In that case the expressions for the coefficients γ^{P}_{mn} (Kozlov, Reznik and Soomere, 1987) are greatly simplified. In particular, we obtain $I_{001}=I_{010}=I_{100}=0$, which reflects the fact that the interactions between two barotropic and

one baroclinic Rossby wave are much less intensive than other resonant interactions.

The boundary between the layers is the main thermocline, usually located at a depth of roughly 1 kilometer (h_1). The mean depth of the ocean (h_1+h_2) is approximately 5 kilometers. With these values, the kinetic equation contains factors $Q_{\pm}^2=(h_2/h_1\pm 1)^2$. It is convenient to remove these coefficients by defining the new time scale $T=\tau Q_+^2$. By replacing $F_0 \Rightarrow F$ and $F_1 \Rightarrow G=\alpha^2 F_1$, we obtain:

$$\frac{\partial F}{\partial T} = 8\pi(I_{000}+\alpha I_{011}); \qquad \frac{\partial G}{\partial T} = 8\pi(1-\alpha)^2 I_{111}+16\pi I_{110}, \tag{9}$$

where $\alpha=h_1/h_2$ and the kernels of the collision integrals are modified in an obvious manner. Eqs. (9) have similar conservation laws as those of the full equations (3). Also, the H-theorem $dH/dT=d(\int \ln F G d\vec{\kappa})/dT \geq 0$ is in force for Eqs. (9). Thus, in this approximation all the main features of the original wave system hold true.

Each collision integral is reduced to a single integral as described in the previous section. Integrals I_{000} and I_{111} are identical (to within the values of the parameters a_0, a_1) with the interaction integrals of the barotropic kinetic equation. In computing the integral I_{011} one must take into account that some of the barotropic waves cannot interact with the baroclinic waves at all. The integrands of I_{011} and I_{110} may contain non-integrable singularities at the double resonance points. The actual behaviour of the spectrum in the vicinity of the double resonance curve is neglected, and its value is estimated after each time step on the basis of its adjacent values. Only a modification of the scheme B is used in the numerical simulations, i.e. the interactions with wave vectors lying outside the computational area are neglected. As in the barotropic case, the mean error of the calculations is less than 1% per unit of time T. Every time step of the described numerical scheme takes about 40 minutes of the CPU time of PC AT/8MHz. The evolution of about 10 initial spectra is investigated during 200-350 time steps.

7. Numerical simulations

7.1. Evolution of barotropic wave systems: amplification of the zonal component

The numerical simulations of the spectral evolution of barotropic

weakly nonlinear Rossby wave systems are mostly performed with the scaling $a=1$ (Soomere and Reznik, 1984a,b, Soomere, 1983). The results are reviewed in some detail by Reznik (1986).

An interesting coexistence of two tendencies was established in all numerical experiments. First, a portion of the energy is concentrated near the l-axis to form a well-defined peak in the spectrum, thereby enhancing the zonal component of the flow. Second, in a region sufficiently far from the l-axis the spectrum tends to become isotropic. However, the intensity of the peak referred to depends crucially on the initial conditions.

The phenomenon of reinforcement of the zonal component of motion is analogous to the well-known phenomenon of negative viscosity, consisting in energy transfer from turbulent fluctuations to an average state. Actually, energy transfer to the wave vectors of almost meridional orientation means the existence of an energy flow from motion components which change rapidly in time to slowly changing components here. This phenomenon is created only by the β-effect and nonlinearity, and, in our opinion, is of fundamental significance, since it points to the possibility of energy transfer from synoptic eddies to the average flows. This mechanism may also play a direct role in the formation and maintenance of zonal currents in the atmospheres of the Earth and other planets, and also of the ring currents occurring in plasma (Hasegawa, Maclennan and Kodama 1979).

Beginning at some instant, something like saturation of the zonal component of the motion occurs. The rest of the energy is distributed more or less isotropically over the wave vectors. As the frequency of the Rossby waves $\omega \sim k$, a rather interesting "singular" equilibrium is noticeable between the slowly-changing, almost zonal component and the rapidly-changing component of the motion responsible for the isotropic portion of the spectrum.

7.2. Evolution of barotropic wave systems: general features

The evolution of an asymmetric spectrum was also investigated. The initial asymmetry quite rapidly tends towards a symmetrical state. This tendency is closely related to the effect of the generation of the nearly zonal current. Nevertheless, by virtue of the maintenance of the meridional component of the momentum, complete symmetrization of the spectrum is not possible.

Some of the calculations were repeated in the so-called "rigid lid" approximation (i.e. $a=0$). The introduction of this approximation does not

change the results qualitatively, but only increases the rate of energy transfer to some extent.

Most of the calculations were carried out in parallel, using schemes A and B. This permits the evaluation of the effect of weak dissipation on the nonlinear evolution of the spectrum. Dissipation has almost no effect (during the time of calculation) on the energy distribution for $\kappa \sim 1$ (i.e. in those regions where energy differs considerably from zero) and becomes significant only for large κ. Although the dissipation itself is isotropic, its effect is highly anisotropic, leading to a damping of the fairly short wave components, whose propagation direction is far from meridional. The intensity of the short-wave components, propagating primarily along a meridian, remains practically unchanged. Thus, dissipation is an additional factor enhancing the concentration of energy near the l-axis.

7.3. Numerical simulations versus thermodynamics

The tendency of the Rossby wave system to evolve towards a thermodynamically equilibrium state was investigated on the basis of the integral intensity of the interactions $I=\int | \partial F/\partial \tau | \, d\vec{\kappa}$ as well as on the behaviour of the entropy and its time derivative (in simulations by using scheme B only). The evolution of an initial spectrum was computed normally unless the quantity I decreased 10 times as compared to its value at $\tau=0$. Apparently by that time the wave system had reached a nearly "final" state of evolution. During the computational time, the entropy derivative of the wave system decreases by three to five orders of magnitude. Accordingly, the entropy at the origin increases very rapidly, and it finally becomes nearly constant. Such behaviour of the entropy and its derivative makes it possible to conclude that the system in question actually approaches the thermodynamical equilibrium. From the general picture of the evolution described in the preceding section it follows that the spectrum of weakly nonlinear Rossby waves evidently tends to a distribution consisting of the sum of a delta-shaped spectrum corresponding to the zonal flow and an isotropic spectrum (Reznik and Soomere, 1984b).

7.4. Evolution of baroclinic wave systems

Numerical experiments with the baroclinic kinetic equation are performed mostly in cooperation with K.Rannat (Ph.D. student). Preliminary results of these experiments are published by Soomere and Rannat (1990,1991); a review of the results is in preparation. The main con-

clusions of the experiments have also been presented at several meetings and lectures.

Let us discuss in some detail the evolution of the initial spectra $f(k,l)=p_0\sin^4\varphi\exp(-\kappa^4)$; $g(k,l)=p_1\sin^4\varphi\exp(-\kappa^4)$ in case $a_0=0$; $a_1=1$; $\alpha=0.2$. The values of p_0,p_1 are selected so that the full energy of the system $E=1$ at $T=0$. These initial conditions correspond to the system of synoptic motions in which the zonal component of motion is predominating. Of interest is the behaviour of both the spectra and their time derivatives (Figures 4,5). At the moment $T=0$ the field $\partial F/\partial T$ is divided into four parts by the curves $\partial F/\partial T=0$; these parts are marked with Roman numbers I-IV. There are two areas of energy inflow I,III near the l- and k-axes, respectively, and two areas of energy outflow II,IV. The absolute values $|\partial F/\partial T|$ in area IV are at least by an order smaller than the analogous values in area II, and this area does not play any essential role in the total energy transfer. Physically, there exist two directions of energy redistribution: into an almost-zonal flow with $\kappa\sim 1$ and into an almost-meridional flow, which are generated by the rest of the wave field. In the course of time areas I,III as well as the values of $\partial F/\partial T$ in these areas decrease and relatively far from the origin there arises a new inflow area V. Area II increases to some extent and stretches out in the direction of the l-axis.

In the case of the baroclinic component G of the motion we see a somewhat different distribution of its time derivative. At $T=0$ there is an area of intensive energy inflow I in the lower half of the area represented in Figure 5, and an area of energy outflow II, approximately of the same size as area I. Area III of weak outflow in the close vicinity of the l-axis plays a negligible role in the total energy redistribution. This distribution of $\partial G/\partial T$ indicates that only the almost-meridional baroclinic flow will be generated. In the course of time the outflow area decreases and stretches out in the direction of the l-axis and, in the end, divides the inflow area into two parts I and IV. By the moment $T=2.5$ the integral intensity of nonlinear interactions $I=\int(|\partial F/\partial T|+\alpha|\partial G/\partial T|)d\vec{\kappa}$ has decreased 10 times as compared to its value at $T=0$. This feature allows us to conclude that the distributions of F,G at $T=2.5$ are quite close to the final state of the evolution.

In conformity with the distribution of $\partial F/\partial T$, a narrow and high barotropic spectral peak arises in the very near vicinity of the l-axis. The peak, corresponding to the almost-zonal flow, appears to be much higher and nearer to the l-axis as compared to the pure barotropic case. Physi-

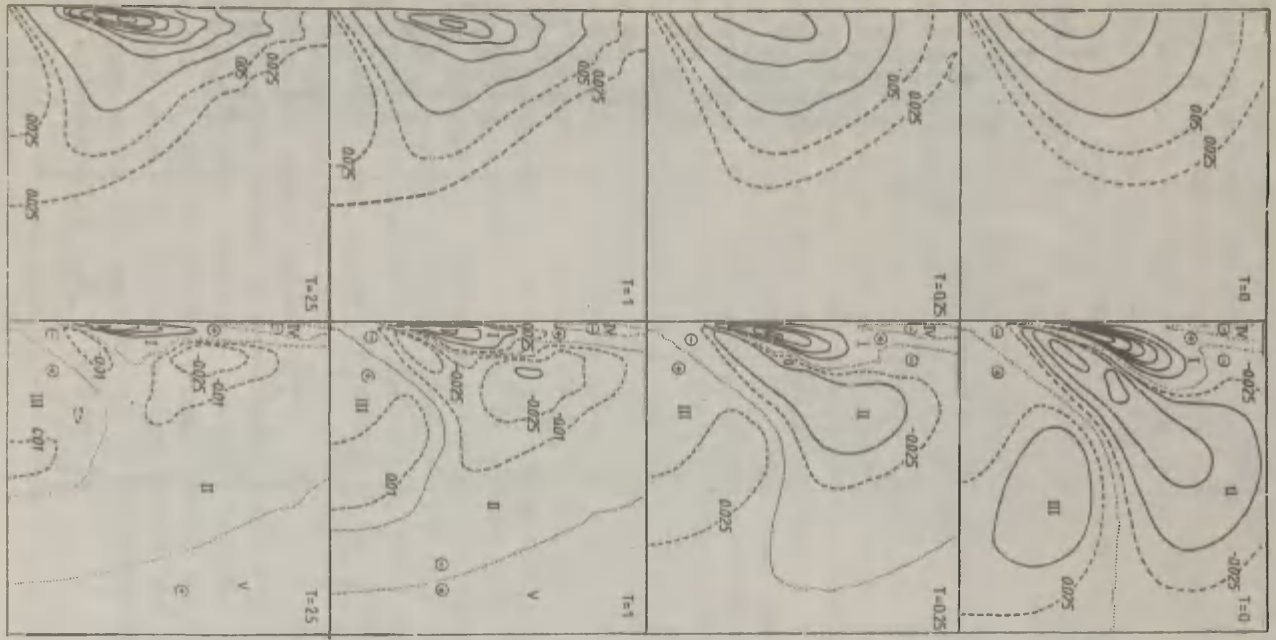

Figure 4. Instantaneous fields of the spectrum F (left column) of the barotropic mode and its time derivative $\partial F/\partial T$ (right column) at the time moments T=0; 0.25; 1; 2.5. Distance between main isolines (*continuous lines*) is 0.1 (F) or +0.05 ($\partial F/\partial T$). Area $0 \le k,l \le 2$ is represented in every box. *Dotted line* marks the curves $\partial F/\partial T=0$. The energy inflow and outflow areas are marked by symbols "+" and "-", respectively.

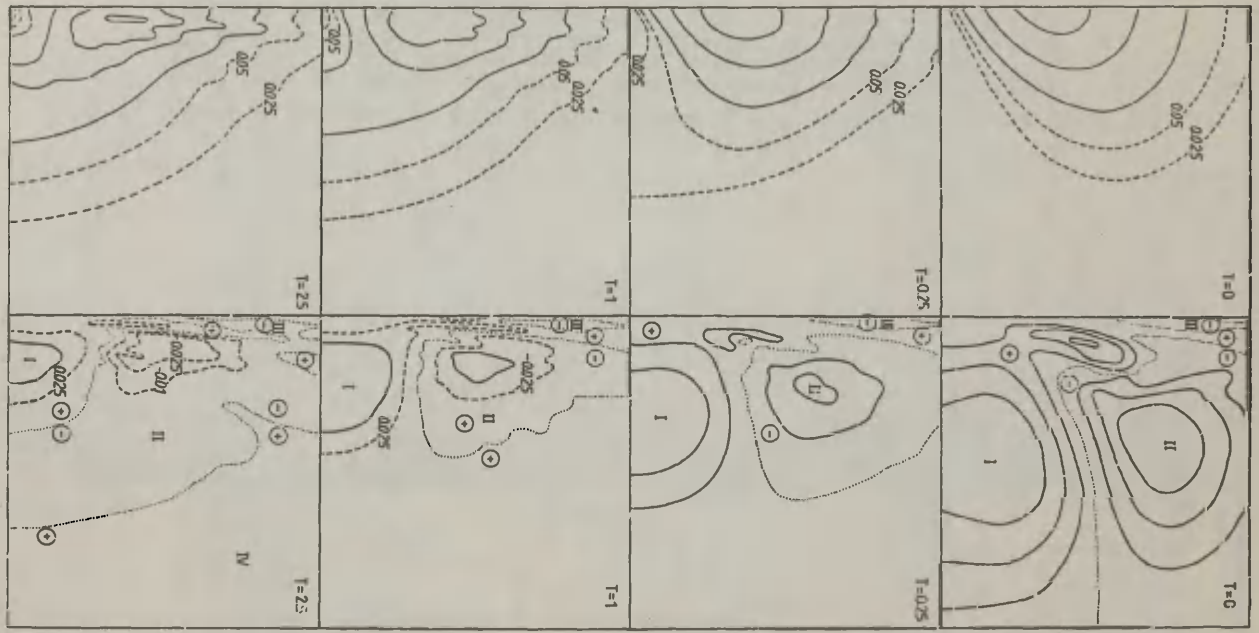

Figure 5. Instantaneous fields of the spectrum G (left column) of the baroclinic mode and its time derivative $\partial G/\partial T$ (right column) at the time moments T=0; 0.25; 1; 2.5. Distance between main isolines (*continuous lines*) is 0.1 (F) or +0.05 ($\partial F/\partial T$). Area $0 \le k,l \le 2$ is represented in every box. *Dotted line* marks the curves $\partial G/\partial T=0$. The energy inflow and outflow areas are marked by symbols "+" and "-", respectively.

cally, this means that a quite powerful practically zonal flow is generated by the nonlinear interactions. In the area, far from this axis, the spectrum F apparently tends towards some isotropic state. Again, a singular equilibrium is established between the nearly stationary zonal component and the rapidly-changing component of the motion.

The zonal component of the baroclinic part of the motion does not increase significantly and the whole "final" spectrum is nearly isotropic. Further, there occurs no significant energy exchange between the barotropic and baroclinic components during the whole computational period. This result is in good agreement with the theoretical prediction concerning the essential baroclinicity of the final state of the evolution.

The main features of the spectral evolution of the other initial spectra are qualitatively similar to those described above and may be summarized as follows. (1) In the course of time an intensive almost-zonal nearly barotropic flow arises. The peak in the spectrum of the barotropic mode is significantly higher and nearer to the l-axis than in the purely barotropic case. This quality suggests that the baroclinic mode may play the role of a "catalyst" in amplifying the effect of negative viscosity and speeding up the generation of large-scale zonal barotropic flows in the ocean. Hence, the existence of more powerful currents may be anticipated in relatively well-stratified regions of oceans. (2) Although the zonal component of the baroclinic motions has increased to some extent in several experiments, the relative height of the spectral peak near the l-axis remains essentially lower than the analogous peak in the spectrum of the barotropic mode. Thus, the almost-zonal flow generated by weakly nonlinear interactions of Rossby waves turns out to be practically barotropic. (3) In the areas far from the l-axis the spectra of both modes finally become almost isotropic. (4) During all our numerical experiments the full spectrum of the motion retains its essentially baroclinic nature. The last two features suggest that the effect of the barotropization of synoptic motions observed in oceans mainly results from the formation of intensive large-scale almost-zonal practically barotropic flows. However, the field of relatively small-scale synoptic motions may retain its depth-dependant nature. The listed features also indicate that an arbitrary initial wave system will apparently tend to the anisotropic thermodynamically equilibrium state F_{eq}, G_{eq}.

8. Summary

The main results of the investigations presented are the following:

(1) the kinetic equation for weakly nonlinear Rossby waves has been generalized for the baroclinic case within the bounds of the two-layer model;

(2) a theory of stationary and thermodynamically equilibrium smooth and generalized solutions of the kinetic equation has been worked out;

(3) the geometrical conditions for Rossby wave harmonics to form a double resonance triplet are studied in detail and the set of double resonance waves as well as the topology of the resonance curves is described;

(4) the applicability of the kinetic equation for a wave system with double resonance is discussed and the correctness of this equation for baroclinic Rossby waves has been established;

(5) an effective numerical method for solving the Cauchy problem for the kinetic equation has been constructed;

(6) a rather detailed picture of the spectral evolution of barotropic Rossby wave systems has been given;

(7) the general features of the evolution of baroclinic wave systems within the bounds of the two-layer model have been explained.

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Abstract

The subject of the thesis is the kinetic theory of stochastic weakly nonlinear wave systems, mostly on the example of Rossby waves. The main object of the investigations is the kinetic equation - a nonlinear integro-differential equation (similar to the Boltzmann equation) describing the slow temporal evolution of the spectral density of the energy (spectrum) of the wave system.

The kinetic equation for baroclinic Rossby waves within the bounds of two-layer model is derived. The existence and uniqueness of the smooth and generalized stationary as well as thermodynamically equilibrium solutions of this equation and the stability of these solutions is established. The topology of the set of double resonance points for Rossby waves is studied in some detail. The correctness of the kinetic equation for wave systems with double resonance is discussed and its applicability in case of Rossby waves is proved.

An effective numerical method for solving the Cauchy problem for barotropic and the simplest baroclinic (including only barotropic and the first baroclinic modes) kinetic equation is proposed. A fairly detailed picture of the main features of the free spectral evolution (without energy sources and sinks) of barotropic and two-modal Rossby wave systems is given on the basis of the computer simulations with the mentioned numerical method.

Rossby lainete kineetiline teooria

Annotatsioon

Väitekirjas käsitletakse pideva energiaspektriga nõrgalt mittelineaarsete stohhastiliste lainesüsteemide teooria küsimusi peamiselt Rossby lainete näitel (Rossby lained e. planetaarsed lained kujutavad endast teatavaid laineliisi liikumisi õhukeses vedelikukihis pöörleva kera pinnal). Selliste lainesüsteemide energiaspektri aeglast evolutsiooni kirjeldab nn. kineetiline võrrand (teatav mittelineaarne integro-diferentsiaalvõrrand; Boltzmanni võrrandi analoog). Käsitletakse nii barotroopset (vedelikukihti vaadeldakse vertikaalsuunas homogeensena) kui ka barokliinset (stratifitseeritud vedelikukiht) juhtu.

Töös tuletatakse kineetiline võrrand barokliinsete Rossby lainete jaoks ookeani (atmosfäärі) kahekihilise mudeli raames. Tõestatakse kineetilise võrrandi statsionaarsete ja tasakaaluliste lahendite (sh. üldistatud lahendite) eksisteerimine, ühesus ning stabiilsus teatavas üldistatud funktsioonide klassis. Demonstreeritakse Rossby lainete topeltresonantsi ning esitatakse resonantsi- ning topeltresonantsikõverate geomeetria ning topoloogia põhjalik kirjeldus. Uuritakse kineetilise võrrandi kasutatavust topeltresonantsiga lainesüsteemide puhul ning tõestatakse selle võrrandi korrektsus barokliinsete Rossby lainete puhul.

Väitekirja arvutuslikus osas konstrueeritakse efektiivne arvutusskeem Cauchy ülesande lahendamiseks kineetilise võrrandi jaoks ühe- ja kahekihilise ookeani (atmosfäärі) mudeli puhul ning esitatakse nii barotroopsete kui ka barokliinsete Rossby lainete süsteemide vaba evolutsiooni põhijoonte detailne kirjeldus.