DISSERTATIONES
MATHEMATICAE
UNIVERSITATIS
TARTUENSIS
105

MD. RAKNUZZAMAN

Noncommutative Galois Extension Approach to Ternary Grassmann Algebra and Graded *q*-Differential Algebra

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
105

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

105

MD. RAKNUZZAMAN

Noncommutative Galois Extension Approach to Ternary Grassmann Algebra and Graded q-Differential Algebra

UNIVERSITY OF TARTU
Press

Faculty of Science and Technology, University of Tartu, Tartu, Estonia.

Dissertation has been accepted for the commencement of the Degree of Philosophy (PhD) in Mathematics on June 8, 2016 by the Council of Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu.

Supervisor:

Prof. Viktor Abramov
University of Tartu
Tartu, Estonia

Opponents:

Prof. Alexandre Luis Trovon de Carvalho
Departamento de Matemática, Universidade Federal do Paraná
Curitiba, Brazil

Prof. Sergei Silvestrov
Division of Applied Mathematics, School of Education, Culture and Communication (UKK), Mälardalen University,
Västeras, Sweden

Commencement will take place on August 24, 2016 at 11.00 in J. Liivi 2-403.

ISSN 1024-4212
ISBN 978-9949-77-164-6 (print)
ISBN 978-9949-77-165-3 (pdf)

University of Tartu Press
www.tyk.ee

Contents

Acknowledgements

I would like to express my sincere appreciation to my Supervisor Professor Viktor Abramov. He has been a tremendous mentor to me. I humbly thank him for the way he encouraged me to think scientifically and to know the partialities of teaching and conducting a scientific research.

Professor Viktor Abramov has also been a guardian to me. I remember, when I first arrived in Estonia, it was just my second time abroad and I was not well aware of the cultures and customs here in Estonia. Professor Abramov guided me into all these aspects and helped me to get used to with a new environment.

I would like to thank all the participants who I met during our doctoral seminar sessions and discussed scientific aspects of different Mathematical Phenomena. These discussion sessions helped me to rethink and reconstruct my scientific research, and to develop my ideas more precisely; which I believe helped me greatly in my PhD studies.

I am very much grateful to the Institute of Mathematics and Statistics at the University of Tartu for providing me an excellent research environment to continue my graduate studies. Thanks to the friends and colleagues of the Institute of Mathematics for their enthusiastic supports and sometimes indirectly taking care of my daily life during my studies here as a foreigner.

I would like to give a special thank to the Prof. Eve Oja. She has been inspirational for me. She was the one who always asked me regarding my research proceedings and how to get forward; her query and instructions drove me to do something better.

During the time framework of my PhD study period, I got many friends in and outside of our Institute. I am very much thankful to one of my Estonian friends JaanVajakas for providing me necessary study materials as well as technical supports. His way of Mathematical thinking made me interested to think scientifically and to think how to love Mathematics.

I would like to thank Aminul Islam, who has been a friend and mate to me and encouraged me whenever I felt dawn part. He is the one who made me to think positively and his way of positive thinking derived me to do something better.

I would like to remember my father, who has passed away during my doctoral study. I could not attend to my father's funeral due to my studies and being abroad. My father always had a dream to see me as a doctorate. I would like to dedicate it to my father and I hope from heaven he would see that his dream has come true.

My mother, who has just become a widow, has been my inspiration. She still is helping me whichever way she could. I would like to show my gratitude to my mother. I hope that I could make her proud in my future endeavor.

This research financially supported by the Estonian Science Foundation under the Grant No. ETF9328 and institutional research funding IUT20-57 of the Estonian Ministry of Education and Research. Finally, I am really glad and thankful to the Doctoral School of Mathematics and Statistics for the financial support they provided to pursue my graduate studies here at this Institute.

Chapter 1

Introduction

The outstanding French mathematician and philosopher René Descartes developed his famous method of coordinates which allowed to identify a point of a geometric space with an ordered set of real numbers and this made possible the use of the concepts and methods of algebra in studying and solving geometric problems. Later when the differential and integral calculus was developed a possibility to endow a geometric space with coordinates made possible to apply the notions and methods of differential calculus to study surfaces, and this led to development of a new field of mathematics which is now called a differential geometry. From the time of F. Gauss and until our days the development of differential geometry has been closely related to development of theoretical physics. General theory of relativity developed by A. Einstein and Riemannian geometry created by B. Riemann can serve as the most well known fact of interaction between differential geometry and theoretical physics. Later in the second half of the previous century when C. Yang and R. Mills proposed the first non-abelian gauge theory to describe strong interactions it turned out that this theory (and all others gauge theories) can be formulated in a very elegant way in terms of the theory of connections on a fiber bundles. Physicists began to learn the basics of the theory of connections such as a notion of a principal fiber bundle, a Lie algebra valued connection 1-form, its curvature, Chern characteristic classes. The gauge theories allowed physicists to make a significant step (electroweak theory) towards unified field theory which is a dream of all physicists beginning from the time of A. Einstein. It is believed that the strong interactions will be unified with the electroweak theory in the nearest future. The theory of gravitation is a difficult problem on this way of unifying four different interactions, and even till now there is no quantum theory of gravitation that would suit all physicists. From the time of A. Einstein we know that the theory of gravitation is closely related to geometry of space. Thus in order to construct a quantum theory of gravitation it is natural to find a new concept of a geometric space. It is well known that quantities which commute at a classical level anticommute after quantization, for instant coordinate and momenta in quantum mechanics. One possible approach to quantum theory of gravitation developed by A. Connes is based on a concept of a noncommutative space [16]. The peculiar property of a noncommutative space is that the notion of points of a space gives way to that of coordinates (coordinate functions), of functions on a space and of the algebra of functions. This is similar to what happens in theoretical physics, where the dynamics of point particles described by the action at a distance gives way to that of fields (which can be considered as functions) with a local action functional. Thus in noncommutative geometry approach to a concept of a space we do not consider a space as a set of points, instead of this we consider the algebra of functions on this space, and this algebra is our main tool in studying the geometry of a space, and we assume that this algebra is a noncommutative algebra. It is remarkable that given the algebra of functions on a space we can develop the structures of differential geometry related to this space. We know that vector fields provide a very important tool in studying the geometry of a space. But we can use the notion of a vector field in noncommutative geometry because the space of vector fields is the space of derivations of the algebra of functions.

As it is formulated above, in noncommutative geometry approach we do not consider a space as a set of points, but instead of this we consider the algebra of functions on a space. By other words we are given an algebra, which we consider as an algebra of functions on our noncommutative space, and this algebra is the starting point for developing structures of differential geometry. It is well known that the algebra of smooth functions on n-dimensional space $\mathbb{R}^n$ (or more generally on a smooth n-dimensional manifold) is unital associative commutative algebra. Hence we assume that our algebra, which plays the role of algebra of functions on a noncommutative space, is unital associative algebra, but it is not necessarily commutative. Thus we choose a noncommutative algebra, but we do not want it structure to be very different from the algebra of functions on a commutative space. Hence we need to have analogs of coordinate functions. This can be achieved if we consider a unital associative algebra generated by the set of variables $x^1, x^2, \ldots, x^n$ which obey commutation relations (but do not commute!). We can assume the following commutation relations

$$x^i x^j = p^{ij}\, x^j x^i, \tag{1.0.1}$$

where p^{ij} are numbers not necessarily equal to one (in this case we would have a classical commutative space). We consider the algebra generated by the variables $x^1, x^2, \ldots, x^n$ subjected to the commutation relations (1.0.1) as the algebra of functions of noncommutative space, and this space will be referred to as a quantum space. In order to develop the structures of differential geometry for this quantum space first we need to develop a first order differential calculus with partial derivatives. We would like to point out that physicists prefer to develop a first order differential calculus on a quantum space figuratively speaking by "trial-and-error" method beginning with partial derivatives which would match the previously set requirements. In the present thesis our goal is to construct a first order differential calculus for several algebras, including a quantum space, by using a general approach based on a concept of coordinate first order differential calculus developed by A. Borowiec, V.K. Kharchenko and Z. Oziewicz [15]. This approach can be briefly formulated as follows: a first order differential calculus over a unital associative algebra is a bimodule over this algebra together with a linear mapping which maps algebra to a bimodule over this algebra and satisfies the Leibniz rule with respect to algebra multiplication. This mapping is referred to as a differential of a first order differential calculus. If algebra is generated by a finite set of variables which obey commutation relations then a first order differential calculus over this algebra is called a coordinate first order differential calculus (because we can view the generators as coordinate functions). We see that this approach is different from the one used in physical papers because according to this approach we should first construct a differential and then after this the partial derivatives. A definition of partial derivatives is based on the assumption that right module of a bimodule of first order differential calculus is freely generated by the differentials of generators or coordinate functions. Then if we apply a differential to any element of an algebra (this can be viewed as a function) we get the element of a bimodule and due to our assumption we can write it as a linear combination of differentials of generators multiplied from the right by elements of algebra. These coefficients are called the partial derivatives of first order coordinate differential calculus. We see that in this approach the partial derivatives are induced by a differential. It can be easily proved that if the right module over an algebra is a finite module freely generated by the differentials of generators then the left module is related to the right one by a homomorphism from an algebra to the algebra of matrices over this algebra. This homomorphism determines uniquely a first order differential calculus. It is evident that if an algebra is generated by a set of variables which obey commutation relations then a differential of first order differential calculus should be consistent with these relations. Our goal in the present thesis is to find consistency conditions for a differential in the case when generators of an algebra obey quadratic commutation relations (we point out that quantum space is the particular case of an algebra with quadratics relations). These conditions are given in Proposition 2.1.7. Then we apply this result to a quantum space and solve the corresponding equations in a general form (Theorem 2.2.3) obtaining a family of first order differential calculuses.

A Grassmann algebra plays an important part in a modern differential geometry and theoretical physics. A Grassmann algebra is used to construct an exterior calculus over a vector space. If this exterior calculus is applied to each tangent space of a smooth finite dimensional manifold then we get the vector bundle of exterior algebras over a manifold and smooth sections of this bundle give us smooth differential forms which form the algebra under the wedge product and endowing it with exterior differential we finally obtain the graded differential algebra of differential forms. A Grassmann algebra is also used in modern differential geometry in order to develop a theory of supermanifolds and super Lie groups. In modern theoretical physics a Grassmann algebra is used in supersymmetric field theories. Recently there has been proposed an analog of Grassmann algebra which can be called a 3-Grassmann algebra with cubic relations or ternary Grassmann algebra with cubic relations. This analog of Grassmann algebra has been developed in connection with quarks, and it provides an algebraic approach to quarks possible combinations (color confinement of quarks) [1, 29, 30, 32]. We remind that generators of Grassmann algebra anticommute or, equivalently, if we take the sum of all possible binary products of any two generators then we get zero. This principle can be generalized if we consider triple products of generators. Indeed we can consider the ternary condition, where the sum of all possible triple products of any three generators is equal to zero. Evidently the condition of Grassmann algebra (the sum of all binary products of any two generators is equal to zero) is a particular case of the ternary condition. In the case of a Grassmann algebra we can solve the equation (the sum of all binary products of any two generators is equal to zero) by postulating that any two generators anticommute. It was shown [1] that in the ternary case we also can solve the corresponding equation (the sum of all possible triple products of any three generators is equal to zero) with the help of a primitive 3rd order root of unity. We assume that any product of three generators is equal to cyclic permutation of generators in this product multiplied by a primitive cubic root of unity. It is easy to see that we indeed solve the equation (the sum of all possible triple products of any three generators is equal to zero) because the sum of all possible products can be split into two sums of cyclic permutations, but the sum of cyclic permutations is equal to zero, and this follows immediately from the fact that the sum of all powers of a primitive cubic root of unity is zero. In the present thesis our goal is to construct a ternary analog of Grassmann algebra with involution and to construct a ternary analog of Grassmann algebra which will contain a Grassmann algebra and a ternary Grassmann algebra with cubic relations as subalgebras.

In the present thesis we also study a generalization of graded differential algebra which is known under the name of graded q-differential algebra, where q is a primitive Nth root of unity. A graded q-differential algebra is equipped with a differential d which satisfies $d^N = 0$, $N \geq 2$. The first, who began the study of cohomological complexes with differential d satisfying the equation $d^N = 0$, was M. Kapranov [25]. In the same paper M. Kapranov made an attempt to construct an algebra of differential forms on a n-dimensional space $\mathbb{R}^n$ with exterior differential satisfying $d^N = 0$ (M. Kapranov used the ordinary commutative differential calculus), but later it was shown that this algebra degenerates. A definition of graded q-differential algebra was given by M. Dubois-Violette in [19]. Later the structure of a graded q-differential algebra, a method of its construction and applications in a theory of connections have been studied in the series of papers [3, 4, 5, 6, 7, 8]. In the present thesis our goal is to construct and study a graded q-differential algebra of analogs of differential forms. Unlike M. Kapranov we think that this algebra should be constructed by means of a noncommutative differential calculus. For this purpose we develop and study a noncommutative differential calculus on the algebra of complex square matrices of order N.

Our next goal in the present thesis is to show that a noncommutative Galois extension introduced and studied in [28, 31, 32, 37] can be viewed and studied from the point of view of the theory of graded q-differential algebras. Suppose $\tilde{\mathscr{A}}$ is an associative unital $\mathbb{C}$-algebra, $\mathscr{A} \subset \tilde{\mathscr{A}}$ is its subalgebra, and there is an element $\tau \in \tilde{\mathscr{A}}$ which satisfies $\tau \notin \mathscr{A}$, $\tau^N = \mathbb{1}$, where $N \geq 2$ is an integer and $\mathbb{1}$ is the identity element of $\tilde{\mathscr{A}}$. A noncommutative Galois extension of $\mathscr{A}$ by means of τ is the smallest subalgebra $\mathscr{A}[\tau] \subset \tilde{\mathscr{A}}$ such that $\mathscr{A} \subset \mathscr{A}[\tau]$, and $\tau \in \mathscr{A}[\tau]$. It should be pointed out that a concept of noncommutative Galois extension can be applied not only to associative unital algebra with a binary multiplication law but as well as to the algebra with a ternary multiplication law, for instant to a ternary analog of Grassmann and Clifford algebra [2, 32, 37], and this approach can be used in particle physics to construct an elegant algebraic model for quarks. In the present thesis we will study a special case of noncommutative Galois extension which is referred to as a semi-commutative Galois extension. A noncommutative Galois extension is called a semi-commutative Galois extension [37] if for any element $x \in \mathscr{A}$ there exists an element $x' \in \mathscr{A}$ such that $x\tau = \tau x'$. In the present thesis we propose the method of endowing a semi-commutative Galois extension with the structure of a graded q-differential algebra. Our method is based on the theorem which states that if there exists an element v of graded associative unital $\mathbb{C}$-algebra which satisfies the relation $v^N = \mathbb{1}$ then this algebra can be endowed with the structure of graded q-differential algebra. We can apply this theorem to a semi-commutative Galois extension because we have an element τ with the property $\tau^N = \mathbb{1}$, and this allows us to equip a semi-commutative Galois extension with the structure of graded q-differential algebra. Then we study the first and higher order noncommutative differential calculus induced by the N-differential of graded q-differential algebra. We introduce a derivative and differential with the help of first order noncommutative differential calculus developed in the papers [5, 15]. We also study the higher order noncommutative differential calculus and in this case we consider a differential d as an analog of exterior differential and the elements of higher order differential calculus as analogs of differential forms. Finally we apply our calculus to reduced quantum plane [17].

Chapter 2 is devoted to a first order differential calculus of noncommutative geometry. Here we consider coordinate calculus and later we apply this coordinate calculus to describe first order differential calculus over graded associative unital algebra $\mathscr{A}$. We use right $\mathscr{A}$-module, twisted Leibniz rule to derive a matrix $A^i_j(f)$ which is square matrix of order N. This is not a conventional matrix because here we a superscript i as an index of column and a subscript j as an index of row. Next we show that the matrix $A^i_j(f)$ determines a homomorphism from an algebra $\mathscr{A}$ to the algebra of matrices over $\mathscr{A}$. After that we define partial derivatives of coordinate differential calculus of an algebra with relations. In Section 2.2.1 we present Grassmann algebra and its first order differential calculus. Here in this section we use twisted Leibniz rule to describe the derivatives of monomial and get the result

$$\frac{\partial \xi^I}{\partial \xi^i} = \sum_{r=1}^{p} (-1)^{r-1} \delta_i^{i_r} \xi^{i_1} \xi^{i_2} \cdots \hat{\xi}^{i_r} \cdots \xi^{i_p},$$

where $\hat{\xi}^{i_r}$ stands for a generator ξ^{i_r} which is removed from monomial ξ^I. In Section 2.2.2 we present quantum space and its first order differential calculus. Proposition 2.2.2 shows that for any function f the second order derivatives of the first order differential calculus $x^i dx^j = p^{ij} dx^j x^i$ of a quantum space satisfy the relations $\frac{\partial^2 f}{\partial x^i \partial x^j} = p^{ji} \frac{\partial^2 f}{\partial x^j \partial x^i}$ where $i, j = 1, 2, \cdots, n$. In Section 2.2.4 we present ternary Grassmann algebra with cubic relations, here in this section we assume that product of any triple of generators of an algebra obey the relation $\{\zeta^\mu, \zeta^\nu, \zeta^\kappa\}_{\mathbb{Z}_3} = 0$. Proposition 2.2.6 asserts that the product of any four generators of ternary Grassmann algebra vanishes and the vector space of a cyclic 3-Grassmann algebra is a finite dimensional and the dimension of vector space is $(N^3 + 3N^2 + 2N + 3)/3$. We also describe some more propositions about 3-Grassmann algebras and one of them is Proposition 2.2.8 which deduce that the dimension of a cyclic 3-Grassmann algebra with involution generated by $\theta^1, \theta^2, \cdots, \theta^N$ and $\overline{\theta}^{\dot{A}}, \overline{\theta}^{\dot{B}}, \cdots, \overline{\theta}^{\dot{N}}$ is equal to the square of the dimension of a cyclic 3-Grassmann algebra generated by $\theta^1, \theta^2, \cdots, \theta^N$. Finally we present differential calculus over ternary Grassmann algebra.

Chapter 3 is devoted to graded differential algebras. In the first part of this chapter we describe graded associative unital algebras with the help of additive Abelian group which is used to label the subspaces of graded algebra. We know that graded differential algebras plays a important role in noncommmutative geometry so here in this chapter we emphasize superalgebras as $\mathbb{Z}_2$-graded structure. We firstly denote $\mathbb{Z}_2$-graded vector space as $\mathscr{V} = \mathscr{V}^{\overline{0}} \oplus \mathscr{V}^{\overline{1}}$ and this is called super vector space where $\mathscr{V}^{\overline{0}}$ referred to as the even elements of the subspace and $\mathscr{V}^{\overline{1}}$ as the odd elements of super vector space. After that we give a definition of superalgebra by means of $\mathbb{Z}_2$-graded structure and here it is defined as $\mathscr{A} = \mathscr{A}^{\overline{0}} \oplus \mathscr{A}^{\overline{1}}$. As we know that superalgebras are widely used in the supersymmetric field theories of modern theoretical physics so here we firstly describe some notations and examples of superalgebras by using graded $\mathbb{Z}_2$-graded structure. After that we present a section devoted to superalgebras and super Lie algebras. In Section 3.1.3 we define a graded structure of a 3-Grassmann algebra and propose its extension to the ternary Grassmann algebra with involution. In the second part of this chapter we consider some examples and notations of cochain complexes such as the Chevalley-Eilenberg complex of a Lie algebra. We present some definitions and those definitions show that graded algebra is the special case of graded q-differential algebra.

Chapter 4 is devoted to graded q-differential algebras and Galois noncommutative extension. We divide the chapter into two parts and in the first part of the chapter we describe the notations, theorems, propositions and examples of graded q-differential algebras and the second part of this chapter is devoted to semi-commutative Galois extension. We know that graded q-differential algebra is very important topic in modern differential geometry, quark theory and theoretical physics. Here we firstly remind some notations, characteristics, properties of q-calculus. We also remind the reader the notations such as graded q-commutator, graded q-derivation of degree m, graded q-Leibniz rule and inner graded q-derivation. Next we present a notion of generalized cohomologies which is based on the notion of homological algebra such as differential module, graded module, cochain complex, cohomologies of cochain complex and cosimplical module. We propose the method of construction of graded q-differential algebra. This method is based on the theorem asserts that if there exists an element v of grading one which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{A})$ where $\mathscr{Z}(\mathscr{A})$ denotes the graded center of the associative unital algebra $\mathscr{A}$, then the inner graded q-derivation $d = ad_v^q$ of degree 1 is an N-differential. In Section 4.1.5 we deduce graded q-differential algebra of matrices to describe the matrix structure $\mathrm{Mat}_N(\mathbb{C})$. Then we calculate partial derivatives of graded q-differential algebra. We also study higher order derivatives of q-differential algebra which is similar to the exterior calculus. Second part of this chapter is devoted to the relation between semi-commutative Galois extension and graded q-differential algebras.

Chapter 2

First Order Differential Calculus of Noncommutative Geometry

2.1 Differential Calculus over Associative Unital Algebra

It is well known that the set of smooth functions $C^\infty(M^n)$ of a finite dimensional smooth manifold M^n is the associative unital algebra. The vector space of smooth differential forms of degree one of this manifold $\Omega^1(M^n)$ can be viewed as the module over the algebra of smooth functions $C^\infty(M^n)$. We remind that in modern differential geometry of smooth manifolds functions commute with differential forms of degree one. Locally the module of differential forms of degree one is the finite module freely generated by the differentials of local coordinates $dx^1, dx^2, \ldots, dx^n$. The differential d maps the algebra of smooth functions to the module of differential forms of degree one, i.e. $d: C^\infty(M^n) \longrightarrow \Omega^1(M^n)$. The triple $(C^\infty(M^n), \Omega^1(M^n), d)$ is referred to as the first order differential calculus of a smooth manifold. In this section we will describe noncommutative geometry approach to first order differential calculus.

2.1.1 Coordinate first order differential calculus over algebra

Let $\mathscr{A}$ be a unital associative algebra which is not necessarily commutative and $\mathscr{M}$ be a $\mathscr{A}$-bimodule.

Definition 2.1.1. A linear mapping $d:\mathscr{A}\longrightarrow\mathscr{M}$ is said to be a differential if it satisfies the Leibniz rule

$$d(ab)=da\cdot b+a\cdot db,\quad a,b\in\mathscr{A}. \tag{2.1.1}$$

The triple $(\mathscr{A},\mathscr{M},d)$ is referred to as a first order differential calculus over an algebra $\mathscr{A}$.

The definition of first order differential calculus is quite general and, in order to have a structure more similar to first order differential calculus of a smooth manifold than one given by definition (2.1.1), we can use a coordinate first order differential calculus. For this purpose we assume that algebra $\mathscr{A}$ is generated by finite set of variables $x^1,x^2,\dots,x^n$ either freely (no relations) or the variables $x^1,x^2,\dots,x^n$ obey some relations.

Definition 2.1.2. Let $(\mathscr{A},\mathscr{M},d)$ be a first order differential calculus, where algebra $\mathscr{A}$ is generated by finite set of variables $x^1,x^2,\dots,x^n$. If the right $\mathscr{A}$-module $\mathscr{M}$ is freely generated by $dx^1,dx^2,\dots,dx^n$, where $d:x^i\longrightarrow dx^i$, then the triple $(\mathscr{A},\mathscr{M},d)$ is referred to as coordinate first order differential calculus over an algebra $\mathscr{A}$.

Let $(\mathscr{A},\mathscr{M},d)$ be a coordinate first order differential calculus over an algebra $\mathscr{A}$. In order to stress an analogy with functions and differential 1-forms of a smooth manifold we will denote the elements of algebra $\mathscr{A}$ by f,g,h and the elements of $\mathscr{A}$-bimodule $\mathscr{M}$ by ω,θ calling them differential 1-forms of first order differential calculus. From definition of coordinate calculus it follows that any element ω of right $\mathscr{A}$-module can be expressed in terms of differentials of variables $dx^1,dx^2,\dots,dx^n$ as follows

$$\omega=dx^1 f_1+dx^2 f_2+\ldots+dx^n f_n=dx^i f_i, \tag{2.1.2}$$

where $f_i\in\mathscr{A}$. Given an element $f\in\mathscr{A}$ and making use of the left $\mathscr{A}$-module structure of $\mathscr{M}$, we can consider the product $f\,dx^i$. This product is a differential 1-form, and consequently can be expressed in terms of differentials

$$f\,dx^i=dx^j A^i_j(f),\quad A^i_j(f)\in\mathscr{A}, \tag{2.1.3}$$

where the coefficients of the differential 1-form at the right-hand side depend on f. Thus given an element $f \in \mathscr{A}$ we can relate to it the $n \times n$-matrix $f \mapsto (A^i_j(f))$, whose entries are the elements of $\mathscr{A}$. Let us denote this mapping by A. It is convenient to arrange the elements of the matrix $A^i_j(f)$ so, that superscript i will be the column index and subscript j will be the row index. Thus the matrix has the form

$$A(f) = (A^i_j(f)) = \begin{pmatrix} A^1_1(f) & A^2_1(f) & \dots & A^n_1(f) \\ A^1_2(f) & A^2_2(f) & \dots & A^n_2(f) \\ \dots & \dots & \dots & \dots \\ A^1_n(f) & A^2_n(f) & \dots & A^n_n(f) \end{pmatrix}.$$

The square matrices of order n, whose entries are the elements of unital associative algebra $\mathscr{A}$, form the unital associative algebra. Let us denote this algebra by $\mathrm{Mat}_n(\mathscr{A})$.

Proposition 2.1.3. *The mapping $A : \mathscr{A} \longrightarrow Mat_n(\mathscr{A})$, which relates to any element $f \in \mathscr{A}$ the nth order square matrix $A(f) \in Mat_n(\mathscr{A})$ by means of the relation (2.1.3), is the homomorphism of algebras.*

Proof. Let $f, g \in \mathscr{A}$. We have $(f\,g)\,dx^i = dx^j A^i_j(f\,g)$. Making use of the $\mathscr{A}$-bimodule structure of $\mathscr{M}$ and applying twice the relation (2.1.3) we obtain

$$(f\,g)\,dx^i = f\,(g\,dx^i) = f\,(dx^k A^i_k(g)) = (f\,dx^k) A^i_k(g) = dx^j\,(A^k_j(f) A^i_k(g)).$$

Thus $A^i_j(f\,g) = A^k_j(f) A^i_k(g) \Rightarrow A(f\,g) = A(f)A(g)$.

An advantage of a coordinate first order differential calculus is that one can define and use partial derivatives with respect to variables $x^1, x^2, \dots, x^n$.

2.1.2 Partial derivatives of coordinate differential calculus

Definition 2.1.4. Let $f \in \mathscr{A}$. Then $df \in \mathscr{M}$ is differential 1-form and taking into consideration that the right $\mathscr{A}$-module $\mathscr{M}$ is freely generated by the differentials $dx^1, dx^2, \dots, dx^n$, one can write df in the form of linear combination of the differentials (2.1.2). The coefficients of this linear combination will be referred to as the partial derivatives of f with respect to variables $x^1, x^2, \dots, x^n$ and denoted by $\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \dots, \frac{\partial f}{\partial x^n}$. Hence

$$df = dx^1\,\frac{\partial f}{\partial x^1} + dx^2\,\frac{\partial f}{\partial x^2} + \dots + dx^n\,\frac{\partial f}{\partial x^n} = dx^i\,\frac{\partial f}{\partial x^i}.$$

Making use of the Leibniz rule (2.1.1) one can obtain a rule for taking partial derivative of product of $f\,g$, where $f,g \in \mathscr{A}$.

Proposition 2.1.5. *The partial derivatives defined in (2.1.4) satisfy the twisted Leibniz rule, i.e. for any two elements $f,g \in \mathscr{A}$ it holds*

$$\frac{\partial}{\partial x^i}(f\,g) = \frac{\partial f}{\partial x^i}\,g + A_i^j(f)\,\frac{\partial g}{\partial x^j}.$$

Proof. Applying the Leibniz rule (2.1.1) to the product of two elements $f\,g$ and then the definition of partial derivatives (2.1.4) we get

$$\begin{aligned} dx^i\,\frac{\partial}{\partial x^i}(f\,g) &= dx^i\,\frac{\partial f}{\partial x^i}\,g + f\,dx^i\,\frac{\partial g}{\partial x^j} = dx^i\,\frac{\partial f}{\partial x^i}\,g + dx^i A_i^j(f)\frac{\partial g}{\partial x^j} \\ &= dx^i\big(\frac{\partial f}{\partial x^i}\,g + A_i^j(f)\,\frac{\partial g}{\partial x^j}\big). \end{aligned}$$

2.1.3 Differential calculus over algebra with relations

If algebra is a free algebra generated by a set of variables then the variables of this free algebra will be denoted by $\tilde{x}^1,\tilde{x}^2,\ldots,\tilde{x}^n$ and the algebra will be denoted $\tilde{\mathscr{A}}$. Let $\mathscr{A}$ be an algebra generated by $x^1,x^2,\ldots,x^n$ which obey relations $f_\alpha(x^1,x^2,\ldots,x^n)=0$, where $\alpha = 1,2,\ldots,r$. Assume that we have a first order differential calculus $(\mathscr{A},\mathscr{M},d)$ over an algebra $\mathscr{A}$ and $A:\mathscr{A}\longrightarrow \mathrm{Mat}_n(\mathscr{A})$ is a homomorphism of this first order differential calculus. Let $\mathscr{I}$ is the two-sided ideal of $\tilde{\mathscr{A}}$ generated by the elements $f_\alpha(\tilde{x}^1,\tilde{x}^2,\ldots,\tilde{x}^n)$. Then $\tilde{\mathscr{A}}/\mathscr{I} \equiv \mathscr{A}$. Define a homomorphism $\pi:\tilde{\mathscr{A}}\longrightarrow\mathscr{A}$ by $\pi(\tilde{x}^i)=x^i$. Obviously $\mathrm{Ker}\,\pi = \mathscr{I}$. Let $\tilde{\mathscr{M}}$ be the right $\tilde{\mathscr{A}}$-module freely generated by the elements $\tilde{dx}^1,\tilde{dx}^2,\ldots,\tilde{dx}^n$. We define the differential $\tilde{d}:\tilde{\mathscr{A}}\longrightarrow\tilde{\mathscr{M}}$ by $\tilde{d}(\tilde{x}^i)=\tilde{dx}^i$, extend it to the whole free algebra $\tilde{\mathscr{A}}$ by means of Leibniz rule and define the $\tilde{\mathscr{A}}$-bimodule structure of $\tilde{\mathscr{M}}$ by

$$g\,\tilde{dx}^i = \tilde{dx}^k\,\tilde{A}_k^i(g),$$

where $\tilde{A}_k^i(\tilde{x}^j) = A_k^i(\tilde{x}^j)+R_k^i(\tilde{x}^j)$, $R_i^k(\tilde{x}^j)\in\mathscr{I}$.

Theorem 2.1.6. *The triple $(\tilde{\mathscr{A}},\tilde{d},\tilde{\mathscr{M}})$ is the first order differential calculus over the free algebra where $\tilde{d}$ satisfies $\psi\circ\tilde{d} = d\circ\pi$ and*

$$\psi:\tilde{\mathscr{M}}\longrightarrow\mathscr{M},\;\; \psi(\tilde{dx}^i\,g) = dx^i\,\pi(g),\;\; g\in\tilde{\mathscr{A}}.$$

Let $\mathscr{A}$ be unital associative algebra generated by $x^1, x^2, \dots, x^n$ which obey relations $f_\alpha(x^1, x^2, \dots, x^n) = 0$, $\alpha = 1, 2, \dots m$. Let $(\mathscr{A}, \mathscr{M}, d)$ be a first order differential calculus over an algebra $\mathscr{A}$, where $d(\mathbb{1}) = 0$, $d(x^i) = dx^i$, the right $\mathscr{A}$-module $\mathscr{M}$ is freely generated by $dx^1, dx^2, \dots, dx^n$, and the $\mathscr{A}$-bimodule structure of $\mathscr{M}$ is determined by a homomorphism $A : \mathscr{A} \longrightarrow \mathrm{Mat}_n(\mathscr{A})$. Hence we have two different structures, where one of them is a unital associative algebra with relations, and the second is a first order differential calculus over this algebra. Obviously these two structures should be consistent, and this leads to consistency conditions for $\mathscr{A}$-bimodule structure determining homomorphism A. It is evident that for any $\alpha = 1, 2, \dots, m$ the following consistency conditions must be satisfied

$$d f_\alpha = 0, \quad A(f_\alpha) = 0. \tag{2.1.4}$$

The first condition means that commutation relations of algebra induce differential 1-forms $\omega_\alpha = d f_\alpha$ which must vanish. Making use of partial derivatives of first order differential calculus one can write the first condition in the form

$$\frac{\partial f_\alpha}{\partial x^k} = 0,\ k = 1, 2, \dots, n,\ \alpha = 1, 2, \dots, m.$$

We study these conditions in the case of algebras with quadratic commutation relations because several important algebras have this kind of commutation relations. We assume

$$f_\alpha(x^1, x^2, \dots, x^n) = \lambda_{\alpha,ij}\, x^i x^j, \tag{2.1.5}$$

where $\lambda_{\alpha,ij} \in \mathbb{C}$. The first consistency condition (2.1.4) gives

$$\begin{aligned} d f_\alpha(x^1, x^2, \dots, x^n) &= \lambda_{\alpha,ij}\, (dx^i x^j + x^i dx^j) \\ &= \lambda_{\alpha,ij}\, (dx^i x^j + dx^k A^j_k(x^i)) \\ &= dx^k\, [\lambda_{\alpha,kj} x^j + \lambda_{\alpha,ij} A^j_k(x^i)]. \end{aligned}$$

It is evident that

$$\frac{\partial f_\alpha}{\partial x^k} = \lambda_{\alpha,kj} x^j + \lambda_{\alpha,ij} A^j_k(x^i).$$

Taking into consideration that the right $\mathscr{A}$-module $\mathscr{M}$ is freely generated by the differentials $dx^1, dx^2, \dots, dx^n$ we obtain the equations for the entries of the matrix of homomorphism $A : \mathscr{A} \longrightarrow \mathrm{Mat}_n(\mathscr{A})$ in the case of quadratic commutation relations

$$\lambda_{\alpha,kj}\, x^j + \lambda_{\alpha,ij}\, A^j_k(x^i) = 0. \tag{2.1.6}$$

Proposition 2.1.7. *If $\mathscr{A}$ is a unital associative algebra generated by $x^1, x^2, \ldots, x^n$ which obey quadratic commutation relations $\lambda_{\alpha,ij}\, x^i x^j = 0$, and $(\mathscr{A}, \mathscr{M}, d)$ is a first order differential calculus over an algebra $\mathscr{A}$ with $\mathscr{A}$-bimodule structure of $\mathscr{M}$ determined by the relations $x^i dx^j = dx^k A^j_k(x^i)$ then the entries $A^j_k(x^i)$ of the matrix of a homomorphism $A : \mathscr{A} \longrightarrow Mat_n(\mathscr{A})$ satisfy the equations*

$$\lambda_{\alpha,kj}\, x^j + \lambda_{\alpha,ij}\, A^j_k(x^i) = 0, \quad \lambda_{\alpha,ij}\, A^m_r(x^i) A^s_m(x^j) = 0.$$

2.2 Grassmann Algebras and Quantum Spaces

2.2.1 Grassmann algebra and its first order differential calculus

Let us consider a finite dimensional Grassmann algebra. It should be mentioned that a first order differential calculus described the in previous paragraph can be constructed over infinite dimensional Grassmann algebra ([14]) but in this thesis we will consider only finite dimensional case. Grassmann algebra is a unital associative algebra generated by a finite set of variables which obey some relations. Consequently we can apply the structure described in the previous paragraph to construct a first order differential calculus over Grassmann algebra.

Let $\mathscr{G}^n$ be a Grassmann algebra generated by $\xi^1, \xi^2, \ldots, \xi^n$ over $\mathbb{C}$, where the variables obey the commutation relations

$$\xi^i \xi^j = -\xi^j \xi^i. \tag{2.2.1}$$

The unit element of Grassmann algebra will be denoted by $\mathbb{1}$. Let us denote the set of integers $1, 2, \ldots, n$ by N, i.e. $N = \{1, 2, \ldots, n\}$, and $I = \{i_1, i_2, \ldots, i_p\} \subset N$ be a subset, where $1 \leq i_1 < i_2 \ldots < i_p \leq n$ and $1 \leq p \leq n$. Let us denote the number of integers in I by $|I|$, i.e. for $I = \{i_1, i_2, \ldots, i_p\}$ we have $|I| = p$. For any subset I we define the monomial of Grassmann algebra

$$\xi^I = \xi^{i_1} \xi^{i_2} \ldots \xi^{i_p}, \quad \xi^{\emptyset} = \mathbb{1}.$$

Then $\mathscr{G}^n$ is the 2^n-dimensional vector space spanned by the monomials ξ^I and for any element $f \in \mathscr{G}$ we have

$$f = \sum_I f_I\, \xi^I,$$

where the coefficients $f_I = f_{i_1 i_2 \ldots i_n}$ are complex numbers. It is evident that a Grassmann algebra is a unital associative algebra with quadratic commutation relations $f^{ij}(\xi) = \lambda^{ij}_{kl}\, \xi^k \xi^l$, where

$$\lambda^{ij}_{kl} = \delta^i_k\, \delta^j_l + \delta^i_l\, \delta^j_k. \tag{2.2.2}$$

Hence we can apply the proposition proved at the end of previous section.

In order to construct a first order differential calculus over a Grassmann algebra $\mathscr{G}$ we should define a $\mathscr{G}$-bimodule $\mathscr{M}$ and a differential $d : \mathscr{G} \longrightarrow \mathscr{M}$. Let us introduce a set of new variables $d\xi^1, d\xi^2, \ldots, d\xi^n$ which will be referred to as the differentials of Grassmann algebra variables. Now we use these differentials as elements of a basis to generate left (right) $\mathscr{G}$-module by multiplying $d\xi^1, d\xi^2, \ldots, d\xi^n$ by the elements of $\mathscr{G}$ from the left (right). Let us assume that $\mathscr{G}$-bimodule structure is determined by a homomorphism $A : \mathscr{G} \longrightarrow \mathrm{Mat}_n(\mathscr{G})$, where $f\, d\xi^i = d\xi^j\, A^i_j(f)$. Let us denote this $\mathscr{G}$-bimodule by $\mathscr{M}$. According to Proposition 2.1.4 the entries A^i_j of the matrix of homomorphism must satisfy two equations

$$\lambda^{ij}_{kl}\, \xi^l + \lambda^{ij}_{rl}\, A^l_k(\xi^r) = 0,\;\; \lambda^{ij}_{kl} A^m_r(\xi^k) A^s_m(\xi^l) = 0.$$

Taking into account the expressions (2.2.2) for the coefficients of quadratic relations we see that these equations can be written in the form

$$A^j_k(\xi^i) + A^i_k(\xi^j) + \delta^i_k\, \xi^j + \delta^j_k\, \xi^i \;=\; 0, \tag{2.2.3}$$

$$A^m_r(\xi^i) A^s_m(\xi^j) + A^m_r(\xi^j) A^s_m(\xi^i) \;=\; 0. \tag{2.2.4}$$

It is worth mentioning that the first equation can be obtained by a straightforward computation with the help of differential. Indeed we define a differential $d : \mathscr{G} \longrightarrow \mathscr{M}$ in the case of unit element and generators by

$$d(\mathbb{1}) = 0,\; d(\xi^i) = d\xi^i.$$

In order to extend this differential to any monomial ξ^I we will use the Leibniz rule for a differential d and a matrix A. Differentiating commutation relations of Grassmann algebra (2.2.1) we get

$$d\xi^i\, \xi^j + d\xi^j\, \xi^i + \xi^i\, d\xi^j + \xi^j\, d\xi^i = 0.$$

Making use of a homomorphism A we can write the above equation in the form

$$A^j_k(\xi^i) + A^i_k(\xi^j) + \delta^j_k \xi^i + \delta^i_k \xi^j = 0. \tag{2.2.5}$$

Consequently differentiating the commutation relations of Grassmann algebra and making use of the definition of homomorphism A we obtain the consistency condition (2.2.5) for the entries of a matrix A. We can solve the equations (2.2.3,2.2.4) by putting

$$A^i_k(\xi^j) = -\delta^i_k\,\xi^j.$$

Thus the matrix of homomorphism $A:\mathscr{G}\longrightarrow \mathrm{Mat}_n(\mathscr{G})$ in the case of generators has the form

$$A(\xi^i) = \begin{pmatrix} -\xi^i & 0 & \dots & 0 \\ 0 & -\xi^i & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\xi^i \end{pmatrix}. \tag{2.2.6}$$

From this it follows that $\xi^i\,d\xi^j = -d\xi^j\,\xi^i$. Extending these relations to any monomial ξ^I we obtain

$$\xi^I\,d\xi^i = (-1)^{|I|}d\xi^i\,\xi^I. \tag{2.2.7}$$

The derivatives of this first order differential calculus is defined by $d\xi^I = dx^i\,\frac{\partial\xi^I}{\partial x^i}$. We can calculate a derivative of monomial ξ^I with the help of twisted Leibniz rule (4.2.7). Indeed

$$\begin{aligned}
\frac{\partial\xi^I}{\partial\xi^i} &= \frac{\partial\,\xi^{i_1}}{\partial\xi^i}\,\xi^{i_2}\dots\xi^{i_p} + A^k_i(\xi^{i_1})\,\frac{\partial(\xi^{i_2}\dots\xi^{i_p})}{\partial\xi^k} \\
&= \delta^{i_1}_i\,\xi^{i_2}\dots\xi^{i_p} - \delta^k_i\,\xi^{i_1}\,\frac{\partial(\xi^{i_2}\dots\xi^{i_p})}{\partial\xi^k} \\
&= \delta^{i_1}_i\,\xi^{i_2}\dots\xi^{i_p} - \xi^{i_1}\,\frac{\partial(\xi^{i_2}\dots\xi^{i_p})}{\partial\xi^i} \\
&= \sum_{r=1}^{p}(-1)^{r-1}\,\delta^{i_r}_i\,\xi^{i_1}\xi^{i_2}\dots\hat{\xi}^{i_r}\dots\xi^{i_p},
\end{aligned} \tag{2.2.8}$$

where $\hat{\xi}^{i_r}$ stands for a generator ξ^{i_r} which is removed from monomial ξ^I. The above formula shows that the first order differential calculus over a Grassmann algebra constructed in this paragraph leads to the well known partial derivatives of Grassmann algebra ([14]).

Proposition 2.2.1. *The partial derivatives of Grassmann algebra (2.2.8) anticommute, i.e. for any two generators ξ^i,ξ^j we have*

$$\frac{\partial^2}{\partial\xi^i\partial\xi^j} = -\frac{\partial^2}{\partial\xi^j\partial\xi^i}. \tag{2.2.9}$$

Proof. Consider any monomial $\xi^I = \xi^{i_1}\xi^{i_2}\ldots\xi^{i_p}$. It follows from (2.2.8) that if there is no either ξ^i or ξ^j in this monomial then the both second order derivatives in (2.2.9) are zeros, and the formula (2.2.9) is proved. Assume that a monomial ξ^I contains both generators ξ^i, ξ^j, where ξ^i stands in this monomial on rth position (from the left) and ξ^j stands on sth position (also from the left). Let $r < s$. Making use of the formula (2.2.8) we find

$$\begin{aligned}
\frac{\partial^2 \xi^I}{\partial \xi^i \partial \xi^j} &= (-1)^{r+s-3}\,\xi^{i_1}\xi^{i_2}\ldots\hat{\xi}^{i_r}\ldots\hat{\xi}^{i_s}\ldots\xi^{i_p},\\
\frac{\partial^2 \xi^I}{\partial \xi^j \partial \xi^i} &= (-1)^{r+s-2}\,\xi^{i_1}\xi^{i_2}\ldots\hat{\xi}^{i_r}\ldots\hat{\xi}^{i_s}\ldots\xi^{i_p},
\end{aligned}$$

and we see that the partial derivatives anticommute.

2.2.2 Quantum space and its first order differential calculus

An algebra of functions of a quantum space with coordinate functions $x^1, x^2, \ldots, x^n$ is a unital associative algebra $\mathscr{A}$ over complex numbers $\mathbb{C}$ generated by variables $x^1, x^2, \ldots, x^n$ which obey the following commutation relations

$$x^i x^j = p^{ij}\, x^j x^i, \quad i, j = 1, 2, \ldots, n, \tag{2.2.10}$$

where p^{ij} are complex numbers such that $p^{ii} = 1, p^{ji} = (p^{ij})^{-1}$. If $p^{ij} = 1$ for any i, j then the coordinate functions $x^1, x^2, \ldots, x^n$ of a space commute $x^i x^j = x^j x^i$, and we have a classical (commutative) space. In this section we assume $p^{ij} \neq 1$ for $i \neq j$.

The commutation relations of a quantum space show that we have the algebra of functions with quadratic commutation relation. Hence we can study a first order differential calculus of a quantum space by applying results obtained in the section for algebras with quadratic commutation relations. It is easy to see that the coefficients of polynomials of commutation relations are given by the expressions

$$\lambda^{ij}_{kl} = \delta^i_k\, \delta^j_l - p^{ij}\, \delta^i_l\, \delta^j_k.$$

Hence the consistency conditions for the entries of a matrix of homomorphism (Proposition 2.1.7)

$$\lambda^{ij}_{kl}\, x^l + \lambda^{ij}_{rl}\, A^l_k(x^r) = 0,$$

take on the form of the equations

$$A^j_k(x^i) - p^{ij}\, A^i_k(x^j) = p^{ij}\, \delta^j_k\, x^i - \delta^i_k\, x^j. \tag{2.2.11}$$

As in the case of a Grassmann algebra, here we can find a similar particular solution of consistency equation

$$A_k^j(x^i) = p^{ij}\, \delta_k^j\, x^i. \tag{2.2.12}$$

This solution means the following commutation relations between coordinate functions and their differentials

$$x^i\, dx^j = p^{ij}\, dx^j\, x^i. \tag{2.2.13}$$

Now we can find a partial derivative of any product $x^i\, x^j$ which can be written as

$$\frac{\partial (x^i x^j)}{\partial x^k} = \delta_k^i\, x^j + A_k^l(x^i)\ \delta_l^j = \delta_k^i\, x^j + p^{ij}\ \delta_k^j\, x^i. \tag{2.2.14}$$

From the above formula it is easy to get that for any power function $(x^i)^r$ we have

$$\frac{\partial (x^i)^r}{\partial x^j} = r\, \delta_j^i\, (x^i)^{r-1}.$$

Obviously each function of a quantum space can be written as a linear combination of a product of power functions $f_{r_1 r_2 \ldots r_n}(x) = (x^1)^{r_1} (x^2)^{r_2} \ldots (x^n)^{r_n}$. Thus the general formula for a first order derivative of a function of a quantum space is

$$\begin{aligned} \frac{\partial f_{r_1 r_2 \ldots r_n}(x)}{\partial x^k} &= \sum_{k=1}^{n} r_k \delta_j^k\, (p^{1j})^{r_1} (p^{2j})^{r_2} \ldots (p^{k-1,j})^{r_{k-1}} \times \\ &\qquad \times (x^1)^{r_1} (x^2)^{r_2} \ldots (x^k)^{r_k - 1} \ldots (x^n)^{r_n}. \end{aligned}$$

Proposition 2.2.2. *For any function f the second derivatives of the first order differential calculus (2.2.13) of a quantum space satisfy the relations*

$$\frac{\partial^2 f}{\partial x^i \partial x^i} = p^{ji} \frac{\partial^2 f}{\partial x^j \partial x^i}, \quad i, j = 1, 2, \ldots, n.$$

Proof. First we prove this statement for any product $x^i x^j$ of coordinate functions. We have

$$\begin{aligned} \frac{\partial^2 (x^i x^j)}{\partial x^k \partial x^m} &= \delta_k^i\, \delta_m^j + p^{ij}\, \delta_k^j \delta_m^i, \\ \frac{\partial^2 (x^i x^j)}{\partial x^m \partial x^k} &= \delta_m^i\, \delta_k^j + p^{ij}\, \delta_m^j \delta_k^i. \end{aligned}$$

Hence

$$\frac{\partial^2(x^ix^j)}{\partial x^k \partial x^m} - p^{mk}\frac{\partial^2(x^ix^j)}{\partial x^m \partial x^k} = \delta^i_k\,\delta^j_m + p^{ij}\,\delta^j_k\,\delta^i_m - p^{mk}\,\delta^i_m\,\delta^j_k - p^{mk}\,p^{ij}\,\delta^j_m\delta^i_k. \quad (2.2.15)$$

If $k = m$ then

$$\frac{\partial^2(x^ix^j)}{(\partial x^k)^2} - p^{kk}\frac{\partial^2(x^ix^j)}{(\partial x^k)^2} = \delta^i_k\,\delta^j_k + p^{ij}\,\delta^j_k\,\delta^i_k - p^{kk}\delta^i_k\,\delta^j_k - p^{kk}p^{ij}\,\delta^j_k\,\delta^i_k = 0.$$

If $k \neq m$ then we have only two non-vanishing combinations in (2.2.15), where either $i = k,\ j = m$ or $i = m,\ j = k$. If $i = k,\ j = m$ then the second and the forth term in (2.2.15) vanish, and we have

$$\frac{\partial^2(x^ix^j)}{\partial x^k \partial x^m} - p^{mk}\frac{\partial^2(x^ix^j)}{\partial x^m \partial x^k} = \delta^k_k\,\delta^m_m - p^{mk}\,p^{km}\,\delta^m_m\delta^k_k = 0,$$

where we used $p^{mk}\,p^{km} = 1$. The case $i = m,\ j = k$ can be proved analogously. It can be shown that if we consider a triple product (or higher order) of coordinate functions then a proof can be reduced to the previously proved case of binary products of coordinate functions. Indeed let us consider a triple product $x^ix^jx^s$. Then

$$\frac{\partial^2(x^ix^jx^s)}{\partial x^k \partial x^m} = \delta^i_k\delta^j_m\,x^s + p^{js}\delta^i_k\delta^s_m\,x^j + p^{ij}\,\delta^j_k\delta^i_m\,x^s + p^{ij}p^{is}\,\delta^j_k\delta^s_m\,x^i + p^{ik}p^{js}\,\delta^i_m\delta^s_k\,x^j + p^{ik}p^{js}p^{ij}\,\delta^s_k\delta^j_m\,x^i. \quad (2.2.16)$$

Now we can calculate the expression

$$\frac{\partial^2(x^ix^j)}{\partial x^k \partial x^m} - p^{mk}\frac{\partial^2(x^ix^j)}{\partial x^m \partial x^k},$$

and collecting correspondingly all coefficients of functions x^i, x^j, x^s we get the expressions

$$\begin{aligned}
\delta^i_k\delta^j_m + p^{ij}\delta^j_k\,\delta^i_m - p^{mk}\,\delta^i_m\delta^j_k - p^{ij}p^{mk}\,\delta^i_k\delta^j_m, &\ \ (\text{for } x^s),\\
p^{is}\delta^j_k\,\delta^s_m + p^{ik}p^{js}\,\delta^s_k\delta^j_m - p^{mk}p^{is}\,\delta^j_m\delta^s_k - p^{im}p^{mk}p^{js}\,\delta^j_k\,\delta^s_m, &\ \ (\text{for } x^i),\\
\delta^i_k\delta^s_m + p^{ik}\delta^s_k\delta^i_m - p^{mk}\,\delta^i_m\delta^s_k - p^{im}p^{mk}\,\delta^s_m\delta^i_k, &\ \ (\text{for } x^j).
\end{aligned}$$

The first expression is exactly the expression (2.2.15) in the binary case, and hence it vanishes. The second and third expression also vanish and this can be proved either by the method similar to one used in binary case or by rearranging indexes.

Let us find the general solution of the equations (2.2.11), where unknowns are the entries of matrices $A(x^i), i = 1, 2, \ldots, n$ of homomorphism A"$\mathscr{A} \longrightarrow \mathrm{Mat}_n(\mathscr{A})$. First we observe that if we assume $i = j$ then the equations (2.2.11) become identities. Indeed if $i = j$ then

$$A_k^i(x^i) - A_k^i(x^i) = \delta_k^i x^i - \delta_k^i x^i \equiv 0,$$

and the both sides of equations vanish. This shows that for any $i = 1, 2, \ldots, n$ there are no conditions for the entries $A_1^i(x^i), A_2^i(x^i), \ldots, A_n^i(x^i)$ of the matrix of homomorphism $A(x^i)$. These entries form ith column of the matrix $A(x^i)$, and they determine the commutation relation for the product $x^i\, dx^i$, i.e.

$$x^i\, dx^i = dx^1 A_1^i(x^i) + dx^2 A_2^i(x^i) + dx^3 A_3^i(x^i) + \ldots + dx^n A_n^i(x^i).$$

Thus there are no conditions for the coefficients of the right-hand side of the above relation, and we can consider them as free parameters of our first order differential calculus of a quantum space.

Next important observation is that the equations (2.2.11) do not change if we substitute $i \leftrightarrow j$. Thus we get the independent equations considering $i < j$. It follows from the structure of the equations (2.2.11) that the general solution can be written in the form

$$A_k^i(x^j) = p^{ji} A_k^j(x^i) + p^{ji}\, \delta_k^i x^j - \delta_k^j x^i, \quad i < j. \tag{2.2.17}$$

Let us fix $i = 1$ and $2 \le j \le n$. Then the above solution takes on the form

$$A_k^1(x^j) = p^{j1} A_k^j(x^1) + p^{j1}\, \delta_k^1\, x^j - \delta_k^j\, x^1,$$

which shows that the first column of every matrix $A(x^j)$ starting from the second ($2 \le j \le n$) is expressed in terms of the entries of jth column of the first matrix $A(x^1)$. Taking into consideration that the entries of the first column of the first matrix $A(x^1)$ can be considered as free parameters (there are no relations for them) we conclude that the entries of the first matrix $A(x^1)$ can be taken as free parameters of a first order differential calculus. Hence we have n^2 free parameters. Analogously if we fix $i = 2$ and $3 \le j \le n$ then the solution (2.2.17) can be written as

$$A_k^2(x^j) = p^{j2} A_k^j(x^2) + p^{j2}\, \delta_k^2\, x^j - \delta_k^j\, x^2.$$

Hence the second column of every matrix $A(x^j)$ starting from the third ($3 \le j \le n$) is expressed in terms of the entries of jth column of the second matrix $A(x^2)$. We can consider the entries of the second matrix except for the first column as free parameters, and the second matrix gives us $n^2 - n$ free parameters. Continuing this process we find the number of free parameters

$$n^2 - n + n^2 - 2n + \ldots n^2 - (n-1)n = \frac{n^2(n+1)}{2}.$$

Theorem 2.2.3. *Let $L_A(\mathscr{A})$ be the vector space spanned by the column-vectors*

$$A^1(x^1), A^2(x^1), \ldots, A^1(x^2), A^2(x^2), \ldots, A^n(x^n)$$

of the matrices $A(x^1), A(x^2), \ldots, A(x^n)$ of a homomorphism $A : \mathscr{A} \longrightarrow Mat_n(\mathscr{A})$. Then the consistency equations

$$A^j(x^i) - p^{ij} A^i(x^j) = p^{ij} E^j x^i - E^i x^j, \; (i < j),$$

where E^i is the ith column of unit matrix E, determine the subspace $\tilde{L}_A(\mathscr{A}) \subset L_A(\mathscr{A})$ and

$$dim\, \tilde{L}_A(\mathscr{A}) = \frac{n(n+1)}{2}.$$

We can apply this general result to a 2-dimensional quantum space which is usually referred to as a quantum plane. In this case $n = 2$, and the coordinate functions of a quantum plane are denoted by x, y. They obey the commutation relation

$$xy = q\, yx, \tag{2.2.18}$$

where $q \neq 1$ is a complex number. Let us denote the algebra of functions of a quantum plane by $\mathscr{A}_{xy}$. A homomorphism $A : \mathscr{A}_{xy} \longrightarrow \mathrm{Mat}_2(\mathscr{A}_{xy})$ is entirely determined by two matrices

$$A(x) = \begin{pmatrix} A_1^1(x) & A_1^2(x) \\ A_2^1(x) & A_2^2(x) \end{pmatrix}, \quad A(y) = \begin{pmatrix} A_1^1(y) & A_1^2(y) \\ A_2^1(y) & A_2^2(y) \end{pmatrix}.$$

We have only one consistency equation which can be written in the terms of column-vectors as follows

$$A^2(x) - q\, A^1(y) = q\, E^2 x - E^1 y.$$

We can solve this equation by expressing the first column $A^1(y)$ of the second matrix $A(y)$ in the terms of the second column $A^2(x)$ of the first matrix $A(x)$

$$A^1(y) = q^{-1}\, A^2(x) + q^{-1}\, E^1\, y - E^2\, x.$$

Hence the matrices of a homomorphism A for a quantum plane have the form

$$A(x) = \begin{pmatrix} A_1^1(x) & A_1^2(x) \\ A_2^1(x) & A_2^2(x) \end{pmatrix}, \quad A(y) = \begin{pmatrix} q^{-1}A_1^2(x) + q^{-1}y & A_1^2(y) \\ q^{-1}A_2^2(x) - x & A_2^2(y) \end{pmatrix}.$$

Consequently the commutation relation between the coordinate function y and the differential dx has the form

$$y\, dx = dx\, (q^{-1}A_1^2(x) + q^{-1}\, y) + dy\, (q^{-1}A_2^2(x) - x).$$

2.2.3 Ternary Grassmann algebras with cubic relations

In the first subsection of this section we reminded the definition of Grassmann algebra whose generators $\xi^1, \xi^2, \ldots, \xi^n$ anticommute, i.e. they obey the following commutation relations

$$\xi^i \xi^j = -\xi^j \xi^i. \tag{2.2.19}$$

These commutation relations are quadratic commutation relations and can be written in the equivalent form

$$\xi^i \xi^j + \xi^j \xi^i = 0. \tag{2.2.20}$$

We can read these relations by saying that for any two generators ξ^i, ξ^j of Grassmann algebra the sum of all permutations of generators in the binary product $\xi^i \xi^j$ is zero. This shows a way to generalization of Grassmann algebra by means of transition from binary products of generators to products of higher order. We would like to point out that it is not evident which group of permutations should be used in a transition from binary products to products of higher order. In this case of Grassmann algebra the relations (2.2.19) and (2.2.1) are equivalent which is directly related to the fact that in the case of two elements the subgroup of cyclic permutations coincides with the group of all permutations, i.e. we have $S_2 \equiv \mathbb{Z}_2$. Taking higher order groups of permutations $S_k, k \geq 3$ we have non-trivial subgroups, for example the group S_3 has the subgroup of cyclic permutations $\mathbb{Z}_3$, and $\mathbb{Z}_3 \subset S_3$. Evidently the whole group S_k gives the most general condition, and we will use this group of permutations to construct a generalization of Grassmann algebra with commutation relations of kth order.

Definition 2.2.4. A k-Grassmann algebra $\mathscr{G}^{k,n}$ is an associative unital algebra over $\mathbb{C}$ generated by the variables $\zeta^1, \zeta^2, \ldots, \zeta^n$ which for any sequence of integers $1 \leq \mu_1 \leq \mu_2 \leq \ldots \mu_k \leq n$ are subjected to the relations

$$\sum_{\sigma \in S_k} \zeta^{\mu_{\sigma(1)}} \zeta^{\mu_{\sigma(2)}} \ldots \zeta^{\mu_{\sigma(k)}} = 0, \tag{2.2.21}$$

where S_k is a group of all permutations of k elements.

In the present thesis we will study the case of a 3-Grassmann algebra. Hence taking $k = 3$ in the above definition we get a 3-Grassmann algebra $\mathscr{G}^{3,n}$ generated by $\zeta^1, \zeta^2, \ldots, \zeta^n$ which obey the cubic relations

$$\zeta^\mu \zeta^\nu \zeta^\kappa + \zeta^\nu \zeta^\kappa \zeta^\mu + \zeta^\kappa \zeta^\mu \zeta^\nu + \zeta^\kappa \zeta^\nu \zeta^\mu + \zeta^\nu \zeta^\mu \zeta^\kappa + \zeta^\mu \zeta^\kappa \zeta^\nu = 0. \tag{2.2.22}$$

The basic relation (2.2.22) of 3-Grassmann algebra can be written in the form

$$(\zeta^\mu \zeta^\nu + \zeta^\nu \zeta^\mu)\, \zeta^\kappa + (\zeta^\nu \zeta^\kappa + \zeta^\kappa \zeta^\nu)\, \zeta^\mu + (\zeta^\mu \zeta^\kappa + \zeta^\kappa \zeta^\mu)\, \zeta^\nu = 0,$$

or

$$\{\zeta^\mu, \zeta^\nu\}\, \zeta^\kappa + \{\zeta^\nu, \zeta^\kappa\}\, \zeta^\mu + \{\zeta^\mu, \zeta^\kappa\}\, \zeta^\nu = 0, \tag{2.2.23}$$

where $\{\zeta^\mu, \zeta^\nu\}$ is the anti-commutator of the generators ζ^μ, ζ^ν. It follows immediately from this relation that the generators $\xi^1, \xi^2, \ldots, \xi^n$ of Grassmann algebra satisfy this relation because $\{\xi^i, \xi^j\} = \xi^i \xi^j + \xi^j \xi^i = 0$. Hence every Grassmann algebra is 3-Grassmann algebra.

It is remarkable that we can find a subclass of 3-Grassmann algebras which is different from the subclass of 3-Grassmann algebras induced by Grassmann algebras. For this purpose we introduce the notion of cyclic ternary anti-commutator

$$\{\zeta^\mu, \zeta^\nu, \zeta^\kappa\}_{\mathbb{Z}_3} = \zeta^\mu \zeta^\nu \zeta^\kappa + \zeta^\nu \zeta^\kappa \zeta^\mu + \zeta^\kappa \zeta^\mu \zeta^\nu,$$

and making use of this notion we write the basic relation (2.2.22) of 3-Grassmann algebra in the following form

$$\{\zeta^\mu, \zeta^\nu, \zeta^\kappa\}_{\mathbb{Z}_3} + \{\zeta^\kappa, \zeta^\nu, \zeta^\mu\}_{\mathbb{Z}_3} = 0, \tag{2.2.24}$$

The above relation suggests that if we require for any triple of generators of an algebra to obey the relation

$$\{\zeta^\mu, \zeta^\nu, \zeta^\kappa\}_{\mathbb{Z}_3} = 0, \tag{2.2.25}$$

then this algebra is 3-Grassmann algebra. But this equation can be solved in the sense that a triple product $\zeta^\mu\zeta^\nu\zeta^\kappa$ can be expressed in terms of cyclic permutations of generators of this product. Indeed we can solve this relations with the help of a primitive cubic root of unity by assuming that for any triple product of generators $\zeta^\mu\zeta^\nu\zeta^\kappa$ we have the following cubic commutation relations

$$\zeta^\mu\zeta^\nu\zeta^\kappa = j\,\zeta^\nu\zeta^\kappa\zeta^\mu = j^2\,\zeta^\kappa\zeta^\mu\zeta^\nu. \tag{2.2.26}$$

Indeed, taking into consideration the property of a primitive cubic root of unity $1+j+j^2=0$, we find

$$\{\zeta^\mu,\zeta^\nu,\zeta^\kappa\}_{\mathbb{Z}_3} = \zeta^\mu\zeta^\nu\zeta^\kappa + \zeta^\nu\zeta^\kappa\zeta^\mu + \zeta^\kappa\zeta^\mu\zeta^\nu = (1+j+j^2)\,\zeta^\mu\zeta^\nu\zeta^\kappa = 0.$$

Definition 2.2.5. A cyclic 3-Grassmann algebra $\mathscr{G}_c^{3,N}$ or ternary Grassmann algebra with cyclic cubic relations is an associative unital algebra over complex numbers generated by variables $\theta^1,\theta^2,\ldots,\theta^N$ which obey the cubic commutation relations

$$\theta^A\theta^B\theta^C = j\,\theta^B\theta^C\theta^A = j^2\,\theta^C\theta^A\theta^B,$$

where $A,B,C = 1,2,\ldots,N$ and j is a primitive cubic root of unity. There are no quadratic relations between generators $\theta^1,\theta^2,\ldots,\theta^N$, i.e. all binary products $\theta^A\theta^B$ are linearly independent.

We remind that in the case of a classical Grassmann algebra the square of every generator is zero. We have a similar relation in the case of cyclic 3-Grassmann algebra. It follows immediately from the cubic commutation relations (2.2.26) that the cube of any generator vanishes, i.e.

$$(\theta^A)^3 = 0.$$

Proposition 2.2.6. *A product of any four generators of ternary Grassmann algebra vanishes*

$$\theta^A\theta^B\theta^C\theta^D = 0.$$

The vector space of a cyclic 3-Grassmann algebra is finite dimensional and the dimension of this vector is

$$dim\,\mathscr{G}_c^{3,N} = \frac{N^3+3N^2+2N+3}{3}$$

Proof. The first statement can be proved by the following sequence of transformations by means of cubic commutation relations (2.2.26)

$$\begin{aligned}(\theta^A\theta^B\theta^C)\,\theta^D &= j\,\theta^B(\theta^C\theta^A\,\theta^D)\\ &= j^2\,(\theta^B\theta^A\theta^D)\,\theta^C = \theta^A(\theta^D\theta^B\,\theta^C) = j\,\theta^A(\theta^B\theta^C\,\theta^D).\end{aligned}$$

Because $1-j\neq 0$ we conclude $\theta^A\theta^B\theta^C\theta^C=0$. We can count a number of linearly independent products of generators as follows

$$\mathbb{1},\ \theta^A\ \ (N),\ \theta^A\theta^B\ \ (N^2),\ (\theta^A)^2\theta^B\ \ (N^2-N),\ \theta^A\theta^B\theta^C\ \ (2C_3^N),$$

where in triple products we have $A\neq B\neq C$. Taking into consideration that

$$C_3^N=\frac{N(N-1)(N-2)}{6},$$

we finally get

$$\dim\mathscr{G}_T^N = 1+N+N^2+N^2-N+\frac{N(N-1)(N-2)}{3}=\frac{N^3+3N^2+2N+3}{3}$$

The simplest example of ternary Grassmann algebra is the ternary Grassmann algebra $\mathscr{G}_c^{3,1}$ with one generator θ. The generator satisfies $\theta^3=0$, and any element of this algebra can be written as a linear combination of monomials $\mathbb{1},\theta,\theta^2$. If $N=2$, i.e we consider the algebra $\mathscr{G}_c^{3,2}$ generated by two variables θ^1,θ^2, then the vector space of the ternary Grassmann algebra is spanned by

$$\mathbb{1},\ \theta^1,\theta^2,\ (\theta^1)^2,\theta^1\theta^2,\theta^2\theta^1,(\theta^2)^2,\ (\theta^1)^2\theta^2,(\theta^1)(\theta^2)^2.$$

In order to get an algebra which can be useful in theoretical physics we need to equip a cyclic 3-Grassmann algebra with involution. We can do this by adding to the generators $\theta^1,\theta^2,\ldots,\theta^N$ of a cyclic 3-Grassmann algebra the set of conjugate generators $\bar\theta^{\dot 1},\bar\theta^{\dot 2},\ldots,\bar\theta^{\dot N}$, where we use dotted superscripts to stress an analogy with spinors in physics.

Definition 2.2.7. A 3-Grassmann algebra with involution $\bar{\mathscr{G}}_c^{3,N}$ is an associative unital algebra over $\mathbb{C}$ generated by the variables $\theta^1,\theta^2,\ldots,\theta^N$, and $\bar\theta^{\dot A},\bar\theta^{\dot B},\ldots,\bar\theta^{\dot N}$ which obey the cyclic cubic commutation relations for triple products of generators

$$\theta^A\theta^B\theta^C = j\,\theta^B\theta^C\theta^A = j^2\,\theta^C\theta^A\theta^B, \tag{2.2.27}$$

$$\bar\theta^{\dot A}\bar\theta^{\dot B}\bar\theta^{\dot C} \;=\; j\,\bar\theta^{\dot B}\bar\theta^{\dot C}\bar\theta^{\dot A} = j^2\,\bar\theta^{\dot C}\bar\theta^{\dot A}\bar\theta^{\dot B}, \tag{2.2.28}$$

and the quadratic relations for binary products of generators

$$\theta^A\bar\theta^{\dot B} = j^2\,\bar\theta^{\dot B}\theta^A. \tag{2.2.29}$$

As in the case of a cyclic 3-Grassmann algebra $\mathscr{G}_c^{3,N}$ there are no commutation relations for binary products of generators of the same type, i.e. all products $\theta^A\theta^B$ and $\bar\theta^{\dot A}\bar\theta^{\dot B}$ are linearly independent. An involution $\star : \bar{\mathscr{G}}_c^{3,N} \longrightarrow \bar{\mathscr{G}}_c^{3,N}$ is defined as follows

$$(\theta^A)^\star \;=\; \bar\theta^{\dot A},\;\; (\theta^A\theta^B)^\star = \bar\theta^{\dot B}\bar\theta^{\dot A},\;\; (\theta^A\theta^B\theta^C)^\star = \bar\theta^{\dot C}\bar\theta^{\dot B}\bar\theta^{\dot A} \tag{2.2.30}$$
$$(\bar\theta^{\dot A})^\star \;=\; \theta^A,\;\; (\bar\theta^{\dot A}\bar\theta^{\dot B})^\star = \theta^B\theta^A,\;\; (\bar\theta^{\dot A}\bar\theta^{\dot B}\bar\theta^{\dot C})^\star = \theta^C\theta^B\theta^A. \tag{2.2.31}$$

Let $\mathscr{G}_c^{3,N}(\theta)$ be the cyclic 3-Grassmann subalgebra of $\bar{\mathscr{G}}_c^{3,N}$ generated only by the variables $\theta^1,\theta^2,\ldots,\theta^N$, and $\mathscr{G}_c^{3,N}(\bar\theta)$ be the cyclic 3-Grassmann subalgebra of $\bar{\mathscr{G}}_c^{3,N}$ generated by the conjugate variables $\bar\theta^{\dot A},\bar\theta^{\dot B},\ldots,\bar\theta^{\dot N}$. Evidently both subalgebras are cyclic 3-Grassmann algebras. But it turns out that mixed triple products of generators of different type do not satisfy the cyclic relation (2.2.25), but they satisfy the basic relation of 3-Grassmann algebra (2.2.22). Indeed for a triple $\theta^A,\theta^B,\bar\theta^{\dot C}$ we have

$$\begin{aligned}\theta^A\theta^B\bar\theta^{\dot C} \;&+\; \theta^B\bar\theta^{\dot C}\theta^A+\bar\theta^{\dot C}\theta^A\theta^B+\bar\theta^{\dot C}\theta^B\theta^A+\theta^B\theta^A\bar\theta^{\dot C}+\theta^A\bar\theta^{\dot C}\theta^B\\ &=\; \theta^A\theta^B\bar\theta^{\dot C}(1+j+j^2)+\theta^B\theta^A\bar\theta^{\dot C}(1+j+j^2)=0,\end{aligned}$$

and for triple $\theta^A,\bar\theta^{\dot B},\bar\theta^{\dot C}$ we get

$$\begin{aligned}\theta^A\theta^{\dot B}\theta^{\dot C} \;&+\; \theta^{\dot B}\bar\theta^{\dot C}\theta^A+\bar\theta^{\dot C}\theta^A\theta^{\dot B}+\bar\theta^{\dot C}\theta^{\dot B}\theta^A+\theta^{\dot B}\theta^A\bar\theta^{\dot C}+\theta^A\bar\theta^{\dot C}\theta^{\dot B}\\ &=\; \theta^A\theta^{\dot B}\bar\theta^{\dot C}(1+j+j^2)+\theta^A\bar\theta^{\dot C}\theta^{\dot B}(1+j+j^2)=0.\end{aligned}$$

Thus a 3-Grassmann algebra with involution $\bar{\mathscr{G}}_c^{3,N}$ is not a cyclic 3-Grassmann algebra.

We can verify that the involution of an algebra $\bar{\mathscr{G}}_c^{3,N}$ defined by (2.2.30), (2.2.31) is consistent with the commutation relations of this algebra. Indeed on one hand we have

$$(\theta^A\theta^B\theta^C)^\star = \bar\theta^{\dot C}\bar\theta^{\dot B}\bar\theta^{\dot A}.$$

On the other hand by means of commutation relations we arrive to the same result

$$(\theta^A\theta^B\theta^C)^\star = (j\ \theta^B\theta^C\theta^A)^\star = j^2\ \bar\theta^{\dot A}\bar\theta^{\dot C}\bar\theta^{\dot B} = j^3\ \bar\theta^{\dot C}\bar\theta^{\dot B}\bar\theta^{\dot A}.$$

A consistency of involution with the commutation relations in the case of a product $\bar\theta^{\dot A}\bar\theta^{\dot B}\bar\theta^{\dot C}$ can be verified analogously. It is obvious that the highest degree monomials of an algebra $\mathscr{G}_c^{\bar 3,N}$ which we can form by taking products of its generators can be written in the form

$$\theta^A\theta^B\theta^C\bar\theta^{\dot D}\bar\theta^{\dot E}\bar\theta^{\dot F}.$$

Taking this into consideration we can prove

Proposition 2.2.8. *The dimension of 3-Grassmann algebra $\mathscr{G}_c^{\bar 3,N}$ with involution generated by $\theta^1,\theta^2,\ldots,\theta^N$, and $\bar\theta^{\dot A},\bar\theta^{\dot B},\ldots,\bar\theta^{\dot N}$ is equal to the square of the dimension of a cyclic 3-Grassmann algebra $\mathscr{G}_c^{3,N}(\theta)$, i.e.*

$$dim\, \mathscr{G}_c^{\bar 3,N} = (dim\, \mathscr{G}_c^{3,N})^2.$$

Proof. Let us denote $\dim \mathscr{G}_T^N = k(N)$. It is obvious that the ternary Grassmann algebra $\mathscr{G}_T^N$ generated only by variables $\theta^1,\theta^2,\ldots,\theta^N$ is the subalgebra (we will denote it by $\mathscr{G}_T^N(\theta)$) of the whole algebra $\mathscr{G}_T^{\bar N}$, and the dimension of the vector space of this subalgebra is $k(N)$. Analogously the conjugate variables $\bar\theta^{\dot A},\bar\theta^{\dot B},\ldots,\bar\theta^{\dot N}$ generate the subalgebra (we will denote it by $\mathscr{G}_T^N(\bar\theta)$) whose dimension is also $k(N)$ because there is no fundamental difference between the cubic commutation relations for θ's and $\bar\theta$'s. It is evident that if $f(\theta)$ is a monomial from a basis for the vector space of $\mathscr{G}_T^N(\theta)$, and $g(\bar\theta)$ is a monomial from a basis for the vector space of $\mathscr{G}_T^N(\bar\theta)$ then the product $f(\theta)g(\bar\theta)$ can be taken as the monomial of a basis for the vector space of $\mathscr{G}_T^{\bar N}$, and adding all possible product of this kind to the bases of $\mathscr{G}_T^N(\theta)$ and $\mathscr{G}_T^N(\bar\theta)$ we get the basis for the vector space of whole ternary Grassmann algebra $\mathscr{G}_T^{\bar N}$. Thus

$$\dim \mathscr{G}_T^{\bar N} = 1+2\ (k(N)-1)+(k(N)-1)^2 = k(N)^2.$$

We know that a Grassmann algebra can be considered as a 3-Grassmann algebra because the commutation relations of generators of Grassmann algebra satisfy the basic relation of 3-Grassmann algebra (2.2.22). But a Grassmann algebra is not a cyclic 3-Grassmann algebra because its commutation relations do not satisfy the cyclic condition (2.2.25). On the other hand we have a cyclic 3-Grassmann algebra constructed with the help of cyclic cubic commutation relations (2.2.26). In the next part of this subsection we will address the question of whether it is possible to construct a 3-Grassmann algebra so that it would contain a Grassmann algebra as well as a cyclic 3-Grassmann algebra as its subalgebras. By other word our aim in the rest of this subsection is to extend a Grassmann algebra to 3-Grassmann algebra by means of cyclic commutation relations of a cyclic 3-Grassmann algebra. Let $\xi^1, \xi^2, \ldots, \xi^n$ be generators of Grassmann algebra, and $\theta^1, \theta^2, \ldots, \theta^N$ be generators of cyclic 3-Grassmann algebra. If our aim is to combine them into a single 3-Grassmann algebra then we have to find commutation relations between ξ's and θ's in such a way that they will satisfy the basic relation for 3-Grassmann algebras (2.2.22). Because the left-hand side of basic relation is the sum of triple products of generators we have two possibilities, where the first one is a triple ξ^i, ξ^j, θ^A and the second is $\xi^i, \theta^A, \theta^B$. We have two equations to find a commutation relation between ξ's and θ's, which can be written as follows

$$\begin{aligned}
\xi^i\xi^j\theta^A + \xi^j\theta^A\xi^i + \theta^A\xi^i\xi^j + \theta^A\xi^j\xi^i + \xi^j\xi^i\theta^A + \xi^i\theta^A\xi^j &= 0,\\
\xi^i\theta^A\theta^B + \theta^A\theta^B\xi^i + \theta^B\xi^i\theta^A + \theta^B\theta^A\xi^i + \theta^A\xi^i\theta^B + \xi^i\theta^B\theta^A &= 0.
\end{aligned}$$

Making use of the commutation relations of the generators of Grassmann algebra ξ^i we can write the first equation in the form

$$\xi^j\theta^A\xi^i + \xi^i\theta^A\xi^j = 0 \quad \Longrightarrow \quad \xi^i\theta^A\xi^j = -\xi^j\theta^A\xi^i. \tag{2.2.32}$$

Now let us suppose that commutation relations between ξ's and θ's have the form

$$\xi^i\theta^A = q\,\theta^A\xi^i,$$

where $q \in \mathbb{C}, q \neq 0$. It is easy to see that the equation (2.2.32) is identically satisfied giving no equation for q (it holds for any q). The second equation takes on the form

$$(1+q+q^2)(\theta^A\theta^B + \theta^B\theta^A)\xi^i = 0.$$

Since there are no relations between binary products of generators θ^A of cyclic 3-Grassmann algebra, we conclude

$$1+q+q^2 = 0,$$

which gives two solutions $q = j$ or $q = j^2$. Thus we can give the following definition

Definition 2.2.9. A 3-Grassmann algebra extension of Grassmann algebra by means of cyclic relations (2.2.26) is an associative unital algebra over $\mathbb{C}$ generated by the variables $\xi^1, \xi^2, \ldots, \xi^n$ and $\theta^1, \theta^2, \ldots, \theta^N$, where the variables obey the following commutation relations

$$\begin{aligned} \xi^i \xi^j &= -\xi^j \xi^i, && (2.2.33) \\ \theta^A \theta^B \theta^C &= j\, \theta^B \theta^C \theta^A = j^2\, \theta^C \theta^A \theta^B, && (2.2.34) \\ \xi^i \theta^A &= q\, \theta^A \xi^i, && (2.2.35) \end{aligned}$$

where q is either j or j^2.

2.2.4 Differential calculus over ternary Grassmann algebra

A very important question is a first order differential calculus over a ternary Grassmann algebra with cyclic cubic relations because this calculus will give us a partial derivatives with respect to generators θ^A of ternary Grassmann algebra and making use of these derivatives one can construct an analog of supersymmetry operators. It should be pointed out that there are no relations between binary products of generators of ternary Grassmann algebra, i.e. all binary products $\theta^A \theta^B$ are linearly independent. Hence a ternary Grassmann algebra with cyclic cubic relations does not belong to class of unital associative algebras with quadratic relations, and we can not apply results obtained in the previous subsections to a ternary Grassmann algebra. In this subsection we will develop an approach to first order noncommutative differential calculus for ternary Grassmann algebras with cubic relations.

Let $\mathscr{G}_c^{3,N}$ be a ternary Grassmann algebra with cyclic cubic relations generated by θ^A, $A = 1, 2, \ldots, N$. Then generators satisfy the relations

$$\theta^A \theta^B \theta^C = j\, \theta^B \theta^C \theta^A, \qquad (2.2.36)$$

where j is a primitive 3rd order root of unity. Our aim in this subsection is to construct a first order noncommutative differential calculus over this ternary Grassmann algebra. Hence we assume that $\mathscr{M}$ is a bimodule over a ternary Grassmann algebra $\mathscr{G}_c^{3,N}$, $d : \mathscr{G}_c^{3,N} \longrightarrow \mathscr{M}$ is a differential of first order differential calculus, $d(\theta^A) = d\theta^A$, and the right $\mathscr{G}_c^{3,N}$-module $\mathscr{M}_R$ is freely generated by $d\theta^A$, $A = 1, 2, \ldots, N$. We also assume that the left $\mathscr{G}_c^{3,N}$-module $\mathscr{M}_L$ is related to the right $\mathscr{G}_c^{3,N}$-module by a homomorphism of algebras $\Phi : \mathscr{G}_c^{3,N} \longrightarrow \mathrm{Mat}_N(\mathscr{G}_c^{3,N})$, where $\mathrm{Mat}_N(\mathscr{G}_c^{3,N})$ is the algebra of matrices of order N with entries from $\mathscr{G}_c^{3,N}$. Thus we have

$$\theta^A\,d\theta^B = d\theta^C\,\Phi_C^{BA}, \tag{2.2.37}$$

where $\Phi_C^{BA} = \Phi_C^B(\theta^A)$ are elements of $\mathscr{G}_c^{3,N}$. Differentiating relations (3.1.11) we get

$$\begin{aligned} &d\theta^A(\theta^B\theta^C) + \theta^A d\theta^B\theta^C + (\theta^A\theta^B)d\theta^C \\ &\quad = j\,d\theta^B(\theta^C\theta^A) + j\,\theta^B d\theta^C\theta^A + j\,(\theta^B\theta^C)\,d\theta^A. \end{aligned} \tag{2.2.38}$$

This is the consistency equation between differential d and cyclic cubic commutation relations of a ternary Grassmann algebra $\mathscr{G}_c^{3,N}$. We can solve this equation by assuming

$$d\theta^A\,(\theta^B\theta^C) = j\,(\theta^B\theta^C)\,d\theta^A, \tag{2.2.39}$$

$$(\theta^A\theta^B)\,d\theta^C = j\,\theta^B\,d\theta^C\,\theta^A, \tag{2.2.40}$$

$$\theta^A\,d\theta^B\,\theta^C = j\,d\theta^B\,(\theta^C\theta^A). \tag{2.2.41}$$

Making use of the entries of a homomorphism Φ we can write the right hand side of the first equation (2.2.39) in the form

$$(\theta^B\theta^C)\,d\theta^A = d\theta^K\,\Phi_K^A(\theta^B\theta^C),$$

which gives us the first equation for the entries of a homomorphism

$$\Phi_K^A(\theta^B\theta^C) = j^2\,\delta_K^A\theta^B\theta^C. \tag{2.2.42}$$

Taking into consideration that Φ is a homomorphism we can write this equation in the equivalent form

$$\Phi_K^L(\theta^B)\,\Phi_L^A(\theta^C) = j^2\,\delta_K^A\theta^B\theta^C. \tag{2.2.43}$$

Analogously the second condition (2.2.40) leads to the following equation for the entries of the matrix of a homomorphism Φ

$$\Phi_K^C(\theta^A\theta^B) = j\,\Phi_K^C(\theta^B)\,\theta^A. \tag{2.2.44}$$

Similarly the third condition (2.2.41) gives the equation

$$\Phi_K^L(\theta^A)\,\theta^C = j\,\delta_K^L\,\theta^C\theta^A,$$

and it is easy to see that this is exactly the second equation (2.2.44). Hence we have two independent equations

$$\Phi^L_K(\theta^B)\,\Phi^A_L(\theta^C) \;=\; j^2\,\delta^A_K\,\theta^B\theta^C, \tag{2.2.45}$$

$$\Phi^L_K(\theta^B)\,\theta^C \;=\; j\,\delta^L_K\,\theta^C\theta^B. \tag{2.2.46}$$

Hence we have a first order noncommutative differential calculus over a ternary Grassmann algebra with cyclic cubic relations with partial derivatives

$$\begin{aligned}\frac{\partial}{\partial\theta^B}(\theta^A) &= \delta^A_B,\\ \frac{\partial}{\partial\theta^C}(\theta^A\theta^B) &= \delta^A_C\,\theta^B+\Phi^{KA}_A\,\frac{\partial}{\partial\theta^K}(\theta^B)=\delta^A_C\,\theta^B+\Phi^{BA}_C.\end{aligned}$$

In our first order noncommutative differential calculus we have the parameters $\Phi^{BA}_C=\Phi^B_C(\theta^A)$. It is remarkable that although we have parameters in our differential calculus we can construct an analog of supersymmetry operator which can be explicitly calculated in the case of linearly independent binary products of generators. Making use of the rules for partial derivatives and of the equation (2.2.46) we find

$$\Big(\frac{\partial}{\partial\theta^K}(\theta^A\theta^B)\Big)\theta^C=\delta^A_K\;\theta^B\theta^C+\Phi^B_K(\theta^A)\,\theta^C=\delta^A_K\;\theta^B\theta^C+j\delta^B_K\;\theta^C\theta^A. \tag{2.2.47}$$

We end this subsection with the rules for partial derivatives of triple products of generators. It is remarkable that again although we have parameters in our differential calculus the rules for partial derivatives for triple products of generators have explicit form and they do not depend on parameters Φ^{AB}_C. Indeed let us consider a triple product $\theta^A\theta^B\theta^C$. We have

$$\frac{\partial}{\partial\theta^K}(\theta^A\theta^B\theta^C)=\frac{\partial}{\partial\theta^K}(\theta^A\theta^B)\;\theta^C+\Phi^L_K(\theta^A\theta^B)\;\frac{\partial}{\partial\theta^L}(\theta^C).$$

Now applying the first equation (2.2.45) and the formula (2.2.47) we get

$$\frac{\partial}{\partial\theta^K}(\theta^A\theta^B\theta^C)=\delta^A_K\;\theta^B\theta^C+j\delta^B_K\;\theta^C\theta^A+j^2\,\delta^C_K\;\theta^A\theta^B. \tag{2.2.48}$$

As in the case of ordinary Grassmann algebra, we see that our first order differential calculus is consistent with cubic commutation relations of a ternary Grassmann algebra. Indeed the above formula shows that taking a partial derivative of a triple product we put each generator of this product to the first position by means of cyclic relations and then replace it with the corresponding Kronecker symbol. Hence our calculus of partial derivatives is consistent with the well known rules of taking partial derivatives in associative unital algebras generated by a set of variables which obey commutation relations.

Chapter 3

Graded Differential Algebras

In this chapter we consider the important class of graded differential algebras. First we will describe a notion of a graded vector space. Then we will give a definition of a unital associative graded algebra and give a short description of its structure. A general notion of a unital associative graded algebra is usually defined with the help of additive Abelian group which is used to label the subspaces of graded algebra. In the present chapter we will not give a definition of a unital associative graded algebra in this general way, we will use the additive group of integers $\mathbb{Z}$ or its subgroups of residue classes of modulo by some fixed integer. Particularly we will stress an importance of $\mathbb{Z}_2$-graded algebras which are known under the name of superalgebras. We will show that Grassmann, Clifford and ternary Grassmann algebras defined in the previous chapter can be endowed either with $\mathbb{Z}_2$-graded or $\mathbb{Z}_3$-graded structure. Then we will define a basic structure of the present thesis which we will use in the next chapter to generalize an algebra of differential forms of a smooth manifold. This structure is called a graded differential algebra, and it will be defined by means of a notion of differential graded vector space. We will also define cohomologies of a differential graded vector space because they play an important role in the theory of graded differential algebras.

3.1 Graded Associative Unital Algebras

Let us remind that a unital associative algebra can be considered as a graded algebra if it can be split into a direct sum of its subspaces labelled by integers or residue classes by modulo some fixed integer, and a product of two homogeneous elements should be consistent with their labels (degrees). Graded algebras play an important role not only in modern differential geometry but also in theoretical physics. With regard to applications of graded differential algebras in modern differential geometry, we must point out an algebra of differential forms together with exterior differential on a smooth manifold. Definitely the unital associative graded algebras play an important role in the theory of Lie algebras, where they can be used to construct a Chevalley-Eilenberg complex. As it was mentioned previously a very important class of graded algebras is the class of $\mathbb{Z}_2$-graded algebras which are also known under the name of superalgebras. Superalgebras are widely used in the supersymmetric field theories of modern theoretical physics. Why physicist are so interested in superalgebras? The reason for this can be explained to some extent by the fact that in modern physics of elementary particles there are two large classes of elementary particles which are called bosons and fermions. Figuratively speaking, this division into two large classes gives rise to two subspaces in a Hilbert space of quantum states of particles, and bosons may be viewed as an even subspace of this Hilbert space, and fermions may be viewed as an odd part of the same Hilbert space.

3.1.1 Graded vector space and graded algebra

Let $\mathscr{V}$ be a vector space over the complex numbers $\mathbb{C}$. A vector space $\mathscr{V}$ is said to be a $\mathbb{Z}$-graded vector space, where $\mathbb{Z}$ are integers, if $\mathscr{V}$ is a direct sum of its subspaces $\mathscr{V}^i$, $i \in \mathbb{Z}$ labeled by integers. Hence a graded structure of a vector space is determined by its decomposition into direct sum of subspaces $\mathscr{V} = \oplus_{i\in\mathbb{Z}}\mathscr{V}^i$. The elements of subspace $\mathscr{V}^i$ are called homogeneous elements of degree i of a graded vector space. The degree of a homogeneous element v will be denoted by $|v|$. Hence if $v \in \mathscr{V}^i$ then $|v| = i$. We will also consider $\mathbb{Z}_n$-graded vector spaces, where $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$ is the additive group of residue classes of integers modulo n. Let us point out the special case of $\mathbb{Z}_2$ which plays an important role in supermathematics and supersymmetric field theories. In this case $\mathbb{Z}_2$-graded vector space

$$\mathscr{V} = \mathscr{V}^{\bar{0}} \oplus \mathscr{V}^{\bar{1}}$$

is called a super vector space, the elements of subspace $\mathscr{V}^{\bar{0}}$ are referred to as the even elements and the elements of subspace $\mathscr{V}^{\bar{1}}$ as odd elements of super vector space. We can give an example of a graded vector space if we consider a vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) multilinear mappings of a vector space V to real or complex numbers. Indeed we can associate the integer to each multilinear mapping which is the number of arguments of this multilinear mapping. Obviously this defines the structure of graded vector space in the vector space of all multilinear mappings.

Now we will give the definition of a unital associative graded algebra.

Definition 3.1.1. Let $\mathscr{A}$ be a unital associative algebra over the complex numbers $\mathbb{C}$. An algebra $\mathscr{A}$ will be referred to as a $\mathbb{Z}$-graded algebra if the vector space of this algebra is $\mathbb{Z}$-graded vector space $\mathscr{A} = \oplus_{i\in\mathbb{Z}}\,\mathscr{A}^i$, and for any two homogeneous elements $a, b \in \mathscr{A}$ we have

$$|a\,b| = |a| + |b|. \tag{3.1.1}$$

If $\mathscr{A} = \mathscr{A}^{\bar{0}} \oplus \mathscr{A}^{\bar{1}}$ is a $\mathbb{Z}_2$-graded algebra then $\mathscr{A}$ is usually referred to as a superalgebra.

Remark 3.1.2. If $\mathbb{1}$ is the unit element of algebra $\mathscr{A}$ then $\mathbb{1}\,a = a\,\mathbb{1} = a$. Hence $|\mathbb{1}\,a| = |a\,\mathbb{1}| = |\mathbb{1}| + |a| = |a|$. Consequently $|\mathbb{1}| = 0$ and $\mathbb{1} \in \mathscr{A}^0$.

Now we emphasize a basic properties of the structure of graded associative algebra which we will use in what follows. Consider the subspace $\mathscr{A}^0$ of elements of degree zero. It follows from the definition of graded algebra that the product of any two elements $a, b \in \mathscr{A}^0$ of degree zero is the element of degree zero $|a\,b| = |a| + |b| = 0$. Thus $\mathscr{A}^0$ is the subalgebra of graded algebra $\mathscr{A}$. Consider a subspace $\mathscr{V}^i$ of elements of degree i. If we multiply an element b of this subspace from the left (right) by an element $a \in \mathscr{A}^0$ of degree zero then the degree of product is $|a\,b| = |a| + |b| = |b|$ ($|b\,a| = |b| + |a| = |b|$). Hence $a\,b \in \mathscr{A}^i$ ($b\,a \in \mathscr{A}^i$), and we have two mappings $\mathscr{A}^0 \times \mathscr{A}^i \longrightarrow \mathscr{A}^i$, $\mathscr{A}^i \times \mathscr{A}^0 \longrightarrow \mathscr{A}^i$. Taking into account that $\mathscr{A}^i$ is a vector space, $\mathscr{A}^0$ is the algebra (subalgebra of $\mathscr{A}$) and $\mathscr{A}$ is an associative algebra, one can easily verify that two mentioned above mappings induce the $\mathscr{A}^0$-bimodule structure of $\mathscr{A}^i$. In what follows we will need the following statement

Proposition 3.1.3. *Let $\mathscr{A} = \oplus_{i\in\mathbb{Z}}\,\mathscr{A}^i$ be a graded algebra. Then the subspace of elements of degree zero $\mathscr{A}^0$ is the subalgebra of $\mathscr{A}$ and for each $i \in \mathbb{Z}$ the subspace $\mathscr{A}^i$ of elements of degree i is the $\mathscr{A}^0$-bimodule with respect to left and right multiplication by elements of degree zero.*

The graded subspace $\mathscr{Z}(\mathscr{A}) \subset \mathscr{A}$ generated by homogeneous elements $u \in \mathscr{A}^k$, which for any degree l and any homogeneous element $v \in \mathscr{A}^l$ satisfy

$$u\,v = (-1)^{kl}\,v\,u,$$

is called a graded center of a graded algebra $\mathscr{A}$.

Definition 3.1.4. Let $\mathscr{A} = \oplus_{k\in\mathbb{Z}}\mathscr{A}^k$ be a graded associative unital algebra over $\mathbb{C}$ and $u \in \mathscr{A}^k, v \in \mathscr{A}^l$ be homogeneous elements. The *graded commutator*

$$[\,,\,] : \mathscr{A}^k \otimes \mathscr{A}^l \longrightarrow \mathscr{A}^{k+l}$$

is defined by

$$[u,v] = u\,v - (-1)^{kl}\,v\,u. \tag{3.1.2}$$

Definition 3.1.5. A graded derivation of degree m of a graded algebra $\mathscr{A}$ is a linear mapping $\delta : \mathscr{A} \longrightarrow \mathscr{A}$ of degree m, i.e. $\delta : \mathscr{A}^i \longrightarrow \mathscr{A}^{i+m}$, which satisfies the graded Leibniz rule

$$\delta(u\,v) = \delta(u)\,v + (-1)^{ml}\,u\,\delta(v), \tag{3.1.3}$$

where u is a homogeneous element of grading l, i.e. $u \in \mathscr{A}^l$. If m is even integer then δ is a derivation of an algebra $\mathscr{A}$, and if m is odd integer then δ is called an antiderivation of a graded algebra $\mathscr{A}$.

Given a homogeneous element $v \in \mathscr{A}^m$ of degree m one can define with the help of graded commutator a graded derivation of degree m, which is denoted by ad_v, as follows

$$\mathrm{ad}_v(u) = [v,u] = v\,u - (-1)^{ml}\,u\,v, \tag{3.1.4}$$

where $v \in \mathscr{A}^l$. The graded derivation ad_v is called an inner graded derivation of an algebra $\mathscr{A}$.

3.1.2 Superalgebras and Lie superalgebras

In this subsection we give few examples of $\mathbb{Z}_2$-graded algebras which are usually called superalgebras. We also give definitions of Lie algebra and Lie superalgebra [36]. The two examples of superalgebras which we will describe in this subsection are Grassmann algebra and Clifford algebra. It should be pointed out that Lie superalgebras are very important in the field of supersymmetric field theories, and classification of Lie superalgebras was developed by V. Kac [24].

A Grassmann algebra $\mathscr{G}$ described in Section 2.2 can be considered as a superalgebra. Indeed let $\{\xi^1, \xi^2, \ldots, \xi^n\}$ be the set of generators of Grassmann algebra $\mathscr{G}$. We define the degree of each generator by $|\xi^i| = 1$, and extend this degree to any monomial with the help of condition (3.1.1), that is we define the degree of a product of generators as the sum of degrees of its factors modulo 2. By other words when we multiply generators their degrees add up (modulo 2). Hence the degree of a monomial ξ^I is $|\xi^I| = |I| \pmod 2$. The subspace $\mathscr{G}^{\bar{0}} \subset \mathscr{G}$ ($\mathscr{G}^{\bar{1}} \subset \mathscr{G}$) spanned by the even (odd) degree monomials ξ^I is the subspace of homogeneous elements of degree $\bar{0}$ ($\bar{1}$) and

$$\mathscr{G} = \mathscr{G}^{\bar{0}} \oplus \mathscr{G}^{\bar{1}}.$$

If $I \cap J \neq \emptyset$ then $\xi^I \xi^J = 0$. Obviously for $I \cap J = \emptyset$ we have $|\xi^I \xi^J| = |\xi^I| + |\xi^J|$, and this shows that the condition (3.1.1) is satisfied, and a Grassmann algebra $\mathscr{G}$ is a $\mathbb{Z}_2$-graded algebra or superalgebra. A Grassmann algebra is a basic structure of supermathematics, where it is considered as a $\mathbb{Z}_2$-graded algebra or superalgebra.

Let M^m be a smooth (real) m-dimensional manifold, and $C^\infty(M)$ be the algebra of smooth functions of this manifold. Let $\mathscr{G}$ be a Grassmann algebra generated by n variables $\xi^1, \xi^2, \ldots, \xi^n$, and let us consider the algebra of $\mathscr{G}$-valued functions of a manifold M which will be denoted by $C^\infty_{m,n}(M)$. Any $\mathscr{G}$-valued function f can be written in the form

$$f(x) = f_I(x)\, \xi^I,$$

where $x \in M$ is a point of a manifold M, and $f_I(x) \in C^\infty(M)$. We can define a degree of any $\mathscr{G}$-valued function by saying that $\mathscr{G}$-valued function is a function of even degree (odd degree) if any monomial ξ^I in the above expression is even degree monomial (odd degree monomial). Obviously $C^\infty_{m,n}(M)$ is the superalgebra because it can be decomposed into the direct sum of even degree subspace and the odd degree subspace. This superalgebra is called a superalgebra of $\mathscr{G}$-valued functions on a manifold M, and this superalgebra is a basic structure for (m,n)-dimensional supermanifold over a smooth manifold M. In Section 2.2 we defined the partial derivatives with respect to generators of Grassmann algebra, and this can be used to develop a calculus of vector fields on a supermanifold constructed by means of a Grassmann algebra $\mathscr{G}$.

A vector space $\mathscr{L}$ is called a Lie algebra if it is equipped with a bilinear mapping

$$[\ ,\ ] : \mathscr{L} \times \mathscr{L} \longrightarrow \mathscr{L},$$

which is skew-symmetric, i.e. for any two elements $a, b \in \mathscr{L}$ we have $[a,b] = -[b,a]$, and satisfies the Jacoby identity

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$$

A Lie superalgebra is an extension of a structure of Lie algebra to a $\mathbb{Z}_2$-graded case. A super vector space $\mathscr{L} = \mathscr{L}^{\bar{0}} \oplus \mathscr{L}^{\bar{1}}$ is called a Lie superalgebra if it is endowed with a bilinear mapping

$$[\,,\,] : \mathscr{L} \times \mathscr{L} \longrightarrow \mathscr{L},$$

which is $\mathbb{Z}_2$-graded skew-symmetric, i.e. for any two elements $a, b \in \mathscr{L}$ we have $[a,b] = -(-1)^{|a||b|}[b,a]$, and it satisfies the $\mathbb{Z}_2$-graded Jacoby identity

$$[a,[b,c]] + (-1)^{|a||b|+|a||c|}[b,[c,a]] + (-1)^{|b||c|+|a||c|}[c,[a,b]] = 0.$$

Given a unital associative algebra $\mathscr{A}$ one can construct a Lie algebra with the help of commutator $[a,b] = ab - ba$, where $a, b \in \mathscr{A}$. It should be mentioned that in this case the Jacoby identity holds due to associativity of an algebra $\mathscr{A}$. Similarly given a superalgebra $\hat{\mathscr{A}}$ one can construct a Lie superalgebra with the help of $\mathbb{Z}_2$-graded commutator.

Next example of a superalgebra is a Clifford algebra. Let us remind that a Clifford algebra $\mathscr{C}_n$ is the unital associative algebra over $\mathbb{C}$ generated by the variables $\gamma_1, \gamma_2, \ldots, \gamma_n$ which obey the relations

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\,\delta_{ij}\,\mathbb{1}, \quad i, j = 1, 2, \ldots, n, \tag{3.1.5}$$

where $\mathbb{1}$ is the unit element of Clifford algebra. Let $\mathscr{N} = \{1, 2, \ldots, n\}$ be the set of integers from 1 to n. If I is a subset of $\mathscr{N}$, i.e. $I = \{i_1, i_2, \ldots, i_k\}$ where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, then this subset I determines the monomial of Clifford algebra $\gamma_I = \gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_k}$. For $I = \emptyset$ we put $\gamma_\emptyset = \mathbb{1}$. The number of elements of a subset I will be denoted by $|I|$. Evidently the vector space of Clifford algebra $\mathscr{C}_n$ is spanned by the monomials γ_I, where $I \subseteq \mathscr{N}$. The dimension of this vector space is $\dim \mathscr{C}_n = 2^n$, and any element $x \in \mathscr{C}_n$ can be written in terms of these monomials as

$$x = \sum_{I \subseteq \mathscr{N}} a_I \gamma_I,$$

where coefficients $a_I = a_{i_1 i_2 \ldots i_k}$ are complex numbers. In order to endow a Clifford algebra $\mathscr{C}_n$ with a $\mathbb{Z}_2$-graded structure we define the degree of any monomial γ_I by $|\gamma_I| = |I| \,(\mathrm{mod}\, 2)$. Then a Clifford algebra $\mathscr{C}_n$ can be considered as the superalgebra because for any two monomials we have $|\gamma_I \gamma_J| = |\gamma_I| + |\gamma_J|$.

Another way to construct this superalgebra which does not contain explicit reference to Clifford algebra is given by the following theorem [12].

Theorem 3.1.1. Let I be a subset of $\mathscr{N} = \{1,2,\ldots,n\}$, and γ_I be a symbol associated to I. Let C_n be the vector space spanned by the symbols γ_I. Define the degree of γ_I by $|\gamma_I| = |I| (\bmod 2)$, where $|I|$ is the number of elements of I, and the product of γ_I, γ_J by

$$\gamma_I\,\gamma_J = (-1)^{\sigma(I,J)}\gamma_{I\Delta J}, \tag{3.1.6}$$

where $\sigma(I,J) = \sum_{j\in J}\sigma(I,j)$, $\sigma(I,j)$ is the number of elements of I which are greater than $j \in J$, and $I\Delta J$ is the symmetric difference of two subsets. Then C_n is the unital associative superalgebra, where the unit element e is $\gamma_\emptyset$.

This theorem can be proved by means of the properties of symmetric difference of two subsets. We remind a reader that the symmetric difference is commutative $I\oplus J = J\oplus I$, associative $(I\Delta J)\Delta K = I\Delta(J\Delta K)$ and $I\Delta\emptyset = \emptyset\Delta I$. The latter shows that $\gamma_\emptyset$ is the unit element of this superalgebra. The symmetric difference also satisfies $|I\Delta J| = |I| + |J|\ (\bmod 2)$. Hence C_n is the superalgebra.

The superalgebra $\mathscr{C}_n$ can be considered as a Lie superalgebra if for any two homogeneous elements x, y of this superalgebra we make use of the graded commutator $[x,y] = xy - (-1)^{|x||y|}yx$ and extend it by linearity to a whole superalgebra $\mathscr{C}_n$. We will denote this Lie superalgebra by $\mathfrak{C}_n$. It is evident that $\{\gamma_I\}_{I\subseteq\mathscr{N}}$ can be used as the generators of this Lie superalgebra $\mathfrak{C}_n$, and the structure of this Lie superalgebra is uniquely determined by the graded commutators of γ_I. Then for any two generators γ_I, γ_J we have

$$[\gamma_I, \gamma_J] = f(I,J)\ \gamma_{I\Delta J}, \tag{3.1.7}$$

where $f(I,J)$ is the integer-valued function of two subsets of $\mathscr{N}$ defined by

$$f(I,J) = (-1)^{\sigma(I,J)}\big(1-(-1)^{|I\cap J|}\big),$$

It is easy to verify that the degree of graded commutator is consistent with the degrees of generators, i.e. $[\gamma_I, \gamma_J] = |\gamma_I| + |\gamma_J|$. Indeed the function $\sigma(I,J)$ satisfies

$$\sigma(J,I) = |I||J| - |I\cap J| - \sigma(I,J),$$

and

$$\begin{aligned}
f(J,I) &= (-1)^{\sigma(J,I)}\big(1-(-1)^{|I\cap J|}\big)\\
&= (-1)^{|I||J|-|I\cap J|-\sigma(I,J)}\big(1-(-1)^{|I\cap J|}\big)\\
&= (-1)^{|I||J|}(-1)^{\sigma(I,J)}\big((-1)^{|I\cap J|}-1\big) = -(-1)^{|I||J|}f(I,J).
\end{aligned}$$

Hence $[\gamma_I, \gamma_J] = -(-1)^{|I||J|}[\gamma_J, \gamma_I]$ which shows that the relation (3.1.7) is consistent with the symmetries of graded commutator. It is obvious that if the intersection of subsets I, J contains an even number of elements then $f(I,J) = 0$, and the graded commutator of γ_I, γ_J is trivial. Particularly if at least one of two subsets I, J is the empty set then $f(I,J) = 0$. Thus any graded commutator (3.1.7) containing e is trivial.

As an example, let us consider the Lie superalgebra $\mathfrak{C}_2$. The dimension of its vector space is four, and $\mathfrak{C}_2$ is generated by two even degree generators e, γ_{12} and two odd degree generators γ_1, γ_2. The non-trivial relations of this Lie superalgebra are given by

$$[\gamma_1, \gamma_1] = [\gamma_2, \gamma_2] = 2\,e,\ [\gamma_1, \gamma_{12}] = 2\,\gamma_2,\ [\gamma_2, \gamma_{12}] = -2\,\gamma_1. \tag{3.1.8}$$

Now we assume that $n = 2m, m \geq 1$ is an even integer. The Lie superalgebra $\mathfrak{C}_n$ has a matrix representation which can be described as follows. Fix $n = 2$ and identify the generators γ_1, γ_2 with the Pauli matrices σ_1, σ_2, i.e.

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.1.9}$$

Then $\gamma_{12} = \gamma_1 \gamma_2 = i\,\sigma_3$ where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let S^2 be the 2-dimensional complex super vector space $\mathbb{C}^2$ with the odd degree operators (3.1.9), where the $\mathbb{Z}_2$-graded structure of S^2 is determined by $\sigma_3 = i^{-1}\gamma_{12}$. Then $C_2 \simeq \mathrm{End}\,(S^2)$, and S^2 can be considered as a supermodule over the superalgebra C_2. Let $S^n = S^2 \otimes S^2 \otimes \ldots \otimes S^2 (m-\text{ times})$. Then S^n can be viewed as a supermodule over the m-fold tensor product of C_2, which can be identified with C_n by identifying γ_1, γ_2 in the jth factor with $\gamma_{2j-1}, \gamma_{2j}$ in C_n. This C_n-supermodule S^n is called the supermodule of spinors [34]. Hence we have the matrix representation for the Clifford algebra C_n, and this matrix representation or supermodule of spinors allows one to consider the supertrace, and it can be proved [34] that

$$\mathrm{Str}(\gamma_I) = \begin{cases} 0 & \text{if } I < \mathscr{N}, \\ (2i)^m & \text{if } I = \mathscr{N}. \end{cases} \tag{3.1.10}$$

Now we have the Lie superalgebra $\mathfrak{C}_n$ with the graded commutator defined in (3.1.7) and its matrix representation based on the supermodule of spinors.

3.1.3 Graded structure of ternary Grassmann algebras

In Section 2.2 we introduced a cyclic 3-Grassmann algebra with cubic relations, a cyclic 3-Grassmann algebra with involution and a 3-Grassmann algebra extension of Grassmann algebra by means of cyclic cubic relations. We remind that a cyclic 3-Grassmann algebra with cyclic cubic relations is a unital associative algebra generated by θ^A, $A = 1, 2, \ldots, \theta^N$ which are subjected to the cyclic cubic relations

$$\theta^A \theta^B \theta^C = j\, \theta^B \theta^C \theta^A = j^2\, \theta^C \theta^A \theta^B, \tag{3.1.11}$$

where j is a primitive cubic root of unity. We proved the theorem which states that a product of any four generators of this algebra vanishes. We can endow a cyclic 3-Grassmann algebra with a structure of a graded algebra as follows: we assign degree zero to the identity element $\mathbb{1}$, degree 1 to every generator θ^A and, as usual, we define the degree of any product of generators as the sum of degrees of its factors modulo 3. Hence we will have

$$|\mathbb{1}| = 0,\ \ |\theta^A| = 1,\ \ |\theta^A \theta^B| = 2,\ \ |\theta^A \theta^B \theta^C| = 0.$$

It is clear that our definition of degrees of generators and their products yields the $\mathbb{Z}_3$-graded structure for a cyclic 3-Grassmann algebra. The subspace of degree zero $(\mathscr{G}_c^{3,N})_0$ will be spanned by the identity element $\mathbb{1}$ and all linear independent triple products of generators, the subspace of degree one $(\mathscr{G}_c^{3,N})_1$ will be spanned by the generators, and finally the subspace of degree two $(\mathscr{G}_c^{3,N})_2$ will be spanned by all binary products of generators. Thus

$$\mathscr{G}_c^{3,N} = (\mathscr{G}_c^{3,N})_0 \oplus (\mathscr{G}_c^{3,N})_1 \oplus (\mathscr{G}_c^{3,N})_2.$$

Hence a cyclic 3-Grassmann algebra with cubic relations is the $\mathbb{Z}_3$-graded algebra if we associate degrees to its elements as it is explained above. It is worth mentioning here that we can make a comparison of $\mathbb{Z}_3$-graded structure of a cyclic 3-Grassmann algebra with cubic relations to three color charges of quarks (labelled by blue, green and red) in quark model of theoretical physics.

In order to make this analogy with the color charges of quarks more exact we will extend previously defined $\mathbb{Z}_3$-graded structure of a cyclic 3-Grassmann algebra to a 3-Grassmann algebra with involution. Let us remind that a 3-Grassmann algebra with involution is a unital associative algebra $\mathscr{G}_c^{\bar{3},N}$ generated by $\theta^A, \bar{\theta}^{\dot{B}}$, where $A = 1, 2, \ldots, N$, $\dot{B} = \dot{1}, \dot{2}, \ldots, \dot{N}$, which obey the relations

$$\theta^A \theta^B \theta^C \ \ = \ \ j\, \theta^B \theta^C \theta^A = j^2\, \theta^C \theta^A \theta^B,$$

$$\begin{aligned}\bar\theta^{\dot A}\bar\theta^{\dot B}\bar\theta^{\dot C} &= j\,\bar\theta^{\dot B}\bar\theta^{\dot C}\bar\theta^{\dot A} = j^2\,\bar\theta^{\dot C}\bar\theta^{\dot A}\bar\theta^{\dot B},\\ \theta^A\bar\theta^{\dot B} &= j^2\,\bar\theta^{\dot B}\theta^A.\end{aligned}$$

Now we extend the $\mathbb{Z}_3$-graded structure of a cyclic 3-Grassmann algebra to 3-Grassmann algebra with involution by defining degree of conjugate generators as $|\bar\theta^{\dot A}| = 2$. As before we define the degree of product of generators as the sum of degrees of its components. Now the subspace of elements of degree zero is spanned by the identity element $\mathbb{1}$, triple products of generators, triple products of conjugate generators and binary products of generators and conjugate generators. Hence we have for the subspace of degree zero

$$\mathbb{1},\ \theta^A\bar\theta^{\dot B},\ \theta^A\theta^B\theta^C,\ \bar\theta^{\dot A}\bar\theta^{\dot B}\bar\theta^{\dot C},\ \theta^A\theta^B\bar\theta^{\dot C}\bar\theta^{\dot D},\ \theta^A\theta^B\theta^C\bar\theta^{\dot A}\bar\theta^{\dot B}\bar\theta^{\dot C}.$$

Let us develop further the analogy between $\mathbb{Z}_3$-graded structure of cyclic 3-Grassmann algebra and color charges of quarks. We know that in the quark model of theory of strong interactions a single quark with color charge a attracts the anti-quark with the anti-color charge $-a$ (anti-blue, anti-green, anti-red), and the result of this attraction is the combination of quark-antiquark (meson) which has neutral color (white). In our algebra this corresponds to a product $\theta^A\bar\theta^{\dot B}$, and this suggests that we should draw an analogy between products of generators of degree zero and the combinations of quark-antiquark of neutral color charge. Analogously a combination of three quarks (different color charge) with neutral color charge and a combination of three antiquarks (different anti-color charges) with neutral color charge will give a baryon and anti-baryon. In our algebra this corresponds to triple products of generators $\theta^A\theta^B\theta^C$ and conjugate generators $\bar\theta^{\dot A}\bar\theta^{\dot B}\bar\theta^{\dot C}$. It should be pointed out that there has been conjectured an existence of "exotic" hadrons with tetraquarks combinations, and this would correspond to degree zero products $\theta^A\theta^B\bar\theta^{\dot C}\bar\theta^{\dot D}$ in our algebra.

A 3-Grassmann algebra extension of a Grassmann algebra with involution can be also endowed with a graded structure. A 3-Grassmann algebra extension with involution of a Grassmann algebra is a unital associative algebra generated by $\xi^i, \bar\xi^j, \theta^A, \bar\theta^{\dot B}$, where $i, j = 1, 2, \ldots, n$ and $A, \dot B = 1, 2, \ldots, N$, which satisfy the relations

$$\begin{aligned}\xi^i\xi^j &= -\xi^j\xi^i,\ \ \bar\xi^i\bar\xi^j = -\bar\xi^j\bar\xi^i,\ \ \xi^i\bar\xi^j = -\bar\xi^j\xi^i,\ \theta^A\bar\theta^{\dot B} = j^2\,\bar\theta^{\dot B}\theta^A,\\ \theta^A\theta^B\theta^C &= j\,\theta^B\theta^C\theta^A = j^2\,\theta^C\theta^A\theta^B,\ \ \bar\theta^{\dot A}\bar\theta^{\dot B}\bar\theta^{\dot C} = j\,\bar\theta^{\dot B}\bar\theta^{\dot C}\bar\theta^{\dot A} = j^2\,\bar\theta^{\dot C}\bar\theta^{\dot A}\bar\theta^{\dot B},\\ \xi^i\theta^A &= q\,\theta^A\xi^i,\ \xi^i\bar\theta^{\dot A} = p\,\bar\theta^{\dot A}\xi^i,\ \bar\xi^i\theta^A = p\,\theta^A\bar\xi^i,\ \bar\xi^i\bar\theta^{\dot A} = q\,\bar\theta^{\dot A}\bar\xi^i,\end{aligned}$$

where p,q is either j or j^2. Evidently in this case we need six different degrees to have a consistent graded structure which means that we should construct a $\mathbb{Z}_6$-graded structure for a 3-Grassmann extension of Grassmann algebra with involution. We define a $\mathbb{Z}_6$-graded structure of this algebra by assigning the degree zero to $\mathbb{1}$, the degree two to every generator θ^A, the degree four to every conjugate generator $\bar{\theta}^{\dot{A}}$, and the degree three to every generator ξ^i and conjugate generator $\bar{\xi}^i$. As usual the degree of a product of generators is defined as the sum of degrees of its factors. Then an element of degree zero of this algebra can be written in the form

$$
\begin{aligned}
F \;=\; & \mu\,\mathbb{1} + f_{A\dot{B}}(\xi,\bar{\xi})\,\theta^A\bar{\theta}^{\dot{B}} + g_{ABC}(\xi,\bar{\xi})\,\theta^A\theta^B\theta^C \\
& + h_{\dot{A}\dot{B}\dot{C}}(\xi,\bar{\xi})\,\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}} + l_{AB\dot{C}\dot{D}}(\xi,\bar{\xi})\,\theta^A\theta^B\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{D}},
\end{aligned}
$$

where $\mu \in \mathbb{C}$ and $f_{A\dot{B}}(\xi,\bar{\xi})$, $g_{ABC}(\xi,\bar{\xi})$, $h_{\dot{A}\dot{B}\dot{C}}(\xi,\bar{\xi})$, $l_{AB\dot{C}\dot{D}}(\xi,\bar{\xi})$ are even elements of the Grassmann subalgebra generated by $\xi^i,\bar{\xi}^j$.

3.2 Graded Differential Algebras

Roughly speaking if we endow a graded algebra with a differential d, where differential d is a linear mapping which maps an element of degree i to the element of degree $i+1$, satisfies the graded Leibniz rule and the equation $d^2=0$, then we get a notion of graded differential algebra. A theory of graded differential algebras originated from the theory of differential forms on a smooth manifold developed by French mathematicians E. Cartan and G. de Rham. In this section we will describe the algebra of differential forms on a smooth manifold, and here we give only a brief description of its structure. If M is a smooth finite dimensional manifold then the algebra of differential forms $\Omega(M)=\oplus_p\Omega^p(M)$ together with exterior differential d is a commutative graded differential algebra. If M,N are two smooth finite dimensional manifolds then a smooth mapping $\phi: M \longrightarrow N$ induces with the help of pull-back of differential forms the homomorphism of graded differential algebras $\phi^*: \Omega(N) \longrightarrow \Omega(M)$. The cohomologies of the commutative graded differential algebra $\Omega(M)$ is called *the de Rham cohomologies* of a manifold M, and they play an important role in differential topology of manifolds.

The Chevalley-Eilenberg cochain complex of V-valued cochains on a Lie algebra $\mathfrak{g}$, where V is a vector space of representation of $\mathfrak{g}$ can be viewed as a graded differential algebra. Given a Lie group G and its Lie algebra $\mathfrak{g}$ we can construct the graded vector space $C^n(\mathfrak{g})$ of $\mathbb{C}$-valued n-cochains, i.e. if $\omega \in C^n(\mathfrak{g})$ then ω is a skew-symmetric linear mapping $\omega : \mathfrak{g}\otimes\mathfrak{g}\otimes\ldots\otimes\mathfrak{g}$ (n times) $\longrightarrow \mathbb{C}$. If we equip this graded vector space with an associative product of cochains, we get the graded algebra. It is well known that the graded algebra of $\mathbb{C}$-valued n-cochains $C^n(\mathfrak{g})$ can be identified with the graded algebra $\wedge^n\mathfrak{g}^*$. The graded algebra $\wedge^n\mathfrak{g}^*$ can be identified with the graded algebra $\Omega^n_{\mathrm{inv}}(G)$ of left-invariant differential n-forms on a Lie group G. Then the exterior differential d induces the differential on the graded algebra $C(\mathfrak{g}) = \oplus_n C^n(\mathfrak{g})$, and we get the graded differential algebra of cochains.

3.2.1 Notion of graded differential algebra and its structure

Definition 3.2.1. *A graded differential algebra* is a graded associative unital algebra $\mathscr{A} = \oplus_{i\in\mathbb{Z}}\mathscr{A}^i$ over $\mathbb{C}$ together with a linear mapping $d : \mathscr{A} \longrightarrow \mathscr{A}$, where

i) $d : \mathscr{A}^i \longrightarrow \mathscr{A}^{i+1}$ for any integer $i \in \mathbb{Z}$, i.e. d is of degree 1,

ii) $d(uv) = d(u)\,v + (-1)^{|u|}u\,d(v)$ for any homogeneous $u \in \mathscr{A}$ and any $v \in \mathscr{A}$, i.e. d satisfies the graded Leibniz rule,

iii) $d^2u = 0$ for any $u \in \mathscr{A}$.

A linear mapping $d : \mathscr{A} \longrightarrow \mathscr{A}$ is called a differential of a graded differential algebra $\mathscr{A}$. The properties i) and ii) show that differential d is an antiderivation of a graded algebra $\mathscr{A}$.

Definition 3.2.2. Let $\mathscr{A},\mathscr{B}$ be two graded differential algebras with differentials correspondingly d,d'. A linear mapping $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *a homomorphism of graded differential algebras* if

i) $\phi : \mathscr{A}^i \longrightarrow \mathscr{B}^i$ for any integer $i \in \mathbb{Z}$,

ii) $\phi(uv) = \phi(u)\,\phi(v)$ for any $u,v \in \mathscr{A}$,

iii) $\phi\circ d = d'\circ\phi$.

Definition 3.2.3. A graded differential algebra $\mathscr{A}$ is referred to as *commutative graded differential* if it is commutative graded algebra, i.e. the graded commutator of any two homogeneous elements $u,v \in \mathscr{A}$ vanishes $[u,v] = uv - (-1)^{|u||v|}vu = 0$.

3.2.2 Algebra of differential forms of a manifold

In this paragraph we describe a very important graded differential algebra which arises on a smooth manifold M. Let $TM = \bigcup_{x\in M} T_xM$ be a tangent bundle and let $T^*M = \bigcup_{x\in M} T_x^*M$ be a cotangent bundle on a smooth n-dimensional manifold M. The smooth sections of TM are the vector fields $X : M \longrightarrow TM$, with local representation $X|_U = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$, where $U \subset M$ is a neighborhood of an arbitrary point $x \in M$ and $\left\{ \frac{\partial}{\partial x^i}|_x \right\}_{i=1}^{n}$ is a basis of a fiber T_xM. The smooth sections of a cotangent bundle T^*M are differential 1-forms (or simply 1-forms) and they can be locally written as $\alpha|_U = \sum_{i=1}^{n} \alpha_i dx^i$, where basis $\{dx^i\}_{i=1}^n$ is a dual basis to $\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^{n}$. Let us consider the exterior algebra bundle $\wedge^k T^*M$, whose fiber at $x \in M$ is the antisymmetric tensor product of degree k of vector spaces T_x^*M. Obviously $\wedge T^*M = \bigoplus_{k=0}^{n} \wedge^k T^*M$. We use $A^k(M)$ to denote the space of the smooth sections of the $\wedge^k T^*M$, i.e. $A^k(M) = \Gamma(M, \wedge^k T^*M)$. The elements of the space $A(M) = \bigoplus_{k=0}^{n} A^k(M)$ are called the differential forms of degree k or briefly k-forms. In particular, 0-forms are smooth functions, $A^0(M) = C^\infty(M)$. Wedge product of exterior forms can be extended to the space of differential forms $\wedge : A^k(M) \times A^l(M) \longrightarrow A^{k+l}(M)$ such that for the arbitrary $\alpha_1 \in A^k(M)$ and $\alpha_2 \in A^l(M)$

$$\alpha_1 \wedge \alpha_2 = (-1)^{kl} \alpha_2 \wedge \alpha_1.$$

The wedge product is associative and bilinear. There exists an unique mapping $d : A(M) \longrightarrow A(M)$, called exterior differential, which satisfies the following conditions:

1. $d : A^k(M) \longrightarrow A^{k+1}(M)$ is a linear mapping with a property $d^2 = 0$,
2. if f is a smooth function on a manifold M, then $df \in A^1(M)$ is a 1-form on M such that $(df)(X) = X(f)$ for a vector field X on M,
3. $d(\alpha_1 \wedge \alpha_2) = (d\alpha_1) \wedge \alpha_2 + (-1)^k \alpha_1 \wedge (d\alpha_2)$, where $\alpha_1 \in A^k(M)$ and $\alpha_2 \in A^l(M)$.

It is easy to see that the space of differential forms has a structure of a graded differential algebra with respect to wedge product and exterior differential of forms.

A differential k-form α can be uniquely expressed in local coordinates of a smooth manifold $x^1, x^2, \ldots, x^n$ by an expression

$$\alpha|_U = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \omega_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

If we use the notion of subset of the set of first positive n integers developed for description of the structure of a Grassmann algebra $I = (i_1, i_2, \ldots, i_k) \subset \mathcal{N}$, where$1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, and define

$$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k},$$

then locally a differential k-form can be written in the compact form

$$\alpha|_U = \sum_{I \subset \mathcal{N}} \omega_I \, dx^I,$$

where $\omega_I = \omega_{i_1 i_2 \ldots i_k}$, and the sum is taken over all subsets of the set $\mathcal{N}$ which contain k integers. The exterior differential d for a differential k-form $\alpha|_U$ written in local coordinates of a smooth manifold is given by

$$d(\alpha|_U) = \sum_{I \subset \mathcal{N}} d\omega_I \wedge \ dx^I,$$

where $d\omega_I$ is the exterior differential of a function. This formula can be used (locally) for an explicit proof of the peculiar property of the exterior differential $d^2 = 0$. Indeed we have

$$d^2(\alpha|_U) = d\,(\sum_{I \subset \mathcal{N}} d\omega_I \wedge \ dx^I) = \sum_{I \subset \mathcal{N}} (d^2\omega_I \wedge \ dx^I + \omega_I \, d^2 x^I).$$

For basic differential k-forms dx^I we have

$$d^2 x^I = \sum_{r=1}^{k} (-1)^{r-1} \, dx^{i_1} \wedge \ldots \wedge d^2 x^{i_r} \wedge \ldots \wedge dx^{i_k}.$$

Thus we only need to prove that $d^2 = 0$ in the case of functions (degree zero forms). But this can be proved in a very easy way

$$d^2 f = d(\sum_i \frac{\partial f}{\partial x^j} \, dx^j) = \sum_{i<j} (\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}) \, dx^i \wedge dx^j = 0,$$

where we use the commutativity property of second order derivatives

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}.$$

This property of exterior differential gives rise to a very important concept of cohomology. Let us remind that a differential k-form α is said to be closed if $d\alpha = 0$, and differential k-form α is said to be exact if there exists a differential $(k-1)$-form β such that $d\beta = \alpha$. The quotient of the subspace of closed k-forms by exact k-forms is called the k-dimensional cohomologies of a smooth manifold M. The cohomologies are very important in differential topology of manifolds.

Chapter 4

Graded q-Differential Algebras and Galois Extension

4.1 Graded q-Differential Algebra and Applications

An idea to generalize the concept of a differential module and to develop a corresponding algebraic structures, by assuming for a differential d a more general equation $d^N = 0, N \geq 2$ than the fundamental equation $d^2 = 0$ of a differential of graded differential algebra, seems to be very natural. Taking the equation $d^N = 0$ as a starting point one should choose a space where a calculus with $d^N = 0$ will be constructed. As a calculus with $d^N = 0$ may be considered as a generalization of $d^2 = 0$, and, taking into account that we have the exterior calculus of differential forms with exterior differential $d^2 = 0$ on a smooth manifold, one way to construct $d^N = 0$ is to take a smooth manifold and to consider objects on this manifold more general than the differentials forms [13]. In our approach we use notions of q-deformed structures such as graded q-Leibniz rule, graded q-commutator, graded inner q-derivation, where q is a primitive Nth root of unity [3, 4, 7, 8, 5, 10, 11].

A notion of graded q-differential algebra was introduced in [19] (see also in [18, 20, 21, 22, 23] and it may be viewed as a generalization of a graded differential algebra. Let us mention that a concept of graded q-differential algebra is closely related to the monoidal structure introduced in [25] for the category of N-complexes and it is proved in [23] that the monoids of the category of N-complexes can be identified as the graded q-differential algebras. It is well known that a connection and its curvature are basic elements of the theory of fiber bundles and they play an important role not only in a modern differential geometry but also in theoretical physics namely in a gauge field theory. A basic algebraic structure used in the theory of connections on modules is a graded differential algebra. A graded differential algebra is an algebraic model for the de Rham algebra of differential forms on a smooth manifold. Consequently considering a concept of graded q-differential algebra which is more general structure than a graded differential algebra we can develop a generalization of the theory of connections on modules [35]. One of the aims of this paper is to present and study algebraic structures based on the relation $d^N = 0$ and to generalize a concept of connection and its curvature applying a concept of graded q-differential algebra to the theory of connections on modules.

4.1.1 Calculus of q-numbers

A number q in a concept of graded q-differential is a primitive Nth root of unity, and a calculus of q-numbers is widely used in a theory of graded q-differential algebras. In this subsection we briefly remind several important q-analogs of numbers and their properties. It turns out that the well known notions such as factorial and binomial coefficient can be defined in the case of primitive Nth root of unity [26].

Let q be a primitive Nth root of unity. One can easily show that a primitive Nth root of unity has the property

$$1+q+q^2+\cdots+q^{N-1}=0, \tag{4.1.1}$$

which we used in the previous chapters in the particular case of a primitive cubic root of unity j. Let us denote $1+q+q^2+\cdots+q^{N-1}=S$. Multiplying both sides of this equality by q we obtain

$$q+q^2+\cdots+q^N+1=Sq.$$

Subtracting this equality from the initial equation $1+q+q^2+\cdots+q^{N-1}=S$ we get

$$0=Sq-S \Rightarrow S(1-q)=0 \ \Rightarrow S=0 \ \text{ as } \ 1-q\neq 0.$$

A number $[k]_q$, where k is an integer $k \geq 0$, is defined by the formula

$$[0]_q = 0, \ \ [k]_q = \frac{1-q^k}{1-q} = 1+q+q^2+\cdots+q^{k-1}.$$

Evidently if q is a primitive Nth root of unity then $[N]_q = 0$, which immediately follows from (4.1.1). One can define the q-analogs of factorials of integers by the formula

$$[0!]_q = 1,$$
$$[n!]_q = [1]_q[2]_q \cdots [n]_q.$$

If p,n are integers satisfying $0 \leq p \leq n$, $n \geq 1$ then the Gaussian q-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ p \end{array} \right]_q = \frac{[n!]_q}{[p!]_q[(n-p)!]_q}$$

It can be proved that the Gaussian q-binomial coefficients satisfy the recursion relation

$$\left[\begin{array}{c} n+1 \\ p \end{array} \right]_q = q^k \left[\begin{array}{c} n \\ p \end{array} \right]_q + \left[\begin{array}{c} n \\ p-1 \end{array} \right]_q$$

or

$$\left[\begin{array}{c} n+1 \\ p \end{array} \right]_q = \left[\begin{array}{c} n \\ p \end{array} \right]_q + q^{n+1-k} \left[\begin{array}{c} n \\ p-1 \end{array} \right]_q .$$

The notions of graded commutator and graded derivation of a graded algebra can be generalized within the framework of noncommutative geometry and the theory of quantum groups with the help of q-deformations. In general q may be any complex number different from 1 but for our purpose we need q to be a primitive Nth root of unity.

Definition 4.1.1. Let $\mathscr{A} = \oplus_{k \in \mathbb{Z}_N} \mathscr{A}^k$ be a graded associative unital algebra over $\mathbb{C}$ and $u \in \mathscr{A}^k, v \in \mathscr{A}^l$ be homogeneous elements. The *graded q-commutator* $[\ ,\ ]_q : \mathscr{A}^k \otimes \mathscr{A}^l \longrightarrow \mathscr{A}^{k+l}$ is defined by

$$[u,v]_q = uv - q^{kl} vu, \tag{4.1.2}$$

where q is a primitive Nth root of unity.

Definition 4.1.2. *A graded q-derivation of degree m* of a graded algebra $\mathscr{A}$ is a linear mapping $\delta : \mathscr{A} \longrightarrow \mathscr{A}$ of degree m with respect to graded structure of $\mathscr{A}$, i.e. $\delta : \mathscr{A}^i \longrightarrow \mathscr{A}^{i+m}$, which satisfies the graded q-Leibniz rule

$$\delta(uv) = \delta(u)\,v + q^{ml} u\,\delta(v), \tag{4.1.3}$$

where u is a homogeneous element of grading l, i.e. $u \in \mathscr{A}^l$.

In analogy with an inner graded derivation one defines an inner graded q-derivation of degree m of a graded algebra $\mathscr{A}$ associated to an element $v \in \mathscr{A}^m$ by the formula

$$\mathrm{ad}_v^q(u) = [v,u]_q = v\,u - q^{ml} u\,v, \tag{4.1.4}$$

where $u \in \mathscr{A}^l$. It is easy to verify that an inner graded q-derivation is a graded q-derivation.

4.1.2 N-complex and its cohomologies

In this paragraph we give an brief overview of N-structures such as N-differential module, N-cochain complex. The section is based heavily on [6, 21].

Let K be a commutative ring with a unit and E be a left K-module. A module E endowed with an endomorphism $d : E \longrightarrow E$ is referred to as a differential module with differential d if endomorphism d satisfies $d^2 = 0$. If K is a field, then the differential module E is said to be a differential vector space. It is easy to see that from the property $d^2 = 0$ of differential $d : E \longrightarrow E$ follows $\mathrm{Im}\,d \subset \mathrm{Ker}\,d \subset E$ and one can measure the non-exactness of the sequence $E \xrightarrow{d} E \xrightarrow{d} E$ by the quotient module $H(E) = \mathrm{Ker}\,d/\mathrm{Im}\,d$.

Let us consider two differential modules E_1 and E_2 with differentials

$$d_1 : E_1 \longrightarrow E_1, \quad d_2 : E_2 \longrightarrow E_2.$$

A homomorphism of modules $\phi : E_1 \longrightarrow E_2$ is said to be a homomorphism of differential modules E_1, E_2 if it satisfies $\phi \circ d_1 = d_2 \circ \phi$. It is easy to show that if ϕ is a homomorphism of differential modules then $\phi\,(\mathrm{Im}\,d_1) \subset \mathrm{Im}\,d_2$ and $\phi\,(\mathrm{Ker}\,d_1) \subset \mathrm{Ker}\,d_2$. It means that a homomorphism of differential modules ϕ induces the homomorphism of homologies $\phi_* : H(E_1) \longrightarrow H(E_2)$ of differential modules E_1, E_2.

For an exact sequence of differential modules

$$0 \to E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3 \to 0,$$

i.e. $\operatorname{Im}\phi = \operatorname{Ker}\psi, \operatorname{Ker}\phi = 0, \operatorname{Im}\psi = E_3$, there exists a homomorphism of homologies $\partial : H(E_3) \to H(E_1)$ such that $\operatorname{Ker}(\phi_*) = \operatorname{Im}(\partial)$, $\operatorname{Ker}(\psi_*) = \operatorname{Im}(\phi_*)$ and $\operatorname{Ker}(\partial) = \operatorname{Im}(\psi_*)$, where $\phi_* : H(E_1) \longrightarrow H(E_2)$ and $\psi_* : H(E_2) \longrightarrow H(E_3)$ are homomorphisms of homologies induced by $\phi : E_1 \longrightarrow E_2$ and $\psi : E_2 \longrightarrow E_3$. A module E is said to be a $\mathbb{Z}$-graded module if it is given as a direct sum of submodules $E^i \subset E$ labeled by integers $i \in \mathbb{Z}$, i.e. $E = \oplus_{i\in\mathbb{Z}} E^i$. We will call an element $u \in E^i$ a homogeneous element of $\mathbb{Z}$-graded module of degree i. A cochain complex is a $\mathbb{Z}$-graded differential module E with differential d of degree 1 with respect to a $\mathbb{Z}$-graded structure of E, which means $d : E^i \to E^{i+1}$.

A $\mathbb{Z}$-graded structure of a cochain complex $E \oplus_{i\in\mathbb{Z}} E^i$ induces the $\mathbb{Z}$-graded structure of its homology $H(E)$, i.e. $H(E) = \oplus_{i\in\mathbb{Z}} H^i(E)$, where $H^i(E) = \operatorname{Ker} d \cap E^i / \operatorname{Im} d \cap E^i$. A homology $H(E)$ is said to be a cohomology of a cochain complex E. It can be proved [21] that for an exact sequence of cochain complexes

$$0 \to E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3 \to 0$$

there exists a homomorphism of homologies $\partial : H^i(E_3) \longrightarrow H^{i+1}(E_1)$ such that the sequence

$$\ldots \xrightarrow{\partial} H^n(E_1) \xrightarrow{\phi_*} H^n(E_2) \xrightarrow{\psi_*} H^n(E_3) \xrightarrow{\partial} H^{n+1}(E_1) \to \ldots$$

is exact.

A cochain complex $E = \oplus_{i\in\mathbb{Z}} E^i$ with differential d is said to be a positive cochain complex if for every $i < 0$ a submodule E^i is trivial, i.e. $E^i = 0$. We construct positive cochain complex using a notion of a pre-cosimplicial module. Let us remind that a sequence of modules $(E^n)_{n\in\mathbb{N}} = (E^0, E^1, \ldots, E^n, \ldots)$ with homomorphisms $\phi_0, \phi_1, \ldots, \phi_{n+1}$ such that each ϕ_i determines a sequence

$$E^0 \xrightarrow{\phi_i} E^1 \xrightarrow{\phi_i} E^2 \ldots \xrightarrow{\phi_i} E^n \xrightarrow{\phi_i} E^{n+1} \xrightarrow{\phi_i} \ldots,$$

where $\phi_i : E^n \longrightarrow E^{n+1}$ is a homomorphism of modules, is said to be a pre-cosimplicial module if

$$\phi_j \circ \phi_i = \phi_i \circ \phi_{j-1},$$

where $i, j \in \{0, 1, \ldots, n+1\}$ and $i < j$. For a pre-cosimplitial module we construct a positive cochain complex E with differential d by setting $E = \oplus_{n\in\mathbb{N}} E^n$ and $d =$

$\sum_{i=0}^{n+1}(-1)^i\phi_i$. We shall call this positive cochain complex the pre-cosimplicial complex and its differential d the simplicial differential.

The relation $d^2=0$, which determines the structure of a differential module and allows to define the important characteristics of a non-exactness of a differential module, can be considered in a more general form $d^N=0$, where $N\geq 2$ is an positive integer. This generalization was proposed and considered in the framework of noncommutative geometry almost at the same time in the papers [25, 21, 23]. Now we describe a notion of an N-differential module, which can be viewed as a generalization of a notion of differential module, we also define the generalized homologies of an N-differential module.

Let N be an integer greater or equal to two and E be a left K-module. A left K-module E is said to be a N-differential module with N-differential d if $d: E\longrightarrow E$ is an endomorphism of E satisfying relation $d^N=0$. If K is a field an N-differential module E will be referred to as a N-differential vector space. Obviously N-differential module can be viewed as a generalization of a concept of differential module to any integer $N\geq 2$. For each integer $1\leq m\leq N-1$ we can define the submodules $Z_m(E)=\mathrm{Ker}\,(d^m)\subset E$ and $B_m(E)=\mathrm{Im}\,(d^{N-m})\subset E$. From the relation $d^N=0$ it follows that $B_m(E)\subset Z_m(E)$ and the quotient modules $H_m(E):=Z_m(E)/B_m(E)$ are called the generalized homology of the N-differential module E.

An N-differential module E with N-differential d is said to be a N-cochain complex of modules or simply a N-complex if E is a $\mathbb{Z}$-graded module $E=\oplus_{k\in\mathbb{Z}}E^k$ and its N-differential d has degree 1, i.e. $d: E^k\longrightarrow E^{k+1}$.

If E is an N-complex then its cohomologies $H_m(E)$ are $\mathbb{Z}$-graded modules, i.e. $H_m(E)=\oplus_{n\in\mathbb{Z}}H_m^n(E)$, where

$$H_m^n(E)=\mathrm{Ker}\,(d^m: E^n\longrightarrow E^{n+m})/d^{N-m}(E^{n+m-N}).$$

Let us consider an example of a positive N-complex [23]. Let $q\in\mathbb{C}$ be a Nth root of unity and $((E^n)_{n\in\mathbb{N}}; f_0,f_1,\ldots,f_{n+1})$ be a pre-cosimplicial module, where $f_0,f_1,\ldots,f_{n+1}$ are coface homomorphisms. This pre-cosimplicial module has a structure of the positive N-complexes if we construct the positively graded or $\mathbb{N}$-graded module $E=\oplus_{n\in\mathbb{N}}E^n$ with the endomorphism $d_m: E\longrightarrow E,\, i\leq n+1$ of degree 1 defining it by

$$d_m=\delta_{m+1}+q^{n-m+1}\sum_{r=0}^{m}(-1)^r f_{n-m+r+1}, \tag{4.1.5}$$

where

$$\delta_m = \sum_{i=0}^{n-m+1} q^i f_i. \tag{4.1.6}$$

For $m = 0, 1$ we get

$$d_0 = \sum_{i=0}^{n+1} q^i f_i, \quad d_1 = \sum_{i=0}^{n} q^i f_i - q^n f_{n+1}. \tag{4.1.7}$$

It is easy to see that $d_m : E^n \longrightarrow E^{n+1}$, $d_m^N = 0$ which means that d_m is the N-differential, and we get the N-complexes $(E, d_0), (E, d_1), \ldots, (E, d_{n+1})$.

4.1.3 Graded q-differential algebra

If we endow a cochain complex with an associative unital law of multiplication (by other words we endow a cochain complex with a structure of graded algebra) in such a way that a differential of a cochain complex satisfies the graded Leibniz rule with respect to this multiplication then we obtain a concept of a graded differential algebra. Analogously if we equip a N-complex with an associative unital law of multiplication in such a way that a N-differential of a N-complex satisfies the q-graded Leibniz rule, where q is a primitive Nth root of unity, then we obtain a generalization of a graded differential algebra which is called graded q-differential algebra. In this subsection we give a definition of a graded q-differential algebra, where q is a primitive Nth root of unity. The peculiar property of a graded q-differential algebra is $d^N = 0$, where $N \geq 2$. This equation is a generalization of the equation $d^2 = 0$ of a differential of a graded differential algebra, and this generalization was proposed by M. Kapranov in [25] in the framework of noncommutative geometry. It should be mentioned that the goal of approach proposed in [25] is to study the generalized cohomologies induced by a differential $d^N = 0$ and it does not have a notion of a graded q-differential algebra. A concept of graded q-differential algebra was introduced in [19] and later developed in the series of papers [3, 4, 5, 6, 8, 27]. The next aim of this subsection is to prove the lemma which gives an expansion of kth power of N-differential. We will use this lemma in the next subsection to prove the theorem which gives a way of construction a graded q-differential algebra if we have a unital associative graded algebra.

Let $\mathscr{A} = \oplus_{n\in\mathbb{Z}} \mathscr{A}^n$ be an associative unital graded $\mathbb{C}$-algebra. The subspace of elements of degree zero $\mathscr{A}^0 \subset \mathscr{A}$ is the subalgebra of $\mathscr{A}$. Any subspace $\mathscr{A}^k \subset \mathscr{A}$ of elements of degree $k \in \mathbb{Z}$ is the $(\mathscr{A}^0, \mathscr{A}^0)$-bimodule.

Definition 4.1.3. A graded q-differential algebra over $\mathbb{C}$ is an associative unital $\mathbb{Z}$-graded (or $\mathbb{Z}_N$-graded) $\mathbb{C}$-algebra $\mathscr{A} = \oplus_{k\in\mathbb{Z}}\mathscr{A}^k$ endowed with a linear mapping d of degree one such that the sequence

$$\ldots \xrightarrow{d} \mathscr{A}^{k-1} \xrightarrow{d} \mathscr{A}^k \xrightarrow{d} \mathscr{A}^{k+1} \xrightarrow{d} \ldots$$

is an N-complex with N-differential d satisfying the graded q-Leibniz rule

$$d(\omega\cdot\theta) = d\omega\cdot\theta + q^k\omega\cdot d\theta, \tag{4.1.8}$$

where $\omega \in \mathscr{A}^k, \theta \in \mathscr{A}$.

Obviously the subspace of elements of degree zero $\mathscr{A}^0 \subset \mathscr{A}$ is the subalgebra of a graded q-differential algebra $\mathscr{A}$. Clearly the triple $(\mathscr{A}^0, d, \mathscr{A}^1)$ is the first order differential calculus over the algebra $\mathscr{A}^0$. The triple $(\mathscr{A}^0, d, \mathscr{A}^1)$ will be referred to as an N-differential calculus over the algebra $\mathfrak{A}$.

Let us remind that a graded center of an associative unital graded $\mathbb{C}$-algebra $\mathscr{A} = \oplus_{k\in\mathbb{Z}}\mathscr{A}^k$ is the graded subspace $Z(\mathscr{A}) = \oplus_{k\in\mathbb{Z}}Z^k(\mathscr{A})$ of $\mathscr{A}$ generated by the homogeneous elements $v \in \mathscr{A}^k$, where $k \in \mathbb{Z}$, satisfying $vw = q^{km}wv$ for any $w \in \mathscr{A}^m, m \in \mathbb{Z}$. It is easy to verify that the graded center $Z(\mathscr{A})$ is the graded subalgebra of $\mathscr{A}$.

It is well known that given a homogeneous element $v \in \mathscr{A}^k$ of $\mathscr{A}$ we can associate to it the graded q-derivation of degree k by means of a graded q-commutator $[\ ,\ ]_q$ as follows

$$v \mapsto \mathrm{ad}^q_v(v)\,w = [v,w]_q = vw - q^{km}wv, \tag{4.1.9}$$

where $w \in \mathscr{A}^m$. Clearly $\mathrm{ad}^q_v : \mathscr{A}^m \longrightarrow \mathscr{A}^{m+k}$, and the graded q-derivation ad^q_v associated to a homogeneous element v is referred to as an inner graded q-derivation of degree k of $\mathscr{A}$. Let $v \in \mathscr{A}^1$ be a homogeneous element of degree one and $d_v = \mathrm{ad}^q_v : \mathscr{A}^m \longrightarrow \mathscr{A}^{m+1}$ be the inner graded q-derivation associated to v. The kth power of this inner graded q-derivation can be expanded as follows [33]:

Lemma 4.1.1. For any integer $k \geq 2$ it holds

$$d^k_v w = \sum_{i=0}^{k} p^{(k)}_i v^{k-i} w v^i, \tag{4.1.10}$$

where w is a homogeneous element of $\mathscr{A}$ and

$$p^{(k)}_i \quad = \quad (-1)^i q^{|w|_i}\frac{[k]_q!}{[i]_q![k-i]_q!} = (-1)^i q^{|w|_i}\begin{bmatrix} k \\ i \end{bmatrix}_q, \tag{4.1.11}$$

$$|w|_i \quad = \quad i|w| + \frac{i(i-1)}{2}. \tag{4.1.12}$$

4.1.4 Associative algebra approach to graded q-differential algebra

The aim of this section is to show that given a unital associative graded algebra one can endow it with a structure of graded q-differential algebra if there exists an element v of this algebra whose Nth power belong to the graded center of graded associative algebra. By other words we propose a method of how to construct a graded q-differential algebra with the help of a graded associative algebra, and this method opens a way to constructing a class of graded q-differential algebras. It is worth mentioning that this method can be applied to generalized Clifford algebras because each generator γ^i of a generalized Clifford algebra satisfies $(\gamma^i)^N = \mathbb{1}$, where $\mathbb{1}$ is the identity element of a generalized Clifford algebra. This method was proposed in [3] and later developed and applied to generalized Clifford algebra to construct a graded q-differential algebra of forms on a quantum plane [4],[33]

Theorem 4.1.2. Let $\mathscr{A}$ be a graded associative unital algebra $\mathscr{A} = \oplus_k \mathscr{A}^k$, and q be a primitive Nth root of unity. If there exists an element of grading one $v \in \mathscr{A}^1$ which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{A})$, where $\mathscr{Z}(\mathscr{A})$ is the graded center of $\mathscr{A}$, then the graded algebra $\mathscr{A}$ endowed with the inner graded q-derivation $d = \mathrm{ad}_v^q$ is a graded q-differential algebra (d is its N-differential).

Proof. Our proof of this theorem is based on the lemma proved at the end of previous subsection. Thus we will use the power expansion (4.1.10). Indeed making use of (4.1.10) we can express the Nth power of d as follows

$$d^N u = \sum_{i=0}^{N} (-1)^i p_i \left[\begin{array}{c} N \\ i \end{array} \right]_q v^{k-i} u v^i. \tag{4.1.13}$$

Taking into account that q is a primitive Nth root of unity we get

$$\left[\begin{array}{c} N \\ i \end{array} \right]_q = 0, \quad i \in \{1, 2, \ldots, N-1\}.$$

Hence the terms in (4.1.13), which are numbered with $i = 1, 2, \ldots, N-1$, vanish, and we are left with two terms

$$d^N u = v^N u + (-1)^N q^{\sigma(N)} u v^N.$$

As v^N is the element of grading zero (modulo N) of the graded center $\mathscr{Z}(\mathscr{A})$ we can rewrite the above formula as follows

$$d^N u = (1+(-1)^N q^{\sigma(N)})\, u\, v^N, \quad \sigma(N) = \frac{N(N-1)}{2}.$$

In order to show that the multiplier in the above formula vanish for any $N \geq 2$ we consider separately two cases for N to be an odd or even positive integer. If N is an odd positive integer then the multiplier $1+(-1)^N q^{\sigma(N)}$ vanish because in this case

$$1+(-1)^N q^{\sigma(N)} = 1-(q^N)^{\frac{N-1}{2}} = 0.$$

If N is an even positive integer then

$$1+(-1)^N q^{\sigma(N)} = 1+(q^{\frac{N}{2}})^{N-1} = 1+(-1)^{N-1} = 0.$$

Hence for any $N \geq 2$ we have $d^N = 0$, and this ends the proof of the theorem. $\square$

Let us fix a positive integer $m \in \{1,2,\ldots,N-1\}$ and split up the Nth power of N-differential d as follows $d^N = d^m \circ d^{N-m}$. We remind a concept of generalized cohomologies of N-complex. Then the equation $d^N = 0$ can be given in the form $d^N = d^m \circ d^{N-m} = 0$ and this leads to possible generalization of a concept of cohomology. For each integer $1 \leq m \leq N-1$ one can define the submodules

$$Z_m(E) = \{x \in E : d^m x = 0\} \subset E, \tag{4.1.14}$$
$$B_m(E) = \{x \in E : \exists y \in E,\, x = d^{N-m} y\} \subset E. \tag{4.1.15}$$

From $d^N = 0$ it follows that $B_m(E) \subset Z_m(E)$. For each $m \in \{1,2,\ldots,N-1\}$ the quotient module $H_m(E) := Z_m(E)/B_m(E)$ is said to be a *generalized homology of order m* of N-differential module E. The following lemma [20] gives a very useful criteria for the triviality of the generalized cohomologies of an N-differential module.

Lemma 4.1.3. Let E be an N-differential vector space over complex numbers $\mathbb{C}$, $N \geq 2$ be an integer and q be a complex number satisfying the conditions $[N]_q = 0$ and $[n]_q$ is invertible for any integer $1 \leq n \leq N-1$. If there is a vector space endomorphism $f : E \longrightarrow E$ satisfying $f \circ d - q\, d \circ f = \mathrm{Id}_E$ then the generalized cohomologies of an N-differential vector space E are trivial, i.e. for any integer $1 \leq n \leq N-1$ it holds $H_n(E) = 0$.

Based on this lemma we can prove that the generalized cohomologies of the cochain N-complex described in Theorem 4.1.2 are trivial. It is worth mentioning that the same argument is used in [20] to show that the generalized cohomologies of the N-differential module constructed by means of the algebra of $N \times N$-matrices $\mathrm{Mat}_N(\mathbb{C})$ are trivial.

Theorem 4.1.4. Let q be a primitive Nth root of unity, $\mathscr{A} = \oplus_{i \in \mathbb{Z}_N} \mathscr{A}^i$ be a graded associative unital algebra with an element $v \in \mathscr{A}^1$ satisfying $v^N = \lambda\, \mathbb{1}$, where $\lambda \neq 0$. Then the generalized cohomologies $H_n(\mathscr{A})$ of the cochain N-complex of Theorem 4.1.2

$$\mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \ldots \xrightarrow{d} \mathscr{A}^{N-1} \tag{4.1.16}$$

with N-differential $d = \mathrm{ad}_v^q$, induced by an element v, are trivial, i.e. for any $n \in \{1, 2, \ldots, N-1\}$ we have $H_n(\mathscr{A}) = 0$.

4.1.5 Matrix representation of graded q-differential algebra

Consider the algebra $\mathrm{Mat}_N(\mathbb{C})$ of $N \times N$ matrices. Evidently this algebra is unital associative algebra over $\mathbb{C}$ if we take a product of matrices as a multiplication law for this algebra. The unit matrix will be denoted by $\mathbb{1}$. The basis for vector space of $\mathrm{Mat}_N(\mathbb{C})$ is the matrices $\{E_l^k\}, k, l = 1, 2, \cdots, N$ where

$$(E_l^k)_j^i = \delta_l^k \delta_j^i. \tag{4.1.17}$$

We would like to point out that we use conventional notations for the matrices of $\mathrm{Mat}_N(\mathbb{C})$ which means that superscript i stands for ith row and subscript j stands for jth column. Obviously $\dim \mathrm{Mat}_N(\mathbb{C}) = N^2$. It follows from (4.1.17) that

$$E_l^k E_n^m = \delta_n^k E_l^m. \tag{4.1.18}$$

In the matrix E_l^k we have 1 on the intersection of the k-th column and l-th row and all other entries are zeros.

In what follows we will use two descriptions for the algebra $\mathrm{Mat}_N(\mathbb{C})$. The first is based on the basis E_l^k. In this case any matrix of $\mathrm{Mat}_N(\mathbb{C})$ can be written uniquely as a linear combination of E_l^k by the formula

$$u = \sum_{k,l=1}^{N} \lambda_l^k E_l^k,$$

where $u \in \mathrm{Mat}_N(\mathbb{C})$ and $\lambda_l^k \in \mathbb{C}$. The equation (4.1.18) gives the structural constants of the algebra $\mathrm{Mat}_N(\mathbb{C})$.

The algebra of square matrices of Nth order can be endowed with a graded structure. Indeed graded structure of $\mathrm{Mat}_N(\mathbb{C})$ can be defined by $|E_l^k| = k - l(\mathrm{mod}\, N)$. Then we have the following degrees $0, 1, \cdots, N-1$. Let us denote a subspace of elements of degree k by M^k then

$$\begin{aligned}\mathrm{Mat}_N(\mathbb{C}) &= M_N^0 \oplus M_N^1 \oplus \cdots \oplus M_N^{N-1} \\ &= \oplus_{k=0}^{N-1} M_N^k.\end{aligned}$$

Proposition 4.1.4. *The matrices $E_k^k, k = 1, 2, \cdots, N$ form the basis for the subspace M_N^0. This subspace of matrices of degree zero is the subalgebra of $Mat_N(\mathbb{C})$. Any subspace M^k of $Mat_N(\mathbb{C})$, where $k > 0$, is M^0-bimodule.*

Proof. Any element of M_N^0 can be written in the form

$$\lambda_1 E_1^1 + \lambda_2 E_2^2 + \cdots + \lambda_N E_N^N = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

This is diagonal matrix. If we add two diagonal matrices that will be diagonal matrix and if we multiply two diagonal matrices that is also diagonal matrix. It shows that $M_N^0 \subset \mathrm{Mat}_N(\mathbb{C})$ is subalgebra of $\mathrm{Mat}_N(\mathbb{C})$. It follows from the graded structure of $\mathrm{Mat}_N(\mathbb{C})$ that any M^k, $k > 0$ is a M^0-bimodule.

As we mentioned before the matrices E_l^k satisfy the relation

$$E_l^k E_n^m = \delta_n^k E_l^m \tag{4.1.19}$$

Let us denote the degrees of these matrices by a and b, i.e.

$$|E_l^k| = a \quad \text{and} \quad |E_n^m| = b.$$

According to the definition of graded structure of $\mathrm{Mat}_N(\mathbb{C})$ we have

$$|E_l^k| = a = k - l(\mathrm{mod}\, N) \;\; \text{and} \;\; |E_n^m| = b = m - n(\mathrm{mod}\, N).$$

Now we can calculate the degree of the product of two matrices with the help of (4.1.18), and we obtain

$$|E_l^k E_n^m| = |\delta_n^k E_l^m| = |E_l^m| = m - l (\operatorname{mod} N)$$

and

$$|E_l^k||E_n^m| = k - l + m - n (\operatorname{mod} N) = m - l\ (\operatorname{mod} N) (as\ k = n) = a + b (\operatorname{mod} N)$$

$\square$

Now our aim is to equip the algebra of square matrices of Nth order with a structure of a graded q-differential algebra. In order to construct a graded q-differential algebra we will use Theorem 4.1.2. Actually we can apply this theorem because the algebra of square matrices of order N can be considered as the generalized Clifford algebra, and a possible application of Theorem 4.1.2 to generalized Clifford algebras was mentioned in the previous subsection. It is worth to remind that a generalized Clifford algebra is a unital associative algebra over $\mathbb{C}$ generated by $\gamma_1, \gamma_2, \ldots, \gamma_n$ which obey the relations

$$\gamma_i \gamma_j = q\ \gamma_j \gamma_i \ \ (i < j), \quad (\gamma_i)^N = \mathbb{1},$$

where q is a primitive Nth root of unity. Particularly taking $N = 2$ we see that this definition is the definition of Clifford algebra. As it was mentioned before in the case of algebra of matrices $\mathrm{Mat}_N(\mathbb{C})$ we have two approaches to describe the structure of $\mathrm{Mat}_N(\mathbb{C})$. The first one is based on the basis E_l^k for the vector space of $\mathrm{Mat}_N(\mathbb{C})$, i.e any matrix of $\mathrm{Mat}_N(\mathbb{C})$ can be written uniquely as a linear combination of E_l^k by the formula

$$u = \sum_{k,l=1}^{N} \lambda_l^k E_l^k \ \text{ where } A \in \mathrm{Mat}_N(\mathbb{C}) \text{ and } \lambda_l^k \in \mathbb{C},$$

and the equation (4.1.18) gives the structural constants of $\mathrm{Mat}_N(\mathbb{C})$. The second approach uses the following two matrices

$$x = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & q^{-(N-1)} \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where q is N-th root of unity. It can be proved that the algebra of square matrices of Nth order is generated by these matrices, and x, ξ can be taken as generators of $\mathrm{Mat}_N(\mathbb{C})$. The generators x, y satisfy

$$x\xi = q\,\xi\,x,\; x^N = \xi^N = \mathbb{1}.$$

Thus the algebra of square matrices $\mathrm{Mat}_N(\mathbb{C})$ is the generalized Clifford algebra with two generators ($n = 2$). For any matrix $u \in \mathrm{Mat}_N(\mathbb{C})$ can be expressed as a power expansion of generators

$$\begin{aligned} u &= \sum_{i,j=0}^{N-1} \lambda_{ij} x^i \xi^j \\ &= \lambda_{00}\mathbb{1} + \lambda_{10}x + \lambda_{01}\xi + \lambda_{20}x^2 + \lambda_{11}x\xi + \\ &\quad \lambda_{02}\xi^2 + \cdots\cdots + \lambda_{N-1,N-1}x^{N-1}\xi^{N-1} \end{aligned}$$

Proposition 4.1.5. *The degree of E_l^k is defined by $|E_l^k| = k - l \ (mod\, N)$, the residue classes of module N by $\{\bar{0}, \bar{1}, \cdots, \overline{N-1}\}$ and $Mat_N(\mathbb{C})$ is the graded algebra with respect to the degree defined by $|E_l^k| = k - l \ (mod\, N)$.*

Proof. We only to show that for any two homogenous elements A, B it holds $|AB| = |A| + |B|$. Consider two homogenous elements A, B such that $|A| = a, |B| = b$ where

$$A = \sum_{k,l}^{N} \lambda_l^k E_l^k, \quad B = \sum_{m,n}^{N} \alpha_n^m E_n^m$$

Hence $|E_l^k| = a = k - l (\mathrm{mod}\, N)$ and $|E_n^m| = b = m - n(\mathrm{mod}\, N)$. Then the product of two elements

$$\begin{aligned} AB &= \sum_{k,l}^{N} \sum_{m,n}^{N} \lambda_l^k \alpha_n^m E_l^k E_n^m \\ &= \sum_{k,l}^{N} \sum_{m,n}^{N} \lambda_l^k \alpha_n^m \delta_n^k E_l^m \\ &= \sum_{n,l,m}^{N} \lambda_l^n \alpha_n^m E_l^m \end{aligned}$$

The degree of a product is $|AB| = |\lambda_l^n \alpha_n^m E_l^m| = |E_l^m| = m - l(\mathrm{mod}\, N)$ and the sum of degrees of two elements is $|A| + |B| = a + b = k - l + m - n(\mathrm{mod}\, N) = m - l(\mathrm{mod}\, N)$ as $k = n$. $\square$

Let us remind that the subspace of elements of degree k was denoted by M^k. Then we have

$$\mathrm{Mat}_N(\mathbb{C}) = M^0 \oplus M^1 \oplus \cdots \oplus M^{N-1} = \oplus_{k=0}^{N-1} M^k.$$

Hence the algebra of square matrices of Nth order is the graded unital associative algebra, and it is even the generalized Clifford algebra. We can apply our results obtained in the previous section to construct a graded q-differential algebra. Let us remind that for a differential calculus of first order over a unital associative algebra we used the terminology of function theory calling the elements of subalgebra of degree zero functions and the elements of degree one differential 1-form. We will use the same terminology here in the case of the algebra of matrices. Hence the elements of the subalgebra M^0 of matrices of degree zero we will called functions, and the elements of the subspace of degree will be called differential 1-forms. Given a function $f \in M^0$ we can write it either in terms of generators x, ξ of the algebra of matrices or in the basis E^k_l. Hence we have

1. $f(x) = \lambda_0 \mathbb{1} + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{N-1} x^{N-1}$, and this way of writing of element of degree zero shows that our terminology from a theory of functions is justified because this element has a form of a power function of the variable x;

2. $f(x) = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_N e_N$ where $e_1 = E^1_1,\ e_2 = E^2_2,\ \cdots, e_N = E^N_N$.

Let us define the linear mapping $d : \mathrm{Mat}_N(\mathbb{C}) \longrightarrow \mathrm{Mat}_N(\mathbb{C})$ by

$$du = [\xi, u]_q, \tag{4.1.20}$$

where graded q-commutator is defined by $[\xi, u]_q = \xi u - q^{|u|} u \xi$. Though it follows from the properties of inner graded q-derivation that d is the graded q-differential we can prove this directly.

Proposition 4.1.6. *The linear mapping d defined in (4.1.20) is the N-differential of the algebra of square matrices of order N, i.e. d is the graded q-differential of the algebra $Mat_N(\mathbb{C})$ satisfying $d^N = 0$.*

Proof. First of all we have

$$\begin{aligned} d(uv) &= \xi uv - q^{|uv|} uv\xi \\ &= \xi uv - q^{|u|+|v|} uv\xi. \end{aligned}$$

On the other hand

$$\begin{aligned} du\,v + q^{|u|}u\,dv & \\ &= (\xi u - q^{|u|})v + q^{|u|}u(\xi v - q^{|v|}v\xi) \\ &= \xi uv - q^{|u|}u\xi v + q^{|u|}u\xi v - q^{|u|+|v|}uv\xi \\ &= \xi uv - q^{|u|+|v|}uv\xi \end{aligned}$$

Consequently $d(uv) = du\ v + q^{|u|}u\ dv$ and it is proved that d is the graded q-differential. The second property $d^N = 0$ follows immediately from Theorem 4.1.2. □

Evidently d is the inner graded derivation of the algebra of matrices, and because $\xi^N = \mathbb{1}$ it follows immediately from Theorem 4.1.2 that the algebra of square matrices of order N equipped with the graded structure and inner graded derivation d is the graded q-differential algebra, where inner graded derivation d can be considered as N-differential, i.e. d satisfies $d^N = 0$. For function $f \in M_N^0$ we have

$$df = [\xi, f]_q = \xi f - f\xi.$$

It is obvious that if we restrict the N-differential d to degree zero and one subspaces then we get the first order differential calculus $d : M_N^0 \longrightarrow M_N^1$ or this can be written as the triple (M_N^0, d, M_N^1), where M_N^0 is the algebra of functions of x, and M_N^1 is the M_N^0-bimodule of differential 1-forms. It is easily verify that the differential d satisfies the graded q-Leibniz rule. It follows from the graded structure of the algebra of Nth order matrices that the basis for the subspace M_N^1 is $\{E_1^2, E_2^3. \cdots, E_{N-1}^N, E_N^1\}$. Let us denote the basis for the subspace of differential 1-forms by $\omega_k = E_k^{k+1}$, $(k = 1, 2, \cdots, N)$. Then any differential one form can be expressed in terms of basic differential forms in the form $\omega = \lambda_k \omega_k = \sum_{k=1}^N \lambda_k \omega_k$ where $\lambda_k \in \mathbb{C}$, and the dimension of the subspace of one forms is $\dim M_N^1(\mathbb{C}) = N$.

From Proposition (4.1.4) it follows that the subspace of differential 1-forms M_N^1 is the bimodule over the algebra of functions M_N^0. Thus we can multiply a differential one form ω by a function f either from the right or from the left. We see in Chapter 2 devoted to noncommutative first order differential calculus over a unital associative algebra that these two products are related to each other by means of endomorphism $\phi : M_N^0 \longrightarrow M_N^0$ of the algebra of functions M_N^0. Thus if $\omega = dx\, f(x)$ then

$$f(x)\,dx = dx\,\phi(f),\ \ \phi : M_N^0 \longrightarrow M_N^0.$$

In order to find this endomorphism in the case of algebra of matrices we write

$$\begin{aligned} f(x)dx &= dx\varphi(f) \\ (\lambda_0\mathbb{1}+\lambda_1 x+\lambda_2 x^2+\cdots+\lambda_{N-1}x^{N-1})(1-q)\xi x &= dx\ \varphi(f), \\ (1-q)(\lambda_0\xi x+\lambda_1 x\xi x+\lambda_2 x^2\xi x+\cdots+\lambda_{N-1}x^{N-1}\xi x) &= dx\ \varphi(f), \\ (1-q)(\lambda_0\xi x+\lambda_1 qx\xi xx+\lambda_2 xq\xi xx+\cdots\cdots) &= dx\ \varphi(f), \\ (1-q)(\lambda_0\xi x+\lambda_1 q\xi xx+\lambda_2 q^2\xi xxx+\cdots\cdots) &= (1-q)\ \xi\ x\ \varphi(f). \end{aligned}$$

From these relations it follows that

$$\varphi(f)=\lambda_0+\lambda_1 qx+\lambda_2 q^2x^2+\cdots+\lambda_{N-1}q^{N-1}x^{N-1},$$

Taking into consideration that any function is the power expansion $f(x)=\lambda_0\mathbb{1}+\lambda_1 x+\lambda_2 x^2+\cdots+\lambda_{N-1}x^{N-1}$ we conclude

$$\begin{aligned} \varphi(f(x)) =& \lambda_0+\lambda_1 qx+\lambda_2 q^2x^2+\cdots+\lambda_{N-1}q^{N-1}x^{N-1} \\ =& \lambda_0+\lambda_1(qx)+\lambda_2(qx)^2+\cdots+\lambda_{N-1}(qx)^{N-1} \\ =& f(qx), \end{aligned}$$

so we have

$$\varphi(f(x))=f(qx).$$

Proposition 4.1.7. *The mapping $\varphi: M^0(\mathbb{C})\longrightarrow M^0(\mathbb{C})$ defined by*

$$\varphi(f(x))=f(qx)$$

is the automorphism of the algebra $M^0(\mathbb{C})$.

Proof. Let us consider two functions

$$f(x)=\sum_{i=0}^{N-1}\lambda_i x^i, \qquad h(x)=\sum_{j=0}^{N-1}\mu_j x^j.$$

The linear mapping φ is defined by

$$\varphi(f)=\sum_{m=0}^{N-1}q^m\lambda_m x^m, \qquad \varphi(h)=\sum_{n=0}^{N-1}q^n\mu_n x^n.$$

It is obvious that the mapping φ is bijection. Hence we need to show that

$$\varphi(f(x)h(x)) = \varphi(f(x))\varphi(h(x)).$$

The product of two functions can be expressed as follows

$$f(x)h(x) = \sum_{i,j=0}^{N-1} \lambda_i \mu_j x^{i+j} = \sum_{k=0}^{N-1} \left(\sum_{i+j=k} \lambda_i \mu_j \right) x^k,$$

and

$$\varphi(f(x)h(x)) = \sum_{k=0}^{N-1} q^k \left(\sum_{i+j=k} \lambda_i \mu_j \right) x^k$$

On the otherhand

$$\begin{aligned}\varphi(f)\varphi(h) &= \sum_{l=0}^{N-1} \left(\sum_{i+j=l} q^i \lambda_i q^j \mu_j \right) x^l = \sum_{l=0}^{N-1} \left(\sum_{i+j=l} q^{i+j} \lambda_i \mu_j \right) x^l \\ &= \sum_{l=0}^{N-1} q^l \left(\sum_{i+j=l} \lambda_i \mu_j \right) x^l,\end{aligned}$$

which ends the proof. □

Making use of the definition of N-differential d we can find the differential of coordinate function x. Indeed we obtain

$$dx = \xi x - x\xi = \xi x - qx\xi = (1-q)\xi x. \tag{4.1.21}$$

Our next step in developing this first order noncommutative differential calculus of matrices is to find derivative with respect to x induced by the N-differential d. Applying the definition of partial derivatives of noncommutative first order differential calculus given in Chapter 2 we obtain in this case

$$df = dx \, \frac{\partial f}{\partial x}, \tag{4.1.22}$$

where f is a function. Now taking the powers of the variable x, i.e. $x^2, x^3, \cdots, x^k$, we can calculate the derivative of these functions with the help of definition of differential and formulae (4.1.21),(4.1.22). For instant taking $f(x) = x^2$ and making use of the definition of derivative we can write

$$dx^2 = dx \, \frac{\partial x^2}{\partial x}. \tag{4.1.23}$$

On the other hand we can find dx^2 by applying the definition of differential in terms of graded q-commutator. This gives

$$\begin{aligned} dx^2 &= \xi x^2 - x^2\xi \\ &= \xi x^2 - qx\xi x \\ &= \xi x^2 - q^2\xi x^2 \\ &= (1-q^2)\xi x^2 \end{aligned}$$

Now substituting the obtained expression for dx^2 into the equation (4.1.23) we get

$$(1-q^2)\xi x^2 = (1-q)\xi x\,\frac{\partial x^2}{\partial x},$$

Thus we find the derivative of square of x

$$\frac{\partial x^2}{\partial x} = (1+q)x.$$

Analogously taking $f = x^3$ and calculating the derivative with respect to x as it is shown above we find the general formula for the derivative of a power function

$$dx^3 = \xi x^3 - x^3\xi = \xi x^3 - qx\xi xx = \xi x^3 - q^2x^2\xi x = \xi x^3 - q^3\xi x^3 = (1-q^3)\xi x^3.$$

Now substituting the expression for dx^3 into the equation

$$dx^3 = dx\,\frac{\partial x^3}{\partial x},$$

we obtain

$$\frac{\partial x^3}{\partial x} = (1+q+q^2)x^2.$$

Similarly for any integer k we find

$$\frac{\partial x^k}{\partial x} = (1+q+q^2+\cdots+q^{k-1})\,x^{k-1} = [k]_q\,x^{k-1}.$$

Now we are describing higher order derivatives of q-differential algebra which is similar to the exterior calculus.

$$df^1 = [\xi_\lambda, f^1]$$

$$
\begin{aligned}
&=\xi_\lambda f^1 - f^1\xi_\lambda\\
&= (\lambda_1 E_N^1 + \lambda_1 E_1^2 + \cdots)E_1^1 - E_1^1\ (\lambda_1 E_N^1 + \lambda_1 E_1^2 + \cdots)\\
&=\lambda_1 E_N^1 - \lambda_2 E_1^2,
\end{aligned}
$$

$$
\begin{aligned}
df^2 =&[\xi_\lambda, f^2]\\
=&\xi_\lambda f^2 - f^2\xi_\lambda\\
=& (\lambda_1 E_N^1 + \lambda_1 E_1^2 + \cdots)E_2^2 - E_2^2\ (\lambda_1 E_N^1 + \lambda_1 E_1^2 + \cdots)\\
=&\lambda_2 E_1^2 - \lambda_3 E_2^3.
\end{aligned}
$$

Similarly if we go for further process we can write the following equation

$$
df^N = \lambda_N E_{N-1}^N - \lambda_1 E_N^1.
$$

After that we can make a conclusion of above formula

$$
\begin{aligned}
dx^\mu =&\lambda_{\bar\mu}\zeta^{\bar\mu} - \lambda_{\overline{\mu+1}}\zeta^{\overline{\mu+1}} \ \text{ when } \ 1 \le \mu \le N-1\\
dx^\mu =&\lambda_N\zeta^N - \lambda_1\zeta^1 \ \text{ when } \ \mu = N.
\end{aligned}
$$

Again

$$
\begin{aligned}
d^2 f^1 =&[\xi_\lambda, \lambda_1 E_N^1 - \lambda_2 E_1^2]\\
=&\xi_\lambda(\lambda_1 E_N^1 - \lambda_2 E_1^2) - q(\lambda_1 E_N^1 - \lambda_2 E_1^2)\xi_\lambda\\
=&\lambda_1\lambda_N E_{N-1}^1 - (1+q)\lambda_1\lambda_2 E_N^2 + q\lambda_2\lambda_3 E_1^3,
\end{aligned}
$$

$$
\begin{aligned}
d^2 f^2 =&[\xi_\lambda, \lambda_2 E_1^2 - \lambda_3 E_2^3]\\
=&\xi_\lambda(\lambda_2 E_1^2 - \lambda_3 E_2^3) - q(\lambda_2 E_1^2 - \lambda_3 E_2^3)\xi_\lambda\\
=&\lambda_1\lambda_2 E_N^2 - (1+q)\lambda_2\lambda_3 E_1^3 + q\lambda_3\lambda_4 E_2^4,
\end{aligned}
$$

$$
\begin{aligned}
d^2 f^3 =&d(df^3)\\
=&[\xi_\lambda, \lambda_3 E_2^3 - \lambda_4 E_3^4]
\end{aligned}
$$

$$
\begin{aligned}
&=\xi_\lambda(\lambda_3 E_2^3-\lambda_4 E_3^4)-q(\lambda_3 E_2^3-\lambda_4 E_3^4)\xi_\lambda\\
&=\lambda_2\lambda_3 E_3^1-(1+q)\lambda_3\lambda_4 E_2^4+q\lambda_4\lambda_5 E_3^5.
\end{aligned}
$$

Similarly if we go for further process we can write the following equation

$$
d^2 f^k=\lambda_{k-1}\lambda_k\xi^k-(1+q)\lambda_k\lambda_{k+1}\xi^{k+1}+q\lambda_{k+1}\lambda_{k+2}\xi^{k+2}.
$$

For third order derivative

$$
\begin{aligned}
d^3 f^1=&d(d^2 f^1)\\
=&[\xi_\lambda,\lambda_1\lambda_N E_{N-1}^1-(1+q)\lambda_1\lambda_2 E_N^2+q\lambda_2\lambda_3 E_1^3]\\
=&\lambda_1\lambda_{N-1}\lambda_N E_{N-2}^1-(1+q+q^2)\lambda_1\lambda_2\lambda_N E_{N-1}^2+\\
&+q(1+q+q^2)\lambda_1\lambda_2\lambda_3 E_N^3-q^3\lambda_2\lambda_3\lambda_4 E_1^4,
\end{aligned}
$$

$$
\begin{aligned}
d^3 f^2=&d(d^2 f^2)\\
=&[\xi_\lambda,\lambda_1\lambda_2 E_N^2-(1+q)\lambda_2\lambda_3 E_1^3+q\lambda_3\lambda_4 E_2^4]\\
=&\lambda_1\lambda_2\lambda_N E_{N-1}^2-(1+q+q^2)\lambda_1\lambda_2\lambda_3 E_N^3+\\
&+q(1+q+q^2)\lambda_2\lambda_3\lambda_4 E_1^4-q^3\lambda_3\lambda_4\lambda_5 E_2^5,
\end{aligned}
$$

$$
\begin{aligned}
d^3 f^3=&d(d^2 f^3)\\
=&[\xi_\lambda,\lambda_2\lambda_3 E_3^1-(1+q)\lambda_3\lambda_4 E_2^4+q\lambda_4\lambda_5 E_3^5]\\
=&\lambda_1\lambda_2\lambda_3 E_N^3-(1+q+q^2)\lambda_2\lambda_3\lambda_4 E_1^4+\\
&+q(1+q+q^2)\lambda_3\lambda_4\lambda_5 E_2^5-q^3\lambda_4\lambda_5\lambda_6 E_3^6.
\end{aligned}
$$

If these process continues then we will get the higher order derivative

$$
f^n(x)=f(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(-1)^k x^{\sigma^k})
$$

4.2 Graded q-Differential Algebra Approach to Galois Extension

Let us briefly remind a definition of noncommutative Galois extension [28, 31, 32, 37]. Suppose $\tilde{\mathscr{A}}$ is an associative unital $\mathbb{C}$-algebra, $\mathscr{A} \subset \tilde{\mathscr{A}}$ is its subalgebra, and there is an element $\tau \in \tilde{\mathscr{A}}$ which satisfies $\tau \notin \mathscr{A}$, $\tau^N = \mathbb{1}$, where $N \geq 2$ is an integer and $\mathbb{1}$ is the identity element of $\tilde{\mathscr{A}}$. A noncommutative Galois extension of $\mathscr{A}$ by means of τ is the smallest subalgebra $\mathscr{A}[\tau] \subset \tilde{\mathscr{A}}$ such that $\mathscr{A} \subset \mathscr{A}[\tau]$, and $\tau \in \mathscr{A}[\tau]$. It should be pointed out that a concept of noncommutative Galois extension can be applied not only to associative unital algebra with a binary multiplication law but as well as to the algebra with a ternary multiplication law, for instant to a ternary analog of Grassmann and Clifford algebra [2, 32, 37], and this approach can be used in particle physics to construct an elegant algebraic model for quarks.

A graded q-differential algebra can be viewed as a generalization of a notion of graded differential algebra if we use a more general equation $d^N = 0, N \geq 2$ than the basic equation $d^2 = 0$ of a graded differential algebra. This idea was proposed and developed within the framework of noncommutative geometry [25], where the author introduced the notions of N-complex, generalized cohomologies of N-complex and making use of an Nth primitive root of unity constructed an analog of an algebra of differential forms in n-dimensional space with exterior differential satisfying the relation $d^N = 0$. Later this idea was developed in the paper [19], where the authors introduced and studied a notion of graded q-differential algebra. It was shown [3, 4, 6, 7, 8, 9] that a notion of graded q-differential algebra can be applied in noncommutative geometry in order to construct a noncommutative generalization of differential forms and a concept of connection.

In this section we will study a special case of noncommutative Galois extension which is called a semi-commutative Galois extension. A noncommutative Galois extension is referred to as a semi-commutative Galois extension [37] if for any element $x \in \mathscr{A}$ there exists an element $x' \in \mathscr{A}$ such that $x\tau = \tau x'$. In this paper we show that a semi-commutative Galois extension can be endowed with a structure of a graded algebra if we assign degree zero to elements of subalgebra $\mathscr{A}$ and degree one to τ. This is the first step on a way to construct the graded q-differential algebra if we are given a semi-commutative Galois extension. The second step is the theorem which states that if there exists an element v of graded associative unital $\mathbb{C}$-algebra which satisfies the relation $v^N = \mathbb{1}$ then this algebra can be endowed with the structure of graded q-differential algebra. We can apply this theorem to a semi-commutative Galois extension because we have an element τ with the property $\tau^N = \mathbb{1}$, and this allows us to equip a semi-commutative Galois extension with the structure of graded q-differential algebra. Then we study the first and higher order noncommutative differential calculus induced by the N-differential of graded q-differential algebra. We introduce a derivative and differential with the help of first order noncommutative differential calculus developed in the papers [5, 15]. We also study the higher order noncommutative differential calculus and in this case we consider a differential d as an analog of exterior differential and the elements of higher order differential calculus as analogs of differential forms. Finally we apply our calculus to reduced quantum plane [17].

4.2.1 Semi-commutative Galois extension

In this section we remind a definition of noncommutative Galois extension, semi-commutative Galois extension, and show that given a semi-commutative Galois extension we can construct the graded q-differential algebra.

First of all we remind a notion of a noncommutative Galois extension [28, 31, 32, 37].

Definition 4.2.1. Let $\tilde{\mathscr{A}}$ be an associative unital $\mathbb{C}$-algebra and $\mathscr{A} \subset \tilde{\mathscr{A}}$ be its subalgebra. If there exist an element $\tau \in \tilde{\mathscr{A}}$ and an integer $N \geq 2$ such that

i) $\tau^N = \pm\mathbb{1}$,

ii) $\tau^k \notin \mathscr{A}$ for any integer $1 \leq k \leq N-1$,

then the smallest subalgebra $\mathscr{A}[\tau]$ of $\tilde{\mathscr{A}}$ which satisfies

iii) $\mathscr{A} \subset \mathscr{A}[\tau]$,

iv) $\tau \in \mathscr{A}[\tau]$,

is called the noncommutative Galois extension of $\mathscr{A}$ by means of τ.

In this section we will study a particular case of a noncommutative Galois extension which is called a semi-commutative Galois extension [37]. A noncommutative Galois extension is referred to as a semi-commutative Galois extension if for any element $x \in \mathscr{A}$ there exists an element $x' \in \mathscr{A}$ such that $x\tau = \tau x'$. We will give this definition in terms of left and right $\mathscr{A}$-modules generated by τ. Let

$\mathscr{A}_{\mathrm{l}}^{1}[\tau]$ and $\mathscr{A}_{\mathrm{r}}^{1}[\tau]$ be respectively the left and right $\mathscr{A}$-modules generated by τ. Obviously we have

$$\mathscr{A}_{\mathrm{l}}^{1}[\tau] \subset \mathscr{A}[\tau], \;\; \mathscr{A}_{\mathrm{r}}^{1}[\tau] \subset \mathscr{A}[\tau].$$

Definition 4.2.2. A noncommutative Galois extension $\mathscr{A}[\tau]$ is said to be a right (left) semi-commutative Galois extension if $\mathscr{A}_{\mathrm{r}}^{1}[\tau] \subset \mathscr{A}_{\mathrm{l}}^{1}[\tau]$ ($\mathscr{A}_{\mathrm{l}}^{1}[\tau] \subset \mathscr{A}_{\mathrm{r}}^{1}[\tau]$). If $\mathscr{A}_{\mathrm{r}}^{1}[\tau] \equiv \mathscr{A}_{\mathrm{l}}^{1}[\tau]$ then a noncommutative Galois extension will be referred to as a semi-commutative Galois extension, and in this case $\mathscr{A}^{1}[\tau] = \mathscr{A}_{\mathrm{r}}^{1}[\tau] = \mathscr{A}_{\mathrm{l}}^{1}[\tau]$ is the $\mathscr{A}$-bimodule.

It is well known that a bimodule over an associative unital algebra $\mathscr{A}$ freely generated by elements of its basis induces the endomorphism from an algebra $\mathscr{A}$ to the algebra of square matrices over $\mathscr{A}$. In the case of semi-commutative Galois extension we have only one generator τ and it induces the endomorphism of an algebra $\mathscr{A}$. Indeed let $\mathscr{A}[\tau]$ be a semi-commutative Galois extension and $\mathscr{A}^{1}[\tau]$ be its $\mathscr{A}$-bimodule generated by $[\tau]$. Any element of the right $\mathscr{A}$-module $\mathscr{A}_{\mathrm{r}}^{1}[\tau]$ can be written as τx, where $x \in \mathscr{A}$. On the other hand $\mathscr{A}[\tau]$ is a semi-commutative Galois extension which means $\mathscr{A}_{\mathrm{r}}^{1}[\tau] \equiv \mathscr{A}_{\mathrm{l}}^{1}[\tau]$, and hence each element $x\tau$ of the left $\mathscr{A}$-module can be expressed as $\tau\,\phi_\tau(x)$, where $\phi_\tau(x) \in \mathscr{A}$. It is easy to verify that the linear mapping $\phi : x \longrightarrow \phi_\tau(x)$ is the endomorphism of subalgebra $\mathscr{A}$, i.e. for any elements $x, y \in \mathfrak{A}$ we have $\phi_\tau(xy) = \phi_\tau(x)\phi_\tau(y)$. This endomorphism will play an important role in our differential calculus, and in what follows we will also use the notation $\phi_\tau(x) = x_\tau$. Thus

$$u\,\tau = \tau\,\phi_\tau(x), \quad u\,\tau = \tau\,u_\tau.$$

It is clear that

$$\phi_\tau^N = \mathrm{id}_{\mathscr{A}}, \;\; u_{\tau^N} = u,$$

because for any $u \in \mathscr{A}$ it holds $u\,\tau^N = \tau^N\,\phi^N(u)$ and taking into account that $\tau^N = \mathbb{1}$ we get $\phi_\tau^N(u) = u$.

Proposition 4.2.1. *Let $\mathscr{A}[\tau]$ be a semi-commutative Galois extension of $\mathscr{A}$ by means of τ, and $\mathscr{A}_l^k[\tau], \mathscr{A}_r^k[\tau]$ be respectively the left and right $\mathscr{A}$-modules generated by τ^k, where $k = 1, 2, \ldots, N-1$. Then $\mathscr{A}_l^k[\tau] \equiv \mathscr{A}_r^k[\tau] = \mathscr{A}^k[\tau]$ is the $\mathscr{A}$-bimodule, and*

$$\mathscr{A}[\tau] = \oplus_{k=0}^{N-1} \mathscr{A}^k[\tau] = \mathscr{A}^0[\tau] \oplus \mathscr{A}^1[\tau] \oplus \ldots \oplus \mathscr{A}^{N-1}[\tau],$$

where $\mathscr{A}^0[\tau] \equiv \mathscr{A}$.

Evidently the endomorphism of $\mathscr{A}$ induced by the $\mathscr{A}$-bimodule structure of $A^k[\tau]$ is ϕ^k, where $\phi : \mathscr{A} \longrightarrow \mathscr{A}$ is the endomorphism induced by the $\mathscr{A}$-bimodule $\mathscr{A}^1[\tau]$. We will also use the notation $\phi^k(x) = x_{\tau^k}$.

It follows from Proposition 4.2.1 that a semi-commutative Galois extension $\mathscr{A}[\tau]$ has a natural $\mathbb{Z}_N$-graded structure which can be defined as follows: we assign degree zero to each element of subalgebra $\mathscr{A}$, degree 1 to τ and extend this graded structure to a semi-commutative Galois extension $\mathscr{A}[\tau]$ by determining the degree of a product of two elements as the sum of degree of its factors. The degree of a homogeneous element of $\mathscr{A}[\tau]$ will be denoted by $|\ \ |$. Hence $|u| = 0$ for any $u \in \mathscr{A}$ and $|\tau| = 1$.

Now our aim is to show that given a noncommutative Galois extension we can construct a graded q-differential algebra, where q is a primitive Nth root of unity. First of all we remind some basic notions, structures and theorems of theory of graded q-differential algebras.

Let $\mathscr{A} = \oplus_{k \in \mathbb{Z}_N} \mathscr{A}^k = \mathscr{A}^0 \oplus \mathscr{A}^1 \oplus \ldots \oplus \mathscr{A}^{N-1}$ be a $\mathbb{Z}_N$-graded associative unital $\mathbb{C}$-algebra with identity element denoted by $\mathbb{1}$. Obviously the subspace $\mathscr{A}^0$ of elements of degree 0 is the subalgebra of a graded algebra $\mathscr{A}$. Every subspace $\mathscr{A}^k$ of homogeneous elements of degree $k \geq 0$ can be viewed as the $\mathscr{A}^0$-bimodule. The graded q-commutator of two homogeneous elements $u, v \in \mathscr{A}$ is defined by

$$[v, u]_q = v\,u - q^{|v||u|} u\,v.$$

A graded q-derivation of degree m of a graded algebra $\mathscr{A}$ is a linear mapping $d : \mathscr{A} \longrightarrow \mathscr{A}$ of degree m, i.e. $d : \mathscr{A}^i \longrightarrow \mathscr{A}^{i+m}$, which satisfies the graded q-Leibniz rule

$$d(u\,v) = d(u)\,v + q^{ml} u\,d(v), \tag{4.2.1}$$

where u is a homogeneous element of degree l, i.e. $u \in \mathscr{A}^l$. A graded q-derivation d of degree m is called an inner graded q-derivation of degree m induced by an element $v \in \mathscr{A}^m$ if

$$d(u) = [v, u]_q = v\,u - q^{ml} u\,v, \tag{4.2.2}$$

where $u \in \mathscr{A}^l$.

Now let q be a primitive Nth root of unity, for instant $q = e^{2\pi i/N}$. Then

$$q^N = 1, \;\; 1 + q + \ldots + q^{N-1} = 0.$$

We remind that a graded q-differential algebra is a graded associative unital algebra $\mathscr{A}$ endowed with a graded q-derivation d of degree one which satisfies $d^N = 0$.

In what follows a graded q-derivation d of a graded q-differential algebra $\mathscr{A}$ will be referred to as a graded N-differential. Thus a graded N-differential d of a graded q-differential algebra is a linear mapping of degree one which satisfies a graded q-Leibniz rule and $d^N = 0$. It is useful to remind that a graded differential algebra is a graded associative unital algebra equipped with a differential d which satisfies the graded Leibniz rule and $d^2 = 0$. Hence it is easy to see that a graded differential algebra is a particular case of a graded q-differential algebra when $N = 2, q = -1$, and in this sense we can consider a graded q-differential algebra as a generalization of a concept of graded differential algebra. Given a graded associative algebra $\mathscr{A}$ we can consider the vector space of inner graded q-derivations of degree one of this algebra and put the question: under what conditions an inner graded q-derivation of degree one is a graded N-differential? The following theorem gives answer to this question.

Theorem 4.2.2. *Let $\mathscr{A}$ be a $\mathbb{Z}_N$-graded associative unital $\mathbb{C}$-algebra and $d(u) = [v,u]_q$ be its inner graded q-derivation induced by an element $v \in \mathscr{A}^1$. The inner graded q-derivation d is the N-differential, i.e. it satisfies $d^N = 0$, if and only if $v^N = \pm\mathbb{1}$.*

Now our goal is apply this theorem to a semi-commutative Galois extension to construct a graded q-differential algebra with N-differential satisfying $d^N = 0$.

Proposition 4.2.3. *Let q be a primitive Nth root of unity. A semi-commutative Galois extension $\mathscr{A}[\tau]$, equipped with the $\mathbb{Z}_N$-graded structure described above and with the inner graded q-derivation $d = [\tau, \,]_q$ induced by τ, is the graded q-differential algebra, and d is its N-differential. For any element ξ of semi-commutative Galois extension $\mathscr{A}[\tau]$ written as a sum of elements of right $\mathscr{A}$-modules $\mathscr{A}^k[\tau]$*

$$\xi = \sum_{k=0}^{N-1} \tau^k u_k = \mathbb{1}\, u_0 + \tau u_1 + \tau^2 u_2 + \ldots + \tau^{N-1} u_{N-1}, \;\; u_k \in \mathscr{A},$$

it holds

$$d\xi = \sum_{k=0}^{N-1} \tau^{k+1}(u_k - q^k\,(u_k)_\tau), \tag{4.2.3}$$

where $u_k \longrightarrow (u_k)_\tau$ is the endomorphism of $\mathscr{A}$ induced by the bimodule structure of $\mathscr{A}^1[\tau]$.

A first order differential calculus is a triple $(\mathscr{A},\mathscr{M},d)$ where $\mathscr{A}$ is an associative unital algebra, $\mathscr{M}$ is an $\mathscr{A}$-bimodule, and d, which is called a differential of first order differential calculus, is a linear mapping $d:\mathscr{A}\longrightarrow\mathscr{M}$ satisfying the Leibniz rule $d(fh)=dfh+fdh$, where $f,h\in\mathscr{A}$. A first order differential calculus $(\mathscr{A},\mathscr{M},d)$ is referred to as a coordinate first order differential calculus if an algebra $\mathscr{A}$ is generated by the variables $x^1,x^2,\cdots,x^n$ which satisfy the commutation relations, and an $\mathscr{A}$-bimodule $\mathscr{M}$, considered as a right $\mathscr{A}$-module, is freely generated by $dx^1,dx^2,\cdots,dx^n$. It is worth to mention that a first order differential calculus was developed within the framework of noncommutative geometry, and an algebra $\mathscr{A}$ is usually considered as the algebra of functions of a noncommutative space, the generators $x^1,x^2,\cdots,x^n$ of this algebra are usually interpreted as coordinates of this noncommutative space, and an $\mathscr{A}$-bimodule $\mathscr{M}$ plays the role of space of differential forms of degree one. In this paper we will use the corresponding terminology in order to stress a relation with noncommutative geometry.

Let us consider a structure of coordinate first order differential calculus. This differential calculus induces the differentials $dx^1,dx^2,\cdots,dx^n$ of the generators $x^1,x^2,\cdots,x^n$. Evidently $dx^1,dx^2,\cdots,dx^n\in\mathscr{M}$. $\mathscr{M}$ is a bimodule, i.e. it has a structure of left $\mathscr{A}$-module and right $\mathscr{A}$-module. Hence for any two elements $f,h\in\mathscr{A}$ and $\omega\in\mathscr{M}$ it holds $(f\omega)h=f(\omega h)$. According to the definition of a coordinate first order differential calculus the right $\mathscr{A}$-module $\mathscr{M}$ is freely generated by the differentials of generators $dx^1,dx^2,\cdots,dx^n$. Thus for any $\omega\in\mathscr{M}$ we have $\omega=dx^1f_1+dx^2f_2+\cdots+dx^nf_n$ where $f_1,f_2,\cdots,f_n\in\mathscr{A}$. A coordinate first order differential calculus $(\mathscr{A},\mathscr{M},d)$ is an algebraic structure, which extends to noncommutative case the classical differential structure of a manifold. From the point of view of noncommutative geometry $\mathscr{A}$ can be viewed as an algebra of smooth functions, d is the exterior differential, and $\mathscr{M}$ is the bimodule of differential 1-forms. In order to stress this analogy we will call the elements of algebra $\mathscr{A}$ "functions" and the elements of $\mathscr{A}$-bimodule $\mathscr{M}$ "1-forms".

Because $\mathscr{M}$ is $\mathscr{A}$-bimodule, for any function $f\in\mathscr{A}$ we have two products $f\,dx^i$ and $dx^i f$. Since $dx^1,dx^2,\cdots,dx^n$ is the basis for the right $\mathscr{A}$-module $\mathscr{M}$, each element of $\mathscr{M}$ can be expressed as linear combination of $dx^1,dx^2,\cdots,dx^n$ multiplied by the functions from the right. Hence the element $fdx^i\in\mathscr{M}$ can be expressed in this way, i.e.

$$fdx^i=dx^1r^i_1(f)+dx^2r^i_2(f)+\ldots+dx^nr^i_n(f)=dx^jr^i_j(f), \qquad (4.2.4)$$

where $r^i_1(f),r^i_2(f),\ldots,r^i_n(f)\in\mathscr{A}$ are the functions. Making use of these functions we can compose the square matrix

$$R(f) = (r^i_j(f)) = \begin{pmatrix} r^1_1(f) & r^2_1(f) & \cdots & r^n_1(f) \\ \vdots & \vdots & \vdots & \vdots \\ r^1_n(f) & r^2_n(f) & \cdots & r^n_n(f) \end{pmatrix}.$$

It is worth to point out that an entry $r^i_j(f)$ stands on intersection of i-th column and j-th row. This square matrix determines the mapping $R : \mathscr{A} \longrightarrow \mathrm{Mat}_n(\mathscr{A})$ where $\mathrm{Mat}_n(\mathscr{A})$ is the algebra of n order square matrices over an algebra $\mathscr{A}$. It can be proved

Proposition 4.2.4. *$R : \mathscr{A} \longrightarrow Mat_n(\mathscr{A})$ is the homomorphism of algebras.*

Proof. We need to prove that for any $f, g \in \mathscr{A}$ it holds $R(fg) = R(f)R(g)$. Now according to the equation (4.2.4) we have

$$(fg)dx^i = dx^j r^i_j(fg).$$

The left hand side of the above relation can be written as

$$f(gdx^i) = f(dx^j r^i_j(g)) = (fdx^j)r^i_j(g) = (dx^k r^j_k(f))r^i_j(g) = dx^k(r^j_k(f)r^i_j(g)).$$

Now we can write

$$dx^j r^i_j(fg) = dx^k(r^j_k(f)r^i_j(g)) \Rightarrow r^i_k(fg) = r^j_k(f)r^i_j(g),$$

or in matrix form $R(fg) = R(f)R(g)$ which ends the proof. □

Let $(\mathscr{A}, \mathscr{M}, d)$ be a coordinate first order differential calculus such that right $\mathscr{A}$-module $\mathscr{M}$ is a finite freely generated by the differentials of coordinates $\{dx_i\}^n_{i=1}$. The mappings $\partial_k : \mathscr{A} \longrightarrow \mathscr{A}$, where $k \in \{1, 2, \ldots, n\}$, uniquely defined by

$$df = dx^k\, \partial_k(f), \quad f \in \mathscr{A}, \tag{4.2.5}$$

are called the right partial derivatives of a coordinate first order differential calculus.

Proposition 4.2.5. *If $(\mathscr{A}, \mathscr{M}, d)$ is a coordinate first order differential calculus over an algebra $\mathscr{A}$ such that $\mathscr{M}$ is a finite freely generated right $\mathscr{A}$-module with a basis $\{dx_i\}^n_{i=1}$ then the right partial derivatives $\partial_k : \mathscr{A} \longrightarrow \mathscr{A}$ of this differential calculus satisfy*

$$\partial_k(fg) = \partial_k(f)\, g + r(f)^i_k\, \partial_i(g). \tag{4.2.6}$$

The property (4.2.6) is called the twisted (with homomorphism R) Leibniz rule for partial derivatives.

If $\mathscr{A}$ is a graded q-differential algebra with differential d then evidently the subspace of elements of degree zero $\mathscr{A}^0$ is the subalgebra of $\mathscr{A}$, the subspace of elements of degree one $\mathscr{A}^1$ is the $\mathscr{A}^0$-bimodule, a differential $d : \mathscr{A}^0 \longrightarrow \mathscr{A}^1$ satisfies the Leibniz rule. Consequently we have the first order differential calculus $(\mathscr{A}^0, \mathscr{A}^1, d)$ of a graded q-differential algebra $\mathscr{A}$. If $\mathscr{A}^0$ is generated by some set of variables then we can construct a coordinate first order differential calculus with corresponding right partial derivatives.

4.2.2 First order calculus of Galois extension

It is shown in Subsection 4.1.3 that given a semi-commutative Galois extension we can construct a graded q-differential algebra. In the previous section we described the structure of a coordinate first order differential calculus over an associative unital algebra, and at the end of this section we also mentioned that the subspaces $\mathscr{A}^0, \mathscr{A}^1$ of a graded q-differential algebra together with differential d of this algebra can be viewed as a first order differential calculus over $\mathscr{A}^0$. In this section we apply an approach of first order differential calculus to a graded q-differential algebra of a semi-commutative Galois extension.

Let $\mathscr{A}[\tau]$ be a semi-commutative Galois extension of an algebra $\mathscr{A}$ by means of τ. Thus we have an algebra $\mathscr{A}$ and $\mathscr{A}$-bimodule $\mathscr{A}^1[\tau]$. Next we have the N-differential $d : \mathscr{A}[\tau] \longrightarrow \mathscr{A}[\tau]$ induced by τ, and if we restrict this N-differential to the subalgebra $\mathscr{A}$ of Galois extension $\mathscr{A}[\tau]$ then $d : \mathscr{A} \longrightarrow \mathscr{A}^1[\tau]$ satisfies the Leibniz rule. Consequently we have the first order differential calculus which can be written as the triple $(\mathscr{A}, d, \mathscr{A}^1[\tau])$. In order to describe the structure of this first order differential calculus we will need the vector space endomorphism $\Delta : \mathscr{A} \longrightarrow \mathscr{A}$ defined by

$$\Delta u = u - u_\tau, \quad u \in \mathscr{A}.$$

For any elements $u, v \in \mathscr{A}$ this endomorphism satisfies

$$\Delta(u\,v) = \Delta(u)\,v + u_\tau\,\Delta(v).$$

Let us assume that there exists an element $x \in \mathscr{A}$ such that the element $\Delta x \in \mathscr{A}$ is invertible, and the inverse element will be denoted by Δx^{-1}. The differential dx of an element x can be written in the form $dx = \tau\,\Delta x$ which clearly shows that dx has degree one, i.e. $dx \in \mathscr{A}^1[\tau]$, and hence dx can be used as generator for the right

$\mathscr{A}$-module $\mathscr{A}^1[\tau]$. Let us denote by $\phi_{dx} : u \longrightarrow \phi_{dx}(u) = u_{dx}$ the endomorphism of $\mathscr{A}$ induced by bimodule structure of $\mathscr{A}^1[\tau]$ in the basis dx. Then

$$u_{dx} = \Delta x^{-1}\, u_\tau\, \Delta x = \mathrm{Ad}_{\Delta x}\, u_\tau. \tag{4.2.7}$$

Definition 4.2.6. For any element $u \in \mathscr{A}$ we define the right derivative $\frac{du}{dx} \in \mathscr{A}$ (with respect to x) by the formula

$$du = dx\,\frac{du}{dx}. \tag{4.2.8}$$

Analogously one can define the left derivative with respect to x by means of the left $\mathscr{A}$-module structure of $\mathscr{A}^1[\tau]$. Further we will only use the right derivative which will be referred to as the derivative and often will be denoted by u'_x. Thus we have the linear mapping

$$\frac{d}{dx} : \mathscr{A} \longrightarrow \mathscr{A}, \quad \frac{d}{dx} : u \mapsto u'_x.$$

Proposition 4.2.7. *For any element $u \in \mathscr{A}$ we have*

$$\frac{du}{dx} = \Delta x^{-1}\, \Delta u. \tag{4.2.9}$$

The derivative (4.2.8) satisfies the twisted Leibniz rule, i.e. for any two elements $u, v \in \mathscr{A}$ it holds

$$\frac{d}{dx}(uv) = \frac{du}{dx}\,v + \phi_{dx}(u)\,\frac{dv}{dx} = \frac{du}{dx}\,v + Ad_{\Delta x}\,u_\tau\,\frac{dv}{dx}.$$

We have constructed the first order differential calculus with one variable x, and it is natural to study a transformation rule of the derivative of this calculus if we choose another variable. From the point of view of differential geometry we will study a change of coordinate in one dimensional space. Let $y \in \mathscr{A}$ be an element of $\mathscr{A}$ such that $\Delta y = y - y_\tau$ is invertible.

Proposition 4.2.8. *Let x, y be elements of $\mathscr{A}$ such that $\Delta x, \Delta y$ are invertible elements of $\mathscr{A}$. Then*

$$dy = dx\, y'_x,\ \ \frac{d}{dx} = y'_x\,\frac{d}{dy},\ \ dx = dy\, x'_y,\ \ \frac{d}{dy} = x'_y\,\frac{d}{dx},$$

where $x'_y = (y'_x)^{-1}$.

Indeed we have $dy = \tau\Delta y$, $dx = \tau\Delta x$. Hence $\tau = dx\Delta x^{-1}$ and

$$dy = dx\,(\Delta x^{-1}\Delta y) = dx\,y'_x.$$

If u is any element of $\mathscr{A}$ the for the derivatives we have

$$\frac{du}{dx} = \Delta x^{-1}\,\Delta u = (\Delta x^{-1}\,\Delta y)\,(\Delta y^{-1}\,\Delta u) = y'_x\,\frac{du}{dy}.$$

As an example of the structure of graded q-differential algebra induced by d_τ on a semi-commutative Galois extension we can consider the quaternion algebra $\mathbb{H}$. The quaternion algebra $\mathbb{H}$ is associative unital algebra generated over $\mathbb{R}$ by i, j, k which are subjected to the relations

$$i^2 = j^2 = k^2 = -\mathbb{1},\; i\,j = -j\,i = k,\; j\,k = -k\,j = i,\; k\,i = -i\,k = j,$$

where $\mathbb{1}$ is the unity element of $\mathbb{H}$. Given a quaternion

$$\mathfrak{q} = a_0\,\mathbb{1} + a_1\,i + a_2\,j + a_3\,k$$

we can write it in the form $\mathfrak{q} = (a_0\,\mathbb{1} + a_2\,j) + i\,(a_1 + a_3\,j)$. Hence if we consider the coefficients of the previous expression $z_0 = a_0\,\mathbb{1} + a_2\,j, z_1 = a_1 + a_3\,j$ as complex numbers then $\mathfrak{q} = z_0\,\mathbb{1} + i\,z_1$ which clearly shows that the quaternion algebra $\mathbb{H}$ can be viewed as the semi-commutative Galois extension $\mathbb{C}[i]$. Evidently in this case we have $N = 2, q = -1$, and $\mathbb{Z}_2$-graded structure defined by $|\mathbb{1}| = 0, |i| = 1$. Hence we can use the terminology of superalgebras. It is easy to see that the subspace of odd elements (degree 1) can be considered as the bimodule over the subalgebra of even elements $a\,\mathbb{1} + b\,j$ and this bimodule induces the endomorphism $\phi : \mathbb{C} \longrightarrow \mathbb{C}$, where $\phi(z) = \bar{z}$. Let d be the differential of degree one (odd degree operator) induced by i. Then making use of 4.2.3 for any quaternion $\mathfrak{q}$ we have

$$d\mathfrak{q} = d(z_0\,\mathbb{1} + i\,z_1) = -(\bar{z}_1 + z_1)\,\mathbb{1}.$$

Obviously $d^2\mathfrak{q} = 0$.

4.2.3 Higher order calculus of Galois Extension

Our aim in this section is to develop a higher order differential calculus of a semi-commutative Galois extension $\mathscr{A}[\tau]$. This higher order differential calculus is induced by the graded q-differential algebra structure. In subsection 4.1.3 it is mentioned that a graded q-differential algebra can be viewed as a generalization of a concept of graded differential algebra if we take $N=2, q=-1$. It is well known that one of the most important realizations of graded differential algebra is the algebra of differential forms on a smooth manifold. Hence we can consider the elements of the graded q-differential algebra constructed by means of a semi-commutative Galois extension $\mathscr{A}[\tau]$ and expressed in terms of differential dx as noncommutative analogs of differential forms with exterior differential d which satisfies $d^N=0$. In order to stress this analogy we will consider an element $x \in \mathscr{A}$ as analog of coordinate, the elements of degree zero as analogs of functions, elements of degree k as analogs of k-forms, and we will use the corresponding terminology. It should be pointed out that because of the equation $d^N=0$ there are higher order differentials $dx, d^2x, \ldots, d^{N-1}x$ in this algebra of differential forms.

Before we describe the structure of higher order differentials forms it is useful to introduce the polynomials $P_k(x), Q_k(x)$, where $k=1,2,\ldots,N$. Let us remind that $\Delta x = x - x_\tau \in \mathscr{A}$. Applying the endomorphism τ we can generate the sequence of elements

$$\Delta x_\tau = x_\tau - x_{\tau^2}, \Delta x_{\tau^2} = x_{\tau^2} - x_{\tau^3}, \ldots, \Delta x_{\tau^{N-1}} = x_{\tau^{N-1}} - x.$$

Obviously each element of this sequence is invertible. Now we define the sequence of polynomials $Q_1(x), Q_2(x), \ldots, Q_N(x)$, where

$$Q_k(x) = \Delta x_{\tau^{k-1}} \Delta x_{\tau^{k-2}} \ldots \Delta x_\tau \Delta x.$$

These polynomials can be defined by means of the recurrent relation

$$Q_{k+1}(x) = (Q_k(x))_\tau \Delta x.$$

It should be mentioned that $Q_k(x)$ is the invertible element and

$$(Q_k(x))^{-1} = \Delta x^{-1} \Delta x_\tau^{-1} \ldots \Delta x_{\tau^{k-1}}^{-1}.$$

We define the sequence of elements $P_1(x), P_2(x), \ldots, P_N(x) \in \mathscr{A}$ by the recurrent formula

$$P_{k+1}(x) = P_k(x) - q^k\,(P_k(x))_\tau, \quad k=1,2,\ldots,N-1,$$

and $P_1(x) = \Delta x$. Clearly $P_1(x) = Q_(x)$ and for the $k=2,3$ a straightforward calculation gives

$$\begin{aligned}
P_2(x) &= x - (1+q)x_\tau + q\,x_{\tau^2}, \\
P_3(x) &= x - (1+q+q^2)x_\tau + (q+q^2+q^3)x_{\tau^2} - q^3 x_{\tau^3}.
\end{aligned}$$

Proposition 4.2.9. *If q is a primitive Nth root of unity then there are the identities*

$$P_{N-1}(x)+(P_{N-1}(x))_\tau+\ldots+(P_{N-1}(x))_{\tau^{N-1}}\equiv 0,\quad P_N(x)\equiv 0.$$

Now we will describe the structure of higher order differential forms. It follows from the previous section that any 1-form ω, i.e. an element of $\mathscr{A}^1[\tau]$, can be written in the form $\omega = dx\,u$, where $u\in\mathscr{A}$. Evidently $d:\mathscr{A}\longrightarrow\mathscr{A}^1[\tau], d\omega = dx\,u'_x$. The elements of $\mathscr{A}^2[\tau]$ will be referred to as 2-forms. In this case there are two choices for a basis for the right $\mathscr{A}$-module $\mathscr{A}^2[\tau]$. We can take either τ^2 or $(dx)^2$ as a basis for $\mathscr{A}^2[\tau]$. Indeed we have

$$(dx)^2=\tau^2\,Q_2(x).$$

It is worth mentioning that the second order differential d^2x can be used as the basis for $\mathscr{A}^2[\tau]$ only in the case when $P_2(x)$ is invertible. Indeed we have

$$d^2x=\tau^2\,P_2(x),\quad d^2x=(dx)^2\,Q_2^{-1}(x)P_2(x).$$

If we choose $(dx)^2$ as the basis for the module of 2-forms $\mathscr{A}^2[\tau]$ then any 2-form ω can be written as $\omega=(dx)^2\,u$, where $u\in\mathscr{A}$. Now the differential of any 1-form $\omega = dx\,u$, where $u\in\mathscr{A}$, can be expressed as follows

$$d\omega=(dx)^2\left(q\,u'_x+Q_2^{-1}(x)P_2(x)\,u\right). \tag{4.2.10}$$

It should be pointed out that the second factor of the right-hand side of the above formula resembles a covariant derivative in classical differential geometry. Hence we can introduce the linear operator $D:\mathscr{A}\longrightarrow\mathscr{A}$ by the formula

$$Du=q\,u'_x+Q_2^{-1}(x)P_2(x)\,u,\quad u\in\mathscr{A}. \tag{4.2.11}$$

If $\omega=dv, v\in\mathscr{A}$, i.e. ω is an exact form, then

$$d\omega=d^2v=(dx)^2\,Dv'_x=(dx)^2\left(q\,v''_x+Q_2^{-1}(x)P_2(x)\,v'_x\right).$$

If we consider the simplest case $N=2, q=-1$ then

$$d^2v=0,\;\;P_2(x)\equiv 0,\;\;(dx)^2\neq 0,$$

and from the above formula it follows that $v''_x=0$.

Proposition 4.2.10. *Let $\mathscr{A}[\tau]$ be a semi-commutative Galois extension of algebra $\mathscr{A}$ by means of τ, which satisfies $\tau^2 = \mathbb{1}$, and d be the differential of the graded differential algebra induced by an element τ as it is shown in Proposition 4.2.3. Let $x \in \mathscr{A}$ be an element such that Δx is invertible. Then for any element $u \in \mathscr{A}$ it holds $u''_x = 0$, where u'_x is the derivative (4.2.8) induced by d. Hence any element of an algebra $\mathscr{A}$ is linear with respect to x.*

The quaternions considered as the noncommutative Galois extension of complex numbers provides a simple example for the above proposition. Indeed in this case $\tau = i, \mathscr{A} \equiv \mathbb{C}$, where the imaginary unit is identified with j, $(a\mathbb{1}+b\,j)_\tau = a\mathbb{1}-b\,j$. Hence we can choose $x = a\mathbb{1}+b\,j$ iff $b \neq 0$. Indeed in this case $\Delta x = x - x_\tau = a\mathbb{1}+b\,j - a\mathbb{1}+b\,j = 2b\,j$, and Δx is invertible iff $b \neq 0$. Now any $z = c\,\mathbb{1}+d\,j \in \mathscr{A}$ can be uniquely written in the form $z = \tilde{c}\,\mathbb{1}+\tilde{d}\,x$ iff

$$\begin{vmatrix} 1 & a \\ 0 & b \end{vmatrix} = b \neq 0.$$

Thus any $z \in \mathscr{A}$ is linear with respect to x.

Now we will describe the structure of module of k-forms $\mathscr{A}^k[\tau]$. We choose $(dx)^k$ as the basis for the right $\mathscr{A}$-module $\mathscr{A}^k[\tau]$, then any k-form ω can be written $\omega = (dx)^k u,\ u \in \mathscr{A}$. We have the following relations

$$(dx)^k = \tau^k\,Q_k(x), \quad d^k x = \tau^k\,P_k(x).$$

In order to get a formula for the exterior differential of a k-form ω we need the polynomials $\Phi_1(x), \Phi_2(x), \ldots, \Phi_{N-1}(x)$ which can be defined by the recurrent relation

$$\Phi_{k+1}(x) = \mathrm{Ad}_{\Delta x}(\Phi_k) + q^{k-1}\Phi_1(x), \quad k = 1,2,\ldots,N-1, \tag{4.2.12}$$

where $\Phi_1(x) = Q_2^{-1}(x)P_2(x)$. These polynomials satisfy the relations $d(dx)^k = (dx)^{k+1}\Phi_k(x)$ and given a k-form $\omega = (dx)^k u,\ u \in \mathscr{A}$ we find its exterior differential as

$$d\omega = (dx)^{k+1}\left(q^k\,u'_x + \Phi_k(x)\,u\right) = (dx)^{k+1}\,D^{(k)}u.$$

The linear operator $D^{(k)} : \mathscr{A} \longrightarrow \mathscr{A}, k = 1,2,\ldots,N-1$ introduced in the previous formula has the form

$$D^{(k)}u = q^k\,u'_x + \Phi_k(x)\,u, \tag{4.2.13}$$

and, as it was mentioned before, this operator resembles a covariant derivative of classical differential geometry. It is easy to see that the operator (4.2.11) is the particular case of (4.2.13), i.e. $D^{(1)} \equiv D$.

4.2.4 Galois extension approach to reduced quantum plane

In this section we show that a reduced quantum plane can be considered as a semi-commutative Galois extension. We study a first order and higher order differential calculus of a semi-commutative Galois extension in the particular case of a reduced quantum plane.

Let x,y be two variables which obey the commutation relation

$$x\,y = q\,y\,x, \tag{4.2.14}$$

where $q \neq 0,1$ is a complex number. These two variables generate the algebra of polynomials over the complex numbers. This algebra is an associative algebra of polynomials over $\mathbb{C}$ and the identity element of this algebra will be denoted by $\mathbb{1}$. In noncommutative geometry and theoretical physics a polynomial of this algebra is interpreted as a function of a quantum plane with two noncommuting coordinate functions x,y and the algebra of polynomials is interpreted as the algebra of (polynomial) functions of a quantum plane. If we fix an integer $N \geq 2$ and impose the additional condition

$$x^N = y^N = \mathbb{1}, \tag{4.2.15}$$

then a quantum plane is referred to as a reduced quantum plane and this polynomial algebra will be denoted by $\mathscr{A}_q[x,y]$.

Let us mention that from an algebraic point of view an algebra of functions on a reduced quantum plane may be identified with the generalized Clifford algebra $\mathfrak{C}_2^N$ with two generators x,y. Indeed a generalized Clifford algebra is an associative unital algebra generated by variables $x_1,x_2,\ldots,x_p$ obeying the relations $x_i x_j = q^{\mathrm{sg}(j-i)} x_j x_i$, $x_i^N = 1$, where sg is the sign function.

It is well known that the generalized Clifford algebras have matrix representations, and, in the particular case of the algebra $\mathscr{A}_q[x,y]$, the generators of this algebra x,y can be identified with the square matrices of order N

$$x \quad = \quad \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & q^{-1} & 0 & \ldots & 0 & 0 \\ 0 & 0 & q^{-2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & q^{-(N-2)} & 0 \\ 0 & 0 & 0 & \ldots & 0 & q^{-(N-1)} \end{pmatrix}, \tag{4.2.16}$$

$$
y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{4.2.17}
$$

where q is a primitive Nth root of unity. As the matrices (4.2.17) generate the algebra $\mathrm{Mat}_N(\mathbb{C})$ of square matrices of order N we can identify the algebra of functions on a reduced quantum plane with the algebra of matrices $\mathrm{Mat}_N(\mathbb{C})$.

The set of monomials $B = \{\mathbf{1}, y, x, x^2, yx, y^2, \dots, y^k x^l, \dots, y^{N-1}x^{N-1}\}$ can be taken as the basis for the vector space of the algebra $\mathscr{A}_q[x,y]$. We can endow this vector space with an $\mathbb{Z}_N$-graded structure if we assign degree zero to the identity element $\mathbb{1}$ and variable x and we assign degree one to the variable y. As usual we define the degree of a product of two variables x, y as the sum of degrees of factors. Then a polynomial

$$
w = \sum_{l=0}^{N-1} \beta_l y^k x^l, \quad \beta_l \in \mathbb{C}, \tag{4.2.18}
$$

will be a homogeneous polynomial with degree k. Let us denote the degree of a homogeneous polynomial w by $|w|$ and the subspace of the homogeneous polynomials of degree k by $\mathscr{A}_q^k[x,y]$. It is obvious that

$$
\mathscr{A}_q[x,y] = \mathscr{A}_q^0[x,y] \oplus \mathscr{A}_q^1[x,y] \oplus \dots \oplus \mathscr{A}_q^{N-1}[x,y]. \tag{4.2.19}
$$

In particular a polynomial r of degree zero can be written as follows

$$
r = \sum_{l=0}^{N-1} \beta_l x^l, \qquad \beta_l \in \mathbb{C}, \;\; r \in \mathscr{A}_q^0[x,y]. \tag{4.2.20}
$$

Obviously the subspace of elements of degree zero $\mathscr{A}_q^0[x,y]$ is the subalgebra of $\mathscr{A}_q[x,y]$ generated by the variable x. Evidently the polynomial algebra $\mathscr{A}_q[x,y]$ of polynomials of a reduced quantum plane can be considered as a semi-commutative Galois extension of the subalgebra $\mathscr{A}_q^0[x,y]$ by means of the element y which satisfies the relation $y^N = \mathbb{1}$. The commutation relation $xy = q\,yx$ gives us a semi-commutativity of this extension.

Now we can endow the polynomial algebra $\mathscr{A}_q[x,y]$ with an N-differential d. Making use of Theorem 4.2.2 we define the N-differential by the following formula

$$
dw = [y,w]_q = y\,w - q^{|w|}\,w\,y, \tag{4.2.21}
$$

where q is a primitive Nth root of unity and $w \in \mathscr{A}_q[x,y]$. Hence the algebra $\mathscr{A}_q[x,y]$ equipped with the N-differential d is a graded q-differential algebra.

In order to give a differential-geometric interpretation to the graded q-differential algebra structure of $\mathscr{A}_q[x,y]$ induced by the N-differential d_v we interpret the commutative subalgebra $\mathscr{A}_q^0[x,y]$ of the x-polynomials (4.2.20) of $\mathscr{A}_q[x,y]$ as an algebra of polynomial functions on a one dimensional space with coordinate x. Since $\mathscr{A}_q^k[x,y]$ for $k>0$ is a $\mathscr{A}_q^0[x,y]$-bimodule we interpret this $\mathscr{A}_q^0[x,y]$-bimodule of the elements of degree k as a bimodule of differential forms of degree k and we shall call an element of this bimodule a differential k-form on a one dimensional space with coordinate x. The N-differential d can be interpreted as an exterior differential.

It is easy to show that in one dimensional case we have a simple situation when every bimodule $\mathscr{A}_q^k[x,y], k>0$ of the differential k-forms is a free right module over the commutative algebra of functions $\mathscr{A}_q^0[x,y]$. Indeed if we write a differential k-form w as follows

$$w = y^k \sum_{l=0}^{N-1} \beta_l x^l = y^k r, \quad r = \sum_{l=0}^{N-1} \beta_l x^l \in \mathscr{A}_q^0[x,y], \tag{4.2.22}$$

and take into account that the polynomial $r = (y^k)^{-1} w = y^{N-k} w$ is uniquely determined then we can conclude that $\mathscr{A}_q^k[x,y]$ is a free right module over $\mathscr{A}_q^0[x,y]$ generated by y^k.

As it was mentioned before a bimodule structure of a free right module over an algebra $\mathscr{B}$ generated freely by p generators is uniquely determined by the homomorphism from an algebra $\mathscr{B}$ to the algebra of $(p \times p)$-matrices over $\mathscr{B}$. In the case of a reduced quantum plane every right module $\mathscr{A}_q^k[x,y]$ is freely generated by one generator (for instant we can take y^k as a generator of this module). Thus its bimodule structure induces an endomorphism of the algebra of functions $\mathscr{A}_q^0[x,y]$ and denoting this endomorphism in the case of the generator y^k by $A_k : \mathscr{A}_q^0[x,y] \longrightarrow \mathscr{A}_q^0[x,y]$ we get

$$r\, y^k = y^k A_k(r), \ \text{(no summation over } k) \tag{4.2.23}$$

for any function $r \in \mathscr{A}_q^0[x,y]$. Making use of the commutation relations of variables x,y we easily find that $A_k(x) = q^k x$. Since the algebra of functions $\mathscr{A}_q^0[x,y]$ may be viewed as a bimodule over the same algebra we can consider the functions as degree zero differential forms, and the corresponding endomorphism is the identity mapping of $\mathscr{A}_q[x,y]$, i.e. $A_0 = I$, where $I : \mathscr{A}_q^0[x,y] \longrightarrow \mathscr{A}_q^0[x,y]$ is the identity mapping. Thus the bimodule structures of the free right modules $\mathscr{A}_q^0[x,y], \mathscr{A}_q^1[x,y], \ldots, \mathscr{A}_q^{N-1}[x,y]$ of differential forms induce the associated endomorphisms $A_0, A_1, \ldots, A_{N-1}$ of the algebra $\mathscr{A}_q^0[x,y]$. It is easy to see that for any k it holds $A_k = A_1^k$.

Let us start with the first order differential calculus $(\mathscr{A}_q^0[x,y], \mathscr{A}_q^1[x,y], d)$ over the algebra of functions $\mathscr{A}_q^0[x,y]$ induced by the N-differential d, where

$$d : \mathscr{A}_q^0[x,y] \longrightarrow \mathscr{A}_q^1[x,y],$$

and $\mathscr{A}_q^1[x,y]$ is the bimodule over $\mathscr{A}_q^0[x,y]$. For any $w \in \mathscr{A}_q^0[x,y]$ we have

$$dw = yw - wy = yw - yA_1(w) = y(w - A_1(w)) = y\,\Delta_q(w), \tag{4.2.24}$$

where $\Delta_q = I - A_1 : \mathscr{A}_q^0[x,y] \longrightarrow \mathscr{A}_q^0[x,y]$. It is easy to verify that for any two functions $w, w' \in \mathscr{A}_q^0[x,y]$ the mapping Δ_q has the following properties

$$\Delta_q(ww') = \Delta_q(w)w' + A_1(w)\Delta_q(w'), \tag{4.2.25}$$

$$\Delta_q(x^k) = (1-q)[k]_q\, x^k. \tag{4.2.26}$$

Particularly $dx = y\Delta_q(x)$, and this formula shows that dx can be taken as a generator for the free right module $\mathscr{A}_q^1[x,y]$.

Since the bimodule $\mathscr{A}_q^1[x,y]$ of differential calculus $(\mathscr{A}_q^0[x,y], \mathscr{A}_q^1[x,y], d)$ is a free right module we have a coordinate first order differential calculus over the algebra $\mathscr{A}_q^0[x,y]$, and in the case of a calculus of this kind the differential induces the derivative $\partial : \mathscr{A}_q^0[x,y] \longrightarrow \mathscr{A}_q^0[x,y]$ which is defined by the formula $dw = dx\,\partial w$, $\forall w \in \mathscr{A}_q^0[x,y]$. Using this definition we find that for any function w it holds

$$\partial w = (1-q)^{-1} x^{N-1} \Delta_q(w). \tag{4.2.27}$$

From this formula and (4.2.25),(4.2.26) it follows that this derivative satisfies the twisted Leibniz rule

$$\partial(ww') = \partial(w)\cdot w' + A_1(w)\cdot \partial(w'), \tag{4.2.28}$$

and

$$\partial x^k = [k]_q\, x^{k-1}. \tag{4.2.29}$$

Let us study the structure of the higher order exterior calculus on a reduced quantum plane or, by other words, the structure of the bimodule $\mathscr{A}_q^k[x,y]$ of differential k-forms, when $k>1$. In this case we have a choice for the generator of the free right module. Indeed since the kth power of the exterior differential d is not equal to zero when $k<N$, i.e. $d^k \neq 0$ for $k<N$, a differential k-form w may be expressed either by means of $(dx)^k$ or by means of $d^k x$. Straightforward calculation shows that we have the following relation between these generators

$$d^k x = \frac{[k]_q}{q^{\frac{k(k-1)}{2}}}(dx)^k x^{1-k}. \tag{4.2.30}$$

We will use the generator $(dx)^k$ of the free right module $\mathscr{A}_q^k[x,y]$ as a basis in our calculations with differential k-forms. For any differential k-form $w \in \mathscr{A}_q^k[x,y]$ we have $dw \in \mathscr{A}_q^{k+1}[x,y]$. Let us express these two differential forms in terms of the generators of the modules $\mathscr{A}_q^k[x,y]$ and $\mathscr{A}_q^{k+1}[x,y]$. We have $w=(dx)^k r$, $dw = (dx)^{k+1}\tilde{r}$, where $r,\tilde{r} \in \mathscr{A}_q^0[x,y]$ are the functions. Making use of the definition of the exterior differential d we calculate the relation between the functions $r,\tilde{r}$ which is

$$\tilde{r} = (\Delta_q x)^{-1}(q^{-k}r - q^k A_1(r)), \tag{4.2.31}$$

where A_1 is the endomorphism of the algebra of functions $\mathscr{A}_q^0[x,y]$. This relation shows that the exterior differential d considered in the case of the differential k-forms induces the mapping $\Delta_q^{(k)} : \mathscr{A}_q^0[x,y] \longrightarrow \mathscr{A}_q^0[x,y]$ of the algebra of the function which is defined by the formula

$$dw = (dx)^{k+1}\Delta_q^{(k)}(r), \tag{4.2.32}$$

where

$$w = (dx)^k r. \tag{4.2.33}$$

It is obvious that

$$\Delta_q^{(k)}(r) = (\Delta_q x)^{-1}(q^{-k}r - q^k A_1(r)). \tag{4.2.34}$$

It is obvious that for $k=0$ the mapping $\Delta_q^{(0)}$ coincides with the derivative induced by the differential d in the first order calculus, i.e.

$$\Delta_q^{(0)}(r) = \partial r = (\Delta_q x)^{-1}(r - A_1(r)). \tag{4.2.35}$$

The higher order mappings $\Delta_q^{(k)}$, which we do not have in the case of a classical exterior calculus on a one dimensional space, have the derivation type property

$$\Delta_q^{(k)}(r\,r') = \Delta_q^{(k)}(r)\,r' + q^k A_1(r)\,\Delta_q^{(0)}(r'), \tag{4.2.36}$$

where $k = 0, 1, 2, \ldots, N-1$. A higher order mapping $\Delta_q^{(k)}$ can be expressed in terms of the derivative ∂ as a differential operator on the algebra of functions as follows

$$\Delta_q^{(k)} = q^k \partial + \frac{q^{-k} - q^k}{1-q} x^{-1}. \tag{4.2.37}$$

Thus we see that exterior calculus on a one dimensional space with coordinate x satisfying $x^N = 1$ generated by the exterior differential d satisfying $d^N = 0$ has the differential forms of higher order which are not presented in the case of a classical exterior calculus with $d^2 = 0$.

Bibliography

[1] V. Abramov, R. Kerner, B. Le Roy, *Hypersymmetry: a $\mathbb{Z}_3$-graded generalization of supersymmetry*, J.Math.Phys. 38 (1997) 1650-1669.

[2] V. Abramov, *Noncommutative Galois Extension and Graded q-Differential Algebra*, to appear in the Proceedings of ICCA 10, Advances in Clifford Algebras.

[3] V. Abramov, *On a graded q-differential algebra,* Journal of Nonlinear Mathematical Physics, No. 13 (2006), 1–8.

[4] V. Abramov, *Algebra Forms with $d^N = 0$ on Quantum Plane. Generalized Clifford Algebra Approach,* Advances in Applied Clifford Algebras, No. 17 (2007), 577-588.

[5] V. Abramov, R. Kerner, *Exterior differentials of higher order and their covariant generalization,* Journal of Mathematical Physics, Vol. **41**, No. 8 (2000), 5598–5614.

[6] V. Abramov, O. Liivapuu, *N-complex, graded q-differential algebra and N-connection on modules,* J. Math. Sci. (to appear).

[7] V. Abramov, O. Liivapuu, *Generalization of connection on the concept of graded q-differential algebra,* Proceedings of the Estonian Academy of Sciences, Vol. **59**, No. 4 (2010), 256–264.

[8] V. Abramov, O. Liivapuu, *Graded q-Differential Algebra Approach to Chern-Simons Forms,* Applied Mathematics and Information Sciences, Vol. **9**, No. 11 (2015), 29–38.

[9] V. Abramov, O. Liivapuu, *Geometric approach to BRST-symmetry and $\mathbb{Z}_N$-generalization of superconnection*, J. Nonlinear Math. Phys. 13 (Supplement) (2006), 9-20.

[10] V. Abramov, Md. Raknuzzaman, *Graded q-differential Matrix Algebra,* AIP Conference Proceedings, 1558 (2013), 538-542.

[11] V. Abramov, Md. Raknuzzaman, *Semi-Commutative Galois Extension and Reduced Quantum Plane*, arXiv: 1511.08309, Springer Proceedings in Mathematics and Statistics, Engineering mathematics and algebraic, analysis and stochastic structures for networks, data classification and optimization, Editors: Sergei Silvestrov, Milica Rancic.

[12] V. Abramov, *Super 3-Lie Algebras Induced by Super Lie Algebras*, Advances in Applied Clifford Algebras, Proceedings of ICCA 10 (2015), 1-7.

[13] N. Bazunova, A. Borowiec and R. Kerner, *Quantum de Rham complex with $d^3 = 0$ differential*, Czech. J. **5**1 (12) (2001), 1266-1271.

[14] F. Berezin, *The Method of Second Quantization*, Academic Press, New York 1966

[15] A. Borowiec, V. K. Kharchenko, *Algebraic approach to calculuses with partial derivatives*, Siberian Advances in Mathematics, **5**(2), (1995), 10-37.

[16] A. Connes, *Non-commutative differential geometry*, Publ. Math. IHES **62** (1985), 257–360.

[17] R. Coquereaux, A.O. Garcia and R. Trinchero, *Differential calculus and connections on a quantum plane at a cubic root of unity*, arXiv: math-ph/9807012.

[18] A. E. F. Djemai, H. Samil, *$\mathbb{Z}_N$-Graded Noncommutative Differential Calculus*, International Journal of Theoretical Physics, Vol. **39**, No. 11, 2000, 2533-2543.

[19] M. Dubois-Violette, R. Kerner, *Universal q differential calculus and q analog of homological algebra,* Acta Math. Univ. Comenian, **65**, (1996), 175-188.

[20] M. Dubois-Violette, *Generalized homologies for $d^N = 0$ and graded q-differential algebras*, in Contemporary Mathematics, edited by M. Henneaux, J. Krasilshchik and A. Vinogradov, American Mathematical Society 219 (1998), 69–79.

[21] Dubois-Violette, M., $d^N = 0$. *K-theory*, **14** (1998), 371–404.

[22] M. Dubois-Violette, *Lectures on differentials, generalized differentials and some examples related to theoretical physics,* in Contemporary Mathematics, edited by R. Coquereaux, A. Garcia and R. Trinchero, American Mathematical Society, 2002, 59-94.

[23] Dubois-Violette, M., *Tensor product of N-complexes and generalization of graded differential algebras*, Bulg. J. Phys., **36** (2009), 227–236.

[24] V. Kac, *Lie superalgebras*, Advances in Math., **26** (1997), 1, 8-96.

[25] M. Kapranov M., *On the q-analog of homological algebra,* math.QA/9611005.

[26] C. Kassel, *Quantum groups,* Springer-Verlag, 1995.

[27] R. Kerner, V. Abramov, *On certain realizations of q-deformed exterior differential calculus,* Rep. Math. Phys., Vol. **43**, No. 1-2 (1999), 179-194.

[28] R. Kerner, O. Suzuki, *Internal Symmetry Groups of Cubic Algebras,* Int. J. Geom. Methods Mod. Phy. **09**, (2012), 1261007.

[29] R. Kerner, *The $\mathbb{Z}_3$-generalization of Pauli's principle and a cubic Dirac equation for quarks*, in "Symmetries and groups in contemporary physics" (proceedings of ICGTMP Conference "Group-29", Tianjin, China), World Scientific, eds. Chengming Bai, J.-P.Gazeau, Mo-Lin Ge) (2013), 283-288.

[30] R. Kerner, *Ternary $\mathbb{Z}_3$-graded generalization of Heisenberg algebra*, Journal of Physics, Conference Series **597** (2015), 012049 – 012061.

[31] J. Lawrynowicz, K. Nouno, D. Nagayama, O. Suzuki, *A Method of Noncommutative Galois Theory for Binary and Ternary Clifford Analysis,* In Sivasundaram, S. (editor), 9th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences: ICNAAP 2012, AIP Conf. Proc. **1443**, (2012), 1007.

[32] J. Lawrynowicz, K. Nôno, D. Nagayama, O. Suzuki, *A Method of Noncommutative Galois Theory for Construction of Quark Models (Kobayashi-Masukawa Model) I*, Bulletin de la Société des Science et des Lettres de Łódź, **LXIII** (2013), 95 – 112.

[33] O. Liivapuu, *Graded q-differential algebras and algebraic models in non-commutative geometry*, Dissertationes Mathematicae Universitatis Trartuensis, 68.

[34] V. Mathai and D. Quillen, *Superconnections, Thom classes and equivariant differential forms*, Toplogy. **25** (1986), 85–110.

[35] D. Quillen, *Chern-Simons Forms and Cyclic Cohomology,* The Interface of Mathematics and Particle Physics, edited by D.G. Quillen, G.B. Segal and S.T. Tsou, Clarendon Press, 1990, 117–134.

[36] S. Silvestrov, E. Paal, V. Abramov, A. Stolin, *Generalized Lie theory in Mathematics, Physics and Beyond,* Heidelberg: Springer-Verlaq, Berlin (2009).

[37] O. Suzuki, A. Trovon, *Noncommutative Galois Extensions and Ternary Clifford Analysis,* to appear in the Advances in Applied Clifford Algebras.

Kokkuvõte

Mittekommutatiivse Galois laiendi lähenemine ternaarsele Grassmanni algebrale ja gradueeritud q-diferentsiaalalgebrale

Geomeetrilise ruumi mõistel on kaasaegses teoreetilises füüsikas erakordselt tähtis roll ja sellega seoses kasutatakse diferentsiaalgeomeetria meetodeid teoreetilise füüsika fundamentaalprobleemide lahendamiseks. Kaasaegses teoreetilises füüsikas on neli fundamentaalset vastasmõju (elektromagneetiline, tugev, nõrk ja gravitatsioon) ja igat interaktsiooni kirjeldatakse vastava väljateooria abil. Selleks, et kirjeldada interaktsiooni kvanttasemel peab selle vastasmõju väljateooria olema kvantiseeritud, st peab olema konstrueeritud kvantväljateooria. Kaasaegse teoreetilise füüsika tähtsaks probleemiks on gravitatsiooni üldtunnustatud kvantväljateooria puudumine. Arvatakse, et üks võimalik lähenemine selle probleemi lahendamiseks on seotud geomeetrilise ruumi uue mõiste väljatöötamisega. Motiveerituna gravitatsiooni kvantväljateooria probleemist pani prantsuse matemaatik A. Connes aluse mittekommutatiivsele geomeetriale, mis tugineb mittekommutatiivse ruumi mõistele. Mittekommutatiivse ruumi mõiste pärineb kvantteooriast, kus suurused, mis kommuteeruvad klassikalisel tasemel, ei kommuteeru kvanttasemel, nt koordinaat ja impulss. Mittekommutatiivses ruumis puudub punkti mõiste, punkti mõiste annab teed funktsiooni mõistele ja funktsioonide algebra mõistele. On teada, et lõplikumõõtmelisel muutkonnal moodustavad siledad funktsioonid assotsiatiivse kommutatiivse ühikuga algebra. Mittekommutatiivse ruumi korral on funktsioonide algebra assotsiatiivne ühikuga algebra, kuid ta ei pea olema kommutatiivne. On märkimisväärne, et vaatamata punkti mõiste puudusele mittekommutatiivse ruumi korral, on võimalik rakendada diferentsiaalgeomeetria meetodeid kasutades ainult funktsioonide algebrat. Tõepoolest, diferentsiaalgeomeetria üheks tähtsaks struktuuriks on vektorväljad. Kaasaegses diferentsiaalgeomeetrias näidatakse, et vektorvälja mõistet võib samastada funktsioonide algebra derivatsiooniga. Seega, kui funktsioonide algebra on antud, võime me arendada vektorväljade formalismi kasutades selleks funktsioonide algebra derivatsioone.

Antud väitekirjas kasutatakse Borowiec-Kharchenko-Oziewicz (BKO) lähenemist [15] esimest järku mittekommutatiivse diferentsiaalarvutuse konstrueerimiseks ja uurimiseks Grassmanni algebra, ternaarse Grassmanni algebra ja mittekommutatiivse ruumi (kvantruum) korral. BKO-lähenemises koosneb esimest järku diferentsiaalarvutus kolmest komponendist, kus üheks komponendiks on assotsiatiivne ühikuga algebra (funktsioonide algebra), teiseks komponendiks on bimoodul üle funktsioonide algebra (esimest järku diferentsiaalvormid) ja kolmas komponent on diferentsiaal, st lineaarkujutus funktsioonide algebrast bimoodulisse, mis rahuldab Leibnizi valemit algebra korrutamistehe suhtes. Kui algebra on tekitatud lõpliku arvu moodustajate abil, kusjuures moodustajad rahuldavad kommutatsiooniseoseid ja esimest järku diferentsiaalarvutuse bimooduli parempoolne moodul on vabalt tekitatud moodustajate diferentsiaalide abil, siis vastavat esimest järku diferentsiaalarvutust nimetatakse koordinaatdiferentsiaalarvutuseks. Koordinaatdiferentsiaalarvutuse struktuuri tähtsaks komponendiks on homomorfism algebrast maatriksite algebrasse. Käesolevas väitekirjas on leitud võrrandid ülalpool mainitud homomorfismi jaoks, mis tulenevad algebra kommutatsiooniseostest. Neid võrrandeid on uuritud ja lahendatud teise astme kommutatsiooniseostuste korral. Saadud tulemused on rakendatud esimest järku mittekommutatiivse diferentsiaalarvutuse uurimiseks Grassmanni algebra, ternaarse Grassmanni algebra ja mittekommutatiivse ruumi korral.

Antud väitekirjas uuritakse Grassmanni algebra ternaarset analoogi. Grassmanni algebra moodustajad antikommuteeruvad, või teiste sõnadega, Grassmanni algebra iga kahe moodustaja kõikvõimalike korrutiste summa on null. Me võime eespool sõnastatud tingimust üldistada nõudes, et iga kolme moodustaja kõikvõimalike korrutiste summa on null. Antud tingimus on võimalik lahendada kolmandat järku algjuure abil ja kasutades leitud lahendit defineerime Grassmanni algebra ternaarse analoogi, kus kolme moodustaja korrutise tsükliline permutatsioon on võrdne esialgse korrutisega korrutatud kolmandat järku algjuurega. Mainime, et antud Grassmanni algebra analoog tekkis teoreetilises füüsikas seoses kvarkide mudeliga. Antud väitekirjas on defineeritud ja uuritud Grassmanni algebra ternaarne analoog involutsiooniga. Seega on meil klassikaline (tavaline) Grassmanni algebra ja tema ternaarne analoog. Antud väitekirjas on konstrueeritud algebra nii, et Grassmanni algebra ja tema ternaarne analoog on selle algebra alamalgebrad. Teiste sõnadega antud väitekirjas on leitud Grassmanni algebra ternaarne laiend.

Käesolevas väitekirjas uuritakse mittekommutatiivset Galois' laiendit. Väitekirjas näidatakse seost mittekommutatiivse Galois laiendi ja gradueeritud q-diferentsiaalalgebrate vahel, kus q on N-järku algjuur ühest. Gradueeritud q-diferentsiaalalgebrate teoorias on tõestatud teoreem, mis väidab, et assotsiatiivne ühikuga algebra on gradueeritud q-diferentsiaalalgebra, kui leidub üks selline gradueeringuga element, et astmes N ta on võrdne ühikelemendiga. Antud väitekirjas eespool mainitud teoreem on rakendatud poolkommutatiivsele Galois laiendile ja on tõestatud, et iga poolkommutatiivne Galois laiend on gradueeritud q-diferentsiaalalgebra. On uuritud selle algebra esimest järku ja kõrgemat järku differentsiaalarvutus.

Curriculum Vitae

Name: Md. Raknuzzaman

Date and Place of the Birth: 05.08.1984, Kushtia, Bangladesh

Citizenship: Bangladeshi

Address: Institute of Mathematics and Statistics, University of Tartu, J. Liivi 2, 50409, Tartu, Estonia

Phone: +372 590 36 542

E-mail: raknuzza@ut.ee

Education:

2000-2002: Notre Dame College, Dhaka, Bangladesh

2003-2008: Department of Mathematics, University of Dhaka, Bachelor Studies, B.Sc. in Mathematics

2008-2010: Department of Mathematics, University of Dhaka, Masters Studies, M.Sc. in Applied Mathematics

2012-2016: University of Tartu, Institute of Mathematics, Faculty of Mathematics and Computer science, Doctoral Studies in Mathematics

Employment:

February 2010- February 2012: Department of Arts and Sciences, Ahsanullah University of Science and Technology, Dhaka, Bangladesh, Lecturer of Mathematics

March 2012- September 2012: Department of Mathematics, Islamic University, Kushtia, Bangladesh, Lecturer of Mathematics

Scientific Interest: Differential Geometry and its Applications

Curriculum vitae

Nimi: Md. Raknuzzaman

Sünniaeg ja koht: 05.08.1984, Kushtia, Bangladesh

Kodakondsus: Bangladesh

Aadress: Matemaatika ja statistika instituut, Tartu Ülikool, J. Liivi 2, 50409, Tartu, Eesti

Telefon: +372 590 36 542

E-post: raknuzza@ut.ee

Haridus:

2000-2002: Notre Dame Kolledž, Dhaka, Bangladesh

2003-2008: Matemaatika osakond, Dhaka Ülikool, bakalaureuseõppe, bakalaureusekraad matemaatikas

2008-2010: Matemaatika osakond, Dhaka Ülikool, magistriõppe, magistrikraad rakendusmatemaatikas

2012-2016: Matemaatika instituut, Matemaatika-informaatikateaduskond, Tartu Ülikool, doktoriõppe

Teenistuskäik:

veebruar 2010- veebruar 2012: Kunstide ja teaduste osakond, Ahsanullah Teaduse ja Tehnoloogia Ülikool, Dhaka, Bangladesh, matemaatika lektor

märts 2012- september 2012: Matemaatika osakond, Islami Ülikool, Kushtia, Bangladesh, matemaatika lektor

Teaduslikud huvid: Diferentsiaalgeomeetria ja selle rakendused

List of Original Publications

1. V. Abramov, Md. Raknuzzaman, *Graded q-differential Matrix Algebra*, AIP Conference Proceedings, 1558 (2013), 538-542.

2. V. Abramov, Md. Raknuzzaman, *Semi-Commutative Galois Extension and Reduced Quantum Plane*, Springer Proceedings in Mathematics and Statistics, Engineering mathematics and algebraic, analysis and stochastic structures for networks, data classification and optimization, Editors: Sergei Silvestrov, Milica Rancic, arXiv: 1511.08309

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

1. **Mati Heinloo.** The design of nonhomogeneous spherical vessels, cylindrical tubes and circular discs. Tartu, 1991, 23 p.
2. **Boris Komrakov.** Primitive actions and the Sophus Lie problem. Tartu, 1991, 14 p.
3. **Jaak Heinloo.** Phenomenological (continuum) theory of turbulence. Tartu, 1992, 47 p.
4. **Ants Tauts.** Infinite formulae in intuitionistic logic of higher order. Tartu, 1992, 15 p.
5. **Tarmo Soomere.** Kinetic theory of Rossby waves. Tartu, 1992, 32 p.
6. **Jüri Majak.** Optimization of plastic axisymmetric plates and shells in the case of Von Mises yield condition. Tartu, 1992, 32 p.
7. **Ants Aasma.** Matrix transformations of summability and absolute summability fields of matrix methods. Tartu, 1993, 32 p.
8. **Helle Hein.** Optimization of plastic axisymmetric plates and shells with piece-wise constant thickness. Tartu, 1993, 28 p.
9. **Toomas Kiho.** Study of optimality of iterated Lavrentiev method and its generalizations. Tartu, 1994, 23 p.
10. **Arne Kokk.** Joint spectral theory and extension of non-trivial multiplicative linear functionals. Tartu, 1995, 165 p.
11. **Toomas Lepikult.** Automated calculation of dynamically loaded rigid-plastic structures. Tartu, 1995, 93 p, (in Russian).
12. **Sander Hannus.** Parametrical optimization of the plastic cylindrical shells by taking into account geometrical and physical nonlinearities. Tartu, 1995, 74 p, (in Russian).
13. **Sergei Tupailo.** Hilbert's epsilon-symbol in predicative subsystems of analysis. Tartu, 1996, 134 p.
14. **Enno Saks.** Analysis and optimization of elastic-plastic shafts in torsion. Tartu, 1996, 96 p.
15. **Valdis Laan.** Pullbacks and flatness properties of acts. Tartu, 1999, 90 p.
16. **Märt Põldvere.** Subspaces of Banach spaces having Phelps' uniqueness property. Tartu, 1999, 74 p.
17. **Jelena Ausekle.** Compactness of operators in Lorentz and Orlicz sequence spaces. Tartu, 1999, 72 p.
18. **Krista Fischer.** Structural mean models for analyzing the effect of compliance in clinical trials. Tartu, 1999, 124 p.
19. **Helger Lipmaa.** Secure and efficient time-stamping systems. Tartu, 1999, 56 p.

20. **Jüri Lember.** Consistency of empirical k-centres. Tartu, 1999, 148 p.
21. **Ella Puman.** Optimization of plastic conical shells. Tartu, 2000, 102 p.
22. **Kaili Müürisep.** Eesti keele arvutigrammatika: süntaks. Tartu, 2000, 107 lk.
23. **Varmo Vene.** Categorical programming with inductive and coinductive types. Tartu, 2000, 116 p.
24. **Olga Sokratova.** Ω-rings, their flat and projective acts with some applications. Tartu, 2000, 120 p.
25. **Maria Zeltser.** Investigation of double sequence spaces by soft and hard analitical methods. Tartu, 2001, 154 p.
26. **Ernst Tungel.** Optimization of plastic spherical shells. Tartu, 2001, 90 p.
27. **Tiina Puolakainen.** Eesti keele arvutigrammatika: morfoloogiline ühestamine. Tartu, 2001, 138 p.
28. **Rainis Haller.** *M(r,s)*-inequalities. Tartu, 2002, 78 p.
29. **Jan Villemson.** Size-efficient interval time stamps. Tartu, 2002, 82 p.
30. **Eno Tõnisson.** Solving of expession manipulation exercises in computer algebra systems. Tartu, 2002, 92 p.
31. **Mart Abel.** Structure of Gelfand-Mazur algebras. Tartu, 2003. 94 p.
32. **Vladimir Kuchmei.** Affine completeness of some ockham algebras. Tartu, 2003. 100 p.
33. **Olga Dunajeva.** Asymptotic matrix methods in statistical inference problems. Tartu 2003. 78 p.
34. **Mare Tarang.** Stability of the spline collocation method for volterra integro-differential equations. Tartu 2004. 90 p.
35. **Tatjana Nahtman.** Permutation invariance and reparameterizations in linear models. Tartu 2004. 91 p.
36. **Märt Möls.** Linear mixed models with equivalent predictors. Tartu 2004. 70 p.
37. **Kristiina Hakk.** Approximation methods for weakly singular integral equations with discontinuous coefficients. Tartu 2004, 137 p.
38. **Meelis Käärik.** Fitting sets to probability distributions. Tartu 2005, 90 p.
39. **Inga Parts.** Piecewise polynomial collocation methods for solving weakly singular integro-differential equations. Tartu 2005, 140 p.
40. **Natalia Saealle.** Convergence and summability with speed of functional series. Tartu 2005, 91 p.
41. **Tanel Kaart.** The reliability of linear mixed models in genetic studies. Tartu 2006, 124 p.
42. **Kadre Torn.** Shear and bending response of inelastic structures to dynamic load. Tartu 2006, 142 p.
43. **Kristel Mikkor.** Uniform factorisation for compact subsets of Banach spaces of operators. Tartu 2006, 72 p.

44. **Darja Saveljeva.** Quadratic and cubic spline collocation for Volterra integral equations. Tartu 2006, 117 p.
45. **Kristo Heero.** Path planning and learning strategies for mobile robots in dynamic partially unknown environments. Tartu 2006, 123 p.
46. **Annely Mürk.** Optimization of inelastic plates with cracks. Tartu 2006. 137 p.
47. **Annemai Raidjõe.** Sequence spaces defined by modulus functions and superposition operators. Tartu 2006, 97 p.
48. **Olga Panova.** Real Gelfand-Mazur algebras. Tartu 2006, 82 p.
49. **Härmel Nestra.** Iteratively defined transfinite trace semantics and program slicing with respect to them. Tartu 2006, 116 p.
50. **Margus Pihlak.** Approximation of multivariate distribution functions. Tartu 2007, 82 p.
51. **Ene Käärik.** Handling dropouts in repeated measurements using copulas. Tartu 2007, 99 p.
52. **Artur Sepp.** Affine models in mathematical finance: an analytical approach. Tartu 2007, 147 p.
53. **Marina Issakova.** Solving of linear equations, linear inequalities and systems of linear equations in interactive learning environment. Tartu 2007, 170 p.
54. **Kaja Sõstra.** Restriction estimator for domains. Tartu 2007, 104 p.
55. **Kaarel Kaljurand.** Attempto controlled English as a Semantic Web language. Tartu 2007, 162 p.
56. **Mart Anton.** Mechanical modeling of IPMC actuators at large deformations. Tartu 2008, 123 p.
57. **Evely Leetma.** Solution of smoothing problems with obstacles. Tartu 2009, 81 p.
58. **Ants Kaasik.** Estimating ruin probabilities in the Cramér-Lundberg model with heavy-tailed claims. Tartu 2009, 139 p.
59. **Reimo Palm.** Numerical Comparison of Regularization Algorithms for Solving Ill-Posed Problems. Tartu 2010, 105 p.
60. **Indrek Zolk.** The commuting bounded approximation property of Banach spaces. Tartu 2010, 107 p.
61. **Jüri Reimand.** Functional analysis of gene lists, networks and regulatory systems. Tartu 2010, 153 p.
62. **Ahti Peder.** Superpositional Graphs and Finding the Description of Structure by Counting Method. Tartu 2010, 87 p.
63. **Marek Kolk.** Piecewise Polynomial Collocation for Volterra Integral Equations with Singularities. Tartu 2010, 134 p.
64. **Vesal Vojdani.** Static Data Race Analysis of Heap-Manipulating C Programs. Tartu 2010, 137 p.
65. **Larissa Roots.** Free vibrations of stepped cylindrical shells containing cracks. Tartu 2010, 94 p.

66. **Mark Fišel.** Optimizing Statistical Machine Translation via Input Modification. Tartu 2011, 104 p.
67. **Margus Niitsoo**. Black-box Oracle Separation Techniques with Applications in Time-stamping. Tartu 2011, 174 p.
68. **Olga Liivapuu.** Graded q-differential algebras and algebraic models in noncommutative geometry. Tartu 2011, 112 p.
69. **Aleksei Lissitsin.** Convex approximation properties of Banach spaces. Tartu 2011, 107 p.
70. **Lauri Tart.** Morita equivalence of partially ordered semigroups. Tartu 2011, 101 p.
71. **Siim Karus.** Maintainability of XML Transformations. Tartu 2011, 142 p.
72. **Margus Treumuth.** A Framework for Asynchronous Dialogue Systems: Concepts, Issues and Design Aspects. Tartu 2011, 95 p.
73. **Dmitri Lepp.** Solving simplification problems in the domain of exponents, monomials and polynomials in interactive learning environment T-algebra. Tartu 2011, 202 p.
74. **Meelis Kull.** Statistical enrichment analysis in algorithms for studying gene regulation. Tartu 2011, 151 p.
75. **Nadežda Bazunova.** Differential calculus $d^3 = 0$ on binary and ternary associative algebras. Tartu 2011, 99 p.
76. **Natalja Lepik.** Estimation of domains under restrictions built upon generalized regression and synthetic estimators. Tartu 2011, 133 p.
77. **Bingsheng Zhang.** Efficient cryptographic protocols for secure and private remote databases. Tartu 2011, 206 p.
78. **Reina Uba.** Merging business process models. Tartu 2011, 166 p.
79. **Uuno Puus.** Structural performance as a success factor in software development projects – Estonian experience. Tartu 2012, 106 p.
80. **Marje Johanson.** $M(r, s)$-ideals of compact operators. Tartu 2012, 103 p.
81. **Georg Singer.** Web search engines and complex information needs. Tartu 2012, 218 p.
82. **Vitali Retšnoi.** Vector fields and Lie group representations. Tartu 2012, 108 p.
83. **Dan Bogdanov.** Sharemind: programmable secure computations with practical applications. Tartu 2013, 191 p.
84. **Jevgeni Kabanov.** Towards a more productive Java EE ecosystem. Tartu 2013, 151 p.
85. **Erge Ideon**. Rational spline collocation for boundary value problems. Tartu, 2013, 111 p.
86. **Esta Kägo**. Natural vibrations of elastic stepped plates with cracks. Tartu, 2013, 114 p.
87. **Margus Freudenthal.** Simpl: A toolkit for Domain-Specific Language development in enterprise information systems. Tartu, 2013, 151 p.
88. **Boriss Vlassov.** Optimization of stepped plates in the case of smooth yield surfaces. Tartu, 2013, 104 p.

89. **Elina Safiulina.** Parallel and semiparallel space-like submanifolds of low dimension in pseudo-Euclidean space. Tartu, 2013, 85 p.
90. **Raivo Kolde.** Methods for re-using public gene expression data. Tartu, 2014, 121 p.
91. **Vladimir Šor.** Statistical Approach for Memory Leak Detection in Java Applications. Tartu, 2014, 155 p.
92. **Naved Ahmed.** Deriving Security Requirements from Business Process Models. Tartu, 2014, 171 p.
93. **Kerli Orav-Puurand.** Central Part Interpolation Schemes for Weakly Singular Integral Equations. Tartu, 2014, 109 p.
94. **Liina Kamm.** Privacy-preserving statistical analysis using secure multi-party computation. Tartu, 2015, 201 p.
95. **Kaido Lätt.** Singular fractional differential equations and cordial Volterra integral operators. Tartu, 2015, 93 p.
96. **Oleg Košik.** Categorical equivalence in algebra. Tartu, 2015, 84 p.
97. **Kati Ain.** Compactness and null sequences defined by ℓ_p spaces. Tartu, 2015, 90 p.
98. **Helle Hallik.** Rational spline histopolation. Tartu, 2015, 100 p.
99. **Johann Langemets.** Geometrical structure in diameter 2 Banach spaces. Tartu, 2015, 132 p.
100. **Abel Armas Cervantes.** Diagnosing Behavioral Differences between Business Process Models. Tartu, 2015, 193 p.
101. **Fredrik Milani.** On Sub-Processes, Process Variation and their Interplay: An Integrated Divide-and-Conquer Method for Modeling Business Processes with Variation. Tartu, 2015, 164 p.
102. **Huber Raul Flores Macario.** Service-Oriented and Evidence-aware Mobile Cloud Computing. Tartu, 2015, 163 p.
103. **Tauno Metsalu.** Statistical analysis of multivariate data in bioinformatics. Tartu, 2016, 197 p.
104. **Riivo Talviste.** Applying Secure Multi-party Computation in Practice. Tartu, 2016, 144 p.