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KERLI ORAV-PUURAND

Central Part Interpolation Schemes for Weakly Singular Integral Equations

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Chapter 1

Introduction

An integral equation is a functional equation in which an unknown function appears under one or more integral signs. The early history of integral equations goes back to the special integral equations studied by several mathematicians of the late 18th and early 19th century - Abel, Fourier, Laplace, Liouville, Poisson and others. As written in [15], the terminology "integral equation" was introduced by Paul du Bois-Reymond in 1888. At the end of the nineteenth century the interest in integral equations increased, mainly because of their connection with some of the differential equations of mathematical physics. Systematic study of integral equations started from the works of Volterra [97] and Fredholm [23]. In particular, Fredholm gave the necessary and sufficient conditions for solvability of linear integral equations of the form

$$u(x) = \int_0^1 K(x,y)u(y)\,dy + f(x), \quad x \in [0,1], \tag{1.0.1}$$

which are nowadays often referred to as Fredholm integral equations of the second kind. Here K and f are given real-valued functions, u is the function which we have to find. Function K is called the kernel of the integral equation, the function f is occasionally referred to as free-term, or as forcing function.

There are a number of problems from many different fields, for example chemistry, physics of polymers and mathematical physics, which are directly formulated in terms of integral equations; and there are problems that are represented in terms of differential equations with auxiliary conditions, but which can be reduced to integral equations. Fredholm equations arise in potential problems [53, 32], nuclear physics [12], atmosphere physics [9, 28, 48, 89] and in radiative heat exchange [99]. These equations also arise naturally in the theory of signal processing [79], in linear forward modeling and inverse problems [27], the problem of small deflection of a rotating shaft and radiation transport can also be described as Fredholm integral equations and by a Fredholm integral equation we may represent a boundary value problem associated with a differential equation [33].

If $K(x,y) = 0$ for $0 \le x \le y \le 1$, then equation (1.0.1) takes the form

$$u(x) = \int\limits_0^x K(x,y)u(y)\,dy + f(x), \quad x \in [0,1], \tag{1.0.2}$$

which is usually called Volterra integral equation of the second kind. Such integral equations occur for example in areas as damped vibrations [50], population dynamics [21], study of epidemics [98], viscoelasticity [18, 22, 31], identification of memory kernels in heat conduction [29, 30] and financial mathematics [52]. The relationship between the two varities of equations (1.0.1) and (1.0.2) is a useful one, but it is wrong to infer that the differences between them are minimal. Often a direct study of Volterra equations yields many results which cannot be obtained for Fredholm equations (see, e.g. [91, 92]). There are lots of works on Volterra integral equations. We refer here to the monographs by Brunner [14], Brunner and Houwen [15] and Linz [51]. A reader interested in additional works on Volterra integral equations may consult, for example, papers [16, 39, 42, 43, 44, 56, 57, 58] and PhD theses by Kolk [41], Saveljeva [75] and Tarang [83]. In the present thesis we pay attention to Fredholm integral equations of the form (1.0.1).

There are relatively few integral equations for which we have methods of finding exact solutions, hence, numerical schemes are required for dealing with these equations in a proper manner. Numerical methods for solving Fredholm integral equations have been developed by many researchers in the past. First of all we refer to the monographs by Anselone [3], Atkinson [6], Baker [10], Hackbush [25], Kantorovich and Krylov [40], Krasnoselskii, Vainikko, Zabreiko, Rutitskii, Stetsenko [46], Kress [47], Mikhlin [53], Saranen and Vainikko [74], Vainikko, Pedas and Uba [94], Vainikko [90], see also the survey papers [5, 13, 96] and PhD theses by Hakk [26] and Parts [63].

The main objects of study in the present thesis are high order numerical methods for solving equations (1.0.1), where the functions K and f are at least continuous on $([0,1] \times (0,1)) \setminus diag$ and $[0,1]$, respectively. Here

$$diag = \left\{(x,y) \in \mathbb{R}^2 : x = y\right\}.$$

In particular, we are interested in kernels K that are m-times ($m \ge 0$) continuously differentiable on $([0,1] \times [0,1]) \setminus diag$ and there exists a real number $\nu \in (-\infty; 1)$ such that the inequality

$$\left|\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l K(x,y)\right| \le c \begin{Bmatrix} 1 & \text{for} & \nu + k < 0 \\ 1 + |\log|x-y|| & \text{for} & \nu + k = 0 \\ |x-y|^{-\nu-k} & \text{for} & \nu + k > 0 \end{Bmatrix} \tag{1.0.3}$$

holds for all non-negative integers k and l such that $k + l \le m$. The constant c in (1.0.3) is independent of k and l (it depends on K and m). The set of such functions will henceforth be denoted by

$$\mathcal{S}^{m,\nu} = \mathcal{S}^{m,\nu}(([0,1] \times [0,1]) \setminus diag),$$

where m is a non-negative integer and $\nu \in (-\infty, 1)$. Clearly,

$$\mathcal{S}^{m,\nu} \subset \mathcal{S}^{m_1,\nu_1}, \quad 0 \le m_1 \le m, \quad \nu < \nu_1 < 1.$$

Taking $k = l = 0$, we obtain from (1.0.3) that

$$|K(x,y)| \le c \left\{ \begin{array}{ccc} 1 & \text{for} & \nu < 0 \\ 1 + |\log|x-y|| & \text{for} & \nu = 0 \\ |x-y|^{-\nu} & \text{for} & \nu > 0 \end{array} \right\}, \; (x,y) \in ([0,1] \times [0,1]) \setminus diag.$$

Thus, if $\nu < 0$, then a kernel $K \in \mathcal{S}^{m,\nu}$ itself is bounded on $([0,1] \times [0,1]) \setminus diag$, but the derivatives of $K(x,y)$ may be singular at $x = y$; if $K \in \mathcal{S}^{m,\nu}$ with $0 \le \nu < 1$, then the kernel $K(x,y)$ may have a (weak) singularity at $x = y$. In particular, $K \in \mathcal{S}^{m,\nu}$ may have the form

$$K(x,y) = a(x,y)|x-y|^{-\nu} + b(x,y) \quad (0 < \nu < 1) \tag{1.0.4}$$

or

$$K(x,y) = a(x,y) \log|x-y| + b(x,y), \tag{1.0.5}$$

where a and b are m-times continously differentiable functions on $[0,1] \times [0,1]$.

We also consider a more complicated situation for (1.0.4) and (1.0.5), where $a(x,y)$ and $b(x,y)$ are some differentiable functions for $(x,y) \in [0,1] \times (0,1)$, but their derivatives may have boundary singularities with respect to y (see Chapter 7 in the thesis).

To analyze the convergence of a numerical method for a given integral equation we need information about the smoothness of the exact solution. This becomes more significant when we want to find the maximal order of convergence of a method. For equations (1.0.1) with smooth kernels, the smoothness of the kernel K and the free term f determines the smoothness of the solution u (if it exists) on the closed interval of integration $[0,1]$. If we allow weakly singular kernels, then the resulting solutions are typically non-smooth at the endpoints of the interval of the integration $[0,1]$, where their derivatives become unbounded [24, 68, 72, 76, 93, 94], see also [34, 37, 38, 64, 65, 70, 90] and Theorem 5.0.1 in Chapter 5.

Methods that are often used to solve Fredholm integral equations of the second kind are collocation and product-integration methods. In general a collocation method is a projection method for solving integral equations in which we first choose a finite-dimensional space of candidate solutions (usually, polynomial splines up to a certain degree) and a number of points in the domain (properly chosen collocation points). The collocation solution to an equation is determined by the condition that the equation must be satisfied at the collocation points. Thus, to determine the collocation solution of the equation we need to solve the certain system of the algebraic equations. The principal difficulty with this approach is that there are integrals which must usually be evaluated numerically, resulting in what we call the discrete collocation method [6, 7]. In the case of kernels in the form

$$K(x, y) = K_1(x, y)K_2(x, y),$$

where $K_1(x, y)$ may have some integrable singularities and $K_2(x, y)$ is a regular function of its arguments, these integrals can be evaluated by product integration techinques (see e.g. [4, 6, 47, 48, 54, 77, 80, 85]). In general, product integration is a method for the approximate evaluation of integrals of the form (for example see [8, 80])

$$I_\kappa(\zeta) = \int_0^1 \kappa(t)\zeta(t)dt,$$

where $\zeta(t)$ is 'smooth', but $\kappa(t)$ has an integrable singularity in $[0, 1]$. The essence of the method is to replace $\zeta(t)$ by

$$\zeta_n(t) = \sum_{j=0}^{n} \Psi_j(t)\zeta(t_j),$$

where $\Psi_j(t_i) = \delta_{ij}$ (the Kronecker symbol) and to approximate $I_\kappa(\zeta)$ by

$$I_\kappa(\zeta_n) = \sum_{j=0}^{n} \alpha_j \zeta(t_j),$$

where

$$\alpha_j = \int_0^1 \Psi_j(t)\kappa(t)dt.$$

The functions $\Psi_j(t)$ are chosen so, that α_j can be calculated exactly or, at least, sufficiently accurately. In particular, the functions $\Psi_j(t)$ can be chosen to be piecewise polynomials. For example, if the $\Psi_j(t)$ are the piecewise linear functions satisfying $\Psi_j(t_i) = \delta_{ij}$, where $t_i = ih$ $\left(h = \frac{1}{n}\right)$, $i = 0, 1, 2, \ldots, n$, then $\zeta_n(t)$ is the piecewise linear polynomial.

In collocation and product integration methods the singular behaviour of the exact solution of a weakly singular integral equation (1.0.1) can be taken into account by using polynomial splines and special graded grids with the nodes (cf., e.g. [6, 25, 62, 66, 81, 82, 84, 90, 94, 95])

$$x_i = \frac{1}{2}\left(\frac{i}{N}\right)^r, \quad i = 0, 1, \ldots, N, \quad x_{N+i} = 1 - x_{N-i}, \quad i = 1, 2, \ldots N, \qquad (1.0.6)$$

where N is a positive integer and $r \in [1, \infty)$. In (1.0.6) the parameter r is describing the non-uniformity of the grid: if $r = 1$, then the gridpoints $x_0, \ldots, x_{2N}$ are uniformly located on $[0, 1]$; if $r > 1$, then the gridpoints x_i $(i = 0, \ldots, 2N)$ are more densely clustered near the endpoints of the interval $[0, 1]$ where the solution of (1.0.1) may be singular. High-order methods use large values of r. In particular, in the case of kernels (1.0.4) with $0 < \nu < 1$ we obtain (see, e.g. [94]) a convergence of order $O(N^{-m})$ for

$$r \geq \frac{m}{1-\nu}$$

by using a collocation method based on piecewise polynomials of degree $m-1$ $(m \geq 1)$ and gridpoints (1.0.6). However, the use of strongly non-uniform grids with nodes (1.0.6) with large values of r may cause serious implementation problems of the method and may lead to unstable behaviour of numerical results. Note that the question of the stability of piecewise polynomial collocation methods on nonuniform grids (1.0.6) has been discussed in [35].

In this thesis we propose two new classes of high order numerical methods, which do not need graded grids for solving linear Fredholm integral equations of the second kind with singularities. The methods are developed by means of the 'central part' interpolation by polynomials on the uniform grid and smoothing change of variables. With the help of a change of variables we improve the boundary behaviour of the exact solution of the problem.

We introduce in $\mathbb{R} = (-\infty, \infty)$ the uniform grid

$$\mathbb{R}_h := \{jh : j \in \mathbb{Z}\}, \quad h = \frac{1}{n}, \quad n \in \mathbb{N}, \tag{1.0.7}$$

where, as usual, $\mathbb{N}$ is the set of all positive integers and $\mathbb{Z}$ is the set of integers.

Let $m \in \mathbb{N}$, $m \geq 2$ be fixed. For given interval $[a, b]$, $-\infty < a < b < \infty$, let $C[a, b]$ be a space of continuous functions $f : [a, b] \to \mathbb{R}$. We define a piecewise polynomial interpolant $\Pi_{h,m} f \in C\,[0, 1]$ for a function $f \in C[-\delta, 1+\delta]$, $\delta > 0$, $h \leq \frac{2\delta}{m}$, as follows. On every subinterval

$$[jh, (j+1)\,h]\,, \quad 0 \leq j \leq n-1,$$

the function $\Pi_{h,m} f$ is defined independently from other subintervals as a polynomial $\Pi_{h,m}^{[j]} f$ of degree $\leq m-1$ that interpolates f at m points lh neighbouring jh from two sides:

$$\Pi_{h,m}^{[j]} f\,(lh) = f\,(lh)\,, \quad l = j - \frac{m}{2} + 1, \ldots, j + \frac{m}{2} \quad \text{if } m \text{ is even},$$

$$\Pi_{h,m}^{[j]} f\,(lh) = f\,(lh)\,, \quad l = j - \frac{m-1}{2}, \ldots, j + \frac{m-1}{2} \quad \text{if } m \text{ is odd}.$$

With these interpolants we will guarantee the interpolating at the central parts of the interval. That means it is possible to show, that in the central parts of the interval, the estimates of interpolation error are approximately 2^m times more precise than on the whole interval. In the central parts of the interval, the interpolation process on the uniform grid also has good stability properties as m increases (see Lemmas 3.1.1 and 3.2.1 in Chapter 3). The formula for interpolant is given by

$$\left(\Pi_{h,m}^{[j]} f\right)(t) = \sum_{k\in\mathbb{Z}_m} f\left((j+k)\,h\right) L_{k,m}\left(nt-j\right), \quad j = 0,\dots,n-1.$$

Here $L_{k,m}$ are the Lagrange fundamental polynomials

$$L_{k,m}\left(t\right) = \prod_{l\in\mathbb{Z}_m\setminus\{k\}} \frac{t-l}{k-l}, \quad k \in \mathbb{Z}_m,$$

and

$$\mathbb{Z}_m = \left\{k \in \mathbb{Z} : -\frac{m}{2} < k \le \frac{m}{2}\right\}.$$

For $m \ge 3$, $\Pi_{h,m} f$ uses values of f outside of $[0,1]$. For $f \in C\,[0,1]$, $\Pi_{h,m} f$ obtains a sense after an extension of f onto $[-\delta, 1+\delta]$ with some $\delta > 0$. We will use the simplest extension operator

$$E_\delta : C\,[0,1] \to C\,[-\delta, 1+\delta]\,, \quad (E_\delta f)\,(t) = \left\{ \begin{array}{lll} f\,(0) & \text{for} & -\delta \le t \le 0 \\ f\,(t) & \text{for} & 0 \le t \le 1 \\ f\,(1) & \text{for} & 1 \le t \le 1+\delta \end{array} \right\},$$

that maintains the smoothness of f, and define the operator

$$P_{h,m} := \Pi_{h,m} E_\delta : C\,[0,1] \to C\,[0,1]\,. \tag{1.0.8}$$

To solve equation (1.0.1) with the kernel $K \in \mathcal{S}^{m,\nu}$, $m \ge 2$ and $\nu \in (-\infty, 1)$ we first perform in (1.0.1) a smoothing change of variables,

$$x = \varphi(t), \quad y = \varphi(s), \quad 0 \le t \le 1, \quad 0 \le s \le 1, \tag{1.0.9}$$

where $\varphi : [0,1] \to [0,1]$ is a smooth strictly increasing function such that $\varphi(0) = 0$, $\varphi(1) = 1$; for more complete description of φ and its smoothing properties see Chapter 5. The change of variables is necessary to suppress the singularities of the derivatives of the solution. Equation (1.0.1) takes the form

$$v(t) = \int_0^1 K_\varphi(t,s)v(s)ds + f_\varphi(t), \quad 0 \le t \le 1, \tag{1.0.10}$$

where

$$f_\varphi(t) := f(\varphi(t)),\ K_\varphi(t,s) := K(\varphi(t),\varphi(s))\varphi'(s);$$

the solutions of equations (1.0.1) and (1.0.10) are in the relations

$$v(t) = u(\varphi(t)),\ u(x) = v(\varphi^{-1}(x)),$$

where φ^{-1} is the inverse function of φ. We solve equation (1.0.10) by collocation techniques based on a central part interpolation by polynomials on the uniform grid. Actually, using the interpolation projector $P_{h,m}$ defined in (1.0.8), we approximate equation (1.0.10), i.e. an equation of the form

$$v = T_\varphi v + f_\varphi,$$

by an equation

$$v_h = P_{h,m} T_\varphi v_h + P_{h,m} f_\varphi, \tag{1.0.11}$$

where v_h is a function we have to find and T_φ is the integral operator of (1.0.10), given by the formula

$$(T_\varphi v)(t) = \int_0^1 K_\varphi(t,s) v(s) ds, \quad 0 \leq t \leq 1.$$

Equation (1.0.11) is the operator form of our piecewise polynomial collocation method based on a central part interpolation on the uniform grid (1.0.7). The matrix form of this method is given in Chapter 6 (see (6.2.3)). We study the attainable order of convergence of this method. The obtained results are given by Theorem 6.1.1.

To solve equation (1.0.1) by a product integration method based on the central part interpolation and smoothing change of variables, we consider the kernels with algebraic and logarithmic singularity, (1.0.4) and (1.0.5), respectively. As in the collocation method, we perform the smoothing change of variables (1.0.9) in the initial equations. In the case of the kernel with singularity of the algebraic type we achieve an equation of the form

$$v(t) = \int_0^1 \left[\mathcal{A}(t,s)|t-s|^{-\nu} + \mathcal{B}(t,s)\right] v(s) ds + g(t), \quad 0 \leq t \leq 1, \tag{1.0.12}$$

where $v(t) = u(\varphi(t))$ is the new function we look for,

$$g(t) = f(\varphi(t)) \quad \mathcal{A}(t,s) = a(\varphi(t), \varphi(s)) \Phi(t,s)^{-\nu} \varphi'(s),$$

$$\mathcal{B}(t,s) = b(\varphi(t), \varphi(s)) \varphi'(s),$$

and

$$\Phi(t,s) = \left\{ \begin{array}{ccc} \dfrac{\varphi(t)-\varphi(s)}{t-s} & \text{for} & t \neq s \\ \varphi'(s) & \text{for} & t = s \end{array} \right\}, \quad 0 \leq t, s \leq 1. \tag{1.0.13}$$

In the case of the kernel with singularity of the logarithmic type we achieve an equation of the form

$$v(t) = \int_0^1 (A(t,s) \log|t-s| + B(t,s)) v(s) ds + g(t), \quad 0 \leq t \leq 1, \tag{1.0.14}$$

where $v(t) = u(\varphi(t))$ is the new function we look for,

$$g(t) = f(\varphi(t)), \quad A(t,s) = a(\varphi(t), \varphi(s))\varphi'(s),$$

$$B(t,s) = [a(\varphi(t), \varphi(s)) \log \Phi(t,s) + b(\varphi(t), \varphi(s))]\varphi'(s),$$

and $\Phi(t,s)$ is given by (1.0.13).

Using the interpolation projector $P_{h,m}$, determined by the formula (1.0.8), we approximate equations (1.0.12) and (1.0.14) by equations

$$v_h(t) = \int_0^1 |t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v_h(s))ds + \int_0^1 P_{h,m}(\mathcal{B}(t,s)v_h(s))ds + g(t), \quad 0 \le t \le 1, \tag{1.0.15}$$

and

$$v_h(t) = \int_0^1 \log|t-s| \, P_{h,m}(A(t,s)\, v_h(s))\, ds + \int_0^1 P_{h,m}(B(t,s)\, v_h(s))\, ds + g(t), \quad 0 \le t \le 1. \tag{1.0.16}$$

In these equations operator $P_{h,m}$ is applied to the products $\mathcal{A}(t,s)v_h(s)$, $\mathcal{B}(t,s)v_h(s)$ and $A(t,s)v_h(s)$, $B(t,s)v_h(s)$ as functions of s, treating t as a parameter. With (1.0.15) and (1.0.16) the operator forms of a product integration method corresponding to the piecewise polynomial central part interpolation on the uniform grid $\{ih : i = 0, \ldots, n\}$ are given. The matrix forms of the methods are given by (7.1.56) and (7.2.28), respectively. Such approach is hopeful due to a simpler assembling of the algebraic system of equations compared to collocation method (1.0.11).

We establish the optimal convergence order of methods (1.0.15) and (1.0.16). The obtained results are given by Theorems 7.1.2 and 7.2.2.

The thesis is organised as follows.

In Chapter 2 definitions and basic results used in this work are introduced.

In Chapter 3 we introduce the idea of central part interpolation. The central part interpolation scheme has a suprisingly good error estimate. Additionally, in the central parts of the interval, the interpolation process on the uniform grid has good stability properties.

In Chapters 4 and 5 we introduce the smoothness-singularity class $\mathcal{S}^{m,\nu}$ of kernels and a weighted space of smooth functions $C^{m,\nu}(0,1)$ for describing the smoothness of the solution of a weakly singular integral equation. By $C^{m,\nu}(0,1)$, $m \ge 1$, $\nu < 1$, we denote the space of continuous functions $f : [0,1] \to \mathbb{R}$ which are m-times continuously differentiable on $(0,1)$ such that

$$\left|f^{(j)}(x)\right| \leq c \left\{ \begin{array}{ccc} 1 & \text{for} & j+\nu-1<0 \\ 1+|\log \rho(x)| & \text{for} & j+\nu-1=0 \\ \rho(x)^{-j-\nu+1} & \text{for} & j+\nu-1>1 \end{array} \right\},$$

$$0<x<1, \quad j=0,\ldots,m,$$

where $c=c(f)$ is a positive constant and

$$\rho(x)=\min\{x,1-x\}$$

is the distance from $x\in(0,1)$ to the boundary of the interval $(0,1)$. If $K\in\mathcal{S}^{m,\nu}$, $m\in\mathbb{N}$, $\nu\in\mathbb{R}$, $\nu<1$, then for $f\in C^{m,\nu}(0,1)$ also the solution u of equation (1.0.1) belongs to $C^{m,\nu}(0,1)$. Moreover, we study a class of functions φ for the change of variables and the smoothing properties of such functions (see Theorem 5.2.1).

Chapters 6 and 7 are devoted to the numerical solution of the integral equation (1.0.1) with singularities. Firstly we undertake a change of variables to suppress the singularities - the solution of the transformed equation will be m times continuously differentiable on $[0,1]$ including the boundary points 0 and 1. In Chapter 6 we apply a collocation technique based on a central part interpolation by polynomials on the uniform grid for the numerical solution of equations (1.0.1) with kernels $K\in\mathcal{S}^{m,\nu}$. We study the convergence and the convergence order of this method. In Chapter 7 we use product integration approach with the central part interpolation to solve integral equations (1.0.1) with kernels $K(x,y)$, which may have a diagonal singularity as $y\to x$ and the boundary singularities as $y\to 0$ and/or $y\to 1$. We study the attainable order of the convergence of this method.

In Chapter 8 a series of numerical tests is given. We compare the results of our computational experiments with the theoretical results which have been obtained in Chapters 6 and 7. The numerical results support the theoretical analysis.

Most of the results given in Chapters 3-8 are published in [59, 60, 61], the thesis also contains new results which have not been published yet.

Chapter 2

Notations and Basic Results

In this chapter we introduce some basic notations and formulate some well-known results, which we need later.

2.1 Notations and some results from analysis

In this section we present the notations and some results from analysis. Throughout this work c, c', c_0, ... denote positive constants which may have different values in different occurrences. By $\mathbb{N} = \{1, 2, \dots\}$ we denote the set of all positive integers, by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ the set of non-negative integers, by $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$ the set of integers and by $\mathbb{R} = (-\infty, \infty)$ the set of real numbers.

We denote the set of polynomials of degree not exceeding $m \in \mathbb{N}_0$ by $\mathcal{P}_m$.

By $L^\infty(a, b)$ we denote the set of measurable functions $v : [a, b] \to \mathbb{R}$, such that

$$\inf_{D \subset [a,b]: \mu(D)=0} \sup_{x \in [a,b] \setminus D} |v(x)| < \infty,$$

where $\mu(D)$ is the Lebesgue measure of the set D. The set $L^\infty(a, b)$ is a Banach space with the norm

$$\|v\|_{L^\infty(a,b)} \equiv \|v\|_\infty = \inf_{D \subset [a,b]: \mu(D)=0} \sup_{x \in [a,b] \setminus D} |v(x)|, \quad v \in L^\infty(a, b).$$

By $C^m(D)$ ($D \subset \mathbb{R}^n$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $\mathbb{R}^1 \equiv \mathbb{R}$, $C^0(D) \equiv C(D)$) we denote the space of continuous and m times continuously differentiable functions $v : D \to \mathbb{R}$.

By $C[a, b]$ ($-\infty < a < b < \infty$) we denote the Banach space of continuous functions $v : [a, b] \to \mathbb{R}$ with the norm

$$\|v\|_{C[a,b]} = \|v\|_\infty = \max_{a \le x \le b} |v(x)|.$$

By $C^m[a,b]$ ($m \in \mathbb{N}$, $a < b$) we denote the Banach space of m times continuously differentiable functions $v : [a,b] \to \mathbb{R}$ with the norm

$$\|v\|_{C^m[a,b]} = \sum_{i=0}^{m} \|v^{(i)}\|_\infty.$$

The following result gives a representation of the error that occurs when sufficiently smooth functions are interpolated by polynomials (see e.g [8, 71]).

Theorem 2.1.1. *Assume $f \in C^{m+1}[a,b]$ $m \in \mathbb{N}_0$. Let $\{x_0, \dots, x_m\}$ be any pairwise distinct points in $[a,b]$ and let $p \in \mathcal{P}_m$ be the polynomial with $p(x_k) = f(x_k)$ for $k = 0, \dots, m$. Then for each $x \in [a,b]$ the approximation error is given by*

$$f(x) - p(x) = (x - x_0) \dots (x - x_m) \frac{f^{(m+1)}(\xi)}{(m+1)!}, \tag{2.1.1}$$

with $\xi = \xi(x) \in (a,b)$.

Definition 2.1.1. *A subset $M \subset X$ is called relatively compact in a Banach space X, if any sequence $(x_n) \subset M$ contains a subsequence converging in X.*

Theorem 2.1.2 (Arzelà-Ascoli)**.** *A set $S \subset C[0,1]$ is relatively compact in $C[0,1]$ if and only if the following two conditions are fulfilled:*
(i) the functions $u \in S$ are uniformely bounded, i.e., there is a constant c such that $|u(x)| \leq c$ for all $x \in [0,1]$, $u \in S$;
(ii) the functions $u \in S$ are equicontinuous, i.e., for every $\epsilon > 0$ there is a $\delta > 0$ such that $x_1, x_2 \in [0,1]$, $|x_1 - x_2| \leq \delta$ implies $|u(x_1) - u(x_2)| \leq \epsilon$ for all $u \in S$.

The proof of this result can be found e.g. in [47].

For differentiating compositions we need the following result (see, for example, [45, p. 111]).

Theorem 2.1.3 (Faà di Bruno)**.** *Let u be an m times continuously differentiable function on an interval which contains the values of $\varphi \in C^m[0,1]$. Then the composite function $u(\varphi(x))$ is m times continously differentiable on $[0,1]$ and the derivatives of the composition function at any point $x \in [0,1]$ can be expressed by Faà di Bruno differentiation formula*

$$\left(\frac{d}{dx}\right)^j u(\varphi(x))$$

$$= \sum_{k_1 + 2k_2 + \ldots + jk_j = j} \frac{j!}{k_1! \dots k_j!} u^{(k_1 + \ldots + k_j)}(\varphi(x)) \left(\frac{\varphi'(x)}{1!}\right)^{k_1} \cdots \left(\frac{\varphi^{(j)}(x)}{j!}\right)^{k_j}, \tag{2.1.2}$$

where the sum is taken over all non-negative integers $k_1, \dots, k_j$, $j = 1, \dots, m$, such that $k_1 + 2k_2 + \ldots + jk_j = j$.

2.2 Bounded and Compact Operators

In this section, we present some results from the theory of linear operators (see, for example, [6, 8, 25, 47])

Let X and Y be Banach spaces. A linear operator $A : X \to Y$ is called bounded if there exists a positive constant c such that

$$\|Ax\|_Y \leq c\,\|x\|_X$$

for all $x \in X$. An operator $A : X \to Y$ is said to be continuous if

$$\|x_n - x\|_X \to 0 \ \text{ for } n \to \infty$$

implies

$$\|Ax_n - Ax\|_Y \to 0 \ \text{ for } n \to \infty.$$

A linear operator $A : X \to Y$ is continuous if and only if it is bounded.

One says that a linear operator $A : X \to Y$ has the inverse $A^{-1} : Y \to X$ if $A^{-1}A = I_X$ and $AA^{-1} = I_Y$ where I_X and I_Y are the identity mappings in X and Y, respectively.

For the linear operator $A : X \to Y$ we denote

$$\mathcal{N}(A) = \{x \in X : Ax = 0\} \ \text{ - the null space of } A,$$

$$\mathcal{R}(A) = \{y \in y : y = Ax\} \ \text{ - the range of } A.$$

By $\mathcal{L}(X, Y)$ we denote the Banach space of all linear bounded operators $A : X \to Y$ with the norm

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{x \in X, \|x\|_X \leq 1} \|Ax\|_Y, \quad A \in \mathcal{L}(X, Y).$$

Theorem 2.2.1 (Banach)**.** *Let X and Y be Banach spaces and $A \in \mathcal{L}(X, Y)$. If $\mathcal{N}(A) = 0$ and $\mathcal{R}(A) = Y$ then A has the inverse $A^{-1} \in \mathcal{L}(Y, X)$.*

Theorem 2.2.2 (Banach-Steinhaus)**.** *Let $A : X \to Y$ be a bounded linear operator and let (A_n) be a sequence of linear bounded operators $A_n : X \to Y$ from a Banach space X into a Banach space Y. For pointwise convergence*

$$A_n x \to Ax, \quad n \to \infty \quad \textit{for all } \ x \in X$$

it is necessary and sufficient that

$$\|A_n\|_{\mathcal{L}(X,Y)} \leq c \quad \textit{for all } \ n \in \mathbb{N}$$

with some constant c and that

$$A_n x \to Ax, \quad n \to \infty \quad \textit{for all } \ x \in V,$$

where V is some dense subset of X.

Theorem 2.2.3. *Let X be a Banach space, and let $A \in \mathcal{L}(X,X)$ be a bounded linear operator from X into X with $\|A\|_{\mathcal{L}(X,X)} < 1$. Then there exists $(I-A)^{-1} \in \mathcal{L}(X,X)$, and*

$$\|(I-A)^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{1}{1-\|A\|_{\mathcal{L}(X,X)}},$$

where I is the identity mapping in X.

Theorem 2.2.4. *Let X and Y be Banach spaces. If the operators $A, B \in \mathcal{L}(X,Y)$ are such that A has a bounded inverse $A^{-1} \in \mathcal{L}(Y,X)$ and*

$$\|B\|_{\mathcal{L}(X,Y)}\|A^{-1}\|_{\mathcal{L}(Y,X)} < 1,$$

then $A+B$ has a bounded inverse $(A+B)^{-1} \in \mathcal{L}(X,Y)$ and

$$\|(A+B)^{-1}\|_{\mathcal{L}(Y,X)} \leq \frac{\|A^{-1}\|_{\mathcal{L}(Y,X)}}{1-\|B\|_{\mathcal{L}(X,Y)}\|A^{-1}\|_{\mathcal{L}(Y,X)}}.$$

Definition 2.2.1. *A linear operator $A : X \to Y$ is called compact if A transforms every bounded set of X into a relatively compact set of Y.*

Equivalently, $A : X \to Y$ is compact if for every bounded sequence $(u_n) \subset X$, the sequence (Au_n) contains a subsequence that converges in Y. A linear compact operator $A : X \to Y$ is bounded and continuous.

Theorem 2.2.5. *Let X, Y and V be Banach spaces. Let A be a compact operator mapping X into Y and let $B_n : Y \to V$ be a pointwise convergent sequence of bounded linear operators with limit operator $B : Y \to V$. Then*

$$\|(B_n - B)A\|_{\mathcal{L}(X,V)} \to 0 \quad \textit{for} \;\; n \to \infty.$$

Theorem 2.2.6. *Let $T_n : X \to Y$, $n = 1, 2, \ldots$, be linear compact operators, $T : X \to Y$ a linear bounded operator, and let $\|T_n - T\|_{\mathcal{L}(X,Y)} \to 0$ as $n \to \infty$. Then $T : X \to Y$ is compact.*

Theorem 2.2.7 (Fredholm alternative)**.** *Let X be a Banach space, and let $A \in \mathcal{L}(X,X)$ be a compact operator. Then the equation $x = Ax + f$ with $f \in X$ has a unique solution $x \in X$ if and only if the homogeneous equation $x = Ax$ has only the trivial solution $x = 0$. In such a case, the operator $I - A$ has a bounded inverse $(I-A)^{-1} \in \mathcal{L}(X,X)$.*

Definition 2.2.2. *A linear operator $P : X \to X$ from X to itself, is called a projection operator if $P^2 = P$ i.e. $P(Px) = Px$, for any $x \in X$.*

2.3 Compact convergence

In this section we introduce a concept of compact convergence of operators by Vainikko and present a result from his discrete convergence theory, which in a more general setting can be found in [87] - [90]. In the following simple setting the concept is closely related to the Anselone's concept of collectively compact family of operators, see [3].

Let X be a Banach space.

Definition 2.3.1. *A sequence (T_n) of operators $T_n \in \mathcal{L}(X,X)$ with $n \in \mathbb{N}$ is called compactly converging to $T \in \mathcal{L}(X,X)$ (we write $T_n \longrightarrow T$ compactly) if*

$$\|T_n u - Tu\|_X \to 0 \quad \text{as } n \to \infty \quad \text{for any } \ u \in X, \tag{2.3.1}$$

and for any bounded sequence (u_n) of elements $u_n \in X$, $n \in \mathbb{N}$, it follows that the sequence $(T_n u_n)$ is relatively compact in X (i.e. every subsequence $(T_n u_n)_{n\in\mathbb{N}'\subset\mathbb{N}}$ contains a subsequence $(T_n u_n)_{n\in\mathbb{N}''\subset\mathbb{N}'}$ converging in X).

Let us consider the equation

$$u = Tu + f, \tag{2.3.2}$$

where $f \in X$ and $T \in \mathcal{L}(X,X)$. We approximate (2.3.2) by the equations

$$u_n = T_n u_n + f, \tag{2.3.3}$$

where $n \in \mathbb{N}$, $T_n \in \mathcal{L}(X,X)$. We are interested in the convergence $u_n \to u$ for $n \to \infty$, where $u \in X$ and $u_n \in X$ are the solutions of equations (2.3.2) and (2.3.3), respectively. The following theorem, which we'll need later in Chapter 7, gives us sufficient conditions.

Theorem 2.3.1. *Assume that $T_n \longrightarrow T$ compactly whereby $T_n \in \mathcal{L}(X,X)$ $(n \in \mathbb{N})$ and $T \in \mathcal{L}(X,X)$ are compact operators. Suppose that the homogeneous equation $v = Tv$ has in X only the trivial solution $v = 0_X$.*

Then equation (2.3.2) has a unique solution $u \in X$ and there exists an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, equation (2.3.3) has a unique solution $u_n \in X$; moreover $\|u_n - u\|_X \to 0$ for $n \to \infty$ and the following error estimate holds:

$$c_1 \|T_n u - Tu\|_X \leq \|u_n - u\|_X \leq c_2 \|T_n u - Tu\|_X, \quad n \geq n_0. \tag{2.3.4}$$

Here c_1 and c_2 are some positive constants not depending on n and f.

Chapter 3

Central part interpolation

In this chapter we introduce a "central part interpolation" scheme considered in [61]. The central part interpolation has good error estimates. Additionally, in the central parts of the interval the interpolation process on the uniform grid has good stability properties comparable with the stability of Chebyshev interpolation.

3.1 Central part interpolation by polynomials

Given an interval $[a, b]$ $(a < b)$ and $m \in \mathbb{N}$, introduce the uniform grid consisting of m points

$$x_i = a + \left(i - \frac{1}{2}\right) h, \quad i = 1, \ldots, m, \quad h = \frac{b-a}{m}. \tag{3.1.1}$$

Denote by $\mathcal{P}_{m-1}$ the set of polynomials of degree not exceeding $m-1$ and by Π_m the Lagrange interpolation projection operator assigning to any $f \in C[a, b]$ the polynomial $\Pi_m f \in \mathcal{P}_{m-1}$ that interpolates f at points (3.1.1):

$$(\Pi_m f)(x) = \sum_{j=1}^{m} f(x_j) \prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{x - x_k}{x_j - x_k}, \quad a \leq x \leq b, \quad m \geq 2,$$

$$(\Pi_1 f)(x) = f(x_1), \quad a \leq x \leq b.$$

Lemma 3.1.1. *In the case of interpolation knots* (3.1.1) *with* $m \in \mathbb{N}$, *for* $f \in C^m[a, b]$ *it holds*

$$\max_{a \leq x \leq b} |f(x) - (\Pi_m f)(x)| \leq \theta_m h^m \max_{a \leq x \leq b} \left| f^{(m)}(x) \right|, \tag{3.1.2}$$

with

$$\theta_m = \frac{1 \cdot 3 \cdot \ldots \cdot (2m-1)}{2^m \, m!} = \frac{(2m)!}{2^m \, m!(2 \cdot 4 \cdot \ldots \cdot 2m)} \cong (\pi m)^{-\frac{1}{2}},$$

where $\theta_m \cong \epsilon_m$ *means that* $\theta_m/\epsilon_m \to 1$ *as* $m \to \infty$.

Further, for $m = 2k,\ k \geq 1$,

$$\max_{x_k \leq x \leq x_{k+1}} |f(x) - (\Pi_m f)(x)| \leq \vartheta_m h^m \max_{a \leq x \leq b} \left| f^{(m)}(x) \right|, \tag{3.1.3}$$

with

$$\vartheta_m = 2^{-2m} \frac{m!}{((m/2)!)^2} \cong \sqrt{2/\pi}\, m^{-\frac{1}{2}} 2^{-m}, \tag{3.1.4}$$

whereas for $m = 2k+1,\ k \geq 1$,

$$\max_{x_k \leq x \leq x_{k+2}} |f(x) - (\Pi_m f)(x)| \leq \vartheta_m h^m \max_{a \leq x \leq b} \left| f^{(m)}(x) \right|, \tag{3.1.5}$$

with

$$\vartheta_m = \frac{2\sqrt{3}}{9} \frac{(k!)^2}{(2k+1)!} \cong \frac{2\sqrt{6\pi}}{9}\, m^{-\frac{1}{2}} 2^{-m}. \tag{3.1.6}$$

Proof. These estimates are consequences of the error formula (see (2.1.1))

$$f(x) - (\Pi_m f)(x) = \frac{f^{(m)}(\xi)}{m!}(x - x_1)\dots(x - x_m), \quad x \in [a,b], \quad \xi = \xi(x) \in (a,b),$$

that holds for the interpolation with arbitrary pairwise different knots $x_1, \dots x_m$ of $[a,b]$.

Indeed for points (3.1.1), the maximum of $|(x - x_1)\dots(x - x_m)|$ on $[a,b]$ is attained at the end points of the interval, thus

$$\max_{a \leq t \leq b} |(x - x_1)\dots(x - x_m)| = \frac{h}{2} \cdot \frac{3}{2} h \cdot \dots \cdot \frac{2m-1}{2} h$$
$$= \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)}{2^m} h^m,$$

and (3.1.2) holds with

$$\theta_m = \frac{1 \cdot 3 \cdot \dots \cdot (2m-1)}{2^m\, m!} = \frac{(2m)!}{2^m\, m!(2 \cdot 4 \cdot \dots \cdot 2m)}.$$

The Stirling formula

$$m! \cong \sqrt{2\pi m}\, m^m e^{-m} \tag{3.1.7}$$

yields

$$2 \cdot 4 \cdot 6 \cdot \dots \cdot 2m = 2^m (1 \cdot 2 \cdot \dots \cdot m) \cong 2^m \sqrt{2\pi m}\, m^m e^{-m},$$

and we get

$$\theta_m \cong \frac{2\sqrt{\pi m}\,(2m)^{2m} e^{-2m}}{2^m \sqrt{2\pi m}\, m^m e^{-m} 2^m \sqrt{2\pi m}\, m^m e^{-m}} = \frac{2\sqrt{\pi m}}{2\pi m} = (\pi m)^{-\frac{1}{2}}.$$

Let us prove (3.1.3) and (3.1.4) for $m = 2k$ ($k \in \mathbb{N}$). Note that the maximum of $|(x-x_1)\dots(x-x_{2k})|$ on $[x_k, x_{k+1}]$ is attained at the centre of $[x_k, x_{k+1}]$ (which is also the centre of $[a,b]$, see Figure 3.1) and equals

$$\left(\frac{1}{2}h\right)^2\left(\frac{3}{2}h\right)^2\dots\left(\frac{2k-1}{2}h\right)^2 = \frac{[1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)]^2}{2^{2k}}h^m.$$

Thus, (3.1.3) holds with

$$\vartheta_m = \frac{[1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)]^2}{2^m\, m!} = \frac{[(2k)!]^2}{2^m\, m!(2\cdot 4\cdot 6\cdot\ldots\cdot 2k)^2} = \frac{m!}{2^{2m}\left[\left(\frac{m}{2}\right)!\right]^2}.$$

This together with (3.1.7) yields (3.1.4):

$$\vartheta_m \cong \frac{\sqrt{2\pi m}\, m^m e^{-m}}{2^{2m}\left[\sqrt{2\pi\frac{m}{2}}\,\left(\frac{m}{2}\right)^{\frac{m}{2}}e^{-\frac{m}{2}}\right]^2} = \frac{\sqrt{2}\sqrt{\pi m}\, m^m\, e^{-m}}{2^{2m}\,\pi m\, m^m\, 2^{-m}\, e^{-m}} = 2^{-m}m^{-\frac{1}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.$$

Finally, let us prove (3.1.5) and (3.1.6). Now $m = 2k+1$, $k \ge 1$. We estimate on $[x_k, x_{k+2}]$ separately the product

$$|(x-x_k)(x-x_{k+1})(x-x_{k+2})|$$

and the remaining product

$$|x-x_1|\dots|x-x_{k-1}|\,|(x-x_{k+3})|\dots|(x-x_m)|\,.$$

Undertaking the shift $x - x_{k+1} = y$, we have

$$\max_{x_k\le x\le x_{k+2}}|(x-x_k)(x-x_{k+1})(x-x_{k+2})| = \max_{-h\le y\le h}|(y-h)y(y+h)|$$

$$= \max_{-h\le y\le h}\left|y^3-h^2y\right|.$$

Function $\phi(y) = y^3 - h^2y$ has a local maximum on $[-h,h]$ at $y = -\frac{\sqrt{3}}{3}h$ with $\phi\left(-\frac{\sqrt{3}}{3}h\right) = \frac{2\sqrt{3}}{9}h^3$. Thus

$$\max_{x_k\le x\le x_{k+1}}|(x-x_k)(x-x_{k+1})(x-x_{k+2})| = \frac{2\sqrt{3}}{9}h^3.$$

The maximum of $|x-x_1|\dots|x-x_{k-1}|\,|(x-x_{k+3})|\dots|(x-x_m)|$ on $[x_k, x_{k+2}]$ is attained at the center x_{k+1} of $[x_k, x_{k+2}]$ (see Figure 3.1) and equals

$$(2h)^2(3h)^2\dots(kh)^2 = (k!)^2h^{2(k-1)} = (k!)^2h^{m-3}.$$

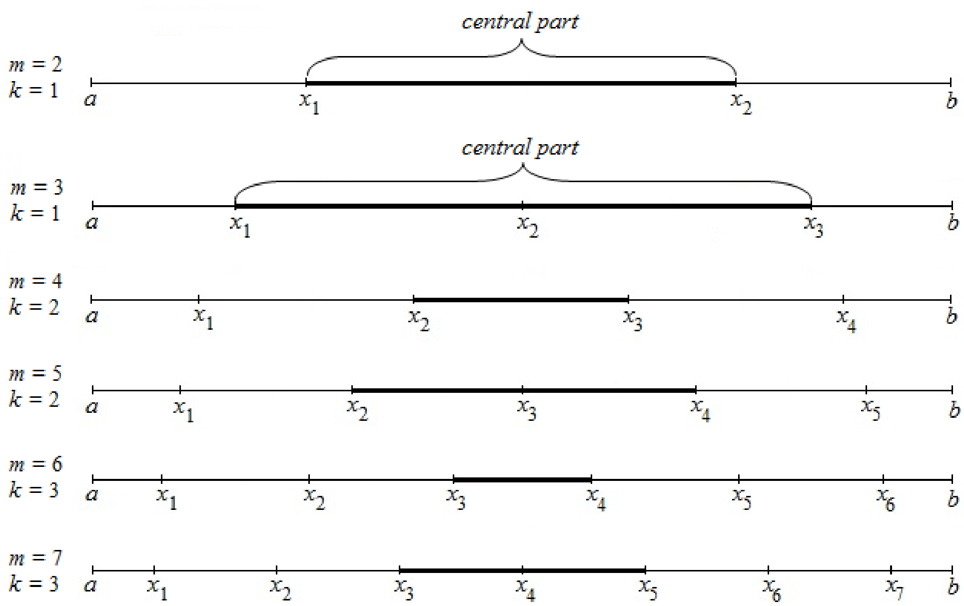

Figure 3.1: Central parts of an interval $[a,b]$ with different values of m

This results to the estimate

$$\max_{x_k \le x \le x_{k+2}} |(x-x_1)\dots(x-x_m)| \le \frac{2\sqrt{3}}{9}(k!)^2 h^m$$

and (3.1.5) with

$$\vartheta_m = \frac{2\sqrt{3}}{9}\frac{(k!)^2}{(2k+1)!}.$$

Due to the Stirling formula (3.1.7)

$$\vartheta_m = \frac{2\sqrt{3}}{9}\frac{(k!)^2}{(2k+1)!} \cong \frac{2\sqrt{3}}{9}\frac{2\pi\, k\, k^{2k}e^{-2k}}{\sqrt{2\pi(2k+1)}\,(2k+1)^{2k+1}e^{-(2k+1)}}$$

$$= \frac{2\sqrt{3}}{9}\frac{\sqrt{2\pi}}{\sqrt{2k+1}\,e^{-1}}\left(\frac{k}{2k+1}\right)^{2k+1}.$$

Since

$$\frac{k}{2k+1} = \frac{k}{2k}\left(\frac{2k}{2k+1}\right) = \frac{1}{2}\left(1-\frac{1}{2k+1}\right),$$

$$\left(\frac{k}{2k+1}\right)^{2k+1} = \left(\frac{1}{2}\right)^{2k+1}\left(1-\frac{1}{2k+1}\right)^{2k+1} \cong \left(\frac{1}{2}\right)^{2k+1} e^{-1},$$

we obtain (3.1.6):

$$\vartheta_m = \frac{2\sqrt{3}}{9}\frac{(k!)^2}{(2k+1)!} \cong \frac{2\sqrt{3}}{9\,e^{-1}}\frac{\sqrt{2\pi}}{\sqrt{2k+1}}\left(\frac{k}{2k+1}\right)^{2k+1} \cong \frac{2\sqrt{6\pi}}{9}\,m^{-\frac{1}{2}}2^{-m}.$$

□

Comparing estimates (3.1.2) - (3.1.5) we observe that in the central parts of $[a,b]$, the estimates for the error $f - \Pi_m f$ are approximately 2^m times preciser than on the whole interval. In the central parts of $[a,b]$, the interpolation process on the uniform grid has also good stability properties as m increases: in contrast to an exponential growth [19] of

$$\|\Pi_m\|_{\mathcal{L}(C[a,b],C[a,b])} \text{ as } m \to \infty,$$

it holds by the Runck's theorem (see [19], [73])

$$\|\Pi_m\|_{\mathcal{L}\left(C[a,b],C\left[\frac{a+b}{2}-rh^{1/2},\frac{a+b}{2}+rh^{1/2}\right]\right)} \leq c_r\,(1+\log m)\,, \quad rh^{\frac{1}{2}} \leq \frac{b-a}{2}, \qquad (3.1.8)$$

where the constant c_r depends only on $r > 0$. As well known (see e.g [19]), logarithmic growth is the best one that holds for projectors $P_m : C[a,b] \to \mathcal{P}_{m-1}$ and, for example, Chebyshev interpolation projectors have this growth.

3.2 Central part interpolation by piecewise polynomials

Introduce in $\mathbb{R}$ the uniform grid

$$\{jh : j \in \mathbb{Z}\}\,, \quad h = \frac{1}{n}, \quad n \in \mathbb{N}. \qquad (3.2.1)$$

Let

$$m \in \mathbb{N}, \quad m \geq 2$$

be fixed. Given a function $f \in C\,[-\delta, 1+\delta]$, $\delta > 0$, we define a piecewise polynomial interpolant $\Pi_{h,m} f \in C\,[0,1]$ for $h = \frac{1}{n} < \frac{2\delta}{m}$ as follows. On every subinterval $[jh, (j+1)\,h]$, $0 \leq j \leq n-1$, the function $\Pi_{h,m} f$ is defined independently from other subintervals as a polynomial $\Pi^{[j]}_{h,m} f \in \mathcal{P}_{m-1}$ of degree $\leq m-1$ by the conditions

$$\Pi^{[j]}_{h,m} f\,(lh) = f\,(lh)\,, \quad l = j - \frac{m}{2} + 1, \ldots, j + \frac{m}{2} \quad \text{if } m \text{ is even},$$

$$\Pi^{[j]}_{h,m} f\,(lh) = f\,(lh)\,, \quad l = j - \frac{m-1}{2}, \ldots, j + \frac{m-1}{2} \quad \text{if } m \text{ is odd}.$$

A unified writing form of these interpolation conditions is

$$\Pi_{h,m}^{[j]} f(lh) = f(lh), \quad \text{for} \quad l \in \mathbb{Z} \quad \text{such that} \quad l - j \in \mathbb{Z}_m, \tag{3.2.2}$$

where

$$\mathbb{Z}_m = \left\{ k \in \mathbb{Z} : -\frac{m}{2} < k \leq \frac{m}{2} \right\}.$$

Observe that $\mathbb{Z}_m$ contains the following m elements (integers):

$$\mathbb{Z}_m = \left\{ -\frac{m}{2} + 1, -\frac{m}{2} + 2, \ldots, \frac{m}{2} \right\} \quad \text{if } m \text{ is even,}$$

$$\mathbb{Z}_m = \left\{ -\frac{m-1}{2}, -\frac{m-1}{2} + 1, \ldots, \frac{m-1}{2} \right\} \quad \text{if } m \text{ is odd.}$$

For an "interior" knot jh, $1 \leq j \leq n-1$, interpolation conditions (3.2.2) contain the condition

$$\left(\Pi_{h,m}^{[j-1]} f \right)(jh) = f(jh)$$

as well as the condition

$$\left(\Pi_{h,m}^{[j]} f \right)(jh) = f(jh),$$

thus $\Pi_{h,m} f$ is uniquely defined at interior knots and $\Pi_{h,m} f$ is continuous on $[0,1]$. Namely, for the "interior" knots jh, $1 \leq j \leq n-1$, interpolation conditions (3.2.2) yield

$$(\Pi_{h,m} f)(jh) = f(jh)$$

for $\Pi_{h,m} f$ as a function on $[(j-1)h, jh]$ as well as a function on $[jh, (j+1)h]$. The one side derivatives of the interpolant $\Pi_{h,m} f$ at the interior knots may be different.

Introduce the Lagrange fundamental polynomials $L_{k,m} \in \mathcal{P}_{m-1}$, $k \in \mathbb{Z}_m$, satisfying $L_{k,m}(l) = \delta_{k,l}$ for $l \in \mathbb{Z}_m$, where $\delta_{k,l}$ is the Kronecker symbol, $\delta_{k,l} = 0$ for $k \neq l$ and $\delta_{k,k} = 1$. An explicit formula for $L_{k,m}$ is given by

$$L_{k,m}(t) = \prod_{l \in \mathbb{Z}_m \setminus \{k\}} \frac{t-l}{k-l}, \quad k \in \mathbb{Z}_m. \tag{3.2.3}$$

We claim that

$$\left(\Pi_{h,m}^{[j]} f \right)(t) = \sum_{k \in \mathbb{Z}_m} f((j+k)h) L_{k,m}(nt - j),$$
$$j = 0, \ldots, n-1, \quad t \in [jh, (j+1)h]. \tag{3.2.4}$$

Indeed, $\Pi_{h,m}^{[j]} f$ defined by (3.2.4) is really a polynomial of degree $\leq m-1$ and it satisfies interpolation conditions (3.2.2): for l with $l - j \in \mathbb{Z}_m$, it holds that

$$\left(\Pi_{h,m}^{[j]} f \right)(lh) = \sum_{k \in \mathbb{Z}_m} f((j+k)h) L_{k,m}(l-j) = \sum_{k \in \mathbb{Z}_m} f((j+k)h) \delta_{k,l-j}$$

$$= f\left(\left(j + (l - j)\right)h\right) = f\left(lh\right).$$

For $m = 2$, the interpolant $\Pi_{h,2}f$ is the usual piecewise linear function joining for $0 \le j \le n - 1$ the pair of points

$$(jh, f(jh)) \in \mathbb{R}^2 \text{ and } ((j+1)h, f((j+1)h)) \in \mathbb{R}^2$$

by a straight line; $\Pi_{h,2}f$ does not use the values of f outside $[0,1]$, and $\Pi_{h,2}f$ is a projection operator in $C[0,1]$, i.e. $\Pi_{h,2}^2 = \Pi_{h,2}$.

For $m \ge 3$, $\Pi_{h,m}f$ uses values of f outside of $[0,1]$. For $f \in C[0,1]$, $\Pi_{h,m}f$ obtains a sense after an extension of f onto $[-\delta, 1+\delta]$ with $\delta \ge \frac{m}{2}h$. In our work we will consider the functions $f \in C^m[0,1]$, that satisfy the boundary conditions

$$f^{(j)}(0) = f^{(j)}(1) = 0, \; j = 1, \ldots, m.$$

Then we are in a lucky situation and the simplest extension operator

$$E_\delta : C[0,1] \to C[-\delta, 1+\delta], \quad (E_\delta f)(t) = \begin{Bmatrix} f(0) & \text{for} & -\delta \le t \le 0 \\ f(t) & \text{for} & 0 \le t \le 1 \\ f(1) & \text{for} & 1 \le t \le 1+\delta \end{Bmatrix} \tag{3.2.5}$$

maintains the smoothness of f. The operator

$$P_{h,m} := \Pi_{h,m}E_\delta : C[0,1] \to C[0,1] \tag{3.2.6}$$

is well defined and $P_{h,m}^2 = P_{h,m}$, i.e., $P_{h,m}$ is a projector in $C[0,1]$.

For $w_h \in \mathcal{R}(P_{h,m})$ (the range of $P_{h,m}$) we have

$$w_h = P_{h,m}w_h = \Pi_{h,m}E_\delta w_h,$$

and due to (3.2.4) we get for $t \in [jh, (j+1)h]$ $(j = 0, \ldots, n-1)$ that

$$w_h(t) = \sum_{k \in \mathbb{Z}_m} (E_\delta w_h)((j+k)h) L_{k,m}(nt - j) \tag{3.2.7}$$

where

$$(E_\delta w_h)(ih) = \begin{Bmatrix} w_h(ih) & \text{for} & i = 0, \ldots, n \\ w_h(0) & \text{for} & i < 0 \\ w_h(1) & \text{for} & i > n \end{Bmatrix}.$$

Thus $w_h \in \mathcal{R}(P_{h,m})$ is uniquely determined on $[0,1]$ by its knot values $w_h(ih)$, $i = 0, \ldots, n$. We conclude, that

$$\dim \mathcal{R}(P_{h,m}) = n + 1.$$

It is also clear, that for a $w_h \in \mathcal{R}(P_{h,m})$ we have $w_h = 0$ if and only if $w_h(ih) = 0$, $i = 0, \ldots, n$.

For $f \in C[-\delta, 1+\delta]$, the interpolant $\Pi_{h,m} f$ is closely related to the central part interpolation of f on the uniform grid treated in Section 3.1. On $[jh, (j+1)h]$, the interpolant $\Pi_{h,m} f = \Pi_{h,m}^{[j]} f$ coincides with the polynomial interpolant $\Pi_m f$ constructed for f on the interval $[a_j, b_j]$ where

$$a_j = \left(j - \frac{m-1}{2}\right) h, \qquad b_j = \left(j + \frac{m+1}{2}\right) h$$

in the case of even m and

$$a_j = \left(j - \frac{m}{2}\right) h, \qquad b_j = \left(j + \frac{m}{2}\right) h$$

in the case of odd m. Moreover, $[jh, (j+1)h]$ is contained in the central part of $[a_j, b_j]$ on which the interpolation error can be estimated by (3.1.3) and (3.1.5). On this way we obtain the following result.

Lemma 3.2.1.

(i) For $f \in C^m[-\delta, 1+\delta]$ $(m \geq 2, \delta > 0, h = \frac{1}{n} < \frac{2\delta}{m})$,

$$\max_{0 \leq t \leq 1} |f(t) - (\Pi_{h,m} f)(t)| \leq \vartheta_m h^m \max_{-\delta \leq t \leq 1+\delta} \left| f^{(m)}(t) \right|, \tag{3.2.8}$$

with ϑ_m*, defined by* (3.1.4) *and* (3.1.6)*, respectively for even and odd* m*.*

(ii) For $f \in V^{(m)} := \left\{ v \in C^m[0,1] : v^{(j)}(0) = v^{(j)}(1) = 0,\ j = 1, \ldots, m \right\}$ *it holds*

$$\max_{0 \leq t \leq 1} |f(t) - (P_{h,m} f)(t)| \leq \vartheta_m h^m \max_{0 \leq t \leq 1} \left| f^{(m)}(t) \right|. \tag{3.2.9}$$

Proof. The claim (i) is a direct consequence of Lemma 3.1.1. Further, to prove the estimate (3.2.9), we have $E_\delta f \in C^m[-\delta, 1+\delta]$ for $f \in V^{(m)}$ and

$$\max_{-\delta \leq t \leq \delta} \left| (E_\delta f)^{(m)}(t) \right| = \max_{0 \leq t \leq 1} \left| f^{(m)}(t) \right|, \quad (E_\delta f)(t) = f(t) \text{ for } 0 \leq t \leq 1.$$

Applying (3.2.8) to $E_\delta f$, it takes the form

$$\max_{0 \leq t \leq 1} |(E_\delta f)(t) - (\Pi_{h,m} E_\delta f)(t)| \leq \vartheta_m h^m \max_{-\delta \leq t \leq 1+\delta} \left| (E_\delta f)^{(m)}(t) \right|.$$

We can rewrite it as

$$\max_{0\le t\le 1} |f(t) - (P_{h,m}f)(t)| \le \vartheta_m h^m \max_{0\le t\le 1} \left| f^{(m)}(t) \right|,$$

which completes the proof. □

From (3.1.8), (3.2.5) and (3.2.6) we obtain that the norms $\|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])}$ are uniformly bounded with respect to n, $h = \dfrac{1}{n}$:

$$\|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])} \le c\,(1 + \log m)\,, \tag{3.2.10}$$

with a constant c which is independent of h (of n).

Together with (3.2.9), noticing that $V^{(m)}$ is dense in $C\,[0,1]$, Theorem 2.2.2 yields the following result.

Lemma 3.2.2. *For any* $f \in C\,[0,1]$,

$$\max_{0\le t\le 1} |f\,(t) - (P_{h,m}f)\,(t)| \to 0, \;\; \textit{as } n \to \infty.$$

Chapter 4

Weakly Singular Integral Operators

This chapter is devoted to the compactness of weakly singular integral operators in $C[0,1]$ and in certain weighted spaces of smooth functions.

4.1 Weakly singular kernels

Consider an integral operator T defined by its kernel function $K(x,y)$ via the formula

$$(Tu)(x) = \int_0^1 K(x,y)u(y)dy, \quad 0 \le x \le 1, \tag{4.1.1}$$

where u is taken from some set of functions, for example, from space $C[0,1]$. In the literature (for example, see [2]), the weak singularity of the kernel K and the corresponding operator T may have different senses. A frequent understanding is that K has the form

$$K(x,y) = a(x,y)\,|x-y|^{-\nu},$$

where $a(x,y)$ is a continuous function on $[0,1]\times[0,1]$ and $0 < \nu < 1$. This kernel has the property

$$\sup_{0\le x\le 1} \int_0^1 |K(x,y)|\,dy < \infty \tag{4.1.2}$$

often used to define the weak singularity in the wide sense: a kernel $K(x,y)$ is at most weakly singular if it is absolutely integrable with respect to y and satisfies (4.1.2). The kernels which we will consider in Chapters 5-6 are somewhere in the middle of these two understandings of the weak singularity. Following [47, 90], we say that a kernel K is weakly singular if K is continuous on $([0,1]\times[0,1])\setminus diag$ and there exist some constants $c>0$ and $\nu\in(0,1)$ such that

$$|K(x,y)|\le c\,|x-y|^{-\nu}, \quad (x,y)\in([0,1]\times[0,1])\setminus diag. \tag{4.1.3}$$

Here *diag* means the diagonal of $\mathbb{R}^2$: $diag=\left\{(x,y)\in\mathbb{R}^2 : x=y\right\}$.

For instance, the kernels

$$K(x,y)=a(x,y)\,|x-y|^{-\nu}, \quad 0<\nu<1, \tag{4.1.4}$$

$$K(x,y)=a(x,y)\log|x-y|, \tag{4.1.5}$$

$$K(x,y)=a(x,y)\,|x-y|^{-\nu}\log^k|x-y|, \quad 0\le\nu<1,\ k\in\mathbb{N}, \tag{4.1.6}$$

with $a\in C([0,1]\times[0,1])$ are weakly singular in the sense of such understanding.

Clearly, the kernel of the form (4.1.4) is weakly singular. The kernel (4.1.5) is a special case of (4.1.6). The kernel (4.1.6) can be estimated as follows:

$$|K(x,y)|=\left|a(x,y)\,|x-y|^{-\nu}\log^k|x-y|\right|\le|a(x,y)|\,|x-y|^{-\nu}\left|\log|x-y|\right|^k$$

$$\le c\,|x-y|^{-\nu-\nu'} \quad \text{for any } \nu'\in(0,1-\nu), \quad (x,y)\in([0,1]\times[0,1])\setminus diag.$$

In what follows we are interested in kernels that are m-times continuously differentiable on $([0,1]\times[0,1])\setminus diag$. Introduce the following smoothness-singularity class $\mathcal{S}^{m,\nu}$ of kernels.

Definition 4.1.1. *For given $m\in\mathbb{N}_0$ and $\nu\in\mathbb{R}$, denote by*

$$\mathcal{S}^{m,\nu} := \mathcal{S}^{m,\nu}(([0,1]\times[0,1])\setminus diag)$$

the set of m times continuously differentiable functions K on $([0,1]\times[0,1])\setminus diag$ that satisfy there for all $k,l\in\mathbb{N}_0$, $k+l\le m$, the inequality

$$\left|\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^l K(x,y)\right|\le c\left\{\begin{array}{ccc} 1 & \text{if} & \nu+k<0 \\ 1+\left|\log|x-y|\right| & \text{if} & \nu+k=0 \\ |x-y|^{-\nu-k} & \text{if} & \nu+k>0 \end{array}\right\}, \tag{4.1.7}$$

where $c=c(K,m)>0$ is a constant.

Taking $k = l = 0$, we obtain from (4.1.7) the following estimate:

$$|K(x,y)| \leq c \left\{ \begin{array}{ccc} 1 & \text{if} & \nu < 0 \\ 1 + |\log|x-y|| & \text{if} & \nu = 0 \\ |x-y|^{-\nu} & \text{if} & \nu > 0 \end{array} \right\}.$$

From here we can see, that for $\nu > 0$, condition (4.1.7) coincides with (4.1.3). Thus a kernel $K \in \mathcal{S}^{m,\nu}$ is weakly singular if $\nu < 1$. A kernel $K \in \mathcal{S}^{m,\nu}$ with $\nu < 0$ is bounded itself, but its derivatives may have singularities on the diagonal; $\nu = 0$ corresponds to a logarithmically singular kernel.

4.2 Compactness of a weakly singular integral operator

A weak singularity of the kernel implies that the corresponding integral operator is compact in the space $C[0,1]$. The proof of the following lemma is standard (cf. [47] or [68]).

Lemma 4.2.1. *A kernel $K \in \mathcal{S}^{m,\nu}$ with $m \geq 0$, $\nu < 1$ defines a compact operator $T : L^\infty(0,1) \to C[0,1]$, hence also a compact operator $T : C[0,1] \to C[0,1]$ and a compact operator $T : L^\infty(0,1) \to L^\infty(0,1)$.*

Proof. Take a "cutting" function $e \in C[0,\infty)$ satisfying the conditions

$$\begin{array}{lll} e(r) = 0 & \text{for} & 0 \leq r \leq \frac{1}{2}, \\ 0 \leq e(r) \leq 1 & \text{for} & r \geq \frac{1}{2}, \\ e(r) = 1 & \text{for} & r \geq 1. \end{array}$$

Define

$$K_n(x,y) = e(n|x-y|)K(x,y), \quad (x,y) \in [0,1] \times [0,1], \quad n \in \mathbb{N},$$

and

$$(T_n u)(x) = \int_0^1 K_n(x,y)u(y)dy, \quad n \in \mathbb{N}.$$

The kernels $K_n(x,y)$ are continuous on $[0,1] \times [0,1]$ - the possible diagonal singularity is "cut" off by the factor $e(n|x-y|)$, $K_n(x,y) = 0$ in a neighbourhood of the diagonal $x = y$. Hence the operators $T_n : L^\infty(0,1) \to C[0,1]$ $(n \in \mathbb{N})$ are compact. Further, for $u \in L^\infty(0,1)$, $0 \le x \le 1$, we have

$$\begin{aligned}(Tu)(x) - (T_n u)(x) &= \int_0^1 [K(x,y) - K_n(x,y)]\, u(y)dy \\ &= \int_0^1 K(x,y)[1 - e(n\,|x-y|)]u(y)dy.\end{aligned}$$

Taking into account that $1 - e(n\,|x-y|) = 0$ for $|x - y| \ge \dfrac{1}{n}$, we therefore obtain for $0 < \nu < 1$ that

$$\begin{aligned}|(Tu)(x) - (T_n u)(x)| &\le \int_0^1 |K(x,y)|\,[1 - e(n\,|x-y|)]dy\,\|u\|_\infty \\ &\le c \int_{|x-y|\le 1/n} |x-y|^{-\nu}\,dy\,\|u\|_\infty \\ &= 2c \int_0^{1/n} z^{-\nu} dz\,\|u\|_\infty \\ &= 2c\frac{(1/n)^{1-\nu}}{1-\nu}\,\|u\|_\infty\,, \quad 0 \le x \le 1.\end{aligned}$$

This implies that $Tu \in C[0,1]$ as a uniform limit of $T_n u \in C[0,1]$, and

$$\|T - T_n\|_{\mathcal{L}(L^\infty(0,1),C[0,1])} \le 2c\frac{(1/n)^{1-\nu}}{1-\nu} \to 0 \text{ as } n \to \infty.$$

In the case $\nu \le 0$ we obtain likewise

$$\|T - T_n\|_{\mathcal{L}(L^\infty(0,1),C[0,1])} \to 0 \text{ as } n \to \infty.$$

Thus T maps $L^\infty(0,1)$ into $C[0,1]$ and $T : L^\infty(0,1) \to C[0,1]$ is compact as a norm limit of compact operators $T_n : L^\infty(0,1) \to C[0,1]$, see Theorem 2.2.6.

□

For describing the smoothness of the solution we need a weighted space of smooth functions with the following properties.

Definition 4.2.1. *For $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$, denote by $C^{m,\nu}(0,1)$ the space of functions $f \in C[0,1] \cap C^m(0,1)$ that satisfy the inequalities*

$$\left|f^{(j)}(x)\right| \leq c \left\{ \begin{array}{ccc} 1 & \text{if} & j+\nu-1<0 \\ 1+|\log\rho(x)| & \text{if} & j+\nu-1=0 \\ \rho(x)^{-j-\nu+1} & \text{if} & j+\nu-1>1 \end{array} \right\}, \quad 0<x<1, \quad j=1,\dots,m, \tag{4.2.1}$$

where $c = c(f) > 0$ is a constant and

$$\rho(x) = \min\{x, 1-x\}$$

is the distance from $x \in (0,1)$ to the boundary of the interval $(0,1)$.

Introduce the weight functions

$$\omega_\lambda(x) = \left\{ \begin{array}{ccc} 1 & \text{for} & \lambda<0 \\ \dfrac{1}{1+|\log\rho(x)|} & \text{for} & \lambda=0 \\ \rho(x)^{\lambda} & \text{for} & \lambda>0 \end{array} \right\}, \quad 0<x<1, \quad \lambda \in \mathbb{R}.$$

Equipped with the norm

$$\|f\|_{C^{m,\nu}(0,1)} = \max_{0\leq x\leq 1}|f(x)| + \sum_{j=1}^{m} \sup_{0<x<1} \omega_{j+\nu-1}(x)\left|f^{(j)}(x)\right|, \quad f \in C^{m,\nu}(0,1),$$

$C^{m,\nu}$ becomes a Banach space.

About the compactness of T given by (4.1.1) the following result holds (cf. [68, 90]).

Theorem 4.2.1. *Let $K \in \mathcal{S}^{m,\nu}$, $m \geq 1$, $\nu < 1$. Then the integral operator T defined by (4.1.1) maps $C^{m,\nu}(0,1)$ into itself and $T : C^{m,\nu}(0,1) \to C^{m,\nu}(0,1)$ is compact.*

4.3 Differentiation of weakly singular integrals

Let us present a differentiation formula for weakly singular integrals with respect to a parameter.

Theorem 4.3.1. *Assume that $g(x,y)$ is a continuously differentiable function on $((0,1)\times[0,1]) \setminus diag$ and satisfies with a $\nu \in (0,1)$ the inequalities*

$$|g(x,y)| \leq c\,|x-y|^{-\nu}, \qquad \left|\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) g(x,y)\right| \leq c\,|x-y|^{-\nu}. \tag{4.3.1}$$

Then the function $x \mapsto \int\limits_0^1 g(x,y)dy$ *is continuously differentiable in* $(0,1)$ *and*

$$\frac{d}{dx}\int\limits_0^1 g(x,y)dy = \int\limits_0^1 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) g(x,y)dy + g(x,0) - g(x,1),\ 0 < x < 1. \quad (4.3.2)$$

Proof. See [68]. □

Chapter 5

Smoothing change of variables

Boundary singularities of the derivatives of a solution of a weakly singular integral equation are typical for such equations. To see that, let us consider the integral equation

$$u(x) = \int_0^1 K(x,y)u(y)\,dy + f(x), \quad 0 \le x \le 1, \tag{5.0.1}$$

where $K \in \mathcal{S}^{m,\nu}$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$ and $f \in C^m[0,1]$. Then, in general, $u \notin C^1[0,1]$.

Indeed, supposing that $u \in C^1[0,1]$, we can differentiate (5.0.1) as an equality due to Theorem 4.3.1 and we obtain on the basis of (4.3.2) that

$$\begin{aligned} u'(x) = \; & \int_0^1 \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) K(x,y)\right] u(y)dy + \int_0^1 K(x,y)u'(y)dy \\ & +K(x,0)u(0) - K(x,1)u(1) + f'(x), \\ & 0 \le x \le 1. \end{aligned} \tag{5.0.2}$$

Since the integral operators with the kernels

$$K(x,y) \quad \text{and} \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) K(x,y)$$

are weakly singular and $u, u' \in C[0,1]$, the first two terms on the r.h.s of (5.0.2) are on the basis of Lemma 4.2.1 continuous on $[0,1]$; the same is true for the term $f'(x)$. On the other hand, the term $K(x,0)u(0)$ has a singularity at $x = 0$ provided that $u(0) \neq 0$ and $K(x,0)$ really has a singularity allowed by inequality (4.1.7), and similarly the term $K(x,1)u(1)$ has a singularity at $x = 1$ if $u(1) \neq 0$ and $K(x,1)$ has a singularity. Thus the assumption $u \in C^1[0,1]$ leads to a contradiction if

$K(x, 0)$ or $K(x, 1)$ is singular and $u(0) \neq 0$, $u(1) \neq 0$; these inequalities hold for most of $f \in C^m[0, 1]$.

We refer to [68] for the proof of the following theorem.

Theorem 5.0.1. *Let $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0, 1)$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$, and let $u \in C[0, 1]$ be a solution of equation* (5.0.1)*. Then $u \in C^{m,\nu}(0, 1)$.*

The derivatives of a solution $u \in C^{m,\nu}(0, 1)$ (see Definition 4.2.1) to equation (5.0.1) may have boundary singularities. Now we undertake a change of variables that kills the singularities - the solution of the transformed equation will be C^m-smooth on $[0, 1]$ including the boundary points. The idea of smoothing the solution by introducing the suitable change of variables has been considered, for example, in [49, 78] to increase the order of convergence of the trapeziodal and midpoint quadrature rule. It has been used also for the numerical solution of weakly singular Fredholm integral equations in [55, 60, 69]. The smoothing change of variables together with the central part interpolation has been considered in [59, 61].

5.1 Change of variables

Let $\varphi : [0, 1] \to [0, 1]$ be a smooth strictly increasing function such that $\varphi(0) = 0$, $\varphi(1) = 1$. Introducing the change of variables

$$x = \varphi(t), \quad y = \varphi(s), \quad 0 \leq t \leq 1, \quad 0 \leq s \leq 1, \tag{5.1.1}$$

equation (5.0.1) takes the form

$$v(t) = \int_0^1 K_\varphi(t, s)v(s)ds + f_\varphi(t), \quad 0 \leq t \leq 1, \tag{5.1.2}$$

where

$$f_\varphi(t) := f(\varphi(t)),\ K_\varphi(t, s) := K(\varphi(t), \varphi(s))\varphi'(s);$$

the solutions on equations (5.0.1) and (5.1.2) are in the relations

$$v(t) = u(\varphi(t)),\ u(x) = v(\varphi^{-1}(x)).$$

Under conditions we have set on $\varphi : [0, 1] \to [0, 1]$, the inverse function $\varphi^{-1} : [0, 1] \to [0, 1]$ exists and is continuous and $\varphi^{-1}(0) = 0$, $\varphi^{-1}(1) = 1$.

5.2 Smoothing Properties

Theorem 5.2.1. *Given $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$, let $p \in \mathbb{N}$ satisfy*

$$p > \left\{ \begin{array}{ccc} m & \text{for} & \nu \leq 0 \\ \dfrac{m}{1-\nu} & \text{for} & 0 < \nu < 1 \end{array} \right\}. \tag{5.2.1}$$

Let $\varphi \in C^p[0,1]$ *satisfy the conditions* $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) > 0$ *for* $0 < t < 1$ *and*

$$\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0, \quad j = 1, \dots, p-1, \quad \varphi^{(p)}(0) \neq 0, \quad \varphi^{(p)}(1) \neq 0. \quad (5.2.2)$$

Then the following claims hold true.

(i) For $f \in C^{m,\nu}(0,1)$, *the function* $f_\varphi(t) = f(\varphi(t))$ *belongs to* $C^m[0,1]$ *and*

$$f_\varphi^{(j)}(0) = f_\varphi^{(j)}(1) = 0, \ j = 1, \dots, m. \quad (5.2.3)$$

(ii) For $K \in \mathcal{S}^{0,\nu}$, *the kernel* $K_\varphi(t,s) = K(\varphi(t), \varphi(s))\varphi'(s)$ *belongs to* $\mathcal{S}^{0,\nu}$ *and hence defines a compact integral operator*

$$T_\varphi : C[0,1] \to C[0,1], \quad (T_\varphi v)(t) = \int_0^1 K_\varphi(t,s) v(s) ds.$$

Proof. (i) Clearly $f_\varphi \in C^m(0,1)$, thus claim (i) concerns only the boundary behaviour of f_φ. Due to the imbedding

$$C^{m,\nu}(0,1) \subset C[0,1], \quad m \geq 1, \nu < 1, \quad (5.2.4)$$

after the extension of f_φ by continuity to points 0 and 1, we have $f_\varphi \in C[0,1]$. We need to show that

$$f_\varphi^{(j)}(0) := \lim_{t\to 0} f_\varphi^{(j)}(t) = 0, \quad f_\varphi^{(j)}(1) := \lim_{t\to 1} f_\varphi^{(j)}(t) = 0, \quad j = 1, \dots, m.$$

We establish these relations for $t \to 0$; for $t \to 1$ the argument is similar. By the formula of Faà di Bruno (see Theorem 2.1.3)

$$f_\varphi^{(j)}(t) = \sum_{k_1 + 2k_2 + \dots + jk_j = j} \frac{j!}{k_1! \dots k_j!} f^{(k_1+\dots+k_j)}(\varphi(t)) \varphi'(t)^{k_1} \dots \varphi^{(j)}(t)^{k_j}, \ 0 < t < 1,$$

where $k_1, \dots, k_j \in \mathbb{N}_0$.

In a vicinity of 0, the inclusion $f \in C^{m,\nu}(0,1)$ yields

$$\left| f^{(j)}(\varphi(t)) \right| \leq c \left\{ \begin{array}{ccc} 1 & \text{for} & j < 1-\nu \\ 1 + |\log \varphi(t)| & \text{for} & j = 1-\nu \\ \varphi(t)^{1-\nu-j} & \text{for} & j > 1-\nu \end{array} \right\}, \quad j = 1, \dots, m. \quad (5.2.5)$$

To estimate the function φ we need Taylor formula which holds for any $f \in C^m[a,b]$, $m \in \mathbb{N}$:

$$f(x) = \sum_{i=0}^{m-1} (x-a)^i \frac{f^{(i)}(a)}{i!} + \frac{1}{(m-1)!} \int_a^x (x-t)^{m-1} f^{(m)}(t) dt, \quad x \in (a,b). \ (5.2.6)$$

Due to (5.2.2) and (5.2.6) with $f = \varphi$ we can achieve

$$\varphi(t) \le ct^p, \quad \varphi^{(i)}(t) \le ct^{p-i} \quad \text{as} \quad t \to 0, \quad i = 0, \dots, p.$$

Hence (see (5.2.5)),

$$\left| f_\varphi^{(j)}(t) \right| = \left| f^{(j)}(\varphi(t)) \right|$$

$$\le c \sum_{k_1+2k_2+\cdots+jk_j=j} \left\{ \begin{array}{cl} 1 & \text{for} \quad k_1 + \cdots + k_j < 1-\nu \\ 1 + |\log t| & \text{for} \quad k_1 + \cdots + k_j = 1-\nu \\ t^{p(1-\nu-k_1-\cdots-k_j)} & \text{for} \quad k_1 + \cdots + k_j > 1-\nu \end{array} \right\} t^{(p-1)k_1} \dots t^{(p-j)k_j}$$

$$= c \sum_{k_1+2k_2+\cdots+jk_j=j} \left\{ \begin{array}{cl} t^{p(k_1+\cdots+k_j)-j} & \text{for} \quad k_1 + \cdots + k_j < 1-\nu \\ (1 + |\log t|) t^{p(k_1+\cdots+k_j)-j} & \text{for} \quad k_1 + \cdots + k_j = 1-\nu \\ t^{p(1-\nu)-j} & \text{for} \quad k_1 + \cdots + k_j > 1-\nu \end{array} \right\},$$

$$1 \le j \le m.$$

For $\nu > 0$, we have $k_1 + \cdots + k_j > 1 - \nu$ and in accordance to lower line

$$\left| f_\varphi^{(j)}(t) \right| \le ct^{p(1-\nu)-j}.$$

For $\nu = 0$, there is one combination of $k_1, \dots, k_j$ such that $k_1 + 2k_2 + \cdots + jk_j = j$ and $k_1 + \cdots + k_j = 1 - \nu$, namely $k_1 = \ldots = k_{j-1} = 0$, $k_j = 1$, yielding

$$\left| f_\varphi^{(j)}(t) \right| \le ct^{p-j}(1 + |\log t|).$$

For $\nu < 0$, the smallest exponent $p(k_1 + \cdots + k_j) - j$ with restrictions $k_1 + 2k_2 + \cdots + jk_j = j$ and $k_1 + \cdots + k_j < 1 - \nu$ again corresponds to the combination $k_1 = \ldots = k_{j-1} = 0$, $k_j = 1$, yielding

$$\left| f_\varphi^{(j)}(t) \right| \le ct^{p-j}$$

from the upper line which dominates over terms in the lower and central lines.

As a summary, in a neighbourhood of 0, it holds

$$\left| f_\varphi^{(j)}(t) \right| \le c \left\{ \begin{array}{cl} t^{p-j} & \text{for} \quad \nu < 0 \\ t^{p-j}(1 + |\log t|) & \text{for} \quad \nu = 0 \\ t^{p(1-\nu)-j} & \text{for} \quad \nu > 0 \end{array} \right\}, \quad j = 1, \dots, m.$$

Now condition (5.2.1) implies that

$$\lim_{t\to 0} f_\varphi^{(j)}(t) = 0 \text{ for } j = 1, \ldots, m.$$

(ii) Claim (ii) is trivial for $\nu < 0$ since then $K_\varphi(t,s)$ is bounded together with $K(x,y)$.

To prove claim (ii) for $0 \le \nu < 1$, we first examine the properties of the function

$$\Phi(t,s) := \left\{ \begin{array}{ccc} \dfrac{\varphi(t)-\varphi(s)}{t-s} & \text{for} & t \ne s \\ \\ \varphi'(t) & \text{for} & t = s \end{array} \right\}, \quad 0 \le t, s \le 1. \tag{5.2.7}$$

Due to the conditions on φ, we have

$$\Phi \in C^{p-1}([0,1]\times[0,1]), \quad \Phi(t,s) > 0$$

for $(t,s) \in ([0,1]\times[0,1]) \setminus \{(0,0),(1,1)\}$.

We show that there exists a positive constant c_0 such that

$$\Phi(t,s) \ge c_0 \min\left\{(t+s)^{p-1}, [(1-t)+(1-s)]^{p-1}\right\}, \quad 0 \le t, s \le 1. \tag{5.2.8}$$

It suffices to establish estimate (5.2.8) in a neighbourhood of the point $(0,0)$; for a neighbourhood of the point $(1,1)$ the estimate follows by the symmetry; on the rest part of $[0,1]\times[0,1]$ function Φ is greater than a positive constant implying (5.2.8) also there, possibly with a smaller but still positive constant c_0.

We choose a neighbourhood

$$U_\delta \subset [0,1]\times[0,1]$$

of $(0,0)$ of a sufficiently small radius $\delta > 0$ such that $\varphi^{(p)}(t) \ne 0$ for $0 \le t \le \delta$ (see (5.2.2)). Then $\varphi^{(p)}(t) > 0$ for $0 \le t \le \delta$, since $\varphi^{(p)}(t) < 0$ for $0 \le t \le \delta$ together with the conditions $\varphi'(0) = \ldots = \varphi^{(p-1)} = 0$ should imply $\varphi'(t) < 0$ for $0 \le t \le \delta$.

Denote

$$d_0 := \min_{0\le t\le\delta} \varphi^{(p)}(t) > 0.$$

Let $0 < s < t \le \delta$. Due to (5.2.2), the Taylor formula with the integral form of the remainder term (see 5.2.6) yields

$$\varphi(t)-\varphi(s) = \frac{1}{(p-1)!}\int_0^t (t-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau - \frac{1}{(p-1)!}\int_0^s (s-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau$$

$$= \frac{1}{(p-1)!}\int_0^s [(t-\tau)^{p-1} - (s-\tau)^{p-1}]\varphi^{(p)}(\tau)d\tau + \frac{1}{(p-1)!}\int_s^t (t-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau.$$

The functions $(t-\tau)^{p-1}-(s-\tau)^{p-1}$ and $(t-\tau)^{p-1}$ under last two integrals are positive. Estimating $\varphi^{(p)}(\tau)\geq d_0>0$ we obtain

$$\varphi(t)-\varphi(s)$$

$$\geq \frac{d_0}{(p-1)!}\left(\int_0^s [(t-\tau)^{p-1}-(s-\tau)^{p-1}]\varphi^{(p)}(\tau)d\tau+\int_s^t (t-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau\right)$$

$$=\frac{d_0}{(p-1)!}\left(\int_0^t (t-\tau)^{p-1}d\tau-\int_0^s (s-\tau)^{p-1}d\tau\right)=\frac{d_0}{p!}(t^p-s^p),$$

and (5.2.8) follows for $0<s<t\leq\delta$:

$$\frac{\varphi(t)-\varphi(s)}{t-s}\geq\frac{d_0}{p!}\frac{t^p-s^p}{t-s}=\frac{d_0}{p!}\sum_{j=0}^{p-1}t^j s^{p-1-j}$$

$$\geq c_0\sum_{j=0}^{p-1}\left(\begin{array}{c}p-1\\ j\end{array}\right)t^j s^{p-1-j}=c_0(t+s)^{p-1}.$$

The case $0<t<s\leq\delta$ is symmetrical to the treated case $0<s<t\leq\delta$. For $0<s=t\leq\delta$, (5.2.8) follows by a limit argument. This completes the proof of (5.2.8).

Let us return to claim (ii) for $0\leq\nu<1$. Consider first the case $0<\nu<1$. Due to (4.1.7) and (5.2.8)

$$|K_\varphi(t,s)|\leq c_K\,|\varphi(t)-\varphi(s)|^{-\nu}\,\varphi'(s)=c_K\left(\frac{\varphi(t)-\varphi(s)}{t-s}\right)^{-\nu}|t-s|^{-\nu}\,\varphi'(s)$$

$$\leq c_K c_0^{-\nu}\,|t-s|^{-\nu}\,\frac{\varphi'(s)}{[\min\{(t+s)^{p-1},[(1-t)+(1-s)]^{p-1}\}]^\nu}\leq c\,|t-s|^{-\nu};$$

on the last step we took into account that

$$\varphi'(s)\leq cs^{p-1}\text{ as }s\to 0,\quad \varphi'(s)\leq c(1-s)^{p-1}\text{ as }s\to 1.$$

Thus $K_\varphi\in\mathcal{S}^{0,\nu}$.

In the case $\nu=0$,

$$|K_\varphi(t,s)|\leq c_K(1+|\log|\varphi(t)-\varphi(s)||)\varphi'(s)$$

$$=c_K\left(1+\left|\log\frac{\varphi(t)-\varphi(s)}{t-s}\right|+|\log|t-s||\right)\varphi'(s)$$

$$\leq c_K(1 + |\log \min \{(t+s), [(1-t)+(1-s)]\}| + |\log |t-s||)\varphi'(s)$$

$$\leq c_1 + c_2(1 + |\log |t-s||),$$

i.e. $K_\varphi \in \mathcal{S}^{0,0}$.

Having established that $K_\varphi \in \mathcal{S}^{0,\nu}$ for $\nu < 1$, the compactness of the operator $T_\varphi : C[0,1] \to C[0,1]$ follows by Lemma 4.2.1. □

Corollary 5.2.1. *Assume the conditions of Theorems 5.0.1 and 5.2.1. Then for the solution $v \in C^m[0,1]$ of equation* (5.1.2) *we have*

$$v^{(j)}(0) = v^{(j)}(1) = 0, \quad j = 1, \ldots, m. \tag{5.2.9}$$

Example 5.2.1. *Let us present an example of function φ that satisfies the conditions on Theorem 5.2.1:*

$$\varphi(t) = c_p \int_0^t \tau^{p-1}(1-\tau)^{p-1} d\tau, \quad c_p = \frac{1}{\int_0^1 \tau^{p-1}(1-\tau)^{p-1} d\tau}, \quad p \in \mathbb{N}. \tag{5.2.10}$$

Note that, for calculating the coefficient c_p we can use the Euler beta function $B(x,y)$:

$$B(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1} d\tau, \quad x > 0, \quad y > 0.$$

It is well known that (see [1])

$$B(1+x, 1+y) = \frac{\Gamma(1+x)\Gamma(1+y)}{\Gamma(2+x+y)}, \quad x, y > -1,$$

where

$$\Gamma(x) = \int_0^\infty \tau^{x-1} e^{-\tau} d\tau \quad (x > 0)$$

is the Euler gamma function and in case of positive integers n we can calculate:

$$\Gamma(n) = (n-1)!.$$

Thus

$$c_p = \frac{1}{B(p,p)} = \frac{(2p-1)!}{[(p-1)!]^2}, \quad p \in \mathbb{N}.$$

Chapter 6

Collocation method based on the central part interpolation

Following [61] we construct in this chapter a high-order method for the numerical solution of weakly singular integral equations of the second kind: we perform in (5.0.1) a smoothing change of variables and solve the resulting equation by collocation techniques based on a central part interpolation by polynomials on the uniform grid.

6.1 Error estimate of the collocation method

Consider equation (5.0.1),

$$u(x) = \int_0^1 K(x,y)u(y)\,dy + f(x), \quad 0 \le x \le 1,$$

and its smoothed counterpart (5.1.2),

$$v(t) = \int_0^1 K_\varphi(t,s)v(s)ds + f_\varphi(t), \quad 0 \le t \le 1.$$

Denote by T the integral operator of equation (5.0.1),

$$(Tu)(x) = \int_0^1 K(x,y)u(y)\,dy, \quad 0 \le x \le 1,$$

and by T_φ the integral operator of equation (5.1.2),

$$(T_\varphi v)(t) = \int_0^1 K_\varphi(t,s)v(s)ds, \quad 0 \le t \le 1.$$

We rewrite (5.1.2) in the operator form

$$v = T_\varphi v + f_\varphi. \tag{6.1.1}$$

Using the interpolation projector $P_{h,m}$ defined in (3.2.6), we approximate equation (6.1.1) by equation

$$v_h = P_{h,m} T_\varphi v_h + P_{h,m} f_\varphi. \tag{6.1.2}$$

This is the operator form of our piecewise polynomial collocation method based on a central part interpolation on the uniform grid.

Denote

$$\mathcal{N}(I-T) = \{u \in C[0,1] : u = Tu\}.$$

Theorem 6.1.1. *Let $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 2$, $\nu < 1$, and let the smoothing transformation $\varphi : [0,1] \to [0,1]$ satisfy the assumptions of Theorem 5.2.1. Let $\mathcal{N}(I-T) = \{0\}$ or equivalently, $\mathcal{N}(I-T_\varphi) = \{0\}$.*

Then equation (6.1.1) *(equation* (5.1.2)*) has a unique solution $v \in C[0,1]$ and there exists an n_0 such that for $n \ge n_0$, the collocation equation* (6.1.2) *has a unique solution v_h. The accuracy of v_h can be estimated by*

$$\|v - v_h\|_\infty \le ch^m \left\|v^{(m)}\right\|_\infty, \quad n = \frac{1}{h} \ge n_0, \tag{6.1.3}$$

where c is a positive constant not depending on n and f (it depends on K, m and p). Moreover,

$$v(t) = u(\varphi(t)), \quad 0 \le t \le 1,$$

with $u(x)$, the solution to (5.0.1).

Proof. By Theorem 5.2.1, $T_\varphi : C[0,1] \to C[0,1]$ is compact. Since $\mathcal{N}(I-T_\varphi) = \{0\}$, the bounded inverse $(I-T_\varphi)^{-1} : C[0,1] \to C[0,1]$ exists due to Fredholm alternative (Theorem 2.2.7); denote

$$\kappa := \left\|(I-T_\varphi)^{-1}\right\|_{\mathcal{L}(C[0,1],C[0,1])}.$$

The compactness of $T_\varphi : C[0,1] \to C[0,1]$ and the pointwise convergence $P_{h,m}$ to I in $C[0,1]$ (see Lemma 3.2.2) imply by Theorem 2.2.5 the norm convergence

$$\epsilon_h := \|T_\varphi - P_{h,m}T_\varphi\|_{\mathcal{L}(C[0,1],C[0,1])} \to 0, \text{ as } n \to \infty \quad (\text{as } h = \frac{1}{n} \to 0).$$

Hence there is an n_0 such that $\kappa\epsilon_h < 1$ for $n > n_0$. With the help of Theorem 2.2.4 we conclude that $I - P_{h,m}T_\varphi$ is invertible in $C[0,1]$ for $n \geq n_0$ and

$$\kappa_h := \left\|(I - P_{h,m}T_\varphi)^{-1}\right\|_{\mathcal{L}(C[0,1],C[0,1])} \to \kappa \text{ as } n \to \infty \quad (\text{as } h = \frac{1}{n} \to 0). \quad (6.1.4)$$

Indeed,

$$\left\|(I - P_{h,m}T_\varphi)^{-1}\right\|_{\mathcal{L}(C[0,1],C[0,1])}$$

$$\leq \frac{\left\|(I - T_\varphi)^{-1}\right\|_{\mathcal{L}(C[0,1],C[0,1])}}{1 - \|T_\varphi - P_{h,m}T_\varphi\|_{\mathcal{L}(C[0,1],C[0,1])} \left\|(I - T_\varphi)^{-1}\right\|_{\mathcal{L}(C[0,1],C[0,1])}}$$

$$= \frac{\kappa}{1 - \kappa\epsilon_h} \to \kappa \text{ as } n \to \infty.$$

This proves the unique solvability of the collocation equation (6.1.2) for $n \geq n_0$.

Let v and v_h be the solutions of (6.1.1) and (6.1.2) respectively. Then

$$(I - P_{h,m}T_\varphi)(v - v_h) = v - P_{h,m}T_\varphi v - P_{h,m}f_\varphi = v - P_{h,m}v,$$

$$v - v_h = (I - P_{h,m}T_\varphi)^{-1}(v - P_{h,m}v)$$

and

$$\|v - v_h\|_\infty \leq \kappa_h \|v - P_{h,m}v\|_\infty, \quad n = \frac{1}{h} \geq n_0. \quad (6.1.5)$$

By Theorem 5.0.1, for the solution u of (5.0.1) we have $u \in C^{m,\nu}(0,1)$; by Corollary 5.2.1, for $v(t) = u_\varphi(t) = u(\varphi(t))$ we have $v \in C^m[0,1]$ and $v^{(j)}(0) = v^{(j)}(1) = 0$, $j = 1, \ldots, m$; by Lemma 3.2.1(ii),

$$\|v - P_{h,m}v\|_\infty \leq \vartheta_m h^m \left\|v^{(m)}\right\|_\infty.$$

Now (6.1.5) yields

$$\|v - v_h\|_\infty \leq \kappa_h \vartheta_m h^m \left\|v^{(m)}\right\|_\infty$$

that together with (6.1.4) implies (6.1.3). □

6.2 Matrix form of the collocation method

The solution v_h of equation (6.1.2) belongs to $\mathcal{R}(P_{h,m})$, so the knot values

$$v_h(ih) \ (i = 0, \ldots, n)$$

determine v_h uniquely. Equation (6.1.2) is equivalent to a system of linear algebraic equation with respect to $v_h(ih)$, $i = 0, \ldots, n$, and our task is to write down this system.

For $w_h \in \mathcal{R}(P_{h,m})$ we have $w_h = 0$ if and only if $w_h(ih) = 0,\ i = 0, \dots, n$. Since $(P_{h,m}w)(ih) = w(ih),\ i = 0, \dots, n$, equation (6.1.2) is equivalent to the conditions

$$v_h(ih) = (T_\varphi v_h)(ih) + f_\varphi(ih), \quad i = 0, \dots, n,$$

i.e. $v_h \in \mathcal{R}(P_{h,m})$ satisfies equation (5.1.2) at the knots $ih,\ i = 0, \dots, n$. Using for v_h the representation (3.2.7) we obtain

$$(T_\varphi v_h)(ih) = \int_0^1 K_\varphi(ih, s) v_h(s) ds = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} K_\varphi(ih, s) v_h(s) ds$$

$$= \sum_{j=0}^{n-1} \sum_{k \in \mathbb{Z}_m} \int_{jh}^{(j+1)h} K_\varphi(ih, s) L_{k,m}(ns - j) ds (E_\delta v_h)((j+k)h)$$

$$= \sum_{j=0}^{n-1} \sum_{k \in \mathbb{Z}_m} \alpha_{i,j,k} \cdot \left\{ \begin{array}{ccc} v_h(0) & \text{for} & j+k \le 0 \\ v_h((j+k)h) & \text{for} & 1 \le j+k \le n-1 \\ v_h(1) & \text{for} & j+k \ge n \end{array} \right\}$$

$$= \sum_{l=0}^{n} b_{i,l} v_h(lh), \quad i = 0, \dots, n,$$

where we denoted

$$\alpha_{i,j,k} = \int_{jh}^{(j+1)h} K_\varphi(ih, s) L_{k,m}(ns-j) ds, \quad i = 0, \dots, n, \quad j = 0, \dots, n-1, \quad k \in \mathbb{Z}_m, \tag{6.2.1}$$

$$b_{i,l} = \left\{ \begin{array}{lll} \displaystyle\sum_{k \in \mathbb{Z}_m} \sum_{\{j: 0 \le j \le n-1,\ j+k \le 0\}} \alpha_{i,j,k}, & \text{for} & l = 0 \\ \displaystyle\sum_{k \in \mathbb{Z}_m} \sum_{\{j: 0 \le j \le n-1,\ j+k = l\}} \alpha_{i,j,k}, & \text{for} & l = 1, \dots, n-1 \\ \displaystyle\sum_{k \in \mathbb{Z}_m} \sum_{\{j: 0 \le j \le n-1,\ j+k \ge n\}} \alpha_{i,j,k}, & \text{for} & l = n \end{array} \right\}, \tag{6.2.2}$$

$$i, l = 0, \dots, n.$$

Thus the matrix form of method (6.1.2) is given by

$$v_h(ih) = \sum_{l=0}^{n} b_{i,l} v_h(lh) + f_\varphi(ih), \quad i = 0, \ldots, n, \tag{6.2.3}$$

with $b_{i,l}$ defined by (6.2.2). Having determined $v_h(ih)$, $i = 0, \ldots, n$, through solving the system (6.2.3), the collocation solution $v_h(t)$ at any intermediate point $t \in [jh, (j+1)h]$, $j = 0, \ldots, n-1$, is given by

$$v_h(t) = \sum_{k \in \mathbb{Z}_m} \left\{ \begin{array}{ccc} v_h(0) & \text{for} & j+k \leq 0 \\ v_h((j+k)h) & \text{for} & 1 \leq j+k \leq n-1 \\ v_h(1) & \text{for} & j+k \geq n \end{array} \right\} \cdot L_{k,m}(nt-j), \tag{6.2.4}$$

where L_k, $k \in \mathbb{Z}_m$, are the Lagrange fundamental polynomials defined in (3.2.3).

Chapter 7

A product integration method based on the central part interpolation

The product integration method for solving weakly singular integral equations of the second kind is more simply realizable numerically than the corresponding collocation method, see [6, 25, 77]. The product integration method on the uniform grid seems to be hopeful, due to cheaper assembling of the corresponding system of linear algebraic equations.

In this chapter we introduce the idea of a product integration method based on the central part interpolation and smoothing change of variables by considering the kernels with algebraic and logarithmic singularity, respectively:

$$K(x,y) = a(x,y)|x-y|^{-\nu} + b(x,y), \quad 0<\nu<1, \tag{7.0.1}$$

and

$$K(x,y) = a(x,y)\log|x-y| + b(x,y). \tag{7.0.2}$$

The assumptions about the functions $a(x,y)$ and $b(x,y)$ are given in the following sections.

7.1 Equation with algebraic singularity

Let the kernel $K(x,y)$ of equation (1.0.1) be in the form (7.0.1) in which case the equation reads as

$$u(x) = \int_0^1 [a(x,y)|x-y|^{-\nu} + b(x,y)]u(y)dy + f(x), \quad 0 \le x \le 1, \quad 0<\nu<1, \tag{7.1.1}$$

where $f \in C[0,1]$. About the coefficient functions $a(x,y)$ and $b(x,y)$ we assume that $a, b \in C([0,1] \times (0,1))$ and they may have some boundary singularities with respect to y (see Lemmas 7.1.1 and 7.1.2 below).

Denote by T the integral operator of equation (7.1.1):

$$(Tu)(x) = \int_0^1 [a(x,y)|x-y|^{-\nu} + b(x,y)]u(y)dy \quad 0 \le x \le 1, \quad 0 < \nu < 1. \quad (7.1.2)$$

We refer to [68] for the proofs of the following two lemmas.

Lemma 7.1.1. *Let the operator T be defined by the formula* (7.1.2) *with a fixed $\nu \in (0,1)$. Let $\lambda_0,\ \lambda_1 \in \mathbb{R},\ \lambda_0 + \nu < 1,\ \lambda_1 + \nu < 1$. Assume that $a,\ b \in C([0,1] \times (0,1))$ and*

$$|a(x,y)| + |b(x,y)| \le c y^{-\lambda_0}(1-y)^{-\lambda_1}, \quad (x,y) \in [0,1] \times (0,1)$$

where $c = c(a,b)$ is a positive constant.

Then T maps $C[0,1]$ into $C[0,1]$ and $T : C[0,1] \to C[0,1]$ is compact.

For $m \in \mathbb{N}$, $\theta_0, \theta_1 \in \mathbb{R}$, $\theta_0 < 1$, $\theta_1 < 1$, denote by $C^{m,\theta_0,\theta_1}(0,1)$ the weighted space of functions $u \in C[0,1] \cap C^m(0,1)$ such that

$$\sum_{k=1}^{m} \sup_{0<x<1} \omega_{k-1+\theta_0}(x)\omega_{k-1+\theta_1}(1-x) \left| u^{(k)}(x) \right| < \infty,$$

where

$$\omega_\rho(r) = \left\{ \begin{array}{ccc} 1 & \text{for} & \rho < 0 \\ \dfrac{1}{1+|\log r|} & \text{for} & \rho = 0 \\ r^\rho & \text{for} & \rho > 0 \end{array} \right\}, \quad r,\ \rho \in \mathbb{R},\ r > 0.$$

Equipped with the norm

$$\|u\|_{C^{m,\theta_0,\theta_1}(0,1)} := \max_{0 \le x \le 1} |u(x)|$$

$$+ \sum_{k=1}^{m} \sup_{0<x<1} \omega_{k-1+\theta_0}(x)\omega_{k-1+\theta_1}(1-x) \left| u^{(k)}(x) \right|, \quad u \in C^{m,\theta_0,\theta_1}(0,1),$$

$C^{m,\theta_0,\theta_1}(0,1)$ is a Banach space.

Thus, if $u \in C^{m,\theta_0,\theta_1}(0,1)$, $m \in \mathbb{N}$, $\theta_0, \theta_1 \in (0,1)$, then $u \in C[0,1] \cap C^m(0,1)$ and

$$\left| u^{(k)}(x) \right| \le c x^{1-\theta_0-k}(1-x)^{1-\theta_1-k}, \quad 0 < x < 1, \quad k = 1, \ldots, m,$$

where $c = c(u) > 0$ is a constant. Note also that

$$C^m[0,1] \subset C^{m,\theta_0,\theta_1}(0,1) \quad \text{for any} \quad m \in \mathbb{N}, \quad \theta_0 < 1, \quad \theta_1 < 1.$$

In what follows we will use the notation

$$\partial_x^k \partial_y^l = \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^l, \quad k, l \in \mathbb{N}_0.$$

Lemma 7.1.2. *Let T be defined by (7.1.2) with $\nu \in (0,1)$. Let $m \in \mathbb{N}$ and $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 + \nu < 1$, $\lambda_1 + \nu < 1$. Assume that a, $b \in C^m([0,1] \times (0,1))$ and satisfy*

$$\left|\partial_x^k \partial_y^l a\,(x,y)\right| + \left|\partial_x^k \partial_y^l b\,(x,y)\right| \le c y^{-\lambda_0 - l}(1-y)^{-\lambda_1 - l}, \quad (x,y) \in [0,1] \times (0,1), \tag{7.1.3}$$

with a positive constant $c = c(a,b)$ for all $k, l \in \mathbb{N}_0$ such that $k + l \le m$.

Then operator T maps $C^{m,\theta_0,\theta_1}(0,1)$ with $\theta_0 = \lambda_0 + \nu$ and $\theta_1 = \lambda_1 + \nu$ into $C^{m,\theta_0,\theta_1}(0,1)$ and $T : C^{m,\theta_0,\theta_1}(0,1) \to C^{m,\theta_0,\theta_1}(0,1)$ is compact.

Denote

$$\mathcal{N}(I - T) = \{u \in C[0,1] : u = Tu\}.$$

The following theorem is a consequence of Lemmas 7.1.1 and 7.1.2.

Theorem 7.1.1. *Assume the conditions of Lemma 7.1.2 and $\mathcal{N}(I-T) = \{0\}$. Let $f \in C^{m,\theta_0,\theta_1}(0,1)$, $\theta_0 = \lambda_0 + \nu$, $\theta_1 = \lambda_1 + \nu$. Then equation (7.1.1) has a solution $u \in C^{m,\theta_0,\theta_1}(0,1)$ which is unique in $C[0,1]$ and*

$$\|u\|_{C^{m,\theta_0,\theta_1}(0,1)} \le c\,\|f\|_{C^{m,\theta_0,\theta_1}(0,1)}, \tag{7.1.4}$$

with a constant c which is independent of f.

The main results of the following subsections are established under assumptions of Theorem 7.1.1.

7.1.1 Operator form of the method, convergence and error estimate

Let $p_0, p_1 \in \mathbb{N}$ be some given numbers. We perform in the equation (7.1.1) the change of variables (5.1.1),

$$x = \varphi(t), \quad y = \varphi(s), \quad 0 \le t \le 1, \quad 0 \le s \le 1,$$

where $\varphi : [0,1] \to [0,1]$ is defined by the formula (cf. (5.2.10))

$$\begin{aligned} \varphi(t) &= \frac{1}{c_*}\int_0^t \sigma^{p_0-1}(1-\sigma)^{p_1-1}d\sigma, \quad 0 \le t \le 1, \\ c_* &= \int_0^1 \sigma^{p_0-1}(1-\sigma)^{p_1-1}d\sigma = \frac{\Gamma(p_0)\Gamma(p_1)}{\Gamma(p_0+p_1)}, \end{aligned} \tag{7.1.5}$$

with Γ, the Euler gamma function. Observe that

$$\begin{aligned} &0 \le \varphi(t) \le c_0 t^{p_0}, \qquad 0 \le 1-\varphi(t) \le c'(1-t)^{p_1}, \quad 0 \le t \le 1, \\ &\left|\varphi^{(k)}(t)\right| \le c_k t^{p_0-k}(1-t)^{p_1-k}, \quad 0 < t < 1, \quad k = 1,\dots,m, \quad m \in \mathbb{N}. \end{aligned} \tag{7.1.6}$$

Note also that the integral $\int_0^t \sigma^{p_0-1}(1-\sigma)^{p_1-1}d\sigma$ in (7.1.5) can be evaluated in a stable way by an exact Gauss rule, since the integrand is a polynomial of degree p_0+p_1-2.

Clearly, $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(t)$ is strictly increasing in $[0,1]$. Hence we have for $s \ne t$ that

$$\begin{aligned} \frac{\varphi(t)-\varphi(s)}{t-s} &> 0, \\ |\varphi(t)-\varphi(s)| &= \frac{\varphi(t)-\varphi(s)}{t-s}|t-s|, \\ |\varphi(t)-\varphi(s)|^{-\nu} &= \left[\frac{\varphi(t)-\varphi(s)}{t-s}\right]^{-\nu}|t-s|^{-\nu}. \end{aligned}$$

After change of variables equation (7.1.1) takes the form

$$v(t) = \int_0^1 \left[\mathcal{A}(t,s)|t-s|^{-\nu} + \mathcal{B}(t,s)\right]v(s)ds + g(t), \quad 0 \le t \le 1, \quad 0 < \nu < 1, \tag{7.1.7}$$

where $v(t) = u(\varphi(t))$ is the new function we look for,

$$g(t) = f(\varphi(t)), \tag{7.1.8}$$

$$\mathcal{A}(t,s) = a(\varphi(t),\varphi(s))\Phi(t,s)^{-\nu}\varphi'(s), \tag{7.1.9}$$

$$\mathcal{B}(t,s) = b(\varphi(t),\varphi(s))\varphi'(s), \tag{7.1.10}$$

and

$$\Phi(t,s) = \left\{ \begin{array}{lll} \dfrac{\varphi(t)-\varphi(s)}{t-s} & \text{for} & t \ne s \\ \varphi'(s) & \text{for} & t = s \end{array} \right\}, \quad 0 \le t,s \le 1. \tag{7.1.11}$$

Note that $\Phi(t,s) > 0$ everywhere in the square $0 \le t, s \le 1$ except two points $(0,0)$ and $(1,1)$ in which Φ vanishes causing singularities of $\Phi(t,s)^{-\nu}$. According to (7.1.5), (7.1.6) and (7.1.11),

$$\Phi(t,s) = \Phi(s,t), \quad 0 \le t, s \le 1,$$

$$\Phi \in C^{m-1}([0,1] \times [0,1]) \quad \text{for} \quad p_0, p_1 \ge m \in \mathbb{N},$$

and as function φ defined by (7.1.5) is a polynomial for $p_0, p_1 \in \mathbb{N}$ so is Φ a polynomial for $p_0, p_1 \in \mathbb{N}$.

Further, we have (cf. (5.2.8))

$$\begin{gathered} \Phi\,(t,s) \asymp (t+s)^{p_0-1}\,((1-t)+(1-s))^{p_1-1} \\ \text{as } t,s \to 0 \text{ or as } t,s \to 1, \end{gathered} \tag{7.1.12}$$

where $\Xi(t) \asymp \Psi(t)$ as $t \to 0$ means that $\dfrac{\Xi(t)}{\Psi(t)}$ and $\dfrac{\Psi(t)}{\Xi(t)}$ are bounded as $t \to 0$.

Indeed, let $0 \le s < t \le \dfrac{1}{2}$. According to (7.1.5) and (7.1.11) it holds

$$\Phi(t,s) = \frac{\varphi(t) - \varphi(s)}{t-s} = \frac{1}{t-s} \int_s^t \varphi'(\sigma) d\sigma = \frac{1}{c_*} \frac{1}{t-s} \int_s^t \sigma^{p_0-1} (1-\sigma)^{p_1-1} d\sigma.$$

Thus

$$\Phi(t,s) \le \frac{1}{c_*} \frac{1}{t-s} \int_s^t \sigma^{p_0-1} d\sigma = \frac{1}{c_*} \frac{1}{p_0} \frac{t^{p_0} - s^{p_0}}{t-s}.$$

By Lagrange's mean value theorem we can estimate

$$\frac{t^{p_0} - s^{p_0}}{t-s} \le p_0 t^{p_0-1} \le p_0 (t+s)^{p_0-1},$$

and therefore

$$\Phi(t,s) \le \frac{1}{c_*} (t+s)^{p_0-1}.$$

We also see that

$$\Phi(t,s) \ge c' (t+s)^{p_0-1}$$

for a positive constant $c' > 0$ and $0 \le s < t \le \dfrac{1}{2}$.

Indeed, for $0 \le s < t \le \delta < 1$ it holds

$$\begin{aligned}\Phi(t,s) &= \frac{1}{t-s}\int\limits_s^t \varphi'(\sigma)d\sigma \\ &= \frac{1}{c_*(t-s)}\int\limits_s^t \sigma^{p_0-1}(1-\sigma)^{p_1-1}d\sigma \\ &\ge \frac{(1-\delta)^{p_1-1}}{c_* p_0}\left(\frac{t^{p_0}-s^{p_0}}{t-s}\right).\end{aligned}$$

Since for $0 \le s < t$, $p_0 \ge 1$,

$$\frac{t^{p_0}-s^{p_0}}{t-s} \ge t^{p_0-1},$$

we get

$$\Phi(t,s) \ge c_\delta t^{p_0-1} \ge \frac{c_\delta}{2^{p_0-1}}(t+s)^{p_0-1} \quad \text{with} \quad c_\delta = \frac{(1-\delta)^{p_1-1}}{c_* p_0}.$$

Therefore

$$\Phi\,(t,s) \asymp (t+s)^{p_0-1} \quad \text{as} \quad t,s \to 0. \tag{7.1.13}$$

In a similar way we obtain that

$$\Phi\,(t,s) \asymp ((1-t)+(1-s))^{p_1-1} \quad \text{as} \quad t,s \to 1.$$

This together with (7.1.13) yields (7.1.12).

Using (7.1.6), we obtain

$$\begin{gathered}\left|\partial_s^k \Phi(t,s)\right| \le c(t+s)^{p_0-k-1}((1-t)+(1-s))^{p_1-k-1}, \\ 0 \le s,t \le 1, \quad k = 1,\dots,m, \quad m \in \mathbb{N}.\end{gathered} \tag{7.1.14}$$

This together with (7.1.12) and the formula of Faà di Bruno (2.1.2) implies the following result.

Lemma 7.1.3. *For $j = 0,\dots,m$, $m \in \mathbb{N}_0$, $0 \le t \le 1$, $0 < s < 1$, it holds*

$$\left|\partial_s^j\left(\Phi(t,s)^{-\nu}\right)\right| \le c(t+s)^{-\nu(p_0-1)-j}((1-t)+(1-s))^{-\nu(p_1-1)-j}. \tag{7.1.15}$$

Since the factor $\varphi'(s) = \dfrac{1}{c_*}s^{p_0-1}(1-s)^{p_1-1}$ damps the singularities, it holds

$$\left|\partial_s^j\left(\Phi(t,s)^{-\nu}\right)\varphi'(s)\right| \le c s^{(p_0-1)(1-\nu)-j}(1-s)^{(p_1-1)(1-\nu)-j}, \quad j = 0,\dots,m.$$

Let us characterise the boundary behaviour of functions v, $\mathcal{A}$ and $\mathcal{B}$ in equation (7.1.7). For the proof of the following lemma we refer to [85].

Lemma 7.1.4. *Let $m \in \mathbb{N}$. If $u \in C^{m,\theta_0,\theta_1}(0,1)$, $\theta_0, \theta_1 \in \mathbb{R}$, $\theta_0 < 1$, $\theta_1 < 1$, and $v(t) = u(\varphi(t))$, then for $j = 1, \ldots, m$, $0 < t < 1$,*

$$\left|v^{(j)}(t)\right| \le c\,\|u\|_{C^{m,\theta_0,\theta_1}(0,1)} \left\{ \begin{array}{ll} t^{p_0-j} & \text{for} \quad \theta_0 < 0 \\ t^{p_0-j}(1+|\log t|) & \text{for} \quad \theta_0 = 0 \\ t^{(1-\theta_0)p_0-j} & \text{for} \quad \theta_0 > 0 \end{array} \right\} \times$$
$$\times \left\{ \begin{array}{ll} (1-t)^{p_1-j} & \text{for} \quad \theta_1 < 0 \\ (1-t)^{p_1-j}(1+|\log(1-t)|) & \text{for} \quad \theta_1 = 0 \\ (1-t)^{(1-\theta_1)p_1-j} & \text{for} \quad \theta_1 > 0 \end{array} \right\}. \tag{7.1.16}$$

The following lemma is a consequence of (7.1.3), (7.1.5) and formula of Faà di Bruno (2.1.2).

Lemma 7.1.5. *Let a and b satisfy the conditions of Lemma 7.1.2 and let φ be defined by* (7.1.5). *Then for $j = 0, \ldots, m$, $0 \le t \le 1$, $0 < s < 1$, it holds*

$$\left|\partial_s^j a(\varphi(t), \varphi(s))\right| + \left|\partial_s^j b(\varphi(t), \varphi(s))\right| \le c s^{-p_0\lambda_0 - j}(1-s)^{-p_1\lambda_1 - j}. \tag{7.1.17}$$

Next we present estimates for the functions $\mathcal{A}(t,s)$, $\mathcal{B}(t,s)$ (see (7.1.9), (7.1.10)) and $\partial_s^m[\mathcal{A}(t,s)v(s)]$, $\partial_s^m[\mathcal{B}(t,s)v(s)]$ in somewhat specific form for the needs of Theorem 7.1.2 below.

Corollary 7.1.1. *Let a and b satisfy the conditions of Lemma 7.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be defined by the formulas* (7.1.9) *and* (7.1.10), *respectively. Let φ be defined by* (7.1.5).

Then the following holds true.

(i) If

$$p_0, p_1 \ge 1, \quad p_0 > (1-\nu)/(1-\nu-\lambda_0), \quad p_1 > (1-\nu)/(1-\nu-\lambda_1), \tag{7.1.18}$$

then with $\delta_0 := (1-\nu-\lambda_0)p_0 - (1-\nu) > 0$, $\delta_1 := (1-\nu-\lambda_1)p_1 - (1-\nu) > 0$, it holds

$$|\mathcal{A}(t,s)| \le c s^{\delta_0}(1-s)^{\delta_1}, \quad (t,s) \in [0,1] \times (0,1). \tag{7.1.19}$$

(ii) If

$$p_0, p_1 \ge 1, \quad p_0 > 1/(1-\lambda_0), \quad p_1 > 1/(1-\lambda_1), \tag{7.1.20}$$

then with $\delta_0 := (1-\lambda_0)p_0 - 1 > 0$, $\delta_1 := (1-\lambda_1)p_1 - 1 > 0$, it holds

$$|\mathcal{B}(t,s)| \le c s^{\delta_0}(1-s)^{\delta_1}, \quad (t,s) \in [0,1] \times (0,1). \tag{7.1.21}$$

Proof. By inequalities (7.1.6), (7.1.17) and (7.1.15) we have

$$|\mathcal{A}(t,s)| \le cs^{-p_0\lambda_0+p_0-1}(t+s)^{-\nu(p_0-1)}(1-s)^{-p_1\lambda_1+p_1-1}(2-t-s)^{-\nu(p_1-1)}$$

that for p_0, p_1 satisfying (7.1.18) yields (7.1.19). Similarly, by (7.1.6) and (7.1.17)

$$|\mathcal{B}(t,s)| \le cs^{-p_0\lambda_0+p_0-1}(1-s)^{-p_1\lambda_1+p_1-1}$$

that for p_0, p_1 satisfying (7.1.20) yields (7.1.21). □

Corollary 7.1.2. *Let a and b satisfy the conditions of Lemma 7.1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be defined by the formulas* (7.1.9) *and* (7.1.10)*, respectively. Let φ be defined by* (7.1.5)*. Finally assume, that $u \in C^{m,\theta_0,\theta_1}(0,1)$, $m \in \mathbb{N}$, $\theta_0 = \lambda_0 + \nu$, $\theta_1 = \lambda_1 + \nu$ and let $v(t) = u(\varphi(t))$.*

Then the following estimates hold true for $(t,s) \in [0,1] \times (0,1)$.

(i) If

$$p_0, p_1 \ge 1, \quad p_0 > m/(1-\nu-\lambda_0), \quad p_1 > m/(1-\nu-\lambda_1), \tag{7.1.22}$$

then with $\delta_0 := (1-\nu-\lambda_0)p_0 - m > 0$, $\delta_1 := (1-\nu-\lambda_1)p_1 - m > 0$,

$$|\mathcal{A}(t,s)| \le cs^{m-(1-\nu)+\delta_0}(1-s)^{m-(1-\nu)+\delta_1} \tag{7.1.23}$$

and

$$|\partial_s^m[\mathcal{A}(t,s)v(s)]| \le cs^{-(1-\nu)+\delta_0}(1-s)^{-(1-\nu)+\delta_1}\,\|u\|_{C^{m,\theta_0,\theta_1}}\,. \tag{7.1.24}$$

(ii) If

$$p_0, p_1 \ge 1, \quad p_0 > m/(1-\lambda_0), \quad p_1 > m/(1-\lambda_1), \tag{7.1.25}$$

then with $\delta_0 := (1-\lambda_0)p_0 - m > 0$, $\delta_1 := (1-\lambda_1)p_1 - m > 0$, it holds

$$|\mathcal{B}(t,s)| \le cs^{m-1+\delta_0}(1-s)^{m-1+\delta_1} \tag{7.1.26}$$

and

$$|\partial_s^m[\mathcal{B}(t,s)v(s)]| \le cs^{-1+\delta_0}(1-s)^{-1+\delta_1}\,\|u\|_{C^{m,\theta_0,\theta_1}}\,. \tag{7.1.27}$$

Proof. These estimates are direct consequences of Lemmas 7.1.3 - 7.1.5. □

Due to (7.1.19) and (7.1.21) we can define $\mathcal{A}(t,s) = 0$ and $\mathcal{B}(t,s) = 0$ for $s = 0$ and for $s = 1$. Moreover, we extend $\mathcal{A}(t,s)$ and $\mathcal{B}(t,s)$ with respect to s outside $[0,1]$ by the zero value. The corresponding extensions of $\mathcal{A}$ and $\mathcal{B}$ will be denoted again by $\mathcal{A}$ and $\mathcal{B}$. Thus, under conditions (7.1.18) and (7.1.20), we obtain that

$$\mathcal{A}, \mathcal{B} \in C([0,1] \times [-\delta, 1+\delta]) \quad \text{for any} \quad \delta \ge 0.$$

We determine the approximate solution v_h for equation (7.1.7) by solving the following problem

$$v_h(t) = \int_0^1 |t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v_h(s))ds + \int_0^1 P_{h,m}(\mathcal{B}(t,s)v_h(s))ds + g(t), \quad 0 \le t \le 1. \tag{7.1.28}$$

Here $P_{h,m}$ (see (3.2.6)) is applied to the products $\mathcal{A}(t,s)v_h(s)$ and $\mathcal{B}(t,s)v_h(s)$ as a functions of s treating t as a parameter. This is the operator form of a product integration method corresponding to the piecewise polynomial "central part" interpolation on the uniform grid $\{ih : i = 0, \dots, n\}$.

Below we will use the integral operators $\mathcal{T}$ and $\mathcal{T}_h$ defined by the following formulas:

$$(\mathcal{T}v)(t) = \int_0^1 [|t-s|^{-\nu}\, \mathcal{A}(t,s) + \mathcal{B}(t,s)]v(s)ds, \quad 0 \le t \le 1, \tag{7.1.29}$$

$$(\mathcal{T}_h v)(t) = \int_0^1 [|t-s|^{-\nu}\, P_{h,m}(\mathcal{A}(t,s)v(s)) + P_{h,m}(\mathcal{B}(t,s)v(s))]ds, \quad 0 \le t \le 1. \tag{7.1.30}$$

The convergence behavior of method (7.1.28) is characerised by the following theorem.

Theorem 7.1.2.

(i) Let $0 < \nu < 1$, λ_0, $\lambda_1 \in \mathbb{R}$, $\lambda_0 + \nu < 1$, $\lambda_1 + \nu < 1$. *Let* $f \in C[0,1]$, a, $b \in C([0,1] \times (0,1))$ *and satisfy*

$$|a(x,y)| + |b(x,y)| \le cy^{-\lambda_0}(1-y)^{-\lambda_1}, \quad (x,y) \in [0,1] \times (0,1),$$

where $c = c(a,b)$ *is a positive constant. Let* $\mathcal{N}(I-T) = 0$, *with* T, *given by* (7.1.2). *Finally, let* φ *be defined by the formula* (7.1.5) *with parameters* $p_0, p_1 \in \mathbb{N}$ *such that*

$$p_0 > (1-\nu)/(1-\nu-\lambda_0), \quad p_1 > (1-\nu)/(1-\nu-\lambda_1)$$

and

$$p_0 > 1/(1-\lambda_0), \quad p_1 > 1/(1-\lambda_1).$$

Then for sufficiently large $n = \dfrac{1}{h}$, *say* $n \ge n_0$, *equation* (7.1.28) *has a unique solution* $v_h \in C[0,1]$, *and*

$$\|v - v_h\|_\infty = \max_{t\in[0,1]} |v(t) - v_h(t)| \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{7.1.31}$$

where $v \in C[0,1]$ *is the solution of* (7.1.7).

(ii) Let $m \in \mathbb{N}$, $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 + \nu < 1$, $\lambda_1 + \nu < 1$, $0 < \nu < 1$. Assume that a, $b \in C^m([0,1] \times (0,1))$ and satisfy

$$\left|\partial_x^k \partial_y^l a\,(x,y)\right| + \left|\partial_x^k \partial_y^l b\,(x,y)\right| \le c y^{-\lambda_0 - l}(1-y)^{-\lambda_1 - l}, \quad (x,y) \in [0,1] \times (0,1),$$

with a positive constant $c = c(a,b)$ for all $k, l \in \mathbb{N}_0$ such that $k + l \le m$. Let $f \in C^{m,\theta_0,\theta_1}(0,1)$ with $\theta_0 = \lambda_0 + \nu$, $\theta_1 = \lambda_1 + \nu$. Let $\mathcal{N}(I - T) = \{0\}$, with T, given by (7.1.2). Finally, let φ be defined by the formula (7.1.5) with parameters $p_0, p_1 \in \mathbb{N}$ such that

$$p_0, p_1 \ge 1, \quad p_0 > m/(1 - \nu - \lambda_0), \quad p_1 > m/(1 - \nu - \lambda_1).$$

Then it holds

$$\|v - v_h\|_\infty \le c h^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad n \ge n_1. \tag{7.1.32}$$

Here $n_1 = \max\{n_0, 2m\}$, and the constant c is independent of $n = \dfrac{1}{h}$ and f.

Proof. We consider equations (7.1.7) and (7.1.28) as operator equations

$$v = \mathcal{T} v + g \tag{7.1.33}$$

and

$$v_h = \mathcal{T}_h v_h + g, \tag{7.1.34}$$

where $\mathcal{T}$ and $\mathcal{T}_h$ are defined by the formulas (7.1.29) and (7.1.30), respectively. Since $f \in C[0,1]$, it follows from (7.1.5) and (7.1.8) that $g \in C[0,1]$. It is clear that $\mathcal{T}$ and $\mathcal{T}_h$ are linear operators. Since $\mathcal{A}, \mathcal{B} \in C([0,1] \times [0,1])$, we obtain that $\mathcal{T}$ and $\mathcal{T}_h$ are compact as operators from $C[0,1]$ into $C[0,1]$.

Next we show, that $\mathcal{T}_h \to \mathcal{T}$ compactly in $C[0,1]$, i.e. (see Definition 2.3.1)

$$\|\mathcal{T}_h v - \mathcal{T} v\|_\infty \to 0 \text{ for every } v \in C[0,1] \text{ as } h = 1/n \to 0, \tag{7.1.35}$$

$$(v_h) \subset C[0,1],\ \|v_h\|_\infty \le 1 \Rightarrow (\mathcal{T}_h v_h) \text{ is relatively compact in } C[0,1]. \tag{7.1.36}$$

First we observe that the sets $\{\mathcal{A}(t,\cdot) : 0 \le t \le 1\}$ and $\{\mathcal{B}(t,\cdot) : 0 \le t \le 1\}$ are relatively compact in $C[-\delta, 1+\delta]$, with a fixed $\delta > 0$. Therefore we get by Lemma 3.2.2 for a fixed $v \in C[0,1]$ extended by $v(s) = v(0)$ for $-\delta \le s \le 0$ and $v(s) = v(1)$ for $1 \le s \le 1 + \delta$ that

$$\sup_{0 \le t \le 1} \max_{0 \le s \le 1} |\mathcal{A}(t,s)\, v(s) - P_{h,m}(\mathcal{A}(t,s) v(s))| \to 0 \text{ as } n \to \infty, \tag{7.1.37}$$

$$\sup_{0 \le t \le 1} \max_{0 \le s \le 1} |\mathcal{B}(t,s) v(s) - P_{h,m}(\mathcal{B}(t,s) v(s))| \to 0 \text{ as } n \to \infty. \tag{7.1.38}$$

Further, we have

$$\int_0^1 |t-s|^{-\nu}\, ds \le \frac{2}{1-\nu}, \quad 0 \le t \le 1, \quad 0 < \nu < 1. \tag{7.1.39}$$

Therefore,

$$\begin{aligned}
&\|\mathcal{T}_h v - \mathcal{T} v\|_\infty \\
&= \max_{0\le t\le 1} \left| \int_0^1 [|t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v(s)) + P_{h,m}(\mathcal{B}(t,s)v(s))] ds \right. \\
&\qquad \left. - \int_0^1 [|t-s|^{-\nu} \mathcal{A}(t,s) + \mathcal{B}(t,s)] v(s) ds \right| \\
&\le \max_{0\le t\le 1} \int_0^1 |t-s|^{-\nu} \left| \mathcal{A}(t,s)v(s) - P_{h,m}(\mathcal{A}(t,s)v(s)) \right| ds \\
&\quad + \max_{0\le t\le 1} \int_0^1 \left| \mathcal{B}(t,s)v(s) - P_{h,m}(\mathcal{B}(t,s)v(s)) \right| ds \\
&\le \left(\frac{2}{1-\nu} \right) \max_{0\le t\le 1} \max_{0\le s\le 1} \left| \mathcal{A}(t,s)v(s) - P_{h,m}(\mathcal{A}(t,s)v(s)) \right| \\
&\quad + \max_{0\le t\le 1} \max_{0\le s\le 1} \left| \mathcal{B}(t,s)v(s) - P_{h,m}(\mathcal{B}(t,s)v(s)) \right|.
\end{aligned}$$

This together with (7.1.37) and (7.1.38) yields (7.1.35).

The proof of (7.1.36) can be built by Arzelà-Ascoli theorem (see Theorem 2.1.2).

Observe that the uniform boundedness of $(\mathcal{T}_h v_h) \subset C[0,1]$ with $(v_h) \subset C[0,1]$, $\|v_h\|_\infty \le 1$ for $n = \dfrac{1}{h} \in \mathbb{N}$ is a consequence of the estimate

$$\|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])} \le c\,(1 + \log m)\,, \tag{7.1.40}$$

where c is a positive constant not depending on $h = \dfrac{1}{n}$ (see (3.2.10)). Indeed, on the basis of (7.1.39), (7.1.40) and the notations

$$\|\mathcal{A}\|_\infty = \max_{(t,s)\in[0,1]\times[0,1]} |\mathcal{A}(t,s)|\,, \quad \|\mathcal{B}\|_\infty = \max_{(t,s)\in[0,1]\times[0,1]} |\mathcal{B}(t,s)|$$

we get

$$
\begin{aligned}
|(\mathcal{T}_h v_h)(t)| &= \left| \int_0^1 \left[|t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v_h(s)) + P_{h,m}(\mathcal{B}(t,s)v_h(s)) \right] ds \right| \\
&\le \int_0^1 |t-s|^{-\nu} \left| P_{h,m}(\mathcal{A}(t,s)v_h(s)) \right| ds + \int_0^1 \left| P_{h,m}(\mathcal{B}(t,s)v_h(s)) \right| ds \\
&\le \|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])} \|v_h\|_\infty \left[\|\mathcal{A}\|_\infty \int_0^1 |t-s|^{-\nu} + \|\mathcal{B}\|_\infty \right] \\
&\le c, \quad 0 \le t \le 1 \quad \text{for any} \quad n = \frac{1}{h} \in \mathbb{N},
\end{aligned}
$$

with a constant $c > 0$ which is independent of $n = \dfrac{1}{h}$.

For the equicontinuity of

$$
(\mathcal{T}_h v_h), \quad v_h \in C[0,1], \quad \|v_h\|_\infty \le 1, \quad h = \frac{1}{n}, \quad n \in \mathbb{N},
$$

we must show that for any $\epsilon > 0$ there is a $\eta = \eta(\epsilon) > 0$ such that $t, t' \in [0,1]$, $|t-t'| \le \eta$ implies $|(\mathcal{T}_h)v_h(t) - (\mathcal{T}_h v_h)(t')| \le \epsilon$ for all $h = \dfrac{1}{n}$, $n \in \mathbb{N}$.

Let $\epsilon > 0$ be fixed. According to the definition (7.1.30) of $\mathcal{T}_h$ we have:

$$
\begin{aligned}
&\left| (\mathcal{T}_h v_h)(t) - (\mathcal{T}_h v_h)(t') \right| \\
&\le \left| \int_0^1 \left[|t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v_h(s)) - |t'-s|^{-\nu} P_{h,m}(\mathcal{A}(t',s)v_h(s)) \right] ds \right| \\
&\quad + \left| \int_0^1 \left[P_{h,m}(\mathcal{B}(t,s)v_h(s)) - P_{h,m}(\mathcal{B}(t',s)v_h(s)) \right] ds \right|, \quad 0 \le t, t' \le 1.
\end{aligned} \tag{7.1.41}
$$

Function $\mathcal{B}$ is uniformly continous as a continous function on a closed set $[0,1] \times [0,1]$. Therefore using (7.1.40), we obtain for the second integral on the r.h.s of (7.1.41) that

$$
\begin{aligned}
&\left| \int_0^1 \left[P_{h,m}(\mathcal{B}(t,s)v_h(s)) - P_{h,m}(\mathcal{B}(t',s)v_h(s)) \right] ds \right| \\
&\le \int_0^1 \left| P_{h,m} \left[\mathcal{B}(t,s) - \mathcal{B}(t',s) \right] \right| |v_h(s)| \, ds
\end{aligned}
$$

$$\leq \|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])} \|v_h\|_\infty \max_{0\leq s\leq 1} \left|\mathcal{B}(t,s)-\mathcal{B}(t',s)\right|$$

$$\leq c(1+\log m) \max_{0\leq s\leq 1} \left|\mathcal{B}(t,s)-\mathcal{B}(t',s)\right|$$

$$\leq \frac{\epsilon}{2} \quad \text{for} \quad \left|t-t'\right| \leq \eta', \quad 0\leq t,t'\leq 1, \quad \forall n=\frac{1}{h}\in\mathbb{N}, \tag{7.1.42}$$

with a sufficiently small $\eta' > 0$. To estimate the first integral on the r.h.s of the (7.1.41) we add and subtract under the sign of integral a term of the form

$$\left|t'-s\right|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v_h(s)).$$

Then we have:

$$\left|\int_0^1 \left[|t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v_h(s)) - \left|t'-s\right|^{-\nu} P_{h,m}(\mathcal{A}(t',s)v_h(s))\right] ds\right|$$

$$\leq \int_0^1 \left|t'-s\right|^{-\nu} \left|P_{h,m}\left[(\mathcal{A}(t,s)-\mathcal{A}(t',s))v_h(s)\right]\right| ds \tag{7.1.43}$$

$$+\int_0^1 \left||t-s|^{-\nu} - \left|t'-s\right|^{-\nu}\right| \left|P_{h,m}\left[\mathcal{A}(t,s)v_h(s)\right]\right| ds, \quad 0\leq t,t'\leq 1.$$

Function $\mathcal{A}$ is uniformely continous on $[0,1]\times[0,1]$ and using (7.1.40), we obtain for the first integral on the r.h.s of (7.1.43) that

$$\int_0^1 \left|t'-s\right|^{-\nu} \left|P_{h,m}\left[(\mathcal{A}(t,s)-\mathcal{A}(t',s))v_h(s)\right]\right| ds$$

$$\leq \|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])} \|v_h\|_\infty \frac{2}{1-\nu} \max_{0\leq s\leq 1} \left|\mathcal{A}(t,s)-\mathcal{A}(t',s)\right|$$

$$\leq c(1+\log m)\frac{2}{1-\nu} \max_{0\leq s\leq 1} \left|\mathcal{A}(t,s)-\mathcal{A}(t',s)\right|$$

$$\leq \frac{\epsilon}{4} \quad \text{for} \quad \left|t-t'\right| < \eta'', \quad 0\leq t,t'\leq 1, \quad \forall n=\frac{1}{h}\in\mathbb{N}, \tag{7.1.44}$$

with a $\eta'' > 0$ which is sufficiently small. Since there exists a number $\eta''' > 0$ such that (assume that $\|\mathcal{A}\|_\infty \neq 0$)

$$\int_0^1 \left||t-s|^{-\nu} - \left|t'-s\right|^{-\nu}\right| ds$$

$$\leq \frac{1}{c(1+\log m)\,\|\mathcal{A}\|_\infty}\frac{\epsilon}{4} \quad \text{for} \quad \left|t-t'\right| < \eta''', \quad 0 \leq t, t' \leq 1,$$

we obtain that

$$\int_0^1 \left| |t-s|^{-\nu} - \left|t'-s\right|^{-\nu} \right| |P_{h,m}\left[\mathcal{A}(t,s)v_h(s)\right]| \, ds$$

$$\leq \|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])} \|\mathcal{A}\|_\infty \int_0^1 \left| |t-s|^{-\nu} - \left|t'-s\right|^{-\nu} \right| ds$$

$$\leq c(1+\log m)\,\|\mathcal{A}\|_\infty \int_0^1 \left| |t-s|^{-\nu} - \left|t'-s\right|^{-\nu} \right| ds$$

$$\leq \frac{\epsilon}{4} \quad \text{for} \quad \left|t-t'\right| < \eta''', \quad 0 \leq t, t' \leq 1, \quad \forall n = \frac{1}{h} \in \mathbb{N}.$$

This together with (7.1.42) and (7.1.44) yields

$$\left|(\mathcal{T}_h v_h)(t) - (\mathcal{T}_h v_h)(t')\right| \leq \epsilon \quad \text{for} \;\; 0 \leq t, t' \leq 1, \quad \left|t-t'\right| < \eta, \quad \forall n = \frac{1}{h} \in \mathbb{N},$$

where $\eta = \min\left\{\eta', \eta'', \eta'''\right\}$. The proof of (7.1.36) is completed and thus $\mathcal{T}_h \to \mathcal{T}$ compactly in $C[0,1]$.

Due to the condition $\mathcal{N}(I-T) = \{0\}$ also $\mathcal{N}(I-\mathcal{T}) = \{0\}$. Now it follows from Theorem 2.3.1 that equation (7.1.33) (equation (7.1.7)) has a unique solution $v \in C[0,1]$ and there exists a $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, equation (7.1.34) (equation (7.1.28)) has a unique solution $v_h \in C[0,1]$ and

$$\|v - v_h\|_\infty \leq c\,\|\mathcal{T}v - \mathcal{T}_h v\|_\infty, \quad n \geq n_0, \tag{7.1.45}$$

with a constant $c > 0$ not depending on n (on $h = \frac{1}{n}$). The convergence (7.1.31) is a consequence of (7.1.35).

Next we establish the estimate (7.1.32). For the solutions u and v of equations (7.1.1) and (7.1.7) we have $v(t) = u(\varphi(t))$ and $u \in C^{m,\theta_0,\theta_1}(0,1)$ by Theorem 7.1.1. To prove (7.1.32), it remains to show that (see 7.1.45)

$$\|\mathcal{T}v - \mathcal{T}_h v\|_\infty \leq ch^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad n \geq n_1. \tag{7.1.46}$$

According to the definitions of operators $\mathcal{T}$ and $\mathcal{T}_h$ (see (7.1.29) and (7.1.30)) we have

$$(\mathcal{T}v)(t) - (\mathcal{T}_h v)(t) = \int_0^1 \left[|t-s|^{-\nu}\,\mathcal{A}(t,s) + \mathcal{B}(t,s)\right] v(s)ds$$

$$-\int_0^1 \left[|t-s|^{-\nu} P_{h,m}\left(\mathcal{A}(t,s)v(s)\right) + P_{h,m}\left(\mathcal{B}(t,s)v(s)\right)\right] ds$$

$$= \int_0^1 |t-s|^{-\nu} \left(I - P_{h,m}\right) \left(\mathcal{A}(t,s)v(s)\right) ds$$

$$+ \int_0^1 (I - P_{h,m})(\mathcal{B}(t,s)v(s)) ds, \quad 0 \le t \le 1.$$

Therefore

$$\begin{aligned} |(\mathcal{T}v)(t) - (\mathcal{T}_h v)(t)| &\le \left| \int_0^1 |t-s|^{-\nu} \left(I - P_{h,m}\right) \left(\mathcal{A}(t,s)v(s)\right) ds \right| \\ &+ \left| \int_0^1 (I - P_{h,m})(\mathcal{B}(t,s)v(s)) ds \right|, \quad 0 \le t \le 1. \end{aligned} \tag{7.1.47}$$

Let us estimate the first integral on the r.h.s of the inequality (7.1.47). We divide the integration into four subintervals: $[0, mh]$, $[mh, 1/2]$, $[1/2, 1-mh]$ and $[1-mh, 1]$, where $mh \le \frac{1}{2}$ or equivalently $n \ge 2m$. Thus, first we estimate

$$\left| \int_0^{mh} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s)) ds \right|$$

$$\le \int_0^{mh} |t-s|^{-\nu} \left|(I - P_{h,m})(\mathcal{A}(t,s)v(s))\right| ds$$

$$\le \left(1 + \|P_{h,m}\|_{\mathcal{L}(C[0,1], C[0,1])}\right) \max_{0 \le s \le mh} |\mathcal{A}(t,s)| \, \|v\|_\infty \int_0^{mh} |t-s|^{-\nu} ds, \quad 0 \le t \le 1. \tag{7.1.48}$$

It follows from (7.1.23) by $\delta_0 := (1-\nu-\lambda_0)p_0 - m > 0$, that

$$\max_{0 \le s \le mh} |\mathcal{A}(t,s)| \le c \max_{0 \le s \le mh} s^{m-(1-\nu)+\delta_0} \le c(mh)^{m-(1-\nu)+\delta_0}, \quad 0 \le t \le 1.$$

Since

$$\int_0^{mh} |t-s|^{-\nu} ds \le \frac{2m^{1-\nu}}{1-\nu} h^{1-\nu}, \quad 0 \le t \le 1,$$

we now get

$$\max_{0\le s\le mh} |\mathcal{A}(t,s)| \int_0^{mh} |t-s|^{-\nu}\, ds \le c_1 h^m, \quad 0\le t\le 1,$$

with a constant $c_1 = c_1(m,\nu,\delta_0) > 0$ which is independent of $h = \frac{1}{n}$. This together with (7.1.40), (7.1.48) and $\|v\|_\infty = \|u\|_\infty \le \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}$ yields

$$\left| \int_0^{mh} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))ds \right|$$
$$\le c_2 h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad 0\le t\le 1, \tag{7.1.49}$$

where c_2 is a positive constant which does not depend on $h = \frac{1}{n}$.

On the subinterval $[mh, 1/2]$ we use (3.2.9) to estimate

$$|(I-P_{h,m})\mathcal{A}(t,s)v(s)| \le \vartheta_m h^m \left|\partial_s^m [\mathcal{A}(t,s)v(s)]\right|, \quad 0\le t\le 1, \quad mh \le s\le \frac{1}{2},$$

and using (7.1.24) we get that

$$\left| \int_{mh}^{1/2} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))ds \right|$$
$$\le c_3 h^m \int_0^{1/2} |t-s|^{-\nu} s^{-(1-\nu)+\delta_0} ds\, \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}$$
$$\le c_4 h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad 0\le t\le 1, \tag{7.1.50}$$

with some positive constants c_3 and c_4 which are independent of $h = \frac{1}{n}$.

In a similar way we get

$$\left| \int_{1/2}^{1-mh} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))ds \right|$$
$$\le c_5 h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad 0\le t\le 1, \tag{7.1.51}$$

$$\left| \int_{1-mh}^{1} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))ds \right|$$

$$\le c_6 h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad 0 \le t \le 1, \tag{7.1.52}$$

where c_5 and c_6 are some constants which do not depend on $h = \dfrac{1}{n}$.

Due to the estimates (7.1.49) - (7.1.52) and (7.1.4) we obtain finally that

$$\begin{aligned} &\left| \int_0^1 |t-s|^{-\nu} (I - P_{h,m}) (\mathcal{A}(t,s)v(s)) ds \right| \\ &\le ch^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)} \\ &\le c' h^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad 0 \le t \le 1, \end{aligned} \tag{7.1.53}$$

with some constants c and c' not depending on $h = \dfrac{1}{n}$.

To estimate the second integral on the r.h.s. of the inequality (7.1.47) we use (3.2.9), (7.1.4) and (7.1.27) and get

$$\left| \int_0^1 (I - P_{h,m})(\mathcal{B}(t,s)v(s)) ds \right|$$

$$\le ch^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad 0 \le t \le 1,$$

with a constant c not depending on $h = \dfrac{1}{n}$.

This together with (7.1.47) and (7.1.53) proves (7.1.46) and completes the proof of Theorem 7.1.32. □

With respect to

$$u_h(x) := v_h(\varphi^{-1}(x))$$

estimate (7.1.32) reads as

$$\max_{0 \le x \le 1} |u(x) - u_h(x)| = \max_{0 \le t \le 1} |v(t) - v_h(t)| \le ch^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad n \ge n_1.$$

An advantage of the product integration method, compared to the collocation method, is that the number of integrals which must be computed numerically are respectively, of order $2mn$ and mn^2.

7.1.2 Matrix form of the method

Let us derive the matrix form of the product interpolation method (7.1.28). This method is of Nyström type - the solution v_h of equation (7.1.28) is uniquely determined by its knot values $v_h(ih)$, $i = 0, \ldots, n$, through the Nyström extension (derived from (7.1.28) by (3.2.4) and (3.2.6))

$$
\begin{aligned}
v_h(t) = & \sum_{j=0}^{n-1} \int\limits_{jh}^{(j+1)h} |t-s|^{-\nu} \sum_{k\in\mathbb{Z}_m} \mathcal{A}\left(t,(j+k)h\right) v_h\left((j+k)h\right) L_{k,m}(ns-j)ds \\
& + \sum_{j=0}^{n-1} \int\limits_{jh}^{(j+1)h} \sum_{k\in\mathbb{Z}_m} \mathcal{B}\left(t,(j+k)h\right) v_h\left((j+k)h\right) L_{k,m}(ns-j)ds \\
& + g(t), \quad 0 \le t \le 1.
\end{aligned} \tag{7.1.54}
$$

We obtain an algebraic system of linear equations w.r.t. the grid values $v_h(ih)$, $i = 0,\dots,n$, by collocating (7.1.54) at the points $t = ih$:

$$
\begin{aligned}
v_h(ih) = & \sum_{j=0}^{n-1} \sum_{k\in\mathbb{Z}_m} \Bigg\{ \mathcal{A}\left(ih,(j+k)h\right) \int\limits_{jh}^{(j+1)h} |ih-s|^{-\nu} L_{k,m}(ns-j)ds \\
& + \mathcal{B}(ih,(j+k)h) \int\limits_{jh}^{(j+1)h} L_{k,m}(ns-j)ds \Bigg\} v_h((j+k)h) \\
& + g(ih), \quad i = 0,\dots,n.
\end{aligned} \tag{7.1.55}
$$

We extended $\mathcal{A}(t,s)$ and $\mathcal{B}(t,s)$ with respect to s outside $[0,1]$ by the zero value, thus

$$
\mathcal{A}(ih,(j+k)h = 0, \quad \mathcal{B}(ih,(j+k)h = 0 \ \text{ for } \ j+k \le 0 \ \text{ and for } \ j+k \ge n,
$$

therefore in the r.h.s. of (7.1.55) the values $v_h(lh)$ with $l \le 0$ and $l \ge n$ actually are not exploited.
Occuring here integrals depend on the difference $i-j$: with the change of variables $ns - j = \sigma$ we see that

$$
\int\limits_{jh}^{(j+1)h} |ih-s|^{-\nu} L_{k,m}(ns-j)ds = h^{1-\nu} \int\limits_{0}^{1} |i-j-\sigma|^{-\nu} L_{k,m}(\sigma)d\sigma.
$$

System (7.1.55) takes the form

$$
v_h(ih) = h^{1-\nu} \sum_{j=0}^{n-1} \sum_{k\in\mathbb{Z}_m} \{\mathcal{A}(ih,(j+k)h)\alpha_{i-j,k}
$$

$$
+\mathcal{B}(ih,(j+k)h)\beta_k\} \, v_h((j+k)h) + g(ih), \quad i = 0,\dots,n,
$$

or, collecting in the r.h.s. the coefficients by $v_h((j+k)h)$ with fixed $j+k=l$,

$$v_h(ih) = \sum_{l=1}^{n-1} c_{i,l} v_h(lh) + g(ih), \quad i = 0, \dots, n, \tag{7.1.56}$$

where

$$c_{i,l} = h^{1-\nu} \left[\mathcal{A}(ih, lh) \sum_{\{k \in \mathbb{Z}_m : 0 \le l-k \le n-1\}} \alpha_{i-l+k,k} + \mathcal{B}(ih, lh) \sum_{k \in \mathbb{Z}_m} \beta_k \right], \tag{7.1.57}$$

$$i = 0, \dots, n, \ l = 1, \dots, n-1,$$

$$\alpha_{i',k} := \int_0^1 |i' - \sigma|^{-\nu} L_{k,m}(\sigma) d\sigma, \quad i' = -n+1, \dots, n, \quad k \in \mathbb{Z}_m, \tag{7.1.58}$$

and

$$\beta_k := h^\nu \int_0^1 L_{k,m}(\sigma) d\sigma, \quad k \in \mathbb{Z}_m. \tag{7.1.59}$$

We took into account that $\mathcal{A}(ih, lh) = 0$ for $l \le 0$ and for $l \ge n$.

Having found the solution $\{v_h(ih)\}$, $(i = 0, \dots, n)$ of system (7.1.56), we can use (7.1.54) to find the solution at any point $t \in [0, 1]$.

Note that we can also find an approximate solution $\tilde{v}_h(t)$ by (6.2.4):

$$\tilde{v}_h(t) = \sum_{k \in \mathbb{Z}_m} \left\{ \begin{array}{ccc} v_h(0) & \text{for} & j+k \le 0 \\ v_h((j+k)h) & \text{for} & 1 \le j+k \le n-1 \\ v_h(1) & \text{for} & j+k \ge n \end{array} \right\} \cdot L_{k,m}(nt - j),$$

where $L_{k,m}$, $k \in \mathbb{Z}_m$, are the Lagrange fundamental polynomials defined in (3.2.3) and $0 \le t \le 1$.

7.2 Equation with logarithmic singularity

In this section we construct a product integration method based on the central part interpolation and smoothing change of variables for the numerical solution of integral equations of the form

$$u(x) = \int_0^1 [a(x,y) \log|x-y| + b(x,y)] u(y) dy + f(x), \quad 0 \le x \le 1, \tag{7.2.1}$$

where $f \in C[0,1]$. The coefficient functions $a, b \in C([0,1] \times (0,1))$ and they may have certain boundary singularities with respect to y characterised by Lemma 7.2.1 and Lemma 7.2.2 below. Due to diagonal and boundary singularities of the kernel

$$K(x,y) = a(x,y) \log|x-y| + b(x,y),$$

the solution $u(x)$ of equation (7.2.1), as a rule, has certain singularities at $x = 0$ and/or $x = 1$, even if a, b and f are sufficiently smooth functions on $[0,1] \times (0,1)$ and $[0,1]$ respectively. In order to characterize the possible singular behaviour of a solution u to (7.2.1) we introduce similarly to Sections 4.2 and 7.1 a suitable weighted space of functions $C_{log}^{m,\lambda_0,\lambda_1}(0,1)$. Showing that u belongs to this space the growth of the derivatives of u near the boundary of $[0,1]$ will be described (see Theorem 7.2.1 below). In Section 7.2.1 we reduce equation (7.2.1) with the help of a smoothing change of variables to a similar equation

$$v(t) = \int_0^1 (A(t,s) \log|t-s| + B(t,s))v(s)ds + g(t), \quad 0 \le t \le 1,$$

in which the coefficients $A(t,s)$ and $B(t,s)$ have no sigularities and vanish for $s = 0$ and $s = 1$; the logarithmic diagonal singularity of the kernel still remains to be present. However, the singularities of the solution of the transformed equation will be milder or disappear at all for suitable parameters of the change of variables.

Note that integral equations with a logartihmic diagonal singularity often arise in modelling physical processes. For example, they occur in radiative transfer theory, where the Milne integral equation (see, e.g. [12, 17, 28, 89]) has the form

$$u(x) = \frac{1}{2} \int_0^H a(y) E(|x-y|) u(y) dy + f(x), \quad 0 \le x \le H.$$

Here E is the integral exponent function:

$$\begin{aligned} E(\tau) &= \int_\tau^\infty \frac{e^{-\sigma}}{\sigma} d\sigma \\ &= -\log\tau + c + \tau - \frac{\tau^2}{2 \cdot 2!} + \frac{\tau^3}{3 \cdot 3!} - \frac{\tau^4}{4 \cdot 4!} + \ldots, \quad \tau > 0, \end{aligned}$$

where

$$c = \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) \approx 0.5772$$

is the Euler constant.

Denote by T the integral operator of equation (7.2.1):

$$(Tu)(x) = \int_0^1 [a(x,y) \log|x-y| + b(x,y)] u(y) dy, \quad 0 \le x \le 1. \tag{7.2.2}$$

For $m \in \mathbb{N}$, $\lambda_0, \lambda_0 \in \mathbb{R}$, $\lambda_0,\ \lambda_1 < 1$, denote by $C_{log}^{m,\lambda_0,\lambda_1}(0,1)$ the weighted space of functions $u \in C[0,1] \cap C^m(0,1)$ such that

$$\sum_{k=1}^{m} \sup_{0<x<1} \omega_{k-1+\lambda_0}(x)\omega_{k-1+\lambda_1}(1-x)\left|u^{(k)}(x)\right| < \infty,$$

where

$$\omega_\rho(r) = \left\{ \begin{array}{cc} 1 & \text{for} \quad \rho < 0 \\ \dfrac{r^\rho}{1+|\log r|} & \text{for} \quad \rho \geq 0 \end{array} \right\}, \quad r,\ \rho \in \mathbb{R},\ r > 0.$$

Thus, if $\lambda_0, \lambda_1 \in (0,1)$, then the inclusion $u \in C_{log}^{m,\lambda_0,\lambda_1}(0,1)$ with a $m \in \mathbb{N}$ yields that $u \in C[0,1] \cap C^m(0,1)$ and

$$\left|u^{(k)}(x)\right| \leq c x^{1-\lambda_0-k}(1+|\log x|)(1-x)^{1-\lambda_1-k}(1+|\log(1-x)|),$$
$$0 < x < 1, \quad k = 1, \dots, m,$$

where $c = c(u) > 0$ is a constant.

Equipped with the norm

$$\|u\|_{C_{log}^{m,\lambda_0,\lambda_1}(0,1)} := \max_{0 \leq x \leq 1} |u(x)|$$
$$+\sum_{k=1}^{m} \sup_{0<x<1} \omega_{k-1+\lambda_0}(x)\omega_{k-1+\lambda_1}(1-x)\left|u^{(k)}(x)\right|, \quad u \in C_{log}^{m,\lambda_0,\lambda_1}(0,1),$$

$C_{log}^{m,\lambda_0,\lambda_1}(0,1)$ is a Banach space. Clearly,

$$C^m[0,1] \subset C_{log}^{m,\lambda_0,\lambda_1}(0,1) \quad \text{for} \quad m \in \mathbb{N}, \quad \lambda_0, \lambda_0 \in \mathbb{R}, \quad \lambda_0, \lambda_1 < 1.$$

Lemma 7.2.1 (see [68])**.** *Let the operator T be defined by the formula* (7.2.2). *Let $a,\ b \in C([0,1] \times (0,1))$ satisfy for $(x,y) \in [0,1] \times (0,1)$ the inequality*

$$|a(x,y)| + |b(x,y)| \leq c y^{-\lambda_0}(1-y)^{-\lambda_1}, \quad \lambda_0,\ \lambda_1 \in \mathbb{R}, \quad \lambda_0 < 1, \quad \lambda_1 < 1,$$

where $c = c(a,b)$ is a positive constant.

Then T maps $C[0,1]$ into $C[0,1]$, and $T : C[0,1] \to C[0,1]$ is compact.

Lemma 7.2.2 (see [68])**.** *Let T be defined by* (7.2.2). *Let $m \in \mathbb{N}$ and $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 < 1, \lambda_1 < 1$. Assume that $a,\ b \in C^m([0,1] \times (0,1))$ and*

$$\left|\partial_x^k \partial_y^l a(x,y)\right| + \left|\partial_x^k \partial_y^l b(x,y)\right| \leq c y^{-\lambda_0-l}(1-y)^{-\lambda_1-l},\ (x,y) \in [0,1] \times (0,1),$$

with a positive constant $c = c(a,b)$ for all $k, l \in \mathbb{N}_0$ such that $k + l \leq m$.

Then T maps $C_{log}^{m,\lambda_0,\lambda_1}(0,1)$ into itself, and $T : C_{log}^{m,\lambda_0,\lambda_1}(0,1) \to C_{log}^{m,\lambda_0,\lambda_1}(0,1)$ is compact.

The following theorem is a consequence of Lemmas 7.2.1 and 7.2.2.

Theorem 7.2.1. *Assume the conditions of Lemma 7.2.2 and* $\mathcal{N}(I-T)=\{0\}$, *with* T, *defined by the formula* (7.2.2). *Let* $f \in C^{m,\lambda_0,\lambda_1}_{log}(0,1)$, $m \in \mathbb{N}$, $\lambda_0, \lambda_1 < 1$.

Then equation (7.2.1) *has a solution* $u \in C^{m,\lambda_0,\lambda_1}_{log}(0,1)$ *which is unique in* $C[0,1]$ *and*

$$\|u\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)} \le c\,\|f\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)}, \tag{7.2.3}$$

with a constant $c>0$ *which is independent of* f.

The main results of the present section (see Theorem 7.2.2 below) are established under assumptions of Theorem 7.2.1.

7.2.1 Operator form of the method, convergence and error estimate

In the integral equation (7.2.1) we perform the change of variables

$$x=\varphi(t), \quad y=\varphi(s), \quad 0\le t\le 1, \quad 0\le s\le 1,$$

where $\varphi:[0,1]\to[0,1]$ is defined by the formula (7.1.5), with $p_0, p_1 \in \mathbb{N}$.

Equation (7.2.1) takes the form

$$\begin{aligned}
u(\varphi(t)) &= \int_0^1 \left[a(\varphi(t),\varphi(s))\log|\varphi(t)-\varphi(s)| + b(\varphi(t),\varphi(s))\right] u(\varphi(s))\varphi'(s)ds \\
&+ g(\varphi(t)) \\
&= \int_0^1 \big[a(\varphi(t),\varphi(s))\left(\log|\varphi(t)-\varphi(s)| + \log|t-s| - \log|t-s|\right)\varphi'(s) \\
&+ b(\varphi(t),\varphi(s))\varphi'(s)\big]\, u(\varphi(s))ds + g(\varphi(t)), \quad 0\le t\le 1,
\end{aligned}$$

or

$$v(t)=\int_0^1 (A(t,s)\log|t-s| + B(t,s))v(s)ds + g(t), \quad 0\le t\le 1, \tag{7.2.4}$$

where $v(t)=u(\varphi(t))$ is the new function we look for,

$$g(t)=f(\varphi(t)), \quad 0\le t\le 1,$$

$$A(t,s) = a(\varphi(t),\varphi(s))\varphi'(s), \quad (t,s) \in [0,1] \times [0,1], \tag{7.2.5}$$

$$B(t,s) = [a(\varphi(t),\varphi(s)) \log \Phi(t,s) + b(\varphi(t),\varphi(s))]\varphi'(s), \quad (t,s) \in [0,1] \times [0,1], \tag{7.2.6}$$

and

$$\Phi(t,s) = \left\{ \begin{array}{ccc} \dfrac{\varphi(t)-\varphi(s)}{t-s} & \text{for} & t \neq s \\ & & \\ \varphi'(s) & \text{for} & t = s \end{array} \right\}, \quad (t,s) \in [0,1] \times [0,1],$$

is the same function, which was defined in subsection 7.1.1. Its behaviour in the vicinity of $(0,0)$ and $(1,1)$ was also studied there.

Lemma 7.2.3. *Let a and b satisfy the conditions of Lemma 7.2.2. Then for $j = 0,\dots,m,\ 0 \le t \le 1,\ 0 < s < 1$, it holds*

$$\left|\partial_s^j a\left(\varphi\left(t\right),\varphi\left(s\right)\right)\right| + \left|\partial_s^j b\left(\varphi\left(t\right),\varphi\left(s\right)\right)\right| \le c s^{-p_0\lambda_0 - j}\left(1-s\right)^{-p_1\lambda_1 - j}. \tag{7.2.7}$$

The proof of (7.2.7) is based on the formula of Faà di Bruno, see (2.1.2).

The derivatives of the function $\Phi(t,s)$ have singularities at $(0,0)$ and $(1,1)$, the only zeroes of $\Phi(t,s)$ in $[0,1]\times[0,1]$. As shown in Section 7.1.1,

$$\Phi\left(t,s\right) \asymp \left(t+s\right)^{p_0-1}\left(\left(1-t\right)+\left(1-s\right)\right)^{p_1-1}$$
$$\text{as } t,s \to 0 \text{ or as } t,s \to 1,$$

and

$$\left|\partial_s^k \Phi\left(t,s\right)\right| \le c\left(t+s\right)^{p_0-k-1}\left(\left(1-t\right)+\left(1-s\right)\right)^{p_1-k-1},$$
$$0 \le s,t \le 1, \quad k = 1,\dots,m,$$

that together with the formula of Faà di Bruno implies the following result.

Lemma 7.2.4. *For $j = 0,\dots,m,\ 0 \le t \le 1,\ 0 < s < 1$, it holds*

$$\left|\partial_s^j\left(\log\left(\Phi\left(t,s\right)\right)\right)\right| \le c\left(t+s\right)^{-j}\left(\left(1-t\right)+\left(1-s\right)\right)^{-j}. \tag{7.2.8}$$

Next we present some estimates for the functions $A(t,s)$, $B(t,s)$ (see Lemmas 7.2.5 and 7.2.6).

Lemma 7.2.5. *Let a and b satisfy the conditions of Lemma 7.2.1. Then the following holds true: if $p_0, p_1 \ge 1$ satisfy*

$$p_0 > 1/\left(1-\lambda_0\right), \quad p_1 > 1/\left(1-\lambda_1\right), \tag{7.2.9}$$

then for $(t,s) \in [0,1]\times(0,1)$ it holds

$$\left|A\left(t,s\right)\right| \le c s^{\delta_0}\left(1-s\right)^{\delta_1}, \quad \left|B\left(t,s\right)\right| \le c s^{\delta_0}\left(1-s\right)^{\delta_1}\left|\log s\left(1-s\right)\right|, \tag{7.2.10}$$

with $\delta_0 := \left(1-\lambda_0\right)p_0 - 1 > 0,\ \delta_1 := \left(1-\lambda_1\right)p_1 - 1 > 0$.

Proof. Following the definition (7.2.5) and approximations (7.2.7) and (7.1.6) we can estimate

$$|A(t,s)| = \left|a(\varphi(t),\varphi(s))\varphi'(s)\right| \leq \left|cs^{-p_0\lambda_0}(1-s)^{-p_1\lambda_1}s^{p_0-1}(1-s)^{p_1-1}\right|$$

$$= \left|cs^{p_0-p_0\lambda_0-1}(1-s)^{p_1-p_1\lambda_1-1}\right| = cs^{\delta_0}(1-s)^{\delta_1}.$$

The second estimation can be achieved in a similar way using definition (7.2.6) and estimates (7.2.7) and (7.1.6). □

Lemma 7.2.6 (see [85]). *Let the conditions of Lemma 7.2.2 be fulfilled. If* $p_0, p_1 \geq 1$ *satisfy*

$$p_0 > m/(1-\lambda_0), \quad p_1 > m/(1-\lambda_1), \tag{7.2.11}$$

then for $(t,s) \in [0,1]\times(0,1)$ *it holds*

$$|A(t,s)| \leq cs^{m-1+\delta_0}(1-s)^{m-1+\delta_1},$$

$$|B(t,s)| \leq cs^{m-1+\delta_0}(1-s)^{m-1+\delta_1}|\log s(1-s)|,$$

with $\delta_0 := (1-\lambda_0)p_0 - m > 0, \delta_1 := (1-\lambda_1)p_1 - m > 0.$

About the boundary behaviour of $v(t) = u(\varphi(t))$ see [86]: for $u \in C^{m,\lambda_0,\lambda_1}_{log}(0,1)$, $j = 1,\dots,m,\ 0 < t < 1$, it holds

$$\left|v^{(j)}(t)\right| \leq c\,\|u\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)} \left\{\begin{array}{ccc} t^{p_0-j} & \text{for} & \lambda_0 < 0 \\ t^{(1-\lambda_0)p_0-j}\,|\log t| & \text{for} & 0 \leq \lambda_0 < 1 \end{array}\right\} \times$$

$$\times \left\{\begin{array}{ccc} (1-t)^{p_1-j} & \text{for} & \lambda_1 < 0 \\ (1-t)^{(1-\lambda_1)p_1-j}\,|\log(1-t)| & \text{for} & 0 \leq \lambda_1 < 1 \end{array}\right\}. \tag{7.2.12}$$

We see, that under conditions (7.2.11) on p_0 and p_1, it holds

$$v^{(j)}(0) = v^{(j)}(1) = 0, \quad j = 1,\dots,m, \tag{7.2.13}$$

if $0 \leq \lambda_0 < 1,\ 0 \leq \lambda_1 < 1$. For $\lambda_0 < 0, \lambda_1 < 0$ the conditions (7.2.13) hold if $p_0,\ p_1 > m$.

Moreover, we have

$$|\partial_s^m[A(t,s)\,v(s)]| \leq cs^{-1+\delta_0}(1-s)^{-1+\delta_1}\,\|u\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)}, \tag{7.2.14}$$

$$|\partial_s^m[B(t,s)\,v(s)]| \leq cs^{-1+\delta_0}(1-s)^{-1+\delta_1}\,|\log s(1-s)|\,\|u\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)}. \tag{7.2.15}$$

Following (7.2.10) we can define $A(t,s) = 0$ and $B(t,s) = 0$ for $s = 0$ and $s = 1$. Moreover, we extend $A(t,s)$ and $B(t,s)$ with respect to s outside $(0,1)$ by the zero value. The corresponding extensions of A and B will again be denoted by A and B. Thus, under conditions (7.2.9), we obtain that

$$A, B \in C([0,1] \times [-\delta, 1+\delta]) \quad \text{for any} \quad \delta \geq 0.$$

We determine the approximate solution v_h of equation (7.2.4) by solving the following problem

$$\begin{gathered} v_h(t) = \int_0^1 \log|t-s|\, P_{h,m}(A(t,s)\, v_h(s))\, ds \\ + \int_0^1 P_{h,m}(B(t,s)\, v_h(s))\, ds + g(t), \quad 0 \leq t \leq 1, \end{gathered} \tag{7.2.16}$$

with $P_{h,m}$, defined by the formula (7.1.40). In (7.2.16) $P_{h,m}$ is applied to the products $A(t,s)v_h(s)$ and $B(t,s)v_h(s)$ as functions of s, treating t as a parameter. This is the operator form of a product interpolation method corresponding to the picewise polynomial "central part" interpolation on the uniform grid $\{ih : i = 0, \ldots, n\}$.

Below we will use the following notations for the integral operators of equations (7.2.4) and (7.2.16):

$$(\mathcal{T}v)(t) = \int_0^1 [A(t,s)\log|t-s| + B(t,s)]\, v(s)\, ds, \quad 0 \leq t \leq 1, \tag{7.2.17}$$

$$(\mathcal{T}_h v)(t) = \int_0^1 [\log|t-s| P_{h,m}(A(t,s)v(s)) + P_{h,m}(B(t,s)v(s))] ds, \quad 0 \leq t \leq 1. \tag{7.2.18}$$

Theorem 7.2.2.

(i) Let $\lambda_0,\ \lambda_1 \in \mathbb{R},\ \lambda_0 < 1,\ \lambda_1 < 1$. *Let* $f \in C[0,1]$, $a,\ b \in C([0,1] \times (0,1))$ *and satisfy*

$$|a(x,y)| + |b(x,y)| \leq c y^{-\lambda_0}(1-y)^{-\lambda_1}, \quad (x,y) \in [0,1] \times (0,1)$$

where $c = c(a,b)$ *is a positive constant. Assume that* $\mathcal{N}(I - T) = \{0\}$, *with* T, *defined by* (7.2.2). *Finally, let* φ *be defined by the formula* (7.1.5) *with parameters* $p_0, p_1 \in \mathbb{N}$ *such that*

$$p_0 > 1/(1-\lambda_0) \ \text{ and } \ p_1 > 1/(1-\lambda_1).$$

Then for sufficiently large $n = \frac{1}{h}$, *say* $n \geq n_0$, *equation* (7.2.16) *has a unique solution* $v_h \in C[0,1]$ *and*

$$\|v - v_h\|_\infty := \max_{t\in[0,1]} |v(t) - v_h(t)| \longrightarrow 0, \tag{7.2.19}$$

where $v \in C[0,1]$ *is the unique solution of equation* (7.2.4).

(ii) Let $m \in \mathbb{N}$ *and* $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 < 1, \lambda_1 < 1$. *Assume that* $a, b \in C^m([0,1] \times (0,1))$ *and satisfy*

$$\left|\partial_x^k \partial_y^l a(x,y)\right| + \left|\partial_x^k \partial_y^l b(x,y)\right| \leq c y^{-\lambda_0 - l}(1-y)^{-\lambda_1 - l},\ (x,y) \in [0,1] \times (0,1),$$

with a positive constant $c = c(a,b)$ *for all* $k, l \in \mathbb{N}_0$ *such that* $k + l \leq m$. *Let* $f \in C_{log}^{m,\lambda_0,\lambda_1}(0,1)$ *and* $\mathcal{N}(I - T) = \{0\}$, *with* T, *defined by* (7.2.2). *Finally, let* φ *be defined by the formula* (7.1.5) *with parameters* $p_0, p_1 \in \mathbb{N}$ *such that*

$$p_0 > m/(1-\lambda_0), \quad p_1 > m/(1-\lambda_1). \tag{7.2.20}$$

Then it holds

$$\|v - v_h\|_\infty \leq c h^m \|f\|_{C_{log}^{m,\lambda_0,\lambda_1}(0,1)}, \quad n \geq n_0, \tag{7.2.21}$$

with a constant c *which is independent of* $n = \frac{1}{h}$ *and* f.

Proof.

(i) The proof of the claim about the convergence of the method and the uniqueness of the solution can be built similarly to the proof of Theorem 7.1.2. We shall not repeat it here.

(ii) Let us prove the error estimate (7.2.21) under conditions (7.2.20) on p_0 and p_1.

For the solution u of (7.2.1) we have by the Theorem 7.2.1 that $u \in C_{log}^{m,\lambda_0,\lambda_1}(0,1)$. According to the definitions of the integral operators $\mathcal{T}$ and $\mathcal{T}_h$ (see (7.2.17) and (7.2.18)) we have

$$(\mathcal{T}v)(t) - (\mathcal{T}_h v)(t) = \int_0^1 [A(t,s)\log|t-s| + B(t,s)]v(s)ds$$

$$-\int_0^1 [\log|t-s| P_{h,m}(A(t,s)v(s)) + P_{h,m}(B(t,s)v(s))]ds$$

$$= \int_0^1 \log|t-s|(I-P_{h,m})(A(t,s)v(s))ds$$

$$+ \int_0^1 (I-P_{h,m})(B(t,s)v(s)ds, \quad 0 \le t \le 1.$$

Therefore

$$\begin{aligned} |(\mathcal{T}v)(t) - (\mathcal{T}_h v)(t)| \;&\le\; \left| \int_0^1 \log|t-s|(I-P_{h,m})(A(t,s)v(s))ds \right| \\ &+\; \left| \int_0^1 (I-P_{h,m})(B(t,s)v(s)ds \right|, \quad 0 \le t \le 1. \end{aligned} \tag{7.2.22}$$

Let us estimate the first integral on the r.h.s. of the inequality (7.2.22). Using (3.2.9) with (7.2.13) and (7.2.14) we get

$$\left| \int_0^1 \log|t-s|(I-P_{h,m})(A(t,s)v(s))ds \right|$$

$$\le \int_0^1 |\log|t-s||\,|(I-P_{h,m})(A(t,s)v(s))|\,ds$$

$$\le c_1 h^m \int_0^1 |\log|t-s||\, s^{-1+\delta_0}(1-s)^{-1+\delta_1} ds \|u\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)}$$

$$\le c_2 h^m \|u\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)}, \quad 0 \le t \le 1, \tag{7.2.23}$$

where c_1 and c_2 are some positive constants that are independent of $h = \dfrac{1}{n}$.

To estimate the second integral on the r.h.s of the inequality (7.2.22) we use (3.2.9) with (7.2.13) and (7.2.15) and get

$$\left| \int_0^1 (I-P_{h,m})(B(t,s)v(s)ds \right|$$

$$\le c_3 h^m \int_0^1 s^{-1+\delta_0}(1-s)^{-1+\delta_1} |\log s(1-s)| ds \|u\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)}$$

$$\leq c_4 h^m \|u\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)}, \quad 0 \leq t \leq 1, \tag{7.2.24}$$

with some constants c_3 and c_4 that do not depend on $h = \frac{1}{n}$.

Due to (7.2.22) - (7.2.24) and (7.2.3) we obtain that

$$|(\mathcal{T}v)(t) - (\mathcal{T}_h v)(t)| \leq ch^m \|f\|_{C^{m,\lambda_0,\lambda_1}}, \quad 0 \leq t \leq 1,$$

with a positive constant c that does not depend on $h = \frac{1}{n}$.

Considering that (for more detailed argument we refer to the proof of Theorem 7.1.2)

$$\|v - v_h\|_\infty \leq c \|\mathcal{T}v - \mathcal{T}_h v\|_\infty,$$

we have established (7.2.21). The proof of Theorem 7.2.2 is completed. □

With respect to

$$u_h(x) := v_h(\varphi^{-1}(x))$$

estimate (7.2.21) reads as

$$\max_{0\leq x\leq 1} |u(x) - u_h(x)| = \max_{0\leq t\leq 1} |v(t) - v_h(t)| \leq ch^m \|f\|_{C^{m,\lambda_0,\lambda_1}_{log}(0,1)}, \quad n \geq n_0.$$

7.2.2 Matrix form of the method

Let us derive the matrix form of method (7.2.16). It follows from the definition of the operator $P_{h,m}$ (see (3.2.4)) that

$$\begin{aligned} v_h(t) = &\sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \log|t-s| \sum_{k\in\mathbb{Z}_m} A\left(t,(j+k)h\right) v_h\left((j+k)h\right) L_{k,m}(ns-j)ds \\ &+ \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \sum_{k\in\mathbb{Z}_m} B\left(t,(j+k)h\right) v_h\left((j+k)h\right) L_{k,m}(ns-j)ds \\ &+ g(t), \quad 0 \leq t \leq 1. \end{aligned} \tag{7.2.25}$$

We obtain an algebraic system of linear equations with respect to the grid values $v_h(ih)$, $i = 0, \ldots, n$, by collocating (7.2.25) at the points $t = ih$:

$$v_h(ih) = \sum_{j=0}^{n-1} \sum_{k \in \mathbb{Z}_m} \left\{ A\left(ih, (j+k)h\right) \int_{jh}^{(j+1)h} \log|ih - s| L_{k,m}(ns - j) ds \right.$$
$$\left. + B\left(ih, (j+k)h\right) \int_{jh}^{(j+1)h} L_{k,m}(ns - j) ds \right\} v_h((j+k)h) \tag{7.2.26}$$
$$+ g(ih), \quad i = 0, \ldots, n.$$

Note that

$$A(ih, (j+k)h) = 0 \text{ and } B(ih, (j+k)h) = 0 \text{ for } j+k \leq 0 \text{ and for } j+k \geq n,$$

thus in the r.h.s of the (7.2.26) the values $v_h(lh)$ with $l \leq 0$ and $l \geq n$ actually are not exploited.

With the change of variables $ns - j = \sigma$ we see that

$$\int_{jh}^{(j+1)h} \log|ih - s| L_{k,m}(ns - j) ds$$
$$= h \left[\log h \int_0^1 L_{k,m}(\sigma) d\sigma + \int_0^1 \log|i - j - \sigma| L_{k,m}(\sigma) d\sigma \right]$$

and

$$\int_{jh}^{(j+1)h} L_{k,m}(ns - j) ds = h \int_0^1 L_{k,m}(\sigma) d\sigma, \quad j = 0, \ldots, n-1, \quad k \in \mathbb{Z}_m,$$

so we have to compute integrals

$$\alpha_{i',k} := \log h \int_0^1 L_{k,m}(\sigma) d\sigma + \int_0^1 \log|i' - \sigma| L_{k,m}(\sigma) d\sigma,$$
$$i' = -n+1, \ldots, n, \quad k \in \mathbb{Z}_m,$$

and

$$\beta_k := \int_0^1 L_{k,m}(\sigma) d\sigma, \quad k \in \mathbb{Z}_m.$$

Thus, system (7.2.26) takes the form:

$$v_h(ih) = h\sum_{j=0}^{n-1}\sum_{k\in\mathbb{Z}_m}\{A(ih,(j+k)h)\alpha_{i-j,k}$$

$$+B(ih,(j+k)h)\beta_k\}\, v_h((j+k)h) + g(ih), \quad i = 0,\dots,n, \tag{7.2.27}$$

or, collecting in the r.h.s. of (7.2.27) the coefficients by $v_h((j+k)h)$ with fixed $j+k=l$,

$$v_h(ih) = \sum_{l=1}^{n-1} c_{i,l} v_h(lh) + g(ih), \quad i = 0,\dots,n, \tag{7.2.28}$$

where

$$c_{i,l} = h\left[A(ih,lh)\sum_{\{k\in\mathbb{Z}_m : 0\le l-k\le n-1\}}\alpha_{i-l+k,k} + B(ih,lh)\sum_{k\in\mathbb{Z}_m}\beta_k\right],$$

$$i = 0,\dots,n,\ l = 1,\dots,n-1.$$

We took into account that $A(ih,lh) = 0$ and $B(ih,lh) = 0$ for $l \le 0$ and $l \ge n$.

Having found the solution $\{v_h(ih)\}$ $(i = 0,\dots,n)$ of system (7.2.28), we can use (7.2.25) to find the approximate solution at any point $t \in [0,1]$.

However, we can also find an approximate solution $\tilde{v}_h(t)$ by (6.2.4):

$$\tilde{v}_h(t) = \sum_{k\in\mathbb{Z}_m}\left\{\begin{array}{ccc} v_h(0) & \text{for} & j+k\le 0 \\ v_h((j+k)h) & \text{for} & 1\le j+k\le n-1 \\ v_h(1) & \text{for} & j+k\ge n \end{array}\right\}\cdot L_{k,m}(nt-j),$$

where $0 \le t \le 1$ and $L_{k,m}$ are the Lagrange fundamental polynomials defined by (3.2.3).

Chapter 8

Numerical Examples

In order to test the collocation method and product integration method described in our thesis we solve in this chapter some weakly singular linear integral equations with kernels of type

$$K(x,y) = |x-y|^{-1/2}$$

and

$$K(x,y) = y^{-\lambda_0}(1-y)^{-\lambda_1}\log|x-y|,$$

where $\lambda_0 < 1$, $\lambda_1 < 1$.

8.1 Collocation Method

In this section we present some numerical results obtained by the collocation method based on central part interpolation.

Example 1. We consider integral equation (1.0.1) (equation (5.0.1)) with the kernel $K(x,y) = |x-y|^{-1/2}$:

$$u(x) = \int_0^1 |x-y|^{-1/2} u(y)dy + f(x), \quad 0 \le x \le 1. \tag{8.1.1}$$

We put

$$u(x) = 1 + x^{1/2} + (1-x)^{1/2}$$

to be the solution of (8.1.1), it corresponds to the free term

$$\begin{aligned} f(x) = 1 - \frac{\pi}{2} - 2x^{1/2} - 2(1-x)^{1/2} - x\log\left(1+(1-x)^{1/2}\right) \\ -(1-x)\log\left(1+x^{1/2}\right) + \frac{1}{2}x\log x + \frac{1}{2}(1-x)\log(1-x). \end{aligned} \tag{8.1.2}$$

To solve equation (8.1.1), we perform the change of variables $x = \varphi(t)$, $y = \varphi(s)$, where φ is given by (5.2.10). As a result we get the equation

$$v(t) = \int_0^1 |t-s|^{-1/2}\, \Phi(t,s)^{-\nu} \varphi'(s) v(s) ds + f_\varphi(t), \quad 0 \le t \le 1, \tag{8.1.3}$$

where $f_\varphi(t) = f(\varphi(t))$, $\Phi(t,s)$ is given by (5.2.7) and $v(t) = u(\varphi(t))$ is the function we look for.

For solving equation (8.1.3) by collocation method (6.1.2), we need to assemble the system (6.2.3). For that we have to calculate the coefficients (6.2.2). For those the integrals (6.2.1) are computed by the exact m point Gauss rule. To achieve the expected convergence order of our method, the parameter p in the definition of φ must be $p \ge 2m+1$ (see Theorem 5.2.1).

In Tables 1 - 4 the errors

$$\epsilon_{m,n,p} := \max_{0 \le i \le n} |v(ih) - v_h(ih)| \tag{8.1.4}$$

are presented. Here v is the exact solution of equation (8.1.3) and v_h is the approximate solution to v obtained by method (6.1.2). Moreover, in Tables 1 - 4 the quotients

$$\frac{\epsilon_{m,n/2,p}}{\epsilon_{m,n,p}}$$

for $m = 2, 3, 4, 5$, different n and $p = 2m+1$ are presented. The expected limit value of $\frac{\epsilon_{m,n/2,p}}{\epsilon_{m,n,p}}$ is 2^m.

Table 1: $m = 2, p = 5$

n	$\epsilon_{2,n,5}$	$\frac{\epsilon_{2,n/2,5}}{\epsilon_{2,n,5}}$
4	4.25E-02	
8	2.05E-02	2.07
16	8.02E-03	2.55
32	2.63E-03	3.05
64	7.58E-04	3.47
128	2.03E-04	3.73
256	5.29E-05	3.84
512	1.36E-05	3.89
1024	3.46E-06	3.93

Table 2: $m = 3, p = 7$

n	$\epsilon_{3,n,7}$	$\frac{\epsilon_{3,n/2,7}}{\epsilon_{3,n,7}}$
4	2.06E-01	
8	2.95E-02	6.98
16	3.95E-03	7.48
32	5.47E-04	7.23
64	7.16E-05	7.63
128	9.17E-06	7.81
256	1.17E-06	7.86
512	1.48E-07	7.89
1024	9.38E-08	1.57

Table 3: $m = 4, p = 9$

n	$\epsilon_{4,n,9}$	$\frac{\epsilon_{4,n/2,9}}{\epsilon_{4,n,9}}$
4	1.94E-02	
8	1.01E-02	2.07
16	1.31E-03	7.74
32	1.09E-04	12.01
64	7.62E-06	14.27
128	5.07E-07	15.04
256	3.27E-07	1.55
512	3.55E-07	0.92

Table 4: $m = 5, p = 11$

n	$\epsilon_{5,n,11}$	$\frac{\epsilon_{5,n/2,11}}{\epsilon_{5,n,11}}$
4	8.33E-02	
8	1.99E-02	10.4
16	8.38E-04	23.7
32	2.89E-05	28.9
64	9.31E-07	31.1
128	3.69E-07	2.52
256	3.60E-07	1.02

Example 2. We consider a problem of the form

$$u(x) = \int_0^1 y^{-\lambda_0}(1-y)^{-\lambda_1} \log|x-y|dy + f(x), \quad 0 \le x \le 1. \tag{8.1.5}$$

This is an equation of the form (7.2.1), where

$$b(x,y) \equiv 0, \quad a(x,y) = y^{-\lambda_0}(1-y)^{-\lambda_1}, \quad \lambda_0 < 1, \quad \lambda_1 < 1.$$

We set

$$u(x) = x^{\lambda_0}(1-x)^{\lambda_1}, \quad \lambda_0 < 1, \quad \lambda_1 < 1$$

to be the exact solution of (8.1.5), it corresponds to the free term

$$f(x) = x^{\lambda_0}(1-x)^{\lambda_1} - x\log x - (1-x)\log(1-x) + 1, \quad 0 < x < 1. \tag{8.1.6}$$

After the change of variables $x = \varphi(t)$, $y = \varphi(s)$ with φ given by (7.1.5), the equation takes the form (7.2.4). In definition (7.1.5) we use different values of smoothing parameters p_0 and p_1, with $\max\{p_0, p_1\} = p \in \mathbb{N}$. And again, to assemble the system (6.2.3) we had to calculate the coefficients (6.2.2). For those the integrals (6.2.1) were computed by Romberg's method, the basic idea behind this method is adaptive Romberg extrapolation combined with cautious error estimation. First such combination was introduced by de Boor in the program CADRE (see [20]).

In Tables 5 - 8 the errors (8.1.4) and the quotients $\frac{\epsilon_{m,n/2,p}}{\epsilon_{m,n,p}}$ for $m = 2, 3$, different n, λ_0, λ_1 and p for equation (8.1.5) are presented. The expected limit value of $\frac{\epsilon_{m,n/2,p}}{\epsilon_{m,n,p}}$ is 2^m.

Table 5: $m = 2, p = 5,$
$\lambda_0 = \lambda_1 = 0.5$

n	$\epsilon_{2,n,5}$	$\frac{\epsilon_{2,n,5}}{\epsilon_{2,2n,5}}$
4	4.52E-02	
8	1.30E-02	3.47
16	3.33E-03	3.91
32	8.26E-04	4.03
64	2.05E-04	4.03
128	5.10E-05	4.02
256	1.27E-05	4.01
512	3.17E-06	4.01
1024	7.93E-07	4.01

Table 6: $m = 3, p = 7,$
$\lambda_0 = \lambda_1 = 0.5$

n	$\epsilon_{3,n,7}$	$\frac{\epsilon_{3,n,7}}{\epsilon_{3,2n,7}}$
4	6.25E-02	
8	1.00E-02	6.24
16	1.42E-03	7.03
32	1.75E-04	8.14
64	2.14E-05	8.15
128	2.66E-06	8.06
256	3.30E-07	8.05
512	4.11E-08	8.04
1024	5.16E-09	7.96

Table 7: $m = 2, p = 11,$
$\lambda_0 = \lambda_1 = 0.25$

n	$\epsilon_{2,n,11}$	$\frac{\epsilon_{2,n,11}}{\epsilon_{2,2n,11}}$
4	1.09E-01	
8	2.29E-02	4.72
16	6.50E-03	3.54
32	1.64E-03	3.96
64	4.09E-04	4.01
128	1.02E-04	4.00
256	2.55E-05	4.00
512	6.38E-06	4.00
1024	1.59E-06	4.00

Table 8: $m = 2, p = 3,$
$\lambda_0 = \lambda_1 = 0.75$

n	$\epsilon_{2,n,3}$	$\frac{\epsilon_{2,n,3}}{\epsilon_{2,2n,3}}$
4	2.93E-02	
8	7.91E-03	3.71
16	1.97E-03	4.02
32	4.85E-04	4.05
64	1.20E-04	4.04
128	2.99E-05	4.02
256	7.46E-06	4.01
512	1.86E-06	4.00
1024	4.65E-07	4.00

8.2 Product Integration Method

As in previous section we solve the equations with singularities of algebraic and logarithmic type, but now by product integration method.

Example 3. To test the algorithm (7.1.56) (the matrix form of product integration method (7.1.28)) we again consider equation (8.1.1). For solving the corresponding system of equations we need to calculate coefficients (7.1.57) and for these the integrals (7.1.58) and (7.1.59) are computed by the exact m point Gauss rule.

In tables 9 - 12 the errors (8.1.4) and the quotients $\frac{\epsilon_{m,n/2,p}}{\epsilon_{m,n,p}}$ for $m = 2, 3, 4, 5$, different n and $p = 2m + 1$ for the equation (8.1.1) are presented. The expected limit value of $\frac{\epsilon_{m,n/2,p}}{\epsilon_{m,n,p}}$ is 2^m.

Table 9: $m = 2, p = 5$

n	$\epsilon_{2,n,5}$	$\frac{\epsilon_{2,n/2,5}}{\epsilon_{2,n,5}}$
4	1.07E-01	
8	3.23E-01	3.31
16	1.15E-01	2.81
32	3.66E-02	3.14
64	1.03E-02	3.54
128	2.74E-03	3.77
256	7.07E-04	3.87
512	1.80E-04	3.92
1024	4.57E-05	3.94

Table 10: $m = 3, p = 7$

n	$\epsilon_{3,n,7}$	$\frac{\epsilon_{3,n/2,7}}{\epsilon_{3,n,7}}$
4	2.94E-00	
8	3.41E-01	8.65
16	5.10E-02	6.68
32	7.34E-03	6.95
64	9.53E-04	7.70
128	1.22E-04	7.82
256	1.55E-05	7.86
512	1.97E-06	7.88
1024	2.83E-07	6.96

Table 11: $m = 4, p = 9$

n	$\epsilon_{4,n,9}$	$\frac{\epsilon_{4,n/2,9}}{\epsilon_{4,n,9}}$
4	1.00E-00	
8	2.49E-01	4.02
16	3.02E-02	8.28
32	2.57E-03	11.74
64	1.79E-04	14.34
128	1.19E-05	15.06
256	7.71E-07	15.42
512	3.31E-07	2.33

Table 12: $m = 5, p = 11$

n	$\epsilon_{5,n,11}$	$\frac{\epsilon_{5,n/2,11}}{\epsilon_{5,n,11}}$
4	2.33E-00	
8	8.91E-01	8.65
16	1.99E-02	44.8
32	7.25E-04	27.4
64	2.19E-05	33.0
128	7.06E-07	31.1
256	3.71E-07	1.90

Example 4. For testing the algorithm (7.2.28) (the matrix form of product integration method (7.2.16)) we again consider equation (8.1.5). We compose system (7.2.28) for $m = 2, 3, 4, 5$, different values of n and λ_0, λ_1, p_0, p_1 with

$$p_0 > m/(1 - \lambda_0), \quad p_1 > m/(1 - \lambda_1).$$

In Tables 13 - 20 the errors (8.1.4) and the quotients $\frac{\epsilon_{m,n/2,p}}{\epsilon_{m,n,p}}$ with $p = \max\{p_0, p_1\}$ are presented. The expected limit value of $\frac{\epsilon_{m,n/2,p}}{\epsilon_{m,n,p}}$ is 2^m.

Table 13: $m = 2, p = 9$, $\lambda_0 = \lambda_1 = 0.75$

n	$\epsilon_{2,n,9}$	$\frac{\epsilon_{2,n,9}}{\epsilon_{2,2n,9}}$
4	3.26E-02	
8	1.84E-02	1.77
16	5.77E-03	3.19
32	1.48E-03	3.89
64	3.69E-04	4.01
128	9.19E-05	4.02
256	2.29E-05	4.01
512	5.71E-06	4.01
1024	1.42E-06	4.02

Table 14: $m = 3, p = 12$, $\lambda_0 = \lambda_1 = 0.75$

n	$\epsilon_{3,n,12}$	$\frac{\epsilon_{3,n,12}}{\epsilon_{3,2n,12}}$
4	1.71E-01	
8	3.11E-02	5.47
16	4.39E-03	7.08
32	5.42E-04	8.10
64	6.72E-05	8.07
128	8.37E-06	8.03
256	1.04E-06	8.05
512	1.30E-07	8
1024	1.62E-08	8.02

Table 15: $m = 4, p = 17$, $\lambda_0 = \lambda_1 = 0.75$

n	$\epsilon_{4,n,9}$	$\frac{\epsilon_{4,n,17}}{\epsilon_{4,2n,17}}$
4	0.14	
8	1.42E-02	9.86
16	2.55E-03	5.92
32	2.13E-04	11.51
64	1.43E-05	14.76
128	9.06E-07	15.78
256	5.66E-08	16.01
512	3.53E-09	16.03
1024	2.18E-10	16.19

Table 16: $m = 5, p = 21$, $\lambda_0 = \lambda_1 = 0.75$

n	$\epsilon_{5,n,21}$	$\frac{\epsilon_{5,n,21}}{\epsilon_{5,2n,21}}$
4	0.20	
8	2.39E-02	8.37
16	3.46E-03	6.91
32	1.27E-04	27.24
64	4.53E-06	28.04
128	1.43E-07	31.68
256	4.46E-09	32.06
512	1.43E-10	31.19
1024	6.52E-12	21.93

Table 17: $m = 2, p = 9$,
$\lambda_0 = \lambda_1 = 0.25$

n	$\epsilon_{2,n,9}$	$\frac{\epsilon_{2,n,9}}{\epsilon_{2,2n,9}}$
4	0.10	
8	4.32E-02	2.31
16	1.17E-02	3.69
32	2.99E-03	3.91
64	7.55E-04	3.96
128	1.90E-04	3.97
256	4.77E-05	3.98
512	1.19E-05	4.01
1024	2.99E-06	3.98

Table 18: $m = 3, p = 15$,
$\lambda_0 = \lambda_1 = 0.25$

n	$\epsilon_{3,n,15}$	$\frac{\epsilon_{3,n,15}}{\epsilon_{3,2n,15}}$
4	0.16	
8	3.11E-02	5.14
16	6.01E-03	5.10
32	8.56E-04	7.13
64	1.11E-04	7.71
128	1.40E-05	7.93
256	1.76E-06	7.95
512	2.21E-07	7.96
1024	2.76E-08	8.01

Table 19: $m = 2, p = 21$,
$\lambda_0 = 0.2,\ \lambda_1 = 0.9$

n	$\epsilon_{2,n,21}$	$\frac{\epsilon_{2,n,21}}{\epsilon_{2,2n,21}}$
4	0.24	
8	8.51E-02	5.67
16	2.32E-02	2.82
32	6.18E-03	3.67
64	1.59E-03	3.89
128	4.06E-04	3.92
256	1.02E-04	3.98
512	2.56E-05	3.98
1024	6.40E-06	4

Table 20: $m = 3, p = 31$,
$\lambda_0 = 0.2,\ \lambda_1 = 0.9$

n	$\epsilon_{3,n,31}$	$\frac{\epsilon_{3,n,31}}{\epsilon_{3,2n,31}}$
4	0.35	
8	0.12	4.91
16	2.29E-02	2.92
32	2.92E-03	7.84
64	3.82E-04	7.64
128	4.87E-05	7.84
256	6.17E-06	7.89
512	7.75E-07	7.96
1024	9.71E-08	7.98

The convergence orders in numerical examples are in a quite good accordance with theoretical ones; for great n the convergence order sometimes vanishes. This circumstance is worth the subject of further study.

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Sisukokkuvõte

Keskosa interpolatsioonil põhinevad meetodid nõrgalt singulaarsete integraalvõrrandite lahendamiseks

Käesolevas töös vaatleme lineaarset teist liiki Fredholmi integraalvõrrandit kujul

$$u(x) = \int_0^1 K(x,y)u(y)\,dy + f(x), \quad x \in [0,1], \tag{9.1.1}$$

kus otsitavaks funktsiooniks on u.

Vabaliikme f kohta seame eeldused, mis on täidetud kõigi lõigus $[0,1]$ pidevate ja m korda pidevalt diferentseeruvate funktsioonide korral ning võimaldavad vaadelda ka selliseid funktsioone, mille tuletised mingist järgust alates võivad olla tõkestamata punkti 0 ja/või 1 läheduses.

Võrrandi tuuma $K(x,y)$ kohta eeldame, et K on m korda ($m \geq 0$) pidevalt diferentseeruv hulgal $([0,1] \times [0,1]) \setminus diag$, kus

$$diag = \left\{(x,y) \in \mathbb{R}^2 : x = y\right\}$$

ning et leidub selline reaalarv $\nu \in (-\infty; 1)$, nii et kõigi tingimust $k + l \leq m$ rahuldavate mittenegatiivsete täisarvude k ja l korral kehtib võrratus

$$\left|\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l K(x,y)\right| \leq c \left\{ \begin{array}{ccc} 1, & \text{kui} & \nu + k < 0 \\ 1 + |\log|x-y||\,, & \text{kui} & \nu + k = 0 \\ |x-y|^{-\nu-k}, & \text{kui} & \nu + k > 0 \end{array} \right\}, \tag{9.1.2}$$

kus $c = c(K)$ on mingi positiivne konstant ja $(x,y) \in ([0,1] \times [0,1]) \setminus diag$. Tingimustest (9.1.2) järeldub $k = l = 0$ korral, et kehtib hinnang

$$|K(x,y)| \leq c \left\{ \begin{array}{ccc} 1, & \text{kui} & \nu < 0 \\ 1 + |\log|x-y||\,, & \text{kui} & \nu = 0 \\ |x-y|^{-\nu}, & \text{kui} & \nu > 0 \end{array} \right\}, \quad (x,y) \in ([0,1] \times [0,1]) \setminus diag.$$

Seega, kui $\nu < 0$, siis tuum $K(x,y)$ ise on tõkestatud hulgal $([0,1] \times [0,1]) \setminus diag$, kuid tema tuletised võivad olla tõkestamata, kui $y \to x$. Kui $0 \leq \nu < 1$, siis võib tuum $K(x,y)$ omada astmelist või logaritmilist iseärasust diagonaalil $x = y$. Muu hulgas võib tuum $K(x,y)$ omada kuju

$$K(x,y) = a(x,y)|x-y|^{-\nu} + b(x,y) \quad (0 < \nu < 1) \tag{9.1.3}$$

või

$$K(x,y) = a(x,y)\log|x-y| + b(x,y), \tag{9.1.4}$$

kus $(x, y) \in ([0,1] \times [0,1]) \setminus diag$ ning a ja b on m korda pidevalt diferentseeruvad funktsioonid ruudus $[0,1] \times [0,1]$.

Vaatleme ka juhtu, kus funktsioonid a ja b tuumade (9.1.3) ja (9.1.4) avaldistes on m korda ($m \geq 0$) pidevalt diferentseeruvad hulgal $[0,1] \times (0,1)$ ning nendel võib esineda rajaiseärasusi muutuja y suhtes. Täpsemalt, tuuma (9.1.3) korral eeldame, et $a,\, b \in C^m([0,1] \times (0,1))$ ja kõigi mittenegatiivsete täisarvude k ja l korral, mille puhul $k + l \leq m$,

$$\left|\partial_x^k \partial_y^l a\,(x,y)\right| + \left|\partial_x^k \partial_y^l b\,(x,y)\right| \leq c y^{-\lambda_0 - l}(1-y)^{-\lambda_1 - l}, \quad (x,y) \in [0,1] \times (0,1),$$

kus $c = c(a,b)$ on mingi positiivne konstant ning

$$\lambda_0, \lambda_1 \in \mathbb{R}, \quad \lambda_0 < 1 - \nu, \quad \lambda_1 < 1 - \nu, \quad 0 < \nu < 1.$$

Tuuma (9.1.4) puhul eeldame, et funktsioonid $a,\, b \in C^m([0,1] \times (0,1))$ ja kõigi mittenegatiivsete täisarvude k ja l korral, mille puhul $k + l \leq m$,

$$\left|\partial_x^k \partial_y^l a\,(x,y)\right| + \left|\partial_x^k \partial_y^l b\,(x,y)\right| \leq c y^{-\lambda_0 - l}(1-y)^{-\lambda_1 - l},\ (x,y) \in [0,1] \times (0,1),$$

kus $c = c(a,b)$ on mingi positiivne konstant ning $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 < 1, \lambda_1 < 1$.

Sileda tuumaga integraalvõrrandite korral tagab tuuma K ja vabaliikme f siledus ka lahendi u (kui see leidub) sileduse kinnisel lõigul $[0,1]$. Nõrgalt singulaarse tuumaga integraalvõrrandi lahendi tuletised võivad integreerimislõigu $[0,1]$ otspunktide läheduses olla tõkestamata, mis komplitseerib kiirete lähismeetodite konstrueerimist selliste võrrandite lahendamiseks.

Nõrgalt singulaarse tuumaga integraalvõrrandi (9.1.1) ligikaudsel lahendamisel tükiti polünomiaalse kollokatsioonimeetodiga saab lahendi u iseärast käitumist arvesse võtta, kui kasutada selliseid ebaühtlaseid võrke, kus võrgupunktid asuvad tihedamalt lõigu $[0,1]$ otspunktide ümbruses. Kuid tugevalt ebaühtlaste võrkude kasutamine võib soodustada ümardamisvigade kuhjumist ning praktiliste arvutuste läbiviimisel põhjustada teatavat numbrilist ebastabiilsust, kui võrgupunktide arv on küllalt suur.

Singulaarsustega integraalvõrrandite (9.1.1) numbriliseks lahendamiseks on doktoritöös käsitletud keskosa interpolatsioonil põhinevaid kollokatsiooni- ja korrutise integreerimise meetodeid, mis ei kasuta ebaühtlast võrku. Täpsemalt, me vaatleme reaalarvude hulgal ühtlast võrku $\{jh : j \in \mathbb{Z}\}$, kus $h = \dfrac{1}{n}$, $n \in \mathbb{N}$. Siin $\mathbb{Z}$ on kõigi täisarvude hulk ja $\mathbb{N}$ on positiivsete täisarvude hulk. Olgu $m \geq 2$ fikseeritud. Suvalise funktsiooni $f \in C\,[-\delta, 1+\delta]$ ($\delta > 0$) korral defineerime tükiti polünomiaalse interpolandi $\Pi_{h,m} f \in C\,[0,1]$ järgmiselt. Igal osalõigul

$$[jh, (j+1)\,h]\,, \quad 0 \leq j \leq n-1,$$

defineerime funktsiooni $\Pi_{h,m} f$ teistest osalõikudest sõltumatult ülimalt $(m-1)-$ astme polünoomina $\Pi_{h,m}^{[j]} f \in \mathcal{P}_{m-1}$, mis interpoleerib funktsiooni f võrgupunkti

jh naabruses mõlemal pool asuvates sõlmedes lh:

$$\Pi_{h,m}^{[j]} f(lh) = f(lh), \quad l = j - \frac{m}{2} + 1, \ldots, j + \frac{m}{2}, \quad \text{kui } m \text{ on paarisarv,}$$

$$\Pi_{h,m}^{[j]} f(lh) = f(lh), \quad l = j - \frac{m-1}{2}, \ldots, j + \frac{m-1}{2}, \quad \text{kui } m \text{ on paaritu arv.}$$

Need interpolandid tagavad interpoleerimise lõigu keskosas ning on võimalik näidata, et lõigu keskosas on interpolatsioonivea hinnang ligikaudu 2^m korda täpsem kui kogu lõigul. Lisaks on lõigu keskosas interpolatsiooniprotsess suuruse m kasvades stabiilne (vt Lemmad 3.1.1 ja 3.2.1 peatükis 3). Valem interpolandi arvutamiseks on esitatav järgmiselt:

$$\left(\Pi_{h,m}^{[j]} f\right)(t) = \sum_{k \in \mathbf{Z}_m} f((j+k)h) L_{k,m}(nt - j), \quad j = 0, \ldots, n-1,$$

kus

$$L_{k,m}(t) = \prod_{l \in \mathbb{Z}_m \setminus \{k\}} \frac{t-l}{k-l}, \quad k \in \mathbb{Z}_m,$$

on Lagrange`i fundamentaalpolünoomid ja

$$\mathbb{Z}_m = \left\{ k \in \mathbb{Z} : -\frac{m}{2} < k \leq \frac{m}{2} \right\}.$$

Kui $m \geq 3$, siis $\Pi_{h,m} f$ kasutab funktsiooni f väärtusi väljastpoolt lõiku $[0,1]$. Seega funktsiooni $f \in C[0,1]$ korral omab interpolant $\Pi_{h,m} f$ tähendust pärast funktsiooni f laiendust lõigule $[-\delta, 1+\delta]$, $\delta > 0$, $h < \dfrac{2\delta}{m}$.

Kui funktsioon f on lõigul $[0,1]$ m korda pidevalt diferentseeruv ning on täidetud tingimused $f^{(j)}(0) = f^{(j)}(1) = 0$, $j = 1, \ldots, m$, siis lihtsaim laiendusoperaator

$$E_\delta : C[0,1] \to C[-\delta, 1+\delta], \quad (E_\delta f)(t) = \left\{ \begin{array}{lll} f(0), & \text{kui} & -\delta \leq t \leq 0 \\ f(t), & \text{kui} & 0 \leq t \leq 1 \\ f(1), & \text{kui} & 1 \leq t \leq 1+\delta \end{array} \right\}$$

säilitab funktsiooni f sileduse.

Võrrandi (9.1.1) lahendamiseks defineerime operaatori

$$P_{h,m} := \Pi_{h,m} E_\delta : C[0,1] \to C[0,1]. \tag{9.1.5}$$

Et parendada iseärasustega võrrandi (9.1.1) lahendi käitumist integreerimislõigu otspunktides, teeme kõigepealt võrrandis muutujate vahetuse

$$x = \varphi(t), \quad y = \varphi(s), \quad 0 \leq t \leq 1, \quad 0 \leq s \leq 1, \tag{9.1.6}$$

kus $\varphi : [0,1] \to [0,1]$ on selline sile rangelt kasvav funktsioon, et $\varphi(0) = 0$ ja $\varphi(1) = 1$. Võrrand (9.1.1) teiseneb kujule

$$v(t) = \int_0^1 K_\varphi(t,s) v(s) ds + f_\varphi(t), \quad 0 \leq t \leq 1, \tag{9.1.7}$$

kus

$$f_\varphi(t) := f(\varphi(t)),\ K_\varphi(t,s) := K(\varphi(t),\varphi(s))\varphi'(s);$$

võrrandite (9.1.1) ja (9.1.7) lahendid on omavahel seotud võrdustega

$$v(t) = u(\varphi(t)),\ u(x) = v(\varphi^{-1}(x)),$$

kus φ^{-1} on funktsiooni φ pöördfunktsioon.

Integraalvõrrandi (9.1.7) ligikaudseks lahendamiseks vaatleme kahte meetodit: kollokatsioonimeetodit tuumade (9.1.2) korral ja korrutise integreerimise meetodit tuumade (9.1.3) ning (9.1.4) korral. Kasutades valemiga (9.1.5) defineeritud interpolatsiooniprojektorit $P_{h,m}$, lähendame võrrandit (9.1.7) võrrandiga (sisuliselt $n+1$ - mõõtmelise võrrandisüsteemiga)

$$v_h = P_{h,m}T_\varphi v_h + P_{h,m}f_\varphi, \tag{9.1.8}$$

kus T_φ on integraaloperaator võrrandist (9.1.7):

$$(T_\varphi v)(t) = \int_0^1 K_\varphi(t,s)v(s)ds, \quad 0 \le t \le 1.$$

Võrrandiga (9.1.8) on esitatud keskosa interpolatsioonil baseeruva tükiti polünomiaalse kollokatsioonimeetodi operaatorkuju ühtlasel võrgul. Nimetatud meetodi maatrikskuju on antud valemiga (6.2.3). Doktoritöös on tõestatud selle meetodi koonduvus ja saadud optimaalset järku lähislahendi veahinnang.

Lahendamaks võrrandit (9.1.1) silendaval muutujate vahetusel ja keskosa interpolatsioonil baseeruva korrutise integreerimise meetodiga, vaatleme algebralise ja logaritmilise iseärasusega tuumasid kujul (9.1.3) ja (9.1.4). Nii nagu kollokatsioonimeetodi puhul teeme ka siin lähtevõrrandites kõigepealt muutujate vahetuse (9.1.6). Algebralise iseärasusega tuuma korral teiseneb võrrand kujule

$$v(t) = \int_0^1 \left[\mathcal{A}(t,s)|t-s|^{-\nu} + \mathcal{B}(t,s)\right] v(s)ds + g(t), \quad 0 \le t \le 1, \tag{9.1.9}$$

kus $v(t) = u(\varphi(t))$ on uus otsitav funktsioon,

$$g(t) = f(\varphi(t)) \quad \mathcal{A}(t,s) = a(\varphi(t),\varphi(s))\Phi(t,s)^{-\nu}\varphi'(s),$$

$$\mathcal{B}(t,s) = b(\varphi(t),\varphi(s))\varphi'(s),$$

ja

$$\Phi(t,s) = \left\{ \begin{array}{lll} \dfrac{\varphi(t)-\varphi(s)}{t-s}, & \text{kui} & t \ne s \\ \varphi'(s), & \text{kui} & t = s \end{array} \right\}, \quad 0 \le t,s \le 1. \tag{9.1.10}$$

Logaritmilise iseärasusega tuuma korral teiseneb võrrand kujule

$$v(t) = \int_0^1 (A(t,s)\log|t-s| + B(t,s))v(s)ds + g(t), \quad 0 \le t \le 1, \tag{9.1.11}$$

kus $v(t) = u(\varphi(t))$ on uus otsitav funktsioon,

$$g(t) = f(\varphi(t)), \quad A(t,s) = a(\varphi(t),\varphi(s))\varphi'(s),$$

$$B(t,s) = [a(\varphi(t),\varphi(s))\log\Phi(t,s) + b(\varphi(t),\varphi(s))]\varphi'(s),$$

ja $\Phi(t,s)$ on antud seosega (9.1.10).

Võrrandite (9.1.9) ja (9.1.11) numbriliseks lahendamiseks kasutame valemiga (9.1.5) esitatud interpolatsiooniprojektorit $P_{h,m}$ ja lähendame võrrandeid (9.1.9) ja (9.1.11) vastavalt võrranditega

$$v_h(t) = \int_0^1 |t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v_h(s))ds + \int_0^1 P_{h,m}(\mathcal{B}(t,s)v_h(s))ds + g(t), \tag{9.1.12}$$

ja

$$v_h(t) = \int_0^1 \log|t-s|\, P_{h,m}(A(t,s)\,v_h(s))\,ds + \int_0^1 P_{h,m}(B(t,s)\,v_h(s))\,ds + g(t), \tag{9.1.13}$$

kus $0 \le t \le 1$. Võrrandites (9.1.12) ja (9.1.13) on $P_{h,m}$ rakendatud vastavalt korrutistele $\mathcal{A}(t,s)v_h(s)$, $\mathcal{B}(t,s)v_h(s)$ ja $A(t,s)v_h(s)$, $B(t,s)v_h(s)$ kui argumendi s funktsioonidele, vaadeldes muutujat t kui parameetrit.

Seostega (9.1.12) ja (9.1.13) oleme esitanud keskosa interpoleerimisel baseeruva korrutise integreerimise meetodi operaatorkuju ühtlasel võrgul $\{ih : i = 0,\dots,n\}$. Meetodite maatrikskujud on toodud vastavalt valemitega (7.1.56) ja (7.2.28). Doktoritöös on tõestatud valemitega (9.1.12) ja (9.1.13) esitatud meetodite koondumine ja saadud optimaalset järku hinnangud lähislahendite vea jaoks (vt Theorem 7.1.2 ja Theorem 7.2.2).

Doktoritöö koosneb kaheksast peatükist. Esimeses kahes peatükis antakse ülevaade tööst ja esitatakse rida abitulemusi, mida läheb vaja lähislahendi vea hindamisel.

Kolmandas peatükis tutvustame keskosa interpolatsiooni, millel on väga hea veahinnang. Lisaks on lõigu keskosas interpolatsiooniprotsessil ka head stabiilsuse omadused.

Neljandas peatükis esitleme tuumade kirjeldamiseks vajalikku siledus-singulaarsuse klassi $\mathcal{S}^{m,\nu}$ ning lahendite kirjeldamiseks vajaliku kaaluruumi $C^{m,\nu}(0,1)$.

Viiendas peatükis vaatleme siledate muutujate vahetuse funktsioonide klassi ning uurime taoliste funktsioonide silendavaid omadusi.

Kuues ja seitsmes peatükk on pühendatud võrrandi (9.1.1) numbrilisele lahendamisele. Kuuendas peatükis käsitleme keskosa interpolatsioonil põhinevat kollokatsioonimeetodit ja seitsmendas peatükis korrutise integreerimise meetodit. On tõestatud vaadeldavate meetodite koondumine ning tuletatud koonduvuskiiruse hinnangud.

Kaheksandas peatükis on teoreetilisi tulemusi testitud numbriliste eksperimentide abil. Testülesannete lahendamisel saadud arvulised tulemused kinnitavad doktoritöös saadud teoreetiliste tulemuste kehtivust.

Enamus antud töö põhitulemustest sisalduvad autori kolmes ilmunud teadusartiklis [59, 60, 61]. Neid tulemusi on tutvustatud kaheksal rahvusvahelisel teaduskonverentsil ja vastavates konverentsiteesides. Osa juba avaldatud tulemusi on laiendatud üldisemale juhule ja osa tulemusi on uued.

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Peamine uurimisvaldkond on efektiivsed numbrilised meetodid integraalvõrrandite ligikaudseks lahendamiseks. Tulemused on publitseeritud kolmes teadusartiklis ja esitatud kaheksal teaduskonverentsil:

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"2nd Dolomites Workshop on Constructive Approximation and Applications" Trentos (2009),

"Fourth International Workshop on Analysis and Numerical Approximation of Singular Problems" Chesteris (2011), "The 17th International Conference Mathematical Modelling and Analysis" Tallinnas (2012),

"The 18th International Conference Mathematical Modelling and Analysis" Tartus (2013),

"The 13th International Conference on Integral Methods in Science and Engineering" Karlsruhes (2014).

List of Publications

1. K. Orav-Puurand. A Central Part Interpolation Scheme for Log-Singular Integral Equations. Mathematical Modelling and Analysis, 18(1) (2013), 136-148.

2. K. Orav-Puurand, A. Pedas, G. Vainikko. Nyström type methods for Fredholm integral equations with weak singularities. Journal of Computational and Applied Mathematics, 234(9) (2010), 2848-2858.

3. K. Orav-Puurand, G. Vainikko. Central part interpolation schemes for integral equations. Numerical Functional Analysis and Optimization, 30 (2009), 352-370.

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

1. **Mati Heinloo.** The design of nonhomogeneous spherical vessels, cylindrical tubes and circular discs. Tartu, 1991, 23 p.
2. **Boris Komrakov.** Primitive actions and the Sophus Lie problem. Tartu, 1991, 14 p.
3. **Jaak Heinloo.** Phenomenological (continuum) theory of turbulence. Tartu, 1992, 47 p.
4. **Ants Tauts.** Infinite formulae in intuitionistic logic of higher order. Tartu, 1992, 15 p.
5. **Tarmo Soomere.** Kinetic theory of Rossby waves. Tartu, 1992, 32 p.
6. **Jüri Majak.** Optimization of plastic axisymmetric plates and shells in the case of Von Mises yield condition. Tartu, 1992, 32 p.
7. **Ants Aasma.** Matrix transformations of summability and absolute summability fields of matrix methods. Tartu, 1993, 32 p.
8. **Helle Hein.** Optimization of plastic axisymmetric plates and shells with piece-wise constant thickness. Tartu, 1993, 28 p.
9. **Toomas Kiho.** Study of optimality of iterated Lavrentiev method and its generalizations. Tartu, 1994, 23 p.
10. **Arne Kokk.** Joint spectral theory and extension of non-trivial multiplicative linear functionals. Tartu, 1995, 165 p.
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