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OPTIMIZATION OF PLASTIC SPHERICAL SHELLS

ERNST TUNGEL

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CONTENTS

INTRODUCTION

Analysis and optimization of non-elastic plates and shells has become a problem of practical interest. There are many books devoted to optimal design of elastic and non-elastic structures. Various problems and methods of optimization have been studied by Banichuk (1990), Bendsoe (1995), Cherkaev (2000), Lepik (1982), G.I.N. Rozvany (1976, 1989), J. Lellep (1991) etc. The basic ideas and methods of direct analysis of rigid-plastic structural elements are accommodated in the books by P. Hodge (1963), N. Jones (1989) and others.

Due to the simplicity of their manufacturing the special significance have the designs of piece wise constant thickness. Circular cylindrical shells of piece wise constant thickness have been treated by C. Cinquini and M. Kouam (1983) in the case of a Tresca material. J. Lellep and S. Hannus (1995) considered the plastic tubes with piece wise constant thickness assuming the material obeyed von Mises yield condition. Optimal designs for stepped plastic shallow shells have been established by J. Lellep and H. Hein (1993a,b, 1994) in the cases of piece wise linear approximations of the exact yield surface corresponding to the original Tresca yield condition on the plane of principal stresses. Employing a lower bound method for determination of the load carrying capacity by J. Lellep and E. Tungel (1998a) defined an optimal design for a stepped spherical shell simply supported at the edge.

Optimization of elastic and non-elastic beams, frames, plates and shells has had the attention of many investigators during the last decades. Comprehensive reviews of problems solved can be found in the books and papers by J. Krużelecki and M. Życzkowski (1985), J. Lellep and Ü. Lepik (1984), G. Rozvany (1976), J. Lellep (1991).

Different approaches to optimization of non-elastic structural elements have been developed by Z. Mróz and A. Gawecki (1975), G. Rozvany (1976), M. Save (1972), J. Lellep (1985, 1991). Mroz and Gawecki (1975) obtained a somewhat unexpected result when studying the post-yield behaviour of rigid-plastic circular plates. It appeared that optimized structures of variable thickness could be even less strong than the structures of constant thickness. The optimization techniques which avoid such unfavourable effect were developed later by Lellep (1991) and Lellep and Majak (1995). Axisymmetric plates and shallow spherical shells of minimum weight are studied by D. Lamblin, G. Guerlement, M. Save (1985) and J. Lellep, H. Hein (1993, 1994) assuming that the thickness is piece-wise constant and that the material obeys Tresca yield condition. Deep spherical shells of Tresca material have been studied by J. Lellep and E. Tungel (1998 a,b),(1999),(2000). Straight plate problems are solved by A. Sawczuk and J. Sokol-Supel (1993) for both, Tresca and Mises materials.

Foundations of the theory of limit analysis and solutions of particular problems are presented in monograph books by Erkhov (1978), Hodge (1963), Ilyshin (1963), Johnson and Mellor (1986), Lin T (1968), Sawczuk (1989), Sawczuk and Sokol-Supel (1993), Zyczkowski (1981).

The new trends in the limit analysis in theory of plasticity and in the application of the methods of plasticity in the structural analysis are presented by Chakrabarty (2000), Save, Massonet and Saxce (1997) in the case of quasistatical loadind. Impulsive and dynamic pressure loading of non-elastic beams, plates and shells is the topic of books by Jones (1989) and Stronge and Yu (1993).

The load carrying capacity of plastic spherical shells is studied by Dumesnil and Nevill (1970), Hodge (1963), Mroz and Bing Ye (1963), Palusamy (1971), Palusamy and Luid (1972), Lee and Onat (1968) and others. Palusamy (1971) considered the plastic collapse of a spherical cap under axial loading, whereas Hodge (1963), Lee and Onat (1968) studied the problems of limit analysis of spherical caps subjected to the uniformly distributed loading.

Spherical caps loaded by the rigid central boss were studied by Yeom and Robinson (1996). Mróz and Bing-Ye (1963) considered the case of loading in the form of loads distributed along the edge of a central hole. Popov (1967a) solved the limit analysis of the spherical shell in the case of combined loading. In these studies the shells of constant thickness are considered. It was assumed that the yield condition was presented in the form of two hexagons on the planes of moments and membrane forces, respectively. The same problem was considered in the further works by Popov (1967b, 1969) in different cases of loading and support conditions. Rozenbljum (1960) developed an approximation of the exact yield surface in the space of membrane forces and bending moments. Later the nonlinear approximation was used in the determination of the load carrying capacity of a spherical cap. Sankaranarayanan (1964) introduced a generalized square yield condition for investigation of plastic spherical shells.

Onat and Prager (1954) have derived the parametrical equations of the exact yield surface in the space of generalized stresses. Making use of these equations the authors have determined the load carrying capacity of a spherical cap subjected to uniformly distributed pressure loading.

Hodge and Lakshmikantham (1962) have defined the load carrying capacity of spherical caps with cutouts.

Later Jones and Ich (1972) suggested a new approximation of the yield surface which consists of two diamonds on the planes of bending moments and membrane forces. The generalized diamond yield surface was successfully used for solution of quasistatic and dynamic problems of plastic plates and shells.

Gabbasov (1963, 1966, 1967) studied the limit analysis of spherical caps making use of kinematical approach. This leads to an upper bound of the exact load carrying capacity. In Gabbasov (1968) a lower bound approach was developed assuming that the yield surface could be presented in the form of hexagons on the planes of membrane forces and moments, respectively.

Gabbasov and Fraint (1968) defined the upper bound of the limit load for a spherical shell with the central hole. The internal edge of the shell was clamped whereas the outer edge was assumed to be absolutely free.

Kulikov and Khomyakov (1976) studied the limit analysis of cylindrical and spherical shells subjected to the distributed internal pressure and concentrated loading.

Ü. Lepik (1972, 1973) was a pioneer in the application of the methods of the theory of optimal control in the optimal design of rigid-plastic plates and shells. It appeared that the principle of maximum of Pontryagin (Bryson and Ho (1975), Pontryagin (1969)) presented a useful tool for optimization of structures with bounds imposed on the thickness. Such an approach was applied by Ü. Lepik (1972, 1973), where the Prager's yield condition was used. The latter papers and the one by Ü. Lepik (1972) considered homogeneous structures. However, Ü. Lepik (1973) studied the sandwich type structures. For the design variable the thickness of the structure (or the thickness of the working sheets) is chosen, this quantity is bounded from below and above. Optimal designs for circular plates were obtained by Ü. Lepik (1972, 1973), whereas axisymmetric cylindrical shells were considered by Ü. Lepik (1982).

Ü. Lepik (1978) has studied the beams with additional supports. The problem of optimal location of an additional support is solved in the case of non-elastic beam. The perfomance index and the constraints are given in a quite general form. The aim of the optimization is to reduce the beam's complience.

J. Kirs (1979, 1984) investigated stepped plates, conical, spherical and cylindrical shells subjected to initial impulsive loading. Kirs studied spherical and conical shells made of an ideal rigid-plastic material obeying Tresca yield condition and associated flow law. The shells under consideration are subjected to the initial impulsive loading. For the cost function which is formed as a combination on the structural weight and the initial acceleration optimal stepped designs have been established.

Circular sandwich Tresca plates subjected to concentrated load were studied by J. Lellep (1977). In this note the load carrying capacity is maximized for given weight.

Axisymmetric shells were considered by Ü. Lepik (1975) taking shear forces into account.

J. Lellep (1977) and Ü. Lepik (1978) demonstrated in their papers the application of the optimal control theory to the optimal design of non-linear elastic, viscous and ideal rigid-plastic beams.

Axisymmetric plates and shallow spherical shells with continuously variable thickness made of a von Mises material were considered by J. Lellep and J. Majak (1989, 1995a, b). In the paper by Lellep and Majak (1995) an optimization technique developed earlier for axisymmetric plates and circular cylindrical shells was accommodated to shallow spherical shells subjected to uniform transverse pressure. It was assumed that the material of the shell was in consistence with a non-linear approximation of the exact yield surface and associated gradientality law corresponding to a von Mises material. Minimum weight designs were established under the condition that the maximal deflection of the shell of variable thickness coincided with that corresponding to the reference shell of constant thickness. J. Lellep and H. Hein (1993, 1994) studied shallow spherical shells of piece wise constant thickness in the case of a Tresca material whereas J. Lellep and E. Tungel (1998, 1999) investigated deep spherical caps with stepped thickness. J. Lellep and S. Hannus (1989, 1995) considered stepped cylindrical shells.

The methods of optimization of plastic shells have been reviewed by G. Rozvany (1989), J. Lellep and Ü. Lepik (1984), J. Lellep (1991), J. Kużelecki and M. Życzkowski (1985). Making use of the methods of the theory of optimal control J. Lellep and J. Majak (1995), J. Lellep and H. Hein (1993) studied rigid-plastic shallow spherical shells.

Lellep and Puman (1994, 1999, 2000) studied stepped conical shells loaded via a rigid central boss or subjected to uniformly distributed external pressure loading. Material of shells under consideration is an ideal rigid-plastic material obeying Tresca or von Mises yield condition. The exact yield surface in the space of generalized stresses corresponding to Tresca condition admits proper approximation with squares or diamonds, respectively, on the planes on membrane forces and bending moments (Lellep, Puman (1994)). Minimum weight designs of stepped shells are established under the condition that the limit loads for the stepped shell and the reference shell of constant thickness, respectively, coincide.

The review papers cited above show that relatively less attention has been paid to the optimization of plates and shells material of which obeys von Mises yield condition. Optimal design for shallow spherical shells of von Mises material have been established by J. Lellep and J. Majak (1995). Circular cylindrical shells of piece-wise constant thickness were studied by J. Lellep and S. Hannus (1995).

In the present work optimization procedures will be developed for plastic spherical shells of piece-wise constant thickness.

The stepped shells clamped or simply supported at the edge and pierced with a central hole are considered. The exact solutions are established for simply supported shells under the assumption that the material of the shells obeys the generalized square yield condition and the associated flow law.

Numerical results are obtained for clamped shells made of a von Mises material. A non-linear approximation of the exact yield surface is used.

CHAPTER 1

FORMULATION OF THE PROBLEM AND GOVERNING EQUATIONS

1.1 PROBLEM FORMULATION

1.1.1 Deep spherical cap

When prescribing the problem to be solved there after we distinguish the cases of a full shell and a shell pierced with a central hole, respectively. The case of a full shell will be studied in the second chapter assuming that the material of the shell obeys the Tresca's yield condition and the associated flow law.

Let us consider a full spherical cap of radius A subjected to the uniformly distributed external pressure of intensity P (Fig. 1). The external edge of the shell is simply supported at $\varphi = \beta$.

The thickness of the shell is assumed to be piece-wise constant, e.g.

$$h = \begin{cases} h_0, & \varphi \in (0, \alpha_1), \\ h_1, & \varphi \in (\alpha_1, \alpha_2), \\ \dots\dots\dots\dots\dots\dots, & \\ h_n, & \varphi \in (\alpha_n, \beta) \end{cases} \tag{1.1}$$

where $h_0, ..., h_n$ and $\alpha_1, ..., \alpha_n$ are treated as previously unknown constant parameters. However, β and n are considered to be given constants.

Weight of the cap may be evaluated by the material volume as

$$V = \sum_{j=0}^{n} (\cos a_j - \cos \alpha_{j+1})(3A^2 h_j + \frac{1}{4h_j^3}). \tag{1.2}$$

Here $V = 3M/2\pi \varrho$ and M is the mass of the shell and ϱ - material density.

We are looking for the design of the cap for which
(i) material volume attains the minimum value for given load carrying capacity,
(ii) load carrying capacity attains the maximum value for fixed weight of the shell.
In the second chapter the main attention will be paid to the problem (ii).

1.1.2 Spherical shell pierced with a hole

In the present work the shells pierced with a central hole will be studied as well. It is assumed that a spherical shell of radius A is subjected to the uniform external

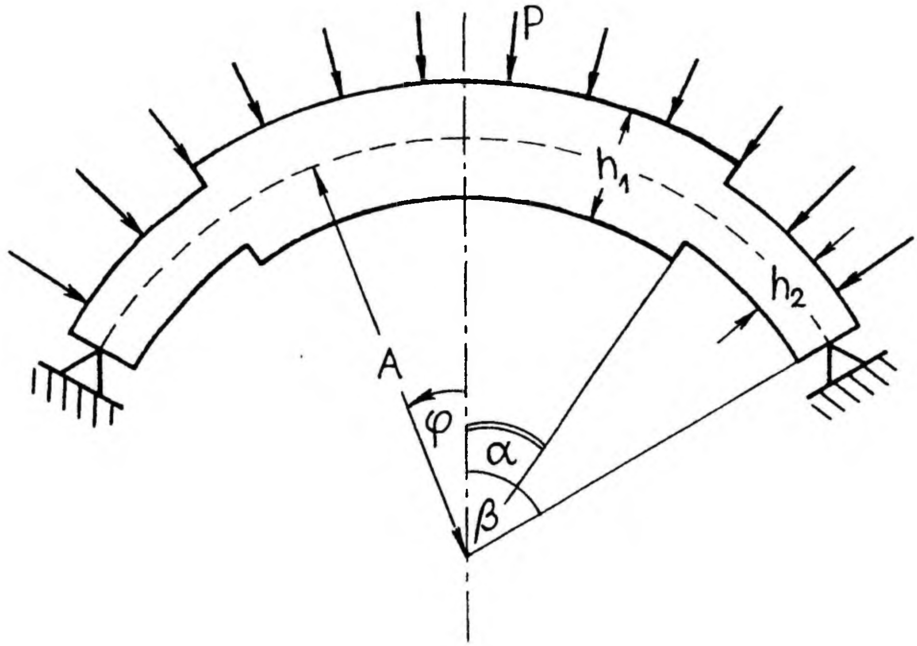

Figure 1.1: Spherical shell of piece wise constant thickness.

pressure of intensity P (Fig. 1). The external edge of the shell (the circle at $\varphi = \beta$) is clamped or simply supported and the inner edge (at $\varphi = \alpha$) is absolutely free. Here the angles α and β are considered as given angles.

In order to get maximal simplicity in the statement of the problem in the case of material obeying Tresca yield condition we are looking for the design of the shell confining our attention to the case of the stepped shell with one step in the thickness. Thus thickness of the shell is

$$h = \begin{cases} h_0, & \varphi \in (\alpha, \alpha_1) \\ h_1, & \varphi \in (\alpha_1, \beta) \end{cases}$$

where h_0, h_1, α_1 are to be considered as unknown constant parameters. These parameters have to be determined so that the load carrying capacity P of the cap attains the maximal value over the set of shells of the same weight (or mass, or material volume).

The volume of the material can be easily defined when considering the spherical bodies with radii $A + h/2$ and $A - h/2$, respectively. Therefore, the weight of the shell can be described by

$$\frac{V'}{2\pi} = (\cos\alpha - \cos\alpha_1)\left(3A^2 h_0 + \frac{h_0^3}{4}\right) + (\cos\alpha_1 - \cos\beta)\left(3A^2 h_1 + \frac{h_1^3}{4}\right).$$

The optimization problem consists in the minimization of the cost function

$$J_0 = -P$$

so that there are satisfied the governing equations of plastic spherical shells and the relation, where V' is considered as a given constant.

However, in the case of the material obeying von Mises yield condition we consider the shell with n different thickness, as in the previous section. The case of the Tresca material will be studied in the Chapter III whereas the case of von Mises material will be investigated in the last chapter.

1.2 GOVERNING EQUATIONS

The set of governing equations consists of the equilibrum equations, geometrical relations and the assotsiated flow law. The equilibrium equations for spherical shells subjected to axisymmetric loading can be presented as

$$
\begin{aligned}
&(N_\varphi \sin\varphi)' - N_\Theta \cos\varphi = S\sin\varphi, \\
&(N_\varphi + N_\Phi + PA)\sin\varphi = -(S\sin\varphi)', \\
&(M_\varphi \sin\varphi)' - M_\Theta \cos\varphi = -AS\sin\varphi,
\end{aligned}
\tag{1.3}
$$

where N_φ, N_Φ stand for membrane forces and M_φ, M_Φ are the principal moments. Here S is the shear force. When deriving (4) it is assumed that the geometry changes of the structure can be neglected, thus the strain components ε_φ, ε_Φ, K_φ, K_Φ and the displacements U, W are small in comparison with unity.

For small strains and displacements the strain rates (geometrical relations) can be presented as

$$
\begin{aligned}
&\dot\varepsilon_\varphi = \frac{1}{A}(\dot U' - \dot W), \qquad \dot\varepsilon_\Phi = \frac{1}{A}(\dot U \cot\varphi - \dot W), \\
&\dot K_\varphi = -\frac{1}{A^2}(\dot U + \dot W')', \qquad \dot K_\Phi = -\frac{1}{A^2}\cot\varphi(\dot U + \dot W').
\end{aligned}
\tag{1.4}
$$

In (1.3), (1.4) and henceforth primes denote differentiation with respect to the current angle φ whereas dots correspond to the derivatives with respect to time or time like parameter. Note that in the limit analysis of plastic shells the role of time can be fulfilled by the loading parameter P.

According to the associated flow law the vector with coordinates (1.4) is to be directed along the external normal to the yield surface at the present point. Since various approximations to exact yield surfaces corresponding to original Tresca or Mises yield conditions will be used the associated flow law will be stated separately in each particular case.

In order to introduce non-dimensional variables let us consider a reference shell of constant thickness h_*. The reference shell has the same middle surface as the shell under consideration. Let the yield moment and yield force for the reference shell be $M_* = \sigma_0 h_*^2/4$ and $N_* = \sigma_0 h_*$, respectively. Here σ_0 stands for the yield stress of the material.

For the sake of convenience let us introduce following non-dimensional quantities

$$
\begin{aligned}
&n_{1,2} = \frac{N_{\varphi,\Theta}}{N_*}, \quad m_{1,2} = \frac{M_{\varphi,\Theta}}{M_*}, \quad s = \frac{S}{N_*}, \quad w = \frac{W}{A}, \quad u = \frac{U}{A}, \\
&\gamma_0 = \frac{h_0}{h_*}, \quad \gamma_1 = \frac{h_1}{h_*}, \quad k = \frac{h_*}{4A}, \quad p = \frac{PA}{N_*}.
\end{aligned}
\tag{1.5}
$$

Making use of (1.5) one can present the equilibrium equations (1.3) as

$$
\begin{aligned}
&(n_1 \sin\varphi)' - n_2 \cos\varphi = s \sin\varphi, \\
&(n_1 + n_2 + p) \sin\varphi = -(s \sin\varphi)', \\
&k\left[(m_1 \sin\varphi)' - m_2 \cos\varphi\right] = s \sin\varphi.
\end{aligned}
\tag{1.6}
$$

The strain rates (1.4) take the form

$$
\begin{aligned}
&\dot{\varepsilon}_\varphi = \dot{u}' - \dot{w}, \qquad \dot{\varepsilon}_\Phi = \dot{u}\cot\varphi - \dot{w}), \\
&\dot{k}_\varphi = -k(\dot{u} + \dot{w}')', \qquad \dot{k}_\Phi = -k\cot\varphi(\dot{u} + \dot{w}').
\end{aligned}
\tag{1.7}
$$

Here the following notation is used:

$$
\dot{k}_\varphi = \frac{h_*}{4A}\dot{K}_\varphi, \quad \dot{k}_\Phi = \frac{h_*}{4A}\dot{K}_\Phi. \tag{1.8}
$$

The boundary conditions for the considered case of geometry of the shell are following

$$
m_1(\alpha) = n_1(\alpha) = s(\alpha) = 0, \quad m_1(\beta) = -\gamma_1^2. \tag{1.9}
$$

1.3 APPROXIMATIONS OF YIELD SURFACES

1.3.1 Approximations of the yield surface corresponding to Tresca condition

It is assumed that the material of shells to be considered is a rigid-plastic material which obeys the Tresca's or von Mises yield condition and associated flow (gradientality) law. The shells of a Tresca material will be studied in Chapters 2, 3 and shells of von Mises material will be treated in Chapter 4.

I is well know that the Tresca's yield condition in its original form is presented as a hexagon on the plane of principal stresses (Fig. 1.4). The yield surface in the space of generalized stresses (membrane forces and moments) can be derived by the use of the method of E. Onat and W. Prager (1954). E. Onat and W. Prager employed theusual assumptions of the theory of thin shells and derived parametrical equations of the yield surface in the space of membrane forces and moments assuming the material obeys Tresca's yield condition. However, the result appeared to be complicated for the practical use. The authors themselves,

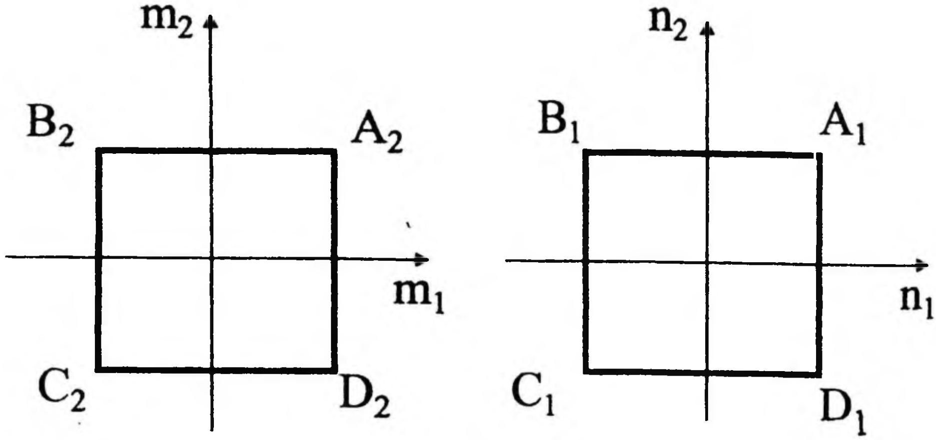

Figure 1.2: Generalized square yield condition

aslo other researchers tried to replace the exact yield surface with a simpler one so that the load carrying capacities obtained on the basis of an approximate yield surface compare favourably with exact ones.

Various aspects of the problems of derivation and the use of different yield surfaces are discussed in the books by P. Hodge (1963), A. Ilyshin (1963), J. Chakrabarty (2000), M. Zyczkowski (1981), N. Jones (1989), M. Save, C. Massonnet, G. Saxce (1997), A. Sawczuk (1989) and others.

P. G. Hodge (1963) suggested so-called two-moment limited interaction yield surface which might be presented in the form of hexagons on the planes of moments and membrane forces, respectively. Later R. Sankaranarayanan (1964), N. Jones and N. T. Ich (1972) suggested further simplifications of the yield surface for rotationally symmetric shells.

In the present study two moment limited interaction yield condition (Fig. 1.4) and the generalized square yield condition (Fig. 1.2) will be used. It is assumed that the vectors $\dot{\kappa} = (\dot{\kappa}_1, \dot{\kappa}_2)$ and $\dot{\varepsilon} = (\dot{\varepsilon}_1, \dot{\varepsilon}_2)$ are normal to the hexagons and squares on planes of bending moments and membrane forces, respectively.

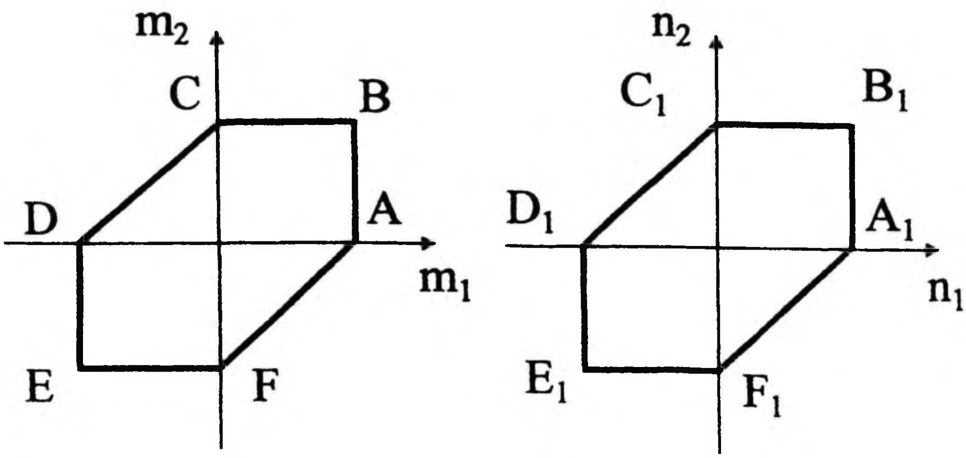

Figure 1.3: Two moment limited interaction yield surface

1.3.2 Approximation of the von Mises yield surface

According to R. Mises plastic yielding starts when $\sigma_i = \sigma_0$ where σ_i stands for the stress intensivity at the current point and σ_0 is the yield stress of the material.

In the case of a plane stress state the von Mises yield condition can be presented as an ellipse on the plane of principal stresses

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \leq \sigma_0^2. \qquad (1.10)$$

A. Ilyshin (1963) derived parametrical equations of the yield surface making use of the concept of thin plates and shells starting from the condition (1.8).

In the theory of thin plates and shells it is more convenient to use the generalized stresses (membrane forces and bending moments). Thus it is desirable to present the yield condition in the space of generalized stresses.

A. Ilyshin succeeded in solving this task. However, due to its complicated structure the exact yield surface is unconvenient for practical calculations.

An approximation to the exact yield surface in the space of membrane forces

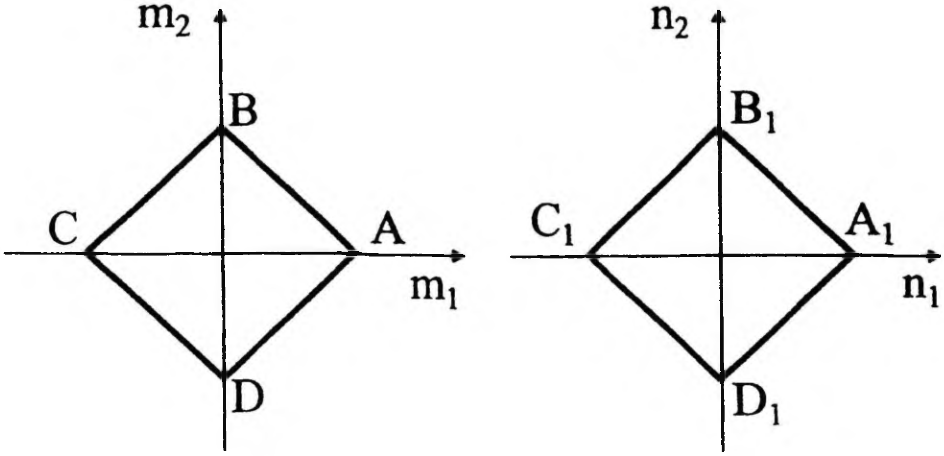

Figure 1.4: Generalized diamond yield condition

and moments was suggested by A. Ilyshin himself. The approximation can be presented as

$$P_n + P_m + \frac{P_{nm}}{\sqrt{3}} = 1, \qquad (1.11)$$

where

$$P_n = n_1^2 - n_1 n_2 + n_2^2,$$
$$P_m = m_1^2 - m_1 m_2 + m_2^2,$$
$$P_{nm} = \frac{1}{2}\Big(2n_1 m_1 + 2n_2 m_2 - n_1 m_2 - n_2 m_1\Big). \qquad (1.12)$$

A relatively simple yield surface

$$P_n + P_m = 1 \qquad (1.13)$$

was suggested by V. I. Rozenbljum (1960). V. Rozenbljum assumed that the stress distribution was linear across the shell thickness and the von Mises yield condition was satisfied in the average across the shell thickness.

It was shown that approximations of the exact yield surface (1.11) and (1.13) lead to the results whose deviations from exact results are of the same order. Since (1.13) is somewhat more simple than (1.11) it is used in the present paper.

The further progres in the simplification of th exact yield surface derived by A. Ilyushin was made by Z. Mroz and X. Bing-Ye (1963) who suggested the surface

$$\sqrt{P_m} + P_n = 1 \tag{1.14}$$

as an approximation to the Ilyshin's surface.

Later G. Ivanov (1967) developed more complicated approximations

$$P_n + \frac{P_m}{2} + \sqrt{\frac{P_m^2}{4} + P_{nm}^2} = 1 \tag{1.15}$$

and

$$P_n + \frac{P_m}{2} - \frac{1/4(P_n P_m - P_{nm}^2)}{P_n + 0.48 P_m} + \sqrt{\frac{P_m^2}{4} + P_{nm}^2} = 1. \tag{1.16}$$

It was shown that (1.15) and (1.16) lead to very good predictions of the limit load for the shell under consideration.

Various approximation were developed by G. Landgraf (1968), M. Robinson (1971), H. M. Haydl and A. N. Sherbourne (1979), which are presented in the book by M. Zyczkowski (1981).

In the present work the approximation (1.13) of the exact yield surface will be used. The approximation of the von Mises yield condition in the dimensionless variables (1.3) can be presented as:

$$m_1^2 - m_1 m_2 + m_2^2 + n_1^2 - n_1 n_2 + n_2^2 = 1. \tag{1.17}$$

The yield surface (1.17) was used in different papers by various authors, for instance, in H. Haydl and A. Sherbourne (1979), J. Lellep and J. Majak (1995), M. Zyczkowski (1981) etc.

CHAPTER 2

OPTIMIZATION OF PLASTIC SPHERICAL SHELLS OF PIECE WISE CONSTANT THICKNESS

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OPTIMIZATION OF PLASTIC SPHERICAL SHELLS OF PIECE WISE CONSTANT THICKNESS

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Abstract. An optimal design procedure is developed for stepped rigid-plastic spherical shells. The shells are subjected to the uniformly distributed external pressure. Material of shells obeys the Tresca yield condition and associated flow law. The problems solved herein consist in the maximization of the load carrying capacity under the condition that the material volume of the shell is fixed and in the weight minimization under given load carrying capacity, respectively.

2.1 INTRODUCTION

The load carrying capacity of plastic spherical shells is studied by Dumesnil and Nevill [1], Hodge [2], Mrozand Bing Ye [4]. Hodge [2] has studied the problems of limit analysis of spherical caps subjected to the uniformly distributed loading. Mróz and Bing-Ye [4] considered the case of loading in the form of loads distributed along the edge of a central hole. Popov [5] solved the same problem in the case of combined loading. In these studies the shells of constant thickness are considered. It was assumed that the yield condition was presented in the form of two hexagons on the planes of moments and membrane forces, respectively. Sankaranarayanan [6] introduced a generalized square yield condition for investigation of plastic spherical shells.

Later Jones and Ich [3] suggested a new approximation of the yield surface which consists of two diamonds on the planes of bending moments and membrane forces.

In the present paper spherical caps of piece-wise constant thickness are considered in the case of the material obeying the yield condition which consists of two hexagons in the planes of moments and membrane forces, respectively.

2.2 PROBLEM FORMULATION

Let us consider a spherical cap of radius A subjected to the uniformly distributed external pressure of intensity P (Fig. 1). The external edge of the shell is simply supported at $\varphi = \beta$.

The thickness of the shell is assumed to be piece-wise constant, e.g.

$$h = \begin{cases} h_0, & \varphi \in (0, \alpha_1), \\ h_1, & \varphi \in (\alpha_1, \alpha_2), \\ \dots\dots\dots\dots, \\ h_n, & \varphi \in (\alpha_n, \beta) \end{cases} \tag{2.1}$$

where $h_0, ..., h_n$ and $\alpha_1, ..., \alpha_n$ are treated as previously unknown constant parameters. However, β and n are considered to be given constants. We are looking for the design of the cap for which
(i) material volume attains the minimum value for given load carrying capacity,
(ii) load carrying capacity attains the maximum value for fixed weight of the shell.

Weight of the cap may be evaluated by the material volume as

$$V = \sum_{j=0}^{n} (\cos a_j - \cos \alpha_{j+1})(3A^2 h_j + \frac{1}{4h_j^3}). \tag{2.2}$$

Here $V = 3M/2\pi\varrho$ and M is the mass of the shell and ϱ - material density.

2.3 GOVERNING EQUATIONS AND BASIC ASSUMPTIONS

For small strains and displacements the equilibrium equations of a shell element have the form [2]

$$\begin{aligned} (N_\varphi \sin\varphi)' - N_\Theta \cos\varphi &= S \sin\varphi \\ (N_\varphi + N_\Theta + PA)\sin\varphi &= -(S\sin\varphi)' \\ (M_\varphi \sin\varphi)' - M_\Theta \cos\varphi &= AS\sin\varphi \end{aligned} \tag{2.3}$$

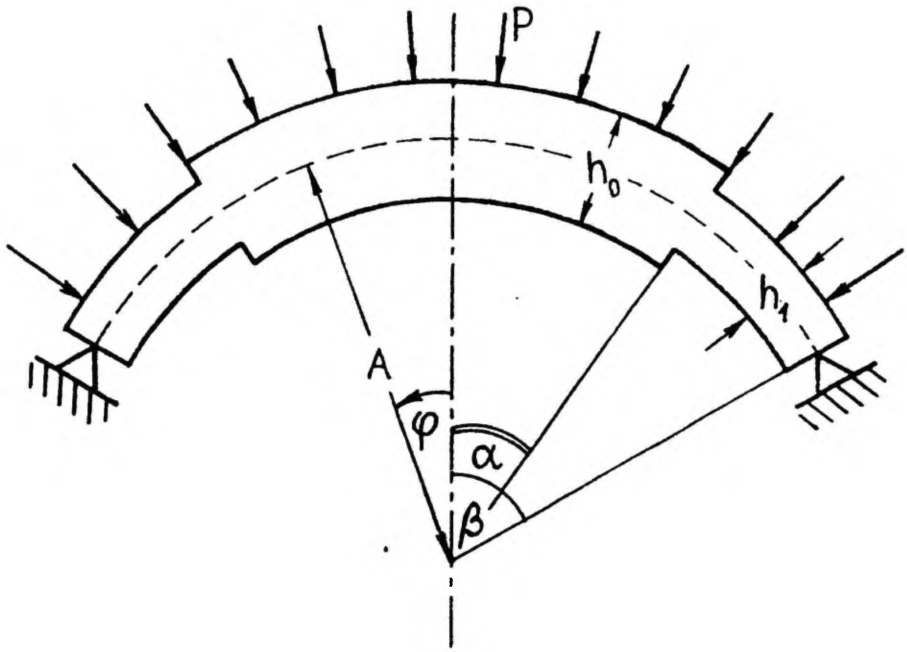

Figure 2.1: Spherical shell of piece wise constant thickness.

In (3) N_φ, N_Θ stand for the membrane forces, M_φ, M_Θ are the moments and S stands for the shear force. Here and henceforth primes denote differentiation with respect to the angle φ.

The strain rate components consistent with (3) are

$$\dot{\varepsilon}_\varphi = \frac{1}{A}(\dot{U}' - \dot{W}), \qquad \dot{\varepsilon}_\Theta = \frac{1}{A}(\dot{U}\cot\varphi - \dot{W}),$$
$$\dot{K}_\varphi = -\frac{1}{A^2}(\dot{U} + \dot{W}')', \qquad \dot{K}_\Theta = -\frac{1}{A^2}\cot\varphi(\dot{U} + \dot{W}') \tag{2.4}$$

where $\dot{U}$ and $\dot{W}$ denote the displacement rates in the meridional and normal direction, respectively.

The material of the shell is assumed to be rigid-plastic obeying the Tresca yield condition. The effects of elastic strains, strain hardening and geometrical non-linearity will be neglected in the present paper.

Yield surfaces in the space of generalized stresses N_φ, N_Θ, M_φ, M_Θ are of complicated structure. Different simplifications have been developed for the yield surface.

In the present study the two moment limited interaction yield surface will be used.

It appears to be convenient to use the following non-dimensional quantities

$$n_{1,2} = \frac{N_{\varphi,\Theta}}{N_*}, \quad m_{1,2} = \frac{M_{\varphi,\Theta}}{M_*}, \quad \gamma_0 = \frac{h_0}{h_*}, \quad \gamma_1 = \frac{h_1}{h_*},$$
$$k = \frac{h_*}{4A}, \quad p = \frac{PA}{N_*}, \quad s = \frac{S}{N_*}, \quad w = \frac{W}{A}, \quad u = \frac{U}{A} \tag{2.5}$$

where $M_* = \sigma_0 h_*^2/4$, $N_* = \sigma_0 h_*$, σ_0 being the yield stress.

Making use of the non-dimensional variables (5) the equilibrium equations (3) may be presented as

$$\begin{aligned} &(n_1 \sin\varphi)' - n_2\cos\varphi = s\sin\varphi \\ &(n_1 + n_2 + p)\sin\varphi = -(s\sin\varphi)' \\ &k[(m_1\sin\varphi)' - m_2\cos\varphi] = s\sin\varphi \end{aligned} \tag{2.6}$$

and the strain rates (4) may be put into the form

$$\dot{\varepsilon}_\varphi = \dot{u}' - \dot{w}, \quad \dot{\varepsilon}_\Theta = \dot{u}\cot\varphi - \dot{w}$$
$$\dot{k}_\varphi = -k(\dot{u} + \dot{w}')', \quad \dot{k}_\Theta = -k\cot\varphi(\dot{u} + \dot{w}') \tag{2.7}$$

where

$$\dot{k}_\varphi = \frac{M_*}{AN_*}\dot{K}_\varphi, \quad \dot{k}_\Theta = \frac{M_*}{AN_*}\dot{K}_\Theta.$$

Boundary conditions for a simply supported spherical cap are

$$\begin{aligned} &m_1(0) = m_2(0), \quad m_1(\beta) = 0, \\ &n_1(0) = n_2(0) \end{aligned} \tag{2.8}$$

It is evident that in the case of the stepped shell the material of the cap is used maximally if the moment M_φ attains its maximal value at $\varphi = \alpha$. Thus in the case $h_1 < h_0$ one has

$$m_1(\alpha) = \gamma_1^2 \tag{2.9}$$

Material volume of the shell (2) may be presented as

$$v = (1 - \cos\alpha)(3\gamma_0 + 4k^2\gamma_0^3) + (\cos\alpha - \cos\beta)(3\gamma_1 + 4k^2\gamma_1^3), \tag{2.10}$$

where $v = V/A^2h_*$.

2.4 LOAD CARRYING CAPACITY OF A SPHERICAL CAP OF A CONSTANT THICKNESS

Consider the spherical cap of constant thickness $h = \delta h_*$. It was shown by Hodge [2] that for small values of the angle β an approximate solution of the posed problem may be obtained if $N_\Theta = 0$, $M_\Theta = M_0$ holds well over the shell. Thus

$$n_2 = 0, \qquad m_2 = \delta^2 \tag{2.11}$$

Integrating the set (6) where (11) is taken into account and satisfying (8) one eventually obtains

$$s = -\frac{p}{2}\varphi$$

$$n_1 = \frac{p}{2}(\varphi \cot \varphi - 1) \tag{2.12}$$

$$m_1 = \delta^2 - \frac{p}{2k}(1 - \varphi \cot \varphi)$$

Substituting $m_1(\beta) = 0$ in (12) gives

$$p = \frac{2k\delta^2}{1 - \beta \cot \beta} \tag{2.13}$$

The value of the load intensity (13) is a lower bound to the load carrying capacity since (13) corresponds to the statically admissible stress distribution (12). For the solution (13) being the exact solution it is necessary that it meets the kinematical requirements. Making use of (7) and the associated flow law one can state that the solution is kinematically admissible for small values of the angle β. Thus for small values of β (13)presents the exaxt limit load. In the case of greather values of angle β the current solution gives the lower bound to the limit load.

2.5 STEPPED SPHERICAL CAP

Consider now the simply supported spherical shell of piece-wise constant thickness (1) whereas non-dimensional thicknesses are γ_0 and γ_1. In this case according to $N_\Theta = 0$, $M_\Theta = M_0$ and (5) $n_2 = 0$ and

$$m_2 = \begin{cases} \gamma_0^2, & \varphi \in [0, \alpha], \\ \gamma_1^2, & \varphi \in [\alpha, \beta] \end{cases} \tag{2.14}$$

Substituting (14) in (6) and integrating under the boundary conditions (8) one easily finds

$$s = -\frac{p}{2}\varphi$$

$$n_1 = \frac{p}{2}(\varphi \cot \varphi - 1) \tag{2.15}$$

for $\varphi \in [0, \beta]$ and

$$m_1 = \gamma_0^2 - \frac{p}{2k}(1 - \varphi \cot \varphi) \tag{2.16}$$

for $\varphi \in [0, \alpha]$. Similarily for $\varphi \in [\alpha, \beta]$ one obtains

$$m_1 = \gamma_1^2 - \frac{p}{2k}(1 - \varphi \cot \varphi) + \frac{\sin \alpha}{\sin \varphi}(\gamma_0^2 - \gamma_1^2), \tag{2.17}$$

where the continuity requirement for m_1 at $\varphi = \alpha$ is taken into account. Satisfying the boundary condition $m_1(\beta) = 0$ in (17) leads to the lower bound of the load carrying capacity of the shell of piece-wise constant thickness

$$p = \frac{2k}{1 - \beta \cot \beta}\left[\gamma_1^2 + \frac{\sin \alpha}{\sin \beta}\left(\gamma_0^2 - \gamma_1^2\right)\right]. \tag{2.18}$$

In order to solve the optimization problem one has to maximize the load carrying capacity under the condition that the material volume of the shell (10) is given. Instead of the exact load carrying capacity the lower bound (18) will be used in present paper. It is reasonable to assume that the shell material is maximally stressed if the condition (9) is satisfied. Thus according to (9), (16)

$$\gamma_0^2 - \gamma_1^2 - \frac{(1 - \alpha \cot \alpha)}{1 - \beta \cot \beta}\left[\gamma_1^2 + \frac{\sin \alpha}{\sin \beta}\left(\gamma_0^2 - \gamma_1^2\right)\right] = 0 \tag{2.19}$$

Assume that the quantity v in (10) is equal to the non-dimensional volume associated with the uniform thickness $\gamma = 1$. This conjecture leads to the relation

$$(1 - \cos \alpha)(3\gamma_0 + 4k^2\gamma_0^3) + (\cos \alpha - \cos \beta)(3\gamma_1 + 4k^2\gamma_1^3) - (1 - \cos \beta)\cdot$$
$$\cdot(3 + 4k^2) = 0(2.20)$$

In order to maximize (18) under constraints (19) and (20) let us introduce the augmented functional

$$J_* = \frac{2k}{\sin\beta - \beta\cos\beta}\left[\gamma_1^2 \sin\beta + \sin\alpha(\gamma_0^2 - \gamma_1^2)\right] +$$

$$+\lambda_1[(1-\cos\alpha)(3\gamma_0 + 4k^2\gamma_0^3) + (\cos\alpha - \cos\beta)(3\gamma_1 + 4k^2\gamma_1^3) -$$

$$-(1-\cos\beta)(3+4k^2)] + \lambda_2\{\gamma_0^2 - \gamma_1^2 -$$

$$-\frac{1-\alpha\cot\alpha}{\sin\beta - \beta\cos\beta}\left[\gamma_1^2\sin\beta + \sin\alpha(\gamma_0^2 - \gamma_1^2)\right]\} \tag{2.21}$$

Necessary conditions of the minimum of (21)

$$\frac{\partial J_*}{\partial\alpha} = 0, \quad \frac{\partial J_*}{\partial\gamma_0} = 0, \quad \frac{\partial J_*}{\partial\gamma_1} = 0$$

may be presented as

$$\begin{cases} \dfrac{2k\cos\alpha(\gamma_0^2 - \gamma_1^2)}{\sin\beta - \beta\cos\beta} + \lambda_1[\sin\alpha(3\gamma_0 + 4k^2\gamma_0^3) - \sin\alpha(3\gamma_1 + 4k^2\gamma_1^3)] + \\ +\dfrac{\lambda_2}{\sin\beta - \beta\cos\beta}\left[\left(\cot\alpha - \dfrac{\alpha}{\sin^2\alpha}\right)(\gamma_1^2\sin\beta - \sin\alpha(\gamma_0^2 - \gamma_1^2)) - \right. \\ \left. -(1-\alpha\cot\alpha)\cdot\cos\alpha(\gamma_0^2 - \gamma_1^2)\right] = 0, \\ \dfrac{4k\gamma_0\sin\alpha}{\sin\beta - \beta\cos\beta} + \lambda_1(1-\cos\alpha)(3 + 12k^2\gamma_0^2) + \\ +2\lambda_2\left[\gamma_0 - \dfrac{1-\alpha\cot\alpha}{\sin\beta - \beta\cos\beta}\cdot\gamma_0\sin\alpha\right] = 0, \\ \dfrac{4k}{\sin\beta - \beta\cos\beta}(\gamma_1\sin\beta - \gamma_1\sin\alpha) + \lambda_1(\cos\alpha - \cos\beta)(3 + 12k^2\gamma_1^2) - \\ -2\lambda_2\left[\gamma_1 + \dfrac{1-\alpha\cot\alpha}{\sin\beta - \beta\cos\beta}(\gamma_1\sin\beta - \gamma_1\sin\alpha)\right] = 0. \end{cases} \tag{2.22}$$

The set of algebraic equations (22) must be solved together with (19), (20) with respect to α, γ_0, γ_1, λ_1, λ_2. This has been done numerically by the aid of the Newton method.

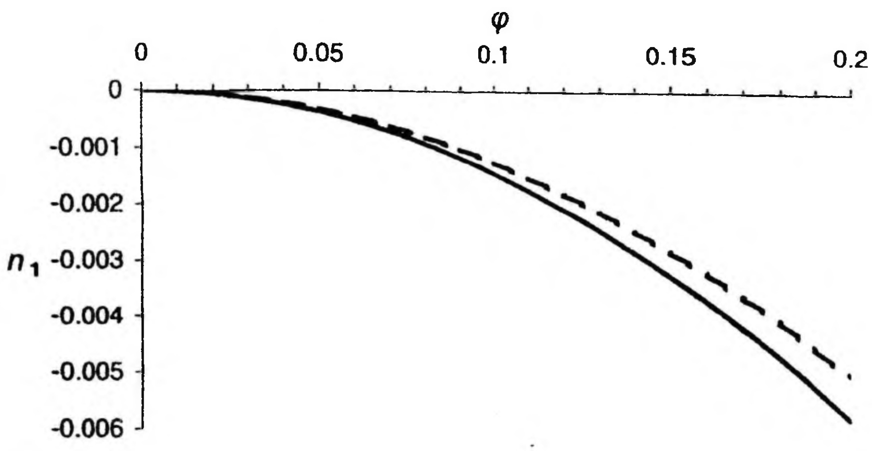

Figure 2.2: Membrane force

2.6 DISCUSSION

The results of calculations are presented Fig. 2,3 and in Tables 1,2 for several values of the angle β. Table 1 corresponds to the case $k = 0,005$, whereas Table 2 is associated with $k = 0,001$. The quantity e in Tables 1,2 can be considered as the economy coefficient defined as

$$e = \frac{p}{p_0}.$$

Here p stands for the lower bound to the load carrying capacity of the stepped shell whereas p_0 is the limit load of the reference shell of constant thickness. In the latter case $\gamma_0 = \gamma_1 = 1$.

Calculations carried out show that the lower bound to the load carrying capacity of the shell can be increased more than 22 % (in the case $\beta = \pi/2$). For smaller values of β the economy coefficient attains smaller values. However, limit load can be increased more than 15 % anyway.

Table 2.1: Optimal values of the design parameters $k = 0,005$.

β	α	γ_0	γ_1	e
0,1	0,08056	1,1395	0,7417	1,15345
0,15	0,12086	1,1394	0,7415	1,1537
0,2	0,16112	1,1393	0,7415	1,1540
0,3	0,24156	1,1390	0,7417	1,1550
0,4	0,32188	1,1386	0;7419	1,1564
0,5	0,40201	1,1380	0,7422	1,1583
0,6	0,48195	1,1373	0,7425	1,1606
0,8	0,64107	1,1355	0,7431	1,1668
1,0	0,7991	1,1330	0,7437	1,1754
1,2	0,9559	1,1298	0,7440	1,1871
1,4	1,1116	1,1257	0,7437	1,2028
$\pi/2$	1,2442	1,1215	0,7428	1,2205

Numerical analysis reveals somewhat unexpected matter that the optimal values of α, γ_0, γ_1 only weakly depend on the geometrical parameter k. For instance, in the case $k = 0,005$ and $\beta = 0,8$ $\alpha = 0,64107$; $\gamma_0 = 1,1355$; $\gamma_1 = 0,7431$. However, if $k = 0,001$ one has $\alpha = 0,6411$; $\gamma_0 = 1,1355$; $\gamma_1 = 0,7432$.

Distributions of the membrane force n_1 and bending moment m_1 are presented in Fig. 2 and 3, respectively. Here $\beta = 0,2$ and $k = 0,005$. According to Table 1 $\alpha = 0,16112$ whereas $\gamma_0 = 1,1393$ and $\gamma_1 = 0,7415$. Note that at $\varphi = \alpha$ the bending moment m_1 has the limit value, e.q $m_1 = \gamma_1^2$. Solid lines in Fig. 2,3 correspond to the optimized shell whereas the dashed lines are due to the reference shell of constant thickness. It can be seen from Fig. 2,3 that the bending moment and membrane force in the optimized structure exceed those corresponding to the reference shell of constant thickness.

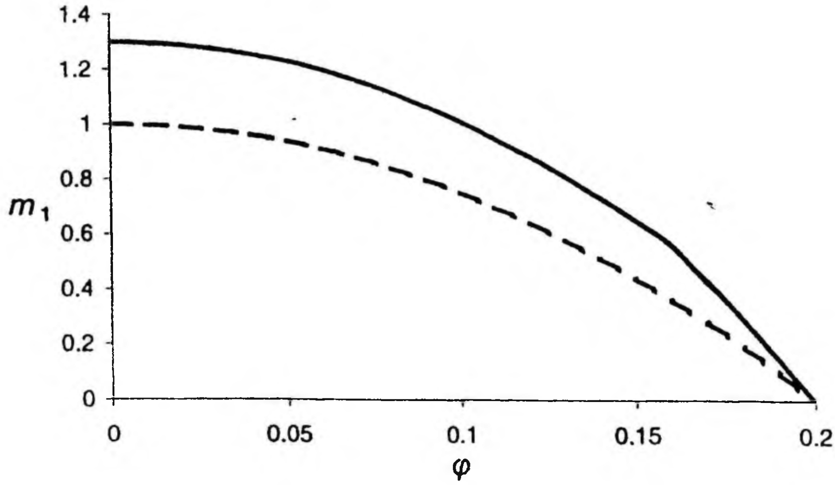

Figure 2.3: Bending moment

Table 2.2: Optimal values of the design parameters $k = 0,001$.

β	α	γ_0	γ_1	e
0,1	0,0814	1,1396	0,7284	1,1533
0,2	0,1611	1,1393	0,7413	1,1541
0,4	0,3219	1,1386	0,7419	1,1565
0,6	0,4820	1,1373	0,7425	1,1606
0,8	0,6411	1,1355	0,7432	1,1660

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CHAPTER 3

OPTIMIZATION OF PLASTIC SPHERICAL SHELLS PIERCED WITH A CENTRAL HOLE

Structural Optimization

OPTIMIZATION OF PLASTIC SPHERICAL SHELLS PIERCED WITH A CENTRAL HOLE

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Abstract. An optimization method regarding to plastic spherical shells is presented. The shells under consideration are clamped at the outer edge and pierced with a central hole. The material of shells obeys generalized square yield condition and associated flow rule. The problem of maximization of the load carrying capacity under the condition that the weight (material volume) of the shell is fixed is transformed into a problem of non-linear programming. The latter is solved with the aid of Lagrangeian multipliers. The obtained solution is compared with the optimal solution of the minimum weight problem for given load carrying capacity.

3.1 INTRODUCTION

Analysis and optimization of non-elastic plates and shells has become a problem of practical interest. Comphensive reviews of problems solved can be found in the works by G.I.N. Rozvany (1976, 1989), M. Życzkowski and Krużelecki (1985), J. Lellep (1991) etc. The methods of direct analysis of rigid-plastic structural elements are accommodated in the books by P. Hodge (1963), N. Jones (1989) and others.

Due to the simplicity of their manufacturing the special significance have the designs of piece wise constant thickness. Circular cylindrical shells of piece wise constant thickness have been treated by C. Cinquini and M. Kouam (1983) in the case of a Tresca material. J. Lellep and S. Hannus (1995) considered the plastic tubes with piece wise constant thickness assuming the material obeyed von Mises yield condition. Optimal designs for stepped plastic shallow shells have been established by J. Lellep and H. Hein (1993, 1994) in the cases of piece wise linear approximations of the exact yield surface corresponding to the original Tresca yield condition on the plane of principal stresses. Employing a lower bound method for determination of the load carrying capacity the authors (1998) defined an optimal design for a stepped spherical shell simply supported at the edge.

In the present paper the stepped shells clamped at the edge and pierced with a central hole are considered. The exact solutions are established under the assumption that the material of the shells obeys the generalized square yield condition and the associated flow law.

3.2 FORMULATION OF THE PROBLEM

Consider a spherical shell of radius A subjected to the uniform external pressure of intensity P (Fig. 1). The external edge of the shell (the circle at $\varphi = \beta$) is clamped and the inner edge (at $\varphi = \alpha$) is absolutely free. Here the angles α and β are considered as given angles.

We are looking for the design of the shell confining our attention to the case of the stepped shell with one step in the thickness. Thus thickness of the shell is

$$h = \begin{cases} h_0, & \varphi \in (\alpha, \alpha_1) \\ h_1, & \varphi \in (\alpha_1, \beta) \end{cases} \tag{3.1}$$

where h_0, h_1, α_1 are to be considered as unknown constant parameters. These parameters have to be determined so that the load carrying capacity P of the cap attains the maximal value over the set of shells of the same weight (or mass, or material volume).

The volume of the material can be easily defined when considering the spherical bodies with radii $A + h/2$ and $A - h/2$, respectively. Therefore, the weight of the shell can be described by

$$\frac{V'}{2\pi} = (\cos\alpha - \cos\alpha_1)\left(3A^2 h_0 + \frac{h_0^3}{4}\right) + (\cos\alpha_1 - \cos\beta)\left(3A^2 h_1 + \frac{h_1^3}{4}\right). \tag{3.2}$$

The optimization problem consists in the minimization of the cost function

$$J_0 = -P \tag{3.3}$$

so that there are satisfied the governing equations of plastic spherical shells and the relation (2), where V' is considered as a given constant.

3.3 BASIC EQUATIONS

The equilibrium equations for spherical shells subjected to axisymmetric loading can be presented as

$$\begin{aligned}
&(N_\varphi \sin\varphi)' - N_\Phi \cos\varphi = S\sin\varphi, \\
&(N_\varphi + N_\Phi + PA)\sin\varphi = -(S\sin\varphi)', \\
&(M_\varphi \sin\varphi)' - M_\Phi \cos\varphi = -AS\sin\varphi,
\end{aligned} \tag{3.4}$$

where N_φ, N_Φ stand for membrane forces and M_φ, M_Φ are the principal moments. Here S is the shear force. When deriving (4) it is assumed that the geometry changes of the structure can be neglected, thus the strain components ε_φ, ε_Φ, K_φ, K_Φ and the displacements U, W are small in comparison with unity.

For small strains and displacements the strain rates can be presented as

$$\begin{aligned}
&\dot\varepsilon_\varphi = \frac{1}{A}(\dot U' - \dot W), \qquad \dot\varepsilon_\Phi = \frac{1}{A}(\dot U\cot\varphi - \dot W), \\
&\dot K_\varphi = -\frac{1}{A^2}(\dot U + \dot W')', \qquad \dot K_\Phi = -\frac{1}{A^2}\cot\varphi(\dot U + \dot W').
\end{aligned} \tag{3.5}$$

In (4), (5) and henceforth primes denote differentiation with respect to the current angle φ whereas dots correspond to the derivatives with respect to time or time like parameter. Note that in the limit analysis of plastic shells the role of time can be fulfilled by the loading parameter P.

In order to introduce non-dimensional variables let us consider a reference shell of constant thickness h_*. The reference shell has the same middle surface as the shell under consideration. Let the yield moment and yield force for the reference shell be $M_* = \sigma_0 h_*^2/4$ and $N_* = \sigma_0 h_*$, respectively. Here σ_0 stands for the yield stress of the material.

For the sake of convenience let us introduce following non-dimensional quantities

$$\begin{aligned}
&n_{1,2} = \frac{N_{\varphi,\Phi}}{N_*}, \quad m_{1,2} = \frac{M_{\varphi,\Phi}}{M_*}, \quad s = \frac{S}{N_*}, \quad w = \frac{W}{A}, \quad u = \frac{U}{A}, \\
&\gamma_0 = \frac{h_0}{h_*}, \quad \gamma_1 = \frac{h_1}{h_*}, \quad k = \frac{h_*}{4A}, \quad p = \frac{PA}{N_*}.
\end{aligned} \tag{3.6}$$

Making use of (6) one can present the equilibrium equations (4) as

$$\begin{aligned}
&(n_1 \sin\varphi)' - n_2 \cos\varphi = s \sin\varphi, \\
&(n_1 + n_2 + p) \sin\varphi = -(s \sin\varphi)', \\
&k\left[(m_1 \sin\varphi)' - m_2 \cos\varphi\right] = s \sin\varphi.
\end{aligned} \tag{3.7}$$

The strain rates (5) take the form

$$\begin{aligned}
&\dot{\varepsilon}_\varphi = \dot{u}' - \dot{w}, \qquad \dot{\varepsilon}_\Phi = \dot{u} \cot\varphi - \dot{w}), \\
&\dot{k}_\varphi = -k(\dot{u} + \dot{w}')', \qquad \dot{k}_\Phi = -k \cot\varphi(\dot{u} + \dot{w}').
\end{aligned} \tag{3.8}$$

Here the following notation is used:

$$\dot{k}_\varphi = \frac{h_*}{4A}\dot{K}_\varphi, \quad \dot{k}_\Phi = \frac{h_*}{4A}\dot{K}_\Phi. \tag{3.9}$$

Material of the shell is assumed to be an isotropic, homogeneous rigid-plastic one obeying the generalized square yield condition suggested by R. Sakaranarayanan (1964). This yield condition has its own application area but it can be handled as an approximation to the Tresca yield condition as well.

The exact yield surface in the space of generalized stresses is of complicated structure even in the case of material obeying the original Tresca yield condition. Due to the complexity of the exact yield surface exact solutions of complicated shell problems are quite rare. As we are rather more interested in the developing an optimization procedure than in solving a particular problem we are seeking in maximal simplicity of the yield surface. Moreover, it is evident that the solutions for simply supported full caps coinside in the cases of materials obeying the generalized square yield condition and the "two moment limited interaction" yield surface, respectively. The latter surface was suggested by P. Hodge (1963).

It is resonable to assume that the stress state of the shell corresponds to the sides AB and $C'D'$ of the squares on the planes of moments and membrane forces, respectively (Fig. 2). Thus

$$M_\Phi = M_0, \quad N_\Phi = -N_0, \tag{3.10}$$

where M_0 and N_0 stand for the yield moment and yield force, respectively, e.g. $M_0 = \sigma_0 h^2/4$, $N_0 = \sigma_0 h$.

According to (6), (10) one has in the case of the stepped shell

$$m_2 = \gamma_0^2, \quad n_2 = -\gamma_0 \tag{3.11}$$

for $\varphi \in (\alpha, \alpha_1)$ and

$$m_2 = \gamma_1^2, \quad n_2 = -\gamma_1 \tag{3.12}$$

for $\varphi \in (\alpha_1, \beta)$.

We are looking for the design of the shell with maximal load carrying capacity. It means that all the sections of the shell must be stressed maximally. The statical restrictions imposed on the yield regime (11), (12) are following

$$-\gamma_0^2 \leq m_1 \leq \gamma_0^2, \quad -\gamma_0 \leq n_1 \leq \gamma_0 \tag{3.13}$$

for $\varphi \in (\alpha, \alpha_1)$ and

$$-\gamma_1^2 \leq m_1 \leq \gamma_1^2, \quad -\gamma_1 \leq n_1 \leq \gamma_1 \tag{3.14}$$

for $\varphi \in (\alpha_1, \beta)$.

Evidently, at the section $\varphi = \alpha_1$ moment m_1 attains its maximal admissible value, e.g. the hinge circle appears at $\varphi = \alpha_1$. Assuming that $h_0 < h_1$ one has an intermediate condition

$$m_1(\alpha_1) = -\gamma_0^2. \tag{3.15}$$

The boundary conditions for the considered case of geometry of the shell are following

$$m_1(\alpha) = n_1(\alpha) = s(\alpha) = 0, \quad m_1(\beta) = -\gamma_1^2. \tag{3.16}$$

Making use of the non-dimensional quantities (6) one can present the weight of the shell as

$$v = (\cos\alpha - \cos\alpha_1)(3\gamma_0 + 4k^2\gamma_0^3) + (\cos\alpha_1 - \cos\beta)(3\gamma_1 + 4k^2\gamma_1^3), \tag{3.17}$$

where $v = V'/2\pi A^2 h_*$.

3.4 THE REFERENCE SHELL OF CONSTANT THICKNESS

Let us consider a spherical shell with a central hole such that the thickness of the shell is $h = h_*\delta$ where δ is a constant.

Assume that the stress strain state of the shell corresponds to the sides AB and $C'D'$ of corresponding squares (Fig. 2). Thus throughout the shell

$$n_2 = -\delta, \quad m_2 = \delta^2. \tag{3.18}$$

Substituting (18) in (7) one can integrate the system of equations (7). It is easy to recheck that the solution of (7) satisfying the boundary conditions (16) has the form

$$\begin{aligned} s &= \frac{1}{2}(2\delta - p)(\varphi - \alpha) + \frac{p}{2}\frac{\sin\alpha}{\sin\varphi}\sin(\varphi - \alpha); \\ n_1 &= -\frac{1}{2}(2\delta - p)\varphi\cot\varphi + \cot\varphi\left[\frac{\alpha}{2}(2\delta - p) + \right. \\ &+ \left. \frac{p}{2}\sin\alpha\cos\alpha\right] - \frac{p}{2}\cos^2\alpha; \\ m_1 &= \delta^2 + \frac{1}{k}\left[\frac{1}{2}(2\delta - p)(1 - \varphi\cot\varphi) + \right. \\ &+ \cot\varphi\left(\frac{1}{2}(2\delta - p)\alpha + \frac{p}{2}\sin\alpha\cos\alpha\right) + \\ &+ \left. \frac{p}{2}\sin^2\alpha + \frac{\sin\alpha}{\sin\varphi}(-k\delta^2 - \delta)\right]. \end{aligned}$$

(3.19)

Finally, substituting $m_1(\beta) = -\delta^2$ in (19) one easily obtains the limit load for the clamped shell

$$p = \frac{2k\left[\frac{\sin\alpha}{\sin\beta}\left(\delta^2 + \frac{\delta}{k}\right) - 2\delta^2\right] - 2\delta\left(1 + (\alpha - \beta)\cot\beta\right)}{\cot\beta(\beta - \alpha + \sin\alpha\cos\alpha) - \cos^2\alpha}. \tag{3.20}$$

According to the associated flow law $\dot{\varepsilon}_\varphi = 0$, $\dot{K}_\varphi = 0$, $\dot{\varepsilon}_\Phi \le 0$, $\dot{K}_\Phi \ge 0$. Thus it follows from (8) that

$$\dot{u}' + \dot{w}'' = 0, \quad \dot{u}' - \dot{w} = 0. \tag{3.21}$$

Integrating (20) and satisfying the boundary conditions one easily obtains

$$\dot{w} = -\frac{\dot{w}_0}{\sin(\beta - \alpha)}\sin(\varphi - \beta) \tag{3.22}$$

and

$$\dot{u} = \frac{\dot{w}_0}{\sin(\beta - \alpha)}\left[\cos(\varphi - \beta) - \cos\beta\right], \tag{3.23}$$

provided $\dot{w}(\beta) = 0$, $\dot{u}(\beta) = 0$ and $\dot{w}(\alpha) = \dot{w}_0$.

Making use of (8) and (22), (23) one can check that $\dot{\varepsilon}_\Phi \leq 0$, $\dot{K}_\Phi \geq 0$. Therefore, the solution (19), (20), (22), (23) is statically and kinematically admissible. It means that (20) presents the exact load carrying capacity for the current problem.

3.5 SPHERICAL CAP OF PIECE WISE CONSTANT THICKNESS

Consider the spherical shell of piece wise constant thickness which is clamped at the outer edge and free at the inner edge. Let the non-dimensional thicknesses be γ_0 and γ_1, respectively.

Guiding by the considerations discussed above we assume that

$$n_2 = -\gamma_0, \quad m_2 = \gamma_0^2 \tag{3.24}$$

for $\varphi \in (\alpha, \alpha_1)$ and

$$n_2 = -\gamma_1, \quad m_2 = \gamma_1^2 \tag{3.25}$$

for $\varphi \in (\alpha_1, \beta)$.

Substituting (24) in (7) and integrating leads to the result

$$\begin{aligned}
s &= \frac{1}{2}(2\gamma_0 - p)(\varphi - \alpha) + \frac{p\sin\alpha}{2\sin\varphi}\sin(\varphi - \alpha); \\
n_1 &= \gamma_0\cot\varphi(\alpha - \varphi) + \frac{p}{2}\cot\varphi(\varphi - \alpha + \sin\alpha\cos\alpha) - \frac{p}{2}\cos^2\alpha; \\
m_1 &= \gamma_0^2 + \frac{1}{k}\left[\frac{1}{2}(2\gamma_0 - p)(1 - \varphi\cot\varphi + \alpha\cot\varphi) + \frac{p}{2}\sin^2\alpha + \right. \\
&+ \left. \frac{p}{2}\sin\alpha\cos\alpha\cot\varphi\right] - \frac{\sin\alpha}{\sin\varphi}\left(\gamma_0^2 + \frac{\gamma_0}{k}\right)
\end{aligned} \tag{3.26}$$

for $\varphi \in (\alpha, \alpha_1)$.

Similarily one obtains

$$\begin{aligned} s &= \frac{1}{2}(2\gamma_1 - p)\varphi - D_1 + (D_2 + \gamma_1).\cot\varphi; \\ n_1 &= \frac{1}{2}(2\gamma_1 - p)(1 - \varphi\cot\varphi) + D_1\cot\varphi + D_2; \\ m_1 &= \gamma_1^2 + \frac{1}{k}\left[\frac{1}{2}(2\gamma_1 - p)(1 - \varphi\cot\varphi) + D_1\cot\varphi + D_2 + \gamma_1\right] + \\ &+ \frac{D_3}{\sin\varphi}, \end{aligned} \tag{3.27}$$

for $\varphi \in (\alpha_1, \beta)$. In (27) D_1, D_2, D_3 stand for arbitrary constants of integration.

Satisfying the continuity requirements for m_1, n_1 and s at $\varphi = \alpha_1$ by the use of (26), (27) one can get

$$D_1 = \gamma_0(\alpha - \alpha_1) + \frac{p}{2}(\sin\alpha\cos\alpha - \alpha) + \alpha_1\gamma_1, \tag{3.28}$$

and

$$D_2 = \frac{p}{2}\sin^2\alpha - \gamma_1, \tag{3.29}$$

also

$$D_3 = \frac{1}{k}\left[(\gamma_0 + k\gamma_0^2)(\sin\alpha_1 - \sin\alpha) - (\gamma_1 + k\gamma_1^2)\sin\alpha_1\right]. \tag{3.30}$$

Two conditions in the set (15), (16) have not used yet. The requirement (15) leads to the load carrying capacity

$$p = \frac{2\sin\alpha(\gamma_0 + k\gamma_0^2) - 2\gamma_0\left(\sin\alpha_1 + (\alpha - \alpha_1)\cos\alpha_1\right) - 4k\gamma_0^2\sin\alpha_1}{(\alpha_1 - \alpha)\cos\alpha_1 + \sin^2\alpha\sin\alpha_1 + \sin\alpha\cos\alpha\cos\alpha_1 - \sin\alpha_1}. \tag{3.31}$$

The last boundary condition in (16) leads to the additional constraint

$$\begin{aligned} &2k\gamma_1^2 + \left(\gamma_1 - \frac{p}{2}\right)(1 - \beta\cot\beta) + \cot\beta\left[\gamma_0(\alpha - \alpha_1) + \right. \\ &+\frac{p}{2}(\sin\alpha\cos\alpha - \alpha) + + \alpha_1\gamma_1\left.\right] + \frac{p}{2}\sin^2\alpha + \\ &+\frac{1}{\sin\beta}\left[(\gamma_0 + k\gamma_0^2)(\sin\alpha_1 - \sin\alpha) - (\gamma_1 + k\gamma_1^2)\sin\alpha_1\right] = 0. \end{aligned} \tag{3.32}$$

It is worthwhile to mention that the associated flow law leads to the equations (21) in the case of the stepped shell as well. Therefore, the solution is kinematically admissible if the displacement rates are defined in the form (22), (23).

3.6 NUMERICAL RESULTS

In order to maximize the load carrying capacity (31) under the constraints (17) and (32) one has to introduce an augmented functional

$$
\begin{aligned}
J_* &= \frac{2\gamma_0(\sin\alpha_1 + (\alpha-\alpha_1)\cos\alpha_1) - 2\sin\alpha(\gamma_0 + k\gamma_0^2) + 4k\gamma_0^2\sin\alpha_1}{(\alpha_1-\alpha)\cos\alpha_1 + \sin^2\alpha\sin\alpha_1 + \sin\alpha\cos\alpha\cos\alpha_1 - \sin\alpha_1} + \\
&+ \lambda_1\left[(\cos\alpha - \cos\alpha_1)(3\gamma_0 + 4k^2\gamma_0^3) + (\cos\alpha_1 - \cos\beta)\cdot\right. \\
&\cdot \left.(3\gamma_1 + 4k^2\gamma_1^3) - v + \lambda_2\left\{2k\gamma_1^2 + \left(\gamma_1 - \frac{p}{2}\right)(1-\beta\cot\beta) + \right.\right. \\
&+ \cot\beta\left[\gamma_0(\alpha-\alpha_1) + \frac{p}{2}(\sin\alpha\cos\alpha - \alpha) + \alpha_1\gamma_1\right] + \frac{p}{2}\sin^2\alpha + \\
&+ \left.\frac{1}{\sin\beta}\left[(\gamma_0 + k\gamma_0^2)(\sin\alpha_1 - \sin\alpha) - (\gamma_1 + k\gamma_1^2)\sin\alpha_1\right]\right\},
\end{aligned}
\tag{3.33}
$$

where p is defined by (31).

Necessary conditions for minimum of (33) can be expressed as

$$
\begin{aligned}
&\frac{\partial p}{\partial\alpha_1}\left[-1 + \frac{\lambda_2}{2}(\sin\alpha\cos\alpha - \alpha + \beta\cot\beta - 1 + \sin^2\alpha)\right] + \\
&+\lambda_1\sin\alpha_1(3\gamma_0 + 4k^2\gamma_0^3 - 3\gamma_1 - 4k^2\gamma_1^3) + \\
&+\frac{\lambda_2}{\sin\beta}\left[\cos\alpha_1(\gamma_0 + k\gamma_0^2 - \gamma_1 - k\gamma_1^2) + \cos\beta(\gamma_1 - \gamma_0)\right] = 0; \\
&\frac{\partial p}{\partial\gamma_0}\left[-1 + \frac{\lambda_2}{2}(\sin\alpha\cos\alpha - \alpha + \beta\cot\beta - 1 + \sin^2\alpha)\right] + \\
&+\lambda_1(\cos\alpha - \cos\alpha_1)(3 + 12k^2\gamma_0^2) + \\
&+\lambda_2\left[\cot\beta(\alpha - \alpha_1) + \frac{\sin\alpha_1 - \sin\alpha}{\sin\beta}(1 + 2k\gamma_0)\right] = 0; \\
&\lambda_1(\cos\alpha_1 - \cos\beta)(3 + 12k^2\gamma_1^2) + \\
&+\lambda_2\left[4k\gamma_1 + 1 - \beta\cot\beta + \cot\beta\cdot\alpha_1 - \frac{\sin\alpha_1}{\sin\beta}(1 + 2k\gamma_1)\right] = 0.
\end{aligned}
\tag{3.34}
$$

Here the following notation is used:

$$
\begin{aligned}
\frac{\partial p}{\partial\alpha_1} &= \left\{\left[-2\gamma_0(-\sin\alpha_1(\alpha-\alpha_1)) - 4k\gamma_0^2\cos\alpha_1\right]\cdot\right. \\
&\cdot \left[(\alpha_1-\alpha)\cos\alpha_1 + \sin^2\alpha\sin\alpha_1 + \sin\alpha\cos\alpha\cos\alpha_1 - \sin\alpha_1\right] - \\
&- \left[2\sin\alpha(\gamma_0 + k\gamma_0^2) - 2\gamma_0(\sin\alpha_1 + (\alpha-\alpha_1)\cos\alpha_1) - \right. \\
&- 4k\gamma_0^2\sin\alpha_1\cdot\left[(\alpha-\alpha_1)\sin\alpha_1 + \sin^2\alpha\cos\alpha_1 - \right. \\
&- \sin\alpha\cos\alpha\sin\alpha_1\cdot\left\{(\alpha_1-\alpha)\cos\alpha_1 + \sin^2\alpha\sin\alpha_1 + \right. \\
&+ \sin\alpha\cos\alpha\cos\alpha_1 - \sin\alpha_1^{-2}, \\
\frac{\partial p}{\partial\gamma_0} &= \frac{2\sin\alpha(1 + 2k\gamma_0) - 2\sin\alpha_1 - 2(\alpha-\alpha_1)\cos\alpha_1 - 8k\gamma_0\sin\alpha_1}{(\alpha_1-\alpha)\cos\alpha_1 + \sin^2\alpha\sin\alpha_1 + \sin\alpha\cos\alpha\cos\alpha_1 - \sin\alpha_1}.
\end{aligned}
\tag{3.35}
$$

The set of algebraic equations (34) is to be solved together with (17) and (32) making use of (35). A standard numerical procedure based on a modification of the Newton method has been employed.

The results of calculations are presented in Fig. 3 and Tables 1-3 for the shells with the geometrical parameter $k = 0,05$; $k = 0,02$ and $k = 0,01$, respectively. Here a simplified version corresponding to $\gamma_1 = 1$ is considered. In this case $h_* = h_1$ and the thickness of the reference shell of constant thickness is denoted by $\gamma_* h_1$. The quantity γ_* is calculated so that the material volumes of the stepped and reference shells of constant thickness, respectively, coincide.

The coefficient of economy in Tables 1-3 is defined as

$$e = \left(-1 + \frac{p}{p_0}\right) 100 \tag{3.36}$$

where p_0 stands for the load carrying capacity of the reference shell and p is the maximal value of the limit load for the shell of piece wise constant thickness.

The economy of the design depends on geometrical parameters of the shell under consideration. It can be seen from Tables 1-3 that in the case $k = 0,05$ and $\alpha = 0,4$; $\beta = 0,6$ the limit load can be increased more than 30% when using the design with one step in the thickness. However, in the case $k = 0,02$ where $\alpha = 0,4$ and $\beta = 0,6$ the load carrying capacity can be increased by 17,5% and in the case $k = 0,01$ ($\alpha = 0,4$; $\beta = 0,6$) by 8,5% in comparison to that corresponding to the reference shell of constant thickness.

The distributions of the membrane force n_1 and bending moment m_1 are presented in Figs. 3 and 4, respectively. Solid lines in Figs. 2 and 3 correspond to the optimized stepped shell, whereas the dashed lines are due to the reference shells of constant thickness.

In similar way one can solve the problem of minimum weight for given load carrying capacity. In this case one has to minimize (17) under the condition that the load intensity p in (31) has a given value.

The results of calculations are presented in Tables 4-6. As p is now fixed instead of (36) the coefficient of economy is defined as

$$e = \left(1 - \frac{v}{v_0}\right) 100. \tag{3.37}$$

In (37) v_0 stands for the material volume for the reference shell of constant thickness. The constant thickness can be determined from (20) under the assumption that the stepped shell and the reference shell, respectively, have the common load carrying capacity. In the case of the shell with $\gamma_* = 1$ one has

$$v_0 = (\cos\alpha - \cos\beta)(3 + 4k^2). \tag{3.38}$$

It can be seen from Tables 4-6 that for the optimized shell with one step in the thickness the outer thickness exceeds unity whereas the inner thickness is less than one. The same feature can be observed when maximizing the load carrying capacity for given material volume. However, the values of design parameters α_1, γ_0, γ_1 are slightly different in these cases.

It is worthwhile to note that for a fixed value of inner angle the eventual material saving is the greater the less is the difference between outer and inner angles, respectively (Tables 4-6).

Similarily, it can be seen from Tables 1-3 that the less is $\beta - \alpha$ the greater is the relative increase of the load carrying capacity for fixed value of the inner angle α. One can observe the same feature when the outer angle β is fixed and the inner angle α is variable.

It is somewhat surprising that the material saving is relatively high even in the case of one step in the thickness. For instance, in the case $k = 0,05$; $\alpha = 0,8$; $\beta = 1,0$ one can save 17,08% of the material when utilizing the design with one step. At the same time the uniform shell and the stepped shell, respectively, have the same value of the load carrying capacity $p_0 = 4,239$.

3.7 CONCLUDING REMARKS

A method of optimization of spherical shells with free internal edge and clamped external edge has been developed. The shells of piece wise constant thickness with one step in the thickness have been considered. Exact solutions have been established assuming the material of shells obeys the generalized square yield condition and associated flow rule.

The results obtained numerically showed that the load carrying capacity of the shell can be increased significantly (even more than 35% in the case $k = 0,05$; $\alpha = 0,8$; $\beta = 1,0$) when using the design of the shell with step wise varying thickness. Similary, the weight of the shell can be reduced as well for fixed load carrying capacity.

It is interesting to remark that the two following problems:

(i) minimization of the weight for given load carrying capacity.

(ii) maximization of the load carrying capacity for fixed weight or material volume of the shell (3.38)

lead to different values of design parameters. Numerical results show that the coefficient of efficiency for the problem (i) is less than that of the problem (ii).

ACKNOWLEDGEMENT

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Table 3.1: Design of maximal load carrying capacity ($k = 0,01$)

α	β	α_1	γ_0	γ_*	e	p	p_0
0,4	0,6	0,5441	0,7514	0,83	8,56	1,639	1,510
0,4	0,8	0,7630	0,9321	0,94	0,84	1,400	1,388
0,6	1,0	0,9586	0,9204	0,93	0,9	1,268	1,256
0,8	1,0	0,9378	0,7174	0,81	9,60	1,397	1,274

Table 3.2: Design of maximal load carrying capacity ($k = 0,02$)

α	β	α_1	γ_0	γ_*	e	p	p_0
0,4	0,6	0,5221	0,6471	0,80	17,54	2,308	1,964
0,4	0,8	0,7384	0,8640	0,89	3,00	1,555	1,510
0,4	1,0	0,9542	0,9445	0,95	0,65	1,488	1,479
0,6	1,0	0,9326	0,8510	0,88	3,18	1,387	1,344
0,8	1,0	0,9277	0,6614	0,79	20,20	1,959	1,630

Table 3.3: Design of maximal load carrying capacity ($k = 0,05$)

α	β	α_1	γ_0	γ_*	e	p	p_0
0,4	0,6	0,5129	0,5956	0,79	30,49	4,531	3,472
0,4	0,8	0,6908	0,7461	0,83	9,99	2,114	1,922
0,4	1,0	0,9060	0,8497	0,88	3,53	1,719	1,660
0,6	1,0	0,8846	0,7322	0,82	10,99	1,842	1,660
0,8	1,0	0,9234	0,6319	0,78	35,83	3,750	2,760

Table 3.4: The design of minimum weight ($k = 0,01$)

α	β	α_1	γ_1	γ_2	p	e
0,4	0,6	0,5363	0,8188	1,1421	1,959	6,54
0,4	0,8	0,7575	0,581	1,0645	1,493	0,79
0,4	1,0	0,9695	0,9986	1,0110	1,438	0,06
0,6	1,0	0,9549	0,9789	1,0727	1,366	0,88
0,8	1,0	0,9313	0,7934	1,1542	1,694	7,62

Table 3.5: The design of minimum weight ($k = 0,02$)

α	β	α_1	γ_1	γ_2	p	e
0,4	0,6	0,5209	0,7174	1,1208	2,785	10,55
0,4	0,8	0,7258	0,9319	1,1143	1,754	2,65
0,4	1,0	0,9458	0,9852	1,0571	1,570	0,63
0,6	1,0	0,9194	0,9245	1,1251	1,578	2,91
0,8	1,0	0,9161	0,6838	1,1221	2,330	12,41

Table 3.6: The design of minimum weight ($k = 0,05$)

α	β	α_1	γ_1	γ_2	p	e
0,4	0,6	0,5118	0,6397	1,0853	5,262	14,40
0,4	0,8	0,6789	0,8157	1,1332	2,538	6,91
0,4	1,0	0,8911	0,9248	1,1196	1,966	3,02
0,6	1,0	0,8675	0,7922	1,1402	2,215	7,80
0,8	1,0	0,9077	0,6016	1,0752	4,239	17,08

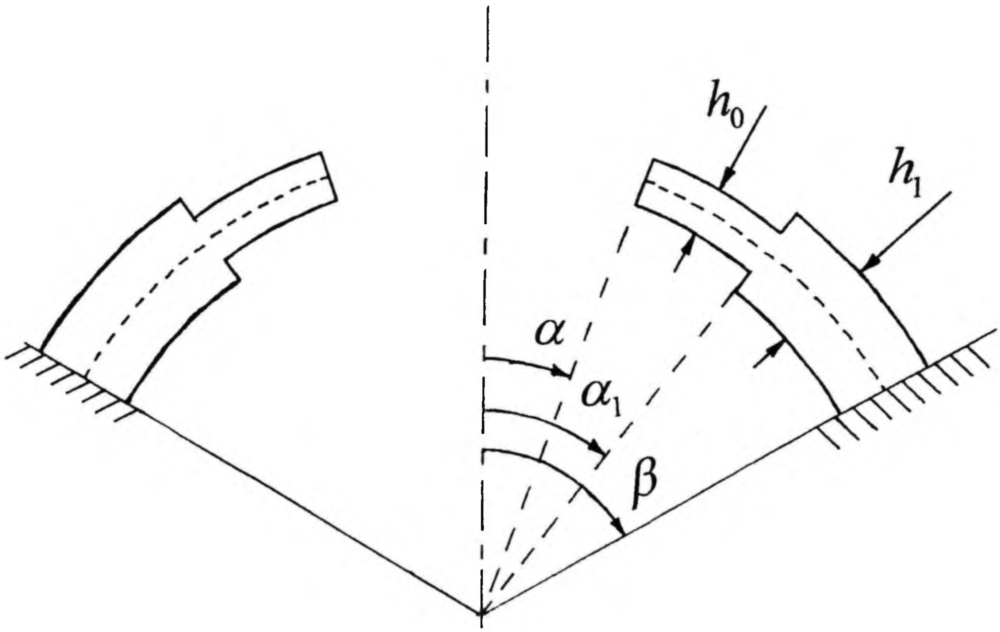

Figure 3.1: Spherical cap

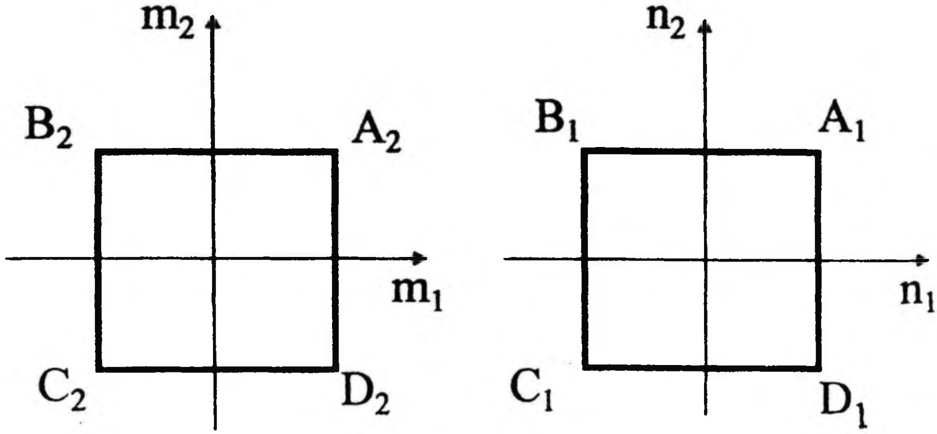

Figure 3.2: Generalized square yield condition

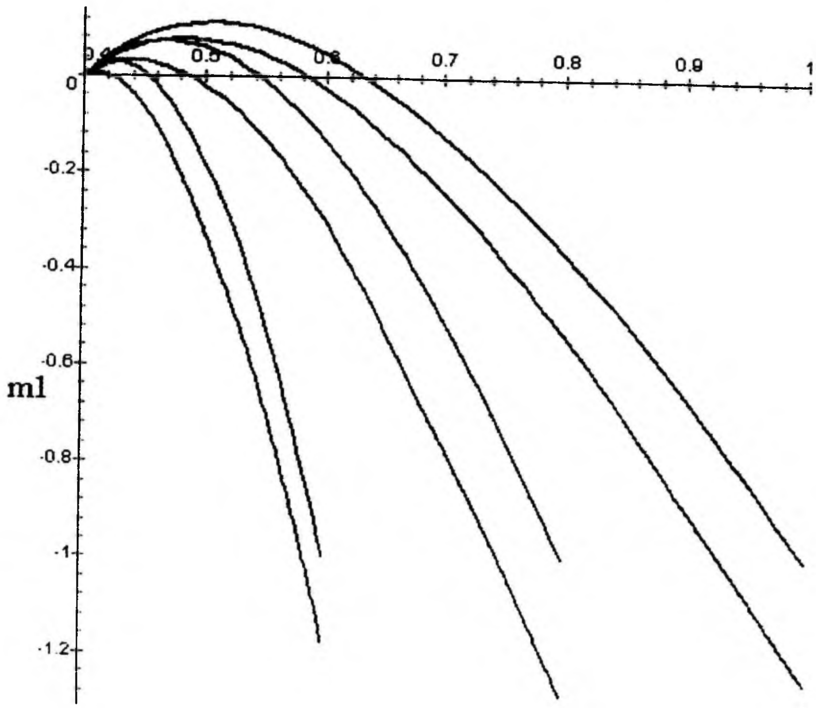

Figure 3.3: Bending moment for a cap ($k = 0,05$)

CHAPTER 4

OPTIMIZATION OF PLASTIC SPHERICAL SHELLS OF VON MISES MATERIAL

OPTIMIZATION OF PLASTIC SPHERICAL SHELLS OF VON MISES MATERIAL

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Abstract. An optimization procedure is developed for spherical shells pierced with a central hole. Outer edge of the shell is simply supported whereas the inner edge is absolutely free. The material of the shell is assumed to be and ideal plastic material obeying von Mises yield condition. Resorting to the lower bound theorem of limit analysis the shells with constant and piece-wise constant thickness are considered. The designs of spherical shells corresponding to maximal load carrying capacity are established for given weight. Necessary optimality conditions are derived with the aid of variational methods of the theory of optimal control. The obtained set of equations is solved numerically.

4.1 INTRODUCTION

Optimization of elastic and non-elastic beams, frames, plates and shells has had the attention of many investigators during the last decades. Comprehensive reviews of problems solved can be found in the books and papers by J. Krużelecki and M. Życzkowski (1985), J. Lellep and Ü. Lepik (1984), G. Rozvany (1976), M. Bendsoe (1995), J. Lellep (1991).

Different approaches to optimization of non-elastic structural elements have been developed by Z. Mróz (1975), G. Rozvany (1976), M. Save (1972), J. Lellep (1985, 1991). Optimal plastic design of shells was discussed by Prager and Rozvany (1980), Nakamura et al. (1981), Dow et. al. (1981). Axisymmetric plates and shallow spherical shells of minimum weight are studied by D. Lamblin, G. Guerlement, M. Save (1985) and J. Lellep, H. Hein (1993, 1994) assuming that the thickness is piece-wise constant and that the material obeys Tresca yield condition. Deep spherical shells of Tresca material have been studied by J. Lellep and E. Tungel (1999). Straight plate problems are solved by A. Sawczuk and J. Sokol-Supel (1993) for both, Tresca and Mises materials.

It is somewhat surprising that relatively less attention has been paid to the optimization of plates and shells material of which obeys von Mises yield condition. Optimal design for shallow spherical shells of von Mises material have been established by J. Lellep and J. Majak (1995). Circular cylindrical shells of piece-wise constant thickness were studied by J. Lellep and S. Hannus (1995).

In the present paper an optimization procedure will be developed for plastic spherical shells of piece-wise constant thickness in the case of von Mises material.

4.2 FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

Let us consider a spherical shell of radius A subjected to the uniformly distributed transverse pressure of intensity P. Assume that the external edge of the shell (the circle corresponding to $\varphi = \beta$) is simply supported and the inner edge at $\varphi = \alpha$ is absolutely free.

Let the shell wall be of piece-wise constant thickness (Fig. 1), e.g.

$$h = h_j, \quad \varphi \in (\alpha_j, \alpha_{j+1}) \tag{4.1}$$

where $\alpha_0 = \alpha,\ \alpha_{n+1} = \beta$ and $j = 0, ..., n$. The number n and angles $\alpha,\ \beta$ are considered as given constants whereas h_j $(j = 0, ..., n)$ and α_j $(j = 1, ..., n)$ are to be defined so that the load carrying capacity P_0 attains the maximal value for given weight or material volume of the shell.

Material of the cap is assumed to be an ideal rigid-plastic material obeying von Mises yield condition. The weight or material volume of the shell can be defined when calculating the volume of a body located between spherical surfaces with radii $A - h/2$ and $A + h/2$ for each region (α_j, α_{j+1}), respectively. However, it is assumed that the shell wall is of ideal sandwich type whereas h stands for the thickness of carrying layers and H is the total thickness.

In the case of a sandwich spherical shell the material volume of a carrying layer can be presented as

$$V = \sum_{j=0}^{n} h_j(\cos \alpha_j - \cos \alpha_{j+1}). \tag{4.2}$$

We are looking for the minimum of the cost function

$$J_1 = -P_0 \tag{4.3}$$

under the condition that $V = V_0$ and that there exists a statically admissible stress field corresponding to the external loading $P = P_0$. In other words, we are using the lower bound approach to the load carrying capacity. According to the lower bound theorem of limit analysis actual limit load corresponds to the maximum of the load factor associated with a statically admissible stress field (see Hodge, 1963).

In the case of spherical shells subjected to an axisymmetric loading the stress resultants contributing to the internal energy are the membrane forces N_φ, N_Θ and bending moments M_φ, M_Θ. The shear force S which influences on the equilibrium of a shell element does not contribute the internal power of the shell.

The equilibrium equations of a spherical shell element can be presented as (here the configuration changes of the shell are neglected)

$$\begin{aligned} (N_\varphi \sin\varphi)' - N_\Theta \cos\varphi &= S \sin\varphi, \\ (N_\varphi + N_\Theta + PA)\sin\varphi &= -(S\sin\varphi)' \\ (M_\varphi \sin\varphi)' - M_\Theta \cos\varphi &= -AS\sin\varphi. \end{aligned} \tag{4.4}$$

In (4) and henceforth primes denote differentiation with respect to current angle φ.

For the sake of convenience the following non-dimensional quantities will be used

$$\begin{aligned} &\gamma_j = \frac{h_j}{h_*}, \quad k = \frac{t}{2A}, \quad p = \frac{PA}{N_*}, \quad \gamma = \frac{h}{h_*}, \\ &n_{1,2} = \frac{N_{\varphi,\Theta}}{N_*}, \quad m_{1,2} = \frac{M_{\varphi,\Theta}}{M_*}, \quad s = \frac{S}{N_*}, \quad v = \frac{V}{h_*}. \end{aligned} \tag{4.5}$$

Here h_* stands for the thickness of layers of the reference shell of constant thickness. The quantities M_* and N_* stand for the yield moment and yield force for the reference shell, respectively. Thus $N_* = 2\sigma_0 h_*$, $M_* = \sigma_0 h_* t$, σ_0 being the yield stress of the material of carrying layers.

Material of the shell (of carrying layers) is assumed to be an ideal rigid-plastic material obeying von Mises yield condition. In its original form the yield condition suggested by von Mises can be presented as

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \le \sigma_0^2 \tag{4.6}$$

σ_1, σ_2 being principal stresses.

The exact yield surface in the space of membrane forces and moments for an axisymmetric shell material of which obeys the plasticity condition (6) was derived by A. Ilyushin [1957]. It was shown later by several authors that there are several non-linear surfaces which present good approximations to the exact yield surface (see H.M. Haydl and A.N. Sherbourne, 1979; also M. Robinson, 1971; M. Życzkowski, 1981).

In the present paper the yield surface

$$\frac{1}{N_0^2}(N_\varphi^2 - N_\varphi N_\Theta + N_\Theta^2) + \frac{1}{M_0^2}(M_\varphi^2 - M_\varphi M_\Theta + M_\Theta^2) = 1 \tag{4.7}$$

will be used. Here M_0, N_0 stand for the yield moment and yield force, respectively, for the current section of the shell, e.g. $M_0 = \sigma_0 th$, $N_0 = 2\sigma_0 h$.

Making use of (5) equilibrium equations (4) can be presented as

$$\begin{aligned} (n_1 \sin\varphi)' - n_2 \cos\varphi &= s \sin\varphi, \\ (n_1 + n_2 + p)\sin\varphi &= -(s\sin\varphi)' \\ k[(m_1 \sin\varphi)' - m_2 \cos\varphi] &= s\sin\varphi. \end{aligned} \tag{4.8}$$

In the similar way the constraint imposed on the weight of the cap takes according to (2) and (5) the form

$$v_0 = \sum_{j=0}^{n} \gamma_j(\cos\alpha_j - \cos\alpha_{j+1}). \tag{4.9}$$

The equation of the yield surface (7) can be put into the form

$$n_1^2 - n_1 n_2 + n_2^2 + m_1^2 - m_1 m_2 + m_2^2 - \gamma_j^2 = 0 \tag{4.10}$$

for the segment $D_j = (\alpha_j, \alpha_{j+1})$ where $j = 0, ..., n$.

Boundary conditions for the shell with simply supported outer edge and inner edge are $s(\alpha) = 0$ and

$$m_1(\alpha) = n_1(\alpha) = 0; \quad m_1(\beta) = 0. \tag{4.11}$$

4.3 NECESSARY CONDITIONS OF OPTIMALITY

The problem set up above will be considered as a particular problem of optimal control with the objective function (3), state equations (8) and additional contraints (9), (10). Variables n_1, m_1, s will be treated as state variables and n_2, m_2 as controls (Lellep, 1991). However, it appears that the variable s can be eliminated from the set (8).

Multiplying the first equation with $\sin\varphi$ and the second one with $\cos\varphi$ adding one to another leads to the equation

$$n_1' \sin^2\varphi + 2n_1 \sin\varphi\cos\varphi + p\sin\varphi\cos\varphi = s\sin^2\varphi + (s\sin\varphi)'\cos\varphi = 0$$

which can be presented as

$$(n_1\sin^2\varphi)' + p\sin\varphi\cos\varphi + (s\sin\varphi\cos\varphi)' = 0. \qquad (4.12)$$

Integrating (12) with respect to φ and taking into account that $n_1(\alpha) = s(\alpha) = 0$ one easily can determine the shear force

$$s = -\left(n_1 + \frac{p}{2}\right)\tan\varphi + \frac{p}{2}\frac{\sin^2\alpha}{\sin\varphi\cos\varphi}. \qquad (4.13)$$

Due to the non-linearity of the constraint (10) the problem will be solved numerically up to the end. For the sake of convenience of calculations it is reasonable to interprete the intensity of the pressure p as a state variable whereas $p' = 0$. Since the loading intensity is equal to be load carrying capacity of the shell one can demand that $p(\alpha) = p_0$, or $p(\beta) = p_0$. In this case $p \equiv p_0$ for $\varphi \in (\alpha, \beta)$. When p is treated as a phase coordinate the objective function (3) is to be replaced with

$$J = -p(\alpha). \qquad (4.14)$$

Substituting (13) in (8) leads to the set

$$\begin{aligned} n_1' &= (n_2 - n_1)\cot\varphi - \left(n_1 + \frac{p}{2}\right)\tan\varphi + \frac{p}{2}\frac{\sin^2\alpha}{\sin\varphi\cos\varphi}, \\ m_1' &= (m_2 - m_1)\cot\varphi - \frac{1}{k}\left(n_1 + \frac{p}{2}\right)\tan\varphi + \frac{p}{2k}\frac{\sin^2\alpha}{\sin\varphi\cos\varphi}, \\ p' &= 0 \end{aligned}$$

$$(4.15)$$

which must be integrated with boundary conditions (11) and additional constraints (9) and (10).

In order to get necessary conditions of optimality let us introduce adjoint variables Ψ_1, Ψ_2, Ψ_3 and create the extended functional (see A. Bryson and Y.C. Ho, 1975; J. Lellep, 1991)

$$\begin{aligned} J_* &= -p(\alpha) + \sum_{j=0}^{n} \int_{D_j} \Big\{ \Psi_1 (n_1' - (n_2 - n_1)\cot\varphi + (n_1 + \frac{p}{2})\tan\varphi - \\ &- \frac{p}{2} \frac{\sin^2\alpha}{\sin\varphi\cos\varphi}) + \Psi_2 (m_1' - (m_2 - m_1)\cot\varphi + \frac{1}{k}(n_1 + \frac{p}{2})\tan\varphi - \\ &- \frac{p}{2k} \frac{\sin^2\alpha}{\sin\varphi\cos\varphi}) + \Psi_3 p' + \nu_j (n_1^2 - n_1 n_2 + n_2^2 + m_1^2 - m_1 m_2 + \\ &+ m_2^2 - \gamma_j^2) d\varphi + \lambda \left\{ \sum_{j=0}^{n} \gamma_j (\cos\alpha_j - \cos\alpha_{j+1}) - v_0 \right\}. \end{aligned} \tag{4.16}$$

In (16) ν_j $(j = 0, ..., n)$ stand for Lagrange'ian multipliers corresponding to constraints (10) and λ is associated with the equality (9).

Calculating the total variation of the functional (16) one obtains

$$\begin{aligned} \Delta J_* &= -\Delta p(\alpha) + \sum_{j=0}^{n} \int_{D_j} \{ -\Psi_1' \delta n_1 + \Psi_1 (\cot\varphi(\delta n_1 - \delta n_2) + \\ &+ (\delta n_1 + \tfrac{\delta p}{2})\tan\varphi - \sigma \frac{\delta p}{2} \frac{\sin^2\alpha}{\sin\varphi\cos\varphi} - \Psi_2' \delta m_1 + \\ &+ \Psi_2 \left((\delta m_1 - \delta m_2)\cot\varphi + \tfrac{1}{k}\left(\delta n_1 + \tfrac{\delta p}{2}\right)\tan\varphi - \tfrac{\delta p}{2k} \tfrac{\sin^2\alpha}{\sin\varphi\cos\varphi} \right) - \\ &- \Psi_3' \delta p d\varphi + \sum_{j=0}^{n} \{ (\Psi_1 \delta n_1 + \Psi_2 \delta m_1 + \Psi_3 \delta p) \big|_{\alpha_{j+1} - 0} - \\ &- (\Psi_1 \delta n_1 + \Psi_2 \delta m_1 + \Psi_3 \delta p) \big|_{\alpha_j + 0} + \int_{D_j} \{ \nu_j (2n_1 - n_2)\delta n_1 + \\ &+ (2n_2 - n_1)\delta n_2 + (2m_1 - m_2)\delta m_1 + (2m_2 - m_1)\delta m_2 - \\ &- 2\gamma_j \Delta\gamma_j) d\varphi + + \lambda \Delta\gamma_j (\cos\alpha_j - \cos\alpha_{j+1}) - \\ &- \lambda\gamma_j (\sin\alpha_j \Delta\alpha_j - \sin\alpha_{j+1}\Delta\alpha_{j+1}) = 0. \end{aligned} \tag{4.17}$$

When deriving (17) two types of variations are distinguished. The quantity δz stands for a weak variation of the variable z whereas $\Delta z(\alpha_*)$ is the total variation of z at the point α_*. It is easy to recheck that

$$\Delta z(\alpha_* \pm) = \delta z(\alpha_* \pm) + z'(\alpha_* \pm)\Delta\alpha_*. \tag{4.18}$$

Here $\Delta\alpha_*$ stands for the increment of the parameter α_*.

In the case of continuous variables in (18) $\Delta z(\alpha_*-) = \Delta z(\alpha_*+)$. However, if z is discontinuous then the quantities $\Delta z(\alpha_*-)$ and $\Delta z(\alpha_*+)$ can be independent variations. Note that n_1, m_1 and p are considered to be continuous for each $\varphi \in (\alpha, \beta)$ whereas n_2 and m_2 may have finite discontinuities.

From (17) one easily obtains the adjoint equations

$$\begin{aligned} \Psi_1' &= \Psi_1(\tan\varphi + \cot\varphi) + \frac{\Psi_2}{k}\tan\varphi + \nu_j(2n_1 - n_2), \\ \Psi_2' &= \Psi_2 \cot\varphi + \nu_j(2m_1 - m_2), \\ \Psi_3' &= \frac{1}{2}\left(\tan\varphi - \frac{\sin^2\alpha}{\sin\varphi\cos\varphi}\right)\left(\Psi_1 + \frac{\Psi_2}{k}\right) \end{aligned} \tag{4.19}$$

for $\varphi \in D_j$ $(j = 0, ..., n)$.

The boundary conditions for adjoint system (transversality conditions) have the form

$$\Psi_1(\beta) = 0, \quad \Psi_3(\alpha) = -1, \quad \Psi_3(\beta) = 0 \tag{4.20}$$

as at $\varphi = \alpha$ and $\varphi = \beta$; $\delta n_1 = \Delta n_1$, $\delta m_1 = \Delta m_1$ and $\delta p = \Delta p$.

Due to arbitraryness of quantities δn_2 and δm_2 in (17) following equations hold good

$$\begin{aligned} -\Psi_1 \cot\varphi + \nu_j(2n_2 - n_1) &= 0, \\ -\Psi_2 \cot\varphi + \nu_j(2m_2 - m_1) &= 0 \end{aligned} \tag{4.21}$$

for $\varphi \in D_j$ $(j = 0, ..., n)$.

Variations $\Delta\gamma_j$ must be considered as constant quantities, therefore

$$\lambda(\cos\alpha_j - \cos\alpha_{j+1}) - 2\gamma_j \int_{D_j} \nu_j d\varphi = 0 \tag{4.22}$$

for $j = 0, ..., n$.

When accounting for (18)-(21) the equation (17) can be cast into the form

$$\begin{aligned} &\sum_{j=1}^{n} -1 \{[\Psi_1(\alpha_j)\delta n_1(\alpha_j)] + [\Psi_2(\alpha_j)\delta m_1(\alpha_j)] + \\ + \quad &[\Psi_3(\alpha_j)\delta p(\alpha_j)]\} + \sum_{j=1}^{n} \lambda(\gamma_{j-1} - \gamma_j)\sin\alpha_j \Delta\alpha_j = 0 \end{aligned} \tag{4.23}$$

where the quadratic brackets stand for finite discontinuities of variables, e.g.

$$[y(\alpha_*)] = y(\alpha_* + 0) - y(\alpha_* - 0). \tag{4.24}$$

According to their physical meaning variables n_1, m_1 and p are continuous at each point $\varphi = \alpha_j$ $(j = 1, ..., n)$. Therefore, according to (18) and (23)

$$\begin{aligned} \Psi_1(\alpha_j - 0) &= \Psi_1(\alpha_j + 0), \\ \Psi_2(\alpha_j - 0) &= \Psi_2(\alpha_j + 0), \\ \Psi_3(\alpha_j - 0) &= \Psi_3(\alpha_j + 0) \end{aligned} \tag{4.25}$$

for each $j = 1, ..., n$ and

$$\lambda \sin \alpha_j(\gamma_{j-1} - \gamma_j) + [H(\alpha_j)] = 0; \quad j = 1, ..., n \tag{4.26}$$

where H is the Hamiltonian function defined as

$$\begin{aligned} H &= \Psi_1 \left((n_2 - n_1) \cot \varphi - \left(n_1 + \frac{p}{2}\right) \tan \varphi + \frac{p}{2} \frac{\sin^2 \alpha}{\sin \varphi \cos \varphi} \right) + \\ &+ \Psi_2 \left((m_2 - m_1) \cot \varphi - \frac{1}{k} \left(n_1 + \frac{p}{2}\right) \tan \varphi + \frac{p}{2k} \frac{\sin^2 \alpha}{\sin \varphi \cos \varphi} \right). \end{aligned} \tag{4.27}$$

According to (25) the adjoint variables are continuous at each $\varphi \in (\alpha, \beta)$. Therefore, making use of (27) one can present (26) as

$$\lambda \sin \alpha_j(\gamma_{j-1} - \gamma_j) + \Psi_1(\alpha_j)[n_2(\alpha_j)] + \Psi_2(\alpha_j)[m_2(\alpha_j)] = 0; \quad j = 1, ..., n. \tag{4.28}$$

Dividing the first equation in (21) with the second one gives

$$2n_2 - n_1 = \frac{\Psi_1}{\Psi_2}(2m_2 - m_1). \tag{4.29}$$

On the other hand, from (10) one easily can find

$$n_2 = \frac{n_1}{2} \pm \sqrt{\frac{-3n_1^2}{4} - m_1^2 - m_2^2 + m_1 m_2 + \gamma_j^2} \tag{4.30}$$

for $\varphi \in D_j$ $(j = 0, ..., n)$. Combining (29) and (30) leads to the relations

$$m_2 = \frac{m_1}{2} \pm \frac{\Psi_2}{2} \sqrt{\frac{-3n_1^2 - 3m_1^2 + 4\gamma_j^2}{\Psi_1^2 + \Psi_2^2}} \tag{4.31}$$

for $\varphi \in D_j$ $(j = 0, ..., n)$ and

$$n_2 = \frac{n_1}{2} + \frac{\Psi_1}{\Psi_2}\left(m_2 - \frac{m_1}{2}\right) = \frac{n_1}{2} \pm \frac{\Psi_1}{2}\sqrt{\frac{-3n_1^2 - 3m_1^2 + 4\gamma_j^2}{\Psi_1^2 + \Psi_2^2}} \tag{4.32}$$

for each $\varphi \in (\alpha, \beta)$.

4.4 SHELL OF CONSTANT THICKNESS

Consider the reference shell of constant thickness h_* associated with non-dimensional thickness $\gamma = 1$. In order to investigate the stress state of the shell at yield point load we shall use the lower bound theorem of limit analysis as above.

In the present case we have to minimize the cost function (3) or (14) so that the equilibrium equations (8) and yield condition (10) are satisfyed. Note that in (10) as well as in (30)-(32) $\gamma_j = 1$. Since we have now only one region for the variable φ we can omit subscripts when speaking about D_j, γ_j, ν_j in (10), (19), (21) and (30)-(32).

For the current optimization problem the necessary conditions derived above hold good as well. The only exceptions are (22) and (26) (or (28)) which are associated with variations of parameters γ_j and α_j, respectively.

From (21) making use of (31) and (32) one can find

$$\nu = \pm \cot \varphi \sqrt{\frac{\Psi_1^2 + \Psi_2^2}{4 - 3m_1^2 - 3n_1^2}} \tag{4.33}$$

whereas (31) and (32) take the form

$$m_2 = \frac{m_1}{2} \pm \frac{\Psi_2}{2}\sqrt{\frac{-3n_1^2 - 3m_1^2 + 4}{\Psi_1^2 + \Psi_2^2}} \tag{4.34}$$

and

$$n_2 = \frac{n_1}{2} \pm \frac{\Psi_1}{2}\sqrt{\frac{-3n_1^2 - 3m_1^2 + 4}{\Psi_1^2 + \Psi_2^2}}, \tag{4.35}$$

respectively.

Substituting (33)-(35) in (15) and (19) leads to the equations

$$
\begin{aligned}
n_1' &= \left(-\frac{n_1}{2} \pm \frac{\Psi_1}{2}\sqrt{\frac{4-3n_1^2-3m_1^2}{\Psi_1^2+\Psi_2^2}}\right)\cot\varphi - \left(n_1 + \frac{p}{2}\right)\tan\varphi + \\
&+ \frac{p}{2}\frac{\sin^2\alpha}{\sin\varphi\cos\varphi}; \\
m_1' &= \left(-\frac{m_1}{2} \pm \frac{\Psi_1}{2}\sqrt{\frac{4-3n_1^2-3m_1^2}{\Psi_1^2+\Psi_2^2}}\right)\cot\varphi - \frac{1}{k}\left(n_1 + \frac{p}{2}\right)\tan\varphi + \\
&+ \frac{p}{2k}\frac{\sin^2\alpha}{\sin\varphi\cos\varphi}; \\
p' &= 0; \\
\Psi_1' &= \frac{\Psi_1}{2}\cot\varphi + \left(\Psi_1 + \frac{\Psi_2}{k}\right)\tan\varphi \pm \frac{3n_1}{2}\cot\varphi\sqrt{\frac{\Psi_1^2+\Psi_2^2}{4-3n_1^2-3m_1^2}}; \\
\Psi_2' &= \frac{\Psi_2}{2}\cot\varphi \pm \frac{3m_1}{2}\cot\varphi\sqrt{\frac{\Psi_1^2+\Psi_2^2}{4-3n_1^2-3m_1^2}}; \\
\Psi_3' &= \frac{1}{2}\left(\Psi_1 + \frac{\Psi_2}{k}\right)\left(\tan\varphi - \frac{\sin^2\alpha}{\sin\varphi\cos\varphi}\right).
\end{aligned}
\tag{4.36}
$$

The system of equations (36) is to be integrated under the boundary conditions (11) and (20). The solution of the boundary value problem results in the limit load p and the stress distribution corresponding to the limit state.

4.5 SHELL OF PIECE WISE CONSTANT THICKNESS

Let us consider now a shell of piece wise constant thickness. In addition to the stress resultants n_1, n_2, m_1, m_2 and adjoint variables Ψ_1, Ψ_2, Ψ_3 we have to determine the design parameters α_j $(j = 1, ..., n)$; γ_j $(j = 0, ..., n)$ as well as the Lagrange'ian multipliers λ and ν_j $(j = 0, ..., n)$. It appears that the Lagrange'ian multipliers λ and ν_j can be eliminated from the equations to be solved numerically.

It easily follows from (21) and (31), (32) that

$$\nu_j = \pm \cot\varphi \sqrt{\frac{\Psi_1^2 + \Psi_2^2}{4\gamma_j^2 - 3n_1^2 - 3m_1^2}}; \quad j = 0, ..., n. \tag{4.37}$$

Combining (37) with (22) and (9) leads to the relations

$$\lambda = \pm \frac{2}{v_0} \sum_{j=0}^{n} \gamma_j^2 \int_{D_j} \cot\varphi \sqrt{\frac{\Psi_1^2 + \Psi_2^2}{4\gamma_j^2 - 3n_1^2 - 3m_1^2}} d\varphi \tag{4.38}$$

and

$$\begin{aligned} & \sum_{i=0}^{n} \gamma_i^2 \int_{D_i} \cot\varphi \sqrt{\frac{\Psi_1^2 + \Psi_2^2}{4\gamma_i^2 - 3n_1^2 - 3m_1^2}} d\varphi (\cos\alpha_j - \cos\alpha_{j+1}) - \\ - \quad & v_0 \gamma_j \int_{D_j} \cot\varphi \sqrt{\frac{\Psi_1^2 + \Psi_2^2}{4\gamma_j^2 - 3n_1^2 - 3m_1^2}} d\varphi = 0, \end{aligned} \tag{4.39}$$

where $j = 0, ..., n$. Substituting (38) in (28) one easily obtains

$$\begin{aligned} \pm \quad & \frac{2}{v_0} \sin\alpha_j (\gamma_{j-1} - \gamma_j) \sum_{i=0}^{n} \gamma_i^2 \int_{D_i \cot\varphi} \sqrt{\frac{\Psi_1^2 + \Psi_2^2}{4\gamma_i^2 - 3n_1^2 - 3m_1^2}} d\varphi + \\ + \quad & \Psi_1(\alpha_j)[n_2(\alpha_j)] + \Psi_2(\alpha_j)[m_2(\alpha_j)] = 0 \end{aligned} \tag{4.40}$$

for each $j = 1, ..., n$.

Equations (39) and (40) serve for determination of design parameters γ_j ($j = 0, ..., n$) and α_j ($j = 1, ..., n$), respectively. For given set of α_j, γ_j one can integrate the equilibrium equations (15) and adjoint equations (19) substituting preliminarily quantities ν_j, m_2, n_2 according to relations (37), (31), (32), respectively. When integrating the set (15), (19) boundary conditions (11) and (20) must be taken into account.

Due to their mechanical background the state variables n_1, m_1, p are to be continuous at each point $\varphi = \alpha_j$ ($j = 1, ..., n$). According to (25) the adjoint variables Ψ_1, Ψ_2, Ψ_3 are continuous as well. Therefore the six boundary conditions in (11), (20) admit to solve the current boundary value problem.

The efficiency of the design established can be assessed by the coefficient

$$e = \left(\frac{p}{p_0} - 1 \right) 100\%. \tag{4.41}$$

In (41) p is the load carrying capacity of the optimized shell whereas p_0 stands for the limit load of the reference shell of constant thickness. Note that both, the optimized shell of piece-wise constant thickness and the reference shell, have common weight (material volume) v_0.

4.6 DISCUSSION

The results of calculations are presented in Fig.2-4 and Tables 1-4. Tables 1-4 correspond to the shell with one step in the thickness. Calculations carried out showed that the results depend on the upper bound γ_0 imposed on the thickness. In Table 1 the values of quantities α_1, γ_1, p, p_0 and e are presented for different values of $\alpha_0 = \alpha$ and $a_2 = \beta$ in the case if $\gamma_0 = 1.5$ It can be seen from Table 1 that for fixed outer radius of the shell the load carrying capacity increases when inner radius decreases. However, eventual effectivity of the stepped design of the shell is greater for a narrow annulus of moderately large inner radius. For instance, in the case $\alpha = 0.8$ and $\beta = 1.0$ the limit load increases 35% with respect to that of the shell of constant thickness (Table 1).

In Tables 2 and 3 the design parameters are presented for shells with fixed inner radii. Here one can see the dependence of quantities α_1, γ_1, p, e on the upper bound γ_0. As it might be expected greater values of the upper bound gave more effective optimal designs. It can be seen fron Table 3 that the limit load can be increased 64 %, if $\gamma_0 = 4$ and $\alpha = 0.8$, $\beta = 1.0$.

Generalized stresses m_1, n_1, m_2, n_2 are depicted in Fig. 2-4. It can be seen from Fig. 2-4 that m_1, n_1 are continous over the domain $\phi \in (\alpha, \beta)$ whereas m_2 and n_2 have finite jumps at the cross-sections where thickness has the step. It is some what surprising that n_1 and n_2 are approximately constants in the neighbourhood to the outer edge.

4.7 CONCLUDING REMARKS

An optimization technique has been developed for plastic spherical shells subjected to the uniformly distributed transverse pressure. Material of the shells obeys Mises yield condition. Resorting to the lower bound theorem of limit analysis

Table 4.1: Optimal design for $k = 0.02$ and $\gamma_0 = 1.5$

α_0	α_2	α_1	γ_1	p	p_*	e
0,8	1,0	0,929	0,159	1,399	1,033	35%
0,6	1,0	0,834	0,417	1,557	1,205	29%
0,4	1,0	0,671	0,716	1,722	1,413	22%
0,4	0,8	0,624	0,523	1,663	1,351	23%
0,4	0,6	0,525	0,303	1,429	1,290	11%

Table 4.2: Optimal design for $k = 0.02$, $\alpha_0 = 0.4$ and $\alpha_2 = 0.6$

γ_0	α_1	γ_1	p	e
1,5	0,525	0,3031	1,429	11%
2,0	0,4905	0,3136	1,591	23,3%
2,5	0,47	0,3321	1,6656	29,1%
3,0	0,457	0,3426	1,7111	32,6%
3,5	0,4475	0.3592	1,7411	35%
4,0	0,4415	0,3545	1,7615	36,5%

and variational methods of the theory of optimal control necessary conditions for optimality are derived.

Numerical results are presented for a spherical cap with unique step in the thickness. Calculations carried out showed that the optimization procedure appeared to be more effective in the case of shells resembling to a narrow annulus, e.g. shells with small difference in values of α and β. Similar matter has been revealed earlier in the case of annular plates and shallow shells (see Lellep, Majak (1995)).

Evidently, the optimization technique can be extended to shells of different shape operating in the limit state.

ACKNOWLEDGEMENT

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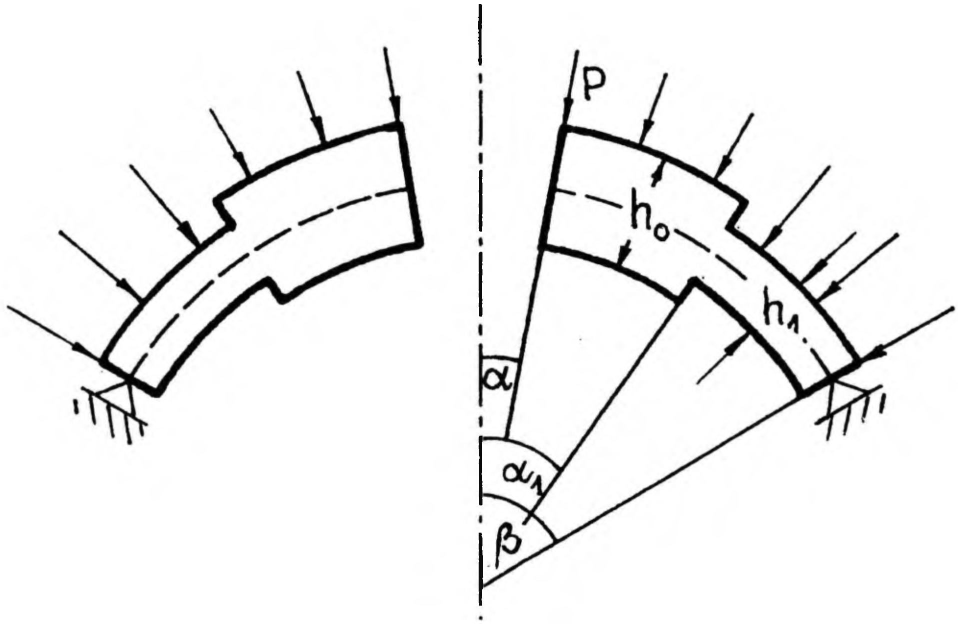

Figure 4.1: Spherical cap

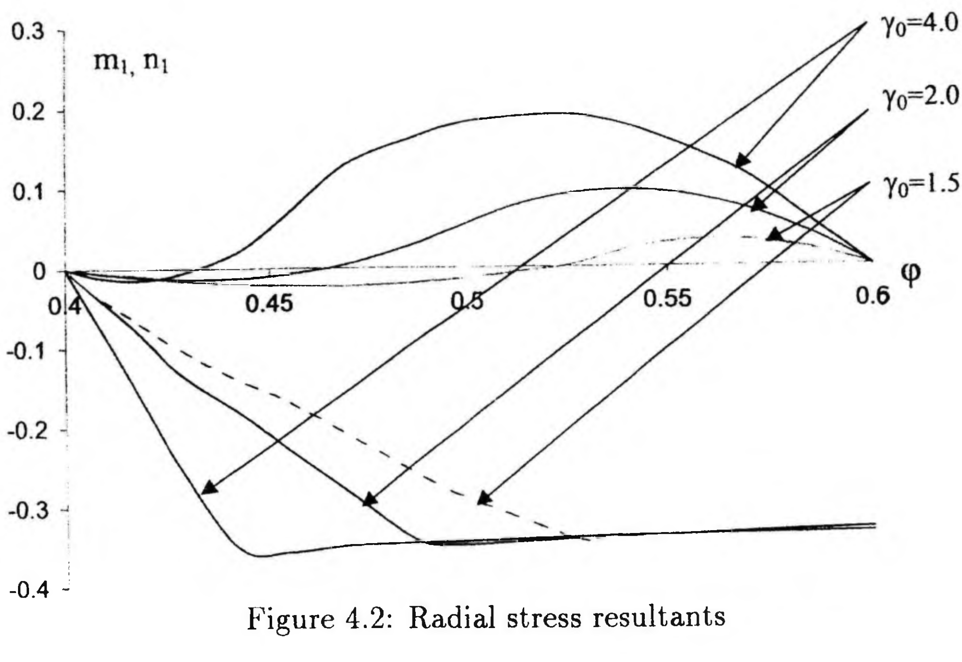

Figure 4.2: Radial stress resultants

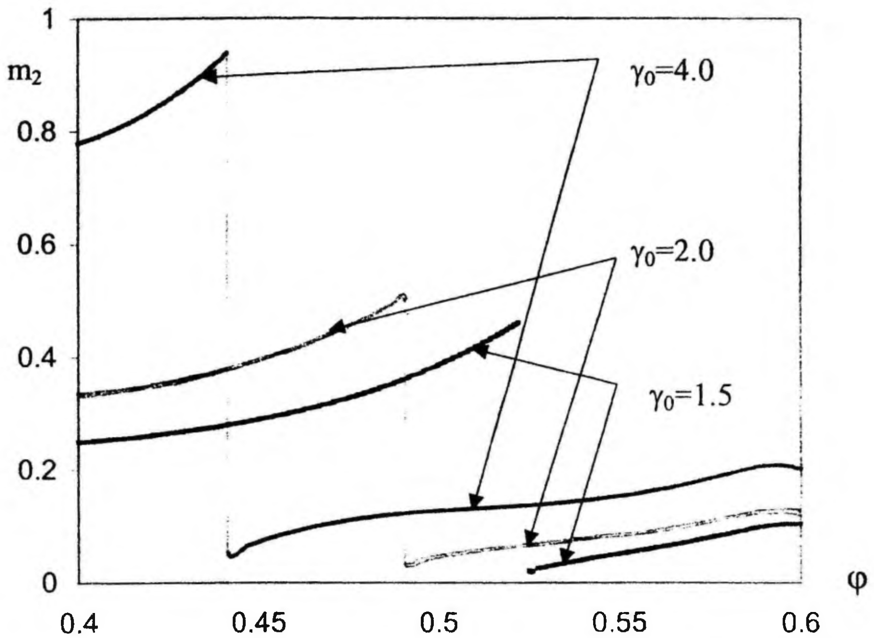

Figure 4.3: Circumferential moment

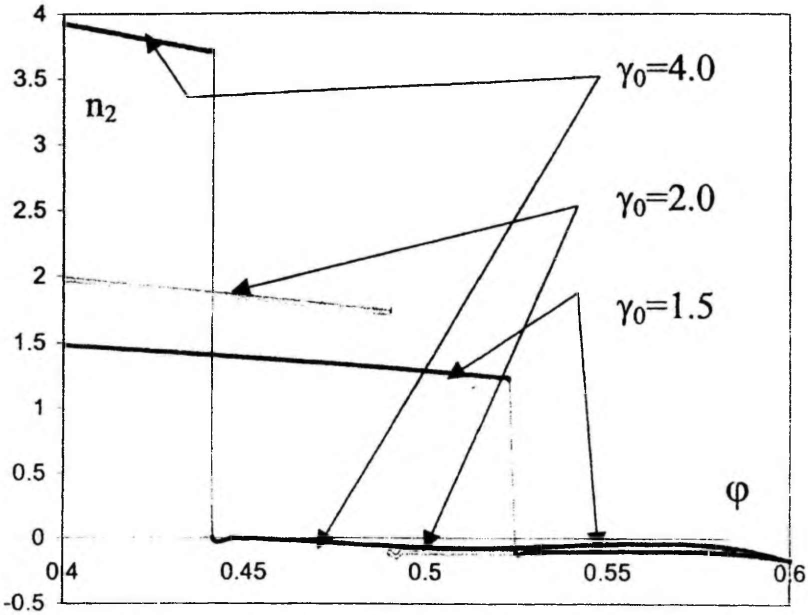

Figure 4.4: Circumferential membrane force

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Table 4.3: Optimal design for $k = 0.02$, $\alpha_0 = 0.8$ and $\alpha_2 = 1.0$

γ_0	α_1	γ_1	p	e
1,5	0,929	0,1594	1,4	35%
2,0	0,893	0,1976	1,5476	49,8%
2,5	0,871	0,2389	1,6169	56,5%
3,0	0,858	0,2476	1,6557	60,3%
3,5	0,848	0.2734	1,6787	62,5%
4,0	0,841	0,2883	1,6937	64%

Table 4.4: Optimal design for $k = 0.02$, $\alpha_0 = 0.2$ and $\alpha_2 = 0.6$

γ_0	α_1	γ_1	p	e
1,5	0,395	0,7078	1,826	24,6%
2,0	0,319	0,7543	1,876	28,1%
2,5	0,284	0,7757	1,892	29,2%
3,0	0,265	0,7857	1,9	29,7%
3,5	0,253	0.7917	1,904	30%
4,0	0,244	0,7996	1,907	30,2%

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SUMMARY

Optimization of plastic spherical shells

In the present work the methods of optimization are developed for spherical shells of piece-wise constant thickness. Shells with various support conditions are considered. The cases of materials obeying Tresca or von Mises yield condition, respectively, are studied in greater detail.

In the introduction a review of existing literature in this area is presented.

In the first chapter an optimal design procedure is developed for stepped rigid-plastic spherical shells. The shells are subjected to the uniformly distributed external pressure. Material of shells obeys the Tresca yield condition and associated flow law. The problems solved herein consist in the maximization of the load carrying capacity under the condition that the material volume of the shell is fixed and in the weight minimization under given load carrying capacity, respectively.

In the second chapter an optimization method regarding to plastic spherical shells pierced with a central hole is presented. The shells under consideration are clamped at the outer edge and absolutely free at the inner edge. The material of shells obeys generalized square yield condition and associated flow rule. The problem of maximization of the load carrying capacity under the condition that the weight (material volume) of the shell is fixed is transformed into a problem of non-linear programming. The latter is solved with the aid of Lagrangeian multipliers. The obtained solution is compared with the optimal solution of the minimum weight problem for given load carrying capacity.

In the third chapter an optimization procedure is developed for shells of von Mises material. It is assumed that the outer edge of the shell is simply supported whereas the inner edge is absolutely free. Resorting to the lower bound theorem of limit analysis the shells with constant and piece-wise constant thickness are considered. The designs of spherical shells corresponding to maximal load carrying capacity are established for given weight. Necessary optimality conditions are derived with the aid of variational methods of the theory of optimal control. The obtained set of equations is solved numerically.

KOKKUVÕTE (Summary in Estonian)

Plastsete sfääriliste koorikute optimeerimine

Reaalse materjali käitumise kirjeldamiseks on loodud palju erisuguseid mudeleid: elastne, elastne-plastne, kalestuv jt. Käesolevas töös on käsitletud ideaalselt jäikplastse materjali mudelit. Ideaalselt jäikplastse materjali korral kuni voolavuspiirini deformatsioonid puuduvad ja see lihtsustab ülesande seadet. Optimeerimisülesannetes on otsitud materjali jaotust, mille korral uurimisobjekti kandevõime saavutab maksimumi etteantud ruumala korral või ruumala saavutab miinimumi fikseeritud kandevõime korral. Lahendamisel on eeldatud, et paksuse jaotus on tükiti konstantne, niisugust konstruktsiooni on lihtsam toota, samuti lihtsustab see optimeerimisülesannet.

Uurimisobjekt on sfääriline koorik, eri osades on vaadeldud nii avausega, kui ka täiskoorikut, mõlemas seades on koorik telgsümmeetriline, st avaus keskel. Kõigis püstitustes on sfäär koormatud ühtlaselt jaotatud väliskoormusega. Erinevad ülesanded on lahendatud mitme kinnitusviisi (nii jäigalt kinnitatud kui ka vabalt toetatud) kooriku jaoks. Kasutatakse nii Tresca kui ka Misese voolavustingimusi.

Teises peatükis uuritakse sfäärilist täiskoorikut ühtlaselt jaotatud ristkoormuse mõju all, koorik on välisservast vabalt toetatud. Eeldatakse, et materjal allub Tresca voolavustingimusele, kasutatakse aproksimatsiooni $N_2 = 0$, $M_2 = M_0$. Püstitatakse ülesanne leida niisugune tükiti konstantne paksuse jaotus, et kandevõime saavutaks maksimumi. Võrdluskoorikuna kasutatakse konstantse paksusega koorikut ja maksimeeritakse kandevõimete suhet. Võrdluskooriku ja muutuva paksusega kooriku kaalud loetakse võrdseks. Paksuse jaotusel on kasutatud ühte astet ja kahesugust paksust, saadud tulemused on esitatud tabelina sõltuvalt sfääri välisnurgast.

Kolmandas peatükis uuritakse avausega sfäärilist koorikut, mis on välisservast jäigalt kinnitatud, siseserv vaba. Koormuseks on ühtlaselt jaotatud väliskoormus. Kasutatakse voolavustingimuse aproksimatsiooni $N_2 = -N_0$, $M_2 = M_0$. Lahendatakse kaks duaalset ülesannet: kandevõime maksimeerimine fikseeritud ruumala korral ja ruumala minimeerimine etteantud kandevõime korral. Paksuse jaotusel on kasutatud kahesugust paksust (üks aste). Numbrilised tulemused on esitatud kuues tabelis sõltuvalt optimeerimisülesande püstitusest ja kooriku geomeetrilisest parameetrist ($k = h/4A$).

Neljandas peatükis uuritakse avausega sfäärilist koorikut, mis on välisservast vabalt toetatud, siseserv vaba. Kasutatakse Misese voolavuspinna aproksimatsiooni. Koorik on koormatud ühtlaselt jaotatud väliskoormusega.

Et töö eri osades on vaadeldud erisuguseid koorikuid erinevate rajatingimustega (kinnitusviisid) ning erisuguseid voolamistingimuse aproksimatsioone, siis pole kahjuks võimalik peatükkide otsene võrdlus. Rajatingimuste erinevused on tingitud paljuski lihtsustustest, nii on Misese voolavustingimuse korral vaadeldud vabalt toetatud, Tresca aproksimatsiooni korral aga jäigalt kinnitatud koorikut. Mõningane võrdlus on siiski võimalik – suurim efekt astme sissetoomisel saavutatakse kolmandas peatükis vabalt toetatud avausega sfäärilise kooriku korral.

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