Topos Semantics for Higher-Order Modal Logic

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Introduction

The connection between algebraic and categorical semantics for S4 modal systems has a long history. Algebraic models for classical propositional S4 modal logic and related systems have already been studied by Tarski [22, 23]. For quantified modal logic the standard approach is through possible worlds semantics, or Kripke frames. A topos-theoretic account of modalities started a few years back with the work of Ghilardi, Reyes, Makkai and others [12, 10, 26, 24, 21]. The present work develops these accounts further w.r.t. higher-order theories. Part of it has been presented in [6], where models of higher-order modal systems in a topos were defined and soundness proved. In addition, we improve on these results by proving two completeness theorems of the system of higher-order logic that was studied in [6]. The first is an elementary completeness theorem w.r.t. models in topoi in general. It is achieved by constructing the syntactic topos $\mathcal{E}_{\mathbb{T}}$ from a higher-order theory $\mathbb{T}$ and defining a generic canonical model in it. The idea follows the construction of the syntactic topos for a higher-order intuitionistic theory [16]. The second is a topological completeness theorem proved abstractly using general topos-theoretic constructions. The idea for that proof follows an idea used in [3] who used it to prove topological completeness of classical higher-order logic w.r.t. models in sheaf categories.

For the second theorem to work we introduce the new notion of *relative model structure* which is studied mostly in sections 2.3, 3.6, and 4.3. The elementary soundness and completeness result concerned models in S4 algebras

$$i : \Omega_{\mathcal{E}} \leftrightarrows H : \tau$$

in a topos $\mathcal{E}$, w.r.t. a complete Heyting algebra H for which the initial frame map i is monic. Many naturally occurring examples are of that kind. On the other hand, a relative model structure is an S4 algebra

$$i : B \leftrightarrows H : \tau,$$

for which the classifying map $\beta : B \to \Omega_{\mathcal{E}}$ of the top element $1 : \top \to B$ of B is a monomorphim. In fact, $\Omega_{\mathcal{E}}$-based structures are instances of this relaxed notion. As such, it will be shown that the canonical model of a higher-order modal theory $\mathbb{T}$ in the syntactic topos $\mathcal{E}_{\mathbb{T}}$ can be embedded as a relative model into a topos of sheaves on a space.

The material is structured in four main parts. In the first one we review a few central notions associated with algebraic modal logic and indicate how internalize these into a topos. We also recall a few aspects of the topos structure, in particular the definition of a subobject classifier which plays crucial role in topos semantics in general. The second part develops the structure that is needed to define a sound notion of model in a category. We chose the hyperdoctrinal approach first propsed by Lawvere [17, 18], as it nicely exhibits the purely algebraic character of the categorical semantics developed. In the third part we revisit the soundness proof and discuss some aspects of the system of higher-order logic studied here, in particular the failure of normal function and propositional extensionality. Moreover, we will show how to develop the familiar Kripke structures from our framework. The material from this chapter is mainly contained in [6]. In the last chapter we finally turn to the completeness results.

We presuppose familiarity with standard category-theoretic notions and results, including topos theory. References for basic categorical background include [19] and [2]. The standard reference for topos-theory is [15].

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1 Preliminary Concepts

1.1 Modal Operators Through Adjoints

Throughout the following work we will be concerned exclusively with modal operators that satisfy the rules of an S4 modal logic:

(M1) $\dfrac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}$

(M2) $\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$

(M3) $\top \vdash \Box\top$

(M4) $\Box\varphi \vdash \varphi$

(M5) $\Box\varphi \vdash \Box\Box\varphi$

The reason to focus on S4 modal operators lies with their mathematically rather well-behaved nature and in their admitting many significant examples, as will be seen presently.

Semantically, a classical S4 modal logic corresponds to a Boolean algebra A with an operator $\Box$ that satisfies the same rules as the modal operator $\Box$ when the syntactic entailment relation $\vdash$ is replaced by the partial ordering $\leq$ on A. The rule M1 expresses that $\Box$ is an order-preserving map. The rules M2 and M3 say that $\Box$ preserves finite meets, while the rules M4 and M5 express that $\Box$ is a comonad on A. The category of coalgebras for the comonad $\Box$ is the set of fixed points for $\Box$

$$\Box A = \{x \in A \mid \Box x = x\}.$$

This set is in general only a Heyting algebra. Finite meets are computed as in A, as they are $\Box$-stable. However, implication between $\Box$-stable elements is not computed in A. This is essentially due to the fact that the algebraic converse of the K-axiom does not hold in general, so that $x \Rightarrow y$ is not in general $\Box$-stable. Instead, $\Box(x \Rightarrow y)$ works, as can be easily verified. That $\Box A$ does not generally admit complements, is most readily seen by a counterexample. Probably the easiest concerns the four element Boolean algebra

$$\begin{tikzcd} & 11 & \\ 01 \arrow[ur] & & 10 \arrow[ul] \\ & 00 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

on which the modal operator is explicitly defined as

$$\Box x = \begin{cases} x, & \text{if } x \in \{00, 10, 11\} \\ 00 & \text{ow.} \end{cases}$$

The inclusion $\Box A \hookrightarrow A$ thus looks like

$$\begin{tikzcd}
11 & & & & 11 & \\
& 10 \arrow[ul] & \hookrightarrow & 01 \arrow[ur] & & 10 \arrow[ul] \\
00 \arrow[ur] & & & & 00 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The left-hand structure evidently fails to have complements. The same example illustrates that i may not preserve implication. On the one hand,

$$10 \Rightarrow 00 = 00$$

in $\Box A = \mathbf{3}$, while

$$10 \Rightarrow 00 = 01$$

in $A = \mathbf{4}$.[1]

There is an obvious map $\tau : A \to \Box A$ defined by

$$\tau(x) = \Box x,$$

for any $x \in A$. This map preserves the order and finite meets by definition. More importantly, τ is right adjoint to the inclusion $i : \Box A \hookrightarrow A$, since

$$i(x) \leq y \quad \text{iff} \quad x \leq \tau(y),$$

for any $x \in \Box A$ and $y \in A$.

The algebraic structure of $\Box A$ can now be formulated more precisely in terms of the maps i and τ. For $x, y \in \Box A$, define:

$$x \wedge' y = \tau(i(x) \wedge i(y))$$

$$x \Rightarrow' y = \tau(i(x) \Rightarrow i(y))$$

$$x \vee' y = \tau(i(x) \vee i(y)).$$

A broad and well-studied class of examples is provided by topological spaces X. The power set $\mathcal{P}(X)$ is a Boolean algebra under the usual set-theoretic operations. The modal operator $\Box$ is defined by

$$\Box U = \bigvee \{V \in \mathcal{O}(X) \mid V \subseteq U\},$$

[1] Recall that for any complete Heyting algebra H the implication $x \Rightarrow y$ can be expressed by $\bigvee\{z \in H \mid z \wedge x \leq y\}$.

i.e. the largest open set within U, also known as the *interior* of U. More precisely, the interior operation is the right adjoint $\tau : \mathcal{P}(X) \to \mathcal{O}(X)$ to the inclusion $\mathcal{O}(X) \hookrightarrow \mathcal{P}(X)$.

Not only does any modal operator $\Box$ on a Boolean algebra define an adjunction

$$i : \Box A \leftrightarrows A : \tau,$$

there is also a converse. Consider a Boolean algebra A, a distributive lattice B, and an adjunction

$$i : B \leftrightarrows A : \tau$$

($i \dashv \tau$) where i is injective and in addition preserves finite meets. This adjunction determines a modal operator $i\tau : A \to A$. The rules M1-3 are satisfied because both i and τ preserve the order and finite meets. The rules M4-5 in turn hold by properties of the adjunction $i \dashv \tau$; the inequality $i\tau(x) \leq x$ is the counit, while $\tau(x) = \tau i\tau(x)$ is one of the triangle identities. Note moreover that $\tau i = 1$, since the other triangle identity $i(x) = i\tau i(x)$ implies $x = \tau i(x)$, because i is injective. Finally, this makes i into an order embedding, since if $i(x) \leq i(y)$ in A, then $x = \tau i(x) \leq \tau i(y) = y$ in B.

Although B was not assumed to be a Heyting algebra, one may define an implication for $x, y \in B$ by setting

$$x \to y := \tau(i(x) \Rightarrow i(y)),$$

where $\Rightarrow$ is the implication of A:

$$\begin{aligned} z \leq \tau(i(x) \Rightarrow i(y)) \ &\text{iff} \ \ i(z) \leq i(x) \Rightarrow i(y) \\ &\text{iff} \ \ i(z) \wedge i(x) \leq i(y) \\ &\text{iff} \ \ i(z \wedge x) \leq i(y) \\ &\text{iff} \ \ z \wedge x \leq y, \end{aligned}$$

using the properties mentioned before. Note that this uniquely determines the Heyting implication of B in any case. Of course, the assumption that i is injective and preserves finite limits is essential in all that. Note, finally, that the argument remains valid if A is merely assumed to be a Heyting algebra. The proof of the following is then essentially obvious.

Proposition 1.1. *There is a one-to-one correspondence between S4 modal operators on a Heyting algebra A and adjunctions*

$$i : B \leftrightarrows A : \tau$$

where $i \dashv \tau$, B is a Heyting algebra, and such that i is injective and preserves finite limits.

Note finally that for any complete A and B the right adjoint τ can be described by the formula

$$\tau(x) = \bigvee\{y \in B \mid i(y) \leq x\}. \tag{1}$$

Since adjoints are unique, the right adjoint, if it exists, must be given by that formula.

Example 1.1. Kripke models for propositional S4 are special kinds of algebraic models. By definition, a Kripke structure consists of a poset $(A, \leq)$ on which the model of a modal theory is defined by suitable forcing conditions for each logical connective. The expression '$x \Vdash \varphi$' is to mean that φ 'holds' at stage/world/point x. Given an extension for all the basic propositional variables, one recursively extends the definition to all the formulas, for instance

$$x \Vdash \varphi \wedge \psi \;\text{ iff }\; x \Vdash \varphi \;\text{ and }\; x \Vdash \psi.$$

In particular, the modal formulas are interpreted as

$$x \Vdash \Box\varphi \;\text{ iff }\; y \Vdash \varphi, \;\text{ for all }\; y \geq x.$$

Every formula φ determines a set $U_\varphi \subseteq A$ where it is defined to hold:

$$x \in U_\varphi \;\text{ iff }\; x \Vdash \varphi.$$

One then shows by induction that

$$\begin{gathered} x \in U_{\varphi\wedge\psi} \;\text{ iff }\; x \in U_\varphi \cap U_\psi \\ x \in U_{\varphi\Rightarrow\psi} \;\text{ iff }\; x \in U_\varphi \Rightarrow U_\psi \\ \vdots \\ x \in U_{\Box\varphi} \;\text{ iff }\; x \in \Box U_\varphi, \end{gathered}$$

where

$$\Box U_\varphi := \bigcup\{V \subseteq U_\varphi \mid y \in V \;\&\; y \leq z \;\text{ implies }\; z \in V\}.$$

The operation $U \mapsto \Box U$ defines a map $\mathcal{P}(A) \to \mathcal{P}(A)$ on the whole powerset that satisfies the rules of an S4 operator on the Boolean algebra $\mathcal{P}(A)$. As such, it may be factored as adjunction

$$i : \Box\mathcal{P}(A) \leftrightarrows \mathcal{P}(A) : \tau.$$

The set of box stable elements for the modal operator $\Box$ on $\mathcal{P}(A)$ is isomorphic to the set of upward closed subsets $\mathcal{U}(A)$ of the poset $(A, \leq)$. The left adjoint $i : \mathcal{U}(A) \to \mathcal{P}(A)$ is the inclusion while the right adjoint τ is given by

$$\tau(U) = \bigcup\{V \in \mathcal{U}(A) \mid V \subseteq U\}, \tag{2}$$

in accordance with the formula (1). Thus, a Kripke model for S4 on a poset $(A, \leq)$ may equivalently be given purely algebraically by an adjunction

$$i : \mathcal{U}(A) \leftrightarrows \mathcal{P}(A) : \tau$$

by recursively assigning to each formula φ an element $U_\varphi \in \mathcal{P}(X)$ and interpreting the modal operator through the composite $i\tau$. We will return to Kripke models in section 3.5.

1.2 Modal operators in a Category

Eventually, we will be concerned not with set-based Heyting algebras, but Heyting algebras in an arbitrary topos $\mathcal{E}$. The definition of a Heyting algebra in $\mathcal{E}$ is a suitable diagrammatic version of the standard one and makes sense in any category with finite limits. It is, by definition, an object H in $\mathcal{E}$ along with maps

$$H \times H \xrightarrow{\wedge, \vee, \Rightarrow} H \xleftarrow{\top, \bot} 1$$

that provide the Heyting structure on H. To this end these maps are to make certain diagrams commute, corresponding to the usual equations defining a Heyting algebra. For instance, the axiom $x \wedge \top = x$, for any $x \in H$, corresponds diagrammatically to the requirement that the diagram

$$\begin{tikzcd} H \times 1 \arrow[r, "1\times\top"] \arrow[dr, "\pi_1"'] & H \times H \arrow[d, "\wedge"] \\ & H \end{tikzcd}$$

commutes. The correspondence between the usual equational definition and commutative diagrams in a topos $\mathcal{E}$ can be made precise using the *internal language* of $\mathcal{E}$.[2]

The induced partial ordering on H is constructed as the equalizer

$$E \rightarrowtail H \times H \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} H,$$

corresponding to the usual definition

$$x \leq y \;\text{ iff }\; x \wedge y = x.$$

Equivalently, it is the pullback of $\top : 1 \to H$ along the implication $H \times H \xrightarrow{\Rightarrow} H$, corresponding to the rule $\top \leq x \Rightarrow y$ iff $x \leq y$. It may be shown that

[2] Cf. e.g. [20].

the relation E thus defined satisfies the diagrammatic versions of a partial ordering. For instance, reflexivity means that the diagonal $\Delta := \langle 1_H, 1_H \rangle : H \to H \times H$ factors through the subobject $E \rightarrowtail H \times H$.

The notion of homomorphism also internalizes. Given posets A and B in $\mathcal{E}$ with orderings $\leq_A \rightarrowtail A \times A$ and $\leq_B \rightarrowtail B \times B$, respectively, an arrow $f : A \to B$ is said to preserve the order, if there is a factorization as in the following:

$$\begin{tikzcd}
\leq_A \arrow[r, dotted] \arrow[d, tail] & \leq_B \arrow[d, tail] \\
A \times A \arrow[r, "f \times f"'] & B \times B
\end{tikzcd}$$

This directly expresses that $x \leq_A y$ implies $f(x) \leq_B f(y)$. Preservation of algebraic structure may be similarly expressed using commutative diagrams. For instance, provided the required structure is present, the following two diagrams describe preservation of finite meets:

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "\top"'] & 1 \arrow[d, "\top"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \times A \arrow[r, "f \times f"] \arrow[d, "\wedge"'] & B \times B \arrow[d, "\wedge"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

One may also define an adjunction $f : A \leftrightarrows B : g$ between posets in $\mathcal{E}$. This may be achieved by saying that $f \dashv g$, if in the internal language of $\mathcal{E}$ it holds that $f(x) \leq y$ iff $x \leq g(y)$. Diagrammatically, this is most easily expressed by requiring the unit and counit inequalities. That is to say, requiring that the map $\langle fg, 1_B \rangle : B \to B \times B$ factors through $\leq_B$ and that $\langle 1_A, gf \rangle : A \to A \times A$ factors through $\leq_A$.

The S4 rules may similarly be expressed by appropriate diagrams. For any Heyting algebra H in $\mathcal{E}$, a map $b : H \to H$ is an S4 modal operator if it is a finite meet preserving map of posets that in addition satisfies $bb = b$ and such that the map

$$H \xrightarrow{\langle b, 1 \rangle} H \times H$$

factors through the partial ordering on H. With this we have:

Proposition 1.2. *For any topos $\mathcal{E}$, there is a one-to-one correspondence between modal operators on a Heyting algebra A and adjunctions*

$$i : B \leftrightarrows A : \tau$$

in $\mathcal{E}$ such that $i \dashv \tau$ and i is a finite limit preserving monomorphism.

Proof. Given such an adjunction, it is readily seen that the composite $i\tau$ defines a modal operator much as in the set case. On the other hand, given an S4 operator $b : H \to H$, define in the internal language an object $\Box H = \{x : H \mid x = bx\} \rightarrowtail H$. The inclusion will be the structure map i. By Kripke-Joyal forcing, the map $b : H \to H$ factors through the subobject $\Box H$ if and only if for all maps $z : Z \to H$, it holds that $bbz = bz$. This is indeed the case by assumption on b. Hence b factors as $b = i\tau$, for some map $\tau : H \to \Box H$.

The object $\Box H$ is a Heyting algebra. As to the top, note that $\top : 1 \to H$ satisfies $\top = b\top$, by assumption on b, and hence factors through i. The factorization thus constructed is the pullback of $\top : 1 \to H$ along i, which makes it a top element of $\Box H$. The meet operation is the composite

$$\Box H \times \Box H \xrightarrow{i\times i} H \times H \xrightarrow{\wedge} H \xrightarrow{\tau} \Box H.$$

In a similar way one obtains the other binary operations, in the same way as were defined in the set-case before. The partial ordering similarly arises by pulling back $\leq_H \rightarrowtail H \times H$ along $i \times i$.

Moreover, $\tau i = 1$, because i factors through $\Box H$ via the identity. Hence, by definition of Kripke-Joyal forcing again (regarding the generalized element i), we get $i(x) = bi(x) = i\tau i(x)$ in the internal language, and so $1 = \tau i$, because i is monic. Therefore also $i \dashv \tau$. The counit is simply the condition $b \leq 1$, while the unit is provided by the identity $1 = \tau i$.

While the map i preserves all finite joins because it is a left adjoint, it moreover preserves finite limits. It preserves the top by construction. It moreover preserves finite meets, as through the following commutative diagram:

$$\begin{tikzcd}
& H \times H \arrow[r, "\wedge"] \arrow[d, "\tau\times\tau"] & H \arrow[d, "\tau"] \\
\Box H \times \Box H \arrow[ur, "i\times i"] \arrow[dr, "i\times i"'] & \Box H \times \Box H \arrow[d, "i\times i"] & \Box H \arrow[d, "i"] \\
& H \times H \arrow[r, "\wedge"'] & H
\end{tikzcd}$$

where the square commutes, because $b = i\tau$ preserves meets of H by assumption.

$\square$

1.3 Complete Heyting Algebras in a Topos

By definition, for any topos $\mathcal{E}$, a Heyting algebra H in $\mathcal{E}$ is complete if and only if for any object I in $\mathcal{E}$, the diagonal map $\Delta_I : H \to H^I$ obtained by exponentiation of the unique map $I \to 1$ has both a left and a right adjoint. Set-theoretically, exponentiation by a map f corresponds to the operation of precomposition with f. In the case $I \to 1$, this reads as

$$\Delta_I(x)(i) = x, \quad \text{for all} \;\; x \in H, i \in I.$$

Here we use that $H^1 \cong H$ and that maps $1 \to H$ from the singleton 1 correspond to elements $x \in H$. Its right and left adjoints $\forall_I$ are given by

$$\forall_I(f) = \bigwedge_{i \in I} f(i),$$

$$\exists_I(f) = \bigvee_{i \in I} f(i),$$

resp. This directly follows from the adjunction, where we use the pointwise ordering on H^I

$$\Delta_I(x) \leq f \;\text{ iff }\; x \leq \forall_I(f),$$
$$\exists_I(f) \leq x \;\text{ iff }\; f \leq \Delta_I(x).$$

The condition $\Delta_I(x) \leq f$ means nothing other than that $x \leq f(i)$, for all $i \in I$; dually for $f \leq \Delta_I(x)$. The biconditionals thus precisely capture the definition of big joins and meets.

The definition might also be seen from a more general viewpoint. Replacing a complete Heyting algebra by a category $\mathbf{C}$, and I by any indexing category, then $\mathbf{C}$ has I-indexed (co)limits if and only if the functor $\Delta_I : \mathbf{C} \to \mathbf{C}^I$ has a (left) right adjoint. This connection is more than an analogy, since every Heyting algebra is a category and every set I is a discrete category, and joins and meets are precisely colimits and limits in an ordered category.

A map $f : A \to B$ between complete Heyting algebras is said to preserve arbitrary meets and joins, if the following commute, for any object I in $\mathcal{E}$:

$$\begin{tikzcd} A^I \arrow[r, "\forall_I"] \arrow[d, "f^I"'] & A \arrow[d, "f"] \\ B^I \arrow[r, "\forall_I"'] & B \end{tikzcd} \qquad \begin{tikzcd} A^I \arrow[r, "\exists_I"] \arrow[d, "f^I"'] & A \arrow[d, "f"] \\ B^I \arrow[r, "\exists_I"'] & B \end{tikzcd}$$

With these observations, H^I is a complete Heyting algebra in its own right whose structure is obtained by exponentiating the structure H with I. For sets, this is the ususal pointwise definition. For instance, the meet map

$$\wedge_I : A^I \times A^I \to A^I$$

is the composite

$$A^I \times A^I \cong (A \times A)^I \xrightarrow{\wedge^I} A^I.$$

Since the functor $(-)^I$ preserves limits all the required commutative diagrams for a meet operation remain commutative. The structural diagonal map $H^I \to (H^I)^J$, for any J, is given by exponentiating Δ_J as in

$$(\Delta_J)^I : H^I \to (H^J)^I \cong (H^I)^J.$$

It is, up to the isomorphism $(H^I)^J \cong H^{I\times J}$, the same as the map $H^I \to H^{I\times J}$ obtained by exponentiating with the projection $I \times J \to I$.

The right and left adjoints to this map are

$$(H^I)^J \cong (H^J)^I \xrightarrow{(\forall_J)^I/(\exists_J)^I} H^I$$

resp. Note also that since the functor $(-)^I$ preserves finite limits the (pointwise) ordering on H^I is the pulllback of $\top^I : 1 \to H^I$ along $\Rightarrow^I$.

Each map $\Delta_I : A \to A^I$ is a map of complete Heyting algebras. For a binary operation β on A observe that each square commutes:

$$\begin{tikzcd}
A \times A \arrow[r, "\beta"] \arrow[d, "\Delta_I \times \Delta_I"'] & A \arrow[d, "\Delta_I"] \\
A^I \times A^I \arrow[r, "\beta^I"'] & A^I
\end{tikzcd}$$

This is because the maps

$$\Delta_I : A \times A \to (A \times A)^I$$

and

$$\Delta_I \times \Delta_I : A \times A \to A^I \times A^I$$

coincide up to the isomorphism $(A \times A)^I \cong A^I \times A^I$. But each square

$$\begin{tikzcd}
A \times A \arrow[r, "\beta"] \arrow[d, "\Delta_I"'] & A \arrow[d, "\Delta_I"] \\
(A \times A)^I \arrow[r, "\beta^I"'] & A^I
\end{tikzcd}$$

commutes by construction of Δ_I, as in the following, where the η's are the unit of the adjuntion $(-) \times I \dashv (-)^I$:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] & (A \times I)^I \arrow[r, "\pi^I"] & A^I \\
A \times A \arrow[u, "\beta"] \arrow[r, "\eta_{A\times A}"'] & ((A \times A) \times I)^I \arrow[u, "(\beta\times 1)^I"] \arrow[r, "\pi^I"'] & (A \times A)^I \arrow[u, "\beta^I"']
\end{tikzcd}$$

Moreover, any map Δ_I also preserves arbitrary indexed joins and meets, because it has both a left and a right adjoint, by the assumption that A is a complete Heyting algebra.

Propositionally speaking, H^I should be viewed as the object of predicates varying over I. The map Δ_I corresponds to variable weakening. It maps a proposition $x \in H$ to a predicate that doesn't actually vary over I, i.e. has a constant truth value x. Similarly, for the maps $(\Delta_J)^I : H^I \to (H^I)^J \cong H^{I\times J}$. The adjointness

$$\exists_I \dashv \Delta_I \dashv \forall_I$$

ensures the maps behave like quantifiers. That is to say,

$$\forall_I(\varphi) = \top \quad \text{iff} \quad \varphi(i) = \top, \quad \text{for all} \quad i \in I,$$

and

$$\exists_I(\varphi) = \top \quad \text{iff} \quad \varphi(i) = \top, \quad \text{for some} \quad i \in I.$$

Note that since Δ_I is a map of complete lattices its adjoints may alternatively described as

$$\forall_I(f) = \bigvee\{a \in H \mid \Delta_I(a) \leq f\},$$

$$\exists_I(f) = \bigwedge\{a \in H \mid f \leq \Delta_I(a)\}.$$

This formulation lends itself to describe examples in other contexts when a suitable definition of Δ_I is known.

Example 1.2. Again, it helps to bear in mind classical two-valued semantics in **Sets**, where $H = \mathbf{2}$ is the two-element set with its usual ordering.[3] Then, for any I, J, and $U \subseteq I$

$$(\Delta_J)^I(U) = U \times J,$$

i.e. inverse image along the projection $I \times J \to I$. When $I = 1$, then $\Delta_J(x)$ is either all of J of empty, depending on whether x is all of 1 or empty. The

[3] For convenience we may consider $\mathbf{2}$ as the powerset of a singleton 1. Then the top element is the subset $1 \subseteq 1$, and the bottom element is the empty set.

adjoints are computed according to the formulas above as follows. For any $V \subseteq I \times J$, where we write π^* for $(\Delta_J)^I$, and $\forall_J/\exists_J$ for its adjoints:

$$\begin{aligned}\forall_J(V) &= \bigcup\{U \subseteq 2^I \mid \pi^*(U) \subseteq V\}\\ &= \bigcup\{U \subseteq 2^I \mid U \times J \subseteq V\}\\ &= \{x \in I \mid \text{ for all } \ y \in J : (x, y) \in V\}.\\ &= \{x \in I \mid \pi^*(x) \cap V = J\}.\end{aligned}$$

Dually for the existential quantifier:

$$\begin{aligned}\exists_J(V) &= \bigcap\{U \subseteq 2^I \mid V \subseteq \pi^*(U)\}\\ &= \bigcap\{U \subseteq 2^I \mid V \subseteq U \times J\}\\ &= \{x \in I \mid \text{ there is } \ y \in J : (x, y) \in V\}.\\ &= \{x \in I \mid \pi^*(x) \cap V \neq \emptyset\}.\end{aligned}$$

For a unary predicate this gives the conditions from before. For then, if $\pi : J \to 1$ is the projection, $\pi^*(x) = J$ iff $x = 1$ and empty otherwise. Hence, for $V \subseteq J$,

$$\forall_J(V) = 1 \ \text{ iff } \ V = J,$$

and

$$\exists_J(V) = 1 \ \text{ iff } \ V \neq \emptyset.$$

Example 1.3. Consider $\mathbf{Sets}^{\mathbf{C}^{op}}$, for a small category $\mathbf{C}$. Products in $\mathbf{Sets}^{\mathbf{C}^{op}}$ are computed pointwise. In particular, a Heyting algebra H in $\mathbf{Sets}^{\mathbf{C}^{op}}$ has pointwise natural structure. That is to say, each $H(C)$, for C in $\mathbf{C}$, is a Heyting algebra in such a way that e.g. for all binary operations $\star$ on H, $H(f) \circ \star_D = \star_C \circ (H(f) \times H(f))$, for any arrow $f : C \to D$ in $\mathbf{C}$. This is because the structure maps, being arrows in $\mathbf{Sets}^{\mathbf{C}^{op}}$, are natural transformations. Hence, for each $f : C \to D$ in $\mathbf{C}$, the map $H(f)$ preserves the Heyting structure. The following sums up what it means to be a complete Heyting algebra in $\mathbf{Sets}^{\mathbf{C}^{op}}$.[4]

Because we will need it later, we spell out the definition of the complete join and meet maps. Recall that exponentials are computed by the formulas

$$\begin{aligned}H^I(C) &= \mathrm{Hom}(\mathbf{y}C \times I, H)\\ H^I(f) &: \eta \mapsto \eta \circ (\mathbf{y}f \times 1_I),\end{aligned}$$

where $\mathbf{y}C$ denotes the contravariant functor $\mathrm{Hom}_{\mathbf{C}}(-, C)$. The induced Heyting structure on H^I is the pointwise one at each component. In particular, for any $\eta, \mu : \mathbf{y}C \times I \to H$,

$$\begin{aligned}\eta \leq \mu \text{ (in } H^I(C)) \ &\text{ iff } \ \eta_D \leq \mu_D, \text{ for each } \ D \in \mathbf{C}\\ &\text{ iff } \ \eta_D(f, b) \leq \mu_D(f, b) \ \text{(in } H(D)), \text{for each } f : D \to C, b \in I(D).\end{aligned}$$

[4] [J,C1.6.9]

Since we are mainly interested in adjoints between ordered structures, for any two order-preserving maps $\eta : H \leftrightarrows G : \mu$ between internal partial orderings H, G in $\mathbf{Sets}^{\mathbf{C}^{op}}$, $\eta \dashv \mu$ means that $\eta_C \dashv \mu_C$ at each component C. That is to say

$$\eta_C(x) \leq y \ \text{ iff } \ x \leq \mu_C(y),$$

for all $x \in H(C)$, $y \in G(C)$.

The natural transformation $\Delta_I : H \to H^I$ (henceforth Δ) determines for each $x \in H(C)$ a natural transformation $\Delta_C(x) : \mathbf{y}C \times I \to H$ with components

$$\Delta_C(x)_D(f, a) = H(f)(x).$$

Its right adjoint $\forall_I : H^I \to H$ (henceforth $\forall$) has components, for any $\eta \in \mathrm{Hom}(\mathbf{y}C \times I, H)$,

$$\begin{aligned} \forall_C(\eta) &= \bigvee \{s \in H(C) \mid \Delta_C(s) \leq \eta\} \\ &= \bigvee \{s \in H(C) \mid H(f)(s) \leq \eta_D(f, b), \text{ for all } f : D \to C, b \in I(D)\}, \end{aligned}$$

where the join is taken in $H(C)$. Dually, the left adjoint $\exists$ of Δ has components

$$\begin{aligned} \exists_C(\eta) &= \bigwedge \{s \in H(C) \mid \eta \leq \Delta_C(s)\} \\ &= \bigwedge \{s \in H(C) \mid \eta_D(f, b) \leq H(f)(s), \text{ for all } f : D \to C, b \in I(D)\}. \end{aligned}$$

Lastly, each $H(C)$ really is a complete Heyting algebra in the usual sense of having arbitrary set-indexed meets and joins (so the previous definitions of $\forall$ and $\exists$ actually make sense). For any set J, the right adjoint $\forall_J : H(C)^J \longrightarrow H(C)$ can be found as follows. Consider the constant J-valued functor ΔJ on $\mathbf{C}$ (and constant value 1_J on arrows in $\mathbf{C}$). For any C in $\mathbf{C}$, there is an isomorphism

$$\mathrm{Hom}_{\mathbf{Sets}}(J, HC) \cong \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathbf{y}C \times \Delta J, H)$$

(natural in J and H). Given a function $h : J \to HC$, define a natural transformation $\nu h : \mathbf{y}C \times \Delta J \to H$ to have components $(\nu h)_D(g, a) = H(g)f(a)$. Conversely, given a natural transformation η on the right, define a function $f\eta : J \to HC$ by $f\eta(a) = \eta_C(1_C, a)$. These assignments are mutually inverse. Moreover, the map that results from composing $\Delta_J : HC \to H^{\Delta J}(C)$ with that isomorphism is computed as

$$f(\Delta_C(x))(a) = \Delta_C(x)_C(1_C, a) = H(1_C)(x) = x,$$

so that for any $x \in HC$, $\Delta_C(x)$ is the constant x-valued map on J. This justifies taking the right adjoint to Δ_J as the sought right adjoint of the diagonal map $HC \to H(C)^J$.

Indeed, for exponents ΔJ the formula for the right adjoint to Δ_C, for instance, takes the familiar form met in the previous example

$$\forall_C(\eta) = \forall_J(f\eta) = \bigwedge_{a\in J} f\eta(a) = \bigwedge_{a\in\Delta J(C)} \eta_C(1_C, a),$$

or

$$\forall_J(h) = \forall_C(\nu h) = \bigwedge_{a\in\Delta J(C)} (\nu h)_C(1_C, a) = \bigwedge_{a\in J} h(a),$$

respectively. □

As for any equationally definable structure, if H is a Heyting algebra in $\mathcal{E}$, then so is the set

$$\mathrm{Hom}_{\mathcal{E}}(A, H),$$

for any object A in $\mathcal{E}$. For instance, the top element is the composite

$$A \xrightarrow{!} 1 \xrightarrow{\top} H.$$

The binary operations $*$ are given, for any two maps $f, g : A \rightrightarrows H$, by the composites

$$A \xrightarrow{\langle f,g\rangle} H \times H \xrightarrow{*} H,$$

resp. The partial ordering is the relation

$$\mathrm{Hom}_{\mathcal{E}}(A, E) \hookrightarrow \mathrm{Hom}_{\mathcal{E}}(A, H \times H) \cong \mathrm{Hom}_{\mathcal{E}}(A, H) \times \mathrm{Hom}_{\mathcal{E}}(A, H),$$

obtained by composition with the ordering relation $E \hookrightarrow H \times H$ in $\mathcal{E}$.

When $\mathcal{E} = \mathbf{Sets}$, then this is the usual pointwise definition of the operations on the set of functions into H. For any arrow $f : A \to B$ in $\mathcal{E}$, there is moreover a function

$$f^* : \mathrm{Hom}_{\mathcal{E}}(B, H) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(A, H),$$

defined by precomposition with f.

1.4 The Subobject Classifier of a Topos

Although the notion of subobject classifier is at the very core of the definition of a topos and standard material, we will recall some of its structure in detail, because we will refer to it extensively in the later parts. At the same time it serves as an example of a complete Heyting algebra in a topos.

To begin with, recall that for each object A in $\mathcal{E}$, the set $\mathrm{Sub}_{\mathcal{E}}(A)$ of subobjects of A is defined to contain equivalence classes of monomorphisms $U \rightarrowtail A$, where two monomorphisms $m : U \rightarrowtail A$ and $n : V \to A$ are equivalent if and only if there is a, necessarily unique, isomorphism $i : U \to V$ such that

$$n \circ i = m,$$

as in

$$\begin{tikzcd}
U \arrow[rr, "i"] \arrow[dr, "m"'] & & V \arrow[dl, "n"] \\
& A &
\end{tikzcd}$$

We will treat these equivalence classes silently, in that we will refer to a subobject in terms of their representing monomorphisms. The set $\mathrm{Sub}_{\mathcal{E}}(A)$ is partially ordered by setting

$$m \leq n \quad \text{iff} \quad \text{there exists } k : U \to V \text{ such that } n \circ k = m.$$[5]

Moreover, we recall that for any arrow $f : B \to A$ the operation of pullback along f is a monotone map of posets

$$f^* : \mathrm{Sub}_{\mathcal{E}}(A) \to \mathrm{Sub}_{\mathcal{E}}(B).$$

The concept of subobject classifier is part of the definition of $\mathcal{E}$. It is, by definition, an object that admits, for any object C in $\mathcal{E}$, an isomorphism

$$\mathrm{Sub}_{\mathcal{E}}(C) \cong \mathrm{Hom}_{\mathcal{E}}(C, \Omega_{\mathcal{E}}),$$

natural in C w.r.t. to pullback of subobjects on the left-hand side and composition on the right. Equivalently, there is a distinguished subobject $\top : 1 \to \Omega_{\mathcal{E}}$ with the property that for any subobject $m : U \rightarrowtail C$ there exists a unique morphism $\mu : C \to \Omega_{\mathcal{E}}$ that fits into a pullback diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, "\top"] \\
C \arrow[r, "\mu"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

The map μ is called the *classifying map* of m.[6] The map $\top$ itself is the classifying map of the identity arrow on the terminal object 1. The identity map on $\Omega_{\mathcal{E}}$, in turn, classifies the monomorphism $\top$. Thus, for instance, it is readily seen that $\Omega_{\mathcal{E}}$ is uniquely determined up to isomorphism. Also, it follows from properties of pullbacks in general that a monomorphism m is an isomorphism if and only if its classifying map μ factors through $\top$.

[5] Thus, equivalently, we could have first defined the preorder of monomorphisms to A in a similar way, and then define $\mathrm{Sub}_{\mathcal{E}}(A)$ to be the poset reflection of that preorder.

[6] The second definition derives from the first by defining the generic subobject $\top : X \to \Omega_{\mathcal{E}}$ to be the subobject corresponding to the identity map on $\Omega_{\mathcal{E}}$. It follows that X is a terminal object.

Logically speaking, $\Omega_{\mathcal{E}}$ is an object of truth values. The existence of a universal, generic, subobject $\top : 1 \to \Omega_{\mathcal{E}}$ may be understood as a separation principle. Regarding a map $\mu : C \to \Omega_{\mathcal{E}}$ as a "property" on C, the universal property of the generic subobject $\top$ states that every property on an object C uniquely determines a sub-object U_μ of V in a canonical way. Adopting set-theoertic notation, one may thus write U_μ as $\{x \in C \mid \mu(x)\}$.

Example 1.4. In the category **Sets** of all sets any two-element set with with a distinguished element is a subobject classifier.[7] Of course, a canonical choice might be $\{\emptyset, \{\emptyset\}\}$, with $\top$ singeling out $\{\emptyset\}$. Slightly more generally, in the following we will simply consider any singleton 1 and the corresponding two-element set $\mathcal{P}(1) = \{\emptyset, 1\}$. Note that

$$\mathrm{Sub}_{\mathbf{Sets}}(A) \cong \mathcal{P}(A) \cong 2^A.$$

In fact, in any topos $\mathcal{E}$, the object $\Omega_{\mathcal{E}}^A$ behaves like an internal powerobject of A.

Example 1.5. The subobject classifier in the topos of I-indexed families of sets, for some fixed set I, is the family with constant components the two element set $\mathbf{2}$. Equivalently, viewed as a functor category $\mathbf{Sets}^I$, the subobject classifier is the functor $\Omega : I \to \mathbf{Sets}$ with components $\Omega(i) = \mathbf{2}$. A subobject of a family of sets is family of inclusion, with characteristic the pointwise charactristic map as in **Sets**.

Example 1.6. Generalizing the previous example, the subobject classifier in $\mathbf{Sets}^{\mathbf{C}^{op}}$, for any small category $\mathbf{C}$, is described as follows. For any object C in $\mathbf{C}$, $\Omega(C)$ is the set of all *sieves* σ on C, i.e. sets of arrows h with codomain C such that $h \in \sigma$ implies $h \circ f \in \sigma$, for all f with $\mathrm{cod}(f) = \mathrm{dom}(h)$. For an arrow $g : D \to C$ in $\mathbf{C}$, $\Omega(g)(\sigma)$ is the restriction of σ along g:

$$\Omega(g)(\sigma) = \{f : X \to D \mid g \circ f \in \sigma\},$$

which is a sieve on D. The mono $\top : 1 \to \Omega$ is the natural transformation whose components pick out the maximal sieve $\top_C$ on C, i.e. the set of all arrows with codomain C (the terminal object 1 being pointwise the singleton). The classifying map χ_m of a subfunctor $m : E \rightarrowtail F$ has components

$$(\chi_m)_C(a) = \{f : X \to C \mid F(f)(a) \in E(X)\}.$$

In particular, if $\mathbf{C}$ is a *preorder*, then $\Omega(C)$ is the set of all downward closed subsets of $\downarrow C$. Since in this case there is at most one arrow $g : D \to C$, the function $F(g)$ may be thought of as the *restriction* of the set $F(C)$ to $F(D)$ along the inequality $D \leq C$.

[7] For a discussion of foundational questions regarding the category **Sets** see e.g. [19]. For instance, one may assume a universe whose elements are sets that are closed under the standard set-forming operations. The elements of this category are then called *small sets*, while subsets of the universe are called *large*. In the following we will continue to refer to **Sets** as the "category of sets", where by "set" we mean small set in an appropriate sense.

Proposition 1.3. *For each object A in $\mathcal{E}$, the set* $\mathrm{Sub}_{\mathcal{E}}(A)$ *of subobjects of A is a Heyting algebra, and for each arrow $f : B \to A$ in $\mathcal{E}$ the operation*

$$f^* : \mathrm{Sub}_{\mathcal{E}}(B) \to \mathrm{Sub}_{\mathcal{E}}(A)$$

of pulling back subobjects along f is a map of Heyting algebras. Moreover, each map f^* has both a left and a right adjoint.

Proof. The top element of each $\mathrm{Sub}_{\mathcal{E}}(A)$ is the identity map on A. The meet of $m : U \to A$ and $n : V \to A$ is the diagonal through the pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & V \arrow[d, "n"] \\
U \arrow[r, "m"'] & A
\end{tikzcd}$$

The join of $m : U \to A$ and $n : V \to A$ is the image of the induced map

$$[m, n] : U + V \longrightarrow A.$$

Finally the bottom element is the unique map $0 \to A$ from the initial object of A, which in a topos is always monic.

For the second part, the map f^* preserves the finite limit structure by instant properties of pullbacks. For joins and the bottom element the statement is essentially the fact that in a topos coproducts and image factorizations are stable under pullback.

The left adjoint $\exists_f$ of f^* is obtained, for any subobject $m : U \rightarrowtail B$, as the image of the composite fm, as in

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "m"'] & \bullet \arrow[d, "\exists_f(m)"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}$$

The right adjoint $\forall_f$ is the restriction to monomorphisms of the right adjoint

$$\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$$

to the pullback functor defined on the entire slice category. For a detailed description of that right adjoint, see e.g. [20]. Regardless of the explicit description of $\forall_f$, however, it follows that implication $m \Rightarrow n$, in each $\mathrm{Sub}_{\mathcal{E}}(A)$, can be expressed by

$$\forall_m m^*(n).$$

For compute, purely by definition of order adjoints, for any $k \in \mathrm{Sub}_{\mathcal{E}}(A)$:

$$\begin{aligned} k \leq \forall_m m^*(n) \ \text{ iff } \ & m^*(k) \leq m^*(n) \\ \text{iff } \ & \exists_m m^*(k) \leq n \\ \text{iff } \ & k \wedge m \leq n \end{aligned}$$

As for the last step, recall that the meet of two subobjects m and k is given by the diagonal composite through their pullback. This is equivalently expressed by first pulling back k along m, and then composing with m again. Since the direct image functor $\exists_m$ along a monomorphism m is essentially composition with m, the composite $\exists_m m^*(k)$ is precisely the meet $k \wedge m$.

With this description of implications one may show that f^* preserves these using the Beck-Chevalley condition for the right adjoints $\forall$. (For the latter, see proposition 2.4 and example 2.3 below.)

□

The proposition essentially shows that the functor

$$\mathrm{Sub}_{\mathcal{E}}(-) : \mathcal{E} \longrightarrow \mathbf{Sets}$$

is Heyting algebra in the category of all functors from $\mathcal{E}$ to **Sets**. We now outline how through the natural isomorphism

$$\mathrm{Sub}_{\mathcal{E}}(-) \cong \mathrm{Hom}_{\mathcal{E}}(-, \Omega_{\mathcal{E}}) \tag{3}$$

the Heyting structure of $\mathrm{Sub}_{\mathcal{E}}(-)$ internalizes to the object $\Omega_{\mathcal{E}}$. Of course, it would be enough to give the structure maps pertaining to $\Omega_{\mathcal{E}}$, and then verify the equations defining a Heyting algebra. However, we give some more detail to show how the Heyting structure of $\Omega_{\mathcal{E}}$ is conceptually linked to the structure of the subobject lattices in $\mathcal{E}$.

Proposition 1.4. *For any topos $\mathcal{E}$, its subobject classifier $\Omega_{\mathcal{E}}$ is a Heyting algebra in $\mathcal{E}$.*

Proof. For any A in $\mathcal{E}$, the operation of forming meets is a function

$$\wedge_A : \mathrm{Sub}_{\mathcal{E}}(A) \times \mathrm{Sub}_{\mathcal{E}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{E}}(A),$$

and therefore by the isomorphism (3) at A a map

$$\wedge_* : \mathrm{Hom}_{\mathcal{E}}(A, \Omega_{\mathcal{E}}) \times \mathrm{Hom}_{\mathcal{E}}(A, \Omega_{\mathcal{E}}) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(A, \Omega_{\mathcal{E}}).$$

This operation is natural in the argument A, since the operation $\wedge_A$ is preserved by pulling back. Hence, by the Yoneda lemma, there exists a unique map

$$\wedge : \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \longrightarrow \Omega_{\mathcal{E}}$$

in $\mathcal{E}$ such that $\wedge_*$ is defined by composition with $\wedge$. Moreover, there is a canonical way of deducing $\wedge$, namely by applying the map $\wedge_*$, modulo the isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(A, \Omega_{\mathcal{E}}) \times \mathrm{Hom}_{\mathcal{E}}(A, \Omega_{\mathcal{E}}) \cong \mathrm{Hom}_{\mathcal{E}}(A, \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}}),$$

to the identity map on $\Omega_{\mathcal{E}}$. In this way the meet operation

$$\wedge : \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \longrightarrow \Omega_{\mathcal{E}}$$

turns out to be the classifying map of $\langle \top, \top \rangle : 1 \longrightarrow \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}}$, which is the classifying map of the pullback of $\langle 1, \top u \rangle$ and $\langle \top u, 1 \rangle$ ($u : \Omega_{\mathcal{E}} \to 1$ is the canonical map):

$$\begin{tikzcd}
1 \arrow[r, "\top"] \arrow[d, "\top"'] & \Omega_{\mathcal{E}} \arrow[d, "\langle 1, \top u \rangle"] \\
\Omega_{\mathcal{E}} \arrow[r, "\langle \top u, 1 \rangle"'] & \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}}
\end{tikzcd}$$

viewed as subobject of $\Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}}$; while $\langle 1, \top u \rangle$ and $\langle \top u, 1 \rangle$ in turn arise as the subobjects classified by π_2 and π_1, respectively. Since the external maps $\wedge_A$ all satisfy the basic equalities defining meets (idempotency, commtuativity, symmetry), so does the internal map $\wedge$.

The induced partial ordering is then again the equalizer

$$E \rightarrowtail \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} \Omega_{\mathcal{E}}.$$

It now follows that the generic subobject $\top : 1 \to \Omega_{\mathcal{E}}$ is the top element of $\Omega_{\mathcal{E}}$, since both composites

$$\Omega_{\mathcal{E}} \times 1 \xrightarrow{1 \times \top} \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \xrightarrow{\pi_1} \Omega_{\mathcal{E}}$$

and

$$\Omega_{\mathcal{E}} \times 1 \xrightarrow{1 \times \top} \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \xrightarrow{\wedge} \Omega_{\mathcal{E}}$$

classify the same subobject, hence are equal. Therefore $1 \times \top$ factors through the ordering E. The bottom element is the classifying map of the subobject $0 \to 1$. The join operation is determined as classifying map of the image of the map

$$[\langle 1, \top u_{\Omega_{\mathcal{E}}} \rangle, \langle \top u_{\Omega_{\mathcal{E}}}, 1 \rangle] : \Omega_{\mathcal{E}} + \Omega_{\mathcal{E}} \longrightarrow \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}}.$$

Finally, implication, if it exists, is necessarily the classifying map of the ordering $E \hookrightarrow \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}}$. The latter of course exists and can be expressed as the following lower horizontal composite

$$\begin{tikzcd}
E \arrow[r] \arrow[d, dotted, tail] & \Omega_{\mathcal{E}} \arrow[r] \arrow[d, "\Delta_{\Omega_{\mathcal{E}}}"] & 1 \arrow[d, "\top"] \\
\Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \arrow[r, "\langle \wedge, \pi_1 \rangle"'] & \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \arrow[r, "\delta_{\Omega_{\mathcal{E}}}"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

following a standard description of equalizers. A Yoneda argument in the style as was employed for meets and joins determines $\Rightarrow$ as the classifying map of the subobject $\forall_{\langle \top u_{\Omega_{\mathcal{E}}}, 1 \rangle}(\top)$ of $\Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}}$, which in turn is precisely the said equalizer.

$\square$

Proposition 1.5. *For any topos $\mathcal{E}$ the subobject classifier $\Omega_{\mathcal{E}}$ is a complete Heyting algebra.*

Proof. Using a standard Yoneda argument, the adjoints to the map $\Delta_I : \Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}^I$ are provided by the fact that, for any topos $\mathcal{E}$, and any arrow $f : A \to B$ in $\mathcal{E}$, the pullback functor

$$f^* : \mathrm{Sub}_{\mathcal{E}}(B) \longrightarrow \mathrm{Sub}_{\mathcal{E}}(A)$$

has both a right and a left adjoint. In the context of an additional parameter X the functor

$$(1_X \times f)^* : \mathrm{Sub}_{\mathcal{E}}(X \times B) \longrightarrow \mathrm{Sub}_{\mathcal{E}}(X \times A)$$

may be written as a functor

$$\mathrm{Hom}_{\mathcal{E}}(X, \Omega_{\mathcal{E}}^B) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, \Omega_{\mathcal{E}}^A),$$

in view of the isomorphisms

$$\mathrm{Sub}_{\mathcal{E}}(X \times Y) \cong \mathrm{Hom}_{\mathcal{E}}(X \times Y, \Omega_{\mathcal{E}}) \cong \mathrm{Hom}_{\mathcal{E}}(X, \Omega_{\mathcal{E}}^Y).$$

These are natural in X and so by Yoneda provide a map

$$\Omega_{\mathcal{E}}^B \longrightarrow \Omega_{\mathcal{E}}^A,$$

which is precisely $\Omega_{\mathcal{E}}^f$. In particular, Δ_I arises in this way from pullback along the projection $\pi_1 : X \times I \to X$:

$$\pi_1^* : \mathrm{Sub}_{\mathcal{E}}(X) \longrightarrow \mathrm{Sub}_{\mathcal{E}}(X \times I),$$

that is by applying the previous argument to the map $u_I : I \longrightarrow 1$, as required. The external adjoints of π_1^* induce the required internal adjoints of $\Omega_{\mathcal{E}}^{u_I} = \Delta_I$.

$\square$

In topos-theoretic contexts complete Heyting algebras are often referred to as *frames*. The reason for the extra terminology lies in the fact that oftentimes one studies not maps of complete Heyting algebras, but only those Heyting maps that preserve arbitrary joins, not necessarily arbitrary meets. An example is the inclusion

$$i : \mathcal{O}(X) \hookrightarrow \mathcal{P}(X)$$

of the open sets of a topological space X into the powerset. It does not in general preserve arbitrary meets.

More generally, for any S4 algebra A, the inclusion $i : \Box A \to A$ is a frame map. We have seen earlier that this inclusion has a right adjoint. In a similar fashion every frame map $f : A \to B$ has a right adjoint $f_* : B \to A$ given, for any $y \in B$, by

$$f_*(y) = \bigvee \{x \in A \mid f(x) \leq y\}.$$

Another class of examples involves the inverse image functor

$$f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$$

of a continuous maps of spaces $f : X \to Y$, which does not in general preserve arbitrary meets.

As a frame, the Heyting algebra $\Omega_{\mathcal{E}}$ has the property that it is the initial frame in $\mathcal{E}$. That is to say, for any frame H in $\mathcal{E}$, there exists a unique frame map

$$i : \Omega_{\mathcal{E}} \to H.$$

We call a complete Heyting algebra H in $\mathcal{E}$ *faithful*, if the canonical frame map i is a monomorphism. As a frame map, i has a right adjoint τ, which is the classifying map of the top element of H. The initial frame map has the special propert that i is a monomorphism if and only if the diagram (which commutes, because i preserves meets)

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "\top"'] & 1 \arrow[d, "\top"] \\
\Omega_{\mathcal{E}} \arrow[r, "i"'] & H
\end{tikzcd}$$

is a pullback diagram. If i is monic, then this is a straightforward verification. On the other hand, assuming it to be a pullback, the following composite diagram is a pullback as well:

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "\top"'] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
\Omega_{\mathcal{E}} \arrow[r, "i"'] & H \arrow[r, "\tau"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

and hence $\tau i = 1$, by uniqueness of classifying maps. It follows that i is monic, as is any map that has a retract.

2 Algebraic Models for First-Order Theories

2.1 First-Order Hyperdoctrines

The following notion is due to Lawvere [17].

Definition 2.1. *A hyperdoctrine is a functor $H : \boldsymbol{C}^{op} \to \boldsymbol{HA}$ from a finitely complete category $\boldsymbol{C}$ to the category of Heyting algebras. The functor H is to satisfy the following conditions:*

- *For every projection $\pi : X \times Y \to X$ in $\boldsymbol{C}$, the map*

  $$H(\pi) : H(X) \to H(X \times Y)$$

  has both a left and a right adjoint

  $$\forall, \exists : H(X \times Y) \rightrightarrows H(X)$$

  which, however, are not required to be Heyting maps.

- *Both adjoints satisfy the Beck-Chevalley condition. That is to say, for any arrow $f : Z \to X$ in $\boldsymbol{C}$, the following commute:*

  $$\begin{tikzcd}
  H(X) \arrow[d, "H(f)"'] & H(X \times Y) \arrow[l, "Q_\pi"'] \arrow[d, "H(f\times 1)"] \\
  H(Z) & H(Z \times Y) \arrow[l, "Q_\pi"]
  \end{tikzcd}$$

  for $Q = \forall, \exists$.

- *For all maps $\Delta_X = \langle 1_X, 1_X \rangle : X \to X \times X$ the morphism $H(\Delta_X)$ has a left adjoint that satisfies the Beck-Chevalley condition for pullback diagrams*

$$\begin{tikzcd}
A \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"'] & A \times A \arrow[d, "1 \times \Delta_A"] \\
A \times A \arrow[r, "\Delta_A \times 1"'] & A \times A \times A
\end{tikzcd}$$

The definition states the minimal amount of structure needed to interpret (intuitionistic) first-order logic. In practice, however, most examples are such that every map $H(f)$ will have adjoints satisfying the Beck-Chevalley condition w.r.t. underlying pullbacks in $\mathbf{C}$. The idea is that $\mathbf{C}$ encodes variable contexts, i.e. sorts, while the arrows are terms. Note that the category $\mathbf{C}$ is not necessarily to have actual meaning in the sense that the objects and arrows in $\mathbf{C}$ are mere indices. Elements in $H(X)$, in turn, are to be thought of propositions in the free variable X, or properties. In particular, the set $H(1)$ encodes propositions. The operation $H(\pi)$ models the operation of variable weakening whose adjoints are quantifiers. An arrow $f : Z \to X$ in $\mathbf{C}$ may in turn be regarded as corresponding to a term $z : Z \mid f : X$, and $H(f)$ to model a substitution function that substitutes $f(z)$ for x in any property in $H(X)$. The Beck-Chevalley condition thus ensures that quantification commutes with substitution.

Before giving examples, we outline the notion of model in a hyperdoctrine of a sorted first-order theory so as to provide some more detailed intuition about the intention behind the notion of hyperdoctrine. We proceed informally, skipping some technical details, in order to convey the guiding idea. A sorted first-order language consists of a set Σ of sorts, basic propositional constants $\top, \bot$, and a relation $=_A$, for each sort $A \in \Sigma$. Moreover, there may be basic relation symbols each R, each equipped with a (possibly empty) sequence $A_1, \ldots, A_n$ of sorts, indicating the sorts over which R is defined, written

$$R : A_1, \ldots, A_n.$$

Empty relation symbols (those which are assigned the empty sequence) are to be thought of as propositions. In a similar way one may assume constants

$$c : A$$

and function symbols

$$f : A_1, \ldots, A_n \to B.^8$$

[8] The arrow "$\to$" is a purely formal notation, with the same syntactical function as

Terms and formulas are going to be written in a *variable context* which indicates the free variables that occur in formula φ, along with sort of the variable. The context may contain more variables than actually occur in φ but it must at least contain all the free variables of φ. A context generally looks like

$$x_1 : A_1, \ldots, x_n : A_n$$

meaning that the variable x_i is of type A_i. For instance, the expression

$$x : A, y : B \mid \varphi$$

means that φ may contain the variables x and y of sorts A and B, resp. Also, the expression

$$\emptyset \mid \varphi$$

means that φ does not contain any free variables whatsoever. Similarly for terms. Most importantly, terms and formulas always occur in such contexts. Terms-in-context and formulas-in-context are thus the basic syntactic entities of the formal system. One and the same formula φ may occur in different contexts, yielding two different formulas-in-context. It is only the latter that are meaningful in the formal system.

The set of terms-in-context and formulas-in-context is defined recursively (henceforth simply "term" and "formula"). For instance one has the following term forming rules

- For every variable x, the expression $x : A \mid x : A$ is a term.
- For each constant $c : A$, the following is a term

$$\emptyset \mid c : A$$

- For any function symbol $f : A_1, \ldots, A_n \to B$, given terms

$$\Gamma \mid t_1 : A_1, \ldots, \Gamma \mid t_n : A_n,$$

  where Γ is any suitable context, there is a term

$$\Gamma \mid f(t_1, \ldots, t_n) : B.$$

One additionally assumes structural rules that specify new terms through manipulation of the contexts. These are permuation of variable declarations, contraction, and weakening. The last one is quite important. It says that if

$$\Gamma \mid t : A$$

e.g. a comma; the choice of the arrow is of course to indicate its intended meaning. As for propositions, one may also treat constants as function symbols where the sequence $A_1, \ldots, A_n$ is empty.

is a term, then for any variable $z : C$ there is a term

$$\Gamma, z : C \mid t : A.$$

Formulas are defined in an analogous way.

- The expressions $\emptyset \mid \bot$ and $\emptyset \mid \top$ are formulas
- If $R : A_1, \ldots, A_n$ is a basic relation symbol, then given terms

$$\Gamma \mid t_1 : A_1, \ldots, \Gamma \mid t_n : A_n,$$

  where Γ is any suitable context, the expression

$$\Gamma \mid R(t_1, \ldots, t_n)$$

  is a formula.
- if $\Gamma \mid s : A$ and $\Gamma \mid t : A$ are terms, then $\Gamma \mid s =_A t$ is a formula.
- if $\Gamma \mid \varphi$ and $\Gamma \mid \psi$ are formulas, then $\Gamma \mid \varphi \wedge \psi$ is a formula, etc., for all the connectives.
- if $\Gamma, y : B \mid \varphi$ is a formula such that $y : B$ does not occur in Γ, then

$$\Gamma \mid \exists_{y:B}\varphi \qquad \text{and} \qquad \Gamma \mid \forall_{y:B}\varphi$$

  are formulas.
- if $\Gamma \mid \varphi$ is a formula, then for any variable $z : C$, the expression $\Gamma, z : C \mid \varphi$ is a formula. (Weakening)

Finally one specifies an intuitionistic deduction relation for formulas. For instance one defines

$$\Gamma \mid \varphi \vdash \psi \wedge \rho \ \text{ iff } \ \Gamma \mid \varphi \vdash \psi \ \&\ \Gamma \mid \varphi \vdash \rho.$$

We will give a precise formulation for higher-order logic in section 3. The idea behind the definition of the connectives and the quantifiers is, however, the same. The purely propositional rules concerning the logical connectives determine a deduction relation $\vdash_\Gamma$ relative to each context. The quantifiers, however, relate different context with each other. They are defined by the following two-way rules, where we assume that the variable that is quantified out does not occur in the context Γ:

$$\frac{\Gamma \mid \psi \vdash \exists_{y:B}\varphi}{\Gamma, y : B \mid \psi \vdash \varphi} \qquad \frac{\Gamma \mid \forall_{y:B}\varphi \vdash \psi}{\Gamma, y : B \mid \varphi \vdash \psi}$$

Note also that we assume that $\Gamma \mid \psi$ is well-written, hence does not contain y freely.

As for interpretations, consider any hypderdoctrine $H : \mathbf{C}^{op} \to \mathbf{HA}$. We first interpret each sort A in Σ by some object $[\![A]\!]$ in $\mathbf{C}$. Any term-in-context

$$x_1 : A_1, \ldots, x_n : A_n \mid t : B$$

is going to be recursively assigned an arrow

$$[\![A_1]\!] \times \cdots \times [\![A_n]\!] \xrightarrow{[\![x_1:A_1,\ldots,x_n:A_n|t:B]\!]} [\![B]\!]$$

in the index category $\mathbf{C}$. Although the assignment is context-sensitive, to ease notation, if the context is clear we will simply write $[\![t]\!]$. Following the previous term-forming rules, a constant $\emptyset \mid c : A$ is interpreted as an arrow

$$[\![c]\!] : 1 \longrightarrow [\![A]\!].$$

A term of the form $x : A \mid x : A$ is interpreted as the identity arrow on $[\![A]\!]$. A term constructed from a functional symbol $f : A_1, \ldots, A_n \to B$ and terms $\Gamma \mid t_1 : A_1, \ldots, t_n : A_n$, as defined above, is interpreted as the composite

$$[\![\Gamma]\!] \xrightarrow{\langle [\![t_1]\!], \ldots [\![t_n]\!] \rangle} [\![A_1]\!] \times \cdots \times [\![A_n]\!] \xrightarrow{[\![f]\!]} [\![B]\!].$$

The object $[\![\Gamma]\!]$ is the cartesian product of all the interpretations of sorts occurring in Γ. Finally, a term $\Gamma, z : C \mid t : B$ obtained by weakening is interpreted as the composite

$$[\![\Gamma]\!] \times [\![C]\!] \xrightarrow{\pi} [\![\Gamma]\!] \xrightarrow{[\![\Gamma|t:B]\!]} [\![B]\!],$$

where π is the projection.

One also defines substitution recursively, which is a bit tricker. However, without going into details, it can be done in such a way that, for instance (to take a simple case), if $x : A \mid t : B$ and $y : B \mid s : C$ are terms, then $x : A \mid s[t/y] : C$ is defined and interpreted as the composite

$$[\![A]\!] \xrightarrow{[\![t]\!]} [\![B]\!] \xrightarrow{[\![s]\!]} [\![C]\!].$$

Formulas-in-context

$$x_1 : A_1, \ldots, x_n : A_n \mid \varphi$$

are recursively interpreted as elements in the Heyting algebra

$$H([\![A_1]\!] \times \cdots \times [\![A_n]\!]).$$

The basic propositions $\perp$ and $\top$ are interpreted as top and bottom:

$$[\![\perp]\!] = \perp_{H(1)}$$

$$[\![\bot]\!] = \top_{H(1)}.$$

Each basic relation $R : A_1, \dots, A_n$ is assigned an element

$$[\![R]\!] \in H([\![A_1]\!] \times \cdots \times [\![A_n]\!]).$$

In particular, the equality predicate $=_A$, for each sort A, is interpreted as

$$[\![x : A, y : A \mid x = y]\!] = \exists_{H(\Delta_{[\![A]\!]})}(\top_{H([\![A]\!])}) \in H([\![A]\!] \times [\![A]\!]),$$

as explained before. A formula $\Gamma \mid R(t_1, \dots, t_n)$, constructed from $\Gamma \mid t_1 : A_1, \dots, t_n : A_n$ is interpreted as

$$H(\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle)([\![R]\!]);$$

that is to say by applying the function

$$H([\![A_1]\!], \dots, [\![A_n]\!]) \xrightarrow{H(\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle)} H([\![\Gamma]\!])$$

to

$$[\![R]\!] \in H([\![A_1]\!], \dots, [\![A_n]\!]).$$

For each context Γ the formulas formed through the rules for the connectives are interpreted by the corresponding algebraic operations in each $H([\![\Gamma]\!])$.

The operation of weakening by $z : C$, for a given formula $\Gamma \mid \varphi$, is defined in a similar way by applying $H(\pi)$ to $[\![\Gamma \mid \varphi]\!] \in H([\![\Gamma]\!])$, where $\pi : [\![\Gamma]\!] \times [\![C]\!] \to [\![\Gamma]\!]$ is the projection.

If $\Gamma, y : B \mid \varphi$ is a formula such that $y : B$ does not occur in Γ, the formulas $\Gamma \mid \exists_{y:B}\varphi$ and $\Gamma \mid \forall_{y:B}\varphi$ are interpreted, resp., as

$$\exists_\pi [\![\Gamma, y : B \mid \varphi]\!] \in H([\![\Gamma]\!])$$

$$\forall_\pi [\![\Gamma, y : B \mid \varphi]\!] \in H([\![\Gamma]\!])$$

Here, $\pi : [\![\Gamma]\!] \times [\![B]\!] \to [\![\Gamma]\!]$ is again the projection. As for terms, one defines substitution of terms in formulas in such a way that one can show – again considering a simple case – that if $x : A \mid t : B$ is a term and $y : B \mid \varphi$ any formula, then $x : A \mid \varphi[t/y]$ is defined and interpreted as the element

$$H([\![t]\!])([\![y : B \mid \varphi]\!]) \in H([\![A]\!]),$$

where, recall, $[\![t]\!] : [\![A]\!] \to [\![B]\!]$, and hence $H([\![t]\!]) : H([\![B]\!]) \to H([\![A]\!])$.

Soundness is obtained almost by construction as the algebraic operations mirror the syntactic ones. That is to say, if one can prove (in intuitionistic first-order logic) that

$$\Gamma \mid \varphi \vdash \psi,$$

then

$$[\![\Gamma \mid \varphi]\!] \leq [\![\Gamma \mid \psi]\!]$$

in $H([\![\Gamma]\!])$. In particular, the biconditional rules for the quantifiers correspond exactly to the adjointness conditions.

Remark 2.2. The definition of equality as a predicate $\exists_{\Delta_X}(\top)$ is due to Lawvere and neatly describes the equality relation as the smallest reflexive relation. If $\varphi \in H(X \times X)$ is any predicate in two variables, then $H(\Delta_X)(\varphi) \in H(X)$ is the same predicate restricted to pairs (x, x). The definition of the equality predicate through the left adjoint of $H(\Delta_X)$ yields the bicondition

$$\frac{\exists_{\Delta_X}(\top) \vdash \varphi}{\top \vdash H(\Delta_X)(\varphi)}.$$

Informally, this means that $\varphi(x, x)$ holds for all $x \in X$ if and only if φ contains the equality predicate.[9]

It follows that the equality relation is symmetric and transitive. For instance, as to symmetry, note that the twist map

$$\theta = \langle \pi_2, \pi_1 \rangle : X \times X \to X \times X$$

is an isomorphism. Hence so is the map $H(\theta)$ which then trivially has a left adjoint (its inverse). Moreover, since $\Delta_X = \theta\Delta_X$,

$$\exists_{\Delta_X} \cong \exists_\theta \exists_{\Delta_X}.$$

Applying $H(\theta)$ yields

$$\exists_{\Delta_X}(\top_X) \cong H(\theta)\exists_\theta \exists_{\Delta_X}(\top_X) \cong H(\theta)\exists_{\Delta_X}(\top_X),$$

because $\exists_\theta$ and $H(\theta)$ are inverses. But that is precisely the symmetry statement.

Remark 2.3. From the point of view of finding models for first-order theories, the category $\mathbf{C}$ may, as hinted at the beginning, be viewed as a mere index category for a structure in which to interpret first-order theories. The interpretation of the sorts, constants, and function symbols need not necessarily have any particular meaning in the sense that they are merely used to define a $\mathbf{C}$-indexed Heyting algebra and operations between the components. Strictly speaking, one would not even need to make the detour through $\mathbf{C}$ but could interpret a sort A directly by a Heyting algebra $H(A)$, and a function symbol $f : A \to B$ by a Heyting map $H(f) : H(B) \to H(A)$ in a suitable way. The functorial approach is, from this perspective, just a convenient way to express this idea of a purely algebraic approach to models.

However, there are advantages to the functorial approach. From the point of view of a finite limit category $\mathbf{C}$, a hyperdoctrine may be seen as endowing $\mathbf{C}$ it with an "external" first-order structure in which to build models. Mostly, this external structure is in fact defined through $\mathbf{C}$ itself. We will see examples of this presently. For instance, if $\mathbf{C}$ is a Heyting category, the functor $\mathrm{Sub}_{\mathbf{C}}(-) : \mathbf{C}^{op} \to \mathbf{HA}$ sends an object X to the poset of subobjects of X, which is a Heyting algebra. We will take a closer look at these examples when $\mathbf{C}$ is a topos.

[9] In particular, $\top \vdash H(\Delta_X)\exists_{\Delta_X}(\top)$, which is the unit at $\top$.

Example 2.1. A hyperdoctrine for classical logic has each $H(X)$ to be a Boolean algebra. For instance, one may put $\mathbf{C} = \mathbf{Sets}$ and $H(X) = \mathcal{P}(X)$, while $H(f)$ is inverse image along the function f. Inverse image in fact preserves all the Boolean structure, so it is a map in the category of Boolean algebras and homomorphisms. The inverse image along a function $f : Y \to X$ always has a left and a right adjoint. The left adjoint is the direct image functor while the right adjoint is the dual image operation:

$$\forall_f(U) = \bigcup\{V \subseteq X \mid f^{-1}(V) \subseteq U\}.$$

Hence the second and third conditions in def. 2.1 are satisfied. For the Beck-Chevalley condition we refer to the next example.

To illustrate the definition of the equality predicate, consider any set $U \subseteq X$ and the direct image along Δ_X:

$$\exists_{\Delta_X}(U) = \{(x, y) \in X \times X \mid x = y \;\&\; x \in U\}.$$

For X itself, which is the top element of $\mathcal{P}(X)$, this boils down to

$$\exists_{\Delta_X}(X) = \{(x, y) \mid x = y\}.$$

Example 2.2. The previous example may be generalized. Let again $\mathbf{C} = \mathbf{Sets}$ and fix any complete Heyting algebra A in $\mathbf{Sets}$. Define a hyperdoctrine by setting

$$H(X) := A^X.$$

For any $f : Y \to X$, the map $H(f) : A^X \to A^Y$ is defined by precomposition with f:

$$H(f)(g) = g \circ f.$$

The maps so defined are Heyting homomorphisms. Each map $H(f) : A^X \to A^Y$ also has a left and right adjoint. The left adjoint $\exists_f$ of $H(f)$, for instance, sends a map $g \in A^Y$ to the function $\overline{g}$ defined on any $x \in X$ as

$$\overline{g}(x) = \bigvee_{f(y)=x} g(y).$$

Dually for the right adjoint. For $f = \Delta_X$, and $g = \top$ (the constant $\top_A$-valued map $X \to 1 \xrightarrow{\top} A$), this reads

$$\overline{g}(x, y) = \bigvee_{x=y} g(x) = \bigvee_{x=y} \top$$

and thus

$$\exists_{\Delta_X}(\top)(x, y) = \top \text{ iff } x = y.$$

Note that if $x \neq y$ the join $\bigvee_{x=y} \top$ is empty, so that $\bigvee_{x=y} \top = \bot$ in that case.

We will verify a slightly more general version of the Beck-Chevalley condition from which the special one involving projections follows. Consider a complete Heyting algebra H and a pullback square

$$\begin{tikzcd}
P \arrow[r, "r"] \arrow[d, "q"'] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Spelling out definitions, for any $a \in A$, and $h \in H^C$:

$$\exists_q H^f(h)(a) = \exists_q(hr)(a) = \bigvee_{q(x)=a} hr(x)$$

$$H^f \exists_g(h)(a) = \exists_g(h)(f(a)) = \bigvee_{g(y)=f(a)} h(y)$$

First, consider any x such that $q(x) = a$. By commutativity of the square above

$$gr(x) = fq(x) = f(a).$$

Hence, for $y = r(x)$,

$$hr(x) \leq \bigvee_{g(y)=f(a)} h(y),$$

and therefore

$$\bigvee_{q(x)=a} hr(x) \leq \bigvee_{g(y)=f(a)} h(y).$$

On the other hand, consider any $y \in C$ for wich $g(y) = f(a)$. By the definition of a pullback, there exists a unique $x \in P$ such that $q(x) = a$ and $r(x) = y$. Hence, since $r(x) = y$:

$$h(y) \leq \bigvee_{q(x)=a} hr(x).$$

Since y was arbitrary

$$\bigvee_{g(y)=f(a)} h(y) \leq \bigvee_{q(x)=a} hr(x).$$

A dual argument can be made for the right adjoint $\forall_f$.

□

The previous example of a hyperdoctrine $H(X) = A^X$, for a complete Heyting algebra A generalizes from **Sets** to an arbitrary locally small topos $\mathcal{E}$. Concretely, for given A, a hyperdoctrine $H : \mathcal{E} \to \mathbf{Sets}$ is defined as

$$X \mapsto \mathrm{Hom}_{\mathcal{E}}(A, X)$$

An arrow $f : Y \to X$ is mapped to the function $(-) \circ f$ of precomposition with f. Using the internal language in a topos, the definitions of the adjoints are logically the same. To this end, the function of precomposing with f corresponds under the isomorphism $\mathrm{Hom}(X, A) \cong \mathrm{Hom}(1, A^X)$ to the map of composition with A^f. We will spell out the adjoints to precomposition with projections, although of course the argument generalizes to arbitrary maps in $\mathcal{E}$.

Proposition 2.4. *For any complete Heyting algebra A in $\mathcal{E}$, and projection $\pi_Y : X \times Y \to X$ in $\mathcal{E}$, the operation of precomposing with π_Y*

$$\pi_Y^* : \mathrm{Hom}_{\mathcal{E}}(X, A) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X \times Y, A)$$

has both a left and a right adjoint satisfying the Beck-Chevalley condition.

Proof. Define the right adjoint on any $g : X \times Y \to A$ as the map

$$X \xrightarrow{\overline{g}} A^Y \xrightarrow{\forall_Y} A,$$

where $\overline{g}$ is the exponential transpose of g, i.e. by transposing along the composite

$$\mathrm{Hom}_{\mathcal{E}}(X \times Y, A) \cong \mathrm{Hom}_{\mathcal{E}}(X, A^Y) \xrightarrow{(\forall_Y)_*} \mathrm{Hom}_{\mathcal{E}}(X, A).$$

The left adjoint is defined in a similar fashion by composition with $\exists_Y$.

The result now follows by noting that the operation π_Y^* is given by transposition along

$$\mathrm{Hom}_{\mathcal{E}}(X, A) \xrightarrow{(\Delta_Y)_*} \mathrm{Hom}_{\mathcal{E}}(X, A^Y) \cong \mathrm{Hom}_{\mathcal{E}}(X \times Y, A).$$

For any $f : X \to A$ the exponential transpose of the composite

$$X \xrightarrow{f} A \xrightarrow{\Delta_Y} A^Y$$

is the map $\pi \circ (f \times 1)$, fitting into the following commutative square:

$$\begin{tikzcd}
X \times Y \arrow[r, "f \times 1"] \arrow[d, "\pi_Y"'] & A \times Y \arrow[d, "\pi"] \\
X \arrow[r, "f"'] & A
\end{tikzcd}$$

which proves the claim. The adjoints thus defined satisfy the Beck-Chevally condition by naturality (in X) of the composition operations and of the product-exponential adjunction.

$\square$

Remark 2.5. In the set-theoretic notation of the internal language, the equality map was given by

$$\exists_{\Delta_X}(\top_X)(x,y) = \bigvee_{x=y} \top.$$

However, in contrast to the **Sets** case, if $x \neq y$ we cannot conclude that the join is empty, i.e. equals $\bot$. For in the context of an arbitrary topos $\mathcal{E}$ the equality relation is not in general decidable unless $\mathcal{E}$ is Boolean. In other words, it is not the case that "either $x = y$ or $x \neq y$" holds in the internal language.

There is another difference. Although A is a complete Heyting algebra in $\mathcal{E}$, the set $\mathrm{Hom}_{\mathcal{E}}(X, A)$ is not in general a complete Heyting algebra. That is to say, it does not admit arbitrary set-indexed meet and joins. It is, on the other had, if for instance $\mathcal{E}$ is cocomplete. For let $\gamma : \mathcal{E} \to \mathbf{Sets}$ be the unique geometric morphism that exists since $\mathcal{E}$ is cocomplete. The direct image is $\gamma_*(X) = \mathrm{Hom}_{\mathcal{E}}(1, X)$, for any X in $\mathcal{E}$. Modulo the isomorphism

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{E}}(X, A)^I &= \mathrm{Hom}_{\mathbf{Sets}}(I, \mathrm{Hom}_{\mathcal{E}}(X, A)) \\
&\cong \mathrm{Hom}_{\mathbf{Sets}}(I, \mathrm{Hom}_{\mathcal{E}}(1, A^X)) \\
&\cong \mathrm{Hom}_{\mathbf{Sets}}(I, \gamma_*(A^X)) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(\gamma^*(I), A^X) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(\gamma^*(I) \times X, A) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(X, A^{\gamma^*(I)})
\end{aligned}$$

the diagonal $\mathrm{Hom}_{\mathcal{E}}(X, A) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, A^{\gamma^*(I)})$ is defined by composition with

$$\Delta_{\gamma^*(I)} : A \to A^{\gamma^*(I)}.$$

Similarly, the meet and join operations are given by composition with $\forall_{\gamma^*(I)}$ and $\exists_{\gamma^*(I)}$, resp. External adjointness $(\exists_{\gamma^*(I)})_* \dashv (\Delta_{\gamma^*(I)})_* \dashv (\forall_{\gamma^*(I)})_*$ follows from $\exists_{\gamma^*(I)} \dashv \Delta_{\gamma^*(I)} \dashv \forall_{\gamma^*(I)}$. [10] If $\mathcal{E} = \mathbf{Sets}$ this recovers the example of hyperdoctrine mentioned at the beginning of the section because $\mathrm{Hom}_{\mathbf{Sets}}(X, A) = A^X$. The geometric morphism γ is simply the identity functor (**Sets** is cocomplete), so that $\mathrm{Hom}_{\mathcal{E}}(X, A)^I \cong \mathrm{Hom}_{\mathcal{E}}(X, A^I)$.

[10] Cocompleteness is slightly stronger than needed. While the functor γ_* always exists, in order to have a left exact left adjoint one needs set-indexed copowers of 1. This is also a necessary condition. Every such left adjoint must have $\gamma^*(I) \cong \coprod_I 1_{\mathcal{E}}$.

Cocompleteness of a topos is, in general, a rather strong condition. Examples include presheaf and Grothendieck toposes. In fact, by Giraud's theorem every locally small cocomplete topos with a small generating set is a Grothendieck topos.

Example 2.3. Another example of a hyperdoctrine is provided by any topos $\mathcal{E}$. The functor

$$\mathrm{Hom}_{\mathcal{E}}(-, \Omega_{\mathcal{E}}) : \mathcal{E} \to \mathbf{HA}$$

discussed earlier assigns to each object A in $\mathcal{E}$ the complete Heyting algebra

$$\mathrm{Sub}_{\mathcal{E}}(A) \cong \mathrm{Hom}_{\mathcal{E}}(A, \Omega_{\mathcal{E}})$$

is a hyperdoctrine. We know already by prop. 1.3 that for each $f : A \to B$ the pullback functor along f has a left and a right adjoint. The Beck-Chevalley condition also holds. For the left adjoints this can be verified directly using the fact that image factorizations are stable under pullback. The Beck-Chevalley condition for right adjoints then follows from this fact, because for any commutative square

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "q"'] & C \arrow[d, "g"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

if

$$\begin{tikzcd}
\mathrm{Sub}(P) \arrow[d, "\exists_q"'] & \mathrm{Sub}(C) \arrow[l, "p^*"'] \arrow[d, "\exists_g"] \\
\mathrm{Sub}(A) & \mathrm{Sub}(B) \arrow[l, "f^*"]
\end{tikzcd}$$

commutes, then so does the square with all the respective right adjoints.

For further illustration, consider the equality predicate. For any A in $\mathcal{E}$, the left adjoint to pullback along the diagaonal

$$\Delta_A = \langle 1_A, 1_A \rangle : A \to A \times A$$

sends a subobject $m : U \to A$ simply to the composite $\Delta_A \circ m$ (because Δ_A is monic). Hence it sends the top element of $\mathrm{Sub}_{\mathcal{E}}(A)$, i.e. the identity arrow on A to the subobject Δ_A in $\mathrm{Sub}_{\mathcal{E}}(A \times A)$, the classifying map of which is $\delta_A : A \times A \to \Omega_{\mathcal{E}}$.

2.2 Modal Hyperdoctrines

In this section we extend the notion of hyperdoctrine so as to allow models for (first- and higher-order) modal logic.[11] Recall that an S4 algebra is an adjunction

$$i : B \leftrightarrows A : \tau$$

where A and B are Heyting algebras, and i is a monic finite meet-preserving left adjoint to τ. It follows that i preserves joins, and thus is a map of distributive lattices. The modal operator on A is the composite $i\tau$. A morphism h of S4 algebras formally is a pair of maps $(h_1 : A \to A', h_2 : B \to B')$ such that $h_1 i = i' h_2$ and $h_2 \tau = \tau' h_1$. A right adjoint of a map of modal algebras (h_1, h_2) is a pair (r_1, r_2) such that $h_1 \dashv r_1$ and $h_2 \dashv r_2$. Similarly for left adjoints.

Definition 2.6. *A modal hyperdoctrine is a functor $P : \boldsymbol{C}^{op} \to \boldsymbol{MA}$ satisfying formally the same conditions as a first-order hyperdoctrine.*

To understand this definition better denote for any X in $\mathbf{C}$ the modal algebra associated with it by the hyperdoctrine by

$$i_X : B_X \leftrightarrows A_X : \tau_X.$$

For every map $f : Y \to X$ in $\mathbf{C}$ label the corresponding map of modal algebras with $f_A : A_X \to A_Y$ and $f_B : B_Y \to B_X$, resp. Then the Beck-Chevalley condition for the first part requires that for each $f : X \to Z$, and projections $\pi : Y \times X \to X$ and $\pi' : Y \times Z \to Z$, the following commutes, where $\exists_Y$ and $\exists'_Y$ are the left adjoints of π_A and π'_A, resp:

$$\begin{tikzcd}
A_{X\times Y} \arrow[r, "\exists_Y"] & A_X \\
A_{Z\times Y} \arrow[u, "f_A \times 1_Y"] \arrow[r, "\exists'_Y"'] & A_Z \arrow[u, "f_A"']
\end{tikzcd}$$

Similarly for $\forall_Y$ and $\forall'_Y$. The same is then required to hold with all A's replaced with B's. This ensures that quantification commutes with substitution. For the modal operators, and any map $f : Y \to X$, the condition

[11] A slightly more narrow notion of modal hyperdoctrine is discussed in [11] w.r.t. the structure induced by a geometric morphism discussed below.

that (f_A, f_B) be a map of modal algebras means that the diagrams

$$
\begin{tikzcd}
A_X \arrow[r, "f_A"] & A_Y \\
B_X \arrow[u, "i_X"] \arrow[r, "f_B"'] & B_Y \arrow[u, "i_Y"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A_X \arrow[r, "f_A"] \arrow[d, "\tau_X"'] & A_Y \arrow[d, "\tau_Y"] \\
B_X \arrow[r, "f_B"'] & B_Y
\end{tikzcd}
$$

commute. These conditions ensure that the modal operator commutes with substitution. For that reason we will informally refer to these two conditions as "Beck-Chevalley-condition for i and τ", although from a technical point of view, they are rather different from the Beck-Chevalley condition of the quantifiers.

Note that the left and right adjoints that are to model the quantifiers are not maps of modal algebras themselves, for this would mean that they commute with the modal operator which would contravene basic principles from quantified modal logic.

Moreover, we observe the following.

Proposition 2.7. *In any modal hyperdoctrine the equality relation is box-stable in the sense that*

$$\exists_{\Delta_X}(\top_X) \leq i\tau\exists_{\Delta_X}(\top_X),$$

with $\exists_{\Delta_X} : A_X \to A_{X\times X}$ the left adjoint of $\Delta_A : A_{X\times X} \to A_X$ (dropping the index X for readibility), and $\top_X$ is the top element of A_X.

Proof. For better readiblity we leave out the indices for the maps i and τ. The inequality

$$\top_X \leq \Delta_A \exists_{\Delta_X}(\top_X),$$

is the unit of the adjunction $\exists_{\Delta_X} \dashv \Delta_A$ at $\top_X$. Hence

$$\top_X = i\tau\top_X \leq i\tau\Delta_A\exists_{\Delta_X}(\top_X) = \Delta_A i\tau\exists_{\Delta_X}(\top_X).$$

The second equality holds because Δ_A is, by assumption, a map of modal algebras. With the above, the inequality

$$\exists_{\Delta_X}(\top_X) \leq i\tau\exists_{\Delta_X}(\top_X),$$

follows by adjointness again.

$\square$

Proposition 2.8. *In any modal hyperdoctrine, existential quantification is box-stable for box-stable elements in the sense that for each projection $\pi : X \times Y \to Y$*

$$\exists^A_\pi i = i\tau\exists^A_\pi i,$$

where $\exists_\pi : A_{X\times Y} \to A_Y$ is the left adjoint to π_A.

Proof. The diagram

$$\begin{tikzcd} A_{X\times Y} \arrow[r, "\exists^A_\pi"] & A_Y \\ B_{X\times Y} \arrow[u, "i"] \arrow[r, "\exists^B_\pi"'] & B_Y \arrow[u, "i"'] \end{tikzcd}$$

commutes, because the corresponding diagram with the respective right adjoints commutes by the Beck-Chevalley condition for τ. Hence

$$\tau\exists^A_\pi i = \tau i \exists^B_\pi = \exists^B_\pi.$$

Then composing with i again:

$$i\tau\exists^A_\pi i = \tau i \exists^B_\pi = i\exists^B_\pi = \exists^A_\pi i.$$

$\square$

Logically, the first proposition expresses that the equality predicate, for each X, is box-stable. The second expresses that the existential quantifier is $\Box$-stable for $\Box$-stable formulas. It corresponds to the principle $\Box\exists_{x:A}\varphi \dashv\vdash \exists_{x:A}\varphi$, provable as long as φ is $\Box$-stable. We will give a proof in the next section that is also valid in first-order logic.

We will mainly be concerned with representable modal doctrines that arise from a complete S4 algebra in a topos $\mathcal{E}$, in particular a faithful Heyting algebra H

$$i : \Omega_{\mathcal{E}} \leftrightarrows H : \tau$$

The components of the functor $\mathcal{E}^{op} \to \mathbf{Sets}$ are given by the external adjunction

$$\tau_* : \mathrm{Hom}_{\mathcal{E}}(X, H) \leftrightarrows \mathrm{Hom}_{\mathcal{E}}(X, \Omega_{\mathcal{E}}) \cong \mathrm{Sub}_{\mathcal{E}}(X) : i_*. \tag{4}$$

The Beck-Chevalley condition for quantifiers holds by prop. 2.4. For i and τ the Beck-Chevalley condition is simply associativity of composition.

The representable modal doctrines that arise from a faithful Heyting algebra have the nice property that the equality predicate as defined in the hyperdoctrinal way by Lawvere's adjointness condition can be expressed in a simpler more direct way.

Lemma 2.9. *For any X in $\mathcal{E}$ the composite*

$$X \times X \xrightarrow{\delta_X} \Omega_{\mathcal{E}} \xrightarrow{i} H$$

is the equality predicate for the hyperdoctrine (4)*. (Here* δ_X *is the classifying map of the diagonal* $\Delta_X : X \to X \times X$*.) That is to say,*

$$\exists_{\Delta_X}(\top) = i\delta_X.$$

Proof. The top element of $\mathrm{Hom}_{\mathcal{E}}(X, H)$ is the composite $X \to 1 \xrightarrow{\top} H$. Hence $\exists_{\Delta_X}(\top)$ is the transpose of the composite

$$1 \xrightarrow{\quad \top \quad} H^X \xrightarrow{\quad \exists_{\Delta_X} \quad} H^{X \times X}$$

Now in the following diagram

$$\begin{tikzcd}
1 \arrow[r, "\top^X"] \arrow[dr, "\top^X"'] & H^X \arrow[r, "\exists_{\Delta_X}"] & H^{X \times X} \\
& \Omega_{\mathcal{E}}^X \arrow[u, "i^X"] \arrow[r, "\exists_{\Delta_X}"'] & \Omega_{\mathcal{E}}^{X \times X} \arrow[u, "i^{X \times X}"']
\end{tikzcd}$$

the left-hand triangle always commutes. The right-hand square commutes, because the corresponding square with the respective right adjoints does so by the Beck-Chevalley condition for τ. The claim now follows because the transpose of the lower composite is precisely

$$X \times X \xrightarrow{\quad \delta_X \quad} \Omega_{\mathcal{E}} \xrightarrow{\quad i \quad} H.$$

The last can be seen for instance by the final remark of the previous section, and the external description of the map $\exists_{\Delta_X} : \Omega_{\mathcal{E}}^X \longrightarrow \Omega_{\mathcal{E}}^{X \times X}$ through images of subobjects along Δ_X.

$\square$

In fact, any map $\partial_X : X \times X \to H$ such that $i\tau\partial_X = \partial_X$ and that fits into a pullback

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "\top"] \\
X \times X \arrow[r, "\partial_X"'] & H
\end{tikzcd}$$

must necessarily equal $i\delta_X$, since then $\tau\partial_X = \delta_X$, by uniqueness of classifying maps.

A wide class of the previous kind of examples arises from surjective geometric morphisms $f : \mathcal{F} \to \mathcal{E}$ with

$$H = f_* \Omega_{\mathcal{F}}.$$

When f is fixed, we will continue to write Ω_* for $f_*\Omega_{\mathcal{F}}$. The next holds for any geometric morphism.

Proposition 2.10. *For any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, the object Ω_* is a complete Heyting algebra in $\mathcal{E}$.*

Proof. The object Ω_* is a Heyting algebra under the image of f_*, since f_* preserves products. The same algebraic structure is equivalently determined through Yoneda by the external Heyting operations on each $\mathrm{Sub}_{\mathcal{F}}(A^*)$ under the natural isomorphisms

$$\mathrm{Sub}_{\mathcal{F}}(A^*) \cong \mathrm{Hom}_{\mathcal{F}}(A^*, \Omega_{\mathcal{F}}) \cong \mathrm{Hom}_{\mathcal{E}}(A, \Omega_*). \tag{5}$$

Completeness means that Ω_* has I-indexed joins and meets, for any object I in $\mathcal{E}$. One way to see this is to first note that there are isomorphisms (natural in E)

$$\mathrm{Hom}(E, (\Omega_*)^I) \cong \mathrm{Hom}(E \times I, \Omega_*) \cong \mathrm{Hom}(E^* \times I^*, \Omega_{\mathcal{F}}) \cong \mathrm{Hom}(E^*, \Omega_{\mathcal{F}}^{I^*}),$$

where we use that f^* preserves finite limits. Composition with

$$\forall_{I^*} : \Omega_{\mathcal{F}}^{I^*} \longrightarrow \Omega_{\mathcal{F}}$$

hence yields a function

$$\mathrm{Hom}(E, (\Omega_*)^I) \xrightarrow{\cong} \mathrm{Hom}(E^*, \Omega_{\mathcal{F}}^{I^*}) \xrightarrow{\forall_{I^*} \circ (-)} \mathrm{Hom}(E^*, \Omega_{\mathcal{F}}) \xrightarrow{\cong} \mathrm{Hom}(E, \Omega_*),$$

all natural in E. Thus, by the Yoneda lemma, there is a unique map

$$\forall_I : (\Omega_*)^I \longrightarrow \Omega_*$$

such that the function

$$\mathrm{Hom}(E, (\Omega_*)^I) \longrightarrow \mathrm{Hom}(E, \Omega_*)$$

from above is induced by composition with $\forall_I$. The internal map $\forall_I$ is indeed right adjoint to $\Delta_I : \Omega_* \to \Omega_*^I$. For $\Delta_{I^*} : \Omega_{\mathcal{F}} \to \Omega_{\mathcal{E}}^{I^*}$ induces, by composition, a function

$$\mathrm{Hom}(E, \Omega_*) \cong \mathrm{Hom}(E^*, \Omega_{\mathcal{F}}) \xrightarrow{\Delta_{I^*} \circ (-)} \mathrm{Hom}(E^*, \Omega_{\mathcal{F}}^{I^*})$$

with

$$\Delta_{I^*} \circ (-) \dashv \forall_{I^*} \circ (-).$$

This adjunction in turn is the one that corresponds by Yoneda under the isomorphism (5) to the adjunction $\pi_1^* \dashv \forall_{\pi_1}$:

$$\forall_{\pi_1} : \mathrm{Sub}_{\mathcal{F}}(E^* \times I^*) \leftrightarrows \mathrm{Sub}_{\mathcal{E}}(E^*) : \pi_1^*,$$

where π_1^* is pulling back along $\pi_1 : E^* \times I^* \to E^*$. I-indexed joins are treated similarly. $\square$

The same line of argument provides a map

$$g^* : \Omega_*^J \to \Omega_*^I,$$

for any arrow $g : I \to J$ in $\mathcal{E}$, as well as both adjoints. The map g^* internalizes the operation of precomposition

$$\mathrm{Hom}_{\mathcal{E}}(J, \Omega_*) \to \mathrm{Hom}_{\mathcal{E}}(I, \Omega_*).$$

In terms of subobjects, it corresponds to the operation of pulling back along f^*g:

$$f^*g : \mathrm{Sub}_{\mathcal{F}}(f^*J) \to \mathrm{Sub}_{\mathcal{F}}(f^*I).$$

(Again, f^* denotes the inverse image part of a geometric morphism $f : \mathcal{F} \to \mathcal{E}$.) Similarly the adjoints. For instance, the operation

$$\exists_g : \mathrm{Sub}_{\mathcal{F}}(f^*I) \to \mathrm{Sub}_{\mathcal{F}}(f^*J)$$

sends a subobject of f^*I to the image along the composite with f^*g.

To account for the modal operator, we must consider surjective geometric morphisms. Before ging into details, we recall a few equivalent characterizations of surjective geometric morphisms, which we state here without proof. There are more conditions, but the ones below will be most useful for us.

Proposition 2.11. *For any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ the following are equivalent:*

(i) The inverse image part f^* is faithful

(iii) The unit of the adjunction $f^* \dashv f_*$ is a monomorphism

(iv) For each A in $\mathcal{E}$ the inverse image f^* induces an injective lattice homomorphism

$$\mathrm{Sub}_{\mathcal{E}}(A) \to \mathrm{Sub}_{\mathcal{F}}(f^*(A)),$$

natural in A; hence a monic frame homomorphism of functors $\mathrm{Sub}_{\mathcal{E}}(-) \to \mathrm{Sub}_{\mathcal{F}}(f^*(-))$.

(vi) *For each monomorphism $m : A \rightarrowtail B$ the square*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "m"'] & f_* f^*(A) \arrow[d, "f_* f^* m"] \\ B \arrow[r, "\eta_B"] & f_* f^*(B) \end{tikzcd}$$

is a pullback.

(vii) *The canonical frame map $i : \Omega_{\mathcal{E}} \to f_* \Omega_{\mathcal{F}}$ is a monomorphism.*

Any geometric morphism that satisfies either of these conditions is called surjective. □

Consider the map $\tau : f_*\Omega_{\mathcal{F}} \to \Omega_{\mathcal{E}}$ that classifies the top element $f_*\top : 1 \to f_*\Omega_{\mathcal{F}}$, right adjoint to the monic frame map $i : \Omega_{\mathcal{E}} \to f_*\Omega_{\mathcal{F}}$. In view of the isomorphism (5), the external adjunction

$$\tau_* : \mathrm{Hom}_{\mathcal{E}}(X, \Omega_*) \leftrightarrows \mathrm{Hom}_{\mathcal{E}}(X, \Omega_{\mathcal{E}}) : i_*$$

defined by composition with τ and i, resp., can be formulated in terms of the corresponding subobject structure

$$\tau_* : \mathrm{Sub}_{\mathcal{F}}(f^*X) \leftrightarrows \mathrm{Sub}_{\mathcal{E}}(X) : i_*$$

In this context the operation i_* corresponds to applying f^* to subobjects. Since i is monic, i_* (in either form) will also be. Transposing any $U \rightarrowtail X$ along

$$\mathrm{Sub}_{\mathcal{E}}(X) \cong \mathrm{Hom}_{\mathcal{E}}(X, \Omega_{\mathcal{E}}) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{E}}(X, f_*\Omega_{\mathcal{F}}) \cong \mathrm{Hom}_{\mathcal{F}}(f^*X, \Omega_{\mathcal{F}}) \cong \mathrm{Sub}_{\mathcal{F}}(f^*X),$$

must yield the subobject of f^*X that is classified by the transpose along $f^* \dashv f_*$ of the composite $i \circ \alpha$, where $\alpha : X \to \Omega_{\mathcal{E}}$ is the classifying map of U. The transpose of $i\alpha$ is the map

$$f^*X \xrightarrow{f^*\alpha} f^*\Omega_{\mathcal{E}} \xrightarrow{f^*i} f^*f_*\Omega_{\mathcal{F}} \xrightarrow{\varepsilon} \Omega_{\mathcal{F}},$$

with ε the counit at $\Omega_{\mathcal{F}}$. Now the composite $\varepsilon \circ f^*i$ classifies the monomorphism

$$f^*\top : 1 \longrightarrow f^*\Omega_{\mathcal{E}}.$$

The claim now follows because pulling back $f^*\top$ along $f^*\alpha$ yields precisely the monomorphism $f^*U \rightarrowtail f^*X$, as f^* preserves pullbacks.[12]

[12] The converse also holds. That is to say the operation of applying f^* to subobjects, as it commutes with pullback, uniquely determines a map $\Omega_{\mathcal{E}} \to f_*\Omega_{\mathcal{F}}$ through the Yoneda lemma which must then be equal to i.

In a similar fashion one spells out the action of

$$\tau_* : \mathrm{Sub}_{\mathcal{F}}(f^*X) \to \mathrm{Sub}_{\mathcal{E}}(X).$$

For any $m : U \rightarrowtail f^*X$ with classifying map $\beta : f^*X \to \Omega_{\mathcal{F}}$, we must construct the subobject classified by the composite

$$X \xrightarrow{\eta_X} f_*f^*(X) \xrightarrow{f_*\beta} f_*\Omega_{\mathcal{F}} \xrightarrow{\tau} \Omega_{\mathcal{E}}.$$

This is done by forming pullbacks:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & f_*U \arrow[r] \arrow[d, "f_*m"] & 1 \arrow[r, equal] \arrow[d, "f_*\top"] & 1 \arrow[d, "\top"] \\
X \arrow[r, "\eta_X"'] & f_*f^*(X) \arrow[r, "f_*\beta"'] & f_*\Omega_{\mathcal{F}} \arrow[r, "\tau"'] & \Omega_{\mathcal{E}}.
\end{tikzcd}$$

The leftmost square succinctly describes the action of $\tau_* : \mathrm{Sub}_{\mathcal{F}}(f^*X) \to \mathrm{Sub}_{\mathcal{E}}(X)$.

Lemma 2.12. *For each A in $\mathcal{E}$, the maps*

$$i_* : \mathrm{Sub}_{\mathcal{E}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{F}}(f^*(A))$$

$$\tau_* : \mathrm{Sub}_{\mathcal{F}}(f^*(A)) \longrightarrow \mathrm{Sub}_{\mathcal{E}}(A)$$

satisfy the Beck-Chevalley condition.

Proof. For i_*, and any arrow $g : B \to A$, commutativity of

$$\begin{tikzcd}
\mathrm{Sub}_{\mathcal{E}}(A) \arrow[r, "i_*"] \arrow[d, "g^*"'] & \mathrm{Sub}_{\mathcal{F}}(f^*(A)) \arrow[d, "g^*"] \\
\mathrm{Sub}_{\mathcal{E}}(B) \arrow[r, "i_*"] & \mathrm{Sub}_{\mathcal{F}}(f^*(B))
\end{tikzcd}$$

follows because f^* preserves pullbacks. For τ_* one invokes the unit of the adjunction $f^* \dashv f_*$:

$$\begin{tikzcd}
f_*f^*A \arrow[r, "f_*f^*g"] & f_*f^*B \\
A \arrow[u, "\eta_A"] \arrow[r, "g"'] & B \arrow[u, "\eta_B"']
\end{tikzcd}$$

so that for every subobject $m : U \rightarrowtail f^*B$ the pullback of f_*m along either composite must be identical, which is precisely saying that

$$\begin{tikzcd}
\mathrm{Sub}_{\mathcal{E}}(A) \arrow[d, "g^*"'] & \mathrm{Sub}_{\mathcal{F}}(f^*(A)) \arrow[l, "\tau_*"'] \arrow[d, "g^*"] \\
\mathrm{Sub}_{\mathcal{E}}(B) & \mathrm{Sub}_{\mathcal{F}}(f^*(B)) \arrow[l, "\tau_*"']
\end{tikzcd}$$

□

Although by lemma 2.9 we know that the equality predicate $M \times M \to \Omega_*$ can be expressed by the map $i \circ \delta_M$, there is an equivalent description (prop. 2.14 below) which was first studied in [29]. Before turning to that, we need a lemma.

Lemma 2.13. *For any map $\alpha : D \to \Omega_*$, it holds that $i\tau \circ \alpha = \alpha$ iff the subobject classified by the transpose $\widetilde{\alpha} : f^*D \to \Omega_{\mathcal{F}}$ of α is of the form $f^*m : f^*U \rightarrowtail f^*D$, for some $m : U \rightarrowtail D$ in $\mathcal{E}$.* □

Proof. Assuming $i\tau \circ \alpha = \alpha$, consider the subobject $m : U \rightarrowtail D$ classified by α. This yields pullbacks (denoting again $f^*(-)$ by $(-)^*$)

$$\begin{tikzcd}
U^* \arrow[r] \arrow[d, "m^*"'] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
D^* \arrow[r, "\alpha^*"'] & (\Omega_*)^* \arrow[r, "\tau^*"'] & \Omega^* \arrow[r, "\tau_\bullet"'] & \Omega_{\mathcal{F}}
\end{tikzcd}$$

where $\tau_\bullet$ is the classifying map of $\top^* : 1 \to \Omega^*$, and Ω^* is short for $\Omega_{\mathcal{E}}{}^*$. The transpose of α is the composite $\varepsilon_{\Omega_{\mathcal{F}}} \circ \alpha^*$. On the other hand, $\tau_\bullet$ is the transpose of $i : \Omega_{\mathcal{E}} \to \Omega_*$, i.e.

$$\tau_\bullet = \varepsilon_{\Omega_{\mathcal{F}}} \circ i^*.$$

Hence the above diagram extends to

$$\begin{tikzcd}
U^* \arrow[r] \arrow[d, "m^*"'] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[rr, equal] \arrow[d, "\top"] & & 1 \arrow[d, "\top"] \\
D^* \arrow[r, "\alpha^*"'] & (\Omega_*)^* \arrow[r, "\tau^*"'] & \Omega^* \arrow[r, "i^*"'] & (\Omega_*)^* \arrow[r, "\varepsilon_{\Omega_{\mathcal{F}}}"'] & \Omega_{\mathcal{F}}
\end{tikzcd}$$

Sicne the lower composite equals $\widetilde{\alpha} = \varepsilon_{\Omega_{\mathcal{F}}} \circ \alpha^*$, the claim follows.

Conversely, suppose $\widetilde{\alpha}$ classifies a subobject $m^* : U^* \rightarrowtail D^*$. Then $\tau\alpha$ classifies $m : U \rightarrowtail D$:

$$\begin{tikzcd}
U \arrow[r, "\eta_U"] \arrow[d, "m"'] & (U^*)_* \arrow[rr] \arrow[d, "(m^*)_*"'] & & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
D \arrow[r, "\eta_D"'] & (D^*)_* \arrow[r, "(\alpha^*)_*"'] & ((\Omega_*)^*)_* \arrow[r, "(\varepsilon_{\Omega_{\mathcal{F}}})_*"'] & \Omega_* \arrow[r, "\tau"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

The leftmost unit square is a pullback, because the geometric morphism f is surjective. Moreover, $\alpha = (\varepsilon_{\Omega_{\mathcal{F}}}\alpha^*)_*\eta_D$ by definition of transposing back and forth. Hence, since $\tau\alpha$ classifies m, the subobject m^* is classified by the composite

$$\tau_\bullet \circ (\tau\alpha)^* = \varepsilon_{\Omega_{\mathcal{F}}} i^* \circ \tau^*\alpha^*,$$

i.e. the transpose of $i\tau\alpha$. Therefore,

$$\widetilde{\alpha} = \widetilde{i\tau\alpha},$$

by uniqueness of classifying maps. So finally $\alpha = i\tau\alpha$ by uniqueness of transposing maps along adjunctions. $\square$

Denote by δ_{M^*} be the classifying map of the diagonal $\langle 1_{M^*}, 1_{M^*}\rangle : M^* \to M^* \times M^*$. We will write its transpose along $f^* \dashv f_*$ simply as

$$M \times M \xrightarrow{\delta_*} \Omega_* \tag{6}$$

when M is clear.

Proposition 2.14. *For any object D in $\mathcal{E}$, and any geometric morphism $f : \mathcal{F} \to \mathcal{E}$:*

$$\delta_* = i \circ \delta_D.$$

Proof. We prove this by showing

$$\tau \circ \delta_* = \delta_D,$$

whence the statement follows from $\delta_* = i \circ \tau \circ \delta_* = i \circ \delta_D$, where the identity $\delta_* = i \circ \tau \circ \delta_*$ holds by applying lemma 2.13 to δ_*.

The proof is essentially contained in the following diagram

$$\begin{tikzcd}
D \arrow[r, "\eta_D"] \arrow[dd, "\Delta_D"'] & (D^*)_* \arrow[r] \arrow[dd, "(\Delta^*)_*"] & 1 \arrow[dd, "\top"] \arrow[dr, equal] & \\
& & & 1 \arrow[dd, "\top"] \\
D \times D \arrow[r, "\eta_D \times \eta_D"] \arrow[rr, bend right, "\delta_*"'] \arrow[drrr, bend right, "\delta_D"'] & (D^*)_* \times (D^*)_* \arrow[r, "(\delta_{D^*})_*"] & \Omega_* \arrow[dr, "\tau"] & \\
& & & \Omega_{\mathcal{E}}
\end{tikzcd}$$

where $\Delta_D = \langle 1_D, 1_D \rangle$, η is the unit of $f^* \dashv f_*$, and δ, τ denote the respective classifying maps. The square in the middle is a pullback, since f_* preserves them. Moreover, by the definition of δ_D, the large outer square is a pullback. Note further that $\delta_* = (\delta_{D^*})_* \circ \eta_{D \times D}$, by the definition of δ_* as the transpose of δ_{D^*} along $f^* \dashv f_*$. Thus the desired equality would follow if the unit square were a pullback, for then

$$\tau \circ (\delta_{D^*})_* \circ \eta_{D \times D} = \tau \circ \delta_*$$

would classify Δ_D, and so $\tau \circ \delta_* = \delta_D$. This is in fact the case. For $f : \mathcal{F} \to \mathcal{E}$ being surjective (i.e. f^* faithful) implies that the unit components, and therefore $\eta_D \times \eta_D$, are monic. A direct verification then shows that the square is a pullback. $\square$

Remark 2.15. The class of examples deriving from geometric morphisms are in a sense representative, since it follows from general topos-theoretic considerations that any faithful Heyting algebra over $\Omega_{\mathcal{E}}$ is of the form $f_* \Omega_{\mathcal{F}}$, for a some geometric morphism $f : \mathcal{F} \to \mathcal{E}$, for some topos $\mathcal{F}$. Specifically, for any complete Heyting algebra H in $\mathcal{E}$ one may form the topos $\mathrm{Sh}_{\mathcal{E}}(H)$ of $\mathcal{E}$-internal sheaves on H. There results a geometric morphism

$$p : \mathrm{Sh}_{\mathcal{E}}(H) \longrightarrow \mathcal{E}$$

with the property that

$$f_* \Omega_{\mathrm{Sh}_{\mathcal{E}}(\mathcal{H})} \cong H.$$

In particular, p is surjective just in case the initial frame map $i : \Omega_{\mathcal{E}} \to H$ is monic.

2.3 Relative Modal Structures

In this section we study a useful weakening of the canonical modal structure studied in the previous section, and develop some of its properties. The difference is that instead of considering modal adjunctions

$$i : \Omega_{\mathcal{E}} \leftrightarrows H : \tau$$

we consider S4 algebras

$$i : B \leftrightarrows H : \tau$$

in a topos $\mathcal{E}$ where B is a Heyting algebra for which the classfiying map $\beta : B \to \Omega_{\mathcal{E}}$ of the top element of B is a monomorphism. We begin by giving some examples.

- In any topos the classifying map of the coproduct inclusion $\top : 1 \to 1 + 1$.

- For any Grothendieck topos $\mathrm{Sh}(\mathbf{C}, J)$ the subobject classifier Ω_J assigns to every object C in $\mathbf{C}$ the set of *closed* sieves on C. The classifying map of the top elementof Ω_J assigns to each closed sieve $\sigma \in \Omega_J(C)$ the set
$$\tau_C(\sigma) = \{Z \to C \mid f^*\sigma = \top_Z\}.$$
Here $\top_Z$ is the maximal sieve on Z, f^* is restriction along f. Since for any sieve, $s \in \sigma$ if and only if $s^*\sigma = \top$, then, if $\tau_C(\sigma) = \tau_C(\rho)$, then for any $s : X \to C$ in σ clearly $s^*\sigma = \top_X$. Hence $s^*\rho = \top_X$, and so $s \in \rho$. Similarly, $\rho \subseteq \sigma$. In fact, τ_C is simply the inclusion of Ω_J as a subfunctor of Ω.

- More generally, for any topos $\mathcal{E}$, and any Lawvere-Tierney topology $j : \Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}$,[13] since $j\top = j$, the map $\top$ factors through the image $m : \Omega_j \rightarrowtail \Omega_{\mathcal{E}}$ of j via a map $\top_j : 1 \to \Omega_j$. It is necessarily a pullback, since m is monic. The map $\top_j : 1 \to \Omega_j$ is the subobject classifier in the topos of j-sheaves, and hence a Heyting algebra. In fact, up to equivalence of categories every geometric embedding is of the form $\mathrm{Sh}_j(\mathcal{E}) \hookrightarrow \mathcal{E}$.

- For any small topos $\mathcal{E}$, the finite epi topology J is subcanonical. Therefore, the Yoneda embedding factors through $\mathrm{Sh}(\mathcal{E}, J)$, i.e. every representable presheaf is a sheaf. The classifying map $\mathbf{y}\Omega_{\mathcal{E}} \to \Omega$ is a monomorphism. In fact, it is (componentwise) the ideal completion of $\mathbf{y}\Omega_{\mathcal{E}}$. We will return to this example later. (This is an example where the Heyting algebra $\mathbf{y}\Omega_{\mathcal{E}}$ in question is actually not complete.)

  For instance, the trivial topology (only maximal sieves cover) is finitely epimophic. Hence the classifying map $\mathbf{y}\Omega_{\mathcal{E}} \to \Omega$ in $\mathbf{Sets}^{\mathcal{E}^{op}}$ is monic, which is also readily checked directly.

[13] Johnstone calls them *local operators*.

- Let $f : \mathcal{F} \to \mathcal{E}$ be any locally connected geometric morphism. Then the classifying map
$$f^*\Omega_{\mathcal{E}} \to \Omega_{\mathcal{F}}$$
of $f^*\top : f^*1 \to f^*\Omega_{\mathcal{E}}$ is a monomorphism.

The next proposition expresses that for any $\beta : B \to \Omega_{\mathcal{E}}$, the complete lattice structure B is completely determined by a choice of a complete Heyting algebra A and an adjunction

$$i : B \leftrightarrows A : \tau$$

in which i is a monomorphism.

Proposition 2.16. *In any typos $\mathcal{E}$, consider any S4 algebra $i : B \leftrightarrows A : \tau$ in $\mathcal{E}$ such that the classifying map $\beta : B \to \Omega_{\mathcal{E}}$ of the top element $\top : 1 \to B$ is a monomorphism. Then, if A is a complete Heyting algebra, so is B. Moreover, the map i exhibits B as a subframe of A.*

Proof. Define the meet operation on B as

$$\forall_I^{(B)} : B^I \xrightarrow{i^I} A^I \xrightarrow{\forall_I} A \xrightarrow{\tau} B.$$

For any $x \in B$, and $g \in B^I$:

$$\begin{aligned} x \leq \tau\forall_I i^I(g) \;\; &\text{iff} \;\; i(x) \leq \forall_I i^I(g) \\ &\text{iff} \;\; \Delta_I i(x) \leq i^I(g) \\ &\text{iff} \;\; i^I\Delta_I^{(B)}(x) \leq i^I(g) \\ &\text{iff} \;\; \Delta_I^{(B)}(x) \leq g. \end{aligned}$$

The last step uses that i^I is an order-embedding given that i is.[14] The equality $\Delta_I i = i^I\Delta_I^{(B)}$ holds because the left-hand square in the following commutes, as it is the exponential transpose of the right-hand one:

$$\begin{tikzcd} A \arrow[r, "\Delta_I"] & A^I \\ B \arrow[u, "i"] \arrow[r, "\Delta_I^{(B)}"'] & B^I \arrow[u, "i^I"'] \end{tikzcd} \qquad \begin{tikzcd} A \times I \arrow[r, "\pi"] & A \\ B \times I \arrow[u, "i\times 1"] \arrow[r, "\pi"'] & B \arrow[u, "i"'] \end{tikzcd}$$

For further use we note that an analogous argument shows that

$$i^I\tau^I\Delta_I = \Delta_I i\tau.$$

[14] The functor $(-)^I$, as a right adjoint, preserves monomorphisms and pullbacks.

However, the map i does not necessarily preserve $\forall_I^{(B)}$. A counterexample is the same as provided below which showed that the sequent $\forall_{x:A}\Box\varphi \vdash \Box\forall_{x:A}\varphi$ is not provable. It will be given later once the required notions are introduced.

The left adjoint $\exists_I^{(B)} : B^I \to B$ exists, since $\Delta_I^{(B)}$ preserves arbitrary meets. To see this note that

$$i^I : B^I \leftrightarrows A^I : \tau^I$$

is an S4 algebra, since exponentiation preserves all the relevant properties. Then $\Delta_I : A \to A^I$ is an S4 algebra map that preserves the complete Heyting structure as it has both a left and right adjoint. Hence we can write $\Delta_I^{(B)}$ as

$$\Delta_I^{(B)} = \tau^I \Delta_I i,$$

in accordance with former observation.

To prove that $\Delta_I^{(B)}$ preserves all meets we show that for every J the following commutes:

$$\begin{tikzcd}
B^J \arrow[r, "\forall_J^{(B)}"] \arrow[d, "(\Delta_I^{(B)})^J"'] & B \arrow[d, "\Delta_I^{(B)}"] \\
(B^I)^J \arrow[r, "\forall_J^{(B^I)}"'] & B^I
\end{tikzcd}$$

To begin with, the diagram translates into the following, unwinding definitions:

$$\begin{tikzcd}
B^J \arrow[r, "i^J"] & A^J \arrow[r, "\forall_J"] \arrow[d, "(\Delta_I)^J"'] \arrow[dr, phantom, "(1)"] & A \arrow[r, "\tau"] \arrow[d, "\Delta_I"] \arrow[dr, phantom, "(2)"] & B \arrow[d, "\Delta_I^{(B)}"] \\
& (A^I)^J \arrow[r, "\forall_J^{(A^I)}"'] \arrow[d, "(\tau^I)^J"'] & A^I \arrow[r, "\tau^I"'] \arrow[d, "\tau^I"'] \arrow[dr, phantom, "(4)"] & B^I \\
& (B^I)^J \arrow[r, phantom, "(3)"] \arrow[d, "(i^I)^J"'] & B^I \arrow[d, "i^I"'] & \\
& (A^I)^J \arrow[r, "\forall_J^{(A^I)}"'] & A^I \arrow[uur, bend right, "\tau^I"'] &
\end{tikzcd}$$

The upper horizontal composite $\tau\forall_J i^J$ is the definition of $\forall_J^{(B)}$. Furthermore,

$$(\tau^I)^J(\Delta_I)^J i^J = (\Delta_I^{(B)})^J,$$

$$\tau^I\forall_J(i^I)^J = \forall_J^{(B^I)}.$$

The square (1) commutes because Δ_I preserves big meets [it has a right adjoint w.r.t. the relevant ordering], while (2) always commutes. While the square (3) does not necessarily commute the composite square of (3) and (4) does.[15] To see this note first that both composites

$$A^J \xrightarrow{\forall_J} A \xrightarrow{\tau} B \xrightarrow{\beta} \Omega_{\mathcal{E}}$$

$$A^J \xrightarrow{(i\tau)^J} A^J \xrightarrow{\forall_J^I} A \xrightarrow{\tau} B \xrightarrow{\beta} \Omega_{\mathcal{E}}$$

classify the top element of A^J, and hence are equal. Therefore, since β is monic, the respective composites without β on the right coincide as well. Exponentiating with I then gives the composite square (3)-(4) drawn before, modolu the isomorphism $(A^I)^J \cong (A^J)^I$ (Recall that modulo that isomorphism the map $\forall_J^{(A^I)}$ was defined as $(\forall_J)^I$).

$\square$

[15] Compare to the fact that the double sequent $\Box\forall_{x:A}\Box\varphi \dashv\vdash \Box\forall_{x:A}\varphi$ is provable in the logic introduced later.

Note also that once $\exists_I^{(B)}$ is in place, the following commutative diagram verifies the equation

$$\tau \exists_I i^I = \exists_I^{(B)} :$$

$$\begin{tikzcd}
B^I \arrow[r, "\exists_I^{(B)}"] \arrow[d, "i^I"'] & B \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, "1_B"] & A \arrow[d, "\tau"] \\
A^I \arrow[r, "\exists_I"'] & A \arrow[r, "\tau"'] & B
\end{tikzcd}$$

Here, the left-hand side commutes, because the square with the respective right adjoints does. In particular, the left-hand-side expresses the fact that i preserves arbitrary joins, as is expected from i having a right adjoint. A consequence of the last observation is that

$$i\tau \exists_I i^I = i\exists_I^{(B)} = \exists_I i^I.$$

We have met this property when we considered modal hyperdoctrines. However, the difference to def. 2.6 is that there we assumed certain properties that we here derived from the mere assumption that $\beta : B \to \Omega_{\mathcal{E}}$ is a monomorphism.

We will later discuss the case where the Heyting algebras are not complete but only complete w.r.t. a certain class of objects $\mathcal{M}$ in $\mathcal{E}$.

Definition 2.17. *In any topos $\mathcal{E}$, and collection $\mathcal{M}$ of objects in $\mathcal{E}$, a Heyting algebra B is called $\mathcal{M}$-complete, if the map $\Delta_M : B \to B^M$ has both a left and a right adjoint, for all $M \in \mathcal{M}$.*

An analogous fact to prop. 2.16 holds for this case.

Proposition 2.18. *Given any S4 algebra $i : B \leftrightarrows A : \tau$ in $\mathcal{E}$, if A is an $\mathcal{M}$-complete Heyting algebra, then to is B. Moreover, i exhibits B as a sub-$\mathcal{M}$-frame of A.*

Note, incidentally, that for any $\mathcal{M}$-complete Heyting algebra B in a topos $\mathcal{E}$, and any map $f : X \to Y$ between objects $X, Y \in \mathcal{M}$, the exponentiated map $f^* : B^Y \to B^X$ will both have a left and a right adjoint. This observation will be important later one when defining representable modal hypderdoctrines on $\mathcal{M}$-complete S4 algebras.[16]

We now turn to the equality predicate for B-relative structures. In proposition 2.9 we gave an explicit description of the abstract definition of equality given through the hyperdoctrinal approach. One may wonder

[16] Cf. example 2.2.

whether it is possible to give a similar characterization of equality for the representable hyperdoctrine associated with a B-relative S4 algebra.

To begin with, consider any X in $\mathcal{E}$ for which the classifying map $\delta_X : X \times X \to \Omega_{\mathcal{E}}$ of Δ_X factors through $\beta : B \hookrightarrow \Omega_{\mathcal{E}}$, by a, necessarily unique, map $\partial_X : X \times X \to B$.[17] Inspecting the following diagram

$$\begin{tikzcd}
& A \arrow[d, "\tau"] & \\
X \times X \arrow[ur, "\exists_{\Delta_X}(\top)"] \arrow[r, "\partial_X"'] & B \arrow[r, tail, "\beta"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

it is easy to see that the left-hand triangle commutes if and only if $\exists_{\Delta_X}(\top)$ fits into a pullback diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "\top"] \\
X \times X \arrow[r, "\exists_{\Delta_X}(\top)"'] & A
\end{tikzcd} \tag{7}$$

This then provides a factorization $\exists_{\Delta_X}(\top) = i \circ \partial_X$, because the equality is box stable. Conversely, $\exists_{\Delta_X}(\top) = i \circ \partial_X$ implies $\tau\exists_{\Delta_X}(\top) = \partial_X$, because i is monic.[18] To sum up:

Fact 2.19. *Consider any X for which the classifying map $\delta_X : X \times X \to \Omega_{\mathcal{E}}$ of Δ_X factors through $\beta : B \hookrightarrow \Omega_{\mathcal{E}}$ via map $\partial_X : X \times X \to B$. Then the following are equivalent:*

- $\exists_{\Delta_X}(\top) = i \circ \partial_X$
- $\tau\exists_{\Delta_X}(\top) = \partial_X$
- (7) *is a pullback*

The following proposition states another equivalent condition that does not directly mention the equality predicate $\exists_{\Delta_X}(\top)$.

[17] For formal reasons, we also assume that the map $\Delta_X^* : B^{X \times X} \to B^X$ has a left adjoint. In later applications this will always be the case. For instance, if X belongs to the class $\mathcal{M}$, for which B is complete.

[18] Note, incidentally, that since $\Delta_X^*\exists_{\Delta_X}(\top) = \top$, by the unit of the adjunction $\exists_{\Delta_X} \dashv \Delta_X^*$, the square (7) always commutes.

Proposition 2.20. *Consider any X for which the classifying map $\delta_X : X \times X \to \Omega_{\mathcal{E}}$ of Δ_X factors through $\beta : B \hookrightarrow \Omega_{\mathcal{E}}$ via map $\partial_X : X \times X \to B$. Then the following are equivalent:*

(i) $\exists_{\Delta_X}(\top) = i \circ \partial_X$

(ii) $\overline{\partial_X} = \exists_{\Delta_X} \top^X$*, where* $\overline{\partial_X}$ *denotes exponential transposition, and* $\exists_{\Delta_X} \top^X$ *is the map*

$$1 \xrightarrow{\top^X} B^X \xrightarrow{\exists_{\Delta_X}} B^{X\times X}$$

(iii) β *preserves the equality relation in the sense that the following commutes:*

$$\begin{tikzcd} 1 \arrow[r, "\top^X"] \arrow[dr, "\top^X"'] & B^X \arrow[r, "\exists_{\Delta_X}"] & B^{X\times X} \arrow[d, "\beta^{X\times X}"] \\ & \Omega_{\mathcal{E}}^X \arrow[r, "\exists_{\Delta_X}"'] & \Omega_{\mathcal{E}}^{X\times X} \end{tikzcd} \tag{8}$$

Proof. Suppose δ_X factors through β via a map $\partial_X : X \times X \to B$ and that (8) holds. We show that (ii) holds. To see this, note that

$$\beta\partial_X = \delta_X$$

along with lemma 2.9 entails that the following commutes:

$$\begin{tikzcd} 1 \arrow[r, "\overline{\partial_X}"] \arrow[d, "\top"'] & B^{X\times X} \arrow[d, "\beta^{X\times X}"] \\ \Omega_{\mathcal{E}}^X \arrow[r, "\exists_{\Delta_X}"'] & \Omega_{\mathcal{E}}^{X\times X} \end{tikzcd} \tag{9}$$

With the assumption (8), it follows that

$$\overline{\partial_X} = \exists_{\Delta_X} \top^X,$$

because $\beta^{X\times X}$ is monic. To show (ii) $\Rightarrow$ (i) observe that

$$\begin{tikzcd} 1 \arrow[r, "\top^X"] \arrow[dr, "\top^X"'] & A^X \arrow[r, "\exists_{\Delta_X}"] & A^{X\times X} \\ & B^X \arrow[u, "i^X"] \arrow[r, "\exists_{\Delta_X}"'] & B^{X\times X} \arrow[u, "i^{X\times X}"'] \end{tikzcd}$$

commutes; the triangle does by the usual properties of i, and the square with the respective right adjoints does by definition of map of modal algebras (see def. 2.6). Hence the transposes of the outer two composites must be equal. Given (ii), this is precisely saying that

$$\exists_{\Delta_X}(\top) = i \circ \partial_X.$$

In the other direction, we show that (i) implies (ii) which in turn implies (iii). In fact, $\overline{\partial_X} = \exists_{\Delta_X} \circ \top^X$ entails (iii) because (9) commutes. So assume that $\exists_{\Delta_X}(\top) = i \circ \partial_X$. To see that exponential transpose of $\exists_{\Delta_X} \circ \top^X$ is ∂_X, we may use the decription of $\exists_{\Delta_X} : B^X \to B^{X\times X}$ given earlier as

$$B^X \xrightarrow{\quad i^X \quad} A^X \xrightarrow{\quad \exists_{\Delta_X} \quad} A^{X\times X} \xrightarrow{\quad \tau^{X\times X} \quad} B^{X\times X}$$

Since $i^X \circ \top^X = \top^X$, we are led to compute the transpose of

$$1 \xrightarrow{\quad \top^X \quad} A^X \xrightarrow{\quad \exists_{\Delta_X} \quad} A^{X\times X} \xrightarrow{\quad \tau^{X\times X} \quad} B^{X\times X}$$

But this is precisely $\tau\exists_{\Delta_X}(\top)$, which equals ∂_X by the previous observations. $\square$

Although we don't have a counterexample, it does not seem that the mere existence of β and the factorization ∂_X are already sufficient to entail either of the foregoing equivalent conditions. Hence, in the case of a B-relative modal structure, where one considers a B-relative S4 algebra $i : B \leftrightarrows A : \tau$ in $\mathcal{M}$ with the associated functor $\mathcal{M} \to \mathbf{HA}$

$$i_* : \mathrm{Hom}_{\mathcal{E}}(X, B) \leftrightarrows \mathrm{Hom}_{\mathcal{E}}(X, A) : \tau_*,$$

the hyperdoctrinal equality predicate $\exists_{\Delta_X}(\top)$ may not in general be described as $i \circ \partial_X$. Thus one might want to impose the additional requirement that for each X in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "\top"] \\ X \times X \arrow[r, "\exists_{\Delta_X}(\top_X)"'] & A \end{tikzcd}$$

is a pullback. Formulated as a requirement on the underlying choice of B, the definition of the equality predicate, for each M in $\mathcal{M}$, as the transpose of

$$1 \xrightarrow{\top} A^M \xrightarrow{\exists_{\Delta_M}} A^{M\times M}$$

then remains intact. All of the representable relative structures that we will meet presently are of that form, which is why we will mostly define equality directly through $i \circ \partial_X$.

We will meet an example of a structure that meets these conditions when considering relative models in the topos of sheaves on a small topos. In this respect, note that in the proof we didn't make use of the fact that either A or B is actually complete, aside from the assumption that the adjoint $\exists_{\Delta_X}$ exists. Therefore the propositions still applies in case A and B are merely complete w.r.t. to certain collection of objects as long as every such object X admits the factorization ∂_X. This is of interest insofar as relative models seem to be generally useful when studying S4 algebras that are complete only w.r.t. certain objects.

Definition 2.21. *For any topos $\mathcal{E}$, consider any Heyting algebra H for which the classifying map $\imath : H \to \Omega_{\mathcal{E}}$ of the top element is a monomorphism. An object A is called* H-standard *if the classifying map $\delta_A : A \times A \to \Omega_{\mathcal{E}}$ of the diagonal $\Delta_A : A \to A \times A$ factors through $\imath : H \to \Omega_{\mathcal{E}}$.*

Recall that such a factorization, if it exists, is necessarily unique. As before, we will denote it be $\partial_A : A \times A \to H$. It follows immediately that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "\top"] \\ A \times A \arrow[r, "\partial_A"'] & H \end{tikzcd}$$

is a pullback. Thus ∂_A behaves very much like the classifying map of the diagonal.

H-standard objects may be used to interpret the types in the language. In particular, ∂_A is used to interpret equality on a type. For this to work, it needs to be verified that H-standard objects are closed under the categorical type forming operations.

Lemma 2.22. *For any $\mathcal{E}$, and any relative algebra $\imath : H \hookrightarrow \Omega_{\mathcal{E}}$ in $\mathcal{E}$, the collection of H-standard objects is closed under finite products and exponentiation by objects in $\mathcal{M}$.*

Proof.

- The terminal object is H-standard, because $1 \times 1 \cong 1$, so that the diagonal coincides with the identity map on 1. The classifying map of the identity is $\top : 1 \to \Omega_{\mathcal{E}}$. Since $\imath : H \to \Omega$ preserves $\top$, the map $\top : 1 \to H$ yields the required factorization.

- Let A and B be H-standard and consider

$$\Delta_{A\times B} = \langle \Delta_A, \Delta_B \rangle : A \times B \to (A \times B) \times (A \times B).$$

Therefore, modulo the isomorphism $(A \times B) \times (A \times B) \cong (A \times A) \times (B \times B)$, there is the following pullback diagram:

$$\begin{tikzcd}
A \times B \arrow[r] \arrow[d, "\Delta_{A\times B}"'] & 1 \arrow[r, equal] \arrow[d, "\langle \top, \top \rangle"] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
(A \times A) \times (B \times B) \arrow[r, "\partial_A \times \partial_B"] & H \times H \arrow[r, "\wedge"] & H \arrow[r, "\imath"] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

Hence the composite at the bottom must equal $\delta_{A\times A}$ and $\wedge \circ \partial_A \times \partial_B$ is the required factorization.

- Suppose A is H-standard, and consider any $M \in \mathcal{M}$. The factorization ∂_{A^M} of δ_{A^M} is given, modulo the isomorphism $A^M \times A^M \cong (A \times A)^M$, through the composite

$$(A \times A)^M \xrightarrow{(\partial_A)^M} H^M \xrightarrow{\forall_M} H \xrightarrow{\imath} \Omega_{\mathcal{E}},$$

by uniqueness of classifying maps as before.

$\square$

The corresponding facts for the interpretation of the type of propositions is contained in the following lemma.

Lemma 2.23. *For any relative complete S4 algebra (H, M, i, θ) in $\mathcal{E}$, both H and M are H-standard.*

Proof. The factorization ∂_H is obtained because the following is a pullback.

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "\top"] \\
H \times H \arrow[r, "\Leftrightarrow"'] & H
\end{tikzcd}$$

So the factorization of δ_H as $\imath \circ \Leftrightarrow$ follows from uniqueness of classifying maps. As for M, there are pullbacks

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
M \times M \arrow[r, "\Leftrightarrow"'] & M \arrow[r, "\theta"'] & H
\end{tikzcd}$$

so the lower composite is a factorization of δ_M. $\square$

Definition 2.24. *Given a topos $\mathcal{E}$, and a collection of objects $\mathcal{M}$ of $\mathcal{E}$, a* relative $\mathcal{M}$-modal structure *is a triple (B, A, i) where*

- *A and B are $\mathcal{M}$-complete Heyting algebras*
- *i is a monic map of $\mathcal{M}$-complete Heyting algebras*
- *the classifying map $\beta : B \to \Omega_{\mathcal{E}}$ of $\top : 1 \to B$ is a monomorphism*
- *the classifying map $\theta : A \to \Omega_{\mathcal{E}}$ of $\top : 1 \to A$ factors through $\imath$ via a map $\tau : A \to B$, and $i \dashv \tau$*
- *each M in $\mathcal{M}$ is B-standard and B satisfies the condition* (8) *of proposition 2.20.*

The adjointness condition $i \dashv \tau$ involves the information that for any B for which β is monic, there is at most one map i that may possibly define a modal structure on A. Moreover, since i is monic and $i \dashv \tau$, it follows that $1 = \tau i$.

We have seen examples of Heyting algberas $B \rightarrowtail \Omega_{\mathcal{E}}$ before. We saw that a significant class of examples was provided by geometric embeddings $e : \mathcal{F} \hookrightarrow \mathcal{E}$. These examples are special because $e_*\Omega_{\mathcal{F}}$ will always be complete, as the direct image part always preserves completeness.[19] By contrast, the inverse image of a locally connected geometric morphism $f : \mathcal{E} \to \mathcal{F}$ does not preserve completeness of Heyting algebras. However, the object $f^*\Omega_{\mathcal{F}}$ is complete w.r.t. objects of the form f^*A. In case $\mathcal{M}$ consists of the objects in the image of a functor F, we refer to a Heyting algebra that is $\mathcal{M}$-complete as being F-complete. Thus, for instance, the Heyting algebra $f^*\Omega_{\mathcal{F}}$ is f^*-complete, for the inverse image part of a locally connected geometric morphism $f : \mathcal{F} \to \mathcal{E}$. Moreover, for each object A in $\mathcal{E}$

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "\top"] \\
f^*A \times f^*A \arrow[r, "f^*\delta_A"'] & f^*\Omega_{\mathcal{F}}
\end{tikzcd}$$

is pullback, since f^* preserves finite limits. Hence every object in the image of f^* satisfies the fourth condition in def. 2.24.

[19] Incidentally, we note that the map β is the right adjoint of the initial frame map $\Omega_{\mathcal{E}} \to e_*\Omega_{\mathcal{F}}$. In fact, since $\mathcal{F} \simeq \mathrm{sh}_j(\mathcal{E})$, for a unique local operator $j : \Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}$, and $e_*\Omega_{\mathcal{F}}$ classifies j-closed subobjects in $\mathcal{E}$, it follows by [15] (C4.3.6) that the $e_*\Omega_{\mathcal{F}}$-standard objects are the j-separated ones.

The slightly more general notion of relative model structures enables one to obtain a sufficiently flexibel notion of model preserving functor between toposes. For in general geometric morphisms do not preserve the subobject classifier, and hence do not preserve Ω-based model structures. We will use this additional generality to study topological models.

3 Higher-Order Modal Logic

3.1 Intuitionistic Higher-Order S4

The formal system of higher-order modal logic considered here is simply the union of the usual axioms for higher-order logic and S4. The higher-order part is a version of type theory (cf. [13, 15, 16]). Types and terms are defined recursively. A higher-order language $\mathcal{L}$ consists of a collection of basic types $A, B, \ldots$ along with basic terms (constants) $a : A, b : B$. To stay close to topos-theoretic formulations, we assume the following type and term forming operations that inductively specify the collection of types and terms of the language:

- There are basic types 1, P
- If A, B are types, then there is a type $A \times B$
- If A, B are types, then there is a type A^B

Terms are recursively constructed as follows. Here we assume, for every type A, an infinite set of variables of type A, written as $x : A$, to be given. We follow [13] in writing $\Gamma \mid t : B$, for $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$, involving at least all the free variables in the term t. A context Γ may also be empty. Formally, every term t always occurs in some variable context Γ and is well-typed only w.r.t. such a context. This is important to understand the recursive clauses below. To simplify notation, however, we omit Γ if it is unspecified and the same throughout a recursive clause.

- There are distinguished terms $\emptyset \mid * : 1$ and $\emptyset \mid \top, \bot : \mathsf{P}$
- If $t : A$ and $s : B$ are terms, then $\langle t, s\rangle : A \times B$ is a term
- If $t : A \times B$ is a term, then there are terms $\pi_1 t : A$ and $\pi_2 t : B$
- If $\Gamma \mid t : A$ is a term and $y : B$ a variable in Γ, then there is a term $\Gamma[y : B] \mid \lambda y.t : A^B$; where $\Gamma[y : B]$ is the context that results from Γ by deleting $y : B$.
- If $t : A^B$ and $s : B$ are terms, then $\mathsf{app}(t, s) : A$ is a term.
- For any two terms $t : \mathsf{P}$, $s : \mathsf{P}$ there are terms $t \wedge s : \mathsf{P}$, $t \vee s : \mathsf{P}$, $t \Rightarrow s : \mathsf{P}$.

- If $\Gamma, y : B \mid t : \mathsf{P}$ is a term, then $\Gamma \mid \forall y.t : \mathsf{P}$ is a term; and similarly for $\Gamma \mid \exists y.t : \mathsf{P}$
- If $t : A$ and $s : A$ are terms, then $s =_A t : \mathsf{P}$ is a term.
- If $t : \mathsf{P}$ is a term, then $\Box t : \mathsf{P}$ is a term.

There may also be additional basic constants $a : A$, for a type A. Moreover, we allow typed function symbols $f : A_1, \ldots, A_n \to B$. For such an f, we declare that if

$$y_1 : B_1, \ldots, y_m : B_m \mid t_1 : A_1$$

$$\vdots$$

$$y_1 : B_1, \ldots, y_m : B_m \mid t_n : A_n$$

are terms, then there is a term

$$y_1 : B_1, \ldots, y_m : B_m \mid f(t_1, \ldots, t_n) : B.$$

Lastly, every expression $x : A \mid x : A$ to be a term.

One also assumes the following structural rules:

- Weakening. If $x_1 : A_1, \ldots, x_n : A_n \mid t : B$ is a term, then so is

  $$x_1 : A_1, \ldots, x_{n+1} : A_{n+1} \mid t : B,$$

  for any $x_{n+1} : A_{n+1}$.
- Permutation. If

  $$x_1 : A_1, \ldots, x_n : A_n, x_{n+1} : A_{n+1}, \ldots, x_{n+m} : A_{n+m} \mid t : B$$

  is a term, then so is

  $$x_1 : A_1, \ldots, x_{n-1} : A_{n-1}, x_{n+1} : A_{n+1}, x_n : A_n, x_{n+2} : A_{n+2}, \ldots, x_{n+m} : A_{n+m} \mid t : B$$
- Contraction. If $\Gamma \mid t : B$ is a term such that $x_i \equiv x_j$ $(1 \leq i, j \leq n)$, then $\Gamma[x_i : A_i] \mid t : B$ is a term.

As usual, we define a deductive system by specifying a relation $\vdash$ between terms of type P. The crucial difference between the standard formulation of intuitionistic higher-order logic and the present one are the modified extensionality principles marked with $(*)$.

- $\varphi \vdash \varphi$
- $\dfrac{\varphi \vdash \psi \quad t : A}{\varphi[t/x] \vdash \psi[t/x]}$, for $x : A$ (similarly for simultaneous substitution)

- $$\frac{\varphi \vdash \psi \quad \psi \vdash \vartheta}{\varphi \vdash \vartheta}$$

- $\top \vdash x =_A x$, where $x : A$
- $\varphi \wedge x =_A x' \vdash \varphi[x'/x]$. where $x : A, x' : A$

(∗) $\Box \forall x (f(x) =_B g(x)) \vdash f =_{B^A} g$, for terms $x : A$ and $f, g : B^A$

(∗) $\Box (p \Leftrightarrow q) \vdash p =_{\mathsf{P}} q$, for terms $p, q : \mathsf{P}$

- $\top \vdash * =_1 x$, where $x : 1$
- $\top \vdash \pi_1 \langle x, y \rangle =_A x$ and $\top \vdash \pi_2 \langle x, y \rangle =_B y$, where $x : A$ and $y : B$
- $\top \vdash \langle \pi_1 w, \pi_2 w \rangle =_{A \times B} w$, for $w : A \times B$
- $\Gamma[x : A] \mid \top \vdash \mathsf{app}(\lambda x.t, x') =_B t[x'/x]$, for $\Gamma \mid t : B$ and $x' : A$
- $\top \vdash \lambda x.\mathsf{app}(w, x) =_{B^A} w$, for $w : B^A$

- $\varphi \vdash \top$, for any $\varphi : \mathsf{P}$
- $\bot \vdash \varphi$, for any $\varphi : \mathsf{P}$
- $\varphi \vdash \psi \wedge \vartheta$ iff $\varphi \vdash \psi$ and $\varphi \vdash \vartheta$
- $\varphi \vee \psi \vdash \vartheta$ iff $\varphi \vdash \vartheta$ and $\psi \vdash \vartheta$
- $\varphi \vdash \psi \Rightarrow \vartheta$ iff $\varphi \wedge \psi \vdash \vartheta$
- $\Gamma \mid \exists x.\varphi \vdash \psi$ iff $\Gamma, x : A \mid \varphi \vdash \psi$
- $\Gamma \mid \varphi \vdash \forall x.\psi$ iff $\Gamma, x : A \mid \varphi \vdash \psi$

Definition 3.1. *A* theory *in a language $\mathcal{L}$ as specified above consists of a set of closed sentences α, i.e. terms of type P with no free variables (well-typed in the empty context), and which may be used as axioms in the form $\Gamma \mid \top \vdash \alpha$.*

Remark 3.2. Adding the axiom

$$\Gamma \mid \top \vdash \forall p.p \vee \neg p$$

makes the logic classical.

As is well-known there are more concise formulations of higher-order systems. The particular one chosen here is very close to the definition of a topos as a cartesian closed category with subobject classifier. One does not really need all exponential types and their constructors, however, but only those of the form P^A, for every type A, which we write $\mathsf{P}A$ and call *powertypes*. Along these lines one may define:

$$\{x : A \mid \varphi\} :\equiv \lambda x.\varphi : \mathsf{P}A,$$

where $x : A \mid \varphi : \mathsf{P}$. On the other hand, for $\sigma : \mathsf{P}A$ and $x : A$, set

$$x \in \sigma :\equiv \mathsf{app}(\sigma, x).$$

According to the axioms for exponential terms, we have

$$\begin{gathered} x' : A \mid \top \vdash x' \in \{x : A \mid \varphi\} = \varphi[x'/x] \\ \mid \top \vdash \{x : A \mid x \in w\} = w. \end{gathered}$$

Thus one could instead take only types of the form $\mathsf{P}A$, and the constructors $\{\cdots \mid -\}$ and $\in$ as basic, along with the last two axioms. For further simplifications see [15, 16].

Finally, the S4 axioms are the usual ones

- $\dfrac{\Gamma \mid \varphi \vdash \psi}{\Gamma \mid \Box\varphi \vdash \Box\psi}$
- $\Gamma \mid \top \vdash \Box\top$
- $\Gamma \mid \Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$
- $\Gamma \mid \Box\varphi \vdash \varphi$
- $\Gamma \mid \Box\varphi \vdash \Box\Box\varphi$

The first three axioms express that $\Box$, viewed as an operator, is a monotone finite meet preserving operation. The other two axioms are the T and 4 axioms, respectively. Further useful rules provable from the axioms are necessitation

$$\frac{\Gamma \mid \top \vdash \varphi}{\Gamma \mid \top \vdash \Box\varphi},$$

and the axiom K:

$$\Gamma \mid \Box(\varphi \Rightarrow \psi) \vdash \Box\varphi \Rightarrow \Box\psi.$$

As far as deductions are concerned, proving a sequent in a context Γ may also be read so as to mean that the sequent holds provided the types in the context are non-empty. To quote an example from [LS], it is straightforward to derive

$$x : A \mid \forall_{x:A} \vdash \exists_{x:A}\varphi,$$

which only makes sense in context $x : A$, i.e. under the assumption that the type A is inhabited. The quantifier rules directly model the essential properties of image ($\exists$) as a left adjoint to weakening (inverse image along product projections) and dual image ($\forall$) as right adjoint to weakening. Due to the order-theoretic nature of the calculus ($\vdash$), in order to show that $\Gamma \mid \varphi \dashv\vdash \psi$, it suffices to show that for an arbitrary $\Gamma \mid \vartheta$

$$\Gamma \mid \varphi \vdash \vartheta \text{ iff } \Gamma \mid \psi \vdash \vartheta.$$

Or, equivalently,

$$\Gamma \mid \vartheta \vdash \varphi \text{ iff } \Gamma \mid \vartheta \vdash \psi.$$

To illustrate the rules for the quantifiers, we will demonstrate a few easy theorems.

- $x : A \mid \varphi \vdash \exists_{x:A}\varphi$.

  This is an immediate application of the $\exists$-rule to the axiom

  $$\emptyset \mid \exists_{x:A}\varphi \vdash \exists_{x:A}\varphi.$$

  An analogous argument shows that the sequent

  $$x : A \mid \forall_{x:A}\varphi \vdash \varphi$$

  is derivable.

- If $x : A \mid \varphi \vdash \psi$, then $\emptyset \mid \exists_{x:A}\varphi \vdash \exists_{x:A}\psi$.

  First we have $\varphi \vdash \exists_{x:A}\psi$ by the previous fact and transitivity of $\vdash$, and thus $\exists_{x:A}\varphi \vdash \exists_{x:A}\psi$ by the $\exists$-rule.

  Dually for $\forall$.

- Suppose the variable $x : A$ does not occur freely in φ. Then $\varphi \dashv\vdash \exists_{x:A}\varphi$.

  Applying the $\exists$-rule gives:

  $$\frac{\exists_{x:A}\varphi \vdash \exists_{x:A}\varphi}{\varphi \vdash \exists_{x:A}\varphi} \qquad\qquad \frac{\varphi \vdash \varphi}{\exists_{x:A}\varphi \vdash \varphi}$$

  Dually for $\forall$.

- $\exists_{x:A}[\varphi \wedge \psi] \dashv\vdash \exists_{x:A}\varphi \wedge \psi$, where $x : A$ does not occur freely in ψ.

  Consider any ϑ that is well-written in the same context as $\exists_{x:A}[\varphi \wedge \psi]$, i.e. that doesn't involve free $x : A$. Then we have the following chain of biconditions, adapting a standard proof of the Frobenius reprocity for projections (weakening):

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists_{x:A}[\varphi \wedge \psi] \vdash \vartheta}{x : A \mid \varphi \wedge \psi \vdash \vartheta}}{x : A \mid \varphi \vdash \psi \Rightarrow \vartheta}}{\exists_{x:A}\varphi \vdash \psi \Rightarrow \vartheta}}{\exists_{x:A}\varphi \wedge \psi \vdash \vartheta}$$

Of course, one may read off a direct proof from the reflexivity axiom by replacing ϑ at the top and bottom by the same formula as on the left, respectively, and then do the argument up (for the left-to-right) or down (for the right-to-left).

However, note that we can't make a dual argument to prove

$$\forall_{x:A}[\varphi \vee \psi] \vdash \forall_{x:A}\varphi \vee \psi,$$

because the operation $\psi \vee (-)$ is not assumed to have a left adjoint (coimplication). In fact, the sequent is not intuitionistically valid. The converse, however, holds:

$$\dfrac{\dfrac{\dfrac{\forall_{x:A}\varphi \vdash \forall_{x:A}\varphi}{x : A \mid \forall_{x:A}\varphi \vdash \varphi}}{x : A \mid \forall_{x:A}\varphi \vee \psi \vdash \varphi \vee \psi}}{\forall_{x:A}\varphi \vee \psi \vdash \forall_{x:A}[\varphi \vee \psi]}$$

The very last step uses the variable condition that $x : A$ does not occur free in ψ.

- $\forall_{x:A}(\varphi \Rightarrow \psi) \vdash \exists_{x:A}\varphi \Rightarrow \psi$, where x does not occur freely in ψ.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall_{x:A}(\varphi \Rightarrow \psi) \vdash \forall_{x:A}(\varphi \Rightarrow \psi)}{x : A \mid \forall_{x:A}(\varphi \Rightarrow \psi) \vdash \varphi \Rightarrow \psi}}{x : A \mid \forall_{x:A}(\varphi \Rightarrow \psi) \wedge \varphi \vdash \psi}}{\exists_{x:A}(\forall_{x:A}(\varphi \Rightarrow \psi) \wedge \varphi) \vdash \exists_{x:A}\psi}}{\forall_{x:A}(\varphi \Rightarrow \psi) \wedge \exists_{x:A}\varphi \vdash \psi}}{\forall_{x:A}(\varphi \Rightarrow \psi) \vdash \exists_{x:A}\varphi \Rightarrow \psi}$$

- $\exists_{x:A}(\varphi \Rightarrow \psi) \vdash \varphi \Rightarrow \exists_{x:A}\psi$, where x does not occur freely in φ:

$$\dfrac{\dfrac{\dfrac{\varphi \wedge (\varphi \Rightarrow \psi) \vdash \psi}{\exists_{x:A}(\varphi \wedge (\varphi \Rightarrow \psi)) \vdash \exists_{x:A}\psi}}{\varphi \wedge \exists_{x:A}(\varphi \Rightarrow \psi) \vdash \exists_{x:A}\psi}}{\exists_{x:A}(\varphi \Rightarrow \psi) \vdash \varphi \Rightarrow \exists_{x:A}\psi}$$

- For any $x : A \mid \varphi$:
$$\exists_{x:A}\Box\varphi \vdash \Box\exists_{x:A}\varphi$$

$$\begin{array}{c} \exists_{x:A}\varphi \vdash \exists_{x:A}\varphi \\ \hline x : A \mid \varphi \vdash \exists_{x:A}\varphi \\ \hline x : A \mid \Box\varphi \vdash \Box\exists_{x:A}\varphi \\ \hline \exists_{x:A}\Box\varphi \vdash \Box\exists_{x:A}\varphi \end{array}$$

  In particular, if $x : A \mid \varphi$ is $\Box$-stable, then

$$\exists_{x:A}\varphi \vdash \Box\exists_{x:A}\varphi$$

  For the universal quantifier we only have, for any formula $x : A \mid \varphi$:

$$\Box\forall_{x:A}\varphi \vdash \forall_{x:A}\Box\varphi :$$

$$\begin{array}{c} \Box\forall_{x:A}\varphi \vdash \varphi \\ \hline \Box\forall_{x:A}\varphi \vdash \Box\varphi \\ \hline \Box\forall_{x:A}\varphi \vdash \forall_{x:A}\Box\varphi \end{array}$$

  but not the other way around.

- For any φ:
$$\Box\forall_{x:A}\Box\varphi \dashv\vdash \Box\forall_{x:A}\varphi.$$

  Proof. For the right-to-left we reason as follows:

$$\begin{array}{c} \Box\forall_{x:A}\varphi \vdash \forall_{x:A}\varphi \\ \hline \Box\forall_{x:A}\varphi \vdash \varphi \\ \hline \Box\forall_{x:A}\varphi \vdash \Box\varphi \\ \hline \Box\forall_{x:A}\varphi \vdash \forall_{x:A}\Box\varphi \\ \hline \Box\forall_{x:A}\varphi \vdash \Box\forall_{x:A}\Box\varphi \end{array}$$

  In the other direction

$$\begin{array}{c} \Box\forall_{x:A}\Box\varphi \vdash \Box\forall_{x:A}\Box\varphi \\ \hline \Box\forall_{x:A}\Box\varphi \vdash \forall_{x:A}\Box\varphi \\ \hline \Box\forall_{x:A}\Box\varphi \vdash \Box\varphi \\ \hline \Box\forall_{x:A}\Box\varphi \vdash \varphi \\ \hline \Box\forall_{x:A}\Box\varphi \vdash \forall_{x:A}\varphi \\ \hline \Box\forall_{x:A}\Box\varphi \vdash \Box\forall_{x:A}\varphi \end{array}$$

  $\Box$

3.2 The Definition of Models

The definition of model of a higher-order modal theory in a topos is mainly joins elements from models of non-modal higher-order logic and modal logic as described in the previous sections. To make the presentation self-contained, we state an explicit definition. We implicitly assume that the topos $\mathcal{E}$ is equipped with a canonical topos structure, i.e. specific choice of products, exponentials, subobject classifier, etc.

Definition 3.3. *A* model *of a higher-order modal type theory in a topos $\mathcal{E}$ consists of a faithful frame H in $\mathcal{E}$, and an assignment $[\![-]\!]$ that assigns to each basic type A in $\mathcal{L}$ an object $[\![A]\!]$ in such a way that*

- $[\![1]\!] = 1_{\mathcal{E}}$
- $[\![\mathsf{P}]\!] = H$
- $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$
- $[\![A^B]\!] = [\![A]\!]^{[\![B]\!]}$.

Moreover, each term $\Gamma \mid t : B$ in $\mathcal{L}$, where $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ is a suitable variable context for t, is assigned an arrow

$$[\![t]\!] : [\![\Gamma]\!] \to [\![B]\!]$$

recursively as follows (where $[\![\Gamma]\!]$ is short for $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$ and $[\![t]\!]$ really means $[\![\Gamma \mid t : B]\!]$).

- *Each constant $c : A$ in $\mathcal{L}$ is assigned an arrow*

  $$[\![c]\!] : 1_{\mathcal{E}} \to [\![A]\!].$$

  In particular:

  $[\![\top : \mathsf{P}]\!] = \top_H : 1_{\mathcal{E}} \longrightarrow H$
  $[\![\bot : \mathsf{P}]\!] = \bot_H : 1_{\mathcal{E}} \longrightarrow H$
  $[\![* : 1]\!] = 1_{1_{\mathcal{E}}}$ *(the identity arrow on the terminal object).*

- *Every function symbol $f : A_1, \ldots, A_n \to B$ is assigned an arrow*

  $$[\![f]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![B]\!]$$

- *A term $x : A \mid x : A$ is assigned the identity arrow on A.*

This extends to arbitrary terms-in-context as follows

- *If $\Gamma \mid s : A$ and $\Gamma \mid t : B$ are terms, then $[\![\Gamma \mid \langle s, t\rangle : A \times B]\!]$ is the map*

  $$\langle [\![s]\!], [\![t]\!] \rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!].$$

- *If* $\Gamma \mid t : A \times B$ *is a term, then* $[\![\Gamma \mid \pi_1 t : A]\!]$ *is*

$$[\![\Gamma]\!] \xrightarrow{[\![t]\!]} [\![A]\!] \times [\![B]\!] \xrightarrow{\pi_1} [\![A]\!],$$

and similarly for $\pi_2 t$.

- *If* $\Gamma \mid t : A$ *is a term and* $y : B$ *a variable in* Γ, *then* $[\![\Gamma[y : B] \mid \lambda y.t : A^B]\!]$ *is*

$$\lambda_{[\![B]\!]}[\![t]\!] : [\![\Gamma[y : B]]\!] \to A^{[\![B]\!]}$$

- *If* $\Gamma \mid t : A^B$ *and* $\Gamma \mid s : B$ *are terms, then* $[\![\Gamma \mid \mathsf{app}(t, s) : A]\!]$ *is*

$$\langle [\![t]\!], [\![s]\!] \rangle : [\![\Gamma]\!] \to A^B \times B \xrightarrow{\varepsilon} A.$$

- *For any two terms* $\Gamma \mid p : \mathsf{P}$, $\Gamma \mid q : \mathsf{P}$, *and* $\star$ *any of the connectives* $\wedge, \vee, \Rightarrow$, $[\![\Gamma \mid p \star q : \mathsf{P}]\!]$ *is*

$$[\![\Gamma]\!] \xrightarrow{\langle [\![p]\!], [\![q]\!] \rangle} H \times H \xrightarrow{\star} H,$$

where in the last line $\star$ *is the evident algebraic operation on* H.

- *If* $\Gamma, y : B \mid t : \mathsf{P}$ *is a term, then* $[\![\Gamma \mid \forall y.t : \mathsf{P}]\!]$ *is*

$$[\![\Gamma]\!] \xrightarrow{\lambda_{[\![B]\!]}[\![t]\!]} H^{[\![B]\!]} \xrightarrow{\forall_{[\![B]\!]}} H$$

and similarly for $[\![\Gamma \mid \exists y.t : \mathsf{P}]\!]$ *via* $\exists_{[\![B]\!]}$.

- *If* $\Gamma \mid t : A$ *and* $\Gamma \mid s : A$ *are terms, then* $[\![\Gamma \mid t =_A s : \mathsf{P}]\!]$ *is the map*

$$[\![\Gamma]\!] \xrightarrow{\langle [\![t]\!], [\![s]\!] \rangle} [\![A]\!] \times [\![A]\!] \xrightarrow{\delta_{[\![A]\!]}} \Omega_{\mathcal{E}} \xrightarrow{i} H,$$

where i *is the unique (monic) frame map.*

- *If* $\Gamma \mid t : \mathsf{P}$ *is a term, then* $[\![\Gamma \mid \Box t : \mathsf{P}]\!]$ *is the map*

$$[\![\Gamma]\!] \xrightarrow{[\![t]\!]} H \xrightarrow{\tau} \Omega_{\mathcal{E}} \xrightarrow{i} H,$$

where τ *is the classifying map of* $\top_H : 1 \to H$, *as described before.*

The structural rules are interpreted in the obvious way.

- *Weakening: given a term* $x_1 : A_1, \ldots, x_n : A_n \mid t : B$, *then the term*

$$x_1 : A_1, \ldots, x_{n+1} : A_{n+1} \mid t : B$$

is interpreted as the arrow

$$[\![A_1]\!] \times \cdots \times [\![A_{n+1}]\!] \xrightarrow{\pi} [\![A_1]\!] \times \cdots \times [\![A_n]\!] \xrightarrow{[\![t]\!]} [\![B]\!],$$

where π *is the projection.*

- *Contraction: without loss of generality (by permutation) we consider the case where given a term*

$$x_1 : A_1, \ldots, x_{n+1} : A_{n+1} \mid t : B$$

such that $x_n \equiv x_{n+1}$ *(and thus* $A_n \equiv A_{n+1}$*) , we form the term*

$$x_1 : A_1, \ldots, x_n : A_n \mid t : B$$

which results from the former by omitting the variable declaration $x_{n+1} : A_{n+1}$*. The latter is interpreted by the composite*

$$[\![A_1]\!] \times \cdots \times [\![A_n]\!] \xrightarrow{\langle 1_{[\![A_1]\!] \times \cdots \times [\![A_n]\!]}, 1_{A_n} \rangle} [\![A_1]\!] \times \cdots \times [\![A_{n+1}]\!] \xrightarrow{[\![t]\!]} [\![B]\!],$$

noting that $[\![A_n]\!] = [\![A_{n+1}]\!]$.

- *Permutation. If*

$$x_1 : A_1, \ldots, x_n : A_n, x_{n+1} : A_{n+1}, \ldots, x_{n+m} : A_{n+m} \mid t : B$$

is a term, then the term

$$x_1 : A_1, \ldots, x_{n+1} : A_{n+1}, x_n : A_n, \ldots, x_{n+m} : A_{n+m} \mid t : B$$

is interpreted by composing the interpretation of the upper term with the isomorphism

$$1_{[\![A_1]\!]} \times \cdots \times 1_{[\![A_{n-1}]\!]} \times \tau \times 1_{[\![A_{n+2}]\!]} \times \ldots 1_{[\![A_{n+m}]\!]},$$

where $\tau = \langle \pi_n, \pi_{n+1} \rangle : [\![A_{n+1}]\!] \times [\![A_n]\!] \to [\![A_n]\!] \times [\![A_{n+1}]\!]$ *is the twist map.*

Substitution is defined by composition in the obvious way.

We note that a constant $a : A$ in a context $\Gamma \mid a : A$ is always interpreted by the composite

$$[\![\Gamma]\!] \to 1_{\mathcal{E}} \xrightarrow{[\![a]\!]} [\![A]\!].$$

One must be careful here. Strictly speaking, by the nature of the recursive definition one does interpreted terms $\Gamma \mid t : B$ but whole derivation trees. In some cases, such as the above, $\Gamma \mid a : A$, there may be many ways to derive the formula. For instance the term $x : A \mid a : A$ may be constructed either by weakening or substitution of $x : A$ into $\emptyset \mid a : A$. However, although being a tedious formal proof by induction on the construction of the term $\Gamma \mid a : A$, it is quite easily seen that the interpretation is the same in each case, so that $\Gamma \mid a : A$ is always interpreted as the arrow displayed above. Before moving on, let us review some common examples

Examples 3.1.

1. A well-studied class of examples are structures induced by surjective geometric morphisms $f : \mathcal{F} \to \mathcal{E}$. If $\mathcal{F}$ is Boolean, then so is $f_*\Omega_{\mathcal{F}}$. For instance, there are geometric morphisms

$$\mathbf{Sets}^{|\mathbf{C}|} \longrightarrow \mathbf{Sets}^{\mathbf{C}}$$

induced by the inclusion $|\mathbf{C}| \to \mathbf{C}$. When $\mathbf{C}$ is a preorder, then this yields Kripke semantics for first-order modal logic. This case was originally studied in [12, 28].

Similarly, the canonical geometric morphism

$$\mathbf{Sets}/X \longrightarrow \mathrm{Sh}(X)$$

induced by the continuous inclusion $|X| \hookrightarrow X$ gives rise to sheaf models for classical first- (and higher-) order modal logic, studied in [5]. The exact structure of these examples will be discussed in more detail in section **??** below.

2. More generally, by a well-known theorem of Barr, every Grothendieck topos $\mathcal{G}$ can be covered by a Boolean topos $\mathcal{B}$ in the sense that there is a surjective geometric morphism

$$f : \mathcal{B} \longrightarrow \mathcal{G}.$$

For $H = f_*\Omega_{\mathcal{B}}$, this provides models in Grothendieck topoi.[20]

3. Of course, in any topos $\mathcal{E}$ the subobject classifier $\Omega_{\mathcal{E}}$ itself would do. However, as noted e.g. in [25, 27], the resulting modal operator will be the identity on $\Omega_{\mathcal{E}}$.

Remark 3.4. In our definition of model of a higher-order modal theory $\mathbb{T}$ in a topos $\mathcal{E}$ we included the provision of a faithful Heyting algebra H in $\mathcal{E}$. Strictly speaking, however, H is to be regarded as a part of the "logical" algebraic-semantic structure on top of which interpretations are defined. It interprets in an invariant way a certain logical constant in the language, namely the type of propositions. For non-modal higher-order logic, one may speak of models in a topos, because the provision of $\mathcal{E}$ suffices to fix the interpretation of the logical constants of the type theory. Thus, foundational concerns aside, it would seem in accordance with model-theoretic practice to define an interpretation of a higher-order modal theory $\mathbb{T}$ w.r.t. a given pair $(\mathcal{E}, H)$, where H is a faithful Heyting algebra in $\mathcal{E}$. In this way we make the

[20] Cf. e.g. [20], IX.9. Actually, the geometric morphism f can be extended to a surjective geometric morphism $\mathcal{E} \longrightarrow \mathcal{B} \longrightarrow \mathcal{G}$, where $\mathcal{E}$ is the topos of sheaves on a topological space, although $\mathcal{E}$ might not be Boolean [20], IX.11.

choice of a faithful Heyting algebra H part of the structure of the category we define models in. We call a pair $(\mathcal{E}, H)$, where $\mathcal{E}$ is a topos, and H is a faithful Heyting algebra in $\mathcal{E}$, a τ-topos. The foregoing definition of model of a higher-order modal theory $\mathbb{T}$ then essentially remains the same except that one states it as defining a model, or interpretation, of $\mathbb{T}$ in $(\mathcal{E}, H)$. This may look like a mere matter of convention, and indeed, nothing changes about the idea of interpreting a higher-order modal theory $\mathbb{T}$ in a topos. However, we will point out that later (thm. 4.21) we will explicitly need to refer to interpretations in a structure $(\mathcal{E}, H)$, rather than merely $\mathcal{E}$. In fact, for the main theorem (4.21) of functorial semantics to work, it is essential to keep H fixed, and to consider interpretations w.r.t. fixed H. This last observation may be taken as evidence that the notion of model, or interpretation, of $\mathbb{T}$ in $(\mathcal{E}, H)$ is the more appropriate one from a conceptual point of view.

3.3 Propositional Extensionality

The given system of intuitionistic higher-order S4 modal logic is sound w.r.t. the semantics described in def. 3.3. Except for the two extensionality principles, soundness is straightforward following known topos semantics. The reason why plain propositional extensionality fails in our semantics is the interpretation of implication. In the general topos semantics based on $\Omega_{\mathcal{E}}$ Heyting implication on $\Omega_{\mathcal{E}}$ is given by the map

$$\Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \xrightarrow{\langle \pi_1, \wedge \rangle} \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \xrightarrow{\delta} \Omega_{\mathcal{E}}$$

that immediately implies propositional extensionality. By contrast, for an arbitrary frame H we observe:

Lemma 3.5. *For an arbitrary topos $\mathcal{E}$, and a (faithful) frame H in $\mathcal{E}$, it is not in general the case that*

$$\begin{tikzcd}
H \times H \arrow[r, "\Rightarrow"] \arrow[d, "\langle \pi_1, \wedge \rangle"'] & H \\
H \times H \arrow[r, "\delta_H"] & \Omega_{\mathcal{E}} \arrow[u, "i"']
\end{tikzcd}$$

commutes.

Proof. A counterexample may easily be found in the topos **Sets** with subobject classifier $\mathbf{2}$ and $H = \mathcal{P}(X)$, for some set $X \neq 1$. The adjunction

$$i : \mathbf{2} \leftrightarrows \mathcal{P}(X) : \tau$$

$(i \dashv \tau)$ is defined by

$$i(x) = \begin{cases} X, & \text{if } x = 1 \\ \emptyset, & \text{if } x = 0 \end{cases}$$

and

$$\tau(U) = 1 \ \text{ iff } \ U = X.$$

For any $U, V \in \mathcal{P}(X)$,

$$U \Rightarrow V = \bigcup\{W \in \mathcal{P}(X) \mid W \cap U \subseteq V\}.$$

If $U \nsubseteq V$, then $U \neq U \cap V$, and so

$$i\delta\langle \pi_1, \wedge\rangle(U, V) = i\delta_{\mathcal{P}(X)}(U, U \cap V) = i(0) = \emptyset.$$

But $U \nsubseteq V$ does not in general imply $U \Rightarrow V = \emptyset$. (Consider e.g. $V \subseteq U \Rightarrow V$, for $U \cap V \neq \emptyset$.) $\square$

As suggested by the example, the reason for the failure of plain propositional extensionality is that failure to be true (in the sense of $\top = X \nsubseteq U \Rightarrow V$) does not imply equality to $\bot$ in H. On the other hand, note that $\tau(U \Rightarrow V) = 0$, because $X \nsubseteq U \Rightarrow V$. This observation generalizes. Although $i\delta\langle \pi_1, \wedge\rangle = \Rightarrow$ fails in general, we have the following.

Lemma 3.6. *In any topos $\mathcal{E}$, the diagram*

$$\begin{tikzcd} H \times H \arrow[r, "\Rightarrow"] \arrow[d, "\langle \pi_1, \wedge \rangle"'] & H \arrow[d, "\tau"] \\ H \times H \arrow[r, "\delta_H"] & \Omega_{\mathcal{E}} \end{tikzcd}$$

commutes, and thus

$$i\tau\circ \Rightarrow \; = \; i\delta_H\langle \pi_1, \wedge\rangle.$$

Proof. Consider the pullbacks

$$\begin{tikzcd}
(\leq) \arrow[r] \arrow[d] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
H \times H \arrow[r, "\Rightarrow"'] & H \arrow[r, "\tau"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

$$\begin{tikzcd}
(\leq) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, "\Delta"] & 1 \arrow[d, "\top"] \\
H \times H \arrow[r, "\langle \pi_1, \wedge \rangle"'] & H \times H \arrow[r, "\delta_H"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

whence the claim follows from uniqueness of classifying maps. The left-hand square in the first diagram is a pullback by the definition of $\Rightarrow$, while the second diagram is the definition of the induced partial ordering on H as the equalizer of π_1 and $\wedge$. $\square$

This argument neatly exhibits the conceptual role played by the modal operator τ (more exactly, the adjunction $i \dashv \tau$). The soundness proof is essentially a corollary to that.

Corollary 3.7. *Modalized propositional extensionality*

$$p : \mathsf{P}, q : \mathsf{P} \mid \Box(p \Leftrightarrow q) \vdash p =_{\mathsf{P}} q$$

is true in any model $(\mathcal{E}, H)$.

Proof. In view of lemma 3.6, and since τ, i commute with meets, the left-hand side of the above sequent is interpreted as the map

$$i \wedge (\delta_H \times \delta_H)\langle\langle \wedge_H, \pi_1\rangle, \langle \wedge_H, \pi_2\rangle\rangle,$$

with $\wedge$ the meet on $\Omega_{\mathcal{E}}$. The right-hand side is the internal equality on H:

$$i\delta_H : H \times H \to \Omega_{\mathcal{E}} \to H.$$

It is clear from the properties of $\leq_\Omega$ as a partial ordering that

$$\wedge(\delta_H \times \delta_H)\langle\langle \wedge_H, \pi_1\rangle, \langle \wedge_H, \pi_2\rangle\rangle \leq_\Omega \delta_H.$$

Since i preserves that ordering, we have

$$i \wedge (\delta_H \times \delta_H)\langle\langle \wedge_H, \pi_1\rangle, \langle \wedge_H, \pi_2\rangle\rangle \leq_H i\delta_H. \qquad \square$$

3.4 Function Extensionality

The failure of plain function extensionality and its recovering via τ can be analyzed in a similar fashion. For non-modal function extensionality in the standard $\Omega_{\mathcal{E}}$-valued setting essentially holds because $\forall_Y \circ (\delta_X)^Y = \delta_{X^Y}$. However, in our setting we don't in general have $\forall_Y \circ (i\delta_X)^Y = i\delta_{X^Y}$, but rather:

Lemma 3.8. *For any topos $\mathcal{E}$, and any faithful frame in H, the following diagram commutes:*

$$\begin{tikzcd}
\Omega^Y \arrow[r, "i^Y"] & H^Y \arrow[r, "\forall_Y"] \arrow[d, "\tau^Y"] & H \arrow[d, "\tau"] \\
X^Y \times X^Y \arrow[u, "(\delta_X)^Y"] \arrow[r, "(\delta_X)^Y"] \arrow[r, bend right, "\delta_{X^Y}"'] & \Omega^Y \arrow[r, "\forall_Y"] & \Omega
\end{tikzcd}$$

Hence in particular

$$i\delta_{X^Y} = i\tau \circ \forall_Y \circ (i\delta_X)^Y.$$

Proof. The right-hand square of the diagram commutes by uniqueness of classifying maps, while for the left-hand square we have $\tau i = 1$. Similarly, the bottom triangle commutes, because

$$\begin{tikzcd}
X^Y \arrow[r] \arrow[d, "\Delta_{X^Y}"'] & 1 \arrow[r, equal] \arrow[d, "\top^Y"] & 1 \arrow[d, "\top"] \\
X^Y \times X^Y \arrow[r, "(\delta_X)^Y"'] & \Omega^Y \arrow[r, "\forall_Y"'] & \Omega
\end{tikzcd}$$

is a pullback diagram. (Note that the left-hand square is a pullback, because the functor $(-)^Y$, as a right adjoint, preserves these.) $\square$

Corollary 3.9. *Modal function extensionality*

$$f : X^Y, g : X^Y \mid \Box(\forall y : Y.f(y) =_X g(y)) \vdash f =_{X^Y} g.$$

is true in any interpretation $(\mathcal{E}, H)$.

Proof. The left-hand side of the sequent is interpreted by the arrow

$$X^Y \times X^Y \xrightarrow{\lambda_Y(i\delta_X\langle ev\pi_{13}, ev\pi_{23}\rangle)} H^Y \xrightarrow{\forall_Y} H \xrightarrow{\Box} H,$$

where the projections come from $X^Y \times X^Y \times Y$, and $ev : X^Y \times Y \to X$ is the canonical evaluation. The right-hand side is simply

$$X^Y \times X^Y \xrightarrow{\delta_{X^Y}} \Omega_{\mathcal{E}} \xrightarrow{i} H.$$

We need to show that the arrow

$$\langle i\tau\forall_Y \lambda_Y(i\delta_X\langle ev\pi_{13}, ev\pi_{23}\rangle), i\delta_{X^Y}\rangle : X^Y \times X^Y \to H \times H$$

factors through the partial ordering $(\leq) \rightarrowtail H \times H$. Write the left-hand component as $i\varphi$. It is enough to show that

$$\varphi \leq_\Omega \delta_{X^Y} : X^Y \times X^Y,$$

whence the claim follows as before, i being order-preserving.

To show that the subobject (Q, m) classified by the map $\tau\forall_Y\lambda_Y(i\delta_X\langle ev\pi_{13}, ev\pi_{23}\rangle)$ factors through Δ_{X^Y}, as subobjects of $X^Y \times X^Y$, observe first that $\lambda_Y(i\delta_X\langle ev\pi_{13}, ev\pi_{23}\rangle)$ can be written as

$$X^Y \times X^Y \xrightarrow{\eta} (X^Y \times X^Y \times Y)^Y \xrightarrow{\langle ev\pi_{13}, ev\pi_{23}\rangle^Y} (X \times X)^Y \xrightarrow{(\delta_X)^Y} \Omega^Y \xrightarrow{i^Y} H^Y,$$

where η is the unit component (at $X^Y \times X^Y$) of the product-exponential adjunction $(-) \times Y \dashv (-)^Y$. By the previous lemma

$$\tau \circ \forall_Y \circ i^Y \circ (\delta_X)^Y = \delta_{X^Y}.$$

The subobject in question thus arises from pullbacks

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, tail, "m"'] & X^Y \arrow[r] \arrow[d, "\Delta_{X^Y}"] & 1 \arrow[d, "\top"] \\
X^Y \times X^Y \arrow[r, "\langle ev\pi_{13}, ev\pi_{23}\rangle^Y \circ \eta"] & X^Y \times X^Y \arrow[r, "\delta_{X^Y}"] & \Omega
\end{tikzcd}$$

But $\langle ev\pi_{13}, ev\pi_{23}\rangle^Y \circ \eta$ is the identity arrow. For it is the transpose (along the adjunction $(-) \times Y \dashv (-)^Y$) of

$$\langle ev\pi_{13}, ev\pi_{23}\rangle : X^Y \times X^Y \times Y \to X \times X.$$

The latter in turn is the canonical evaluation of $X^Y \times X^Y$ viewed as the exponential $(X \times X)^Y$, i.e. the counit of the adjunction at $X \times X$, transposing which yields the identity. As a result,

$$\tau\forall_Y\lambda_Y(i\delta_X\langle ev\pi_{13}, ev\pi_{23}\rangle) \leq_\Omega \delta_{X^Y},$$

and therefore

$$i\tau\forall_Y\lambda_Y(i\delta_X\langle ev\pi_{13}, ev\pi_{23}\rangle) \leq_H i\delta_{X^Y}. \qquad \square$$

Remark 3.10. Before giving a counterexample to $i\delta_{X^Y} = \forall_Y \circ (i\delta_X)^Y$, let us remark that the equation does actually *hold* in the topos **Sets**. For consider $f \neq g \in X^Y$, i.e. $f(y) \neq g(y)$, for some $y \in Y$. Then for any complete Heyting algebra H, the function $(i\delta_X)^Y(f,g) \in H^Y$ is defined as

$$(i\delta_X)^Y(f,g)(y) = i\delta_X(f(y), g(y)) = \top, \quad \text{if } f(y) = g(y),$$

and $\perp$ otherwise. Thus taking the meet (cf. the definition in example **??**) yields

$$\bigwedge_{y \in Y} (i\delta_X)^Y(f,g)(y) = \perp,$$

because $f(y) \neq g(y)$, for some $y \in Y$, by assumption. In turn the meet equals $\top$ just in case $f(y) = g(y)$, for all $y \in Y$, i.e. if and only if $f = g$.

We now turn to a counterexample of function extensionality.

Proposition 3.11. *It is not in general the case that for a topos $\mathcal{E}$ and a frame H in $\mathcal{E}$:*

$$i\delta_{X^Y} = \forall_Y \circ (i\delta_X)^Y.$$

Proof. To find a counterexample we consider a specific presheaf topos $\mathbf{Sets}^{\mathbf{C}^{op}}$ described below.[21] Let's first recall some general facts. Write $\Omega_{|\mathbf{C}|}$ for the subobject classifier in $\mathbf{Sets}^{|\mathbf{C}|}$ and choose $H = f_*\Omega_{|\mathbf{C}|}$ (henceforth Ω_*), where f is the geometric morphism $f : \mathbf{Sets}^{|\mathbf{C}|} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ induced by the inclusion $|\mathbf{C}| \hookrightarrow \mathbf{C}$ via right Kan extensions. Recall moreover from the beginning that the subobject classifier Ω of $\mathbf{Sets}^{\mathbf{C}^{op}}$ determines for each C the set of all sieves on C. By contrast, $\Omega_*(C)$ is the set of arbitrary sets of arrows with codomain C (cf. also the example from the next section).

Recall that in any category of the form $\mathbf{Sets}^{\mathbf{C}^{op}}$ the evaluation maps $\varepsilon : B^A \times A \to B$ have components

$$\varepsilon_C(\eta, a) = \eta_C(1_C, a),$$

where $\eta \in B^A(C) = \mathrm{Hom}(\mathbf{y}C \times A, B)$ and $a \in A(C)$. The exponential transpose $\overline{\alpha} : Z \to B^A$ of a map $\alpha : Z \times A \to B$ has components

$$\overline{\alpha}_C(z) = \alpha \circ (\zeta \times 1_A), \tag{10}$$

where $\zeta : \mathbf{y}C \to Z$ corresponds under the Yoneda lemma to the element $z \in Z(C)$, i.e. is defined as $\zeta(f) = Z(f)(z)$, for any $f \in \mathbf{y}C(D)$.

For any object A in $\mathbf{C}$, the functor $(-)^A$ acts on arrows $f : C \to D$ as

$$f^A = \overline{f \circ \varepsilon},$$

[21] The counterexample, in particular the choice of $\mathbf{C}$ and the functor $G : \mathbf{C} \to \mathbf{Sets}$ below, follows a slightly different, though equivalent, proof first given in [29].

for evaluation $\varepsilon : C^A \times A \to C$. In particular,

$$(i\delta_B)^A = \overline{i\delta_B \circ \varepsilon},$$

for $\varepsilon : (B \times B)^A \times A \to B \times B$ evaluation at A. Thus, for any pair

$$\langle \eta, \mu \rangle \in (B \times B)^A(C) = \mathrm{Hom}(\mathbf{y}C \times A, B \times B),$$

we have

$$(\overline{i\delta_B \circ \varepsilon})_C(\eta, \mu) = i\delta_B \varepsilon(\langle \eta, \mu \rangle^* \times 1_A) = i\delta_B \langle \eta, \mu \rangle.$$

Here we use that $\langle \eta, \mu \rangle^* : \mathbf{y}C \to (B \times B)^A$ corresponds under Yoneda to the element $\langle \eta, \mu \rangle \in (B \times B)^A(C) = \mathrm{Hom}(\mathbf{y}C \times A, B \times B)$ and that $\langle \eta, \mu \rangle^*$ is equal to the exponential transpose of $\langle \eta, \mu \rangle$. Accordingly,

$$\begin{aligned}
\forall_C (i\delta_B)^A_C(\eta, \mu) &= \forall_C (\overline{i\delta_B \circ \varepsilon})_C(\eta, \mu) \\
&= \forall_C (i\delta_B \langle \eta, \mu \rangle) \\
&= \bigcup \{ s \in \Omega_*(C) \mid \Omega_*(g)(s) \leq i_D(\delta_B)_D(\eta_D(g, b), \mu_D(g, b)), \text{ for all} \\
&\qquad\qquad (g : D \to C, b \in A(D)) \},
\end{aligned}$$

On the other hand, the classifying map of the diagonal on a functor $B : \mathbf{C}^{op} \to \mathbf{Sets}$ is computed as

$$(\delta_B)_C(x, y) = \{ f : D \to C \mid B(f)(x) = B(f)(y) \},$$

for all pairs $(x, y) \in B(C) \times B(C)$. It is the maximal sieve $\top_C$ on C just in case $x = y$.

Now let $\mathbf{C}$ be the finite category

$$C \xrightarrow{g} D,$$

and define a functor $G : \mathbf{C}^{op} \to \mathbf{Sets}$ as follows:[22]

$$G(D) = \{u\},\ G(C) = \{v, w\},\ G(g)(u) = v.$$

Furthermore, choose $\eta, \mu \in G^G(D)$ such that $\eta \neq \mu$. Observe that, while necessarily

$$\eta_D = \mu_D : \mathbf{y}D(D) \times G(D) \to G(D)$$

with assignment

$$(1_D, u) \mapsto u,$$

we can chose η, μ in such a way that $\eta_C(g, x) \neq \mu_C(g, x)$, for some pair $(g, x) \in \mathbf{y}D(C) \times G(C)$. Specifically, since the first component g is fixed,

[22] Although $g : C \to D$ may be seen as the two-element poset with resulting presheaf topos $\mathbf{Sets}^{\to}$, we will not need that description. The objects and arrows in $\mathbf{C}$ merely play the role of indices, so it seems better to use the more neutral notation C, D, g.

the choice is only about $x \in G(C)$ which in turn must concern $w \in G(C)$. For naturality requires that

$$G(g)\eta_D(1_D, u) = \eta_C(\mathbf{y}D(g) \times G(g))_C(1_D, u) = \eta_C(g, v),$$

so that since $G(g)\eta_D(1_D, u) = G(g)(u) = v$, we must have $\eta_C(g, v) = v$; similarly $\mu_C(g, v) = v$. However, no constraint is put on the values $\eta_C(g, w)$ and $\mu_C(g, w)$, respectively.

Then:

$$(\delta_{G^G})_D(\eta, \mu) = \{x : X \to D \mid G^G(x)(\eta) = G^G(x)(\mu)\} = \emptyset. \tag{11}$$

For if $x = g$, observe

$$G^G(g)(\eta) = \eta \circ (\mathbf{y}g \times 1_G) \neq \mu \circ (\mathbf{y}g \times 1_G) = G^G(g)(\mu),$$

because

$$\eta_C(\mathbf{y}g \times 1_G)_C(1_C, w) = \eta_C(g, w) \neq \mu_C(g, w) = \mu_C(\mathbf{y}g \times 1_G)_C(1_C, w),$$

where the inequality holds by construction. But also, if $x = 1_D$, then $G^G(x)(\eta) = \eta \neq \mu = G^G(x)(\mu)$, where the inequality holds by assumption again.

On the other hand,

$$\forall_D(i\delta_G)^G_D(\eta, \mu) = \bigcup\{s \in \Omega_*(D) \mid \Omega_*(x)(s) \leq i_X(\delta_G)_X(\eta_X(x, b), \mu_X(x, b))\} = \{1_D\}. \tag{12}$$

for all pairs $(x : X \to D, b \in G(X))$ from $\mathbf{C}$. It is clear that $s = \{1_D\}$ satisfies the condition on the underlying set of the union, since for $x = 1_D$,

$$\begin{aligned}\Omega_*(1_D)(\{1_D\}) &= \{1_D\} \\ &\subseteq \top_D = i_D(\delta_G)_D(\eta_D(1_D, u), \mu_D(1_D, u)).\end{aligned}$$

On the other hand, for $x = g$, it is trivially always the case that

$$\Omega_*(g)(\{1_D\}) = \emptyset \subseteq (\delta_G)_C(\eta_C(g, b), \mu_C(g, b)),$$

for all $b \in G(C)$.

Furthermore, note that if $g \in s$, for some $s \in \Omega_*(D)$, then

$$\Omega_*(g)(s) = \top_C = \{1_C\}.$$

So if $g \in s$, for some s in the underlying set of the union (12), we had to have

$$\top_C = \Omega_*(g)(s) \leq i_C(\delta_G)_C(\eta_C(g, b), \mu_C(g, b)),$$

for all $b \in G(C)$. However, since by assumption $\eta_C(g, w) \neq \mu_C(g, w)$,

$$(\delta_G)_C(\eta_C(g, w), \mu_C(g, w)) = \emptyset,$$

and so

$$\Omega_*(g)(s) \nleq i_C(\delta_G)_C(\eta_C(g,w), \mu_C(g,w)).$$

Thus $g \notin s$, for all $s \in \Omega_*(D)$ in the underlying set of $\forall_D(i\delta_G)^G_D(\eta,\mu)$. Therefore

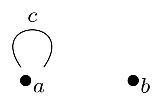
$$\forall_D(i\delta_G)^G_D(\eta,\mu) = \{1_D\},$$

as claimed, and in contrast to (11):

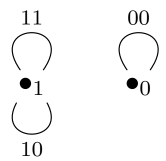
$$i_D(\delta_{G^G})_D(\eta,\mu) = \emptyset.$$

(Of course, $\tau(\{1_D\}) = \emptyset$, as lemma 3.8 predicts.) $\square$

There is an alternative, more combinatorial way of presenting the previous proof. The idea is to formulate the proof in terms of loop graphs rather than presheaves. For presheaves on the category $\{C \xrightarrow{g} D\}$ can equivalently be regarded as labelled graphs that consist only of loops and points, for instance:

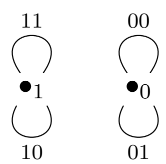


Here, $G(D)$ is the set of edges and $G(C)$ the set of vertices, while $G(g)$ assigns to an edge a point, its “source”. Thus every loop has a unique source but each point may admit several edges on it. Ω is the following graph which is easily seen to classify subgraphs:



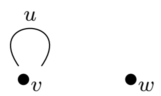
The labelling expresses the imposed algebraic structure of Ω with $0 < 1$ and $xy \leq uv$ iff $x \leq u$ & $y \leq v$. Intuitively, in presheaf terms, 1 stands for the maximal sieve on C and 0 for the empty sieve; similarly pairs xy encode sieves on D, where $x = 1$ if and only if g is in the sieve and $y = 1$ if and only if 1_D is in it. Then the source of an edge xy is just x. For instance, the sieve $\{g\}$ on D is encoded by 10. Then $\Omega(g)(\{g\}) = \{1_C\}$ which is encoded by 1. Note also that the set of edges is the three-element Heyting algebra from example **??**.

By contrast Ω_* is the graph

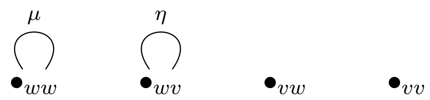


Here the additional edge 01 corresponds to the fact that $\{1_D\} \in \Omega_*(D)$. Thus the set of edges is the four-element Boolean algebra with the source map $2^2 \to 2$ induced by the inclusion $1 \hookrightarrow 2$.

The functor G from before becomes the graph



while G^G is



For recall that a natural transformation $\theta : \mathbf{y}C \times G \to G$ is completely determined by pairs (x, y) representing the component at C, so that xy stands for $\theta_C(x) = y$. In this way, for instance, the vertex vw represents the natural transformation θ with component $\theta_C(v) = w$. Moreover, recall from the example before that there are precisely two edges in the graph G, namely the two natural transformation $\eta, \mu : \mathbf{y}D \times G \to G$ from before, differing only in the values $\eta_C(g, w)$ and $\mu_C(g, w)$. Without loss of generality, set $\eta_C(g, w) = v$ and $\mu_C(g, w) = w$. The source of η is the natural transformation $G^G(\eta) = \eta(\mathbf{y}g \times 1_G)$ which in turn only has non-trivial components at C; similarly for η. Since, as argued before,

$$\eta_C(\mathbf{y}g \times 1_G)_C(1_C, v) = \eta_C(g, v) = v = \mu_C(g, v) = \mu_C(\mathbf{y}g \times 1_G)_C(1_C, v)$$

the source of η and μ are completely determined by the values

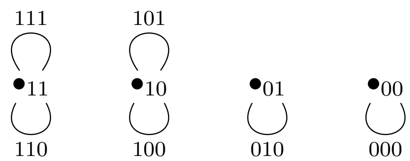
$$\eta_C(\mathbf{y}g \times 1_G)_C(1_C, w) = \eta_C(g, w) = v$$

and

$$\mu_C(\mathbf{y}g \times 1_G)_C(1_C, w) = \mu_C(g, w) = w.$$

Therefore, in the picture above the source of η is wv and that of μ is ww.

The graph Ω^G then looks like this:



again with the pointwise ordering.

The labelling is to be understood as follows. Vertices are $\Omega^G(C) = \mathrm{Hom}(\mathbf{y}C \times G, \Omega) = 2^2$, as there are exactly four natural transformations $\nu : \mathbf{y}C \times G \to \Omega$, each one defined by the pair xy of values of the component at C; understood in such a way that x corresponds to $\nu_C(1_C, v)$ and y corresponds to $\nu_C(1_C, w)$. (Hence, in the expression xy, $y = 1$ iff $1_C \in \nu_C(1_C, w)$, and 0 ow.)

In turn, an edge θ in $\Omega^G(D)$ is uniquely determined by the values $\theta_D(1_D, u)$ and $\theta_C(g, w)$. For the value of $\theta_C(g, v)$ is always determined by

$$\theta_C(g, v) = \theta_C(\mathbf{y}D(g) \times G(g))(1_D, u) = \Omega^G(g)\theta_D(1_D, u),$$

by naturality of θ. The notation xyz is chosen in such a way that the source is xy. Thus, xyz is to be read so as to mean $\theta_D(1_D, u) = xz$ and $\theta_C(g, w) = y$. For by definition the source of an edge θ in Ω^G is $\Omega^G(g)(\theta) = \theta(\mathbf{y}g \times 1_G)$. Its component at D is empty while for C, while for $x = w$

$$\theta_C((\mathbf{y}g)_C \times 1_{GC})(1_C, w) = \theta_C(g, w),$$

and for $x = v$

$$\theta_C((\mathbf{y}g)_C \times 1_{GC})(1_C, v) = \theta_C(g, v) = \theta_C(\mathbf{y}D(g) \times G(g))(1_D, u) = g^*\theta_D(1_D, u),$$

where the last identity holds by naturality of θ. Thus the source is the pair

$$(g^*\theta_D(1_D, u), \theta_C(g, w)).$$

But $g^*\theta_D(1_D, u)$ is the first digit of (the code corresponding to) $\theta_D(1_D, u)$ (which, recall, was 1 iff $g \in \theta_D(1_D, u)$).

The graph Ω^G_* is:

$$\begin{tikzcd}
\bullet_{11} \arrow[loop above, "111"] \arrow[loop below, "110"'] & \bullet_{10} \arrow[loop above, "101"] \arrow[loop below, "100"'] & \bullet_{01} \arrow[loop above, "011"] \arrow[loop below, "010"'] & \bullet_{00} \arrow[loop above, "001"] \arrow[loop below, "000"']
\end{tikzcd}$$

The vertices are the four element Boolean algebra 2^2 with the pointwise ordering, and the same for the edges 2^3. The source map $xyz \mapsto xy$ is the map $2^3 \to 2^2$ induced by the inclusion $2 \hookrightarrow 3$ that projects out the first two arguments of an element of 2^3.

The natural transformation $\delta^G : (G \times G)^G \to \Omega^G$ is computed at the component $X = C, D$, as the composite

$$(\delta^G)_X\langle\theta, \theta'\rangle = \delta\langle\theta, \theta'\rangle : \mathbf{y}X \times G \to G \times G \to \Omega,$$

for $\theta, \theta' : \mathbf{y}X \times G \to G$, resp.; in accordance with exponentiation as explained earlier. Hence, in particular

$$(\delta^G)_D\langle\eta, \mu\rangle = 101.$$

In order to determine what $(\delta^G)_D\langle\eta,\mu\rangle = \delta\langle\theta,\theta'\rangle$ means, we calculate the codings for $(\delta\langle\eta,\mu\rangle)_D(1_D,u)$ and $(\delta\langle\eta,\mu\rangle)_C(g,w)$, resp. As to the second, compute:

$$\begin{aligned}(\delta\langle\eta,\mu\rangle)_C(g,w) &= \delta_C\langle\eta,\mu\rangle_C(g,w)\\ &= \delta_C(\eta_C(g,w),\mu_C(g,w))\\ &= \delta_C(v,w)\\ &= \emptyset.\end{aligned}$$

Therefore $y = 0$.

On the other hand,

$$(\delta\langle\eta,\mu\rangle)_D(1_D,u) = \delta_D(\eta_D(1_D,u),\mu_D(1_D,u)) = \top_D,$$

as $\eta_D(1_D,u) = \mu_D(1_D,u)$. Hence $x = z = 1$

On the other hand, $\Delta_C(x) = xx$ and $\Delta_D(xy) = xxy$, and so

$$\forall_D(xyz) = \bigvee\{st \in \Omega_*(D) \mid sst \leq xyz\},$$

and similarly for Ω. Thus $\forall_D(101) = \bigvee\{00,01\} = 01$, for $\forall_D : \Omega_*^G(D) \to \Omega_*(D)$, while $\forall_D(101) = \bigvee\{00\} = 00$, for $\forall_D : \Omega^G(D) \to \Omega(D)$.

Note finally that function extensionality *is* valid in constant domain models. (See next section for the connection between topos semantics and Kripke models.) For instance, consider a loop graph where $G(D) \cong 2 \cong G(C)$. An element in $\Omega^G(D)$, as a natural transformation $\eta_D : \mathbf{y}D \times G \to \Omega$, is completely determined by the two values $\eta_D(1,a), \eta_D(1,b)$, for $\{a,b\} = G(D)$. Thus, edges in Ω^G can be represented by sequences $xyzw$, where xy and zw are the respective edges $\eta_D(1,a)$ and $\eta_D(1,b)$ in $\Omega(D)$, using the binary notation from before. The source of an edge $xyzw$ is xz. On the other hand, the map $\Delta_D : \Omega(D) \to \Omega^G(D)$ can be computed as $\Delta_D(st) = stst$. Now note that there can be no edge in Ω^G of the form $xy01$ or $01zw$, because 01 is not an edge in Ω (moreover that's the only difference between Ω^G and Ω_*^G). As a result, there is no edge in Ω^G such that applying $\forall$ to it is different from applying $\forall$ to that same edge in Ω_*^G. For the only reason this might happen is because 01 is in the underlying set of the join

$$\forall_D(xyzw) = \bigvee\{st \in \Omega_*(D) \mid stst \leq xyzw\}.$$

However, if $0101 \leq xyzw$, for any edge $xyzw$ in Ω^G, then $xyzw = 1111$. But certainly $\forall$ has the same value on 1111 for both Ω^G and Ω_*^G. Although the argument is for models with domain of cardinality 2, it easily generalizes to any n.

Remark 3.12. A similar style example can be used to show the failure of the sequent $\forall_{x:A}\Box\varphi \vdash \Box\forall_{x:A}\varphi$. In fact, we will show that the square

$$\begin{tikzcd} \Omega^A \arrow[r, "\forall_A"] \arrow[d, "i^A"'] & \Omega \arrow[d, "i"] \\ \Omega_*^A \arrow[r, "\forall_A^*"'] & \Omega_* \end{tikzcd}$$

does not necessarily commute (though note that the analogous one for τ always does). Consider again the finite category $\mathbf{C}$

$$C \xrightarrow{g} D$$

and the functor $A : \mathbf{C}^{op} \to \mathbf{Sets}$ defined by

$$A(D) = \{a\},\ A(C) = \{b, c\},\ A(g)(a) = b.$$

Define the natural transformation $\eta : \mathbf{y}D \times A \to \Omega$ by

$$\eta_D(1_D, a) = \top_D \quad \text{and} \quad \eta_C(g, c) = \emptyset$$

Necessarily $\eta_C(g, b) = \{1_C\} = \top_C$. For by naturality of η the following has to commute:

$$\begin{tikzcd} \mathbf{y}D(D) \times A(D) \arrow[r, "\eta_D"] \arrow[d, "\mathbf{y}D(g)\times A(g)"'] & \Omega(D) \arrow[d, "g^*"] \\ \mathbf{y}D(C) \times A(C) \arrow[r, "\eta_C"'] & \Omega(C) \end{tikzcd}$$

i.e.

$$\eta_C(g, b) = \eta_C(\mathbf{y}D(g) \times A(g))(1_D, a) = g^*(\eta_D(1_D, a)) = g^*\top_D = \top_C.$$

Then:

$$\begin{aligned} \forall_D(\eta) &= \bigvee\{\sigma \in \Omega(D) \mid \Omega(l)(\sigma) \leq \eta_X(l, x), \quad \text{for all} \quad l : X \to D \text{ and } x \in A(X)\} \\ &= \emptyset. \end{aligned}$$

Because $\Omega(g)(\sigma) \leq \eta_X(l, x)$, for all $l : X \to D$ and $x \in A(C)$, only if $\sigma = \emptyset$. On the other hand,

$$\begin{aligned} \forall_D^*(\eta) &= \bigvee\{\sigma \in \Omega_*(D) \mid \Omega(l)(\sigma) \leq \eta_X(l, x), \quad \text{for all} \quad l : X \to D \text{ and } x \in A(X)\} \\ &= \bigvee\{\emptyset, \{1_D\}\} \\ &= \{1_D\}, \end{aligned}$$

because $g^*\{1_D\} = \emptyset$ and $\{1_D\} \leq \eta_D(1_D, a) = \top_D$. We apply this argument to the case where $\eta = \tau_D^A(\eta')$, for some $\eta' \in \Omega_*^A$.

A similar example also shows that the dual statement

$$\Box \exists_{x:A} \varphi \vdash \exists_{x:A} \Box \varphi$$

fails. In the previous setting change the definition of η so that

$$\eta_D(1_D, a) = \emptyset = \eta_C(g, b).$$

Then:

$$\begin{aligned}
\exists_D(\eta) &= \bigwedge\{\sigma \in \Omega(D) \mid \eta_X(l, x) \leq \Omega(l)(\sigma), \quad \text{for all} \;\; l : X \to D \text{ and } x \in A(X)\} \\
&= \bigwedge\{\top_D, \{g\}\} \\
&= \{g\},
\end{aligned}$$

while

$$\begin{aligned}
\exists_D^*(\eta) &= \bigwedge\{\sigma \in \Omega_*(D) \mid \eta_X(l, x) \leq \Omega(l)(\sigma), \quad \text{for all} \;\; l : X \to D \text{ and } x \in A(X)\} \\
&= \bigwedge\{\{g\}, \top_D, \{1_D\}\} \\
&= \emptyset.
\end{aligned}$$

3.5 Kripke Models

In this section we recall how Kripke models are described through the present framework as a special case. As is well known, any functor $F : \mathbf{C} \to \mathbf{D}$ induces a geometric morphism

$$f^* \dashv f_* : \mathbf{Sets}^{\mathbf{C}} \to \mathbf{Sets}^{\mathbf{D}},$$

where f^* is precomposition with F, and f_* is a right Kan extension. Let $\mathbf{C} = |\mathbf{D}|$ and F the inclusion $i : |\mathbf{D}| \to \mathbf{D}$. Then the induced geometric morphism $i^* \dashv i_* : \mathbf{Sets}^{|\mathbf{D}|} \to \mathbf{Sets}^{\mathbf{D}}$ is surjective. The subobject classifier $\Omega_{\mathbf{D}}$ in $\mathbf{Sets}^{\mathbf{D}}$ consists, for each D, of the set of cosieves on D, which can be construed as the functor category

$$2^{D/\mathbf{D}},$$

where 2 is viewed as the poset $\{0 \leq 1\}$; while $\Omega_{|\mathbf{D}|}(D) = 2$, for each D in $\mathbf{D}$.

On the other hand, by the definition of right Kan extension, $i_*\Omega_{|\mathbf{D}|}(D) = \prod_{h \in D/\mathbf{D}} 2 = 2^{|D/\mathbf{D}|}$, as can also be seen from

$$i_*\Omega_{|\mathbf{D}|}(D) \cong \mathrm{Hom}_{\widehat{\mathbf{D}}}(\mathbf{y}D, i_*\Omega_{|\mathbf{D}|}) \cong \mathrm{Hom}_{\widehat{|\mathbf{D}|}}(i^*(\mathbf{y}D), \Omega_{|\mathbf{D}|}).$$

The last set is (isomorphic to) the set of subfamilies of the functor $i^*(\mathbf{y}D) : |\mathbf{D}| \to \mathbf{Sets}$, by the definition of the subobject classifier $\Omega_{|\mathbf{D}|}$: each natural transformation

$$i^*\mathbf{y}D = \mathbf{y}D \circ i = \mathrm{Hom}_{\mathbf{D}}(D, -) \longrightarrow 2$$

determines, for each D' in $\mathbf{D}$, a set of arrows $D \to D'$. On arrows $h : D \to D''$, the functor $i_*\Omega_{|\mathbf{D}|}$ is the function $i_*\Omega_{|\mathbf{D}|}(h) : i_*\Omega_{|\mathbf{D}|}(D) \to i_*\Omega_{|\mathbf{D}|}(D'')$ defined as

$$i_*\Omega_{|\mathbf{D}|}(h)(A) = \{f : D'' \to X \mid f \circ h \in A\}.$$

The components of the (internal) adjunction $i : \Omega_{\mathbf{D}} \leftrightarrows i_*\Omega_{|\mathbf{D}|} : \tau$ then read

$$i_D : 2^{D/\mathbf{D}} \leftrightarrows 2^{|D/\mathbf{D}|} : \tau_D,$$

where $i_D \dashv \tau_D$ "externally". It is not hard to see that i is the inclusion, while

$$\tau_D(A) = \bigvee\{S \in 2^{D/\mathbf{D}} \mid i_D(S) \leq A\},$$

by the definition of right adjoint to the frame map i (cf. (**??**)). In words, τ maps any family of arrows with domain D to the largest cosieve on D contained in it. In particular, when $\mathbf{D}$ is a preorder, then $D/\mathbf{D} = \uparrow(D)$, the upward closure of D; while $2^{D/\mathbf{D}}$ is the set of all monotone maps $\uparrow(D) \to 2$, i.e. upsets of $\uparrow(D)$, while $2^{|D/\mathbf{D}|}$ is the set of arbitrary subsets of $\uparrow(D)$.

An arrow $\varphi : E \to i_*\Omega_{|\mathbf{D}|} = 2^{|-/\mathbf{D}|}$ in $\mathbf{Sets}^{\mathbf{D}}$ defines an indexed subfamily P of the functor F, and conversely. Explicitly, given such $\varphi : E \to i_*\Omega_{|\mathbf{D}|}$, define subsets $P_\varphi(D) \subseteq E(D)$, for each D in $\mathbf{D}$ and $a \in E(D)$, by

$$a \in P_\varphi(D) \ \text{ iff } \ 1_D \in \varphi_D(a). \tag{13}$$

Conversely, given maps $E(D) \to 2$, i.e. components of an arrow $i^*E \to \Omega_{|\mathbf{D}|}$ in $\mathbf{Sets}^{|\mathbf{D}|}$, or equivalently a subfamily P of E, define a natural transformation $\varphi_P : E \to i_*\Omega_{|\mathbf{D}|}$ by

$$(\varphi_P)_D(a) = \{f : D \to C \mid E(f)(a) \in P(C)\}, \tag{14}$$

These constructions are mutually inverse and so describe the canonical isomorphism

$$\mathrm{Hom}(E, i_*\Omega_{|\mathbf{D}|}) \cong \mathrm{Hom}(i^*E, \Omega_{|\mathbf{D}|}) \cong \mathrm{Sub}(i^*E).$$

Note also that the transpose $\overline{\varphi} = \varepsilon\varphi^*$ of $\varphi : E \to \Omega_*$ along the adjunction $f^* \dashv f_*$ actually is the classifying map in $\mathbf{Sets}^{|\mathbf{D}|}$ of the subobject P_φ of f^*E defined in (13):

$$\begin{aligned} \varepsilon_C\varphi^*_C(a) = 1 \ \ &\text{iff} \ \ 1_C \in \varphi^*_C(a) \\ &\text{iff} \ \ 1_C \in \varphi_C(a) \\ &\text{iff} \ \ a \in P_\varphi(C), \end{aligned}$$

for any $a \in E(C)$.

On the other hand, considering $\Omega_{\mathbf{D}} = 2^{D/\mathbf{D}}$ instead of $2^{|D/\mathbf{D}|}$, the same definitions (13) and (14) establish a correspondence between sub*functors* of E and their classifying maps in $\mathbf{Sets}^{\mathbf{D}}$. In particular, the classifying map of a subfunctor of E factors through $i_*\Omega_{|K|}$ via τ.

Thus, when $\mathbf{D}$ is a preorder, algebraic models in the complete Heyting algebra $i_*\Omega_{|K|}$ are precisely Kripke models on $\mathbf{D}$. The "domain" of the model is given by the functor E, while each $E(D)$ is the domain of individuals at each world D. Each formula determines, as an arrow $\varphi : E \to i_*\Omega_{|K|}$, a subfamily of E, that is a family $(P_\varphi(D) \subseteq E(D))$. Then τ determines the largest compatible subfamily of that family, i.e. a family closed under the action of E. Indeed, for $x \in E(D)$,

$$x \in P_{\tau\varphi}(D) \;\text{ iff }\; 1_D \in (\tau\varphi)_D(x).$$

Now $(\tau\varphi)_D(x)$ is the maximal sieve on D just in case $\varphi_D(x)$ is. So, if satisfied, the right-hand side means that $x \in P_\varphi(D)$ and moreover $F(f)(x) \in P_\varphi(C)$, for all $C \geq D$. Semantically speaking, x satisfies $\tau\varphi$ (at D) just in case x (or rather its "counterpart" $F_{CD}(x)$) satisfies φ in all worlds accessible from D.

Thus we recovered the natural adjunction

$$\Delta_E : \mathrm{Sub}(E) \leftrightarrows \mathrm{Sub}(i^*E) : \Gamma_E$$

that succinctly describes the algebraic structure of Kripke models.

Lastly, presheaf semantics reduces to standard Kripke semantics for propositional modal logic in the following sense. In the latter, propositional formulas are recursively assigned elements in $\mathcal{P}(\mathbf{K})$, for a preorder $\mathbf{K}$. Let $\mathcal{P}(\downarrow(-)) = \Omega_*$ be the composite functor

$$\mathbf{K} \xrightarrow{\downarrow} \mathbf{Sets} \xrightarrow{\mathcal{P}(-)} \mathbf{Sets}^{op}.$$

Observe that

$$\mathcal{P}(\mathbf{K}) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{K}^{op}}}(1, \mathcal{P}(\downarrow(-))),$$

via assignments (where $\varphi \subseteq \mathcal{P}(\mathbf{K})$)

$$\varphi \mapsto (\varphi_k = \downarrow(k \cap \varphi) \mid k \in \mathbf{K})$$

and

$$(\varphi_k \mid k \in \mathbf{K}) \mapsto \bigcup_k \varphi_k.$$

Thus modelling formulas (in one variable, say) by maps of presheaves

$$M \longrightarrow \mathcal{P}(\downarrow(-)) = \Omega_*$$

yields precisely the familiar Kripke model idea for propositions, i.e. closed formulas. Moreover, for constant domains:

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathbf{K}^{op}}}(\Delta M, \mathcal{P}(\downarrow(-))) \cong \mathrm{Hom}_{\mathbf{Sets}}(M, \varprojlim \mathcal{P}(\downarrow(-))) \cong \mathrm{Hom}_{\mathbf{Sets}}(M, \mathcal{P}(K)).$$

Here, $\Delta : \mathbf{Sets} \longrightarrow \mathbf{Sets}^{\mathbf{K}^{op}}$ is the functor $\Delta(M)(k) = M$, for any set M and $k \in \mathbf{K}$. A function $\varphi : M \longrightarrow \mathcal{P}(K)$ assigns to each individual in the domain M a set of worlds for which the individual satisfies the formula represented by φ.

Another way of seeing the close relation between presheaf semantics and Kripke semantics is via the notion of "Kripke-Joyal forcing" [20, 16]. For any topos $\mathcal{E}$ one can define a forcing relation $\Vdash$ to interpret intuitionistic higher-order logic . Given an arrow $\varphi : M \to \Omega_{\mathcal{E}}$, let S_φ be the subobject of M classified by φ. Then for any $a : X \to M$, define

$$X \Vdash \varphi(a) \ \text{ iff } \ a \text{ factors through } S_\varphi. \tag{15}$$

This holds iff $\varphi a = \mathtt{t}_X$, where $\mathtt{t}_X$ is the arrow $\top \circ !_X : X \to 1 \to \Omega_{\mathcal{E}}$. The idea is that φ corresponds to a formula, while a is a generalized element of M, thought of as a term $x : X \mid a : M$. In fact, φ and a *are* terms in the internal language of $\mathcal{E}$, reinterpreted into $\mathcal{E}$ by the forcing relation. The relation $\Vdash$ satisfies certain recursive clauses for all the logical connectives [20, 16]. Conversely, starting with an interpretation of the basic symbols of a higher-order type theory in a topos $\mathcal{E}$ (as maps into $\Omega_{\mathcal{E}}$), then these recursive clauses determine when a formula is true ("at an object X"). When a is a closed term, i.e. a constant, for which one may assume $X = 1$, then this says that the two arrows

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[rr, bend right, "\top"'] & M \arrow[r, "\varphi"] & \Omega_{\mathcal{E}} \end{tikzcd}$$

are equal; i.e. the closed sentence $\varphi[a/x]$ is "true". In general, the forcing relation thus defines when formulas are true (at X), much as in Kripke semantics, as we now illustrate.

Consider presheaf toposes of the form $\mathbf{Sets}^{\mathbf{C}^{op}}$. In this case, the forcing relation $X \Vdash \varphi(a)$ can be restricted to objects X in $\mathcal{E}$ forming a generating set.[23] For presheaf toposes $\mathbf{Sets}^{\mathbf{C}^{op}}$ the representable functors $\mathbf{y}C$ form a generating set, so one may assume that $X = \mathbf{y}C$, for some object C in $\mathbf{C}$. Also, by the Yoneda lemma, generalized elements $a : \mathbf{y}C \to M$ may be replaced by actual elements $a \in M(C)$. To say that $a : \mathbf{y}C \to M$ factors through a subobject $S \in \mathrm{Sub}_{\mathcal{E}}(M)$ is then equivalent to saying that the

[23] Cf. [16]. One says that a set S of objects from $\mathcal{E}$ is *generating*, iff for any $f \neq g : A \rightrightarrows B$ in $\mathcal{E}$, there is an arrow $x : X \to A$, for some $X \in S$, such that $fx \neq gx$.

corresponding element $a \in M(C)$ actually lies in $S(C)$. As a result, the forcing condition becomes

$$\mathbf{y}C \Vdash \varphi(a) \ \text{ iff } \ a \in S_\varphi(C),$$

where, as before, φ classifies the subobject S_φ of M. We shall hereafter write $C \Vdash \ldots$ instead of $\mathbf{y}C \Vdash \ldots$.

Now consider the standard Ω_*-valued model for classical higher-order modal logic in a presheaf topos $\mathbf{Sets}^{\mathbf{C}^{op}}$, associated with the canonical geometric morphism $\mathbf{Sets}^{|\mathbf{C}|} \to \mathbf{Sets}^{\mathbf{C}^{op}}$. We define another forcing relation $C \Vdash_* \varphi(a)$ which takes this modal logic into account.

Definition 3.13. *For any presheaf topos* $\mathbf{Sets}^{\mathbf{C}^{op}}$*, define a* forcing relation $\Vdash_*$ *for arrows* $\varphi : M \to \Omega_*$*, objects* C *in* $\mathbf{C}$*, and elements* $a \in M(C)$ *by:*

$$C \Vdash_* \varphi(a) \ \textit{ iff } \ C \Vdash \overline{\varphi}(a), \tag{16}$$

where $\Vdash$ *on the right-hand side is the usual forcing relation w.r.t.* $\mathbf{Sets}^{|\mathbf{C}|}$ *(as defined in* (15)*), and* $\overline{(-)}$ *indicates transposition along* $f^* \dashv f_*$*.*

Further analysing the right-hand side of (16) gives:

$$C \Vdash \overline{\varphi}(a) \ \text{ iff } \ a \in S_{\overline{\varphi}}(C) \tag{17}$$

where $S_{\overline{\varphi}}$ is the subobject of M^* classified by $\overline{\varphi}$ in $\mathbf{Sets}^{|\mathbf{C}|}$.

Proposition 3.14. *Let* $\Vdash_*$ *be the forcing relation of Definition 3.13. Then for all* $\varphi, \psi : M \to \Omega_*$ *and* $a \in M(C)$ *the following hold:*

$C \Vdash_* \top$	*always*	
$C \Vdash_* \bot$	*never*	
$C \Vdash_* \varphi(a) \wedge \psi(a)$	*iff*	$C \Vdash_* \varphi(a)$ *and* $C \Vdash_* \psi(a)$
$C \Vdash_* \varphi(a) \vee \psi(a)$	*iff*	$C \Vdash_* \varphi(a)$ *or* $C \Vdash_* \psi(a)$
$C \Vdash_* \varphi(a) \Rightarrow \psi(a)$	*iff*	$C \Vdash_* \varphi(a)$ *implies* $C \Vdash_* \psi(a)$
$C \Vdash_* \forall x \varphi(x, a)$	*iff*	$C \Vdash_* \varphi(b, a)$ *for all* $b \in M(C)$
$C \Vdash_* \exists x \varphi(x, a)$	*iff*	$C \Vdash_* \varphi(b, a)$ *for some* $b \in M(C)$
$C \Vdash_* \Box \varphi(a)$	*iff*	$D \Vdash_* \varphi(p^* a)$ *for every* $p : D \to C$
$C \Vdash_* t(a) \in u(a)$	*iff*	$(1_C, t_C(a)) \in (u_C(a))_C$, *for* $t : M \to N$ *and* $u : M \to \Omega_*^N$

where $\Box = i\tau$*, and* $\forall x \varphi$ *is the arrow* $M \xrightarrow{\widehat{\varphi}} \Omega_*^M \xrightarrow{\forall_M} \Omega_*$*, with* $\widehat{\varphi}$ *the exponential transpose of* $M \times M \xrightarrow{\varphi} \Omega_*$*, and similarly for* $\exists x \varphi(x, a)$*.*

Remark 3.15. Although $\Vdash_*$ is a relation between objects C and arrows $\varphi : M \to \Omega_*$, it also makes sense to think of the φ as formulas, with the clauses above holding w.r.t. the arrow $[\![\varphi]\!]$ assigned to the formula φ as in section 3.2 For instance, interpreting a syntactic expression $\exists x \varphi(x, y)$ (by 3.3) yields an arrow $\exists_M \widehat{[\![\varphi]\!]}$. When $\mathbf{C}$ is a preorder this is then not merely similar to, but actually *is* the Kripkean satisfaction relation between worlds and formulas, extended to higher-order logic.

Proof. We shall just do a few exemplary cases for the purpose of illustration. Consider $C \Vdash_* \varphi(a) \vee \psi(a)$, which by definition 3.13 means that $a \in S_{\overline{\varphi \vee \psi}}(C)$. Here, $\Omega_* \times \Omega_* \xrightarrow{\vee} \Omega_*$ is the join map. Recall from proposition 2.10 that $\vee$ actually is of the form $\vee_*$, for the join map $\Omega \times \Omega \xrightarrow{\vee} \Omega$ in $\mathbf{Sets}^{|\mathbf{C}|}$. Thus the following commutes, by naturality of the counit ε:

$$\begin{tikzcd}
M^* \times M^* \arrow[r, "\langle \varphi^*, \psi^* \rangle"] \arrow[dr, "\langle \overline{\varphi}, \overline{\psi} \rangle"'] & (\Omega_*)^* \times (\Omega_*)^* \arrow[r, "(\vee_*)^*"] \arrow[d, "\varepsilon \times \varepsilon"] & (\Omega_*)^* \arrow[d, "\varepsilon"] \\
& \Omega \times \Omega \arrow[r, "\vee"'] & \Omega
\end{tikzcd}$$

That is to say,

$$\overline{\varphi \vee \psi} = \overline{\varphi} \vee \overline{\psi},$$

and so $S_{\overline{\varphi} \vee \overline{\psi}} = S_{\overline{\varphi \vee \psi}}$. Since $\mathbf{Sets}^{|\mathbf{C}|}$ is a Boolean topos, by the definition of $S_{\overline{\varphi} \vee \overline{\psi}}$ in $\mathbf{Sets}^{|\mathbf{C}|}$ we have:

$$a \in S_{\overline{\varphi} \vee \overline{\psi}}(C) \text{ iff } a \in S_{\overline{\varphi}}(C) \text{ or } a \in S_{\overline{\psi}}(C),$$

i.e. if and only if $C \Vdash_* \varphi(a)$ or $C \Vdash_* \psi(a)$. The argument for the other logical connectives is similar.

For $\forall$, by definition,

$$C \Vdash_* \forall x \varphi(x, a) \text{ iff } a \in S_{\forall_M \widehat{\varphi}}(C),$$

with

$$S_{\forall_M \widehat{\varphi}}(C) = \{a \in M(C) \mid 1_C \in (\forall_M \widehat{\varphi})_C(a)\}$$

defined as in (13). By the definition of $\forall_M$, and because $|\mathbf{C}|$ is discrete:

$$\begin{aligned}
1_C \in (\forall_M \widehat{\varphi})_C(a) \ \text{ iff } \ & 1_C \in \bigcup\{s \in \Omega_*(C) \mid \Omega_*(f)(s) \leq \widehat{\varphi}_C(a)_D(f,b), \\
& \qquad \text{for all } f : D \to C, b \in M(D)\} \\
\text{iff } \ & 1_C \in \bigcup\{s \in \Omega_*(C) \mid s \leq \widehat{\varphi}_C(a)_C(1_C,b), \ \text{for all } b \in M(C)\} \\
\text{iff } \ & 1_C \in \varphi_C(a,b), \ \text{for all } b \in M(C) \\
\text{iff } \ & (a,b) \in S_\varphi, \ \text{for all } b \in M(C) \\
\text{iff } \ & C \Vdash_* \varphi(a,b), \ \text{for all } b \in M(C).
\end{aligned}$$

The last two equivalences hold by the definition of S_φ and $\Vdash_*$. To see the third equivalence, let $\alpha : \mathbf{y}C \to M$ be the map that corresponds under Yoneda to $a \in M(C)$. Then, by the definition of $\widehat{\varphi}$ (cf. (10)):

$$\widehat{\varphi}_C(a)_C(1_C,b) = \varphi_C(\alpha \times 1_M)_C(1_C,b) = \varphi_C(\alpha_C(1_C),b) = \varphi_C(a,b).$$

Then, if 1_C is in the union, it is in one of the $s \in \Omega_*(C)$, and thus $1_C \in \varphi_C(a,b)$, for all $b \in M(C)$. On the other hand, if $1_C \in \varphi_C(a,b)$, for all $b \in M(C)$, then 1_C is in the union for $s = \{1_C\}$.

The clause for $\in$ follows from its definition:

$$\begin{aligned}
S_{\varepsilon\langle s,t\rangle} &= \{a \in M(C) \mid 1_C \in \varepsilon\langle s,t\rangle_C(a)\} \\
&= \{a \in M(C) \mid 1_C \in \varepsilon_C(s_C(a), t_C(a))\} \\
&= \{a \in M(C) \mid 1_C \in (s_C(a))_C(1_C, t_C(a))\},
\end{aligned}$$

using the definition of the evaluation map $\varepsilon : \Omega^A \times A \to \Omega$.

For $\Box$, as before, $i\tau\varphi$ determines a subfamily of M with components

$$S_{i\tau\varphi}(C) = \{a \in M(C) \mid 1_C \in (i\tau\varphi)_C(a)\}.$$

But $(i\tau\varphi)_C(a)$ is a sieve, as it factors through $\Omega(C)$, and so

$$S_{i\tau\varphi}(C) = \{a \in M(C) \mid (i\tau\varphi)_C(a) = \top_C\},$$

for $\top_C$ the maximal sieve on C. However, by the defining properties of τ and i,

$$(i\tau\varphi)_C(a) = \top_C \ \text{ iff } \ \varphi_C(a) = \top_C.$$

Therefore,

$$\begin{aligned}
S_{i\tau\varphi}(C) &= \{a \in M(C) \mid \varphi_C(a) = \top_C\} \\
&= \{a \in M(C) \mid (\chi_{S_\varphi})_C(a) = \top_C\} \\
&= \{a \in M(C) \mid \{p : D \to C \mid p^*a \in S_\varphi(D)\} = \top_C\} \\
&= \{a \in M(C) \mid p^*a \in S_\varphi(D), \ \text{for all } p : D \to C\}.
\end{aligned}$$

In forcing terms:

$$\begin{aligned} C \Vdash_* i\tau\varphi(a) \;\; &\text{iff} \;\; a \in S_{i\tau\varphi}(C) \\ &\text{iff} \;\; p^*a \in S_\varphi(D), \;\text{for all } p : D \to C \\ &\text{iff} \;\; D \Vdash_* \varphi(p^*a), \;\text{for all } p : D \to C. \end{aligned} \qquad \square$$

3.6 Relative Models

This section sums up the notion of model w.r.t. relative model structures.

Definition 3.16. *Consider any topos $\mathcal{E}$, and any $\mathcal{M}$-relative model structure*

$$\begin{tikzcd} H \arrow[r, shift left, "\tau"] \arrow[r, phantom, "\top"] & B \arrow[l, shift left, "i"] \arrow[r, tail, "\beta"] & \Omega_{\mathcal{E}} \end{tikzcd}$$

(for some suitable set $\mathcal{M}$) in $\mathcal{E}$ that satisfies the conditions from def. 2.24. An interpretation of a higher-order modal theory $\mathbb{T}$ (as defined in section 3.1) is same as in 3.3, with the following slight differences

- *For each type A in $\mathbb{T}$, $[\![A]\!] \in \mathcal{M}$, i.e. $[\![A]\!]$ must be B-standard*
- *The type of proposition is interpreted by H*
- *The modal operator is interpreted by $i\tau$*
- *For any type A, the equality relation is interpreted by the composite*

  $$i \circ \partial_A,$$

  where ∂_A is the factorization

  $$\partial_A : [\![A]\!] \times [\![A]\!] \to [\![B]\!]$$

  of the classifying map $\delta_A : [\![A]\!] \times [\![A]\!] \to \Omega_{\mathcal{E}}$ through β.

Note that the last condition can always be met given the first one.

We now record the soundness for relative models. It more or less follows directly from the definitions. In particular, soundness of equality follows from prop. 2.20 and soundness w.r.t. the doctrinal interpretation. Therefore, it is mainly the two extensionality principles that are of interest.

Lemma 3.17. *Modal propositional extensionality is valid in general models $\theta : M \leftrightarrows H : i$.*

Proof. This follows right away because $\partial_M = \theta \circ \Leftrightarrow$ by construction of ∂_M.

Equivalently, in analogy to lemma 3.2 in the LeA paper it would be sufficient to show that the following diagram commutes:

$$\begin{tikzcd}
M \times M \arrow[r, "\Rightarrow"] \arrow[d, "\langle \pi_1, \wedge \rangle"'] & M \arrow[d, "\theta"] \\
M \times M \arrow[r, "\partial_M"'] & H
\end{tikzcd}$$

which is readily seen through the following two pullbacks and uniqueness of classifying maps.

$$\begin{tikzcd}
(\leq) \arrow[r] \arrow[d] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
M \times M \arrow[r, "\Rightarrow"'] & M \arrow[r, "\theta"'] & H \arrow[r, tail, "\imath"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

$$\begin{tikzcd}
(\leq) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\Delta"] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
M \times M \arrow[r, "\langle \pi_1, \wedge \rangle"'] & M \times M \arrow[r, "\partial_M"'] \arrow[rr, bend right, "\delta_M"'] & H \arrow[r, tail, "\imath"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

□

Lemma 3.18. *Function extensionality is valid in general models $\theta : M \leftrightarrows H : i$.*

Proof. Again, the idea is to prove a lemma analogous to lemma 3.4 in the LeA paper. There we showed that the diagram

$$\begin{tikzcd}
\Omega^Y \arrow[r, "i^Y"] & M^Y \arrow[r, "\forall_Y"] & M \arrow[d, "\tau"] \\
X^Y \times X^Y \arrow[u, "(\delta_X)^Y"] \arrow[rr, "\delta_{X^Y}"'] & & \Omega
\end{tikzcd} \tag{18}$$

commutes. Similarly, in the context of general models there exists a commutative diagram:

$$\begin{tikzcd}
H^Y \arrow[r, "i^Y"] & M^Y \arrow[r, "\forall_Y"] & M \arrow[d, "\theta"] \\
X^Y \times X^Y \arrow[u, "(\partial_X)^Y"] \arrow[rr, "\partial_{X^Y}"'] & & H
\end{tikzcd}$$

In fact, the composite $\theta \circ \forall_Y \circ i^Y \circ (\partial_X)^Y = \forall_Y(\partial_X)^Y$ was precisely how the factorization of δ_{X^Y} was obtained earlier.

Inspecting the soundness proof of functional extensionality for standard models, it turns out that commutativity of the latter is precisely what is needed to make the proof work for the present context of general models; just as (18) was essential for the soundness of function extensionality w.r.t. standard models. $\square$

Thus we record:

Proposition 3.19. *Consider any topos $\mathcal{E}$, any $\mathcal{M}$-relative model structure*

$$\begin{tikzcd}
H \arrow[r, shift left, "\tau"] \arrow[r, phantom, "\top"] & B \arrow[l, shift left, "i"] \arrow[r, tail, "\beta"] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

(for some suitable set $\mathcal{M}$) in $\mathcal{E}$ that satisfies the conditions from def. 2.24. Then for any model $[\![-]\!]$ in that structure defined as in 3.16, if

$$\Gamma \mid \varphi \vdash \psi \quad \textit{in} \quad \mathbb{T}$$

in $\mathbb{T}$, then

$$[\![\Gamma \mid \varphi]\!] \leq [\![\Gamma \mid \psi]\!] \quad \textit{in} \quad \mathrm{Hom}_{\mathcal{E}}([\![\Gamma]\!], H).$$

4 Elementary Completeness

By elementary completeness we mean completeness of higher-order modal logic w.r.t. to models in arbitrary toposes. We prove this by constructing the syntactic topos $\mathcal{E}_{\mathbb{T}}$ associated with a higher-order modal theory $\mathbb{T}$. As will turn out, the definition of $\mathcal{E}_{\mathbb{T}}$ is a variation of the non-modal higher-order case. We will give the definition and point out where it differs from the non-modal version.

4.1 The Syntactic Topos of a higher-order theory

Given a higher-order modal theory $\mathbb{T}$, define the category $\mathcal{E}_{\mathbb{T}}$ as follows:

- Objects: closed terms $\alpha : PA$ of the form $\alpha = \{x : A \mid \Box\varphi\}$.
- An arrow $\varphi : \alpha \to \beta$ is a triple (α, φ, β), where $\varphi : P(A \times B)$ is a $\Box$-stable term that is provably a functional relation from α to β:

  $$\vdash \forall_{x:A}(x \in \alpha \Rightarrow \exists!_{y:B}[y \in \beta \wedge \langle x, y\rangle \in \varphi]).$$

  Note that if we can prove this sequent, the formula is automatically $\Box$-stable given that φ is.

  The requirement of a functional relation is equivalent to the provabilty of the following sequents:

  $$x : A, y : B \mid \alpha \vdash \exists_{y:B}\varphi$$

  $$x : A, y : B \mid \varphi \vdash \alpha \wedge \beta$$

  $$x : A, y : B, z : B \mid \varphi \wedge \varphi[z/y] \vdash z = y$$

  It is important to explicitly specify domain and codomain of an arrow, as it may happen that the same term φ defines different arrows. For instance, this is the case whenever an arrow factors through a subobject of its codomain such as for image factorizations described below. Another example are identity arrows and canonical monomorphisms.

- Objects and arrows are equal in case they are $\mathbb{T}$-provably equivalent. Note, however, if we are given two parallel arrows represented by terms φ, φ' resp., it readily follows that

  $$\varphi \dashv\vdash \varphi' \;\text{ iff }\; \top \vdash \varphi = \varphi'.$$

  In the following we will therefore not distinguish between equality of arrows that are represented by terms φ and φ', resp. and internal identity of terms in the sense that $\top \vdash \varphi = \varphi'$. That is to say, we will refer to arrows φ and φ' and say that they are equal in case they are equal as terms $\varphi = \varphi'$.

- Identity arrows $1_\alpha : \alpha \to \alpha$ are given by

  $$\{\langle x, x'\rangle \in A \times A \mid \alpha \wedge x = x'\}.$$

- The composite of two arrows $\varphi : \alpha \to \beta$, $\psi : \beta \to \gamma$ is defined to be the term

  $$\{\langle x, z\rangle : A \times C \mid \exists_{y:B}(\langle x, y\rangle \in \varphi \wedge \langle y, z\rangle \in \psi)\}.$$

Since φ and ψ are $\Box$-stable, it follows that $\exists_{y:B}(\langle x,y\rangle \in \varphi \wedge \langle y,z\rangle \in \psi)$ is $\Box$-stable, because for any stable $x : A \mid \varphi$ it holds that $\Box\exists x.\varphi = \exists x.\varphi$.

This concludes the definition of $\mathcal{E}_{\mathbb{T}}$. We will make a few remarks about monomorphisms and subobjects. Given an arrow $\varphi : \alpha \to \beta$, denote by φ^{-1} the term

$$\{(y,x) : P(B \times A) \mid (x,y) \in \varphi\}.$$

The term φ^{-1} does not necessarily define an arrow itself. However, we can still define the relational composition of the terms φ and φ^{-1} in the same way as the composition of arrows was defined. We will also denote it using $\circ$.

Lemma 4.1. *An arrow $\varphi : \alpha \to \beta$ is a monomorphism, if and only if*

$$\varphi^{-1} \circ \varphi = 1_\alpha,$$

where is the term defining the identity arrow in $\mathcal{E}_{\mathbb{T}}$:

$$1_\alpha = \{(x,x') \mid x = x'\}.$$

Proof. Suppose $\varphi^{-1} \circ \varphi = 1_\alpha$ and consider any two arrows $\psi, \sigma : \gamma \rightrightarrows \alpha$ such that $\varphi \circ \psi = \varphi \circ \sigma$. Evidently,

$$\psi = \varphi^{-1} \circ \varphi \circ \psi = \varphi^{-1} \circ \varphi \circ \sigma = \sigma.$$

Conversely, consider the object

$$\varphi^{-1} \circ \varphi \equiv \{(x,x') \in P(A \times A) \mid \exists_{y:B}.(x,y) \in \varphi \wedge (x',y) \in \varphi\}.$$

and arrows $\psi, \sigma : \varphi^{-1} \circ \varphi \rightrightarrows \alpha$:

$$\psi \equiv \{((x,x'),x'') : P((A \times A) \times A) \mid x'' = x \wedge (x,x') \in \gamma\}.$$

$$\sigma \equiv \{((x,x'),x'') : P((A \times A) \times A) \mid x'' = x' \wedge (x,x') \in \gamma\}.$$

It now follows that $\varphi \circ \psi = \varphi \circ \sigma$, and thus $\psi = \sigma$, because φ is monic. This in turn implies that $\varphi^{-1} \circ \varphi = 1_\alpha$. For details see [16]

$\Box$

Lemma 4.2. *Given $\mathbb{T}$, an arrow $\varphi : \alpha \to \beta$ in $\mathcal{E}_{\mathbb{T}}$ is a monomorphism if and only if*

$$x : A, x' : A, y : B \mid \varphi \wedge \varphi[x'/x] \vdash x = x'$$

is provable in $\mathbb{T}$.

Proof. Assume the condition holds. Then:

$$\begin{aligned}
\varphi^{-1} \circ \varphi &= \{(x, x') : \mathsf{P}(A \times A) \mid \exists_{y:B}.(x, y) \in \varphi \wedge (y, x') \in \varphi'\} \\
&= \{(x, x') : \mathsf{P}(A \times A) \mid \exists_{y:B}.(x, y) \in \varphi \wedge (x', y) \in \varphi\} \\
&= \{(x, x') : \mathsf{P}(A \times A) \mid x = x'\} = 1_\alpha.
\end{aligned}$$

Here, we use that

$$x : A, x' : A \mid \exists_{y:B}.(x, y) \in \varphi \wedge (x', y) \in \varphi \vdash x = x'$$

if and only if

$$x : A, x' : A, y : B \mid (x, y) \in \varphi \wedge (x', y) \in \varphi \vdash x = x',$$

since y doesn't occur free on the right.

Conversely, assume that $varphi^{-1} \circ \varphi = 1_\alpha$. Then from

$$\{(x, x') : \mathsf{P}(A \times A) \mid \exists_{y:B}.(x, y) \in \varphi \wedge (x', y) \in \varphi\} = \{(x, x') : \mathsf{P}(A \times A) \mid x = x'\}$$

the claim follows.

$\square$

For instance, every object $\alpha : PA$ is a subobject of $\{x : A \mid \top\}$ via the map

$$\{(x, x') : A \times A \mid \alpha \wedge x = x'\},$$

easily seen to be a monomorphism. We now show that in a similar way every subobject can be represented in a canonical form.

Lemma 4.3. *The image of a map $\varphi : \alpha \to \beta$ in $\mathcal{E}_{\mathbb{T}}$ is the object*

$$I_\varphi \equiv \{y : B \mid \exists_{x:A}\varphi\}$$

included into β via the map

$$\imath_\varphi \equiv \{(y, z) : B \times B \mid \exists_{x:A}.\varphi \wedge y = z\}.$$

Proof. First, note that both terms are $\Box$-stable. Moreover, the term $\imath_\varphi$ really defines a map $I_\varphi \to \beta$. In particular, the sequent

$$y : B, z : B \mid \exists_{x:A}\varphi \wedge y = z \vdash \beta$$

holds because $x : A, y : B \mid \varphi \vdash \beta$. The map $\imath_\varphi$ is moreover clearly monic. The factorization of φ through $\imath_\varphi$ is given by the term φ itself. We need to check that the formula φ is well-defined as an arrow with codomain I_φ. As to the first condition of a functional relation we have

$$x : A \mid \alpha \vdash \exists_{y:B}\exists_{x:A}\varphi,$$

because from $\alpha \vdash \exists_{y:B}\varphi$ we get $\exists_{x:A}\alpha \vdash \exists_{x:A}\exists_{y:B}\varphi$ and thus $\alpha \vdash \exists_{x:A}\exists_{y:B}\varphi$ by the $\exists$-rule. As to the second condition, the sequent

$$x : A, y : B \mid \varphi \vdash \exists_{x:A}\varphi$$

is a consequence of $y : B \mid \exists_{x:A}\varphi \vdash \exists_{x:A}\varphi$. On the other hand $\varphi \vdash \alpha$ holds by assumption on φ, as does the third condition, uniqueness of values.

Given any subobject $\gamma : \mu \rightarrowtail \beta$, through which φ factors via a map $\eta : \alpha \to \mu$, define a map $\nu : I_\varphi \to \mu$ to be given by the term

$$\nu = \{(y, z) : B \times C \mid \exists_{x:A}(\eta \wedge \varphi)\}.$$

It is a functional relation:

- With $\varphi \vdash \alpha \vdash \exists_{z:C}\eta$ we have $\varphi \vdash \exists_{z:C}\eta \wedge \varphi$, and hence $\exists_{x:A}\varphi \vdash \exists_{x:A}[\exists_{z:C}\eta \wedge \varphi]$, and therefore $\exists_{x:A}\varphi \vdash \exists_{x:A}\exists_{z:C}[\eta \wedge \varphi]$, since z is not free in φ. Hence
$$\exists_{x:A}\varphi \vdash \exists_{z:C}\exists_{x:A}[\eta \wedge \varphi]$$
- From $\varphi \wedge \eta \vdash \varphi \vdash \exists_{x:A}\varphi$ and $\varphi \wedge \eta \vdash \eta \vdash \mu$ obtain $\varphi \wedge \eta \vdash \exists_{x:A}\varphi \wedge \mu$, and so $\exists_{x:A}(\varphi \wedge \eta) \vdash \exists_{x:A}\varphi \wedge \mu$, by the $\exists$-rule, since x is not free in μ. Thus, the domain and codomain of ν is I_φ and μ, resp.
- Uniqueness. Follows from $\varphi \dashv\vdash \exists_{z:C}[\gamma \wedge \eta]$ and that γ is monic.

Moreover, $\imath_\varphi = \gamma \circ \nu$, as arrows in $\mathcal{E}_{\mathbb{T}}$. The composite $\gamma \circ \nu$ is given by the term
$$\{(y, z) : B \times B \mid \exists_{c:C}[\nu \wedge \gamma]\}.$$

Writing

$$x : A, y : B \mid \varphi,$$
$$x : A, c : C \mid \eta,$$
$$c : C, z : B \mid \gamma,$$

observe that in context $y : B, z : B$ the following equivalences hold:

$$\begin{aligned}
\exists_{c:C}[\nu \wedge \gamma] &\dashv\vdash \exists_{c:C}[\exists_{x:A}[\eta \wedge \varphi] \wedge \gamma] \\
&\dashv\vdash \exists_{c:C}\exists_{x:A}[\eta \wedge \varphi \wedge \gamma] \\
&\dashv\vdash \exists_{x:A}\exists_{c:C}[\eta \wedge \gamma] \wedge \varphi] \\
&\dashv\vdash \exists_{x:A}[\varphi[z/y] \wedge \varphi].
\end{aligned}$$

The second equivalence holds because $x : A$ does not occur free in γ, while the third obtains because $c : C$ is not free in φ. The fourth one uses

$$x : A, z : B \mid \varphi[z/y] \dashv\vdash \exists_{c:C}[\gamma \wedge \eta],$$

which holds by assumption that $\varphi : \alpha \to \beta$ factors through μ. Hence, since $\varphi[z/y] \wedge \varphi \vdash z = y$, we finally have

$$\exists_{x:A}[\varphi[z/y] \wedge \varphi] \dashv\vdash \exists_{x:A}\varphi \wedge z = y.$$

$\square$

Since the image $\imath_\varphi : I_\varphi \hookrightarrow \beta$ of a monomorphism $\varphi : \alpha \to \beta$ is an isomorphism, we have the following corollary:

Corollary 4.4. *In the syntactic category $\mathcal{E}_{\mathbb{T}}$ of a higher-order modal theory, every subobject of an object β can be represented by a map $\imath_{\beta'} : \beta' \to \beta$ in such a way such that $\beta' \vdash \beta$ and the inclusion $\imath_{\beta'}$ is given by the term*

$$\imath_{\beta'} = \{(y', y) : P(B \times B) \mid \beta' \wedge y = y'\}.$$

Proposition 4.5. *For any higher-order modal theory $\mathbb{T}$, the category $\mathcal{E}_{\mathbb{T}}$ has all finite limits and exponentials, and hence is*

Proof.

- Terminal Object **1**: obtained is the term

$$\mid \{x : 1 \mid x = *\} : P1$$

Given any term $\alpha : PA$, the unique arrow $\alpha \to 1$ is given by

$$\{(x, y) : A \times 1 \mid x \in \alpha \wedge y = *\}.$$

- The pullback of two maps

$$\alpha \xrightarrow{\quad\varphi\quad} \gamma \xleftarrow{\quad\vartheta\quad} \beta$$

is constructed as the term

$$\{(x, y) : A \times B \mid \exists_{z:C}.(x, z) \in \varphi \wedge (y, z) \in \vartheta\},$$

with the obvious projections, much as for sets:

$$\{((x, y), x') : P(P(A \times B) \times A) \mid (x, y) \in P \wedge x = x'\},$$

$$\{((x, y), y') : P(P(A \times B) \times B) \mid (x, y) \in P \wedge y = y'\}.$$

- Exponentials: For $\alpha : PA$ and $\beta : PB$, define

$$\beta^\alpha \equiv \{w : P(A \times B) \mid w : \alpha \to \beta\}.$$

where $w : \alpha \to \beta$ is the formula

$$w = \Box w \wedge \Box\forall_{x:A}(x \in \alpha \Rightarrow \exists!_{y:B}[y \in \beta \wedge \langle x, y\rangle \in w])$$

The canonical evaluation map $\varepsilon^{\alpha}_{\beta} : \beta^{\alpha} \times \alpha \longrightarrow \beta$ is given by

$$\{\langle w, x\rangle, y\rangle : P(P(A \times B) \times A) \times B \mid w \in \beta^{\alpha} \wedge \langle x, y\rangle \in w\}$$

For an arrow $h : \alpha \times \beta \longrightarrow \gamma$, define its transpose $h^* : \alpha \longrightarrow \gamma^{\beta}$ to be the term

$$\{\langle x, w\rangle : A \times P(B \times C) \mid x \in \alpha \ \wedge \ w \in \gamma^{\beta} \ \wedge \\ \Box\forall_{y:B}(y \in \beta \Rightarrow \exists_{z:C}(\langle\langle x, y\rangle, z\rangle \in h \ \wedge \ \langle y, z\rangle \in w))\}.$$

□

We now proceed to show that $\mathcal{E}_{\mathbb{T}}$ has a subobject classifier. We wil mainly follow [16]. In order to do this we will need a certain version of lemma 12.3 in [16], suitably adapted to the present context.

Lemma 4.6. *Consider any $t : \mathsf{P} \mid \varphi : \mathsf{P}$ such that*

$$t : \mathsf{P} \mid \varphi \vdash t = \Box t \quad \textit{and} \quad t : \mathsf{P} \mid \varphi \vdash \Box\varphi.$$

Suppose, moreover, that $\vdash \exists!_{t:\mathsf{P}}\varphi(t)$. Then

$$\vdash \forall_{t:\mathsf{P}}.(t = \varphi(\top)) \Leftrightarrow \varphi(t)).$$

Proof. For the right-to-left we note first that

$$\varphi(t) \wedge t \vdash \Box t \vdash t = \top,$$

because $\Box t \dashv\vdash t = \top$. Hence $\varphi(t) \wedge t \vdash \varphi(\top)$, whence $\varphi(t) \vdash t \Rightarrow \varphi(\top)$. On the other hand, by the uniqueness assumption on t,

$$\varphi(t) \wedge \varphi(\top) \vdash t = \top \vdash \Box t \vdash t,$$

and so $\varphi(t) \vdash \varphi(\top) \Rightarrow t$. Hence $\varphi(t) \vdash \varphi(\top) \Leftrightarrow t$. Therefore

$$\Box\varphi(t) \vdash \Box(\varphi(\top) \Leftrightarrow t) \vdash \varphi(\top) = t,$$

with modal propositional extensionality. Hence, finally,

$$\varphi(t) \vdash \varphi(\top) = t,$$

because φ is $\Box$-stable by assumption.

The left-to right is exactly as in [16]. Since $\varphi(s)$, for some $s : \Omega$, from $t = \varphi(\top)$ it follows that $t = \varphi(\top) = s$, and hence $\varphi(t)$.

□

The lemma rests in a sense on a trivialization assumption. For the assumption that $t : \mathsf{P} \mid \varphi \vdash t = \Box t$ and $t : \mathsf{P} \mid \varphi \vdash \Box\varphi$ essentially ensures that the modal operator becomes redundant, so that the argument becomes much as in the non-modal case.

Proposition 4.7. *The syntactic category $\mathcal{E}_{\mathbb{T}}$ of a higher-order theory $\mathbb{T}$ has a subobject classifier.*

Proof. The subobject classifier is provided by the term

$$\Omega_{\mathbb{T}} = \{t : \mathsf{P} \mid \Box t = t\},$$

with generic subobject $\top : 1 \to \Omega$ is

$$\{\langle *, t\rangle \mid t = \top\}.$$

Given an arrow $\varphi : \alpha \to \Omega$ ($\alpha : PA$), the pullback of $\top : 1 \to \Omega_{\mathcal{E}}$ along α becomes using the previous definition of pullbacks (assuming the notation from [16]):

$$\begin{aligned}\text{Ker } \varphi &\cong \{(x, *) : A \times 1 \mid \exists_{t:\mathsf{P}}[(x,t) \in \varphi \wedge (*, t) \in \top]\}\\ &\cong \{x : A \mid (x, \top) \in \varphi\}.\end{aligned}$$

As for any pullback of a monomorphism, the projection to α must be monic as well and in the present case reads as

$$\begin{aligned}\ker \varphi &= \{\langle x, x'\rangle : A \times A \mid x \in \text{Ker } \varphi \wedge x = x'\}\\ &= \{\langle x, x'\rangle : A \times A \mid (x, \top) \in \varphi \wedge x = x'\}.\end{aligned}$$

On the other hand, given a subobject $m : \beta \to \alpha$, set

$$\text{char } m := \{\langle x, t\rangle : A \times \mathsf{P} \mid t = (\exists_{y:B}\langle y, x\rangle \in m)\}.$$

Note that since m is $\Box$-stable by assumption, the existential formula is so as well. Hence, this is well-defined as an arrow $\alpha \to \Omega$. That is to say, one can show that

$$(x, t) \in \text{char } m \vdash t = \Box t.$$

Since for any formula φ we have

$$t = \Box\varphi \vdash t \Leftrightarrow \Box\varphi.$$

Then for any arrow $h : \alpha \longrightarrow \Omega$ in the syntactic category:

$$\begin{aligned}\text{char } \ker h &= \{\langle x, t\rangle : A \times \mathsf{P} \mid t = \exists_{y:A}(\langle y, x\rangle \in \ker h)\}\\ &= \{\langle x, t\rangle : A \times \mathsf{P} \mid t = \exists_{y:A}(\langle x, \top\rangle \in h \;\&\; y = x)\}\\ &= \{\langle x, t\rangle : A \times \mathsf{P} \mid t = \langle x, \top\rangle \in h\}\\ &= \{\langle x, t\rangle : A \times \mathsf{P} \mid \langle x, t\rangle \in h\}.\end{aligned}$$

The last identity uses lemma 4.6.

On the other hand, in order to show that $\ker \text{char } m \cong m$, for any subobject $m : \beta \to \alpha$, define an arrow $u : \beta \to \ker \text{char } m$ by $u \equiv \{(y, x) :$

$B \times B \mid (y, x) \in m\}$, i.e. with the same formula as for m. This is well-defined with codomain Ker char m. Moreover,

$$
\begin{aligned}
(\ker \mathrm{char}\ m) \circ u = & \{(x, y) : B \times A \mid \exists_{x':A}((y, x') \in m \wedge (x', x) \in \ker \mathrm{char}\ m)\} \\
& \{(x, y) : B \times A \mid \exists_{x':A}((y, x') \in m \wedge (x', \top) \in \mathrm{char}\ m \wedge x = x')\} \\
& \{(x, y) : B \times A \mid \exists_{x':A}((y, x') \in m \wedge \top = \exists_{y':B}(y', x') \in m \wedge x = x')\} \\
& \{(x, y) : B \times A \mid \exists_{x':A}((y, x') \in m \wedge \exists_{y':B}(y', x') \in m \wedge x = x')\} \\
& \{(x, y) : B \times A \mid (y, x) \in m \wedge \exists_{y':B}(y', x) \in m\} \\
& \{(x, y) : B \times A \mid (y, x) \in m\}.
\end{aligned}
$$

In particular u is monic. On the other hand,

$$
\begin{aligned}
u \circ u^{-1} &= \{(x, x) : A \times A \mid \exists_{y:B}(x, y) \in m\} \\
&= \{(x, x) : A \times A \mid (x, \top) \in \mathrm{char}\ m\} \quad = \{(x, x) : A \times A \mid x \in \mathrm{Ker\ char}\ m\}.
\end{aligned}
$$

The last term is the identity on Ker char m. □

We consider the object $\{t : \mathsf{P} \mid \top\}$ and define an adjunction

We now have to verify that $\{t : \mathsf{P} \mid \top\}$ is a complete Heyting algebra. Before doing that, we collect a notion of internal order adjunction in $\mathcal{E}_{\mathbb{T}}$.

Definition 4.8. *Given two objects α, β equipped with preorderings ρ, σ, respectively, we say w.r.t. two order maps $\varphi : \alpha \leftrightarrows \beta : \psi$ that φ is left adjoint to ψ just in case*

$$\varphi(x, y) \vdash \sigma(y, y') \quad \textit{iff} \quad \psi(y', x') \vdash (x, x') \in \rho.$$

Also, given a preorder β, with ordering ρ, and any two maps $\varphi, \psi : \alpha \to \beta$, for any α, then the pointwise ordering should intuitively be that

$$\varphi \leq \psi \quad \text{iff} \quad (x, y) \in \varphi \wedge (x, y') \in \psi \vdash (y, y') \in \rho.$$

This is the ordering in the set

$$\mathrm{Hom}_{\mathcal{E}_{\mathbb{T}}}(\alpha, \beta).$$

When $\beta \equiv \{t : \mathsf{P} \mid \top\}$, then the ordering ρ on β is the subobject

$$\{(p, q) : \mathsf{P} \times \mathsf{P} \mid p \wedge q = p\}.$$

Proposition 4.9. *The object $\{t : \mathsf{P} \mid \top\}$ is a faithful complete Heyting algebra in $\mathcal{E}_{\mathbb{T}}$.*

Proof. The finite Heyting structure is defined as in the non-modal case, e.g. the meet operation is defined by the term

$$\{((p, q), t) : (\mathsf{P} \times \mathsf{P}) \times \mathsf{P} \mid t = p \wedge q\}.$$

The ordering is provability, i.e. the subobject $\{(p,q) : \mathsf{P} \times \mathsf{P} \mid p \wedge q = q\}$.

The map

$$\Delta_\alpha : \{p : \mathsf{P} \mid \top\} \longrightarrow \{p : \mathsf{P} \mid \top\}^{\{x:A|\alpha\}}$$

is given by the term

$$\Delta_\alpha = \{(x,w) : P(\mathsf{P} \times P(A \times \mathsf{P})) \mid w = \{(a,q) : P(A \times \mathsf{P}) \mid a \in \alpha \wedge q = x\}\}.$$

Its right adjoint

$$\forall_\alpha : \{p : \mathsf{P} \mid \top\}^{\{x:A|\alpha\}} \longrightarrow \{p : \mathsf{P} \mid \top\}$$

is defined by the term

$$\{(w,t) \mid [\forall_{x:A}(x,\top) \in w] = t\}.$$

In a similar spirit its left adjoint:

$$\exists_\alpha \equiv \{(w,t) \mid [\exists_{x:A}(x,\top) \in w] = t\}.$$

The modal adjunction over the subobject classifier reads

$$i : \{t : \mathsf{P} \mid t = \Box t\} \leftrightarrows \{t : \mathsf{P} \mid \top\} : \tau$$

by setting

$$i = \{(t,s) : \mathsf{P} \times \mathsf{P} \mid t = \Box t \wedge t = s\}$$

$$\tau = \{(t,s) : \mathsf{P} \times \mathsf{P} \mid \Box t = s\}.$$

$\Box$

We next define the canonical model of a higher-order modal theory $\mathbb{T}$ in $\mathcal{E}_\mathbb{T}$.

Definition 4.10. *Suppose given a higher-order theory $\mathbb{T}$. We define the canonical model $[-]$ in $\mathcal{E}_\mathbb{T}$ in the following way.*

- *Basic types A are interpreted by terms $\{x : A \mid \top\}$.*

  $$[A] = \{x : A \mid \top\}.$$

  In particular, the terminal type 1 is interpreted by the terminal object

  $$\{x : 1 \mid x = *\}.$$

  The type of propositions P is interpreted by the complete Heyting algebra

  $$[\mathsf{P}] = \{t : \mathsf{P} \mid \top\}.$$

- *A term $\emptyset \mid t : B$ is sent to the arrow $1 \to [B]$*

$$\{(*, t)\}.$$

A basic function symbol $f : A_1, \dots, A_n \to B$ is interpreted as the arrow

$$\{(x_1, \dots, x_n, y) : A_1 \times \cdots \times A_n \times B \mid f(x_1, \dots, x_n) = y\}.$$

The definition for the complex types and terms follows the notion of a model of a higher-order theory given before, exploiting the topos structure of $\mathcal{E}_{\mathbb{T}}$. In particular a term of the form $x : A \mid \Box t : \mathsf{P}$ is interpreted by the composite

$$\{x : A \mid \top\} \xrightarrow{[t]} \{t : \mathsf{P} \mid \top\} \xrightarrow{\tau} \{t : \mathsf{P} \mid t = \Box t\} \xrightarrow{i} \{t : \mathsf{P} \mid \top\}.$$

It follows that for every $x : A \mid t : B$ the corresponding arrow $[t] : [A] \to [B]$ in $\mathcal{E}_{\mathbb{T}}$ is well-defined, i.e. defined through a box-stable formula. For instance, a term $x : A \mid f(t) : B$, for a function symbol $f : A \to A'$ and term $x : A \mid t : A'$, will be the arrow

$$\{(x', y) : A' \times A \mid f(x) = y\} \circ \{(x, x') \times A \times A' \mid (x, x') \in |t|\},$$

where we assume by hypothesis that $|t|$ is box-stable. Also note that terms $x : A \mid \varphi : \mathsf{P}$ and $x : A \mid \Box\varphi : \mathsf{P}$ are in general different, since $x : A \mid \Box\varphi : \mathsf{P}$ is interpreted as the arrow

$$\{(s, t) : \mathsf{P} \times \mathsf{P} \mid t = \Box s\} \circ \varphi = \{(x, p) : A \times \mathsf{P} \mid \exists_t.(x, t) \in |\varphi| \wedge p = \Box t\}$$

Theorem 4.11 (Completeness)**.** *For any two formuals $x : A \mid \varphi : P$ and $y : B \mid \psi : P$, if*

$$[\![x : A \mid \varphi : P]\!] \leq [\![y : B \mid \psi : P]\!]$$

in $\mathcal{E}_{\mathbb{T}}$, then

$$x : A \mid \varphi \vdash \psi$$

Proof. To begin with, consider a single formula $x : A \mid \varphi : \mathsf{P}$ such that

$$[\![x : A \mid \varphi : \mathsf{P}]\!] = \{(x, p) : A \times \mathsf{P} \mid \varphi = p\}$$

in $\mathcal{E}_{\mathbb{T}}$ that factors through $\top : 1 \to \{p : \mathsf{P} \mid \top\}$. That is to say

$$\{(x, p) : A \times \mathsf{P} \mid \varphi(x) = p\} = \{(x, p) : A \times \mathsf{P} \mid p = \top\}$$

It follows that

$$\top \vdash \varphi(x) = \top \vdash \Box\varphi(x) \vdash \varphi(x).$$

Similarly, suppose the pair

$$\langle [\![x : A \mid \varphi : \mathsf{P}]\!], [\![y : B \mid \psi : \mathsf{P}]\!] \rangle : \{x : A \mid \top\} \longrightarrow \{t : \mathsf{P} \mid \top\} \times \{t : \mathsf{P} \mid \top\}$$

factors through the partial ordering on $\{t : \mathsf{P} \mid \top\}$. This entails that

$$\{(x, p) : A \times \mathsf{P} \mid p = \top\} = \{(x, p) : A \times \mathsf{P} \mid p = (\varphi \wedge \psi = \varphi)\}$$

It the follows as in the first case that

$$\top \vdash \top = (\varphi \wedge \psi = \varphi) \vdash (\varphi \wedge \psi = \varphi)$$

and thus

$$\varphi \vdash \psi.$$

□

Remark 4.12. Note that in contrast to non-modal intuitionistic higher-order logic it is not in general the case that every functional relation

$$\varphi : \{x : A \mid \alpha\} \to \{p : \mathsf{P} \mid \top\}$$

is of the form $f(x) = p$, for some term $x : A \mid f : \mathsf{P}$ in $\mathbb{T}$, i.e. lies in the image of the canonical model. It is only true for those arrows $\mathcal{E}_{\mathbb{T}}$ that factor through the subobject classifier. However, every proposition $\mathcal{E}_{\mathbb{T}}$ that is internally true in $\mathcal{E}_{\mathbb{T}}$, is of such a form, since then it is box-stable.

4.2 Functorial Semantics

In this section we show that the correspondence between logical functors $\mathcal{E}_{\mathbb{T}} \to \mathcal{E}$ and models in a topos $\mathcal{E}$ that exists for a higher-order intuitionistic theory $\mathbb{T}$ also works, in a slightly modified form, for modal higher-order logic.[24] Specifically, we intend to show that for any faithful complete Heyting algebra H in $\mathcal{E}$ there is an equivalence

$$\mathrm{Log}_{\tau}(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H)) \simeq \mathrm{Mod}_{\mathbb{T}}(\mathcal{E}, H)$$

between the category of suitable logical functors (which we will call τ-logical later on) and models, or interpretations, of $\mathbb{T}$ in a τ-topos $(\mathcal{E}, H)$, i.e. in the structure $i : \Omega_{\mathcal{E}} \leftrightarrows H : \tau$.[25] To formulate and prove this in detail, we need to make some preliminary considerations. To begin with, we need to define the two categories.

[24] For the higher-order intuitionistic case see e.g. [1].
[25] For the notion of τ-topos, confer remark 3.4.

Definition 4.13. *Consider a higher-order modal theory* $\mathbb{T}$ *and any two models* $[\![-]\!]_M$ *and* $[\![-]\!]_N$ *in a* τ*-topos* $(\mathcal{E}, H)$*. An* isomorphism of models $h : [\![-]\!]_M \to [\![-]\!]_N$ *is a family of isomorphisms*

$$h_A : [\![A]\!]_M \longrightarrow [\![A]\!]_N,$$

indexed by the basic non-logical types in $\mathcal{L}(\mathbb{T})$*. We extend the family* (h_A) *to all types as follows:*

$$h_{A\times B} = h_A \times h_B,$$

$$h_{A^B} = (([\![A]\!]_N)^{h_B})^{-1} \circ h_A^{([\![B]\!]_M)} :$$

$$[\![A]\!]_M^{[\![B]\!]_M} \xrightarrow{h_A^{([\![B]\!]_M)}} [\![A]\!]_N^{[\![B]\!]_M} \xrightarrow{(([\![A]\!]_N)^{h_B})^{-1}} [\![A]\!]_N^{[\![B]\!]_N},$$

$$h_1 = 1_{\mathcal{E}},$$

where 1 *is the terminal type in* $\mathbb{T}$*, and* $1_{\mathcal{E}}$ *is the identity arrow on the terminal object in* $\mathcal{E}$*;*

$$h_{\mathsf{P}} = 1_H.$$

As for terms, we require that for every constant $c : A$*, the following commutes:*

$$\begin{tikzcd}
1_{\mathcal{E}} \arrow[d, "{[\![c]\!]_M}"'] \arrow[dr, "{[\![c]\!]_N}"] & \\
{[\![A]\!]_M} \arrow[r, "h_A"'] & {[\![A]\!]_N}
\end{tikzcd}$$

For every function symbol $f : A_1, \dots, A_n \to B$*, the diagram*

$$\begin{tikzcd}
{[\![A_1]\!]_M \times \cdots \times [\![A_n]\!]_M} \arrow[r, "{[\![f]\!]_m}"] \arrow[d, "h_{A_1}\times\cdots\times h_{A_n}"] & {[\![B]\!]_M} \arrow[d, "h_B"] \\
{[\![A_1]\!]_N \times \cdots \times [\![A_n]\!]_N} \arrow[r, "{[\![f]\!]_N}"'] & {[\![B]\!]_N}
\end{tikzcd}$$

is to commute.

Lemma 4.14. *For every term $\Gamma \mid t : B$ the following diagram commutes:*

$$\begin{tikzcd}
[\![A_1]\!]_M \times \cdots \times [\![A_n]\!]_M \arrow[r, "[\![t]\!]_m"] \arrow[d, "h_{A_1}\times\cdots\times h_{A_n}"'] & [\![B]\!]_M \arrow[d, "h_B"] \\
[\![A_1]\!]_N \times \cdots \times [\![A_n]\!]_N \arrow[r, "[\![t]\!]_N"'] & [\![B]\!]_N
\end{tikzcd}$$

Proof. This is shown by induction. For instance, assume that by induction hypothesis for any two terms $\Gamma \mid t_1 : A$ and $\Gamma \mid t_2 : A$ it holds that $h_A \circ [\![t_i]\!]_M = [\![t_i]\!]_N \circ h_\Gamma$, for $i = 1, 2$. It follows that

$$\begin{aligned}
h_{A\times B} \circ [\![\langle t_1, t_2\rangle]\!]_M &= h_A \times h_B \circ \langle [\![t_1]\!]_M, [\![t_2]\!]\rangle \\
&= \langle h_A \circ [\![t_1]\!]_M, h_B \circ [\![t_2]\!]\rangle \\
&= \langle [\![t_1]\!]_N \circ h_\Gamma, [\![t_2]\!]_N \circ h_\Gamma\rangle \\
&= \langle [\![t_1]\!]_N, [\![t_2]\!]_N\rangle \circ h_\Gamma \\
&= [\![\langle t_1, t_2\rangle]\!]_N \circ h_\Gamma,
\end{aligned}$$

using the rules for interpreting pairing terms and general properties of products in a category. In a similar fashion on shows analogous statements for the other term constructors connected with products and exponentials. For propositions the claim follows because h_{P} is a map of complete Heyting algebras. Thus, for instance, we obtain commutative diagrams

$$\begin{tikzcd}
[\![\Gamma]\!]_M \arrow[r, "\langle [\![\varphi]\!]_M, [\![\varphi]\!]_N\rangle"] \arrow[d, "h_\Gamma"] & [\![\mathsf{P}]\!]_M \times [\![\mathsf{P}]\!]_M \arrow[r, "\wedge"] \arrow[d, "h_{\mathsf{P}}\times h_{\mathsf{P}}"] & [\![\mathsf{P}]\!]_M \arrow[d, "h_{\mathsf{P}}"] \\
[\![\Gamma]\!]_N \arrow[r, "\langle [\![\varphi]\!]_N, [\![\varphi]\!]_N\rangle"] & [\![\mathsf{P}]\!]_N \times [\![\mathsf{P}]\!]_N \arrow[r, "\wedge"] & [\![\mathsf{P}]\!]_N
\end{tikzcd}$$

and

$$\begin{tikzcd}
[\![\Gamma[x_i : A_i]]\!] \arrow[r, "\lambda_{A_i}.[\![\Gamma|\varphi]\!]_M"] \arrow[d, "h_{\Gamma[x_i:A_i]}"] & [\![\mathsf{P}]\!]_M^{[\![A_i]\!]_M} \arrow[r, "\forall_{[\![A_i]\!]_M}"] \arrow[d, "h_{\mathsf{P}}^{[\![A_i]\!]_M}"] & [\![\mathsf{P}]\!]_M \arrow[d, "h_{\mathsf{P}}"] \\
[\![\Gamma[x_i : A_i]]\!] \arrow[r, "\lambda_{A_i}.[\![\Gamma|\varphi]\!]_N"] & [\![\mathsf{P}]\!]_N^{[\![A_i]\!]_N} \arrow[r, "\forall_{[\![A_i]\!]_N}"] & [\![\mathsf{P}]\!]_N
\end{tikzcd}$$

where the map $h_{\Gamma[x_i:A_i]}$ is defined in the obvious way by omitting the ith component of the map $h_{A_1} \times \cdots \times h_{A_n}$. For the modal operator, of course,

$$\tau' i' h_{\mathsf{P}} = h_{\mathsf{P}} i \tau,$$

by the remark after the last definition.

$\square$

Definition 4.15. *For any higher-order modal theory $\mathbb{T}$, and any τ-topos $(\mathcal{E}, H)$, the category $Mod_{\mathbb{T}}(\mathcal{E}, H)$ has as objects all the models of $\mathbb{T}$ in $(\mathcal{E}, H)$, and as arrows model isomorphisms in the sense of def. 4.13.*

Definition 4.16. *A τ-logical functor $(\mathcal{E}, H) \to (\mathcal{F}, H')$ between τ-toposes $(\mathcal{E}, H)$ and $(\mathcal{F}, H)$ is a logical functor $F : \mathcal{E} \to \mathcal{F}$ equipped with an isomorphism of Heyting algebras*

$$\imath_F : F(H) \cong H'$$

w.r.t. the Heyting structure induced by F.

We note that in order to show that $\imath_F$ is an isomorphism of Heyting algebras, it suffices to show that $\imath_F$ is an isomorphism of the underlying posets. Thus, since F preserves the top element and is logical it follows that

$$\tau' \circ \imath_F = F\tau.$$

We define the category

$$\mathrm{Log}_\tau(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H))$$

to have objects τ-logical functors $\mathcal{E}_{\mathbb{T}} \to (\mathcal{E}, H)$ and arrows natural isomorphisms between them. Here, we implicitly regard $\mathcal{E}_{\mathbb{T}}$ as equipped with the canonical model structure given by the faithful Heyting algebra $\{t : \mathsf{P} \mid \top\}$ in $\mathcal{E}_{\mathbb{T}}$.

Let $F : (\mathcal{E}, H) \longrightarrow (\mathcal{F}, K)$ be any τ-logical functor. It follows from the properties of a τ-logical functor that given any model $[\![-]\!]$ in $(\mathcal{E}, H)$, the image of F determines a model $F[\![-]\!]$ in $(\mathcal{F}, K)$. Hence F defines a functor

$$\mathrm{Mod}_{\mathbb{T}}(F) : \mathrm{Mod}_{\mathbb{T}}(\mathcal{E}, H) \longrightarrow \mathrm{Mod}_{\mathbb{T}}(\mathcal{F}, K).$$

On the other hand, a τ-logical functor $F : (\mathcal{E}, H) \longrightarrow (\mathcal{F}, K)$ induces a functor

$$F \circ - : \mathrm{Log}_\tau(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H)) \longrightarrow \mathrm{Log}_\tau(\mathcal{E}_{\mathbb{T}}, (\mathcal{F}, K))$$

by composition with F.

Before stating and proving the theorem, we will collect some auxiliary information.

Lemma 4.17. *For any faithful Heyting algebra, and any two arrows $\varphi, \psi : A \to H$, if $\varphi \leq \psi$ (in $\mathrm{Hom}(A, H)$), then $\varphi^s \leq \psi^s$ (in $\mathrm{Sub}_E(A)$), where φ^s is the pullback of $\top : 1 \to H$ along φ, and similarly for ψ.*

Proof. If $\varphi, \psi : A \to H$ then $\tau\varphi \leq \tau\psi$ w.r.t. $\Omega_{\mathcal{E}}$. Since $\tau\varphi$ and $\tau\psi$ classifies the pullbacks φ^s and ψ^s, resp., it immediately follows that $\varphi^s \leq \psi^s$ in $\mathrm{Sub}_{\mathcal{E}}(A)$. $\square$

A translation $\theta : L \to L'$ between higher-order modal theories is an assignment of types an terms satisfying the following requirements. The type constructors are to be preserved. For instance, $\theta(A \times B) = \theta(A) \times \theta(B)$, etc. In particular, the types 1 and P are preserved. We require that θ preserves closed terms and maps a function symbol $f : A_1 \times \cdots \times A_n \to B$ to a function sumbol $\theta(f) : \theta(A_1), \ldots, \theta(A_n) \to \theta(B)$. The map θ is then extended to all terms as usual in such way that it preserves all the cartesian closed term formers. For instance, a term $\Gamma \mid \pi t : A$ is sent to a term $\theta(\Gamma) \mid \pi\theta(t) : \theta(A)$. Moreoover, the modal operator must be preserved. Lastly, θ is to preserve deduction. That is to say, $\Gamma \mid \varphi \vdash \psi$ in L implies $\theta(\Gamma) \mid \theta(\varphi) \vdash \theta(\psi)$ in L'.

Denote by $L(\mathcal{E})$ the theory of the internal language of a topos $\mathcal{E}$. It is a higher-order modal theory for the trivial modal operator.

Definition 4.18. *For any higher-order theory $\mathbb{T}$, and any model $[\![-]\!]$ in $(\mathcal{E}, H)$, define a translation $\theta : \mathbb{T} \to L(\mathcal{E})$ as follows. On basic types, we define*

$$\theta(1) = \ulcorner 1 \urcorner,$$
$$\theta(\mathsf{P}) = \ulcorner \Omega_{\mathcal{E}} \urcorner,$$
$$\theta(A) = \ulcorner [\![A]\!] \urcorner.$$

One then recursively extends this definition to all the types in $\mathbb{T}$ in the expected way. For a closed term $t : B$:

$$\theta(t) \equiv \ulcorner [\![t]\!] \urcorner$$

Similarly,

$$\theta(f(t_1, \ldots, t_n)) \equiv \ulcorner [\![f]\!] \urcorner(\theta(t_1), \ldots, \theta(t_n))$$

A term

$$\theta(\Gamma \mid \varphi : \mathsf{P}) = (\ulcorner \tau[\![\varphi]\!] \urcorner(x_1, \ldots, x_n) : \ulcorner \Omega_{\mathcal{E}} \urcorner),$$

Lemma 4.19. *Suppose $x : A \mid \varphi \vdash \psi$ in $\mathbb{T}$. Then $x : \theta(A) \mid \theta(\varphi) \vdash \theta(\psi)$ in $L(\mathcal{E})$.*

Proof. Suppose $x : A \mid \varphi \vdash \psi$ in $\mathbb{T}$. By soundness of higher-order modal logic and the previous lemma, we have that $[\![\varphi]\!]^s \leq [\![\psi]\!]^s$ in $\mathrm{Sub}_{\mathcal{E}}([\![A]\!])$. Since $\tau[\![\varphi]\!]$ and $\tau[\![\psi]\!]$ are the classifying map of $[\![\varphi]\!]^s$ and $[\![\psi]\!]^s$, resp., it follows that $x : \theta(A) \mid \theta(\varphi) \vdash \theta(\psi)$ in $L(\mathcal{E})$. $\square$

We use this to carry over the following argument to a higher-order modal theory. In non-modal higher-order logic, in any interpretation every functional relation

$$x : A, y : B \mid \theta$$

from $x : A \mid \varphi$ to $y : B \mid \psi$ determines, by soundness, an arrow $U_\varphi \to U_\psi$ between the subobjects classified by the maps $[\![\varphi]\!] : [\![A]\!] \to \Omega_{\mathcal{E}}$ and $[\![\psi]\!] : [\![B]\!] \to \Omega_{\mathcal{E}}$, resp. The condition that

$$x : A, y : B \mid \theta \vdash \varphi \wedge \psi$$

implies, that the subobject U_θ classified by $[\![\theta]\!] : [\![A]\!] \times [\![B]\!] \to \Omega_{\mathcal{E}}$ factors through $U_\varphi \times U_\psi$, say by a map $\langle a, b\rangle$. The condition that

$$x : A \mid \varphi \vdash \exists_{y:B}\theta$$

implies that a is an epimorphism. Finally, the condition

$$x : A, y : B, y' : B \mid \theta \wedge \theta[y'/y] \vdash y = y'$$

implies that a is a monomorphism. Hence a is an isomorphism. The required arrow $U_\varphi \to U_\psi$ then is the composite

$$U_\varphi \xrightarrow{\quad a^{-1} \quad} U_\theta \xrightarrow{\quad b \quad} U_\psi.$$

Moreover, this construction is functorial. If θ is an identity arrow on a formula $x : A \mid \varphi$, then $U_\varphi \to U_\varphi$ is the identity arrow. If θ is a functional relation from $x : A \mid \varphi$ to $y : B \mid \psi$, determining the subobject

$$\langle a, b\rangle : U_\theta \to U_\varphi \times U_\psi,$$

and σ is a functional relation from $y : B \mid \psi$ to $z : C \mid \rho$, determining a subobject

$$\langle c, d\rangle : U_\sigma \to U_\psi \times U_\rho$$

then the arrow determined by the composite

$$x : A, z : C \mid \exists_{y:B}(\theta \wedge \sigma)$$

is precisely the composite $dc^{-1}ba^{-1}$. Hence the construction preserves compositition. The same argument applies to box-stable functional relation in a higher-order modal theory using lemma 4.19:

Lemma 4.20. *For any box-stable formula $x : A, y : B \mid \theta$ that is provably a functional relation between box-stable formulas $x : A \mid \varphi$ and $y : B \mid \psi$, the pullback of $\top : 1 \to H$ along $[\![\theta]\!] : [\![A]\!] \times [\![B]\!] \to H$ factors through $[\![\varphi]\!] \times [\![\psi]\!]$ and determines a unique arrow $[\![\varphi]\!] \to [\![\psi]\!]$.*

We are now in a position to state and prove the theorem.

Theorem 4.21. *For any higher-order modal theory $\mathbb{T}$, and any H-topos $(\mathcal{E}, H)$, there exists an equivalence of categories*

$$Mod_{\mathbb{T}}(\mathcal{E}, H) \simeq Log_{\tau}(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H)),$$

where we regard $\mathcal{E}_{\mathbb{T}}$ w.r.t. to the canonical model structure. Moreover, this equivalence is natural in $(\mathcal{E}, H)$ in that for any τ-logical functor $L : (\mathcal{E}, H) \longrightarrow (\mathcal{F}, K)$ the following commutes (up to canonical isomorphism):

$$\begin{tikzcd}
Mod_{\mathbb{T}}(\mathcal{E}, H) \arrow[r, dash, "\sim"] \arrow[d, "Mod_{\mathbb{T}}(F)"'] & Log_{\tau}(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H)) \arrow[d, "F\circ -"] \\
Mod_{\mathbb{T}}(\mathcal{F}, K) \arrow[r, dash, "\sim"] & Log_{\tau}(\mathcal{E}_{\mathbb{T}}, (\mathcal{F}, K))
\end{tikzcd}$$

Proof. (i) We first define a functor

$$\mathrm{Mod}_{\mathbb{T}}(\mathcal{E}, H) \to \mathrm{Log}_{\tau}(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H)).$$

Consider any model $[\![-]\!]$ in $(\mathcal{E}, H)$. For an object $\{x : A \mid \alpha\}$ in $\mathcal{E}_{\mathbb{T}}$, define $M(\{x : A \mid \alpha\})$ to be the object that arises from pulling back $\top : 1 \to H$ along $[\![\alpha]\!] : [\![A]\!] \to H$.

$$\begin{tikzcd}
M(\{x : A \mid \alpha\}) \arrow[r] \arrow[d, tail, "\imath_{\alpha}"'] & 1 \arrow[d, "\top"] \\
{[\![A]\!]} \arrow[r, "{[\![\alpha]\!]}"'] & H
\end{tikzcd}$$

For an arrow $\{x : A \mid \alpha\} \longrightarrow \{y : B \mid \beta\}$ in $\mathcal{E}_{\mathbb{T}}$, given by a term $\{(x, y) : A \times B \mid \gamma\}$, there is, by lemma 4.20, an arrow

$$M(\{x : A \mid \alpha\}) \longrightarrow M(\{y : B \mid \beta\})$$

in $\mathcal{E}$ between the subobjects classified by $[\![\alpha]\!]$ and $[\![\beta]\!]$, resp. This construction preserves identities and composites as observed earlier.

The functor M is a τ-logical functor. Cartesian closedness mostly follows the non-modal version. For instance:

- As to the terminal object in $\mathcal{E}_{\mathbb{T}}$ we note that

  $$\{z : 1 \mid z = *\} = \{z : 1 \mid \top\},$$

  since $z : 1 \mid \top \vdash * = z$ in $\mathbb{T}$. But $M(\{z : 1 \mid \top\}) = 1_{\mathcal{E}}$.

- Consider a product $\alpha \times \beta = \{(x, y) : A \times B \mid x \in \alpha \wedge y \in \beta\}$ in $\mathcal{E}_{\mathbb{T}}$. The term $x \in \alpha \wedge y \in \beta$ is interpreted in $\mathcal{E}$ as the composite

$$\llbracket A \rrbracket \times \llbracket B \rrbracket \xrightarrow{\llbracket \alpha \rrbracket \times \llbracket \beta \rrbracket} H \times H \xrightarrow{\wedge} H.$$

The pullback of $\top$ along this arrow is just

$$M(\{x : A \mid \alpha\}) \times M(\{y : B \mid \beta\}),$$

with projection $\imath_\alpha \times \imath_\beta$ into $\llbracket A \rrbracket \times \llbracket B \rrbracket$.

- Subobject classifier:

$$M(\{t : \mathsf{P} \mid t = \Box t\}) = \Omega_{\mathcal{E}}.$$

The term $t : \mathsf{P} \mid t = \Box t$ is interpreted under $\llbracket - \rrbracket$ as the arrow

$$H \xrightarrow{\langle 1, i\tau \rangle} H \times H \xrightarrow{\delta_H} \Omega_{\mathcal{E}} \xrightarrow{i} H$$

in $\mathcal{E}$. The claim now follows from the following sequence of pullbacks:

$$\begin{tikzcd}
\Omega_{\mathcal{E}} \arrow[r, "i"] \arrow[d, "i"'] & H \arrow[r] \arrow[d, "\Delta"] & 1 \arrow[r] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
H \arrow[r, "\langle 1, i\tau \rangle"'] & H \times H \arrow[r, "\delta_H"'] & \Omega_{\mathcal{E}} \arrow[r, "i"'] & H
\end{tikzcd}$$

- $$M(\{t : \mathsf{P} \mid \top\}) = H.$$

The term $t : \mathsf{P} \mid \top : \mathsf{P}$ is interpreted under $\llbracket - \rrbracket$ in $\mathcal{E}$ as the arrow $H \xrightarrow{!} 1 \xrightarrow{\top} H$ whose pullback along $\top : 1 \to H$ is the identity arrow on H.

- M preserves the partial ordering on $\{t : \mathsf{P} \mid \top\}$. The latter was defined as the object

$$\{(p, q) : \mathsf{P} \times \mathsf{P} \mid p \wedge q = p\},$$

with the canonical inclusion as a subobject into $\{t : \mathsf{P} \mid \top\} \times \{t : \mathsf{P} \mid \top\}$. The term

$$p : \mathsf{P}, q : \mathsf{P} \mid p \wedge q = p$$

is interpreted under $\llbracket - \rrbracket$ as the following map:

$$H \times H \xrightarrow{\langle \pi_1, \wedge \rangle} H \times H \xrightarrow{\delta_H} \Omega_{\mathcal{E}} \xrightarrow{i} H$$

The pullback of $\top : 1 \to H$ along this map is precisely the equalizer of π_1 and $\wedge$, i.e. the partial ordering of H.

- The arrow

$$\{p : \mathsf{P} \mid \top\} \xrightarrow{\{\langle t,s\rangle : \mathsf{P}\times\mathsf{P} \mid s=\Box t\}} \{p : \mathsf{P} \mid p = \Box p\}$$

in $\mathcal{E}_{\mathbb{T}}$ is mapped by M to the arrow $\tau : H \to \Omega_{\mathcal{E}}$ in $\mathcal{E}$. By the definition of M we form the pullback of $\top$ along the composite

$$H \times H \xrightarrow{i\tau\times 1} H \times H \xrightarrow{\delta_H} \Omega_{\mathcal{E}} \xrightarrow{i} H.$$

as follows

$$\begin{tikzcd} H \arrow[r, "1"] \arrow[d, "\langle 1, i\tau\rangle"'] & H \arrow[r] \arrow[d, "\Delta"] & 1 \arrow[d, "\top"] \\ H \times H \arrow[r, "i\tau\times 1"'] & H \times H \arrow[r, "\delta_H"'] & \Omega_{\mathcal{E}} \end{tikzcd}$$

The subobject $\langle 1, i\tau\rangle$ factors as $(1 \times i)\langle 1, \tau\rangle$. Hence the arrow defined by the functional relation $s = \Box t$ is exactly τ.

- In a similar spirit one shows that $M(\{(p,q) : \mathsf{P} \times \mathsf{P} \mid p = \Box p \wedge p = q\}$ is mapped to $i : \Omega_{\mathcal{E}} \to H$. We need to compute the pullback of $\top : 1 \to H$ along

$$H \times H \xrightarrow{\langle i\delta_H\langle \pi_1, i\tau\pi_1\rangle, i\delta_H\rangle} H \times H \xrightarrow{\wedge} H.$$

The pullback of $\langle \top, \top\rangle$ along $\langle i\delta_H\langle \pi_1, i\tau\pi_1\rangle, i\delta_H\rangle$ is computed componentwise. On the one hand, the pullback of $\top$ along $i\delta_H$ is of course $\Delta : H \to H \times H$. On the other hand, the pullback of $\top$ along $i\delta_H\langle \pi_1, i\tau\pi_1\rangle$ can be computed as the pullback

$$\begin{tikzcd} \Omega_{\mathcal{E}} \times H \arrow[r] \arrow[d, "i\times 1"'] & H \arrow[d, "\Delta"] \\ H \times H \arrow[r, "\langle \pi_1, i\tau\pi_1\rangle"'] & H \times H \end{tikzcd}$$

This can be directly verified since in a pullback of that form the left-hand vertical projection must be the equalizer of π_1 and $i\tau\pi_1$, which is precisely $i \times 1$.

The pullback of $\langle \top, \top \rangle$ along $\langle i\delta_H \langle \pi_1, i\tau\pi_1 \rangle, i\delta_H \rangle$ then is the diagonal composite through the following pullback:

$$\begin{tikzcd}
\Omega_{\mathcal{E}} \arrow[r, "i"] \arrow[d, "\langle 1,i \rangle"'] & H \arrow[d, "\Delta"] \\
\Omega_{\mathcal{E}} \times H \arrow[r, "i\times 1"'] & H \times H
\end{tikzcd}$$

That is to say, the map $\langle i, i \rangle : \Omega_{\mathcal{E}} \longrightarrow H \times H$. We get that $\langle 1, i \rangle : \Omega_{\mathcal{E}} \longrightarrow \Omega_{\mathcal{E}} \times H$ is the graph of the arrow we are looking for, which is i.

Consider any isomorphism of models $h : [\![-]\!]_M \to [\![-]\!]_N$. We define a natural transformation $\eta : M \to N$ as follows. Given any object $\{x : A \mid \alpha\}$ in $\mathcal{E}_{\mathbb{T}}$, we first get the following commutative diagram in $\mathcal{E}$, where we write H_M for the complete Heyting algebra $[\![\mathsf{P}]\!]_M$, and similarly for N:

$$\begin{tikzcd}
{[\![A]\!]_M} \arrow[r, "{[\![\alpha]\!]_M}"] \arrow[d, "h_A"'] & H_M \arrow[d, "h_{\mathsf{P}}"'] \\
{[\![A]\!]_N} \arrow[r, "{[\![\alpha]\!]_N}"'] & H_N
\end{tikzcd}$$

Pulling back $\top : 1 \to H_N$ yields a pullback cube the left-hand face of which is

$$\begin{tikzcd}
M(\{x : A \mid \alpha\}) \arrow[r, dotted, "\eta_\alpha"] \arrow[d] & N(\{x : A \mid \alpha\}) \arrow[d] \\
{[\![A]\!]_M} \arrow[r, "h_A"'] & {[\![A]\!]_N}
\end{tikzcd}$$

Since this is a pullback and h_A is an isomorphism, hence so is η_α.

As for naturality, consider an arrow

$$\{(x, y) : A \times B \mid \varphi\} : \{x : A \mid \alpha\} \longrightarrow \{y : B \mid \beta\}.$$

in $\mathcal{E}_{\mathbb{T}}$. We construct the following pullback

$$\begin{tikzcd}
G \arrow[r, "\gamma"] \arrow[d, "\langle a,b\rangle"] & G' \arrow[d, "\langle c,d\rangle"] \\
M(\{x : A \mid \alpha\}) \times M(\{x : A \mid \beta\}) \arrow[r, "\eta_\alpha \times \eta_\beta"] \arrow[d, "\imath_\alpha \times \imath_\beta"] & N(\{x : A \mid \alpha\}) \times N(\{x : A \mid \beta\}) \arrow[d, "\imath'_\alpha \times \imath'_\beta"] \\
[\![A]\!]_M \times [\![B]\!]_M \arrow[r, "h_A \times h_B"'] & [\![A]\!]_N \times [\![B]\!]_N
\end{tikzcd}$$

Here G is the pullback of $\top : 1 \to H_M$ along the arrow $[\![(x, y) : A \times B \mid \varphi : \mathsf{P}]\!]_M$. Similarly, G' is the pullback of $\top : 1 \to H_M$ along the arrow $[\![(x, y) : A \times B \mid \varphi : \mathsf{P}]\!]_N$. The factorization $\gamma : G \to G'$ is the upper projection of the back face of a pullback cube around the square

$$\begin{tikzcd}
[\![A]\!]_M \times [\![B]\!]_M \arrow[r, "h_A \times h_B"'] \arrow[d, "[\![x:A, y:B \mid \varphi:\mathsf{P}]\!]_M"] & [\![A]\!]_N \times [\![B]\!]_N \arrow[d, "[\![x:A, y:B \mid \varphi:\mathsf{P}]\!]_N"] \\
H_M \arrow[r, "h_{\mathsf{P}}"'] & H_N
\end{tikzcd}$$

From

$$\langle c, d\rangle\gamma = (\eta_\alpha \times \eta_\beta)\langle a, b\rangle$$

it follows that

$$d\gamma = \eta_\beta b$$

$$c\gamma = \eta_\alpha a.$$

Since the projections a and c are isomorphisms, moreover,

$$\gamma a^{-1} = c^{-1}c\gamma a^{-1} = c^{-1}\eta_\alpha a a^{-1} = c^{-1}\eta_\alpha.$$

Hence the following commutes:

$$\begin{tikzcd}
M(\{x : A \mid \alpha\}) \arrow[r, "a^{-1}"] \arrow[d, "\eta_\alpha"'] & G \arrow[r, "b"] \arrow[d, "\gamma"] & M(\{x : A \mid \beta\}) \arrow[d, "\eta_\beta"] \\
N(\{x : A \mid \alpha\}) \arrow[r, "c^{-1}"] & G' \arrow[r, "d"] & N(\{x : A \mid \beta\})
\end{tikzcd}$$

This is precisely the required naturality square.

(ii) We next construct a functor

$$\mathrm{Log}(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H)) \to \mathrm{Mod}_{\mathbb{T}}(\mathcal{E}, H).$$

Suppose $F : \mathcal{E}_{\mathbb{T}} \to (\mathcal{E}, H)$ is a τ-logical functor. So in particular

$$F(\{t : \mathsf{P} \mid \top\}) = H.$$

Define a model $[\![-]\!]_F$ as follows. For a basic type A in T set

$$[\![A]\!]_F = F(\{x : A \mid \top\}).$$

The terminal type 1 is of course interpreted by the terminal object of $\mathcal{E}$ which in fact agrees with the definition just given, as $\{x : 1 \mid \top\}$ is the terminal object in $\mathcal{E}_{\mathbb{T}}$. Moreover,

$$[\![\mathsf{P}]\!]_F = H,$$

by assumption. The other type formers are interpreted in accordance with the topos structure of $\mathcal{E}$ as described for any model.

As for terms, consider a basic constant $c : A$. Applying F to the arrow

$$[c] : 1 \longrightarrow \{x : A \mid \top\}$$

in the canonical model yields an arrow

$$F([c]) : 1 \longrightarrow F(\{x : A \mid \top\}) = [\![A]\!]_F.$$

in $\mathcal{E}$. Hence we set

$$[\![c : A]\!]_F = F([c]).$$

If $f : A_1 \times \cdots \times A_n \to B$ is a function symbol, we have in $\mathcal{E}_{\mathbb{T}}$ the arrow

$$[f] : \{(x_1, \ldots, x_n) : A_1 \times \cdots \times A_n \mid \top\} \longrightarrow \{y : B \mid \top\}$$

given by the term

$$\{(x_1, \ldots, x_n, y) : A_1 \times \cdots \times A_n \times B \mid f(x_1, \ldots, x_n) = y\}.$$

Thus we define

$$[\![f]\!]_F = F([f]).$$

By induction over terms $x : A \mid t : B$, one obtains

$$[\![x : A \mid t : B]\!]_F = F([x : A \mid t : B]),$$

where $[x : A \mid t : B]$ is the interpretation in the canonical model. This defines the object part of a functor

$$\mathrm{Log}(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H)) \to \mathrm{Mod}_{\mathbb{T}}(\mathcal{E}, H).$$

Given a natural isomorphism $\eta : F \to G$ between τ-logical functors $F, G : \mathcal{E}_{\mathbb{T}} \to (\mathcal{E}, H)$ we obtain a morphism of models $h\eta : [\![-]\!]_F \to [\![-]\!]_G$ by setting

$$(h\eta)_A : [\![A]\!]_F = F(\{x : A \mid \top\}) \xrightarrow{\eta_{\{x:A\mid\top\}}} G(\{x : A \mid \top\}) = [\![A]\!]_G,$$

for any basic type. The map $(h\eta)_{\mathsf{P}} = \eta_{\{p:\mathsf{P}\mid\top\}}$ will in fact be a morphism of complete Heyting algebras. For the Heyting operations $\top, \bot, \wedge, \vee, \Rightarrow$ this follows because F and G are logical. Writing H for the object $\{p : \mathsf{P} \mid \top\}$, for conjunction we get

$$\begin{tikzcd}
FH \times FH \arrow[r, equal, "\sim"] \arrow[d, "\eta_H \times \eta_H"] & F(H \times H) \arrow[r, "F(\wedge)"] \arrow[d, "\eta_{H \times H}"] & FH \arrow[d, "\eta_H"] \\
GH \times GH \arrow[r, equal, "\sim"] & G(H \times H) \arrow[r, "G(\wedge)"] & GH
\end{tikzcd}$$

For any constant $c : A$, we have the following naturality square:

$$\begin{tikzcd}
1_{\mathcal{E}} \cong F(\{x : 1 \mid \top\}) \arrow[r, "\eta_{\{x:1\mid\top\}}", "\cong"'] \arrow[d, "F([c])"'] & G(\{x : 1 \mid \top\}) \cong 1_{\mathcal{E}} \arrow[d, "G([c])"] \\
F(\{x : A \mid \top\}) \arrow[r, "\cong", "\eta_{\{x:A\mid\top\}}"'] & G(\{x : 1 \mid \top\}
\end{tikzcd}$$

Similarly for any function symbol $f : A_1, \ldots, A_n \to B$, where the analogous statement follows because F and G preserve products.

(iii) We proceed to show that these two constructions form an equivalence of categories. We start with a τ-logical functor $F : \mathcal{E}_{\mathbb{T}} \to (\mathcal{E}, H)$ from which we define the model $[\![-]\!]_F$ which in turn is used to define a functor $M : \mathcal{E}_{\mathbb{T}} \to \mathcal{E}$.

First consider objects in $\mathcal{E}_{\mathbb{T}}$ of the form $\{x : A \mid \top\}$. By definition of the functor M, the object $M(\{x : A \mid \top\})$ is obtained through pullback as in

$$\begin{tikzcd}
M(\{x : A \mid \top\}) \arrow[rr] \arrow[d] & & 1 \arrow[d, "\top"] \\
\llbracket A \rrbracket_F \arrow[r] & 1 \arrow[r, "\top"'] & F(\{t : \mathsf{P} \mid \top\})
\end{tikzcd}$$

Here the lower composite is $\llbracket x : A \mid \top : \mathsf{P} \rrbracket_F$. Hence

$$M(\{x : A \mid \top\}) \cong \llbracket A \rrbracket_F = F(\{x : A \mid \top\}).$$

Next, consider any closed term $\{x : A \mid \alpha\} : \mathsf{P}^A$, for a box-stable formula α. The model $\llbracket - \rrbracket_F$ provides us with an arrow $\llbracket \alpha \rrbracket_F : \llbracket x : A \mid \alpha : \mathsf{P} \rrbracket_F \to H$ in $\mathcal{E}$. To show that $M(\{x : A \mid \alpha\}) \cong F(\{x : A \mid \alpha\})$ it suffices to show that there is a pullback diagram

$$\begin{tikzcd}
F(\{x : A \mid \alpha\}) \arrow[r] \arrow[d] & 1 \arrow[d, "\top"] \\
F(\{x : A \mid \top\}) \arrow[r, "\llbracket \alpha \rrbracket_F"'] & F(\{t : \mathsf{P} \mid \top\})
\end{tikzcd}$$

as this is how $M(\{x : A \mid \alpha\})$ was defined. To prove this, recall that the following is a pullback in $\mathcal{E}_{\mathbb{T}}$:

$$\begin{tikzcd}
\{x : A \mid \alpha\} \arrow[r] \arrow[d, "\imath_\alpha"'] & 1 \arrow[r] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
\{x : A \mid \top\} \arrow[r, "\mathrm{char}\ \imath_\alpha"'] & \{t : \mathsf{P} \mid t = \Box t\} \arrow[r, "i"'] & \{t : \mathsf{P} \mid \top\}
\end{tikzcd}$$

Here,

$$\imath_\alpha \equiv \{(x, x') : A \times A \mid \alpha \wedge x = x'\}$$

is the canonical inclusion and char $\imath_\alpha$ the classifying map of $\imath_\alpha$ in $\mathcal{E}_{\mathbb{T}}$. Moreover:

$$\begin{aligned}
i \circ \mathrm{char}\ \imath_\alpha &\equiv i \circ \{(x, t) : A \times \mathsf{P} \mid t = \exists_{x' : A}.(x', x) \in \imath_\alpha\} \\
&= i \circ \{(x, t) : A \times \mathsf{P} \mid t = \exists_{x' : A}.\alpha \wedge x = x'\} \\
&= i \circ \{(x, t) : A \times \mathsf{P} \mid t = \alpha\} \\
&= [x : A \mid \alpha],
\end{aligned}$$

where the latter is the interpretation in the canonical model. Hence

$$[\![x : A \mid \alpha]\!]_F = i \circ F(\text{char } \imath_\alpha),$$

and the claim follows by applying F to the pullback above. Note, incidentally, that it follows $F(\imath_\alpha)$ is precisely the left-hand vertical projection in the pullback

$$\begin{tikzcd}
F(\{x : A \mid \alpha\}) \arrow[rr] \arrow[d, "F(\imath_\alpha)"'] & & 1 \arrow[d, "\top"] \\
F(\{x : A \mid \top\}) \arrow[rr, "{[\![\alpha]\!]_F}"'] & & F(\{t : \mathsf{P} \mid \top\})
\end{tikzcd}$$

To show that the isomorphisms $F(\{x : A \mid \alpha\}) \cong M(\{x : A \mid \alpha\})$ are natural, we note that the previous argument holds for any box-stable formula $x_1 : A_1, \ldots, x_n : A_n \mid \alpha : \mathsf{P}$. In particular, α may be a functional relation. So consider an arrow

$$\{x : A \mid \alpha\} \xrightarrow{\{(x,y):A\times B|\varphi\}} \{y : B \mid \beta\}$$

in $\mathcal{E}_{\mathbb{T}}$. In the following we will abbreviate the application of F to objects $\{x : A \mid \alpha\}$ in $\mathcal{E}_{\mathbb{T}}$ by writing $F(\alpha)$. For a functional relation φ representing an arrow in $\mathcal{E}_{\mathbb{T}}$, we will write $F(\varphi^{\rightarrow})$. This is to distinguish the arrow

$$F(\alpha) \xrightarrow{F(\varphi^{\rightarrow})} F(\beta)$$

in $\mathcal{E}$ from the object $F(\varphi)$ which occurs as a subobject of $[\![A]\!]_F \times [\![B]\!]_F$, classified by the map

$$[\![A]\!]_F \times [\![B]\!]_F \xrightarrow{[\![x:A,y:B|\varphi]\!]_F} F(H_{\mathbb{T}}).$$

As for any model, since φ is a functional relation from $x : A \mid \alpha$ to $y : B \mid \beta$ this object $F(\varphi)$ factors through the monomorphism $\imath_\alpha \times \imath_\beta : F(\alpha) \times F(\beta) \to [\![A]\!]_F \times [\![B]\!]_F$ by a map $\langle a, b\rangle$ where a is an isomorphism and determines an arrow $ba^{-1} : F(\alpha) \to F(\beta)$. That is to say, $ba^{-1} = F(\varphi^{\rightarrow})$. Equivalently, we might show that the map $[\![\varphi]\!]_F$ classifies the monomorphism $\langle \imath_\alpha, \imath_\beta F(\varphi^{\rightarrow})\rangle : F(\alpha) \to [\![A]\!]_F \times [\![B]\!]_F$. This is less obvious than it seems, as we don't really know much about the arrow $F(\varphi^{\rightarrow})$. However, we can use an observation from before. For recall that $[\![\varphi]\!]_F = i \circ F(\text{char } \jmath_\varphi)$, where char $\jmath_\varphi$ is the classifying map of the monomorphism

$$\jmath_\varphi : \{(x, y) : A \times B \mid \varphi\} \longrightarrow \{x : A \mid \top\} \times \{y : B \mid \top\}$$

in $\mathcal{E}_{\mathbb{T}}$; given by the usual formula

$$\jmath_\varphi \equiv \{((x,y),(x',y')) : (A \times B) \times (A \times B) \mid \varphi \wedge x = x' \wedge y' = y\}.$$

As such it factors as usual

$$\begin{tikzcd}
\{(x,y) : A \times B \mid \varphi\} \arrow[r, "\langle c,d\rangle"] & \{x : A \mid \alpha\} \times \{y : B \mid \beta\} \arrow[d, "\jmath_\alpha \times \jmath_\beta"] \\
& \{x : A \mid \top\} \times \{y : B \mid \top\},
\end{tikzcd}$$

with

$$c \equiv \{((x,y),x') \mid \varphi \wedge x = x'\},$$

and similarly for d. Moreover, c is an isomorphism with inverse

$$c^{-1} \equiv \{(x',(x,y)) \mid \varphi[x'/x] \wedge x = x'\}.$$

It now follows that $d \circ c^{-1} = \varphi$, as arrows in $\mathcal{E}_{\mathbb{T}}$. That is to say, the object $\{(x,y) : A \times B \mid \varphi\}$ is the graph of the arrow represented by $\{(x,y) : A \times B \mid \varphi\}$. Now any functor $G : \mathbf{C} \to \mathbf{D}$ that preserves finite limits preserves graphs of arrows in the sense that if $\langle m,n\rangle$ is the graph of an arrow f in $\mathbf{C}$, i.e. $f = nm^{-1}$, then $\langle F(m),F(n)\rangle$ is the graph of the arrow $F(f)$ in $\mathbf{D}$, i.e. $F(f) = F(n)F(m)^{-1}$. Hence, for our case it follows that the monomorphism $\langle a,b\rangle : F(\varphi) \to F(\alpha) \times F(\beta)$ is graph of the arrow $F(\varphi^{\to})$, i.e. $ba^{-1} = F(\varphi^{\to})$.

With this observation we conclude that the arrow

$$[\![\varphi]\!]_F : [\![A]\!]_F \times [\![B]\!]_F \to FH_{\mathbb{T}}$$

classifies the subobject $(\imath_\alpha \times \imath_\beta)\langle 1, F(\varphi^{\to})\rangle$ in $\mathcal{E}$, since of course $\langle 1, F(\varphi^{\to})\rangle$ is also a graph of $F(\varphi^{\to})$. In diagrams, the following commutes:

$$\begin{tikzcd}
F(\alpha) \arrow[rr, "a^{-1}"] \arrow[dr, "\langle 1,F(\varphi^{\to})\rangle"'] & & F(\varphi) \arrow[dl, "\langle a,b\rangle"] \\
& F(\alpha) \times F(\beta) &
\end{tikzcd}$$

so that the a^{-1} is the canonical isomorphism between the two pullbacks.

Now consider again the functor $M : \mathcal{E}_{\mathbb{T}} \to \mathcal{E}$ constructed out of the model $[\![-]\!]_F$. By definition of M, we obtain isomorphisms

$$u : F(\varphi) \cong M(\varphi)$$

and

$$u_\alpha : F(\alpha) \cong M(\alpha) \qquad u_\beta : F(\beta) \cong M(\beta).$$

Moreover, because M preserves finite limits, M likewise preserves graphs. That is to say that the monomorphism

$$\langle M(c), M(d)\rangle : M(\varphi) \to M(\alpha) \times M(\beta)$$

is a graph of the arrow $M(\varphi^{\to})$. This means in particular that the following commutes:

$$\begin{tikzcd}
F(\varphi) \arrow[r, "u"] \arrow[d, "\langle a,b\rangle"'] & M(\varphi) \arrow[d, "\langle M(c),M(d)\rangle"] \\
F(\alpha) \times F(\beta) \arrow[r, "u_\alpha \times u_\beta"'] & M(\alpha) \times M(\beta)
\end{tikzcd}$$

Putting everything together results in the commutative square

$$\begin{tikzcd}
F(\alpha) \arrow[r, "u_\alpha"] \arrow[d, "1"'] & M(\alpha) \arrow[d, "M(c)^{-1}"] \\
F(\alpha) \arrow[r, "ua^{-1}"] \arrow[d, "F(\varphi)"'] & M(\varphi) \arrow[d, "M(d)"] \\
F(\beta) \arrow[r, "u_\beta"'] & M(\beta)
\end{tikzcd}$$

This is precisely saying that we have a natural isomorphism $F \cong M$.

In the other direction, consider any model $[\![-]\!]$ in $\mathcal{E}$. We construct the functor $M : \mathcal{E}_{\mathbb{T}} \to \mathcal{E}$ from which we will construct a model $[\![-]\!]_M$. We then show that $[\![-]\!] \cong [\![-]\!]_M$ in the precise sense that there is an isomorphism of models.

We need to show that for or any type A

$$[\![A]\!] \cong [\![A]\!]_M,$$

and for any term $x : A \mid t : B$

$$[\![t]\!] \cong [\![t]\!]_M;$$

where the isomorphism means that the arrows commute with the isomorphism of their domain and codomain, resp. as required in def. 4.13.

For any basic type A, the object $[\![-]\!]_M$ is the pullback of $\top : 1 \to H$ along the composite $[\![A]\!] \to 1 \xrightarrow{\top} H$ which is $[\![A]\!]$ itself. Hence

$$[\![A]\!]_M \cong [\![A]\!].$$

It is readily checked that this holds for all the types.

For any term $x : A \mid t : B$ the arrow

$$[\![x : A \mid t : B]\!]_M : [\![A]\!]_M \longrightarrow [\![B]\!]_M$$

is by definition the map

$$M(\{(x, y) : A \times B \mid t = y\}).$$

The latter in turn is defined to be the arrow in $\mathcal{E}$ whose graph is classified by the morphism

$$[\![x : A, y : B \mid t = y : \mathsf{P}]\!] : [\![A]\!] \times [\![B]\!] \to H.$$

It is now easy to check that the subobject classified by this map is

$$\langle 1, [\![x : A \mid t : B]\!]\rangle : [\![A]\!] \to [\![A]\!] \times [\![B]\!],$$

as in the following pullback diagram

$$\begin{tikzcd}
{[\![A]\!]} \arrow[r, "{[\![t]\!]}"] \arrow[d, "{\langle 1, [\![t]\!]\rangle}"'] & B \arrow[r] \arrow[d, "\Delta_B"] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
{[\![A]\!] \times [\![B]\!]} \arrow[r, "{[\![t]\!]\times 1}"'] & {[\![B]\!] \times [\![B]\!]} \arrow[r, "\delta"'] & \Omega_{\mathcal{E}} \arrow[r, "i"'] & H
\end{tikzcd}$$

where the lower composite is spelling out the definition of $[\![x : A, y : B \mid t = y : \mathsf{P}]\!]$. Hence the arrow we seek is precisely $[\![x : A \mid t : B]\!]$.

(iv) Lastly, naturality of the equivalence

$$\mathrm{Log}_\tau(\mathcal{E}_{\mathbb{T}}, (\mathcal{E}, H)) \simeq \mathrm{Mod}_{\mathbb{T}}(\mathcal{E}, H)$$

w.r.t. τ-logical functors follows since a τ-logical functor $F : (\mathcal{E}, H) \to (\mathcal{F}, H')$ preserves finite limits and satisfies $FH \cong H'$. Thus, chasing a given model

$[\![-]\!]$ of $\mathbb{T}$ in $\mathcal{E}$ either way around the naturality square results in the pullback

$$\begin{tikzcd}
FM(\{x : A \mid \alpha\}) \arrow[r] \arrow[d] & 1 \arrow[d, "\top"] \\
FM([\![A]\!]) \arrow[r, "F([\![\alpha]\!])"'] & FH \cong H'
\end{tikzcd}$$

□

4.3 Relative Models in Sh($\mathcal{E}$)

We now study properties of $\mathbf{y}\Omega_{\mathcal{E}}$-relative models in the category $\mathrm{Sh}(\mathcal{E})$ of sheaves on a small topos $\mathcal{E}$ for the finite epi topology. We start with the connection between the sheaf $\mathbf{y}\Omega_{\mathcal{E}}$ and the subobject classifier Ω in $\mathrm{Sh}(\mathcal{E})$. Recall that the subobject classifier in the topos of sheaves on a site assigns to each C in $\mathcal{E}$ the set $\Omega_{\mathcal{E}}(C)$ of closed sieves on C. A sieve σ on C is closed if it satisfies the following condition: if for any map $f : D \to C$ in $\mathcal{E}$ the restriction $f^*\sigma$ is a covering sieve on D, then $f \in \sigma$.

Let $\imath : \mathbf{y}\Omega_{\mathcal{E}} \to \Omega$ be the classifying map of the top element $1 \cong \mathbf{y}1 \xrightarrow{\mathbf{y}\top} \mathbf{y}\Omega_{\mathcal{E}}$ of the Heyting algebra $\mathbf{y}\Omega_{\mathcal{E}}$. The map $\imath$ has the property that it is the (pointwise) ideal completion of $\mathbf{y}\Omega_{\mathcal{E}}$. That is to say, each set $\Omega(C)$ is isomorphic, as a complete Heyting algebra, to the set of all ideals in the Heyting algebra $\mathbf{y}\Omega_{\mathcal{E}}(C)$. Through the isomorphism $\mathbf{y}\Omega_{\mathcal{E}}(C) \cong \mathrm{Sub}_{\mathcal{E}}(C)$, and denoting the set of ideals by $\mathrm{Idl}(\mathrm{Sub}_{\mathcal{E}}(C))$, the statement reads as follows.

Fact 4.22. *For Ω in Sh($\mathcal{E}$), and any C in $\mathcal{E}$,*

$$\Omega(C) \cong \mathrm{Id}(\mathrm{Sub}_{\mathcal{E}}(C)).$$

For each C in $\mathcal{E}$, the composite

$$\imath_C : \boldsymbol{y}\Omega_{\mathcal{E}}(C) \to \Omega(C) \cong \mathrm{Id}(\mathrm{Sub}_{\mathcal{E}}(C))$$

sends a subobject M of C to $\downarrow M$.

In particular, $\imath : \mathbf{y}\Omega_{\mathcal{E}} \longrightarrow \Omega$, at a component C, assigns to any arrow $g : C \to \Omega_{\mathcal{E}}$ in $\mathcal{E}$ the sieve of all those arrows $f : X \to C$ such that

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & 1 \arrow[d, "\top"] \\
C \arrow[r, "g"'] & \Omega_{\mathcal{E}}
\end{tikzcd}$$

commutes. In terms of subobjects, for any $M \in \mathrm{Sub}_{\mathcal{E}}(C)$:

$$\imath_C(M) = \{f : X \to C \mid f^*M \cong X\}.$$

That is to say, it is the set of all those maps $f : X \to C$ such that in the pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & M \arrow[d, tail] \\
X \arrow[r, "f"'] & C
\end{tikzcd}$$

the projection $P \to X$ is an isomorphism (i.e. represents the top element of the subobject lattice $\mathrm{Sub}_{\mathcal{E}}(X)$). Equivalently, it is the set of all maps $f : X \to C$ that factor through the subobject (represented by) M.

We now wish to shed some more light on potential $\mathbf{y}\Omega_{\mathcal{E}}$-based relative model structures in $\mathrm{Sh}(\mathcal{E})$. The following propositions gives a characterization of the $\mathbf{y}\Omega_{\mathcal{E}}$-standard objects in $\mathrm{Sh}(\mathcal{E})$.

Proposition 4.23. *For any small topos $\mathcal{E}$, and J the finite epi topology on $\mathcal{E}$, an object A in $Sh(\mathcal{E}, J)$ is $\boldsymbol{y}\Omega_{\mathcal{E}}$-standard if and only if for any E in $\mathcal{E}$ and any map $\eta : \boldsymbol{y}E \to A$ the pullback*

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "p"'] & A \arrow[d, "\Delta_A"] \\
\boldsymbol{y}E \arrow[r, "\eta"'] & A \times A
\end{tikzcd}$$

is representable; i.e. $P \cong \boldsymbol{y}E'$ for some E' and $p = \boldsymbol{y}m$ for some map $m : E' \to E$ in $\mathcal{E}$.

Proof. Suppose A is $\mathbf{y}\Omega_{\mathcal{E}}$-standard. Then for any C in $\mathcal{E}$ and any pair $a, b \in AC$, there is a subobject $m : (\partial_A)_C(a, b) \hookrightarrow C$ such that an arrow $f : D \to C$ factors through $(\partial_A)_C(a, b)$ (necessarily uniquely) if and only if $A(f)(a) = A(f)(b)$. Any natural transformation $\eta : \mathbf{y}C \to A \times A$ determines a pair $(a, b) \in AC \times AC$ such that for any $h : D \to C$ by definition $\eta_D(h) =$

$(A(h)(a), A(h)(b))$. So there is a commutative square

$$\begin{tikzcd}
\mathbf{y}\partial_C(a,b) \arrow[r, "q"] \arrow[d, "\mathbf{y}m"'] & A \arrow[d, "\Delta_A"] \\
\mathbf{y}C \arrow[r, "\eta"'] & A \times A
\end{tikzcd}$$

where q is given by $q_D(h) = A(mh)(a) = A(mh)(b)$. Hence

$$\begin{aligned}
(\Delta_A)_D q_D(h) &= (\Delta_A)_D A(mh)(a) \\
&= (A(mh)(a), A(mh)(b)) \\
&= \eta_D(mh) \\
&= \eta_D(\mathbf{y}m)_D(h).
\end{aligned}$$

It is now immediate that for each D the square is a pullback in **Sets**, i.e. any $f : D \to C$ with $A(f)(a) = A(f)(b)$ lifts uniquely to a map $D \to (\partial_A)_C(a, b)$.

Conversely, assume the "small diagonal" condition and consider any pair $(a, b) \in AC \times AC$. There is a corresponding natural transformation $\eta : \mathbf{y}C \to A \times A$ and thus a pullback square

$$\begin{tikzcd}
\mathbf{y}E \arrow[r, "q"] \arrow[d, "\mathbf{y}m"'] & A \arrow[d, "\Delta_A"] \\
\mathbf{y}C \arrow[r, "\eta"'] & A \times A
\end{tikzcd}$$

for some E and m. In particular $m : E \to C$ must be monic, since $\mathbf{y}m$ is. By definition of pullbacks in **Sets**, for each pair $(f : D \to C, c \in AD)$ such that $A(f)(a) = A(f)(b)$ (i.e. $\eta_D(f) = (\Delta_A)_D(c)$), there exists a unique $h : D \to E$ such that $mh = f$. Hence, the inclusion $m : E \hookrightarrow C$ has precisely the property that defines $(\partial_A)_C(a, b)$.

This really defines a natural transformation $\partial_A : A \times A \to \mathbf{y}\Omega_{\mathcal{E}}$. Consider any $f : D \to C$. Just as for $a, b \in AC$, the pair $A(f)(a), A(f)(b) \in AD$ induces by Yoneda a natural transformation $\eta' : \mathbf{y}D \to A \times A$. We need to show that the map $m' : E' \to D$ that comes from the pullback $\mathbf{y}m' : \mathbf{y}E' \to \mathbf{y}D$ of Δ_A along η' coincides with pullback of the subobject $m : E \rightarrowtail C$ along f. In fact, since the isomorphism $\mathrm{Hom}(\mathbf{y}C, A \times A) \cong AC \times AC$ that defines η is natural in C, it follows that $\eta \circ \mathbf{y}f = \eta'$. Hence there are pullbacks

$$\begin{tikzcd}
\mathbf{y}E' \arrow[r] \arrow[d, tail, "\mathbf{y}m'"'] & \mathbf{y}E \arrow[r] \arrow[d, tail, "\mathbf{y}m"'] & A \arrow[d, "\Delta_A"] \\
\mathbf{y}D \arrow[r, "\mathbf{y}f"] \arrow[rr, bend right, "\eta'"'] & \mathbf{y}C \arrow[r, "\eta"] & A \times A
\end{tikzcd}$$

Since $\mathbf{y}$ reflects pullbacks, this proves the claim.

□

It turns out that the $\mathbf{y}\Omega_{\mathcal{E}}$-standard objects are precisely the *ideals* in $\mathrm{Sh}(\mathcal{E})$ that have been studied in [4]. An ideal diagram in $\mathcal{E}$ is a functor $F : I \to \mathcal{E}$ from a directed poset I such that for any inequality $i \leq j$ in I, the map $F(i) \to F(j)$ in $\mathcal{E}$ is a monomorphism. An ideal in $\mathrm{Sh}(\mathcal{E})$ is, by definition, a colimit of the composite functor

$$I \xrightarrow{\quad F \quad} \mathcal{E} \xrightarrow{\quad \mathbf{y} \quad} \mathrm{Sh}(\mathcal{E})$$

where $\mathbf{y}$ is the factorization of the Yoneda embedding through the sheaf topos, which exists because the finite epi topology is subcanonical. The following proposition occurs in [4].

Proposition 4.24. *The following are equivalent:*

- *A sheaf A is an ideal*
- *A satisfies the small diagonal condition from prop. 4.23, i.e. the pullback of Δ_A along any map $\boldsymbol{y}C \to A$ is representable*

Hence prop. 4.23 gives a new characterization of ideal sheaves.

As far as $\mathbf{y}\Omega_{\mathcal{E}}$-relative model structures are concerned, it is, however, the case that $\mathbf{y}\Omega_{\mathcal{E}}$ is generally not complete w.r.t. ideals. Nevertheless, it seems worthwhile to study the connection between ideals and $\mathbf{y}\Omega_{\mathcal{E}}$-relative modal structures in $\mathrm{Sh}(\mathcal{E})$ w.r.t. to the ideal completion $\imath : \mathbf{y}\Omega_{\mathcal{E}} \hookrightarrow \Omega$. We give a characterization of those potential Heyting algebras that admit a $\mathbf{y}\Omega_{\mathcal{E}}$-relative S4 algebra

$$i : \mathbf{y}\Omega_{\mathcal{E}} \leftrightarrows H : \theta$$

As it turns out, H must be an ideal as well.

To begin with, for any such potential adjunction $i \vdash \tau$, the map τ must pull back the top element of $\mathbf{y}\Omega_{\mathcal{E}}$ to the top element of H. For any Heyting algebra H in $\mathrm{Sh}(\mathcal{E}, J)$, the top element $\top : 1 \to H$ has a classifying map

$\tau : H \to \Omega$. Hence, for any map $h : H \to \mathbf{y}\Omega_{\mathcal{E}}$ that preserves the top element of H, the corresponding square

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "\top"'] & 1 \arrow[d, "\top"] \\ H \arrow[r, "h"'] & \mathbf{y}\Omega_{\mathcal{E}} \end{tikzcd}$$

is a pullback if and only if

$$\imath \circ h = \tau.$$

where, $\imath : \mathbf{y}\Omega_{\mathcal{E}} \to \Omega$ is the ideal completion. Hence, for any potential model structure

$$i : \mathbf{y}\Omega_{\mathcal{E}} \leftrightarrows H : \theta,$$

the right adjoint θ of i must necessarily satisfy $\imath \circ \theta = \tau$.

By definition of $\imath$ and τ a necessary and sufficient condition for the existence of θ is that for any C in $\mathcal{E}$, and $a \in H(C)$, there exists a map $\mu : C \to \Omega_{\mathcal{E}}$ such that the set

$$\tau_C(a) = \{f : X \to C \mid H(f)(a) = \top_X\},$$

where $\top_X$ is the top element of $H(X)$, coincides with the set of all those morphisms $f : X \to C$ such that

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & 1 \arrow[d, "\top"] \\ C \arrow[r, "\mu"'] & \Omega_{\mathcal{E}} \end{tikzcd}$$

commutes. Or, equivalently, with the set of arrows $f : X \to C$ that factor through the subobject classified by μ. The map θ then has components

$$\theta_C(a) = \mu.$$

The components of θ defined in this way indeed form the components of a natural transformation, simply because the outer part of the following diagram commutes

$$\begin{tikzcd}
HC \arrow[r, "\theta_C"] \arrow[d, "H(f)"'] \arrow[rr, bend left, "\tau_C"] & \mathbf{y}\Omega_{\mathcal{E}}(C) \arrow[r, "\imath_C"] \arrow[d, "\mathbf{y}\Omega_{\mathcal{E}}(f)"] & \Omega(C) \arrow[d, "\Omega(f)"] \\
HD \arrow[r, "\theta_D"] \arrow[rr, bend right, "\tau_D"'] & \mathbf{y}\Omega_{\mathcal{E}}(D) \arrow[r, "\imath_D"] & \Omega(D)
\end{tikzcd}$$

Then the left-hand square commutes, because each component of $\imath$ is injective.

Of course, if H is of the form $\mathbf{y}H'$, for a complete Heyting algebra H' in $\mathcal{E}$, with its canonical map $\theta' : H' \to \Omega_{\mathcal{E}}$, then $a \in H(C)$ is a map $a : C \to H'$, and μ is the composite

$$C \xrightarrow{\quad a \quad} H \xrightarrow{\quad \theta' \quad} \Omega_{\mathcal{E}}.$$

The requirement for the more general case can also be expressed as follows.

Proposition 4.25. *For any Heyting algebra H in $Sh(\mathcal{E}, J)$, a map $\theta : H \to \boldsymbol{y}\Omega_{\mathcal{E}}$ satisfying*

$$\imath \circ \theta = \tau$$

exists if and only if, for any C in $\mathcal{E}$, the pullback of $\top : 1 \to H$ along any map $a : \boldsymbol{y}C \to H$ is representable. That is to say, there exists a map $m : U \to C$, necessarily a monomorphism, such that

$$\begin{tikzcd}
\boldsymbol{y}U \arrow[r] \arrow[d, "\boldsymbol{y}m"'] & 1 \arrow[d, "\top"] \\
\boldsymbol{y}C \arrow[r, "a"'] & H
\end{tikzcd}$$

is a pullback.

Proof. The pullback, being a subobject of $\mathbf{y}C$, can be identified with a sieve on C. Unwinding definitions, in particular the Yoneda lemma, shows that, for any $a \in H(C)$, it is precisely the sieve of arrows $f : D \to C$ such that $H(f)(a) = \top_D$. The requirement that it is representable then means that there is a monomorphism $m : U \to C$ such that an arrow is in this sieve if and only if it factors through m. $\square$

Specifically, if the condition is satisfied, one may set $\theta_C(a) = \mu$, where $\mu : C \to \Omega_{\mathcal{E}}$ is the classifying map of m in $\mathcal{E}$.

However, note that

$$\begin{tikzcd}
\mathbf{y}C \arrow[r, "a"] \arrow[dr, "\mathbf{y}\mu"'] & H \arrow[d, "\theta"] \\
& \mathbf{y}\Omega_{\mathcal{E}}
\end{tikzcd}$$

does not commute in general. By contrast, for any D in $\mathcal{E}$ and $f : D \to C$, it holds that

$$\theta_D a_D(f) = (\mathbf{y}\mu)_D(f)$$

just in case f factors through U. In fact, $\mathbf{y}m$ is the equalizer of $\mathbf{y}\mu$ and θa. Consider any map $\eta : A \to \mathbf{y}C$ in $\mathrm{Sh}(\mathcal{E}, J)$. If for any D in $\mathcal{E}$ it holds that

$$\theta_D a_D \eta_D(x) = (\mathbf{y}\mu)_D \eta_D(x),$$

for any $x \in A(D)$, then there exists a, necessarily unique, map $\eta' : A \to \mathbf{y}U$ such that

$$\mathbf{y}m \circ \eta' = \eta.$$

This readily follows by observing that the condition means that for any $x \in A(D)$ the map $\eta_D(x) : D \to C$ factors, necessarily uniquely, through m. Hence define $\eta'_D(x)$ to be that factorization.

We thus have the following.

Proposition 4.26. *For any Heyting algebra H in Sh$(\mathcal{E}, J)$, a map $\theta : H \to \mathbf{y}\Omega_{\mathcal{E}}$ fitting into a pullback diagram*

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, "\top"'] & 1 \arrow[d, "\top"] \\
H \arrow[r, "\theta"'] & \mathbf{y}\Omega_{\mathcal{E}}
\end{tikzcd}$$

exists if and only if the sheaf H is an ideal in Sh$(\mathcal{E}, J)$.

Proof. If such a θ exists, then H is $\mathbf{y}\Omega_{\mathcal{E}}$-standard, because of the pullbacks

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "\Delta_H"'] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
H \times H \arrow[r, "\Leftrightarrow"'] & H \arrow[r, "\theta"'] & \mathbf{y}\Omega_{\mathcal{E}} \arrow[r, "\imath"'] & \Omega
\end{tikzcd}$$

so that the lower composite is a classifying map for Δ_H. Hence, by prop. 4.23, H is an ideal. For the converse, suppose the Heyting algebra H is an ideal in $\mathrm{Sh}(\mathcal{E}, J)$. The pullback of $\top : 1 \to H$ along a map $a : \mathbf{y}C \to H$ can be constructed as the equalizer of the two composites in (not commutative) square

$$\begin{tikzcd}
\mathbf{y}C \times 1 \cong \mathbf{y}C \arrow[r, "u"] \arrow[d] & 1 \arrow[d, "\top"] \\
\mathbf{y}C \arrow[r, "a"'] & H
\end{tikzcd}$$

with the evident projections to $\mathbf{y}C$ and 1, resp. This equalizer (or rather the projection to $\mathbf{y}C$) can in turn be expressed as the vertical projection in the following pullback

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "e"'] & H \arrow[d, "\Delta"] \\
\mathbf{y}C \arrow[r, "\langle a, \top u \rangle"'] & H \times H
\end{tikzcd}$$

Since H is an ideal, $E \cong \mathbf{y}U$, for some U in $\mathcal{E}$, and e is of the form $\mathbf{y}m$ for some monomorphism $m : U \to C$. Hence the following is a pullback:

$$\begin{tikzcd}
\mathbf{y}U \arrow[r, "u"] \arrow[d, "\mathbf{y}m"'] & 1 \arrow[d, "\top"] \\
\mathbf{y}C \arrow[r, "a"'] & H
\end{tikzcd}$$

Hence θ exists by prop. 4.25.

□

As a result, $\mathbf{y}\Omega_{\mathcal{E}}$-relative S4 algebras really sit in the full subcategory $\mathrm{Idl}(\mathcal{E}) \hookrightarrow \mathrm{Sh}(\mathcal{E})$ of ideals. One may wonder if $\mathbf{y}\Omega_{\mathcal{E}}$ has a universal property w.r.t. to complete Heyting algebras in $\mathrm{Idl}(\mathcal{E})$ similar to the initial frame $\Omega_{\mathcal{E}}$ in a topos $\mathcal{E}$. Since $\mathbf{y}\Omega_{\mathcal{E}}$ is not itself ideal complete, the question might be whether there exists a unique adjoint structure $\mathbf{y}\Omega_{\mathcal{E}} \leftrightarrows H$ whenever a Heyting algebra H in $\mathrm{Idl}(\mathcal{E})$ is complete w.r.t. representable functors. However, this does not seem to be the case. In fact, it might even be that a monic left adjoint to $\theta : H \to \mathbf{y}\Omega_{\mathcal{E}}$ exists only if H is representable as well. This would of course significantly weaken the previous results.

Remark 4.27. There is an evident necessary condition for i to exist. Suppose $H \cong \varinjlim_I \mathbf{y}C_i$ is an ideal. Since an ideal colimit is a sheaf, it can be computed as a colimit of presheaves. That is to say, it is evaluated pointwise, for any D in $\mathcal{E}$, as a colimit in **Sets**:

$$(\varinjlim_I \mathbf{y}C_i)(D) \cong \varinjlim_I \mathbf{y}C_i D = \varinjlim_I \mathrm{Hom}_{\mathcal{E}}(D, C_i).$$

By construction of directed colimits in **Sets**, each component of λ_i at D maps a map $f : D \to C_i$ to its equivalence class containing all those maps $g : D \to C_j$, for some object C_j in the diagram underlying H, for which there exists there exist a cospan

$$C_i \xrightarrow{\alpha_{ik}} C_k \xleftarrow{\alpha_{jk}} C_j,$$

in the diagram, such that $\alpha_{jk} g = \alpha_{ik} f$. For an arrow $f : B \to D$, the function

$$(\varinjlim_I \mathbf{y}C_i)(D) \to (\varinjlim_I \mathbf{y}C_i)(B)$$

is by precomposition, and it preserves equivalence classes.

Now any map $\eta : \mathbf{y}D \to \varinjlim_I \mathbf{y}C_i$, determines, by the Yoneda lemma, an element $\eta_0 \in (\varinjlim_I \mathbf{y}C_i)(D)$. That is to say, by the previous, an (equivalence class of an) arrow $\eta_0 : D \to C_0$, for some C_0 in the underlying diagram of the colimit. It follows that $\lambda_0 \circ \mathbf{y}(\eta_0) = \eta$, i.e. the following commutes:

$$\begin{tikzcd}
\mathbf{y}C_0 \arrow[r, "\lambda_0"] & \varinjlim_I \mathbf{y}C_i \\
\mathbf{y}D \arrow[u, "\eta_0"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

Hence, every η factors through the base of $\varinjlim_I \mathbf{y}C_i$. This factorization is not unique in general. There is a factorization $\mathbf{y}f : \mathbf{y}D \to \mathbf{y}C_i$ for any arrow f in the equivalence class of η_0.

If a map i, left adjoint to θ, exists, it necessarily factors through a cocone component $\lambda_0 : \mathbf{y}C_0 \to H$, as $\mathbf{y}i_0$, for a monomorphism $i_0 : \Omega_{\mathcal{E}} \to C_0$. However, no non-trivial sufficient condition is known at that point (non-trivial in the sense that $\mathbf{y}H$ is not assumed to be representable).

4.4 Topological Embeddings

In this section we state the topological completeness theorem. For this we will need to consider finite limit preserving functors $f^* : \mathcal{E} \to \mathcal{F}$ between toposes that preserve the model structure. One motivation to study relative model structure in the first place was that most functors of interest $F : \mathcal{E} \to \mathcal{F}$ between topoi do not preserve the subobject classifier, and thus do not map an adjunction

$$i : \Omega_{\mathcal{E}} \leftrightarrows H : \tau$$

to an adjunction

$$i : \Omega_{\mathcal{F}} \leftrightarrows FH : \tau.$$

Moreover, many functors do not preserve completeness of internal Heyting algebras. Relative models provide a means to keep the essential algebraic properties provided by the adjunction over $\Omega_{\mathcal{E}}$, while at the same time allow for a better notion of model-preserving functor, as we will see shortly.

Definition 4.28. *For any B-relative model structure $H \leftrightarrows B \hookrightarrow \Omega_{\mathcal{E}}$ in a topos $\mathcal{E}$, a functor $F : \mathcal{E} \to \mathcal{F}$ is said to preserve the model structure if*

- *F preserves finite limits*
- *For any Heyting algebra in $\mathcal{E}$ the Heyting algebra FH is F-complete*
- *The classifying map $\imath : F\Omega_{\mathcal{E}} \longrightarrow \Omega_{\mathcal{F}}$ of the top element of the Heyting algebra $F\Omega_{\mathcal{E}}$ is a monomorphism*

Since F preserves finite limits, and thus monomorphisms, it follows that the composite

$$FB \xrightarrow{F\beta} F\Omega_{\mathcal{E}} \xrightarrow{\imath} \Omega_{\mathcal{F}}$$

classifies the top element of FB and is moreover monic. Moreover, such a functor takes an interpretation in $\mathcal{E}$ to an interpretation in $\mathcal{F}$ respecting validity.

For instance, the Yoneda embedding $\mathbf{y} : \mathcal{E} \to \mathrm{Sh}(\mathcal{E})$ preserves model structures in this sense, because it is cartesian closed and $\imath : \mathbf{y}\Omega_{\mathcal{E}} \to \Omega$ is monic. Moreover, since $\mathbf{y}$ is an embedding, given a faithful complete Heyting algebra H in $\mathcal{E}$, the resulting $\mathbf{y}\Omega_{\mathcal{E}}$-based structure

$$\mathbf{y}i : \mathbf{y}\Omega_{\mathcal{E}} \leftrightarrows \mathbf{y}H : \mathbf{y}\tau \tag{19}$$

derives completely from the original one, and any model essentially derives from a model in the structure in $\mathcal{E}$. That is to say, any proposition that holds in any model in the relative structure (19) (when it is understood as $\mathbf{y}$-relative in the sense that types are interpreted by representables) holds in the corresponding model in $\mathcal{E}$.

There is another class of functors that has similar properties. Recall that a connected geometric morphism is one whose inverse image part is full and faithful.

Fact 4.29. *The inverse image of any connected locally connected geometric morphism $f : \mathcal{F} \to \mathcal{E}$ preserves and reflects (relative) model structures.*

Proof. For any locally connected morphism $f : \mathcal{F} \to \mathcal{E}$ the map $\tau^\bullet : f^*\Omega_\mathcal{E} \to \Omega_\mathcal{F}$ is a monomorphism,[26] hence so is the composite $f^*H \rightarrowtail f^*\Omega_\mathcal{E} \xrightarrow{\tau^\bullet} \Omega_\mathcal{F}$. Moreover, f^* is cartesian closed and full and faithful. □

The next theorem can be found in [8] or [9].

Theorem 4.30. *For every Grothendieck topos $\mathcal{G}$ with enough points, there exists a topological space X and a connected, locally connected geometric morphism*

$$p : Sh(X) \longrightarrow \mathcal{G}.$$

Theorem 4.31. *There exists a topological space X and a generic relative model in the topos $Sh(X)$ of sheaves on X.*

Proof. Consider the syntactic topos $\mathcal{E}_\mathbb{T}$ of a higher-order rmodal theory $\mathbb{T}$, and the topos $\mathrm{Sh}(\mathcal{E}_\mathbb{T})$ of sheaves on $\mathcal{E}_\mathbb{T}$ for the finite epi topology. Since topos $\mathcal{E}_\mathbb{T}$ is coherent, and thus has enough points, by theorem 4.30 it can be covered by a connected locally connected geometric morphism

$$p : \mathrm{Sh}(X_\mathbb{T}) \longrightarrow \mathrm{Sh}(\mathcal{E}_\mathbb{T}).$$

Hence there is a string of cartesian closed embeddings

$$\mathcal{E} \xrightarrow{\quad \mathbf{y} \quad} \mathrm{Sh}(\mathcal{E}_\mathbb{T}) \xrightarrow{\quad p^* \quad} \mathrm{Sh}(X_\mathbb{T})$$

that transports the canonical model in $\mathcal{E}_\mathbb{T}$ to a faithful relative model in $\mathrm{Sh}(X_\mathbb{T})$. □

[26] e.g. [7], [14]

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