On the Construction of Classical Superstring Field Theories

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Contents

Zusammenfassung

Diese Dissertation behandelt die Konstruktion von klassischen Superstringfeldtheorien basierend auf dem kleinen Hilbertraum. Zuerst wird die traditionelle Konstruktion der störungstheoretischen Superstringtheorie mittels Integration über den Supermodulraum von Typ-II-Weltflächen beschrieben. Die Geometrie dieses Modulraums bestimmt viele algebraische Eigenschaften der Stringfeldtheoriewirkung. Insbesondere ermöglicht sie es, das Konstruktionsproblem für klassische Superstringfeldtheorien zu algebraisieren.

Als nächstes wird eine Lösung des Konstruktionsproblems für offene Superstrings ausgehend von Wittens Sternprodukt beschrieben. Diese Lösung ist rekursiv und hängt von der Wahl eines Homotopieoperators für die Nullmode des η-Geistfeldes ab. Die rekursive Konstruktion lässt sich auf die Neveu-Schwarz-Sektoren aller Superstringtheorien verallgemeinern. Im allgemeinsten Fall wird eine Hierarchie von Stringprodukten mit verschiedenen Picturedefiziten definiert. Obwohl die Konstruktion ist nicht ganz natürlich ist, gehen verschiedene Lösungen des Konstruktionsproblems mittels Feldredefinition auseinander hervor. Für die Erweiterung auf Ramondsektoren ergibt sich eine weitere Komplikation durch die ungeraden Klebemoduli. Anstelle einer Wirkung werden lediglich eichinvariante Bewegungsgleichungen konstruiert.

Der Lösungsraum der Bewegungsgleichungen für offene Superstrings ist supersymmetrisch. Die Supersymmetrietransformationen werden explizit für offene Superstrings angegeben und es wird gezeig, dass die Kombination aus kleiner Hilbertraumbedingung und Bewegungsgleichungen in polynomielle Form gebracht werden kann und dass dieses erweiterte System supersymmetrisch ist. Die Supersymmetriealgebra schließt nur modulo Eichtransformationen, was darauf hindeutet, dass die $\mathcal{N} = 1$ Supersymmetrie lediglich auf dem Lösungsraum realisiert ist.

Eine wichtige Konsistenzbedingung für alle Superstringwirkungen ist die Äquivalenz der feldtheoretischen S-Matrix zur traditionellen störungstheoretischen S-Matrix. Die S-Matrix einer Feldtheorie ist eng mit dem minimalen Modell der assoziierten Homotopiealgebra verknüpft. Durch die rekursive Konstruktion der Feldtheoriewirkung mittels Produkten bei verschiedenen Picturedefiziten ist es möglich die S-Matrizen bei unterschiedlichen Picturedefiziten durcheinander auszudrücken. Letztendlich führt dies zu einem Ausdruck der Superstring-S-Matrix durch die bosonische S-Matrix und Pictureänderungsoperatoren, die auf die externen Zustände wirken.

Beim offenen Superstring ist es weiterhin möglich eine Wirkung für die vollständigen Bewegungsgleichungen zu finden. Die Präsenz der Pictureänderungsoperatoren in den internen Ramondlinien erfordert, dass man entweder den Hilbertraum einschränkt oder dass man ein Hilfsfeld bei Picture $-\frac{3}{2}$ einführt.

Abstract

This thesis describes the construction of classical superstring field theories based on the small Hilbert space. First we describe the traditional construction of perturbative superstring theory as an integral over the supermoduli space of type II world sheets. The geometry of supermoduli space dictates many algebraic properties of the string field theory action. In particular it allows for an algebraisation of the construction problem for classical superstring field theories in terms of homotopy algebras.

Next, we solve the construction problem for open superstrings based on Witten's star product. The construction is recursive and involves a choice of homotopy operator for the zero mode of the η-ghost. It turns out that the solution can be extended to the Neveu-Schwarz subsectors of all superstring field theories. The recursive construction involves a hierarchy of string products at various picture deficits. The construction is not entirely natural, but it is argued that different choices give rise to solutions related by a field redefinition. Due to the presence of odd gluing parameters for Ramond states the extension to full superstring field theory is non-trivial. Instead, we construct gauge-invariant equations of motion for all superstring field theories.

The realisation of spacetime supersymmetry in the open string sector is highly non-trivial and is described explicitly for the solution based on Witten's star product. After a field redefinition the non-polynomial equations of motion and the small Hilbert space constraint become polynomial. This polynomial system is shown to be supersymmetric. Quite interestingly, the supersymmetry algebra closes only up to gauge transformations. This indicates that only the physical phase space realizes $\mathcal{N} = 1$ supersymmetry.

Apart from the algebraic constraints dictated by the geometry of supermoduli space the equations of motion or action should reproduce the traditional string S-matrix. The S-matrix of a field theory is related to the minimal model of the associated homotopy algebra. Because of the recursive nature of the solution and its construction in terms of products of various picture deficits, it is possible to relate the S-matrices of various picture deficits and, therefore, relate the S-matrix calculated from the bosonic string products at highest picture deficit with the physical vertices at lowest picture deficit through a series of descent equations.

For open superstrings one can go beyond the equations of motion. The presence of picture changing operators at internal Ramond lines imposes either a constraint on the Hilbert space or necessitates the introduction of an auxiliary string field at picture $-\frac{3}{2}$. Based on the full equations of motion for the open string field, an action principle is proposed and shown to be gauge-invariant.

Chapter 1

Introduction

1.1 Motivation

Over the last hundred years, progress in the theoretical understanding of nature has been guided by the principle of unification. Unification means roughly postulating larger symmetry groups and extra dimensions in a way that the low energy physics of the model reduces to the well-established experimental results, such as Lorentz invariance, four spacetime dimensions, the correct particle spectrum and their interactions. This approach makes it possible to construct consistent mathematical models of nature while at the same time reproducing experimental data and allowing for genuinely new predictions. The tension between mathematical soundness and observational compatibility has led physicists to identify several core principles. The most prominent such principles are locality, the gauge principle and unitarity. Their conjunct success is intimately tied to the severe restrictions they impose on the mathematical model. Perhaps the two most famous applications of the gauge principle are Maxwell theory and the standard model. The former is invariant under local U(1) transformations, while the latter is invariant under local $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_W \times \mathrm{SU}(3)_c$ transformations.

Originally, Maxwell theory was formulated in terms of the electric and magnetic field strengths. This form of the theory is a successful description of many observed electromagnetic phenomena like electromagnetic waves. While the Maxwell equations are local, it was not possible to couple them to a charged scalar field using only the field strengths and at the same time producing long ranged, Coulomb-like interactions between small perturbations or charged particles of the scalar field. It was only upon rewriting Maxwell theory in terms of the vector potential A_μ and coupling the scalar field minimally that long-ranged interactions could be produced from local equations of motion. Moreover, after the advent of quantum mechanics and the discovery of the Aharonov-Bohm effect it was realised that the vector potential should be regarded as not merely a mathematical tool to describe electrodynamics, but rather taken as the “fundamental” field of Maxwell theory. Because of its success, the gauge formulation of Maxwell theory has been generalised to include other, even non-Abelian gauge groups. The result is nowadays known under the name Yang-Mills theory. Chiral fermions coupled minimally to Yang-Mills theory together with a Higgs sector describe the theoretical basis for the standard model, a model describing the strong, weak and electromagnetic interactions simultaneously. Arguably, it can be regarded as the most successful physical model of fundamental interactions. It is supported by huge experimental evidence accumulated over more than four decades. One of the most recent being the discovery of the Higgs boson at the LHC. This tremendous success can be interpreted as a very good argument in favour of its underlying theoretical foundations and for taking the combination of locality and gauge invariance as a fundamental guiding principle for building viable physical theories and models in the UV. Another very successful application of this combined locality and gauge paradigm is general relativity. In this theory the invariance under general coordinate transformations or diffeomorphisms is postulated, while at the same time the dynamics of the metric $g_{\mu\nu}$ is described through a set of local equations of motion, the Einstein equations. The recent detection of the GW150914 event by the LIGO indicates that gravitational waves do exist and the measured spectrum from the observed merger of two black holes matched the predicted form from general relativity.

If we regard the standard model or general relativity as classical field theories, they are conceptually satisfactory. However, one of the main results of the 1920s is the observation that physics cannot be described by classical equations alone, but we are required to quantise them. Field theories based on scalar fields, fermions and vector potentials did not pose a serious obstacle to quantisation, but revealed yet another deep interplay between locality, gauge-invariance and Lorentz invariance. In Yang-Mills type theories the physical states described by the vector potential are massless spin 1 particles. For those particles a Lorentz invariant, local quantisation requires that the interactions are gauge-invariant. Alternatively one could fix the gauge beforehand at the price of breaking manifest Lorentz-invariance and introducing non-localities. Contrary to the previous situation quantisation of gravity in the form of general relativity turned out to be much more difficult and has not been successful up to the present day, so that presumably new ideas were needed.

In order to deal with strongly coupled theories such as the theory of mesons and hadrons the S-matrix approach was developed. The S-matrix method deals entirely with physical particle states. These states make up the whole spectrum of free particles and the existence of an S-matrix that should be compatible with a prescribed set of symmetries is postulated. Two further requirements are the analyticity of the S-matrix in the external momenta and that the S-matrix should factorise over non-analyticities such as poles or branch-cuts that should occur precisely when a combination of external momenta goes on-shell. One S-matrix satisfying the axioms is known as the dual resonance model which includes the famous Veneziano amplitude as the four-particle S-matrix. Soon it was realised that the dual resonance model can be interpreted in terms of a model where the fundamental objects are not point particles but rather one-dimensional objects, known as open strings. The model was extended to include closed strings as well. Most interestingly, this extension contained massless spin 2 particle states. Consequently, the dual resonance model became a subject of interested in a much broader community, because a successful quantisation of general relativity was expected to contain massless gravitons of spin 2 and the S-matrix approach provided us with a UV-complete description of their interactions. Since moreover open strings contain massless spin 1-particles, string theory was and is still regarded as the most promising candidate for not just a consistent theory of quantum gravity, but also for a complete theory of all fundamental interactions. In the forth coming years string theory received very large attention and underwent rapid development. For example, the spectrum of the bosonic string contains only spacetime bosons and, in addition, tachyonic particle states that indicate an instability in the underlying theory. Both problems were remedied by making the world-sheet theory supersymmetric, turning the string into a superstring, and applying a consistent truncation to the spectrum, the GSO projection. On a flat ten-dimensional Minkowski background one finds exactly five different consistent S-matrices, which are known as type IIA/B, heterotic $E_8 \times E_8/SO(32)$ and type I superstring theories.

At this point the connection of string theory with the previously emphasised locality and gauge paradigm may not be clear. Covariant perturbative string theory realises it manifestly only at the level of the world-sheet theory that is defined in terms of a sigma model path-integral coupled to conformal two-dimensional gravity. Properties like unitarity/factorisation of the string S-matrix can be attributed directly to world-sheet locality, while manifest covariance is implied by the preservation of world-sheet gauge-invariance. As gauge-invariance and locality are very general principles, it might seem that one could define a good S-matrix for any sigma model. However, it is well-known that defining a path-integral while preserving a local symmetry is only possible if no anomalies arise. For the string sigma model this requirement imposes very severe constraints on the background defining the sigma-model. For example, it requires the dimension of spacetime to be 10 and the background metric to be Ricci flat. All in all, this shows that the combination of locality and gauge symmetries can give us very important advice in the search for the correct string background.

But the situation is not so good as it might appear. Conceptually, the choice of world-sheet theory is not restrictive enough to allow for a small set of candidate vacua for our universe. Even if one restricts to vacua that compactify six out of ten dimensions and preserve $\mathcal{N} = 1$ supersymmetry along the uncompactified directions, the consistency conditions tell us that we have to choose a Calabi-Yau threefold for the compact directions and it is even possible to decorate it with stacks of D-branes, orientifolds, etc. Thus, the amount of allowed vacua is still very large, forcing string theory to lose its predictive power. On the other hand, one of the most robust predictions of string theory is spacetime supersymmetry at high energies. Unfortunately, up to the present day no supersymmetric partners have been found at the LHC and many simple supersymmetric extensions of the standard model that give rise to spontaneously broken supersymmetry at low energies could be constrained or even ruled out by additional data gathered from detailed observation of the cosmic microwave background by the WMAP and Planck collaborations. Given these shortcomings of conventional perturbative string theory it would seem that we lack a deep understanding of the symmetry principles underlying string theory. A deeper understanding of perturbation theory would help to study supersymmetry breaking effects that occur at higher loop-level and shrink the landscape of available superstring vacua and thereby increase the predictive power of string theory.

String field theory is one approach towards such an understanding. It comes in many different flavours, such as bosonic or superstring string field theory and open or closed string field theory. Up to now, most of the work in string field theory was concentrated at bosonic open strings with a few results for bosonic closed strings and even fewer for superstrings. The main objective of this thesis is a formulation of open and closed superstring field theory as classical BV-field theories.

1.2 A brief survey of string field theory

In this section we give a quick and most likely incomplete guide to the physical and mathematical foundations of string field theory. Most of the material is well-established since the 80s, but is usually not emphasised in typical string theory courses. The geometric approach to string field theory is perhaps the most convenient as it is closely related to the world-sheet formulation of string theory. Another reason for working with the geometric approach is that many algebraic properties of string field theory can be directly deduced from the world-sheet picture even without performing explicit calculations.

String theory in its present form is not complete. The main reason is that it only provides us with a prescription to calculate the S-matrix and the particle spectrum around a fixed background as a perturbation series in the string coupling constant, but it is unknown if string theory can be given any meaning beyond its S-matrix. Experience with local quantum field theories suggests that one should reformulate string theory as a second quantised theory. Initial steps in this direction were performed in light-cone gauge [1, 2] and eventually led to light-cone string field theory [3, 4]. For a review of the old work on light-cone field theories and their connection with the dual resonance models see [5]. The light-cone formulations are technically simpler, at the price of losing manifest Poincaré invariance. Since covariance is intimately tied to the gauge-invariance on the world-sheet, the BRST method was employed to restore manifest covariance in [6–9]. In the same year, Witten presented his *open bosonic string field theory* [10] that completed the covariant construction for open bosonic strings and identified the important algebraic structure as a non-commutative, associative differential graded algebra equipped with an invariant inner product.

The gauge fixing procedure of world sheet diffeomorphism invariance via the

BRST method, introduces auxiliary ghost fields b and c and an odd operator, the BRST operator, Q that squares to zero, $Q^2 = 0$. Moreover, ghost number induces a grading on the ghost-extended Hilbert space $\mathcal{H}$. Contrary to the conventional application of the BRST method physical states are identified with states $|\psi\rangle \in \mathcal{H}$ at ghost number 1 (instead of ghost number 0) that are killed by Q, i.e. $Q|\psi\rangle = 0$. Two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are gauge-equivalent if they differ by a Q-exact state, i.e. $|\psi_1\rangle - |\psi_2\rangle = Q|\chi\rangle$, where $|\chi\rangle$ has ghost number 0. Mathematically speaking the physical Hilbert space is identified with the first cohomology group $H^1(Q)$ of the operator Q.

In his open bosonic string field theory Witten also introduced an even binary product $*$ of ghost number 0 and a trace operation $\int : \mathcal{H} \to \mathbb{C}$ that fulfil the following axioms, $A, B, C \in \mathcal{H}$,

$$\textit{(nilpotency)} \qquad Q^2 = 0 \tag{1.1a}$$

$$\textit{(derivation)} \qquad Q(A * B) = (QA) * B + (-1)^{\mathrm{gh}(A)} A * (QB) \tag{1.1b}$$

$$\textit{(associativity)} \qquad A * (B * C) = (A * B) * C \tag{1.1c}$$

$$\textit{(invariance)} \qquad \int QA = 0, \tag{1.1d}$$

$$\textit{(symmetry)} \qquad \int A * B = (-1)^{\mathrm{gh}(A)\mathrm{gh}(B)} \int B * A, \tag{1.1e}$$

where $(-1)^{\mathrm{gh}(A)}$ denotes the Grassmannality of the state A that is determined in terms of its ghost number $\mathrm{gh}(A)$. The trace operator turned out to have ghost number -3, i.e. $\int A$ is zero unless A has ghost number 3. Given such an algebraic structure, we can introduce a string field $\Phi \in \mathcal{H}$ at ghost number 1. The open string field theory action takes the form

$$S_{\mathrm{Witten}} = \frac{1}{2} \int \Phi * (Q\Phi) + \frac{1}{3} \int \Phi * \Phi * \Phi,$$

which is readily recognised as an action of Chern-Simons type. For this reason Witten's open string field theory is said to be Chern-Simons like. Upon varying S_{Witten} w.r.t. Φ, one finds the equations of motion and a gauge invariance $\delta\Phi$,

$$Q\Phi + \Phi * \Phi = 0 \tag{1.2a}$$

$$Q\Lambda - \Lambda * \Phi + \Phi * \Lambda = \delta\Phi, \tag{1.2b}$$

where the gauge-parameter Λ is of ghost number 0. The elegance of Witten's construction lies in the definition of the differential Q and the associative product in terms of data provided by the world-sheet theory. $\mathcal{H}$ is the Hilbert space of the world-sheet theory and the grading is defined in terms of ghost number. The operator Q is identified with the BRST operator of the world-sheet theory. This ensures that at the linearised level the space of solutions to the field equations is the same as the spectrum of the physical string. The binary product $*$, which is also known as Witten's star product, is defined in terms of gluing half-strings. The idea is as follows: As the world-sheet theory is a conformal field theory, we can identify

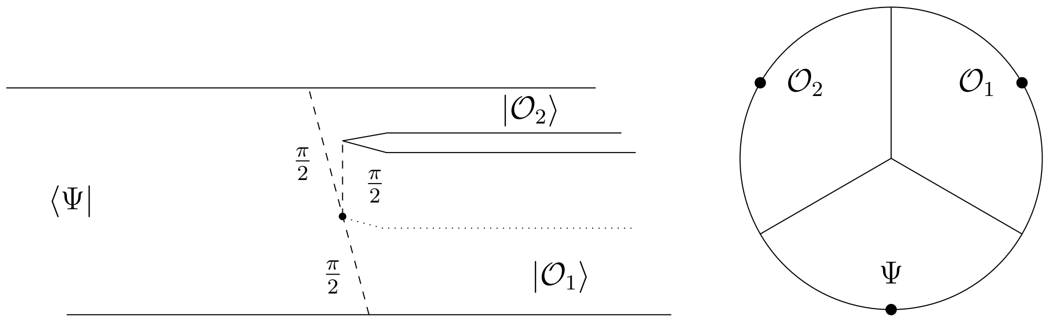

Figure 1.1: Left: Witten's star product $|\mathcal{O}_1\rangle * |\mathcal{O}_2\rangle$ of two states is obtained by evaluating the path-integral on the shown geometry. States are represented by semi-infinite strips of width π equipped with a flat metric and are folded along the dotted line in half. Notice that the metric on the glued world-sheet has a defect angle π at the mid-point. The state $\langle\Psi|$ represents an arbitrary test state. Right: A conformal map of the world-sheet to the unit disc $\mathbb{D}$. The infinitely remote ends of the strips have shrunk to a point and the defect angle has moved to the boundary curvature.

its Hilbert space with the space of local operators via the state-operator correspondence. Invariance under conformal transformations tells us that we can identify each state $|\mathcal{O}\rangle$ with a path-integral evaluated on an semi-infinite strip of width π with boundary conditions at infinity given by the local operator $\mathcal{O}$. The product state $|\mathcal{O}_1\rangle * |\mathcal{O}_2\rangle$ is defined by evaluating the path-integral on a geometry shown in figure 1.1 where the half-strings in $|\mathcal{O}_i\rangle$ are glued such that the natural parametrisations coincide. The geometric definition of Witten's star product makes the associativity condition (1.1c) and the derivation property (1.1b) manifest. Finally, the trace operation $\int$ is defined by folding the semi-infinite strip representing the state and gluing both half-strings together. At this point it remains to explain the connection of Witten's string field theory with the dual resonance model that it is supposed to represent. A crucial consistency condition is that the classical S-matrix calculated from the Witten action agrees with the tree-level amplitudes of the dual resonance model. Because of the gauge-invariance (1.2b) we need to fix a gauge in order to calculate any S-matrix. The most convenient gauge is Siegel gauge in which we require the condition $b_0\Phi = 0$, where $b_0 = \oint \frac{dz}{2\pi \mathrm{i}}\, z b(z)$ denotes a special mode of the Faddeev-Popov antighost field $b(z)$ that describes the fixing of the world-sheet gauge-symmetry and the coordinate z denotes a coordinate on the upper-half-plane. In Siegel gauge the propagator $-Q^\dagger$ takes the form

$$Q^\dagger = \frac{b_0}{L_0} = \int_0^\infty b_0 e^{-\tau L_0}\, d\tau,$$

where $L_0 = [Q, b_0]_+$ is the world-sheet Hamiltonian generating time-evolution along the semi-infinite strip. The second equality is just the Schwinger representation of the propagator. The integrand has a nice geometric interpretation. Since L_0

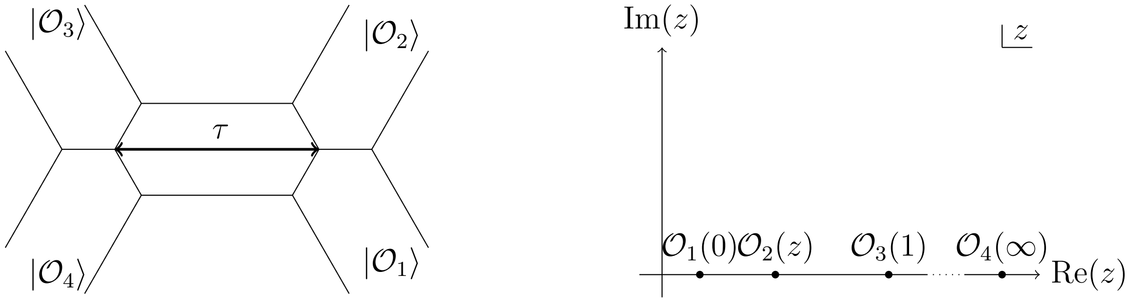

Figure 1.2: Left: s-channel diagram. Because of the defect angle at the mid-point of the Witten-vertex the diagram is slightly distorted and as shown in figure 1.1 near the midpoints. All strips are of width π and the propagator strip has length τ. Right: A conformally equivalent world-sheet is the upper-half-plane with the operators $\mathcal{O}_1$, $\mathcal{O}_3$ and $\mathcal{O}_4$ mapped to 0, 1 and ∞ and the position z of $\mathcal{O}_2$ determined by the value of τ. As τ runs from 0 to ∞, the parameter z runs from $\frac{1}{2}$ to 0. Notice that the orientations induced by τ are different for both channels.

generates translations along the semi-infinite strip, it actually represents the addition of a piece of strip of length τ and width π to a state. The complete propagator is obtained by integrating over the strip lengths τ. As an example we consider the four-point amplitude [11]. The amplitude receives contributions from three distinct colour-orderings and for each ordering we have to consider two diagrams, an s- and a t-channel diagram. Since the external states $|\mathcal{O}_i\rangle$ are supposed to be on-shell, i.e. $Q|\mathcal{O}_i\rangle = 0$, they do not depend on the choice of coordinate frame and the conformal scale factor on the strip. Figure 1.2 sketches the sequence of conformal transformations from the world-sheet constructed from the Feynman rules to the conventional integral over the four-point function of the underlying CFT. At this point several non-trivial things take place. While the mapping between the two diagrams follows directly from the properties of a CFT, the non-trivial features are in the integration measure and integration region. For simplicity we consider a colour-ordered four-point amplitude in which the operators $\mathcal{O}_1$, $\mathcal{O}_3$ and $\mathcal{O}_4$ are mapped to 0, 1 and ∞ and the operators $\mathcal{O}_2$ to z with $0 < z < 1$ as in figure 1.2. The Schwinger representation of the propagator tells us that we need to integrate over the modulus τ. First of all, it is non-trivial that the sum of the s-channel and the t-channel diagram covers the region $0 < z < 1$ completely. For the case at hand this statement is not very hard to see, for if in the s-channel diagram we consider the limit $\tau \to \infty$, the strip becomes infinitely long and we pinch off a thrice-punctured disc with the operators $\mathcal{O}_2$ and $\mathcal{O}_3$ inserted. This means that the operators $\mathcal{O}_2$ and $\mathcal{O}_3$ must collide in this limit. Showing that as $\tau \to \infty$, z will approach 1. Similarly, considering the same limit of the t-channel diagram shows that $z \to 0$. Now, if we set $\tau = 0$, the s-channel and t-channel diagrams coincide because of the associativity of the star product. Therefore, the world-sheets for $\tau = 0$ coincide for both channels and so must the value of z, which turns out to be $z = \frac{1}{2}$. Secondly, we have to ensure that the integration measure reduces to the Veneziano measure dz. In [11] this was shown explicitly and involved some rather intricate identities between elliptic functions.

The question, whether open string field theory produces the correct S-matrix elements, is therefore related to modular invariance in string field theory [12]. Roughly speaking modular invariance in our case means that the diagrams constructed from the Feynman rules as illustrated earlier should cover the complete moduli space of Riemann surfaces with boundaries and punctures on the boundary completely. In [12] it was argued that the Feynman rules should construct a cell decomposition of the underlying moduli space. This became the central idea of the geometric approach to constructing string field theories.

The conceptual simplicity and the elegant solution for the (classical) open string led to analogous extensions to the closed bosonic string. However, it was realised that no cubic vertex exists that can generate a cover of the whole moduli space of punctured Riemann surfaces and that even recovering the tree-level S-matrix is not possible in this way. Figure 1.3 illustrates the uncovered region of the moduli space $\mathcal{M}_{0,4}$. The missing pieces of the moduli space were described explicitly by Saadi and Zwiebach in terms of their polyhedral vertices [13]. Based on this decomposition of the genus 0 moduli space, evidence was given in [14] that closed string field theory should not be just non-cubic but rather non-polynomial. They claimed that the closed string field theory action should schematically take the form

$$S_{\mathrm{CSFT}} = \frac{1}{2}\omega(\Phi, Q\Phi) + \sum_{n\geq 3} \frac{1}{n!}(\Phi, \Phi, \ldots, \Phi)_n, \tag{1.3}$$

where ω denotes some non-degenerate pairing on the closed string Hilbert space obeying the level-matching conditions and $(\cdot, \cdot, \ldots, \cdot)_n$ denotes a completely symmetric n-linear form that is obtained from integrating the CFT correlation function over the uncovered moduli space. They also claimed that S_{CSFT} should enjoy a non-linear gauge-invariance,

$$\delta\Phi = Q\Lambda + \sum_{n\geq 2}[\Phi^n, \Lambda]_{n+1}, \tag{1.4}$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)_n = \omega(\alpha_1, [\alpha_2, \ldots, \alpha_n]_{n-1}),\ \alpha_i \in \mathcal{H}.$$

Up to this point string field theory was an entirely classical theory. Based on a reformulation of the construction of the polyhedral vertices in terms of a minimal area problem [15,16], Zwiebach eventually constructed his *closed bosonic string field theory* in [17]. His construction was remarkable in that it is also consistent at the quantum level. His action looks very similar to (1.3), but all terms receive $\hbar$-corrections. Moreover, the string field is not restricted to ghost number 2, where the conventional physical modes are located, but rather allows for the presence of all ghost numbers. Upon lifting this restriction he shows that the properties of the world-sheet theory imply that S_{CSFT} satisfies a quantum BV-master equation [18–20],

$$-\mathrm{i}\hbar\Delta S_{\mathrm{CSFT}} + \frac{1}{2}(S_{\mathrm{CSFT}}, S_{\mathrm{CSFT}}) = 0,$$

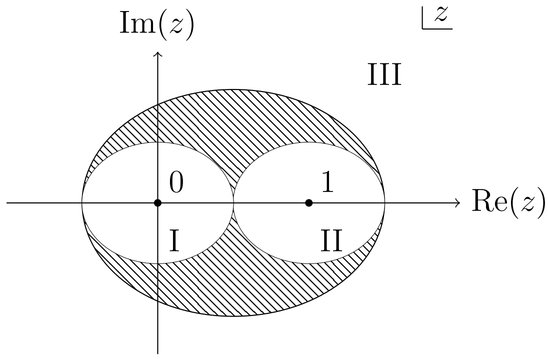

Figure 1.3: The moduli space $\mathcal{M}_{0,4}$ of conformally inequivalent Riemann spheres with four punctures can be identified with the set $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ with complex coordinate z. The regions I, II and III are generated by s-, t- or u-channel Feynman diagrams, while the shaded region is not generated.

where Δ denotes the BV-Laplacian and $(x, y) = (-1)^{|x|}\Delta(xy) - (-1)^{|x|}\Delta(x)y - x\Delta(y)$ the associated BV-antibracket. Moreover, in comparison to the earlier work, the validity of the master equation is independent of the actual decomposition of the moduli space of punctured Riemann surfaces used to construct the interaction terms in S_{CSFT}. It is also interesting to note that the interaction terms involving the "wrong" ghost numbers describe the structure constants of the gauge-transformations and their integrability conditions, such as Bianchi identities.

In the classical limit $\hbar \to 0$, the quantum master equation reduces to the classical master equation,

$$(S_{\text{CSFT}}, S_{\text{CSFT}}) = 0. \tag{1.5}$$

This is the central equation for classical string field theories. After expanding the action in powers of the field variables, the master equation imposes quadratic relations on the coefficients and it turns out that such solutions give rise to the structure of a *cyclic homotopy Lie algebra* or *cyclic L_∞-algebra*[1] on the Hilbert space of the string and, more interestingly, this structure is not preserved or *stable* under passing to the cohomology $H^\bullet(Q)$ of Q. The latter structure is called the *minimal model* and it contains the tree-level Ward identities [21] as well as the S-matrix elements of the theory. Elements of $H^\bullet(Q)$ should be thought of as scattering states. Likewise, in open string field theories the Hilbert space is endowed with the structure of a *cyclic homotopy associative algebra* or *cyclic A_∞-algebra* [22]. We saw in equations (1.1) that Witten's open bosonic string field theory describes a differential associative al-

[1] Roughly speaking, a homotopy Lie algebra is a vector space with a bracket operation $[\cdot,\cdot]$ and differential Q that satisfies all axioms of a (differential graded) Lie algebra, but the Jacobi identity is only required to hold up to Q-exact terms. Cyclicity means that there is also an invariant inner product that generalises the Killing form of a Lie algebra. We refer to section 2.3 for more details.

gebra with a cyclically invariant trace[2]. This structure is indeed a special case of an A_∞ algebra. However, it is not stable under passing to its minimal model [23, 24] so that the stable algebraic structure is indeed an A_∞-algebra [25]. The construction of a decomposition of the moduli space of punctured, closed Riemann surfaces in terms of minimal area metrics generalises to the full open-closed moduli space, so that bosonic open-closed string field theory could be constructed along the same lines [26]. At the classical level the relevant algebraic structure was called an *open-closed homotopy algebra (OCHA)* by Kajiura and Stasheff [27, 28]. An OCHA can be identified with a deformation problem of the open string background that is controlled by the closed string L_∞-algebra. Perhaps the most famous examples for an OCHA is Kontsevich's deformation quantisation [29] and topological strings [30, 31]. In [32] the algebraic properties of solutions to the quantum open-closed master equation were discussed. The authors named that structure *quantum open-closed homotopy algebra (QOCHA)*.

Up to this point the discussion was solely for bosonic string theories. On the other hand, superstring field theories are much less understood. The main obstacle towards progress is the lack of solid mathematical foundations to help one to identify the relevant algebraic structures. Most of the algebraic structure of bosonic string field theories come from geometric properties of the appropriate moduli spaces of punctured Riemann surfaces whose study goes back to Riemann himself. But the superstring world-sheets are subject to a much larger gauge-symmetry that gives rise to a different moduli space, the *supermoduli space*. The prefix *super*- refers to the fact that it is has even and odd directions. The underlying reduced space is isomorphic to the moduli space of spin curves, i.e. the space of pairs of Riemann surfaces equipped with a spin structure, which allows one to define spinors on the world-sheet[3]. This moduli space was first defined and studied by Cornalba [33]. In view of the later chapters we restrict our discussion to punctured superdiscs. Near infinity spin curves can degenerate in two distinct ways that are called Neveu-Schwarz (NS) and Ramond (R)-degenerations. This indicates that one should need two different string fields, the NS string field Φ and the R string field Ψ. In retrospect, superstring field theory was first analysed from the point of this reduced moduli space with the odd directions considered as a decoration. Before the advent of Witten's open string field theory the development of the field theoretical formulation of the dual resonance model and the RNS-model were equally far developed. However, unlike the bosonic version Witten's proposal for open superstring field theory [34] turned out to be ill-defined.

Witten's open string field theory is developed in analogy to bosonic open string field theory. In particular, he postulated a differential graded associative algebra

[2]Another example of a differential graded associative algebra is the cohomology ring $\Omega^\bullet(M)$ of a closed manifold M. Here, grading is given by form degree and the exterior differential d corresponds to the BRST operator Q. The star product $*$ is given by the wedge product $\wedge$ of differential forms. Finally, the trace operation amounts to integrating over the total manifold.

[3]In fact, in the mathematical literature spin curves are defined as algebraic curves equipped with a theta characteristic. The equivalence of spin structures and theta characteristics in two dimensions was established by Atiyah.

with invariant trace operation as in (1.1). However, the actual definition of the operations $*$ and $\int$ and the space of field configurations $\mathcal{H}$ were different. The difficulties can be traced back to the presence of *pictures* [35, 36]. In contrast to the b-c ghost system, the superconformal β-γ system has several unitarily inequivalent representations so that the state space of the world-sheet theory comes in infinitely many copies that are distinguished by an integer for NS-states and a half-integer for R-states called *picture*. Witten takes the string field to take values in $\Phi + \Psi \in \mathcal{H} = \mathcal{H}_{\mathrm{NS},-1} \oplus \mathcal{H}_{\mathrm{R},-\frac{1}{2}}$, i.e. the NS string field has picture -1 and the R string field carries picture $-\frac{1}{2}$. For the bosonic string $\int$ was defined by gluing half-strings. But this operation has picture 2 so that it vanishes unless it is evaluated on states with picture -2. Moreover, the bosonic $*$ carries no picture. But the algebra does not close as the product of two NS states results in a picture -2 state and there are similar problems involving R states. The problems are solved by using the picture-changing operator (PCO) $X(z)$ defined in [36]. This local operator is a world-sheet scalar, carries picture $+1$ and is BRST-invariant. $X(z)$ has an inverse $Y(z)$ in the sense that $\lim_{w\to z} X(z)Y(w) = 1$. $Y(z)$ is called the inverse picture-changing operator. The new trace operation $\oint$ and product $\star$ read

$$\oint \Phi = \int Y(\mathrm{i})\Phi, \qquad \oint \Psi = 0, \tag{1.6a}$$
$$\Phi \star \Phi = X(\mathrm{i})(\Phi * \Phi), \qquad \Phi \star \Psi = X(\mathrm{i})(\Phi * \Psi), \tag{1.6b}$$
$$\Psi \star \Psi = \Psi * \Psi, \qquad \Psi \star \Phi = X(\mathrm{i})(\Psi * \Phi). \tag{1.6c}$$

In this definition the position i denotes the string mid-point. Formally, these definitions satisfy the same axioms as bosonic string field theory. The complete action and equations of motion read

$$S_{\mathrm{OSSFT}} = \frac{1}{2}\int \Phi * Q\Phi + \frac{1}{3}\int X(\mathrm{i})(\Phi * \Phi * \Phi) + \frac{1}{2}\int Y(\mathrm{i})\Psi * Q\Psi + \int \Phi * \Psi * \Psi, \tag{1.7a}$$
$$0 = Q\Phi + X(\mathrm{i})(\Phi * \Phi) + \Psi * \Psi, \tag{1.7b}$$
$$0 = Q\Psi + X(\mathrm{i})(\Phi * \Psi + \Psi * \Phi). \tag{1.7c}$$

At this point one can already see two problems with this action. Let us consider the gauge-invariance of (1.7b) under an infinitesimal bosonic gauge transformation with parameter Λ. According to (1.2b) the fields transform as

$$\delta\Phi = Q\Lambda - X(\mathrm{i})(\Lambda * \Phi - \Phi * \Lambda), \tag{1.8a}$$
$$\delta\Psi = -X(\mathrm{i})(\Lambda * \Psi - \Psi * \Lambda). \tag{1.8b}$$

Upon checking gauge-invariance explicitly, one encounters an ill-defined product of local operators in the form $X(i)^2$. Similarly the same operator is encountered, when studying perturbative solutions to the equations of motion. In [37] this problem was analysed by considering the tree-level four boson amplitude. It was found that adding a suitable counter term to the action reproduces the correct Koba-Nielsen amplitude and at the same time restores gauge-invariance to this order. However, further analysis reveals that the problems reoccur at the next order, so that an infinite number of counter terms is required. In an attempt to solve said problems while at the same time saving Witten's form of the action the *modified theory* was proposed [38, 39]. In the previously used notation, the authors define a new NS string field $\Phi' = X(\mathrm{i})\Phi$ and keep the same R string field. Formally, the equations of motion and gauge-invariance read as (for bosonic gauge-parameter Λ' and fermionic gauge-parameter χ),

$$\begin{aligned}
0 &= Q\Phi' + \Phi' * \Phi' + X(\mathrm{i})(\Psi * \Psi),\\
0 &= Q\Psi + \Phi' * \Psi + \Psi * \Phi',\\
\delta\Phi' &= Q\Lambda' - \Lambda' * \Phi' + \Phi' * \Lambda' + X(\mathrm{i})(\Psi * \chi - \chi * \Psi),\\
\delta\Psi &= Q\chi - \Lambda' * \Psi + \Psi * \Lambda' - \chi * \Phi + \Phi * \chi.
\end{aligned}$$

In this form all tree-level amplitudes involving only bosons are finite and reproduce the correct four-point amplitude. But amplitudes with fermions and gauge transformations with non-zero χ again produce singularities due to operator collisions.

A different approach to open string field theory was developed by Berkovits [40–42]. This approach relies heavily on a chain of embeddings of string vacua with $\mathcal{N} = 0$ supersymmetry into string vacua with $\mathcal{N} = 1$ supersymmetry into $\mathcal{N} = 2$ supersymmetric string vacua [43]. The embedding is constructed by twisting the ghosts of the theory and showing that the original matter+ghost theory can be identified with the matter part of an enlarged supersymmetry algebra. Most interestingly, the central charges are such that one can couple them to a world-sheet supergravity again, but the theory being insensitive to the new geometric structure on the world-sheet, s.a. the spin structure for the first embedding and the $\mathrm{U}(1)_R$-connection for the second embedding. Consequently, the scattering amplitudes calculated in the enlarged theory coincide with the original ones and it is sufficient to develop a string field theory only for $\mathcal{N} = 2$ superstrings. Moreover, $\mathcal{N} = 2$ world-sheet theories with central charge $\hat{c} = 2$[4] automatically enjoy an $\mathcal{N} = 4$ superconformal symmetry [44]. Combining these facts Berkovits proposed an open string field theory that should calculate $\mathcal{N} = 2$ amplitudes of vertex operators that are invariant under R-symmetry transformations. The result takes its simplest form when expressed in terms of $\mathcal{N} = 1$ world-sheet quantities. The string field Φ is an element in the so-called large Hilbert space [35] of picture 0 and is considered a commuting field. In the large Hilbert space there is another operator η of picture -1 whose kernel coincides with the conventional/small Hilbert space and that anticommutes with the BRST-charge Q. The open string field theory action takes a WZW-like form,

$$S_{\mathrm{WZW}} = \frac{1}{2}\int\left((e^{-\Phi}Qe^{\Phi})(e^{-\Phi}\eta e^{\Phi}) - \int_0^1 dt\,(e^{-t\Phi}\frac{\partial}{\partial t}e^{t\Phi})[(e^{-t\Phi}Qe^{t\Phi}),(e^{-t\Phi}\eta e^{t\Phi})]_+\right), \tag{1.9}$$

[4] The central charge represents a quantum anomaly of the classical superconformal symmetry. It commutes with all observables and its value depends on the field contents of the theory.

where $\int$ formally is the same trace operation as in Witten's theory, but carries picture 1 in the large Hilbert space, and string fields are multiplied using Witten's star product introduced earlier. Despite the elegance of this theory, it suffers from some serious short-comings. First, the action only describes bosonic spacetime degrees of freedom. Second, it is not clear that it gives the correct tree-level S-matrix elements. In his original proposal Berkovits only argued that this action gives the correct physical spectrum and three boson amplitude. Third, it is not clear how to quantise this theory. In the traditional approaches to string field theory, after relaxing the ghost number constraint a solution to the quantum master equation was found so that a BV-quantisation should be possible. For S_{WZW} its BV-quantization is less obvious. Last but not least, it obscures the impact of the geometry of the supermoduli space onto the algebraic structure.

As explained earlier, the non-existence of cubic vertices leading to a cover of the moduli space of punctured spheres suggests that extending the construction from open superstring field theories to closed superstring field theories is highly non-trivial. Due to complications with picture changing operators, the effort was concentrated on finding heterotic WZW-like superstring field theory. The main difficulty is that there is no closed expression for a pure-gauge closed string field configuration, which would generalise e^{Φ} in the WZW-like theory. Using an implicit description of such configurations Berkovits, Okawa and Zwiebach eventually constructed a gauge-invariant action in [45, 46]. In [45] the authors made an interesting observation: The elementary vertices do not only receive contributions from the missing regions of the bosonic moduli space, but also the boundaries of the already covered regions give rise to additional vertices. The geometric origin of these corrections is not understood, but seem to require a deeper understanding of the geometry of the supermoduli space of super Riemann surfaces [47–49]. A similar phenomenon was observed in [50] and attributed to a mismatch in the choice of position of the picture changing operators near the boundaries of the cells.

The modern developments of superstring field theory started with [51], in which it was shown that Witten's OSFT and the Berkovits WZW-like action are related by a partial gauge-fixing up to quartic order, and with [52], in which we derived a complete gauge-invariant action for the classical open NS-superstring based on cyclic A_∞-algebras. Thereafter we generalised the latter construction to include heterotic and type II-superstrings as well [53]. Moreover we found gauge-invariant equations of motion for the complete superstring theories, including the Ramond sectors, in [54]. Eventually, we showed in [55] that the newly found formulations reproduce the correct perturbative tree-level S-matrix. The work of [51] was extended and a complete correspondence between the Witten and Berkovits formulation was established [56–58].

From the world-sheet point of view the most recent achievements is the formulation of quantum type IIB closed superstring field theory [59] and the geometric construction of the 1-PI action [60–62] that ultimately led to a proposal for a BV-master action for type IIB-superstrings and heterotic strings in [63]. Quite recently, complete algebraic constructions of gauge-invariant actions for open superstrings based on Witten's star product have been given in [64] and in [65]. In particular [64] can be regarded as an algebraic implementation of the construction from [60].

Finally, we remark that perturbative superstring theory has not been constructed beyond two-loops [66–72]. The recent results of Sen and Witten [50] indicate that it is possible to construct finite scattering amplitudes using the formalism of picture changing operators alone. Together with [73] this seems to imply that string field theory is necessary to make superstring perturbation theory consistent and well-defined to all orders.

1.3 Content of this thesis

In chapter 2 we review important background material. The main objective is to provide a context for the material presented in the forthcoming chapters. In particular we want to draw a connection with conventional superstring theory. In section 2.1, we begin with a review of the geometry of bordered type II world sheets with an emphasis on their deformation theory. Next, in section 2.2, we construct measures on supermoduli space for a Minkowski background. The main theoretical tool is quantum BV theory. Homotopy associative algebras and homotopy Lie algebras are reviewed in section 2.3. We conclude this chapter after section 2.4 with a discussion of integrating the measure over supermoduli space. We discuss the contributions from the various regions of supermoduli space. Most of the construction is entirely analogous to bosonic string field theory. However, the amplitude receives contributions from chains that project to a point in bosonic moduli space, but still have positive even dimension. We argue that these new terms can be absorbed into adding infinitely many vertices to the action and that they satisfy the relations of a cyclic A_∞ algebra. This gives an alternative way to find the correction terms and can be interpreted as constructing patches that fill in the missing regions of supermoduli space, with those holes being topologically a point.

In chapter 3 we give all correction terms for the NS sector of open superstring theory explicitly. The solution is entirely algebraic and employs the large Hilbert space. The final vertices preserve the small Hilbert space. The vertices are constructed recursively starting from Witten's star product. Chapter 4 extends the recursive construction to all consistent decompositions of bosonic moduli space and uses it to construct NS heterotic string theory and NS-NS closed type II superstring theory. During the construction some unnatural choices are made. We discuss the dependence of the final result on these choices in section 4.5.

Inclusion of the Ramond sector for all superstring field theories is achieved in chapter 5. Due to difficulties with inverting the Poisson bracket in the Ramond sector, the results remain restricted to the level of the equations of motion. Inclusion of the Ramond sector allows for a discussion of the realisation of spacetime supersymmetry in open superstring field theory. We discuss this in section 5.6 and show that $\mathcal{N} = 1$ supersymmetry is indeed realised at the level of the equations of motion, but the algebra closes only up to gauge transformations.

As a cross-check that the previously described constructions indeed describe superstring theory, we calculate their classical S-matrix in chapter 6. The S-matrix is constructed using homological perturbation theory and the recursive form of the construction of the string vertices makes evaluating the S-matrix very efficient. We discuss the relation of the field theory S-matrix in Siegel gauge with the minimal model from the theory of homotopy algebras. The proof exploits the recursive nature of the constructions from chapters 3, 4 and 5.

When restricting to the open superstring based on Witten's vertex, one can improve on the results from chapter 5. In chapter 7, we describe some problems arising when inverting the Poisson bracket structure in the R-sector and propose a set of cyclic, combined NS and R vertices. For the kinetic term we offer two alternatives, the first is based on Sen's suggestion [60] and the second uses the restricted Hilbert space, e.g. [59, 74, 75]. Both actions are gauge-invariant and reproduce the correct perturbative S-matrix.

1.4 Published papers

Chapters 3-7 are in parts verbatim reproductions of the content of the author's publications. Some of the results presented in this thesis have been published in the following papers

[52] Erler, T., **Konopka, S.** and Sachs, I., *Resolving Witten's superstring field theory*, **JHEP 1404(2014) 150**, arXiv:1312.2948

[53] Erler, T., **Konopka, S.** and Sachs, I., *NS-NS Sector of Closed Superstring Field Theory*, **JHEP 1408(2014) 158**, arXiv:1403.0940

[54] Erler, T., **Konopka, S.** and Sachs, I., *Ramond Equations of Motion in Superstring Field Theory*, **JHEP 1511(2015) 199**, arXiv:1506.05774

[55] **Konopka, S.**, *The S-Matrix of superstring field theory*, **JHEP 1511(2015) 187**, arXiv:1507.08250

[64] **Konopka, S.** and Sachs, I., *Open superstring field theory on the small Hilbert space*, **JHEP 1604(2016) 164**, arXiv:1602.02583

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Chapter 2

Geometric construction of type II superstring theory

During the long history of string theory several approaches to the perturbative superstring S-matrix have been developed. For the Green-Schwarz [76] and the pure spinor formulations [77] quantisation of the world sheet theory at arbitrary genus is non-trivial and has not been formulated in a covariant way. The Ramond–Neveu–Schwarz (RNS) formulation is the mathematically most robust approach. It expresses the superstring S-matrix as an integral of a particular measure over the supermoduli space of world sheets. Traditionally one integrates over the odd directions first. This procedure modifies the picture of the vertex operators representing the asymptotic states. Then, one performs the integral over the remaining bosonic directions. Bosonic string field theory relies heavily on the factorisation properties of the world sheet near infinity, where a non-trivial cycle pinches off. The shape of the moduli space near infinity turns out to constitute of copies of the moduli space for lower genera or lower number of punctures in a way that reproduces the combinatorics of Feynman graphs when one internal line is cut. This suggests that the S-matrix can be calculated as a perturbation series for an action, the string field theory action. In superstring field theory one would like to pursue a similar line of arguments and calculate the superstring S-matrix as a Feynman perturbation series. Most steps work analogously to the bosonic string, but there are a few additional subtleties.

In this chapter we sketch the construction of type II superstring theory from the supermoduli point of view. Most of the material is standard, but we include it to bridge the gap between the introduction and the actual results presented in the forthcoming chapters. In section 2.1 we review the description of type II world sheets with boundaries and punctures in terms of G-structures and discuss their deformations from a 2d supergravity point of view. In particular we are interested in finding parametrisations of such structures near infinity. In section 2.2 we review the BV-formulation of integration theory on superstacks M/G, where M is a supermanifold and G denotes a Lie supergroup and apply this formalism to the construction of the conventional superstring measure. The main result here is the construction of the pseudoforms $\Omega^{r|s}$ on supermoduli space of type II world sheets with a choice of superconformal frame near each puncture. Sections 2.1 and 2.2 are quite technical and may be omitted on a first reading. Section 2.3 reviews the definition of homotopy associative and homotopy Lie algebras and their connection with solutions to classical BV-master equations. Finally, we merge the geometric and algebraic techniques in section 2.4 and explain how to perform the integration over supermoduli space. In particular we are concerned with establishing a connection with integrals over the reduced moduli space and a choices of odd directions near the split locus inside supermoduli space. Since supermoduli space is not holomorphically fibred over the split locus, we argue that correction terms arise from the boundaries of the vertices. Unfortunately, the purely geometric approach is not completely developed at the moment, so that we restrict to classical open superstring theory, i.e. we work at genus 0, one boundary and no bulk punctures. We describe the expected algebraic structures and their properties and restate the integration problem as an algebraic problem.

2.1 The geometry of type II world sheets

Type II world sheets are the configurations of two dimensional superconformal gravity. This theory is quite unusual as there are no equations of motion and it is purely topological in the sense that locally any infinitesimal deformation of a configuration is pure gauge. It possesses, however, a highly non-trivial configuration space once one includes the global degrees of freedom. For our purposes we consider type II world sheets from the smooth point of view, as it makes describing deformations simpler. Moreover, we only describe the structures that we need. For an in depth review see [48].

2.1.1 Deformations of bordered Riemann surfaces

We begin with describing the configuration space of conformal gravity, which is the non-super symmetric analogue of our configuration space. Configurations are differentiable, two dimensional manifolds equipped with a conformal structure. More precisely, we choose an open cover U_α and on each patch a pair of complex-valued differential forms e^z_α and $e^{\bar{z}}_\alpha$, a conformal frame. On overlaps we require the forms to be related by a conformal transformation, i.e. on overlaps $U_\alpha \cap U_\beta$ there exist complex-valued, nowhere vanishing smooth functions $\lambda_{\alpha\beta}, \bar{\lambda}_{\alpha\beta}$, s.t.

$$e^z_\alpha = \lambda_{\alpha\beta} e^z_\beta, \qquad e^{\bar{z}}_\alpha = \bar{\lambda}_{\alpha\beta} e^{\bar{z}}_\beta. \tag{2.1}$$

Moreover, we require that $e^z_\alpha \wedge e^{\bar{z}}_\alpha$ vanishes nowhere. The last condition ensures that both forms are linearly independent. We also require the torsion constraints,

$$de^z_\alpha \equiv 0 \mod \cdot \wedge e^z_\alpha, \qquad de^{\bar{z}}_\alpha \equiv 0 \mod \cdot \wedge e^{\bar{z}}_\alpha, \tag{2.2}$$

which are empty for conformal gravity. The torsion constraints imply that we can perform a conformal transformation on e^z_α so that the constraint reduces to $de^z_\alpha = 0$, which implies through the Poincaré lemma that $e^z_\alpha = dz_\alpha$ for some complex-valued function z_α. The function z_α is called a local complex coordinate. These coordinates are unique up to conformal transformations, i.e. on double overlaps we have

$$z_\alpha = g_{\alpha\beta}(z_\beta), \tag{2.3}$$

where $g_{\alpha\beta}(z)$ is a holomorphic transformation. Thus, we see that a system of conformal frames determines a complex structure via e^z_α and a second complex structure via $e^{\bar{z}}_\alpha$. Typically we require that e^z_α and $e^{\bar{z}}_\alpha$ are related by complex conjugation so that we just obtain one complex structure. In essence, we have found a one-to-one map between complex structures and conformal frames on a two dimensional manifold.

We now turn to the deformation theory of conformal structures. The most general deformation is

$$\delta e^z_\alpha = \rho_\alpha e^z_\alpha + \mu_\alpha e^{\bar{z}}_\alpha, \tag{2.4}$$

for arbitrary complex valued functions ρ_α and μ_α. Equations (2.1) imply that $\mu_\alpha = \mu_\beta = \mu$. The function ρ_α can always be removed by a local conformal transformation, while μ is invariant under such transformations. Since we can always perform a global reparametrisation, we must divide out deformations of the form

$$\delta e^z_\alpha = \mathcal{L}_V e^z_\alpha,$$

for a complex-valued vector field V. In a local conformal frame we therefore have the ambiguity

$$\mu \sim \mu + \bar{\partial} V^z. \tag{2.5}$$

The last condition is of course nothing else that the defining condition for a Beltrami differential $\mu \in H^0(K^{-1} \otimes \Omega^{(0,1)})/\bar{\partial} H^0(K^{-1} \otimes C^\infty) \cong H^1(K^{-1})$. Thus, we conclude that tangent vectors to the configuration space of conformal gravity are given by the Beltrami differentials. The space of cotangent vectors is given by $H^1(K^{-1})' \cong H^0(K^2)$ by *Serre duality*.

By the *doubling trick*, any bordered or unoriented manifold Σ can be obtained as a quotient of an oriented manifold Σ_0, its double, by the $\mathbb{Z}_2$-action of an orientation reversing diffeomorphism ρ of order 2. The last statement means that it is an involution, $\rho^2 = \mathbf{1}$. In this language the boundary $\partial\Sigma$ is the fixed point set of ρ. It is not hard to see that for every family ρ_t of choices for the involution one can find a family of diffeomorphisms f_t, s.t. $\rho_t = f_t \rho_0 f_t^{-1}$. Thus, introducing a boundary does not introduce any continuous moduli, but only discrete moduli that correspond to the various bordered or unoriented manifolds with the same double Σ_0. Henceforth we assume that a choice for ρ has been made and we only consider boundary preserving diffeomorphisms. By a boundary preserving diffeomorphism we mean a diffeomorphism f of Σ_0 so that $f\rho = \rho f$. Given a conformal frame e^z_α on Σ_0, the involution ρ naturally defines another conformal frame $\rho^* e^z_\alpha$. In general this frame is not equivalent to e^z_α or $e^{\bar{z}}_\alpha$. Since ρ is orientation reversing, it changes the sign of $2ie^z_\alpha \wedge e^{\bar{z}}_\alpha$ and therefore $\rho^* e^z_\alpha$ can never be equivalent to e^z_α. However, if it turns out to be equivalent to $e^{\bar{z}}_\alpha$, the involution ρ is called *antiholomorphic* and e^z_α defines a conformal frame on the bordered surface Σ.

If $p \in \Sigma_0$ is not a fixed-point, we can find disjoint neighbourhoods of p and $\rho(p)$ and any complex coordinate z near p defines a complex coordinate w near $\rho(p)$ via $w = \overline{\rho^* z}$. But locally near p there it looks exactly like a closed Riemann surface. However, if p is a fixed-point of ρ, the situation is more interesting. z and w are complex coordinates near the same point and are therefore related by a conformal map. By a judicious choice of z one can always achieve $z = w$. Hence, near the boundary we can find coordinates in which the antiholomorphic involution is given by $\rho^* z = \bar{z}$. Of course this is just the statement that a Riemann surface with a boundary looks like the upper-half plane near that boundary.

Deformations of bordered Riemann surfaces can be described in terms of its closed double Σ_0. We have already seen that introducing ρ does not add any continuous moduli, so that we only need to consider deformations of the conformal frame on Σ_0 such that ρ stays antiholomorphic. For a Beltrami differential μ this condition reads

$$\rho^* \left(dz + \mu \, d\bar{z}\right) \propto d\bar{z} + \bar{\mu} \, dz, \tag{2.6}$$

from which it follows that $\rho^* \mu = \bar{\mu}$. Beltrami differentials are therefore completely determined by their values on Σ and must be real along the boundary. If we think of μ as K^{-1}-valued $(0,1)$-forms we have to choose K^{-1} as the sheaf of holomorphic vector fields that are tangential to the boundary. (Global) diffeomorphisms have to preserve the involution ρ, which implies that the vector field generating a family of diffeomorphisms has to be tangential to the boundary and that it is completely fixed by its values on Σ. Consequently, we have the same identification of Beltrami differentials as in (2.5), but with V suitably restricted. Eventually, the tangent space near a bordered Riemann surface is $H^1(K^{-1})$ as in the unbordered case, but with K^{-1} interpreted as before.

2.1.2 Deformations of type II world sheets

The description of type II world sheets is a non-trivial generalisation of the description of bosonic world sheets given in section 2.1.1. The basic underlying object for type II world sheets is a real $(2|2)$-dimensional supermanifold Σ. This means that locally Σ is parametrised by two even, real coordinates x^μ, $\mu = 1, 2$ and two odd coordinates θ^i, $i = 1, 2$. More concretely, topologically Σ is just a smooth two-dimensional manifold classified by its genus g, on top of which we choose a real, rank 2 vector bundle $V \to \Sigma$. The supermanifold structure comes from assigning to an open set U the algebra of smooth sections $\Sigma(U) \equiv \Gamma(U, \bigwedge^\bullet V)$. The structure theorem for supermanifolds [49, 78] ensures that this is the most general smooth supermanifold of dimension (2|2). On sufficiently small open sets any vector bundle becomes trivial and we call denote by θ^i some choice of local trivialisation. It is clear that any function f on Σ is locally of the form

$$f(x,\theta) = a(x) + \theta^i b_i(x) + \theta^1\theta^2 c(x),$$

where a, b_i and c are smooth functions. In addition to the topological data, given by the genus g, the construction depends on the choice of rank 2 vector bundle $V \to \Sigma$. For type II world sheets this choice is not arbitrary as we will see later, so that no additional parameters are added. Supermanifolds with boundaries can be defined in several ways. For our purposes it is best to extend the doubling trick from Riemann surfaces and define a supermanifold Σ with boundary in terms of a suitable orientation reversing superdiffeomorphism ρ acting on the double Σ_0. Here, Σ_0 is just a closed oriented (2|2)-dimensional supermanifold built from an orientable vector bundle V. In defining ρ one has to be more careful. Every superdiffeomorphism induces a map between the underlying vector bundles, so that a superdiffeomorphism could change the orientation of the underlying manifold and/or the vector bundle. We choose ρ to reverse the orientation of both the base manifold and the vector bundle. Similar to the discussion in section 2.1.1, this procedure adds no new continuous moduli, as every infinitesimal deformation of ρ can be removed by an infinitesimal superdiffeomorphism. The remaining superdiffeomorphisms are determined by the condition that they keep the involution ρ fixed.

Type II world sheets are the configurations of superconformal gravity. Their definition proceeds along the same lines as we introduced Riemann surfaces through a choice of conformal frame. We define a superconformal frame through the coframe fields e^z_α, e^θ_α and their bared variants $e^{\bar z}_\alpha$ and $e^{\bar\theta}_\alpha$. e^z_α and $e^{\bar z}_\alpha$ are smooth even 1-forms and e^θ_α and $e^{\bar\theta}_\alpha$ are smooth odd 1-forms on Σ. These forms should constitute a basis for all 1-forms. We denote the canonically dual vector fields by ∂, D_θ, $\bar\partial$ and $D_{\bar\theta}$ with the obvious correspondence with the coframe fields[1]. Moreover, we require that the coframe fields satisfy the torsion constraints,

$$de^z_\alpha - e^\theta_\alpha \wedge e^\theta_\alpha \equiv 0 \mod \cdot \wedge e^z_\alpha \tag{2.7a}$$
$$de^{\bar z}_\alpha - e^{\bar\theta}_\alpha \wedge e^{\bar\theta}_\alpha \equiv 0 \mod \cdot \wedge e^{\bar z}_\alpha. \tag{2.7b}$$

[1]There are various sign conventions in supergeometry. We use the homological convention in which we set $AB = (-1)^{p(A,B)}BA$, where $p(A,B) = \sum_i \deg_i(A)\deg_i(B)$ and $\deg_i(A)$ denotes the ith grading of A. In this convention we have

$$\begin{aligned} \iota_V df &= V(f), & \mathcal{L}_V &= [d, \iota_V],\\ V^\mu &= V(x^\mu), & V^\mu\partial_\mu &= V,\\ \omega_\mu &= \iota_{\partial_\mu}\omega, & dx^\mu\,\omega_\mu &= \omega,\\ [\mathcal{L}_V, \iota_W] &= \iota_{[V,W]}, & [\mathcal{L}_V, \mathcal{L}_W] &= \mathcal{L}_{[V,W]},\\ \iota_{\partial_i} e^j &= \delta_i{}^j, & d &= dx^\mu\,\partial_\mu, \end{aligned}$$

where V, W are vector fields (derivations on the algebra of functions), x^μ are coordinates and e^i a coframe field and ∂_i their canonically dual vector fields. The homological convention agrees with the physics convention if there is only one relevant grading and is related to it by a suitable Klein cocycle in the general case.

Two superconformal frames are gauge equivalent, if they are related by a local GL(1|1)-transformation that preserves the torsion constraints and do not mix holomorphic with antiholomorphic coframe fields. The precise form of the transformations are not needed, but it suffices to know that every superconformal frame is gauge equivalent to a frame in which (2.7a) and (2.7b) read

$$de^z_\alpha - e^\theta_\alpha \wedge e^\theta_\alpha = 0 \tag{2.8a}$$
$$de^\theta_\alpha = 0. \tag{2.8b}$$

From this condition we infer the existence of an odd function θ_α with $e^\theta_\alpha = d\theta_\alpha$ and the existence of an even coordinate z_α with $dz_\alpha + \theta_\alpha\, d\theta_\alpha = e^z_\alpha$. The pair $(z_\alpha, \theta_\alpha)$ is called a *superconformal coordinate system*. This coordinate system is not unique, but any other choice of superconformal coordinates $(z'_\alpha, \theta'_\alpha)$ is related to it via a transformation of the form

$$z'_\alpha = f(z_\alpha) + \theta_\alpha \rho(z_\alpha) \tag{2.9a}$$
$$\theta'_\alpha = \theta_\alpha \kappa(z_\alpha) + \lambda(z_\alpha), \tag{2.9b}$$

where f and κ are even holomorphic functions and ρ and λ are odd holomorphic functions and are subject to the constraints

$$\rho(z_\alpha) = \kappa(z_\alpha)\lambda(z_\alpha) \tag{2.10a}$$
$$f'(z_\alpha) = \kappa(z_\alpha)^2 + \lambda'(z_\alpha)\lambda(z_\alpha). \tag{2.10b}$$

In particular, the transformations rule (2.9) describes the relation between superconformal coordinates in different coordinate patches. Restricting to the transition functions f only, we see that every type II world sheet is also endowed with the structure of a Riemann surface. Any manifold endowed with a system of superconformal coordinate frames is called a *super Riemann surface (SRS)*. In general a super Riemann surface is a complex (1|1)-dimensional supermanifold endowed with a maximally non-integrable odd distribution generated locally by D_θ. Maximally non-integrable means that $[D_\theta, D_\theta]$ is everywhere linearly independent from D_θ. In our case the torsion constraints (2.7) ensure this maximal non-integrability. Note that a type II world sheet has more structure than just an SRS. The antiholomorphic analogues of (2.9) give rise to a second SRS structure whose underlying complex structure is the complex conjugate of the first one, but the remaining transition functions may not be related to each other at all.

Type II world sheets with boundary are defined in close analogy to ordinary Riemann surfaces. If ρ denotes the involution defining the bordered supermanifold, we require in addition that ρ be antiholomorphic in the sense that the superconformal frame $(\rho^* e^z_\alpha, \rho^* e^\theta_\alpha)$ should be gauge equivalent to $(e^{\bar z}_\alpha, e^{\bar\theta}_\alpha)$, i.e. it should swap the holomorphic and the antiholomorphic structure. Near the boundary we can find superconformal coordinates (z, θ) and $(\bar z, \bar\theta)$, such that

$$\bar z = \rho^* z + \rho^*\theta\, \kappa(\rho^* z)\lambda(\rho^* z) \tag{2.11a}$$
$$\bar\theta = \rho^*\theta\, \kappa(\rho^* z) + \lambda(\rho^* z). \tag{2.11b}$$

We can assume that $f(z) = z$, since the holomorphic function f would give the relation between z and $\bar{z}$ in the underlying Riemann surface and we know that they can be set to 1 by a suitable change of complex coordinates. The remaining parameter λ can be removed by a superconformal transformation in (z, θ), while keeping $(\bar{z}, \bar{\theta})$ fixed. In summary, near the boundary there is a coordinate frame with

$$\bar{z} = \rho^* z, \qquad \bar{\theta} = \rho^* \theta. \tag{2.12}$$

There is a convenient way to think about SRS. For simplicity we assume that our SRS has no odd moduli. This means that in (2.9) the odd function λ vanishes and we are left with f and κ satisfying $f' = \kappa^2$ on overlaps. We know already that f can be interpreted as holomorphic transition functions between charts. Because of the relation $f' = \kappa^2$ we can think of the functions κ as transition functions for a line bundle $\mathcal{L}$ with $\mathcal{L}^2 \cong K$. A Riemann surface equipped with a square root of its canonical bundle is called a *spin curve*. Therefore, the reduced moduli space of SRS is just the same as the moduli space of spin curves $\mathcal{SM}$. Two square roots $\mathcal{L}$ and $\mathcal{L}'$ differ by tensoring with a square root of the trivial line bundle. On a genus g surface there are 2^{2g} such bundles. The moduli space of genus g spin curves, $\mathcal{SM}_g$, is, hence, a 2^{2g}-sheeted cover of the bosonic moduli space $\mathcal{M}_g$ and consists of two connected components that correspond to whether the spin structure is even or odd. The odd coordinate θ may therefore be thought of as a local section of $\mathcal{L}$. It can be thought of as a real linear combination of the local basis sections θ^1 and θ^2 of the underlying smooth supermanifold. In the bordered case ρ exchanges θ and $\bar{\theta}$. It would induce an isomorphism between $\mathcal{L}$ and $\bar{\mathcal{L}}$ if ρ would preserve the orientation of the vector bundle V. But this is not possible unless $g = 0$ and we work with the R-R spin structure for which $\mathcal{L} = \mathbf{1}$. Hence, ρ must reverse the orientation of V if there are to be non-trivial examples of bordered type II world sheets. If we want to allow for odd parameters in the transition functions, we need to enlarge the dimension of the vector bundle V determining the smooth supermanifold. Then, we can select different lines for different patches over the same point. This means that θ and θ' need not be proportional to each other anymore, but may differ by an odd parameter λ in such a way that D_θ and D'_θ are still proportional to each other.

Deformations of type II world sheets are studied in terms of superconformal gravity. We are interested in deformations $\delta e^z, \delta e^\theta, \delta e^{\bar{z}}, \delta e^{\bar{\theta}}$ of the superconformal frame. A general deformation would have 16 superfield parameters, which make up 64 smooth real parameters. These deformations must preserve the torsion constraint (2.7) which imposes 8 superfield valued algebraic conditions. Moreover, we have to take into account local gauge transformations with 4 superfield valued parameters and 4 generators of superdiffeomorphisms. After a partial gauge-fixing the physical deformations can be parametrised as follows

$$\delta e^z = d\bar{z}(\mu + \theta\chi), \qquad \delta e^\theta = \frac{1}{2} d\bar{z}(\chi + \theta\partial\mu) \tag{2.13a}$$

$$\delta e^{\bar{z}} = dz(\bar{\mu} + \bar{\theta}\bar{\chi}), \qquad \delta e^{\bar{\theta}} = \frac{1}{2} dz(\bar{\chi} + \bar{\theta}\bar{\partial}\bar{\mu}). \tag{2.13b}$$

The *Beltrami differentials* μ and $\bar{\mu}$ parametrise deformations of the complex structure and are smooth even functions independent of θ and $\bar{\theta}$. χ and $\bar{\chi}$ are the *world sheet gravitinos* and are smooth, odd functions independent of θ and $\bar{\theta}$. They parametrise deformations of the type II world sheet in the odd directions. μ and χ are subject to a gauge equivalence,

$$\mu \sim \mu + \bar{\partial} v, \qquad \chi \sim \chi + 2\bar{\partial} s, \tag{2.14a}$$

$$\bar{\mu} \sim \bar{\mu} + \partial \bar{v}, \qquad \bar{\chi} \sim \bar{\chi} + 2\partial \bar{s}, \tag{2.14b}$$

where $v\partial + \bar{v}\bar{\partial}$ is a real smooth vector field generating reparametrisations. s and $\bar{s}$ are smooth sections of $\mathcal{L}^{-1}$ and $\bar{\mathcal{L}}^{-1}$, respectively. We call such sections world sheet spinors. They correspond to local world sheet supersymmetry transformations. From (2.14) it follows that

$$\mu \in H^0(\Sigma, K^{-1} \otimes \Omega^{(0,1)})/\bar{\partial} H^0(\Sigma, K^{-1} \otimes C^\infty) \cong H^1(\Sigma, K^{-1}),$$
$$\chi \in H^0(\Sigma, \mathcal{L}^{-1} \otimes \Omega^{(0,1)})/\bar{\partial} H^0(\Sigma, \mathcal{L}^{-1} \otimes C^\infty) \cong H^1(\Sigma, \mathcal{L}^{-1}).$$

Similar formulas hold for $\bar{\mu}$ and $\bar{\chi}$. The infinitesimal superdiffeomorphism V corresponding to v and s is of the form

$$V = v\partial + \frac{1}{2}\partial v\, \theta D_\theta + s\,(D_\theta - 2\theta\partial) + \text{c.c.} \tag{2.15}$$

More importantly, we can use (2.15) and (2.13) to find the generators the stabiliser group of the standard frame. This requires $\bar{\partial} v = 0$ and $\bar{\partial} s = 0$ so that v and s are holomorphic functions. Denote a vector field of the form (2.15) with $s = 0$ by V_v and one with $v = 0$ by W_s, we find the algebra,

$$[V_v, V_{v'}] = V_{v\partial v' - v'\partial v}, \tag{2.16a}$$

$$[V_v, W_s] = W_{v\partial s - \frac{1}{2}s\partial v}, \tag{2.16b}$$

$$[W_s, W_{s'}] = V_{2ss'}. \tag{2.16c}$$

This algebra is just a version the *super Witt algebra* and these vector fields are just the linearisations of (2.9). In addition there is another copy of this algebra for the antiholomorphic vector fields.

Deformations of bordered type II world sheets can be similarly analysed. The only difference is that this time we have to restrict the superdiffeomorphism invariance to superdiffeomorphisms that preserve the antiholomorphic involution ρ and that our generic deformation should preserve antiholomorphicity of ρ. For a vector field V the first condition gives in local superconformal coordinates $\rho^* V^z = V^{\bar{z}}$ and $\rho^* V^\theta = V^{\bar{\theta}}$. It can be shown that the partial gauge-fixing needed to obtain (2.13) is compatible with this requirement. Eventually, the conditions on the Beltrami differentials and the world sheet gravitinos are

$$\rho^*\mu = \bar{\mu}, \qquad \rho^*\chi = \bar{\chi}, \tag{2.17a}$$

$$\rho^* v = \bar{v}, \qquad \rho^* s = \bar{s}. \tag{2.17b}$$

Contrary to the borderless world sheet, near the boundary these conditions remove one copy of the super Witt algebras.

2.1.3 Parametrisations near infinity

The algebraic structure of string field theory is dictated by the behaviour of the string scattering amplitudes near the unitarity cuts where the amplitude becomes non-analytic when the external momenta allow for the creation of an on-shell intermediate particle. Mathematically, the supermoduli space is not compact and the non-analyticity in the scattering amplitudes can be traced back to not fast enough decay of the integration measure near some regions at infinity. In order to understand those asymptotic regions it is enough to consider deformations of closed type II world sheets on subsets that are topologically a cylinder. In this section we review the concept of punctures and their moduli, describe the geometry of type II world-sheets on a cylinder and give an informal discussion of the Deligne-Mumford compactification of supermoduli space.

Consider a cylindrical region equipped with the structure of a type II world sheet. Recall that on overlaps the transition functions take the form (2.9) and that the functions f endow the cylinder with the structure of a Riemann surface. Hence, we can apply the uniformisation theorem to find a coordinate z that is globally defined on this cylinder and that takes values in an annular region $\{z \in \mathbb{C}|\ r < |z| < R\}$, $r < 1 < R$, in the complex plane. We can assume henceforth that all transition functions in (2.9) have $f(z) = z$. If we have no odd moduli, the only possibility is $\kappa = \pm 1$. Hence, upon appropriately redefining the local coordinates $\theta \to \pm\theta$ on each patch, we can almost find a global chart, depending on whether this $\mathbb{Z}_2$-monodromy is $+$ or $-$ if we go around the unit circle. In the first case the cylinder is of *Neveu-Schwarz type (NS)* and we have a global chart with coordinates z and θ and

$$\partial = \frac{\partial}{\partial z}, \qquad D_\theta = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}. \tag{2.18}$$

If there is a non-trivial monodromy, the cylinder is of *Ramond type (R)*. In this case there is no global superconformal chart, but we can still find another global coordinate chart if we let $z \to z, \theta \to \theta/\sqrt{z}$. In this new coordinate system we have

$$\partial = \frac{\partial}{\partial z}, \qquad D'_\theta = \frac{\partial}{\partial \theta} + z\theta \frac{\partial}{\partial z}, \qquad D'^2_\theta = z\partial, \tag{2.19}$$

where $D'_\theta = \sqrt{z} D_\theta$. The last equation tells us that the distinguished odd distribution fails to be maximally integrable at $z = 0$ so that the SRS structure is singular there. However notice that the point $z = 0$ is not part of our type II world sheet. A cylinder in the bulk of a type II world sheet has two SRS structures that may have independent monodromies, so that there are four distinct types of cylinders NS-NS, NS-R, R-NS and R-R. In the bordered case we can consider cylinders that are left invariant under the antiholomorphic involution ρ. Topologically, they are the double of a strip. Let z be a uniformisation coordinate for the cylinder in the double Σ_0. In this case $z \mapsto \overline{\rho(z)}$ is a holomorphic map of the annulus to itself, so that it must be a rotation by the uniformisation theorem. But then, after possibly multiplying z by a phase, we can assume that $\rho^* z = \bar{z}$. In this case the boundary corresponds to the intersection of the cylinder with real axis. In extending this result to bordered type II world sheets, we only need to notice that compatibility of the coordinate changes with the involution ρ requires us to have the same sign factors for the holomorphic and the antiholomorphic SRS structures so that we can only have NS strips and R strips.

Before we can discuss degenerating type II world sheets we need to introduce the concept of punctures. For the bosonic string a puncture is just a marked point on the world sheet. In the supersymmetric setup the situation is more complicated. In addition to the position on the world sheet a puncture carries more information. Let us choose a point on a type II world sheet and consider a cylinder encircling it. Since the SRS structure is nowhere singular, this cylinder has to be of NS type. Deformations of the SRS structure preserving the underlying Riemann surface structure are parametrised by an odd parameter λ and correspond to changing the local frame as $z \to z + \theta\lambda$, $\theta \to \theta + \lambda$. The new SRS structure is gauge-equivalent to the old structure, but they differ by the choice of line given by θ over the point. A choice of point together with a choice of line over it is called an *NS puncture*. They add one bosonic and one odd modulus to the moduli space and are described by requiring that the vector field v and the spinor s in (2.14) vanish at that point. It also possible to require that s vanishes up to some particular order k. $-1-k$ is called the picture of the NS puncture. Picture 0 punctures may be thought of as an NS puncture for which its line was forgotten. On the other hand, if the cylinder around the chosen point is to be an R cylinder, the SRS structure must degenerate. For topological reasons there must always be an even number of such degenerate points. The underlying Riemann surface should be completely regular near that point and the singularity should only be in the additional line bundle $\mathcal{L}$. Formally, we require that the coordinate system defined in (2.19) should cover an entire neighbourhood of the puncture. In this case we call the singularity an *R puncture*. Geometrically, the presence of an R puncture modifies the line bundle associated to an SRS structure. Thinking of θ as a local trivialisation of $\mathcal{L}$, it is not hard to see that in the presence of R punctures it satisfies $\mathcal{L}^2 \cong K \otimes \mathcal{O}(p_1 + \ldots + p_k)$, where p_i is the position of the ith R puncture. R punctures correspond to generalised spin structures. This way the world sheet gravitino still is a smooth section of $\mathcal{L}^{-1} \otimes \Omega^{(0,1)}$ and the same reasoning as before can be applied[2]. Since a pair of R punctures increases the degree of $\mathcal{L}$ by 1, by the Riemann-Roch theorem the dimension of $H^1(\mathcal{L}^{-1})$ generically increases by 1, which means that an additional odd modulus has appeared. We conclude that every R puncture contributes one even modulus and $\frac{1}{2}$ odd modulus. In this

[2]Let us denote by z, θ local superconformal coordinates on a sliced neighbourhood. By (2.19) we know that z, $\theta' = \theta/\sqrt{z}$ give a global coordinate system. Given a Beltrami differential $\mu = \bar{\partial} v$ and world sheet gravitino $\chi = \bar{\partial} s$, the change in the superconformal coordinates are $\delta z = v - \frac{1}{2}\theta s$ and $\delta\theta = s + \frac{1}{2}\theta\partial v$. If we require this reparametrisation be regular in the z, θ' coordinate frame, we must require that $s' = s/\sqrt{z}$ and v/z are smooth. Since $s'\theta = s\theta'$, s can be regarded as a smooth section of $\mathcal{L}^{-1}$. Elliptic regularity implies that $\chi = \bar{\partial} s$ has a smooth solution for s if and only if χ is a smooth section of $\mathcal{L}^{-1} \otimes \Omega^{(0,1)}$. Similarly, we must interpret μ as a section of $K^{-1} \otimes \mathcal{O}(-p) \otimes \Omega^{(0,1)}$. The latter condition implies that the even position of the puncture is added as a modulus, too.

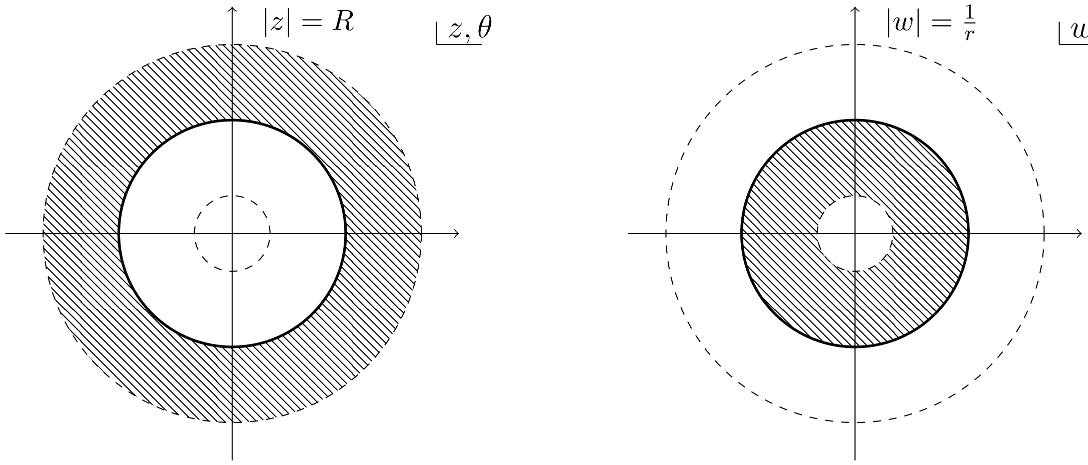


Figure 2.1: Sketch of the topological situation on a cylinder embedded contained in a SRS. z and $w = -1/z$ are uniformising coordinates on the underlying Riemann surface. The shaded regions are mapped into each other and the cylinder lies between the two dashed circles with $|z| = R$ and $|z| = r$. The bold circle denotes the unit circle.

description the puncture is in the $-\frac{1}{2}$ picture. As in the case of the NS puncture, we can restrict the allowed parameters of the local supersymmetry transformations s to have a zero of order k at the puncture. This way we add more odd moduli and obtain R punctures in picture $-\frac{1}{2} - k$. The only interesting case for our purposes is $k = 1$, in which case we distinguish a line above the puncture. In the usual interpretation picture $-\frac{1}{2}$ punctures correspond to a divisor above the puncture, while picture $-\frac{3}{2}$ punctures correspond to a choice of divisor together with a point on it. We denote by $\mathfrak{M}_{g,n,m}$ the supermoduli space of genus g curves with n NS punctures and m R punctures in the canonical pictures. If we consider superdiscs, we mean by $\mathfrak{M}_{0,n,m}$ it discs with n NS punctures and m R punctures at the boundary and no bulk punctures. The precise meaning should be clear from the context.

The main tool to study supermoduli space is the *plumbing fixture method* or *sewing method* in a similar way as for bosonic moduli space. This method constructs a (finite) family of type II world sheets from a given reference world sheet. The main data for this construction is a choice of cylinder or strip in the reference world sheet. It works the same for both SRS structures and we hence restrict our discussion to just the holomorphic sector. We need to distinguish between NS cylinders and R cylinders. As before we can find a uniformising coordinate z on this cylinder and we consider a second coordinate $w = -1/z$. The situation is sketched in figure 2.1. As the superconformal coordinates z, θ are valid down to $z = 0$ on the complex plane and the same is true for w, ψ, we could also forget about the cylinder and fill in the two holes in the surface with a super disc as indicated by the figure. This way we obtain punctured surfaces with a choice of coordinate disc nearby. Let us denote by $\mathfrak{P}_{g,n,m}$ the space of such punctured surfaces and, similarly, by $\mathfrak{P}_{0,n,m}$ the supermoduli space of superdiscs with n NS punctures and m R punctures together with a choice of coordinate disc near each puncture. The central idea is now to deform the coordinates z, θ by a global superconformal transformation of the super Riemann sphere. In order to understand the possible parameters arising this way, we need to treat the NS cylinder and the R cylinder separately.

In the NS case the super Riemann sphere has automorphism group $\mathrm{OSp}(1|2)$ which is of dimension $3|2$. Moving the points $z = 0$ and $z = \infty$ corresponds to changing the position of the centre of the two glued discs, moreover we can also fix a line over the centre of the two discs. Under these restrictions we are left with a $1|0$-parameter group of automorphisms given by the change $z \to \lambda^2 z, \theta \to \lambda\theta$. The family is thus completely determined by giving an NS puncture on each of the two sides of the cylinder and an even parameter, the gluing parameter. The complete identification between the two superconformal frames reads

$$zw = -q_{\mathrm{NS}}^2, \qquad \psi = \frac{q_{\mathrm{NS}}}{z}\theta. \tag{2.20}$$

We therefore obtain either a morphism

$${}_i\circ_j : \mathbb{C}^\times \times \mathfrak{P}_{g_1,n_1+1,m_1} \times \mathfrak{P}_{g_2,n_2+1,m_2} \to \mathfrak{P}_{g_1+g_2,n_1+n_2,m_1+m_2},$$

if the surface splits into two after removing the cylinder, the *separating case*, or

$$\xi_{ij} : \mathbb{C}^\times \times \mathfrak{P}_{g,n+2,m} \to \mathfrak{P}_{g+1,n,m},$$

if it stays connected, the *non-separating case*. The numbers i and j indicate the number of the coordinate discs that should be sewn. The parameter q_{NS} is a coordinate function on supermoduli space and the point $q_{\mathrm{NS}} = 0$ describes the limit in which the cycle along the cylinder shrinks to zero size. The point $e^{\mathrm{i}\pi} q_{\mathrm{NS}}$ gives back the same underlying Riemann surface and can be identified with a *Dehn twist* along the vanishing cycle. However, under this path the SRS may not return to itself, since but leads to a change $\psi \to -\psi$. If the degeneration is separating we can remove this sign by applying a global superconformal transformation on one of the two components, so that the SRS structure is the same. In the non-separating case the two structures are genuinely different. From the perspective of $\mathcal{SM}$ we change the $\mathbb{Z}_2$-monodromy along a cycle going through the cylinder, so that we obtain two different spin structures. The projection $\mathcal{SM} \to \mathcal{M}$ is, hence, ramified over the point $q_{\mathrm{NS}} = 0$ at infinity. Since Dehn twists generate the mapping class group [79], by applying Dehn twists along various different cycles one can reach any spin structure of the same parity. This shows that spin moduli space has just two connected components, $\mathcal{SM}_{\mathrm{even}}$ and $\mathcal{SM}_{\mathrm{odd}}$.

In the case of an R cylinder, we have to consider the Riemann sphere with R punctures at $z = 0$ and $z = \infty$. Without the punctures this SRS has an automorphism group of dimension $3|1$. Upon fixing the punctures we are left with an automorphism group of dimension $1|1$ with parameters λ and α corresponding to $z \to \lambda z(1 + \theta\alpha), \theta \to \pm(\theta + \alpha)$. The complete identification reads

$$zw = q_{\mathrm{R}}(1 - \theta\alpha), \qquad \psi = \pm\mathrm{i}(\theta + \alpha). \tag{2.21}$$

We again obtain a morphism

$$ {}_i\circ_j : G \times \mathfrak{P}_{g_1,n_1,m_1+1} \times \mathfrak{P}_{g_2,n_2,m_2+1} \to \mathfrak{P}_{g_1+g_2,n_1+n_2,m_1+m_2} $$

in the separating case, or

$$ \xi_{ij} : G \times \mathfrak{P}_{g,n,m+2} \to \mathfrak{P}_{g+1,n,m} $$

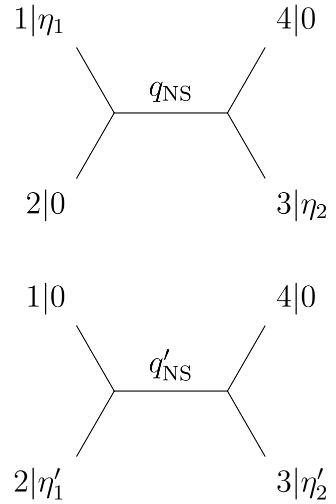
in the non-separating case. Here G is the $1|1$ dimensional supergroup of automorphisms of the super Riemann sphere with two punctures given by transformations of the form (2.21). Unlike for the NS degenerations the projection $\mathcal{SM} \to \mathcal{M}$ is never ramified at $q_{\mathrm{R}} = 0$. Instead of localising the odd moduli outside the cylinder, we are left with a fermionic gluing parameter. The origin of this additional gluing parameter lies in the fact that we required that the two pieces that split off leave behind punctures in the $-\frac{1}{2}$ picture. Alternatively, we could require one of the two punctures to be in the $-\frac{3}{2}$ picture which would eliminate the free gluing parameter from the plumbing fixture construction. From the geometric point of view integrating over the odd gluing parameter explains the origin of the necessity of picture changing operators in the calculation of superstring scattering amplitudes.

If the cylinder in the world sheet lies in the bulk, the holomorphic and the antiholomorphic SRS structures are independent and we have in addition gluing parameters $\bar{q}_{\mathrm{NS}}$ or $\bar{q}_{\mathrm{R}}, \bar{\alpha}$ for the antiholomorphic structure. In the case of a bordered world sheet the plumbing fixture construction has to preserve the antiholomorphic involution. This imposes a reality condition on the parameters, $q_{\mathrm{NS}}^2 = \bar{q}_{\mathrm{NS}}^2$ for an NS cylinder and $q_{\mathrm{R}} = \bar{q}_{\mathrm{R}}, \alpha = \bar{\alpha}$ for an R cylinder.

The important point for superstring field theory is that the coordinates on supermoduli space given by the plumbing fixture construction (2.20) and (2.21) are the ones that ensure a proper separation of the moduli between the two sides of the cylinder and lead to a proper factorisation on the non-analyticities of the scattering amplitude. Although the same logic applies to the bosonic string, there are a few subtleties in the extension to the superstring that were the origin of problems towards progress in constructing consistent superstring field theories (even classically).

Fortunately, these subtleties already arise for open four-point scattering at tree-level, which is quite tractable[3]. For simplicity, we consider a disc with 4 NS punctures at its boundary and consider a parameterisation of its supermoduli space constructed from plumbing fixture of two thrice punctured discs. Topologically, there are two different ways to choose a cylinder on its double. Since all punctures are NS, the cylinders in both cases are NS cylinders. We choose uniformising coordinates z, θ and

[3]Mathematically, the origin of this problem can be traced to the fact that the supermoduli space in question has more than 1 odd dimension. If we compare different coordinate systems the bosonic coordinates may receive contributions from pairs of odd coordinates. One might argue that a clever choice of odd coordinates would remove such terms, but this would mean that the supermoduli space is holomorphically projected. While this is true for genus 0, it has been shown that it fails to be true for genus $g \geq 5$ and no punctures, and even for lower genus if one include punctures [49, 80]. Even if the moduli space were split, it is not known if the choice of global holomorphic projection is compatible with the factorisation at infinity.



$$\psi = \frac{q_{\mathrm{NS}}^2}{1+q_{\mathrm{NS}}^2}\left(1+\frac{q_{\mathrm{NS}}}{1+q_{\mathrm{NS}}^2}\eta_1\eta_2\right)$$
$$\zeta_1 = -\frac{q_{\mathrm{NS}}}{\sqrt{1+q_{\mathrm{NS}}^2}}\eta_1, \quad \zeta_2 = \frac{\eta_2 - q_{\mathrm{NS}}\eta_1}{\sqrt{1+q_{\mathrm{NS}}^2}}$$

$$\psi = \frac{q_{\mathrm{NS}}'^2}{1+q_{\mathrm{NS}}'^2}$$
$$\zeta_1 = -\frac{q'_{\mathrm{NS}}\eta'_1}{\sqrt{1+q_{\mathrm{NS}}'^2}}, \quad \zeta_2 = \frac{\eta'_2}{\sqrt{1+q_{\mathrm{NS}}'^2}}$$

Figure 2.2: Two choices of coordinate systems on $\mathfrak{M}_{0,4,0}$ obtained from plumbing fixture and their relation to the standard coordinate system ψ, ζ_1, ζ_2. The notation $k|\eta$ indicates that NS puncture number k has odd modulus η. q_{NS} and q'_{NS} denote the real gluing parameters for the plumbing fixture construction.

w, ψ on the thrice punctured discs and map the bosonic positions of the punctures to 0, 1 and ∞. The odd moduli of the punctures are specified in this coordinate frame. The coordinate discs used to perform the gluing are the unit discs centred at 0. In order to compare the so produced surface with each other, we need to fix a standard coordinate system on $\mathfrak{M}_{0,4,0}$. Let us enumerate the punctures in ascending order compatible with the orientation of the boundary. We fix the OSp(1|2)-invariance of the disc by sending puncture 1 to the coordinate $0|0$, puncture 2 to $\psi|\zeta_1$, puncture 3 to $1|\zeta_2$ and puncture 4 to $\infty|0$. This fixes the standard coordinate frame ψ, ζ_1, ζ_2 on the supermoduli space. Since the plumbing fixture construction depends on a choice of odd modulus for the two thrice punctured discs, it depends on that parameter η_i and the coordinate system is not unique. Figure 2.2 gives two such choices and the relation of the gluing parameter q_{NS} and odd moduli η_i to the standard OSp(1|2)-invariant coordinate system. From these relations we can deduce the needed change of coordinates,

$$q'_{\mathrm{NS}} = q_{\mathrm{NS}}\left(1+\frac{1}{2}q_{\mathrm{NS}}\eta_1\eta_2\right) \tag{2.22a}$$

$$\eta'_1 = \eta_1, \qquad \eta'_2 = \eta_2 - q_{\mathrm{NS}}\eta_1. \tag{2.22b}$$

These relations demonstrate that the bosonic coordinates on supermoduli space receive nilpotent contributions while being trivial on the reduced moduli space upon going to a different plumbing fixture coordinate system.

2.2 Construction of the superstring measure

After discussing the structure of the supermoduli space $\mathfrak{M}$ of bordered type II world sheets and $\mathfrak{P}$ of bordered type II world sheets with coordinate discs near each puncture, we construct an integration theory on it and use it to define the superstring S-matrix in terms of an integral of suitable smooth sections of the Berezinian $\mathrm{Ber}(\mathfrak{M})$.

In order to aid the reader, we state the main results of this section. If $\mathfrak{M}$ has dimension $r|s$, the smooth sections $\Omega_A^{r|s}$ are defined in equation (2.37),

$$\Omega_A^{r|s} = \langle \prod_m \left(b_i \frac{\partial F^i}{\partial t^m} \right) \prod_\alpha \delta \left(b_i \frac{\partial F^i}{\partial \tau^\alpha} \right) A(x,c) \rangle_\Psi .$$

$\Omega_A^{r|s}$ satisfies a chain map property given in equation (2.38),

$$d\Omega_A^{r|s} + (-1)^{r+1} \Omega_{QA}^{r+1|s} = 0.$$

When applied to superstring theory we obtain a chain map in equation (2.50),

$$\Omega_{g,n+m}^{r|s} : \mathcal{H}^{\otimes(n+m)} \to \Omega^{r|s}(\mathfrak{P}_{g,n,m}).$$

The forms $\Omega_A^{r|s}$ for open superstrings take the familiar form

$$\Omega_A^{r|s} = \langle \prod_m \left(\int d^2 z\, b(z) \mu_m(z,\bar{z}) \right) \prod_\alpha \delta \left(\int d^2 z\, \beta(z) \chi_\alpha(z,\bar{z}) \right) A(x,c) \rangle_\Psi ,$$

where μ_m is the Beltrami differentials for even modulus t^m and χ_α is the gravitino for the odd modulus τ^α. b and β are the Faddeev-Popov antighost fields. The operator $A(x,c)$ is a product of vertex operators.

For a good review of integration theory on supermanifolds we refer the reader to [47] and the references therein.

2.2.1 Equivariant integration

Since the introduction of gauge invariance into local field theory, physicists and mathematicians have developed a very huge set of tools and constructions to describe such theories efficiently. Among the most famous is the Faddeev-Popov method [81]. Historically, this method has been employed by Faddeev and Popov in order to obtain a Lorentz covariant formulation of Yang-Mills perturbation theory at the price of introducing ghost fields. Later [82–85] it was discovered that the ghost extended theory contains a fermionic symmetry, the *BRST symmetry*. This method was analysed from the Hamiltonian and the Lagrangian perspective and generalised to arbitrary gauge symmetries, such as open gauge algebras or redundant gauge parameters. The Hamiltonian formulation is known as *BFV theory* [86, 87] and the Lagrangian formulation [18, 88] as *BV theory*. There is even a combination of the two methods, suitable of manifolds with boundary and corners, called *BFV-BV theory* [89]. Very abstractly one may think of a BFV-BV theory as performing an integral over a subset of the integration variables and study the dependence of the integral on the unintegrated variable, this partial integral is the *wave function* associated to the domain on which the integrated degrees of freedoms live. BFV-BV theory extracts and formalises the algebraic properties of such integrals, so that we may think of a BFV-BV theory as a definition of an integration theory on some quotient space (or superstack) $M/\sim$ of (possibly infinite dimensional) manifold M by an equivalence relation $\sim$, which is just the *gauge symmetry*. We refer the reader to the excellent book [19]. For our purposes the gauge symmetry in question forms a Lie supergroup G and the BV-BFV integration theory simplifies significantly.

The general setup is as follows. We consider a supermanifold M with coordinates x^μ and a Lie supergroup G with its action on M generated by vector fields $X_a^\mu(x)$, $a = 1, 2, \ldots, \dim(G)$. These vector fields satisfy $[X_a, X_b]^\mu = (-1)^{|a|\cdot|b|} f_{ab}{}^m X_m^\mu$ with $f_{ab}{}^m$ being the structure constants of the Lie superalgebra of G. We denote by $|a|$ the internal degree of the generator X_a and by $(-1)^{|a|\cdot|b|}$ the exchange sign obtained upon interchanging X_a with X_b. We introduce new variables c^a of ghost number $\mathrm{gh}(c^a) = 1$ and with internal degree $-|a|$. Let us denote collectively by ϕ^i any field of the theory, x^μ or c^a in our case. To each field ϕ^i we add a conjugate *antifield* ϕ_i^* with opposite internal degree $-|i|$ and ghost number $-1 - \mathrm{gh}(\phi^i)$. In total we have the variables x^μ, c^a, x_μ^* and c_a^*. On top we define a Gerstenhaber bracket $(\cdot, \cdot)$ for which the elementary fields are *Darboux coordinates*, i.e. the non-trivial elementary brackets are

$$(x_\mu^*, x^\nu) = \delta_\mu{}^\nu, \qquad\qquad (c_a^*, c^b) = \delta_a{}^b. \tag{2.23}$$

Let us consider a master action S,

$$S[x, x^*, c, c^*] = S_0[x] + c^a X_a^\mu x_\mu^* - \frac{1}{2} f_{rs}{}^m c^r c^s c_m^*, \tag{2.24}$$

where $S_0[x]$ is an even function of ghost number 0. In total S is even and has ghost number 0. It satisfies the classical master equation $(S, S) = 0$ precisely if S_0 is gauge invariant, i.e. $X_a(S_0) = 0$, $f_{rs}{}^m$ are structure constants of a Lie superalgebra and X_a^μ is a representation of this algebra as vector fields. In order to obtain a BV integration theory, we need to define a compatible BV Laplacian Δ,

$$\Delta = \sum_\mu \frac{\partial^2}{\partial x^\mu \partial x_\mu^*} - \sum_a \frac{\partial^2}{\partial c^a \partial c_a^*} = \sum_i (-1)^{\mathrm{gh}(\phi^i)} \frac{\partial^2}{\partial \phi^i \partial \phi_i^*}. \tag{2.25}$$

We require the Euclidean quantum master equation $\Delta e^{-S/\hbar} = 0$, which is equivalent for the gauge symmetry to be non-anomalous. Necessary counterterms and anomaly cancelling terms should be added to the function $S_0[x]$ so that the quantum master equation is satisfied. BV theory defines correlation functions $\langle \mathcal{O}(x, x^*, c, c^*) \rangle_\Psi$ with the help of a gauge fixing fermion Ψ. Ψ is a function of ghost number -1 and internal degree 0 and must be chosen, s.t. the following definition makes sense

$$\langle O(x, x^*, c, c^*) \rangle_\Psi = \int dx\, dc\, e^{-S/\hbar} O|_{\phi_i^* = \frac{\partial \Psi}{\partial \phi^i}}. \tag{2.26}$$

Moreover, one defines an operator, the BRST operator Q, of ghost number 1 as

$$Q = (S, \cdot) - \Delta(\cdot). \tag{2.27}$$

By virtue of the Euclidean quantum master action we have $Q^2 = 0$. The main theorem of BV theory states that upon changing the gauge-fixing fermion $\Psi \to \Psi + \delta\Psi$, the expectation value $\langle O\rangle_\Psi$ changes as

$$\delta\langle O\rangle_\Psi - \langle \delta\Psi\, Q(O)\rangle_\Psi = 0 \tag{2.28a}$$
$$\langle Q(O)\rangle_\Psi = 0. \tag{2.28b}$$

Equations (2.28) imply that $\langle . \rangle_\Psi$ defines a linear functional on the cohomology $H^\bullet(Q)$ and this functional is invariant under infinitesimal changes in the gauge-fixing fermion Ψ, i.e. the numbers are truly gauge-invariant.

In order to be able to define the superstring measure, we need to consider slightly more general observables than those in $H^\bullet(Q)$. Unfortunately, the minimal variable content given by x^μ and c^a and their antifields is not sufficient and needs to be extended further in order to define the expectation values. To this end we postulate even, ghost number 0, gauge-fixing functions F^i and introduce antighosts b_i of ghost number -1 and internal degree opposite to F^i and Nakanishi-Lautrup fields n_i of ghost number 0 and internal degree equal to F^i. We extend the master action, antibracket and BV-Laplacian as

$$S[x, x^*, c, c^*, b, b^*, n, n^*] = S_0[x] + c^a X_a^\mu x_\mu^* - \frac{1}{2} f_{rs}{}^m c^r c^s c_m^* + n_i b^{i*}, \tag{2.29a}$$
$$(b^{i*}, b_j) = \delta^i{}_j, \tag{2.29b}$$
$$(n^{i*}, n_j) = \delta^i{}_j, \tag{2.29c}$$
$$\Delta \to \Delta + \sum_i \left(\frac{\partial^2}{\partial n_i \partial n^{i*}} - \frac{\partial^2}{\partial b_i \partial b^{i*}} \right). \tag{2.29d}$$

The cohomology of Q is not altered by addition of the new variables. All relevant properties of the superstring measure can be deduced from equations (2.28) and (2.29). The gauge-fixing fermions take the very simple form $\Psi = b_i F^i$ and F^i may only depend on x^μ. The gauge-fixed action reads then

$$S_{\text{gf}} = S_0 + c^a X_a(b_i F^i) + n_i F^i = S_0 + Q(b_i F^i). \tag{2.30}$$

Thus, integration over the Nakanishi-Lautrup fields reduces the integration domain in the path integral to the submanifold Σ defined by $F^i = 0$. Integration over c^a and b_i produces a Faddeev-Popov determinant factor. For κ^a of ghost number 0 and internal degree $|a|$ consider $\langle (S, \kappa^a c_a^*) O\rangle_\Psi$ and use the main identity (2.28) to deduce that for arbitrary observables O one has

$$\langle b_i \kappa^a X_a(F^i) O\rangle_\Psi = \langle \kappa^a \frac{\partial O}{\partial c^a}\rangle_\Psi. \tag{2.31}$$

Similarly, by considering $\langle (S, b_i R^i{}_j n^{j*}) Q(O)\rangle_\Psi$ for arbitrary $R^i{}_j$ of ghost number 0 and internal degree $|i| - |j|$ one finds

$$\langle (S, b_i R^i{}_j)\frac{\partial O}{\partial n_j} + b_i R^i{}_j \frac{\partial O}{\partial b_j}\rangle_\Psi + \langle b_i R^i{}_j F^j Q(O)\rangle_\Psi = 0. \tag{2.32}$$

Equations (2.31) and (2.32) have an important interpretation in terms of submanifolds Σ of M. Every submanifold Σ is the zero set of a section of a vector bundle on the ambient manifold, namely the zero section of the conormal bundle $N^*\Sigma$. So, given F^i, there is a unique submanifold Σ defined by $F^i = 0$. The correspondence is however not one-to-one. An infinitesimal change of F^i that gives rise to the same submanifold is of the form $F^i \to F^i + \delta F^i$ with $\delta F^i = 0$ on the Σ. One can think of such a redundancy as a gauge-invariance. A general correlator $\langle O\rangle_\Psi$ may depend on more than just the submanifold Σ. In general specification of F^i also provides us with a preferred local trivialisation of the conormal bundle $N^*\Sigma$ by dF^i. Equation (2.32) and (2.28a) tell us how a correlator changes under a gauge-transformation $\delta F^i = R^i{}_j F^j$,

$$\delta\langle O\rangle_\Psi = -\langle (S, b_i R^i{}_j)\frac{\partial O}{\partial n_j} + b_i R^i{}_j \frac{\partial O}{\partial b_j}\rangle_\Psi. \tag{2.33}$$

Thus, every insertion of b_i goes to $b_i - b_j R^j{}_i$ or, equivalently, $b_i \delta F^i$ is an invariant quantity, when O does not contain any antifields.

Another interesting change of F^i is $\delta F^i = \kappa^a X_a(F^i)$. It shifts the submanifold Σ along a gauge orbit through Σ. Equation (2.31) implies that a correlator changes as

$$\delta\langle O\rangle_\Psi = \langle \kappa^a \frac{\partial}{\partial c^a} Q(O)\rangle_\Psi. \tag{2.34}$$

We want to consider $\langle O\rangle_\Psi$ as a form defined on the space of maps F^i divided by the action of a gauge group. Since such a space is in general singular, we define it as the space of forms on the space of maps that are invariant and horizontal under the action of the gauge group [90, 91]. More precisely, we consider F^i as a coordinate functional on the space of maps and denote by d the exterior differential. Changes of trivialisation correspond to a vector field R with $\mathcal{L}_R F^i = \iota_R dF^i = R^i{}_j F^j$ and movement along the gauge-orbit to a vector field κ with $\mathcal{L}_\kappa F^i = \iota_\kappa dF^i = \kappa^a X_a(F^i)$. Making use of equation (2.33), we see that insertions of the form $b_i dF^i$ are invariant under R up to terms involving F^i without any derivatives, that do not matter if O contains no insertions of the Nakanishi-Lautrup field n_i. Likewise, by considering $\langle (S, b_i R^i{}_j n^{*j}) O\rangle$ one can see that $b_i dF^i$ is a horizontal form if O does not contain n_i. Consequently, we can build basic forms for the gauge symmetry generated by R if we only use $b_i dF^i$ and do not use n_i in the observable O. We can likewise exploit equations (2.34) and (2.31) to find the failure for a form to be basic and horizontal for the action of the vector fields κ,

$$\iota_\kappa \langle b_i dF^i O\rangle_\Psi = \langle \kappa^a \frac{\partial}{\partial c^a} O\rangle_\Psi + \langle \iota_\kappa O\rangle_\Psi \tag{2.35a}$$

$$d\langle O\rangle_\Psi = \langle b_i dF^i Q(O)\rangle_\Psi + \langle dO\rangle_\Psi \tag{2.35b}$$

$$\mathcal{L}_\kappa \langle O\rangle_\Psi = \langle \kappa^a \frac{\partial}{\partial c^a} Q(O)\rangle_\Psi + \langle \mathcal{L}_\kappa O\rangle_\Psi . \tag{2.35c}$$

Whether or whether not a particular observable is basic and horizontal w.r.t. the gauge symmetries κ depends therefore heavily on the choice of allowed gauge parameters. Moreover, we see that insertions of the ghost field c^a require a restriction of the gauge group and, thus, the corresponding quotient space gets larger.

Physical deformations of Σ are described by δF^i that cannot be written in the form $\kappa^a X_a(F^i)$ for some admissible κ^a. By inspecting the ghost kinetic term in the gauge-fixed action (2.30) one concludes that the zero-modes of the b-ghost can be identified with the linear duals of the physical deformations at a particular point in Σ. Provided the number of zero modes does not jump as we move along Σ, the dimension of the moduli space is equal to that number.

The correlators we need are of the form $\Omega_A = \langle \exp\left(b_i dF^i\right) A(x,c)\rangle_\Psi$. Using (2.35b) one can derive the very important relations

$$d\langle \exp\left(b_i dF^i\right) A(x,c)\rangle_\Psi + \langle \exp\left(-b_i dF^i\right) QA(x,c)\rangle_\Psi = 0, \tag{2.36a}$$

$$\iota_\kappa \langle \exp\left(b_i dF^i\right) A(x,c)\rangle_\Psi - \langle \exp\left(-b_i dF^i\right) \kappa^a \frac{\partial}{\partial c^a} A(x,c)\rangle_\Psi = 0, \tag{2.36b}$$

$$\mathcal{L}_\kappa \langle \exp\left(b_i dF^i\right) A(x,c)\rangle_\Psi + \langle \exp\left(b_i dF^i\right) \left[Q, \kappa^a \frac{\partial}{\partial c^a}\right] A(x,c)\rangle_\Psi = 0. \tag{2.36c}$$

Ω_A is a pseudoform on the space of maps. Pseudoforms are functions on the parity inverted tangent bundle. In order to compare Ω_A with well-known expressions for the superstring measure, we need to convert it into a form of degree $r|s$. Such a conversion is the the *Baranov-Schwartz transform* $\lambda^{r|s}$ [92]. Essentially it performs the integration over the parity inverted fibres. Let us denote by t^m and τ^α the r even and s odd coordinates on the parameter space. The Baranov-Schwartz $\Omega_A^{r|s} = \lambda^{r|s}\Omega_A$ transform is then

$$\begin{aligned}\Omega_A^{r|s} &= \int \mathcal{D}(dt, d\tau)\,\Omega_A = \langle \int \mathcal{D}(dt, d\tau)\, \exp\left(b_i \left(dt^m \frac{\partial F^i}{\partial t^m} + d\tau^\alpha \frac{\partial F^i}{\partial \tau^\alpha}\right)\right) A(x,c)\rangle_\Psi \\ &= \langle \prod_m \left(b_i \frac{\partial F^i}{\partial t^m}\right) \prod_\alpha \delta\left(b_i \frac{\partial F^i}{\partial \tau^\alpha}\right) A(x,c)\rangle_\Psi . \end{aligned} \tag{2.37}$$

At this point we should note a small subtlety concerning the orientation in the variables $(dt, d\tau)$ used to define the above integral. Under changes of variables in dt the measure is multiplied by the determinant of the Jacobian, while changes in the variables $d\tau$ divide by the absolute value of the Jacobian. Thus, the supermoduli space is endowed with a $[+-]$ orientation [59]. Applying the Baranov-Schwartz transform to equation (2.36a), using that it is compatible with the exterior differential and taking into account the change of orientation in dt in the integration, we find that

$$d\Omega_A^{r|s} + (-1)^{r+1}\Omega_{QA}^{r+1|s} = 0. \tag{2.38}$$

2.2.2 The superstring measure

In this section we apply the method of equivariant integration introduced in section 2.2.1 to the supermoduli space of type II world sheets with boundary from section 2.1. Locally, a type II world sheet is defined by a choice of superconformal frame e^z, e^θ and is subject to a gauge-invariance generated by infinitesimal superdiffeomorphisms and local GL(1|1)-transformations preserving the torsion constraint (2.7). The gauge-parameters are an even vector field C, even parameters S and $\bar{S}$ and odd parameters R and $\bar{R}$. On a superconformal frame they act as follows,

$$\delta_C e^i = \mathcal{L}_C e^i, \tag{2.39a}$$

$$\delta_S^{(S)} e^z = 2Se^z, \qquad \delta_S^{(S)} e^\theta = Se^\theta, \qquad \delta_S^{(S)} e^{\bar{z}} = 0, \qquad \delta_S^{(S)} e^{\bar{\theta}} = 0, \tag{2.39b}$$

$$\delta_R^{(R)} e^z = 0, \qquad \delta_R^{(R)} e^\theta = e^z R, \qquad \delta_R^{(R)} e^{\bar{z}} = 0, \qquad \delta_R^{(R)} e^{\bar{\theta}} = 0. \tag{2.39c}$$

We did not specify the actions $\delta_{\bar{S}}^{(\bar{S})}$ and $\delta_{\bar{R}}^{(\bar{R})}$ as they are the antiholomorphic analogues of the above transformations. The non-trivial commutators read, omitting the obvious antiholomorphic analogues,

$$[\delta_C, \delta_C] = \delta_{[C,C]}, \tag{2.40a}$$

$$[\delta_S^{(S)}, \delta_R^{(R)}] = \delta_{SR}^{(R)}, \qquad [\delta_C, \delta_S^{(S)}] = \delta_{C(S)}^{(S)}, \qquad [\delta_C, \delta_R^{(R)}] = \delta_{C(R)}^{(R)}. \tag{2.40b}$$

Locally on the world sheet all deformations are pure gauge, so that any local gauge-fixing condition $F^i = 0$ gives an over-fixing of the gauge symmetry. But this is not a problem, as we may consider families of submanifolds and use the methods from section 2.2.1 to construct forms on the space of such submanifolds modulo gauge-equivalence. Since a local gauge-fixing is a complete gauge-fixing, the path-integral defining the correlators just integrates over a fixed type II world sheet and we can identify the space of submanifolds with the moduli space of type II world sheets. Thus, equation (2.37) defines forms of degree $r|s$ on the corresponding supermoduli space.

Let us fix a reference type II world sheet $\bar{e}^z$, $\bar{e}^\theta$. The superconformal frame corresponding to a nearby world sheet can be decomposed w.r.t. the reference superconformal frame,

$$e^z = \bar{e}^i A_i, \qquad e^\theta = \bar{e}^i B_i \qquad e^{\bar{z}} = \bar{e}^i \bar{A}_i, \qquad e^{\bar{\theta}} = \bar{e}^i \bar{B}_i, \tag{2.41}$$

where the index $i = z, \theta, \bar{z}, \bar{\theta}$. We can use $A, B, \bar{A}, \bar{B}$ as coordinates on the space of superconformal frames. The gauge-fixing condition $e^i = \bar{e}^i$ reduces in those coordinates to

$$
\begin{aligned}
&A_z = 1, && A_\theta = A_{\bar z} = A_{\bar\theta} = 0, && (2.42a)\\
&B_\theta = 1, && B_z = B_{\bar z} = B_{\bar\theta} = 0, && (2.42b)\\
&\bar A_{\bar z} = 1, && \bar A_\theta = \bar A_z = \bar A_{\bar\theta} = 0, && (2.42c)\\
&\bar B_{\bar\theta} = 1, && \bar B_z = \bar B_{\bar z} = \bar B_\theta = 0. && (2.42d)
\end{aligned}
$$

To each of this 16 conditions we add an antighost and a Nakanishi-Lautrup field as described in section 2.1.2. However, most of the terms in the gauge-fixed action turn out to be non-dynamic, so that one can integrate them out immediately. The algebraic conditions are

$$2S + \partial C^z = 0, \qquad C^\theta = \frac{1}{2} D_\theta C^z, \qquad R = -\partial C^\theta, \qquad D_{\bar\theta} C^z|_{\bar\theta=0} = 0, \tag{2.43}$$

and we omitted their antiholomorphic analogues. After this partial gauge-fixing procedure we are left with smooth fields μ, χ, $\bar\mu$ and $\bar\chi$ parametrising a neighbourhood of $\bar e^i$. The residual gauge-invariances are parametrised by ghost fields c, γ, $\bar c$ and $\bar\gamma$ and can be packaged into $C^z = c + 2\theta\gamma$ and $C^{\bar z} = \bar c + 2\bar\theta\bar\gamma$. To first order the nearby superconformal frames are given by equation (2.13),

$$e^z = e^z + d\bar z(\mu + \theta\chi), \qquad e^\theta = e^\theta + d\bar z(\chi + \theta\partial\mu).$$

Using (2.43) one can deduce the BRST variations of μ and χ up to first order,

$$Q\mu = \bar\partial c, \qquad Q\chi = 2\bar\partial\gamma, \qquad + \text{ c.c.} \tag{2.44}$$

The remaining gauge-fixing conditions are $\mu = \chi = 0$. We implement them by introducing antighosts b and β and corresponding Nakanishi-Lautrup fields. b should be a smooth section of K^2 and β a smooth section of $K \otimes \mathcal{L}$. The gauge-fixing fermion is

$$\Psi = \frac{1}{2\pi} \int d^2z \left(b\mu - \frac{1}{2}\beta\chi - \bar b\bar\mu + \frac{1}{2}\bar\beta\bar\chi \right). \tag{2.45}$$

After integrating out the Nakanishi-Lautrup fields, the gauge-fixed action (2.30) takes the form

$$S = S_0 - \frac{1}{2\pi} \int d^2z \left(b\bar\partial c - \beta\bar\partial\gamma - \text{ c.c.} \right). \tag{2.46}$$

Equations (2.43) and (2.40) give rise to the on-shell BRST variations[4]

$$
\begin{aligned}
&Qc = -c\partial c - \gamma^2, && Q\gamma = -c\partial\gamma + \frac{1}{2}(\partial c)\gamma, && (2.47a)\\
&Qb = T, && Q\beta = G, && (2.47b)
\end{aligned}
$$

[4]The BRST operator Q as defined in equation (2.27), only gives $Qb_i = n_i$. However, since we assume that the gauge-fixing conditions $F^i = 0$ give rise to a submanifold, we can regard F^i as coordinates on field space. The gauge-fixing fermion sets $F_i^* = (-1)^{|i|^2} b_i$. Equation (2.28)

the quantity T is the total world sheet stress momentum tensor and G is the total world sheet supersymmetry current. We take equations (2.47b) as their definitions.

We finish the specification of the world sheet theory by giving the matter action S_0. The classical master equation implies that S_0 has to be a gauge-invariant functional of the matter fields and the superconformal frame. In general it is hard to construct such a functional in a manifestly gauge-invariant way. Luckily every superconformal frame is locally equivalent to the default frame (2.8b). It therefore suffices to give the value of S_0 in this frame and just require that it be invariant under superconformal transformations (2.9). For two superconformal coordinates we have the identities

$$dz'\delta(d\theta') = D_\theta\theta'\, dz\delta(d\theta),$$
$$D_\theta = D_\theta\theta' D_{\theta'}.$$

Using these two identities, it follows that

$$S_0[\Phi,\bar\Phi] = \frac{1}{4\pi\alpha'}\int d^2z d^2\theta\, K(\Phi,\bar\Phi)_{\mu\nu}(D_\theta\Phi^\mu)(D_{\bar\theta}\bar\Phi^\nu) \tag{2.48}$$

is an invariant action provided Φ and $\bar\Phi$ transform under superconformal transformations as a scalar. Here we mean $d^2z = -\mathrm{i}dz\wedge d\bar z$. The Kähler metric $K_{\mu\nu}$ is arbitrary in principle, but we will assume that $K_{\mu\nu} = \eta_{\mu\nu}$ for η the flat metric of 9+1 dimensional Minkowski space. The expansions $\Phi^\mu = X^\mu + \theta\psi^\mu + \bar\theta\bar\psi^\mu + \theta\bar\theta F^\mu$ and $\bar\Phi^\mu = X^\mu + \theta\psi^\mu + \bar\theta\bar\psi^\mu + \theta\bar\theta F^\mu$ express S_0 in terms of the ordinary, normalised component fields X^μ and ψ^μ that describe the embedding of the string world sheet into target space. The fields F^μ are auxiliary fields and can be integrated out. The stress-tensor T and the supercurrent G are

$$T = T_m + T_{\mathrm{gh}} = \frac{1}{2\alpha'}\eta_{\mu\nu}\left(\partial X^\mu\partial X^\nu + \psi^\mu\partial\psi^\nu\right) + (2b\partial c + \partial b\, c) - \left(\frac{3}{2}\beta\partial\gamma + \frac{1}{2}\partial\beta\,\gamma\right),$$
$$G = G_m + G_{\mathrm{gh}} = \frac{1}{\alpha'}\eta_{\mu\nu}\partial X^\mu\psi^\nu + \left(\frac{3}{2}\beta\partial c + \partial\beta\, c + 2b\gamma\right).$$

It can be checked that the action (2.30) has BRST symmetry and that the BRST generator Q takes the form,

$$Q = \frac{1}{2\pi\mathrm{i}}\oint dz\,\left(cT_m + \gamma G_m + \frac{1}{2}\left(\gamma G_{\mathrm{gh}} + cT_{\mathrm{gh}}\right)\right) + \text{c.c.}.$$

---

implies then

$$0 = \langle Q(F_i^* O)\rangle = \langle (F_i^*, S)O - (-1)^{|i|^2} b_i(S,O) - \Delta(F_i^* O)\rangle$$
$$= \langle\left(\frac{\partial}{\partial F^i}\left(S_0 + c^a X_a(b_i F^i)\right) - (-1)^{|i|^2} n_i\right) O - \Delta(F_i^* O)\rangle.$$

In the last step we use that O does not depend on the antifields. The term involving the BV-Laplacian vanishes if the observable has no explicit F^i-dependence. The first term gives the desired on-shell identity between insertions of the Nakanishi-Lautrup fields and insertions of variations of (2.46).

As there is no regulator that is superconformally invariant the counter terms added to S_0 may break the superconformal invariance. However, in flat $9+1$ dimensional Minkowski space one can find a suitable modification of S_0 so that S satisfies the Euclidean quantum master action or, equivalently, the BRST operator Q is nilpotent in some renormalisation scheme.

For technical purposes we need to introduce another representation of the β–γ-path integral. This representation is known as the FMS bosonisation representation [35, 36] and has been studied extensively in the literature. Instead of using the bosonic fields β and γ one replaces them formally by a free boson ϕ and an anticommuting ghost system η–ξ, where η has dimension 1 and ξ has dimension 0. The relation between η–ξ and β–γ is

$$\gamma(z) = \eta e^{\phi}(z), \quad \beta(z) = \partial\xi e^{-\phi}(z), \quad \delta(\gamma(z)) = e^{-\phi}(z), \tag{2.49a}$$
$$\delta(\beta(z)) = e^{\phi}(z), \quad \eta(z) = \partial\gamma(z)\delta(\gamma(z)), \quad \partial\xi(z) = \partial\beta(z)\delta(\beta(z)). \tag{2.49b}$$

This representation is particularly well-suited for explicit calculations that are local on the world sheet, like calculating OPEs. But there are global issues at higher genus. For example the zero mode structure is not the same. The β-γ system has a number anomaly $2g-2$ on a compact Riemann surface, while non-vanishing correlators for the bosonised fields require a total ϕ-charge $2g-2$ and in addition one insertion of ξ without any derivatives and g insertions of η. The FMS bosonisation formulas do not specify how to absorb the g η-zero modes. For example, in order to reproduce the explicit form of the correlation functions for β-γ in terms of Riemann theta functions [93], it is necessary to absorb the η-zero modes in a non-local way and perform a projection in each handle onto a fixed picture [94]. The formulas (2.49) should therefore used with care when dealing with global questions. Conversely, we may regard (2.49b) as a definition of the composite operators η and $\partial\xi$ and it can be shown that with an appropriate definition of the β–γ-path integral [93, 95] η and $\partial\xi$ are primaries of weight 1 and have vanishing periods, so that one of the two may be expressed as a derivative of a globally define scalar field. The ambiguity in the zero mode of ξ can be fixed by requiring that it has a zero at a particular position p. This procedure is equivalent to the insertion of $\xi(p)$ into the path-integral. The current η is the generator of translations $\xi \to \xi + c$ and we can therefore use it to probe whether a particular operator depends on the zero mode by integrating η over its boundary. Operators that do not depend on the zero mode of ξ, i.e. are annihilated by $\oint \eta$, are called *small Hilbert space operators*. The totality of all operators are the *large Hilbert space operators*.

Superstring scattering amplitudes are calculated from the pseudoforms Ω_A introduced in section 2.2.1. We are exclusively interested in the case $A = A_1 A_2 \dots A_n$, where A_i are local operators living at a point p_i on the world sheet. The path-integral corresponds to a superconformal field theory, so that by the *state-operator correspondence* we can think of Ω as a map $\Omega_{g,n}$ that assigns to a state in the n-fold SCFT Hilbert space $A_1 \otimes A_2 \otimes \cdots A_n \in \mathcal{H}^{\otimes n}$ a pseudoform. Equations (2.36b) and (2.36c) imply that $\Omega_{g,n}$ constructs a basic pseudoform for the gauge group that is obtained by requiring that the gauge-parameter vanishes at the marked points p_i to all orders. In terms of the gauge-parameters c and γ this means that we only allow diffeomorphisms and supersymmetry transformations with generators vanishing to all orders at the marked points. Suppose that z_i, θ_i are superconformal coordinates near p_i. Then, the restricted gauge group cannot modify this coordinate frame, but can modify coordinate frames near other points. Thus, the pseudoform $\Omega_{g,n+m}$ is defined on the supermoduli space $\mathfrak{P}_{g,n,m}$ of type II world sheets with $n+m$ marked points $p_i, i = 1, 2, \ldots, n+m$ and a fixed choice of coordinate disc near each puncture point p_i. The discussion is the same if we allow the type II world sheet structure to develop a R singularity near m points $p_i, i = n+1, \ldots, n+m$ and no singularity near the remaining points. We require, however, that the type II world sheet structure is regular away from the p_i. The Baranov-Schwartz transform onto $r|s$-forms (2.37) and their main identity (2.38) can, therefore, be reinterpreted as

$$\begin{aligned} \Omega^{r|s}_{g,n+m} :&\mathcal{H}^{\otimes(n+m)} \to \Omega^{r|s}(\mathfrak{P}_{g,n,m}) \\ &d\Omega^{r|s}_{g,n+m} + (-1)^{r+1}\Omega^{r+1|s}_{g,n+m}Q = 0, \end{aligned} \tag{2.50}$$

where Q is the world sheet BRST operator acting on $\mathcal{H}^{\otimes n}$ via extension as a derivation. The forms $\Omega^{r|s}_{g,n+m}$ form the basis for the geometric construction of superstring field theory. Let us remark that $\mathfrak{P}_{g,n,m}$ contains several disconnected components. Each component corresponds to a topologically distinct way to distribute the punctures over the various boundary components and the bulk. Inside the bulk the order of the punctures does not matter as they can be continuously deformed into each other (assuming the absence of defect lines), but punctures cannot go from bulk to boundary or vice versa without going to infinity inside the moduli space. On the other hand, the cyclic ordering of the punctures on a boundary and the particular distribution of the punctures to the boundary components cannot be continuously changed. As the map $\Omega^{r|s}_{g,n+m}$ is single-valued on $\mathfrak{P}_{g,n,m}$ it must be symmetric under exchange of bulk punctures and cyclically symmetric under moving boundary punctures. As we are mainly concerned with the case $g = 0$ with at most one boundary component, the detailled structure for higher genus and multiple boundary components is not important, see e.g. [27, 32, 59] for a detailed discussion of their algebraic properties.

Traditionally on-shell states are identified with vertex operators. In the geometric setup we identify the physical states with the Q-cohomology classes $H^\bullet(Q)$. The chain map property (2.50) ensures that the de Rham cohomology class of $\Omega^{r|s}_{g,n+m}$ does not depend on the particular representative of the physical state. What is the geometric meaning of $\Omega^{r|s}_{g,n+m}$? The precise answer depends crucially on whether the punctures are at a boundary or not and whether the position is regular or singular. To each possibility there corresponds a traditional set of vertex operators. Let us consider regular boundary punctures first. The typical vertex operator is of the form $A_1 = c\delta(\gamma)V$, with V being a superconformal primary of weight 1. States of this form are Q-closed. Consider now condition (2.36b) for being horizontal. It tells us that insertion of A_1 imposes a weaker condition than that the gauge parameters vanish to all orders. It only requires that the parameters should vanish to lowest order. It follows that $\Omega^{r|s}_{g,n+m}$ is defined on a much smaller supermoduli space where puncture

1 is an NS-type puncture in the -1-picture at the boundary by the discussion in section 2.1.2. By the same token, $A_2 = c\bar{c}\delta(\gamma)\delta(\bar{\gamma})V$ with V a superconformal primary of weight $(1,1)$ defines a Q-closed ghost number 2 state and introduces an NS-NS-type puncture at picture $(-1,-1)$ in the bulk. If we have a singularity in the type II world sheet structure at the puncture, we can insert operators of the form $cVe^{-\phi/2}$ at a boundary puncture. $e^{-\phi/2}$ is just the spin operator mapping from periodic to antiperiodic boundary conditions for the β–γ–ghosts. The insertion of c tells us immediately that we have created an R-type puncture at picture $-\frac{1}{2}$ at that point. The remaining cases can be analysed analogously. We have thus seen that for calculating the superstring S-matrix the $\Omega^{r|s}_{g,n+m}$ reduce to smooth elements in $\Omega^{r|s}(\mathfrak{M}_{g,n,m})$ that need to be integrated over $\mathfrak{M}_{g,n,m}$ to get a numerical value for the S-matrix elements. For $9+1$ dimensional Minkowski space with matter action (2.48) the set of physical states at ghost number 1 or 2 at non-vanishing momentum is exhausted by the given set of representatives. For zero-momentum bulk states there are some extra states like the ghost-dilaton that is responsible for changing the coupling constant [96, 97][5]. For closed strings the space of choices of coordinate discs near a point has the homotopy type of S^1. Since S^1 is not contractible, some exact forms on $\mathfrak{P}_{g,n,m}$ will not descend to exact forms on $\mathfrak{M}_{g,n,m}$. The condition that ensures that exact forms stay exact is the *level matching condition*, which implies that $b_0^- A_i = L_0^- A_i = 0$, where $b_0^- = b_0 - \bar{b}_0$, $L_0^- = [Q, b_0^-]$. The level matching condition ensures that all forms are basic w.r.t. rigid rotations of the coordinate disc. Vertex operators in the standard form satisfy this constraint.

2.3 Homotopy algebras and classical BV theories

In its traditional form string theory is not a field theory, but a first quantised theory. Thus, strictly speaking, there is no notion of multistring states or bound states of strings in the sense of states in a Fock space. All that is provided is the S-matrix in the sense of S-matrix theory as a perturbation series around the free theory. While in principle knowledge of the S-matrix should determine all physical coupling constants in a second quantised Hamiltonian, explicit solutions to this inverse scattering problem are only known for some low-energy degrees of freedom

[5]For example, the ghost-dilaton Φ is of the form $\Phi = Q(\partial c - \bar{\partial}\bar{c})$ and produces an exact 2-form on the fibre of $\mathfrak{P}_{g,n+1,m} \to \mathfrak{P}_{g,n,m}$. The fibre is just a copy of the surface with a choice of coordinate disc nearby. The form defined by Φ is horizontal w.r.t. rigid rotations of the coordinate disc and, hence, we can integrate the form generated by Φ over a two-dimensional submanifold of the fibre that consists of points of Σ and a fixed, smooth choice of coordinate disc nearby. Although Φ is exact, the exact form is not smooth on that submanifold as $\partial c - \bar{\partial} c$ is sensitive to the rigid orientation of the coordinate disc. Locally, the submanifold has tangent vectors given by $c = -\delta z/\partial_0 z$, where $\partial_0 = \frac{d}{dz_0}$ with z_0 a reference coordinate system. After covering the submanifold with charts defined by a reference coordinate disc z_i, we can integrate Φ and a left with contributions from the boundaries of the charts, $-\delta z \partial \log(\partial_i z^j)$. It follows that upon integrating the latter form over the boundaries, we actually have integrated the first Chern class of the canonical bundle of the submanifold, which is proportional to its Euler character [96].

in the form of effective actions. Without a second quantised formulation physical questions that go beyond scattering processes are out of reach. Such problems would include the calculation of equilibrium and non-equilibrium thermal properties of string matter at energy scales that are not small compared to the string scale, such as immediately after the big bang. String field theory is an attempt to reformulate traditional string theory in Lagrangian form. The resulting actions are structurally local field theories subject to a gauge-invariance, although they may have infinitely many vertices. Classical BV theories provide a very general framework to describe and construct gauge theories. In this section we describe the expected BV theories algebraically together with the homotopy algebraic structures on the field space. The algebraic structures are more important as they provide simple computational tools.

2.3.1 Open strings and A_∞ algebras

String theories have two sectors: the open string sector and closed string sector. The algebraic structures implied by the classical BV master equation are different in both cases. In this section we are mainly concerned with the description of open strings. Consider a background with N coincident D-branes. Among the massless modes around this background we find N^2 massless vector particles that can be interpreted as gluons. General field theoretical arguments based on locality and Lorentz-invariance imply that the low energy dynamics is governed by an $\mathrm{U}(N)$-gauge theory on the D-branes. Adding more branes that intersect the stack of D-branes gives rise to matter fields in the fundamental representation. An important property of gauge theories of the described type is that tree-level amplitudes can be decomposed into sums over *colour-ordered amplitudes*. The colour-ordered amplitudes are defined as sums over *planar Feynman graphs*. The notion of planarity depends on the details of the underlying field theory. For $\mathrm{U}(N)$-Yang-Mills theories planarity is derived from the isomorphism of the adjoint representation with the tensor product for the fundamental with the antifundamental representation $(\mathbf{N^2}-\mathbf{1})\oplus\mathbf{1}\cong\mathbf{N}\otimes\bar{\mathbf{N}}$ [98].

More abstractly, one can think of colour-ordered amplitudes as open string tree-level amplitudes. A classical BV theory encodes all information about the equations and the gauge structure of a theory in the master action S that solves a classical master equation $(S,S)=0$. S can be regarded as a function on some ambient, well-behaved $\mathbb{Z}$-manifold of fields and the leaf space of the action of the gauge group on the locus $dS=0$ at degree 0 as the underlying phase space of solutions. The phase space is very singular in general and S encodes all information necessary to construct invariant objects on phase space. In particular, it contains all information about the phase space restricted to a formal, perturbative neighbourhood of any point in it. When considering colour-ordered amplitudes the relevant phase space can still be described through a solution to the classical master equation, but the underlying field manifold is non-commutative.

The basic setup is as follows [22, 99]: We assume that the field manifold is locally modelled by a $\mathbb{Z}$-graded vector space $\mathcal{H}$. The homogeneous grading of an element a is called *degree* and is denoted by $\deg(a)$. Locally the ring of functions is the free associative algebra generated from the linear dual $\mathcal{H}'$ and can be thought of as the tensor algebra $T\mathcal{H}'$ with the tensor product as product. Geometrical structures on field space are defined in the sense of non-commutative geometry. For example vector fields are defined as derivations of the algebra $T\mathcal{H}'$ valued in the $T\mathcal{H}'$-bimodule $T\mathcal{H}'$, i.e. as linear maps $D : T\mathcal{H}' \to T\mathcal{H}'$ subject to the (graded) Leibniz rule $D(ab) = D(a)b + (-1)^{\deg(D)\deg(a)} aD(b)$. The bimodule of one forms is the module of Kähler differentials consisting of elements of the form $f(dg)h$ for functions f, g and h. Higher order forms are defined analogously.

Algebraically, the dual space of the function ring $(T\mathcal{H}')'$ plays the role of measures. Among the linear functionals the algebra homomorphisms play the role of commutative points or $\mathbb{C}$-points. Ignoring topological complications, linear functionals can be identified with elements in $T\mathcal{H}$ and commutative points can be identified with *group-like elements* of the form

$$e^a = \sum_{k=0}^{\infty} a^{\otimes k}, \tag{2.51}$$

where $a \in \mathcal{H}$ is arbitrary. Moreover, multiplication on $T\mathcal{H}'$ becomes a *comultiplication* Δ on $T\mathcal{H}$, the unit $1 \in T\mathcal{H}'$ becomes a *counit* ϵ on $T\mathcal{H}$ and derivations D translate to *coderivations* $\mathbf{M}$ on $T\mathcal{H}$ via the usual Kronecker pairing. Formally they enrich $\mathcal{A} = T\mathcal{H}$ to a tensor coalgebra with the axioms,

$$\begin{aligned}(\Delta \otimes' \mathbb{I}_{\mathcal{A}})\Delta &= (\mathbb{I}_{\mathcal{A}} \otimes' \Delta)\Delta \\ (\epsilon \otimes' \mathbb{I}_{\mathcal{A}})\Delta &= (\mathbb{I}_{\mathcal{A}} \otimes' \epsilon)\Delta = \mathbb{I}_{\mathcal{A}}.\end{aligned}$$

With this terminology coderivations $\mathbf{M}$ and group-like elements e^a can be characterised as

$$\begin{aligned}\Delta\mathbf{M} &= (\mathbb{I}_{\mathcal{A}} \otimes' \mathbf{M} + \mathbf{M} \otimes' \mathbb{I}_{\mathcal{A}})\Delta, \\ \Delta e^a &= (e^a \otimes' e^a)\Delta.\end{aligned}$$

It can be shown [22] that every coderivation can be written uniquely as a sum $\mathbf{M} = \sum_{k=0}^{\infty} \mathbf{M}_k$, where $\mathbf{M}_k$ is of the form

$$\mathbf{M}_k = \sum_{r,s\geq 0} \mathbb{I}^{\otimes r} \otimes M_k \otimes \mathbb{I}^{\otimes s} \tag{2.52}$$

with $M_k : \mathcal{H}^{\otimes k} \to \mathcal{H}$. See [56] for a detailed proof. We call M_k the k-string product in $\mathbf{M}$ or simply the k-product and write coderivations always in bold face. Another important property of coderivations is their closure under taking (graded) commutators. We also need the dual notion of an algebra morphism called a *cohomomorphism* $\mathcal{F} : \mathcal{A}_1 \to \mathcal{A}_2$ that is defined as a linear map intertwining the two coalgebra structures,

$$\begin{aligned}\Delta_2\mathcal{F} &= (\mathcal{F} \otimes_2 \mathcal{F})\Delta_1 \\ \epsilon_1 &= \epsilon_2\mathcal{F}.\end{aligned}$$

Any cohomomorphism between tensor coalgebras $T\mathcal{H}_1$ and $T\mathcal{H}_2$ is completely characterised by its projections $f_k = \pi_1 \mathcal{F} \iota_k : \mathcal{H}_1^{\otimes k} \to \mathcal{H}_2$, where π_k and ι_k are the canonical projections $\pi_k : T\mathcal{H}_1 \twoheadrightarrow \mathcal{H}_1^{\otimes k}$ and inclusion maps $\iota_k : \mathcal{H}_1^{\otimes k} \hookrightarrow T\mathcal{H}_1$. The most general form of a cohomomorphism is given by [56]

$$\mathcal{F} = \sum_{n=0}^{\infty} \sum_{r_1, r_2, \ldots, r_n} f_{r_1} \otimes f_{r_2} \otimes \cdots \otimes f_{r_n} = \sum_{n=0}^{\infty} (\pi_1 \mathcal{F})^{\otimes n}. \tag{2.53}$$

If a non-commutative vector field D satisfies $D^2 = 0$, it is called a *differential*. Likewise, we have for the associated coderivation $\mathbf{M}^2 = 0$ and $\mathbf{M}$ is called a *codifferential*. A codifferential $\mathbf{M}$ of degree $+1$ defines a *weak homotopy associative algebra* on $\mathcal{H}$. If the 0-product (or tadpole) vanishes, i.e. $M_0 = 0$, the algebraic structure is called a *strongly homotopy associative algebra* or A_∞ *algebra*. The first few axioms of an A_∞-algebra are

$$\begin{aligned} M_1 M_1 &= 0 \\ M_1 M_2 + M_2(M_1 \otimes \mathbb{I} + \mathbb{I} \otimes M_1) &= 0 \\ M_1 M_3 + M_3(M_1 \otimes \mathbb{I}^{\otimes 2} + \mathbb{I} \otimes M_1 \otimes \mathbb{I} + \mathbb{I}^{\otimes 2} \otimes M_1) + M_2(M_2 \otimes \mathbb{I} + \mathbb{I} \otimes M_2) &= 0. \end{aligned}$$

The solutions to the equations of motion are commutative points e^a on which the cohomological vector field D vanishes. In terms of the coderivation $\mathbf{M}$ this condition is called the *Maurer-Cartan equation*,

$$M_1(a) + M_2(a, a) + M_3(a, a, a) + \ldots = 0. \tag{2.54}$$

The gauge invariance of solutions e^a takes the form

$$\delta a = M_1(\Lambda) + M_2(\Lambda, a) + M_2(a, \Lambda) + \ldots . \tag{2.55}$$

The operator $d_{\mathbf{M}} = [\mathbf{M}, \cdot]$ is a nilpotent derivation on the space of coderivations on $T\mathcal{H}$. It is called the *Hochschild differential* and calculates Hochschild cohomology $HH^\bullet(\mathcal{H}, \mathcal{H})$ of the A_∞-algebra, where the grading is given by the order of the string product. Hochschild cohomology corresponds to non-trivial infinitesimal deformations of $\mathcal{H}$ as an A_∞ algebra and plays an important role in the later chapters. Moreover, together with the Hochschild differential the space of coderivations becomes a differential graded Lie algebra (dgLA) when equipped with the commutator bracket $[\cdot, \cdot]$ of coderivations [100].

A formal manifold equipped with a square-zero vector field D is called a Q-manifold. In order to formulate BV theory for open strings we need a notion of cyclicity. Formally, we need a QP-manifold. A QP-manifold is a Q-manifold together with a D-invariant symplectic form ω of degree -1. Consider the subvector space $(T\mathcal{H}')_c \subset T\mathcal{H}'$ of cyclic functionals. Physically, this subset may be thought of as the space of single trace operators. Likewise, one can define the subspace of cyclic differential forms. We assume that the symplectic form ω is cyclic. Since ω is a D-invariant form, it follows that we can write $dS = \iota_D \omega$. If we can choose the Hamiltonian S as a cyclic functional, we call D compatible with cyclicity. If D has degree 1, S has degree 0. The condition $D^2 = 0$ is then equivalent to the master equation $(S, S) = 0$, where $(\cdot, \cdot)$ is the Poisson bracket associated to the symplectic structure ω. By the non-commutative version of the Darboux theorem [22] we may find a coordinate frame in which the symplectic form is constant

$$\omega = \frac{1}{2}\langle\omega|d\phi\rangle|d\phi\rangle, \tag{2.56}$$

where $\langle\omega| : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ is a graded antisymmetric bilinear form on $\mathcal{H}$ of degree -1 and ϕ denotes the coordinate function. One may translate the condition of cyclicity into an invariance condition for $\langle\omega|$,

$$\langle\omega|(\mathbb{I} \otimes M_k + M_k \otimes \mathbb{I}) = 0. \tag{2.57}$$

If $\mathbf{M}$ defines an A_∞ algebra and satisfies the cyclicity condition (2.57) it is called a *cyclic A_∞ algebra*. Cyclic A_∞ algebras are in one-to-one correspondence with solutions to the classical, cyclic master equation $(S, S) = 0$. For cyclic R, the Hamiltonian vector fields $(R, \cdot)$ are in one-to-one correspondence with coderivations preserving the symplectic form. More importantly, the Poisson bracket $(\cdot, \cdot)$ translates into the commutator of the associated coderivations.

In open string theory the colour-ordered amplitudes are constructed by attaching propagators to planar, cyclic vertices. Each vertex can therefore be identified with a degree 0, cyclic functional. Let R_1 and R_2 be two cyclic functionals encoding two types of vertices. (R_1, R_2) is again a cyclic functional and is explicitly given by

$$(R_1, R_2) = R_1 \frac{\overleftarrow{\partial}}{\partial\phi^i}(\omega^{-1})^{ij}(-1)^{\deg(j)}\frac{\overrightarrow{\partial}}{\partial\phi^j}R_2 + \text{ cyclic }, \tag{2.58}$$

where the inverse ω^{-1} is defined in equation (2.66). If R_i have homogeneous degree n_i, we can visualise this operation as taking the sum over all possible ways to contract a leg from vertex 1 with vertex 2 using the inverse of the symplectic form.

In open string field theory the graded vector space $\mathcal{H}$ is taken as the Hilbert space of the underlying boundary conformal field theory with the degree being related to ghost number as $\deg(a) = \mathrm{gh}(a) - 1$. In homological algebra such a shift in degree is known as a *suspension*. Let us describe the impact of the suspension using Witten's open bosonic string field theory. The action of Witten's open bosonic string field theory [10] is formulated in terms of the world-sheet BPZ inner product $\langle\cdot, \cdot\rangle$, the world-sheet BRST operator Q and a binary product $*$. These algebraic operations act on the Hilbert space of the underlying world-sheet CFT and form a differential graded algebra (DGA). Furthermore $\langle\cdot, \cdot\rangle$ is an invariant, graded-symmetric bilinear form of ghost number -3. The action of the bosonic string reads

$$S = \frac{1}{2}\langle\Phi, Q\Phi\rangle + \frac{1}{3}\langle\Phi, \Phi * \Phi\rangle.$$

The string field Φ is an element in the CFT Hilbert space at ghost number 1. It turns out [22] that gauge-invariance of S is equivalent to $(Q, *)$ forming a DGA, i.e. they verify the following axioms, $a, b, c \in \mathcal{H}$:

$$Q^2 = 0 \tag{2.59a}$$
$$Q(a * b) = Qa * b + (-1)^{\mathrm{gh}(a)}\, a * Qb, \tag{2.59b}$$
$$(a * b) * c = a * (b * c). \tag{2.59c}$$

With these physical conventions Φ is an odd quantity. Let us further recall that Q carries ghost number 1 and $*$ carries no ghost number. Temporarily we denote by $\mathcal{H}[1]$ the graded vector space with grading given by degree. The suspension map $s : \mathcal{H}[1] \to \mathcal{H}$ has degree 1 and reduces to the identity on the underlying vector spaces. We have that $(s \otimes s)(s^{-1} \otimes s^{-1}) = -\mathbb{I} \otimes \mathbb{I}$. Moreover, s is invertible and we can introduce a new string field $\phi = s^{-1}\Phi \in \mathcal{H}[1]$. ϕ is even and carries degree 0. As s is invertible, we can express the DGA axioms equivalently on $\mathcal{H}[1]$,

$$\begin{aligned} M_1 M_1 &= 0 \\ M_1 M_2 + M_2(M_1 \otimes \mathbb{I} + \mathbb{I} \otimes M_1) &= 0 \\ M_2(M_2 \otimes \mathbb{I} + \mathbb{I} \otimes M_2) &= 0, \end{aligned}$$

where $M_1 = s^{-1}Qs$ and $M_2 = s^{-1} * (s \otimes s)$. If we set $M_k = 0, k \geq 3$ it follows that a DGA is a special case of an A_∞ algebra and the effect of the suspension map is the elimination of unnecessary minus signs in the DGA axioms. Furthermore, Witten's bosonic OSFT is also equipped with an invariant bilinear form $\langle \cdot, \cdot \rangle$. The invariance follows from the cyclicity of the action and states that for $a, b, c \in \mathcal{H}$,

$$\begin{aligned} \langle Qa, b \rangle + (-1)^{\mathrm{gh}(a)} \langle a, Qb \rangle &= 0 \\ \langle a, b * c \rangle &= \langle a * b, c \rangle. \end{aligned}$$

In terms of elements in the suspended Hilbert space $\mathcal{H}[1]$ cyclicity can be reexpressed in terms of the symplectic form $\omega = \langle \cdot, \cdot \rangle s^{\otimes 2}$ and reproduces equation (2.57). The suspension converts the formerly graded symmetric form into a graded antisymmetric map of degree -1.

2.3.2 Closed strings and L_∞ algebras

The low energy sector of closed string theory is a flavour of supergravity with additional matter fields. Unlike open strings the amplitudes do not allow for a colour decomposition and are given by sums over Feynman diagrams with totally symmetric vertices. As a reference for this section we give [101, 102]. When formulating a BV theory for closed strings, we impose that all functionals should be totally symmetric. More precisely, the algebra of functions is given by the symmetric algebra $S\mathcal{H}'$. The rest of the construction follows the open string case very closely. On top of the graded manifold of fields we postulate a QP-structure, i.e. a nilpotent vector field D of degree 1 and an invariant symplectic form ω. The space of linear functionals is isomorphic to the symmetric algebra $S\mathcal{H}$. Points on the manifold are described by the group-like elements and take the form

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^{\wedge k}. \tag{2.60}$$

Through the Kronecker pairing $S\mathcal{H}$ can be endowed with a comultiplication Δ and a counit ϵ. The comultiplication is now cocommutative and turns $S\mathcal{H}$ into a cocommutative tensor coalgebra on $\mathcal{H}$. The duals of derivations are again coderivations. The relations satisfied by cohomomorphisms, coderivations and the coalgebra structure are the same as in the open string case. Denote the dual coderivation for D by $\mathbf{L}$.

The analogue of the tensor product of maps might be unfamiliar. We therefore give an explicit formula. If $f_i : \mathcal{H}^{\wedge k_i} \to \mathcal{H}, i = 1, 2, \ldots, N$ are linear maps, we define their product, $M = \sum_i k_i$,

$$\begin{aligned} f_1 \wedge f_2 \wedge \cdots \wedge f_N :&\mathcal{H}^{\wedge M} \to \mathcal{H}^{\wedge N} \\ (f_1 \wedge f_2 \wedge \cdots \wedge f_N)(a_1 a_2 \ldots a_M) =& \sum_{\substack{\{1,2,\ldots,M\} = \amalg_{i=1}^{N} S_i \\ |S_i| = k_i}} (\pm) f_1(a_{S_1}) f_2(a_{S_2}) \cdots f_N(a_{S_N}), \end{aligned}$$

where $a_S = \prod_{i \in S} a_i$ and $(\pm)$ is the usual Koszul sign from changing the order of the objects. The sum runs over all splittings of the set $\{1, 2, \ldots, M\}$ into N disjoint subsets. With this definition, the most general form of a coderivation is $\mathbf{L} = \sum_{k=0}^{\infty} \mathbf{L}_k$, where $\mathbf{L}_k$ is of the form

$$\mathbf{L}_k = \sum_{r=0}^{\infty} L_k \wedge \mathbb{I}_r, \tag{2.61}$$

where $L_k : \mathcal{H}^{\wedge k} \to \mathcal{H}$. We also introduce the notation $n!\, \mathbb{I}_n = \mathbb{I}^{\wedge n}$ for the n-fold identity map. A cohomomorphism $\mathcal{F} : \mathcal{H}_1 \to \mathcal{H}_2$ is completely specified by its projection onto $\mathcal{H}$. Writing $f_k = \pi_k \mathcal{F} \iota_k : \mathcal{H}_1^{\wedge k} \to \mathcal{H}_2$ it takes the form

$$\mathcal{F} = \sum_{n=0}^{\infty} \sum_{\substack{r_1 < r_2 < \ldots < r_n \\ k_1, k_2, \ldots, k_n \geq 1}} \frac{1}{k_1! k_2! \ldots k_n!} f_{r_1}^{\wedge k_1} \wedge f_{r_2}^{\wedge k_2} \wedge \cdots \wedge f_{r_n}^{\wedge k_n}.$$

If the derivation D is a differential, then the associated coderivation is square-zero $\mathbf{L}^2 = 0$. In this case the components L_k satisfy the axioms, assuming $L_0 = 0$,

$$\begin{aligned} L_1 L_1 &= 0 \\ L_2(L_1 \wedge \mathbb{I}) + L_1 L_2 &= 0 \\ L_3(L_1 \wedge \mathbb{I}_2) + L_2(L_2 \wedge \mathbb{I}) + L_1 L_3 &= 0. \\ &\vdots \end{aligned}$$

The set of component maps L_k defines the structure of a *weak homotopy Lie algebra* on $\mathcal{H}$. If the tadpole L_0 vanishes, the structure is called a *strong homotopy Lie* *algebra* or L_∞ *algebra*. The operator $d_{\mathbf{L}} = [\mathbf{L}, \cdot]$ is a nilpotent derivation on the space of coderivations on $S\mathcal{H}$. It is called the *Chevalley-Eilenberg differential* and plays a role analogous to the Hochschild differential in the associative case. Its cohomology is called the *Chevalley-Eilenberg cohomology* or *Lie algebra cohomology* of $\mathbf{L}$.

Locally, the degree -1 symplectic structure ω can be brought into Darboux form. This form is characterised by a graded antisymmetric bilinear form $\langle\omega|$ and the invariance condition reduces to

$$\langle\omega|(\mathbb{I}\otimes L_k + L_k \otimes \mathbb{I}) = 0.$$

In the presence of an invariant symplectic form, the algebraic structure given by $\mathbf{L}$ is called a *cyclic L_∞ structure*. It follows in particular that all conventional classical BV theories are equivalent to cyclic L_∞ algebras on a suitable model space $\mathcal{H}$.

Closed bosonic string backgrounds give rise to a plenty of cyclic L_∞ algebras. The model space is given by the subspace $\mathcal{H}$ of the CFT Hilbert space spanned by states that satisfy the *level-matching constraints*,

$$L_0^- = 0, \qquad\qquad b_0^- = 0. \tag{2.62}$$

The grading on this space is induced by ghost number. Since we are only interested in algebraic properties, we assume that no higher products beyond a binary product are needed for consistency of the theory. The algebraic ingredients are given by a differential Q of ghost number $+1$ and a bracket $[\cdot,\cdot]$ of ghost number -1. We assume further that Q is a derivation of the bracket. The full set of axioms is, $a, b, c \in \mathcal{H}$,

$$\begin{aligned}
0 &= Q^2\\
0 &= Q[a,b] + [Qa,b] + (-1)^{\mathrm{gh}(a)}[a,Qb]\\
0 &= [[a,b],c] + (-1)^{\mathrm{gh}(a)(\mathrm{gh}(b)+\mathrm{gh}(c))}[[b,c],a] + (-1)^{\mathrm{gh}(c)(\mathrm{gh}(a)+\mathrm{gh}(b))}[[c,a],b].
\end{aligned}$$

The invariant symplectic structure is given in terms of the the BPZ inner-product,

$$\langle a,b\rangle = (-1)^{\mathrm{gh}(a)}\langle a, c_0^- b\rangle_{\mathrm{BPZ}}. \tag{2.63}$$

It is a graded antisymmetric form and carries ghost number -5. Translating this to the symmetric coalgebra construction requires that we perform a double suspension to $\mathcal{H}[2]$. Formally this is achieved by introducing the double suspension map $s : \mathcal{H}[2] \to \mathcal{H}$ that carries degree 2 and acts as the identity on the underlying vector space. Since s carries even degree, it does not modify any relative sign factors and does not change the symmetry properties of the symplectic form. With the identifications $L_1 = s^{-1}Qs$, $L_2 = s^{-1}[\cdot,\cdot]s\otimes s$, $\langle\omega| = \langle\cdot,\cdot\rangle s\otimes s$, one checks immediately, that this defines a cyclic L_∞ algebra. The full closed string contains also higher products $L_3, L_4, \ldots$ up to all orders. Let us call this cyclic L_∞ algebra $\mathcal{H}_c$, the closed string L_∞ algebra.

The relevance of $\mathcal{H}_c$ comes from its relation with the dgLA comprised of the Hochschild differential $d_{\mathbf{M}}$ and the Gerstenhaber bracket, which we again call $[\cdot,\cdot]$. The underlying vector space is $\mathrm{Coder}(\mathcal{H}_o)$, the vector space of coderivations on the tensor coalgebra for the open string Hilbert space $\mathcal{H}_o$. After passing to the suspended vector space $\mathrm{Coder}(\mathcal{H}_o)[1]$ the dgLA axioms can be translated into an L_∞ algebra. Open-closed string theory can be described as an L_∞-morphisms from the L_∞ algebra $\mathcal{H}_c$ to this algebra [27]. Kontsevich's deformation quantisation construction provides a concrete example of such an L_∞-morphism that is in fact an L_∞-quasi-isomorphism [27, 103, 104].

Every associative algebra gives rise to a Lie algebra by the commutator bracket. From the homotopy algebraic setup this can be understood as symmetrising the associative product. Conversely, to every Lie algebra one can construct its *universal enveloping algebra*. Taking the Lie algebra of an universal envelope recovers the original Lie algebra. For all practical purposes one may, hence, replace a Lie algebra with its universal envelope. The construction of the universal envelope can be extended to the homotopical setting [102, Theorem 3.3]. The functor $(-)_L : A_\infty \to L_\infty$ from the category of A_∞ algebra to the category of L_∞ algebras that symmetrises the A_∞-algebra has a left-adjoint functor $\mathcal{U} : L_\infty \to A_\infty$, the universal envelope.

2.4 Integration over supermoduli space

The calculation of a superstring scattering amplitude is divided into several steps. First, we need to construct an SCFT that serves as the background, then we have to evaluate the forms $\Omega^{r|s}_{g,n+m}$ for the physical states in question and, finally, we have to integrate the resulting $r|s$-form over supermoduli space to obtain the S-matrix element. Among these steps the last is technically the most difficult and the least well understood. Although many properties can be deduced to arbitrary genus from looking at the behaviour of $\Omega^{r|s}_{g,n+m}$ near infinity, most calculations in the literature have been concerned with tree level or, at most, one loop amplitudes. Moreover, for superstrings one has to take into account the subtleties of the supermoduli space. For a recent, general discussion of superstring scattering amplitudes from the supermoduli point of view see [95].

In this section we describe the integration procedure in detail. We construct local sections of $\mathfrak{P}_{g,n,m} \to \mathfrak{M}_{g,n,m}$ starting from local sections of the underlying bosonic moduli spaces $\mathcal{P}_{g,n,m} \to \mathcal{M}_{g,n,m}$. We discuss the appearance of contact terms at the boundaries between two patches and argue that they are generically present, even if the underlying bosonic sections fit together nicely. We then chop the total integral into smaller parts and cover most of the moduli space with sections constructed by plumbing fixture of topologically simpler world sheets with coordinate discs. The discussion remains restricted to colour-ordered tree level amplitudes. We argue that the partial integrals together with suitable contact terms give rise to a cyclic A_∞ algebra as in the bosonic case and that the total integral can be identified with a Feynman diagram like expansion. It follows that the construction problem can be reduced to a purely algebraic problem. The resulting classical field theory is called *open superstring field theory.*

2.4.1 Local fibrewise integration

If we evaluate $\Omega^{r|s}_{g,n+m}$ on Q-closed states, we obtain a closed $r|s$-form on $\mathfrak{P}_{g,n,m}$ which we can pullback to $\mathfrak{M}_{g,n,m}$ by a local section σ of the fibre bundle $\mathfrak{P}_{g,n,m} \to \mathfrak{M}_{g,n,m}$. Since $\mathfrak{M}_{g,n,m}$ has dimension $3g-3+n+m|2g-2+n+m/2$ we obtain local sections of the Berezinian line bundle $\mathrm{Ber}(\mathfrak{M}_{g,n,m})$ by $\sigma^*\Omega^{3g-3+n+m|2g-2+n+m/2}_{g,n,m}$. The string scattering amplitude is then equal to the integral of this section. By the chain map property (2.50) we see immediately that replacing one of the external states A by $A+Q\alpha$ the section of the Berezinian changes by a total derivative so that the value of the integral does not change for generic external momentum. Unfortunately, this prescription requires that σ must be a global section. Otherwise we receive contributions from the boundaries between the domains where the local sections are defined.

For the bosonic moduli spaces $\mathcal{P}_{g,n,m} \to \mathcal{M}_{g,n,m}$ the construction of a global section is well understood[6]. It is given through the unique solution to a minimal area metric problem [13, 105]. Henceforth we assume that a (local) section σ_0 of the bosonic bundle has been constructed by some method. Suppose that σ and σ' are lifts of σ_0 to local sections of $\mathfrak{P}_{g,n,m} \to \mathfrak{M}_{g,n,m}$. Let (t,τ) be coordinates on $\mathfrak{P}_{g,n,m}$ and (x,χ) coordinates on $\mathfrak{M}_{g,n,m}$. Since σ and σ' have to agree when ignoring the odd coordinates τ and χ, schematically they must take the form

$$\begin{aligned} \sigma^* t &= \sigma_0^* t + \text{nilpotent}, & \sigma^*\tau &= \text{nilpotent} \\ \sigma'^* t &= \sigma_0^* t + \text{nilpotent}, & \sigma'^*\tau &= \text{nilpotent}. \end{aligned}$$

Thus σ and σ' differ only by nilpotent terms. Given a section σ we can construct a new section that is infinitesimally close to σ via changing it by the flow of a vector field V on $\mathfrak{P}_{g,n,m}$ that vanishes when restricted to the underlying bosonic section σ_0. The change in the integrals over σ and $\sigma+\delta\sigma$ is

$$\begin{aligned} \int(\sigma+\delta\sigma)^*\Omega^{3g-3+n+m|2g-2+n+m/2}_{g,n+m} &- \int\sigma^*\Omega^{3g-3+n+m|2g-2+n+m/2}_{g,n+m} \\ &= \int\sigma^*\mathcal{L}_V\Omega^{3g-3+n+m|2g-2+n+m/2}_{g,n+m} \\ &= \int d\sigma^*\iota_V\Omega^{3g-3+n+m|2g-2+n+m/2}_{g,n+m} + \int\sigma^*\iota_V d\Omega^{3g-3+n+m|2g-2+n+m/2}_{g,n+m}. \end{aligned} \tag{2.64}$$

The first term is a total derivative and reduces to an integral over the boundary of the region where the local sections are defined. The second term is Q-exact by

[6] For closed strings this bundle has no global projection. In this case the level matching conditions (2.62) ensure that $\Omega^{r|s}_{g,n+m}$ descends to a form on $\hat{\mathfrak{P}}_{g,n,m}$, where we divided out the rigid rotations of the coordinate discs. It turns out that the underlying bosonic bundle has indeed a global section. Henceforth, when dealing with closed strings we always assume that we work with $\hat{\mathfrak{P}}_{g,n,m}$ instead of $\mathfrak{P}_{g,n,m}$.

equation (2.50). A few comments are in order. We see that infinitesimal changes in the section give rise to boundary terms and Q-exact terms. One might worry that there are global issues with extending this statement to arbitrary pairs of section σ and σ' extending σ_0. Due to the nilpotent nature of the difference between them, at least locally there should be no obstruction to finding a homotopy interpolating between them [47]. A more serious issue is the presence of *spurious poles* [50, 60] in the superstring measure that need to be avoided by the interpolating homotopy. Spurious poles arise from an improper gauge-fixing procedure of the world-sheet theory and occur precisely when the number of zero modes for the $\beta-\gamma$ ghost system becomes non-minimal, i.e. at points where the line bundle $\mathcal{L}$ satisfies $h^0(\mathcal{L}^{-1}) > 0$, a criterion that can be described by the vanishing of a Riemann theta function [50, 93]. In [50] it was argued that one can always avoid the spurious poles by a *vertical integration* procedure. Spurious poles do not occur for tree level amplitudes so that we ignore them henceforth. In general a properly posed analogue of the *minimal area metric problem* should describe a way to circumvent this problem.

2.4.2 Feynman graphs and supermoduli space at infinity

Factorisation of the superstring scattering amplitudes originates in the regions of the supermoduli space near infinity. Near them we can use the plumbing fixture construction from section 2.1.3 to provide us with a well-behaved coordinate system on $\mathfrak{P}_{g,n,m}$. Henceforth we restrict our attention to the moduli space $\mathfrak{P}_{0,n,m}$ of superdiscs endowed with n NS punctures and m R punctures and coordinate systems near them. We call a surface $\Sigma \in \mathfrak{P}_{0,n,m}$ *stable*, if its image in $\mathfrak{M}_{0,n+m}$ has no infinitesimal automorphisms. This is precisely the case if $n+m \geq 3$. The full amplitude receives contributions from the various components in $\mathfrak{M}_{0,n+m}$ corresponding to different colour-orderings. We restrict our attention further to one such connected component in which the punctures are labelled $1, 2, \ldots, n+m$ along the boundary in positive direction. The regions near the Deligne-Mumford compactification divisors are enumerated by the topological different ways to choose inequivalent strips to which plumbing fixture can be applied and such that the components pinched off by the strip remain stable. It can be shown [106, 107] that there is a one-to-one correspondence between the topologically different ways to perform plumbing fixture and rooted planar graphs with puncture 1 being assigned to the root and with each vertex having at least three edges attached to it. We call such diagrams *stable trees*. Figures 2.3 and 2.4 illustrate that correspondence for four- and five-punctured discs.

More concretely, consider a stable tree Γ with set of vertices V. For $v \in V$ we denote by $k_v = n_v + m_v$ the valence of the vertex v, where n_v is the number of NS type legs and m_v the number of R type legs. For each vertex v we choose an element $\Sigma_v \in \mathfrak{P}_{0,n_v,m_v}$, to each NS internal line i a non-negative number τ_i and to each R internal line j a pair $\tau_j|\alpha_j$. The stable tree Γ gives a construction plan for building more complicated world sheets out of these choices. The enumeration and the types of the external legs are completely specified by giving Γ. Algebraically, we obtain a

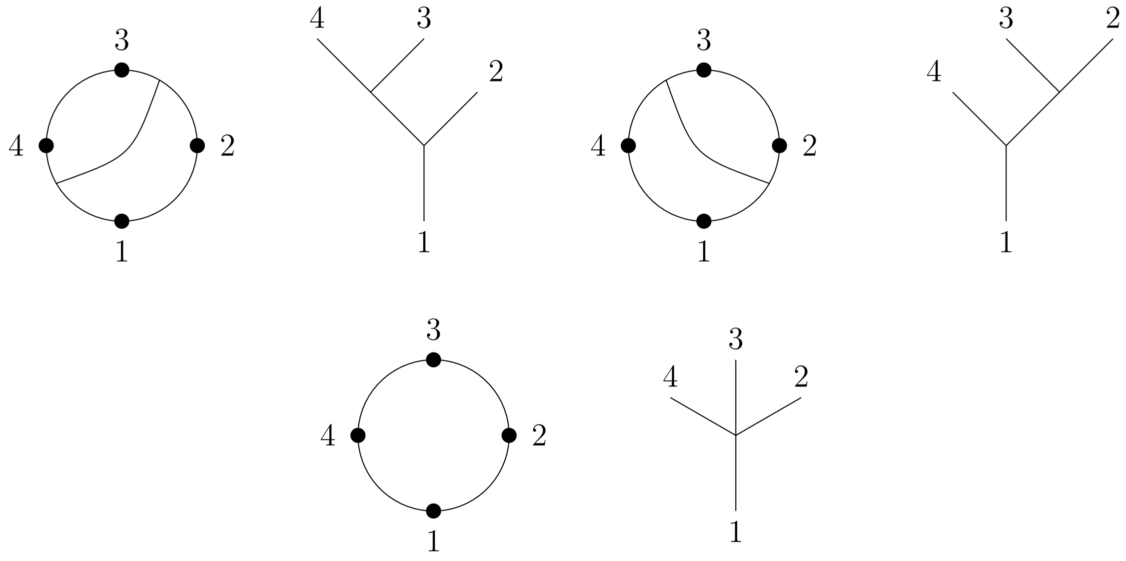


Figure 2.3: Illustration of the correspondence between topologically distinct ways to perform the (iterated) plumbing fixture construction on a four punctured disc and planar rooted trees with three leaves. The topological picture corresponds to the tree left to it.

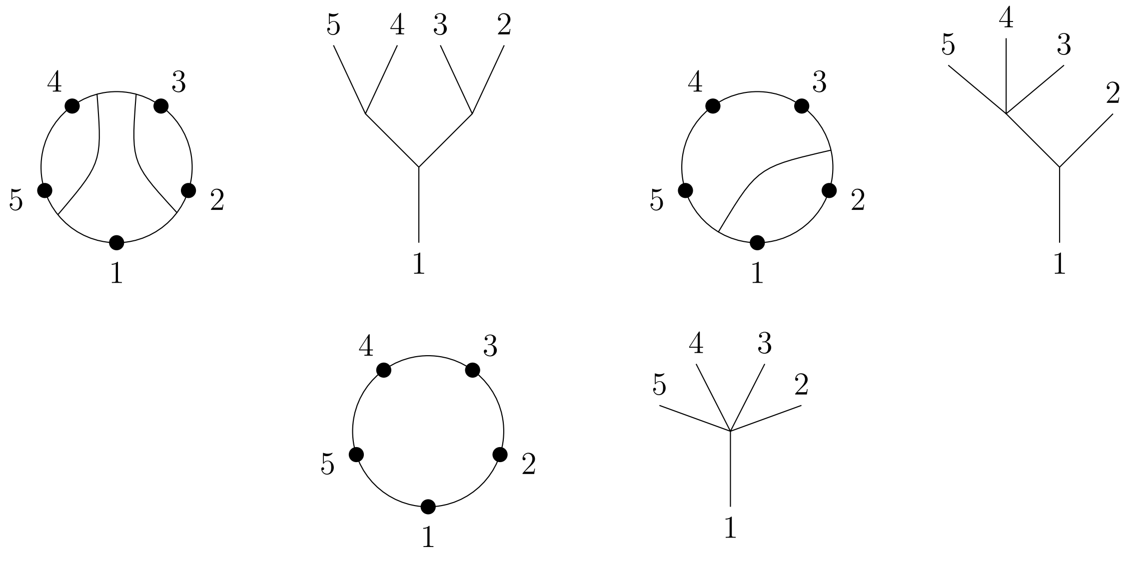


Figure 2.4: Examples of topologically distinct choices for the iterated plumbing fixture construction for the five-punctured disc. There are five diagrams of the first type, five diagram of the second type and one diagram of the third type.

map

$$\Phi_\Gamma : (\mathbb{C}^\times)^{\times N} \times G^{\times M} \times \mathfrak{P}_{0,n_{v_1},m_{v_1}} \times \ldots \to \mathfrak{P}_{0,n,m},$$

where N denotes the total number of NS internal lines and M the total number of R internal lines. $i \mapsto v_i$ is an enumeration of the vertices. At this point there might be an ambiguity in which order the gluing should be performed. The origin is that elements in $\mathfrak{P}_{0,n,m}$ are equivalence classes of type II world sheet by diffeomorphisms preserving the coordinate discs. However, since the individual components do not overlap apart from the coordinate discs, any equivalent representative gives rise to an equivalent type II world sheet after gluing. Thus, the gluing procedure is strictly compatible with grafting trees.

In addition to the continuous parameters t_i or $t_j|\alpha_j$, the plumbing fixture procedure for type II world sheets has one more additional $\mathbb{Z}_2$-valued gluing parameter. This parameter is given by the relative sign used to glue the odd coordinate and determines the sheet of the cover $\mathcal{SM}_{g,n,m} \to \mathcal{M}_{g,n,m}$. If we want to cover the whole of $\mathcal{SM}_{g,n,m}$, we need to sum over all spin structures. From the world sheet perspective the difference is in the insertion of a factor $(-1)^F$, where F denotes world sheet fermion number. If the state space is GSO-projected, we automatically ensure that $\mathcal{SM}_{g,n,m}$ is covered completely if we naively cover $\mathcal{M}_{g,n,m}$.

2.4.3 Pullback and grafting

We can use Φ_Γ to glue various families of type II world sheets. Assume that the glued family has total dimension $r|s$. We are interested in the pullback of the pseudoform $\Omega_A^{r|s}$ along Φ_Γ. Geometrically, the plumbing fixture construction identifies two coordinate discs in a particular way. Unfortunately, this does not tell us immediately how the path integral on such a surface should be evaluated. The procedure does not affect the world sheet structure outside the coordinate discs. Thus, we expect that the path-integral outside is not affected by the plumbing fixture procedure. Consequently, the state that is defined on the unit half circles $|z| = 1$ and $|w| = 1$ is not changed. Since the world sheet induced orientation of the half circles is opposite to the orientation on the coordinate disc, the states living there are actually dual vectors. In order to make use of the operator formalism, we need a way to change the orientation of one of the half circles. In a conformal field theory such a pairing is the *BPZ inner product*. It is a pairing $\langle\omega|$ between two ket states and is equal to the path-integral evaluated on an infinitely thin half annulus with ingoing boundary orientations. With the state operator correspondence, the matrix elements of $\langle\omega|$ are

$$\langle\omega|A_1\rangle|A_2\rangle = \langle A_1(\infty)A_2(0)\rangle, \tag{2.65}$$

where the correlation function is given by the path integral on a genus 0 world sheet with the coordinate discs $z|\theta$ and $-\frac{1}{z}|\frac{\theta}{z}$ for NS states and with coordinate discs $z|\theta$

and $-\frac{1}{z}|\mathrm{i}\theta$ for R states. The inverse pairing $|\omega^{-1}\rangle$ is defined through

$$\mathbf{1} = (-1)^{|\omega|^2}(\mathbf{1} \otimes \langle\omega|)(|\omega^{-1}\rangle \otimes \mathbf{1}), \tag{2.66a}$$
$$\mathbf{1} = (\langle\omega| \otimes \mathbf{1})(\mathbf{1} \otimes |\omega^{-1}\rangle). \tag{2.66b}$$

The conversion between bra and ket states reads (note that $\langle A|$ has internal degree $|\omega| + |A|$, where by $|A|$ we denote that degree of the ket states $|A\rangle$),

$$\langle A| = \langle\omega|\mathbf{1} \otimes |A\rangle, \tag{2.67}$$
$$|A\rangle = (-1)^{(|A|+|\omega|)\cdot|\omega|}\mathbf{1} \otimes \langle A|\omega^{-1}\rangle. \tag{2.68}$$

Denote by $\langle A|$ and $\langle B|$ the two ket states created by the path integral on the half circles.

The region between the circles is given by $|q| < |z| < 1$, where $q = q_{\mathrm{NS}}^2$ or $q = -q_{\mathrm{R}}$, and the path integral defines an operator O. The total path integral is calculated by $\langle A|O|B\rangle$ and can be rewritten as

$$\langle A|O|B\rangle = (-1)^{(|O|+|\omega|)\cdot(|B|+|\omega|)}\langle A|\langle B|O \otimes \mathbf{1}|\omega^{-1}\rangle. \tag{2.69}$$

The particular form of O depends on the type of puncture. In the NS case the remaining rescaling with $z \to qz, \theta \to \sqrt{-q}\theta$ is generated by the mode L_0 of the stress-energy tensor. The corresponding Beltrami differential μ can be written as $\mu = \bar\partial\tilde v$ for a smooth vector field on the world sheet with $|w| = 1$ removed. The difference of $\tilde v$ between both sides of the cut is just $z\partial + \bar z\bar\partial$. In total we find the contribution

$$O = q^{L_0}e^{b_0 d\log q}. \tag{2.70}$$

In the R sector we have one even and one odd modulus. The total remaining transformation is $z \to (-q_R)z(1-\theta\alpha), \theta \to \theta - \alpha$. We interpret this as a concatenation of two superconformal transformations, first we shift the odd coordinate by $-\alpha$, then we perform a rescaling by $-q_{\mathrm{R}}$. The rescaling gives the same operator as for the NS sector. When we look for the generator of the shift in the odd variable, we have to take into account the relation between $z|\theta$ and the local superconformal coordinates given by equation (2.19), that θ provides a local trivialisation of $\mathcal{L}^{-1}$ and that s transforms as a section of $\mathcal{L}^{-1}$. The generator is given by the spinor $s = \sqrt{z}(dz)^{-1/2} = \theta$ and the corresponding modes are G_0 and β_0. Since the modes β_0 and G_0 do not commute, the definition of the exponential $e^{b_i dF^i}$ is ambiguous. We fix this ambiguity in the operator formalism by declaring the insertion to be

$$O = q^{L_0}e^{b_0 d\log q}e^{G_0\alpha+\beta_0 d\alpha}. \tag{2.71}$$

It is instructive to perform the integration over the odd directions in O. In terms of the operator X_0, defined as

$$X_0 = \int \mathcal{D}(\alpha, d\alpha)\, e^{G_0\alpha+\beta_0 d\alpha} = G_0\delta(\beta_0) + b_0\delta'(\beta_0), \tag{2.72}$$

we find that

$$O = q^{L_0} e^{b_0 d \log q} X_0. \tag{2.73}$$

The operator X_0 is called a *picture changing operator (PCO)*. By construction it commutes with the BRST operator Q. Consequently, integration over the odd moduli gives rise to the insertion of picture changing operators. On the other hand, integrating out the even modulus $0 < q < 1$ gives rise to the factor

$$\int \mathcal{D}(q, dq)\, q^{L_0} e^{b_0 d \log q} = \frac{b_0}{L_0}.$$

Thus, upon integration of $\Omega_A^{r|s}$ over a family in the image of Φ_Γ assigns the operator $b_0 L_0^{-1}$ of Klein-Gordon type to each NS edge of the graph and the operator $b_0 L_0^{-1} X_0 = b_0 G_0^{-1} \delta(\beta_0)$ of Dirac-Ramond type to each R edge. These factors may be interpreted as propagators for the two sectors of open string theory. If, on the other hand, we include the odd shifting for R strips onto one of the two coordinate discs, that disc gives rise to a picture $-\frac{3}{2}$ puncture, but the propagator is of Klein-Gordon type in both sectors.

Let us denote by $|e_a\rangle$ a basis of the Hilbert space $\mathcal{H}$ and by $\langle e^a|$ a canonically dual basis. For a general state $|\phi\rangle = |e_a\rangle \phi^a$ one can recover its components w.r.t. the chosen basis via $\phi^a = \langle e^a|\phi\rangle$. Let us further denote the components of the inverse BPZ inner product as $(\omega^{-1})^{km} = \langle e^k|\langle e^m|\omega^{-1}\rangle$. We introduce a formal bidifferential operator $(\cdot,\cdot)$ via

$$(\phi^k, \phi^m) = (-1)^{|\omega|\cdot|m|} (\omega^{-1})^{km}. \tag{2.74}$$

We extend $(\cdot,\cdot)$ to all polynomials in ϕ^k as a right derivation in the first argument and as a left derivation in the second argument. For open strings $\langle\omega|$ is graded symmetric and satisfies $(-1)^{|\omega|^2} = -1$, as it is of ghost number -3. It follows that ω^{-1} is graded antisymmetric and $(\cdot,\cdot)$ is graded symmetric and has degree $|\omega|$.

Using equation (2.69) and the bracket (2.74) the path integral over two world sheets Σ_1 and Σ_2 connected via the plumbing fixture construction can be rewritten in a very compact form,

$$\langle \Sigma_1 \circ \Sigma_2 \rangle = \left(\langle \Sigma_1 | O\phi \rangle, \langle \Sigma_2 | \phi \rangle \right), \tag{2.75}$$

where the operator O has been introduced earlier and represents the influence of the moduli of the plumbing fixture procedure.

When a world sheet separates upon removing the plumbing fixture cylinder, the moduli arise from deformations induced by Beltrami differentials or gravitinos localised on one of the two components or on the cylinder. Therefore, the term $b_i dF^i$ in the pseudoform $\Omega_{0,n+m}$ can be written as a sum $b_i dF^i = b_i dF^i_{(1)} + b_i dF^i_{(2)} + b_i dF^i_{(gl)}$, where the gauge-fixing conditions $F^i_{(1)}$ and $F^i_{(2)}$ arise from Beltrami differentials or gravitinos localised on component 1 or 2, respectively. The gauge-fixing conditions $F^i_{(gl)}$ are completely localised on the plumbing fixture cylinder. Consequently,

$\Omega_{0,n+m}$ has a similar representation as in equation (2.75) in terms of Ω_{0,n_1+m_1+1} and Ω_{0,n_2+m_2+1}, where the additional punctures are joined as in (2.75). The Baranov-Schwartz transform produces a composition of forms of fixed degree $r_1|s_1$, $r_2|s_2$ and O gives rise to a form of degree $1|0$ for NS cylinders and of degree $1|1$ for R cylinders.

In summary, the pullbacks $\Phi_\Gamma^*\Omega_{0,n+m}^{r|s}$ are given by tensor products of $\Omega_{0,n_v+m_v+k_v}^{r_v|s_v}$ for the dimension $r_v|s_v$ of the family of world sheet inserted at a vertex $v \in \Gamma$, k_v being the number of internal lines attached to that vertex. Each internal line in Γ gives rise to contraction with the bracket (2.74) and the insertion of $q_i^{L_0} b_0$ for each NS internal line and $q_j^{L_0} b_0(X_0\,\alpha_j + \delta(\beta_0))$ for each R internal line. This fixes the pullback up to a sign that rises from the relative orientations of source and domain of Φ_Γ.

2.4.4 Relative orientations and suspension

The precise sign in front of $\Phi_\Gamma^*\Omega_{0,n+m}^{r|s}$ depends on the ordering of the factors in the domain of Φ_Γ and on the precise form of Γ. Signs arising from the first ambiguity are treated correctly, if vertices and legs appear in the same order in the domain of Φ_Γ and in the associated algebraic expression. We illustrate this procedure for $\mathfrak{M}_{0,4,0}$. The supermoduli space $\mathfrak{M}_{0,3,0}$ has dimension $0|1$ and, hence, is topologically a point. We take this point as the Witten vertex and choose an arbitrary extension to a section σ_3 of $\mathfrak{P}_{0,3,0} \to \mathfrak{M}_{0,3,0}$. We define a three vertex,

$$S_3 = C_3 = \langle\cdot,\cdot\rangle(\mathbb{I}\otimes\tilde{m}_2) = \int \sigma_3^*\Omega_{0,3+0}^{0|1}, \tag{2.76}$$

which evaluates the three point amplitude. The operator $\tilde{m}_2$ carries ghost number 0. $\mathcal{M}_{0,4}$ is the unit interval and decomposes into three regions labelled by three trees, c.f. figure 2.3. By considering how the global coordinate changes upon varying the gluing parameters in figure 2.3, one deduces that the orientations of the upper left and the lower tree agree and are opposite to the orientation for the upper right tree. The naive total four-point amplitude is therefore

$$\begin{aligned} \tilde{S}_4 &= \langle\cdot,\cdot\rangle\left(\mathbb{I}\otimes\tilde{m}_2(T\tilde{m}_2\otimes\mathbb{I} - \mathbb{I}\otimes T\tilde{m}_2)\right), \\ T &= \int_0^\infty dt\, e^{-tL_0} b_0, \end{aligned} \tag{2.77}$$

where we interpret T as the propagator and have taken into account the relative orientations of the trees. $\tilde{S}_4$ is not gauge-invariant, we discuss the necessary correction terms later, but we note that the full amplitude takes the form

$$S_4 = \langle\cdot,\cdot\rangle\left(\mathbb{I}\otimes\tilde{m}_2(T\tilde{m}_2\otimes\mathbb{I} - \mathbb{I}\otimes T\tilde{m}_2) + \tilde{m}_3\right), \tag{2.78}$$

where the operator $\tilde{m}_3$ carries ghost number -1 and is related to the four vertex C_4 by

$$C_4 = \langle\cdot,\cdot\rangle(\mathbb{I}\otimes\tilde{m}_3). \tag{2.79}$$

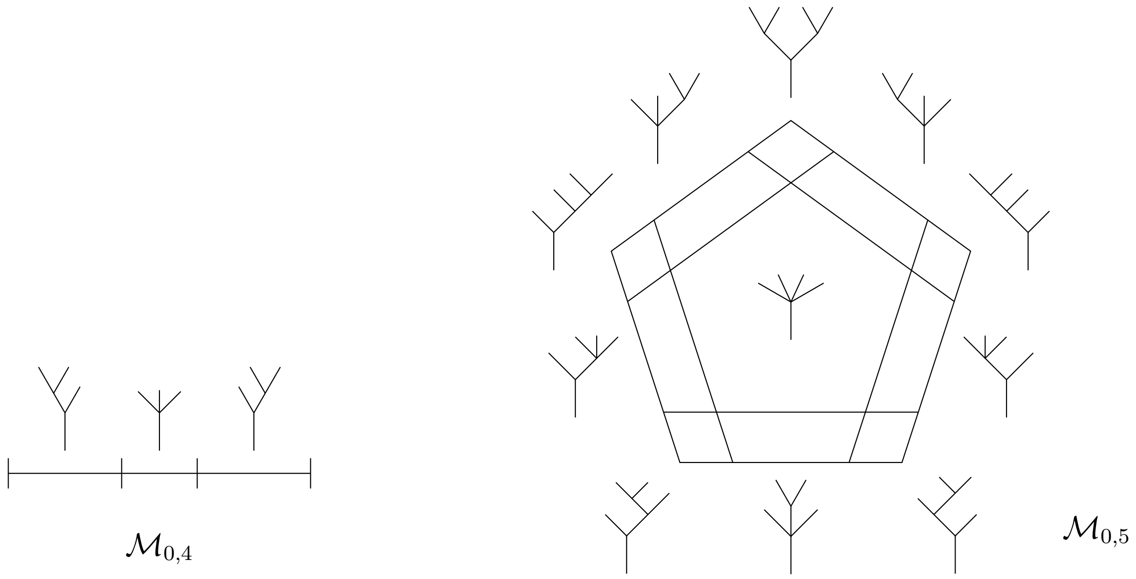
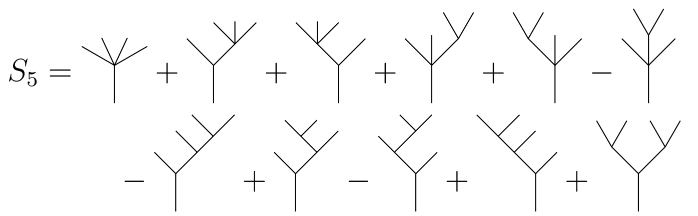


Figure 2.5: Decomposition of the bosonic moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ into various regions at infinity. The trees give the topological form of the plumbing fixture construction covering that particular region of the moduli space. The compactification divisor corresponds to the limit $q \to 0$, in which the moduli space splits into two moduli spaces with fewer number of punctures.

Next we consider the moduli space $\mathfrak{M}_{0,5,0}$. The bosonic moduli space $\mathcal{M}_{0,5}$ is sketched in 2.5. The contributions from the various regions come with the relative signs



$$= \langle \cdot, \cdot \rangle \, (\mathbb{I} \otimes (\tilde{m}_4 + \tilde{m}_3(T\tilde{m}_2 \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes T\tilde{m}_2 \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes T\tilde{m}_2) \\ + \tilde{m}_2(T\tilde{m}_3 \otimes \mathbb{I} + \mathbb{I} \otimes T\tilde{m}_3) \\ + \tilde{m}_2(T\tilde{m}_2 \otimes T\tilde{m}_2 + T\tilde{m}_2(T\tilde{m}_2 \otimes \mathbb{I} - \mathbb{I} \otimes T\tilde{m}_2) \otimes \mathbb{I}) \\ + \tilde{m}_2(\mathbb{I} \otimes T\tilde{m}_2(T\tilde{m}_2 \otimes \mathbb{I} - \mathbb{I} \otimes T\tilde{m}_2) \otimes \mathbb{I})) \;)) \,, \tag{2.80}$$

where the maps $\tilde{m}_k$ carry ghost number $2 - k$ and are defined in terms of the vertex C_{k+1} by

$$C_{k+1} = \langle \cdot, \cdot \rangle (\mathbb{I} \otimes \tilde{m}_k). \tag{2.81}$$

Keeping track of the various relative signs is quite cumbersome. However, it turns out that the suspension map s introduced in section 2.3 automatically keeps track of those signs [108–110]. We recall the symplectic form $\langle\omega| = \langle\cdot,\cdot\rangle s^{\otimes 2}$ and introduce products $M_k, k \geq 2$,

$$M_k = (-1)^k s^{-1}\tilde{m}_k s^{\otimes k}, \qquad \langle\omega|\mathbb{I}\otimes M_k = C_{k+1}s^{\otimes(k+1)}. \tag{2.82}$$

Let us introduce the operator $Q^\dagger = s^{-1}Ts$ and promote M_k to coderivations $\mathbf{M}_k$. With these definitions, we can express S_3, S_4 and S_5 as

$$S_3\, s^{\otimes 3} = \langle\omega|\mathbb{I}\otimes\mathbf{M}_2 \tag{2.83a}$$

$$S_4\, s^{\otimes 4} = \langle\omega|\mathbb{I}\otimes(\mathbf{M}_3 + \mathbf{M}_2(-Q^\dagger\mathbf{M}_2\otimes\mathbb{I} + \mathbb{I}\otimes -Q^\dagger\mathbf{M}_2)) \tag{2.83b}$$

$$\begin{aligned} S_5\, s^{\otimes 5} = \langle\omega|\mathbb{I}\otimes \Big(&\mathbf{M}_4 + \mathbf{M}_3(-Q^\dagger\mathbf{M}_2\otimes\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes -Q^\dagger\mathbf{M}_2\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{I}\otimes -Q^\dagger\mathbf{M}_2) \\ &+ \mathbf{M}_2(-Q^\dagger\mathbf{M}_3\otimes\mathbb{I} + \mathbb{I}\otimes -Q^\dagger\mathbf{M}_3) \\ &+ \mathbf{M}_2(-Q^\dagger\mathbf{M}_2\otimes -Q^\dagger\mathbf{M}_2 - Q^\dagger\mathbf{M}_2(-Q^\dagger\mathbf{M}_2\otimes\mathbb{I} + \mathbb{I}\otimes -Q^\dagger\mathbf{M}_2)\otimes\mathbb{I}) \\ &+ \mathbf{M}_2(\mathbb{I}\otimes -Q^\dagger\mathbf{M}_2(-Q^\dagger\mathbf{M}_2\otimes\mathbb{I} + \mathbb{I}\otimes -Q^\dagger\mathbf{M}_2))\ \Big) \end{aligned} \tag{2.83c}$$

The higher point amplitudes can be constructed in a similar way.

2.4.5 Gauge invariance and contact terms

Decoupling of Q-exact states is a crucial requirement for consistency of the theory. We denote the suspended operator $s^{-1}Qs$ by the same letter. Since Q is a BPZ-odd operator, we conclude that

$$C_{k+1}Qs^{\otimes(k+1)} = \langle\omega|\mathbb{I}\otimes[\mathbf{M}_k, \mathbf{Q}]. \tag{2.84}$$

Also notice that $[Q, Q^\dagger] = \mathbb{I} - e^{-\infty L_0}$. The operator $P = e^{-\infty L_0}$ denotes the contribution arising from infinity. We assume here that the external momenta a generic so that we can ignore these contributions. Algebraically, it follows from equation (2.83) that gauge-invariance requires that the A_∞ relations (2.54) should be satisfied by $\mathbf{M}_k$. Cyclicity of the S-matrix requires moreover the cyclicity condition (2.57).

We investigate the failure of gauge-invariance of the naive four point amplitude (2.77). Algebraically it is given by the associator of the binary product $\tilde{m}_2$. Geometrically, the plumbing fixture construction gives rise to two local sections σ_0 and σ_1 of $\mathfrak{P}_{0,4,0} \to \mathfrak{M}_{0,4,0}$ living over the appropriate domains in $\mathcal{M}_{0,4}$. The failure arises from evaluating the integral of $\Omega^{0|2}_{0,4+0}$ over two different extensions of the same underlying bosonic section. Such differences were discussed in section 2.4.1 and give rise to a Q-exact term. More precisely, we find

$$\tilde{S}_4 Q = -\int(\sigma_1^* - \sigma_0^*)\Omega^{0|2}_{0,4+0} = -\int\int_0^1 dt\,\frac{\partial}{\partial t}\sigma_t^*\Omega^{0|2}_{0,4+0} = -\int\sigma_4^*\Omega^{1|2}_{0,4+0}Q, \tag{2.85}$$

where σ_4 denotes the homotopy that interpolates between the two sections interpreted as a $1|2$ dimensional submanifold in $\mathfrak{P}_{0,4,0}$. Thus, if we introduce the four vertex

$$C_4 = \int\sigma_4^*\Omega^{1|2}_{0,4+0}, \tag{2.86}$$

we restore gauge-invariance. C_4 represents a contact term although it is an integral over a chain with non-vanishing bosonic dimension. However, the chain is located entirely over one point in $\mathcal{M}_{0,4}$.

Denote by $\tilde{S}_5$ the naive five point amplitude that is calculated from the cubic and the quartic vertex. Making use of the suspended form (2.83) of the S-matrix to keep track of all signs, one can easily deduce that

$$\tilde{S}_5 Q = -\langle\cdot,\cdot\rangle(\mathbb{I}\otimes(\tilde{m}_2(\tilde{m}_3\otimes\mathbb{I}+\mathbb{I}\otimes\tilde{m}_3) \\ -\tilde{m}_3(\tilde{m}_2\otimes\mathbb{I}\otimes\mathbb{I}-\mathbb{I}\otimes\tilde{m}_2\otimes\mathbb{I}+\mathbb{I}\otimes\mathbb{I}\otimes\tilde{m}_2)). \quad (2.87)$$

The five terms represent a one-dimensional bosonic integral each interpolating between corner values of local sections built from σ_1 and σ_0 above using plumbing fixture. By carefully following the path of interpolations, one observes that it represents a closed loop. Following the discussion in section 2.4.1 we conclude that we can fill this loop with a disc given by a section σ_5 that maps out a 2|3-dimensional chain over a point in $\mathcal{M}_{0,5}$. Taking care of the orientation change, we conclude that

$$\tilde{S}_5 Q = \int \sigma_5^* d\Omega_{0,5+0}^{1|3} = -\int \sigma_5^*\Omega_{0,5+0}^{2|3} Q. \quad (2.88)$$

Introducing the five vertex as $C_5 = \int \sigma_5^*\Omega_{0,5+0}^{2|3}$, the full five-point amplitude S_5 becomes gauge-invariant. Again C_5 is a contact term as it sits over a single point in $\mathcal{M}_{0,5}$.

2.4.6 Algebraisation of the problem

In principle, the higher order vertices could be constructed in a similar way. The only requirement is that the fibre sitting over a single point in $\mathcal{M}_{0,5}$ has vanishing homotopy groups. If the bosonic cover of the moduli space requires elementary higher vertices, the geometric construction becomes more cumbersome as we would also have to take into account lower codimension boundaries of the elementary vertices. It is not clear that the correction terms are contact terms anymore. Another drawback is that this field theory is not entirely constructive. In order to be able to do meaningful calculations, we need to construct $\sigma_k, k \geq 4$ explicitly, which seems not to be feasible. Alternatively, the procedure indicates that we may solve the A_∞-relations algebraically and take them as a replacement for the construction of σ_k. The only drawback is that we need to show that the correct S-matrix is indeed reproduced by (2.83) and their higher analogues. In chapter 3 we complete the construction and show indeed that all higher order vertices can be chosen as contact vertices if we start with Witten's vertex.

The whole procedure can be reinterpreted within classical BV theory. After the suspension, the Poisson bracket is given by equation (2.74). Let us introduce a generating function S_{int},

$$S_{\text{int}} = \sum_{k\geq 3}\frac{1}{k}\int \sigma_k^*\Omega_{0,k}^{k-3|k-2}\phi^k. \quad (2.89)$$

S_{int} is a cyclic functional of degree 0. As the products $\mathbf{M}_k$ constitute an A_∞ algebra, we have

$$\sum_{k\geq 3}\frac{1}{k}\int \sigma_k^*\Omega_{0,k}^{k-3|k-2}Q\phi^k+\frac{1}{2}(S_{\text{int}},S_{\text{int}})=0. \tag{2.90}$$

If the Poisson bracket $(\cdot,\cdot)$ comes from a symplectic structure ω, we can also express the first term in terms of the Poisson bracket,

$$S=\frac{1}{2}\langle\omega|\phi\otimes Q\phi\rangle+S_{\text{int}}, \tag{2.91a}$$

$$0=\frac{1}{2}(S,S). \tag{2.91b}$$

Invoking the results from section 2.3, we conclude that finding the maps induced by the integrals of σ_k is equivalent to finding a symplectic structure inducing the Poisson bracket (2.74), solving the cyclic master equation (2.91) and showing that the S-matrix calculated from the master action S indeed reproduces the usual perturbative string S-matrix. For the open superstring these problems are tackled in chapters 3, 5, 6 and 7.

With these results in mind, it is straightforward to generalise the construction problem to include Ramond fields and also to closed strings. If Ramond fields are present, the BV bracket does not come from a naive symplectic form. Giving up cyclicity, we can still solve (2.90) or equivalently the L_∞- or A_∞-relations. Formally, one may still consider the S-matrix given by (2.83) and show that it coincides with the traditional perturbative S-matrix. Thus the same reasoning can be applied to open superstrings based on a decomposition of the bosonic moduli space based on Witten's star product with stubs, to heterotic strings and to closed type II superstrings. Due to some problems with the invertibility of the Poisson bracket when including Ramond fields, the generalisation is easiest when restricting to pure NS fields. Without the P structure it is still possible to construct a Q-manifold structure giving rise to gauge-invariant equations of motion. The results are discussed in chapter 4 for the NS subsectors and in chapter 5 for the Q-structure for all fields, chapter 6 evaluates the S-matrix.

Chapter 3

Resolving Witten's open superstring field theory

Classical open string field theories are determined by cyclic A_∞ algebras. In this chapter we construct such an algebra for the NS sector of open superstring theory. The construction starts with Witten's singular open superstring field theory and regulates it by replacing the picture-changing insertion at the midpoint with a contour integral of picture changing insertions over the half-string overlaps of the cubic vertex. The resulting product between string fields is non-associative, but we provide a solution to the A_∞ relations defining all higher vertices. The result is an explicit covariant superstring field theory which by construction satisfies the classical BV master equation.

This chapter is based on the paper **Resolving Witten's open superstring field theory** by T. Erler, the author and I. Sachs [52].

3.1 Introduction

For the bosonic string, the construction of covariant string field theories is more or less well understood. We know how to construct an action, quantise it, and prove that the vertices and propagators cover the the moduli space of Riemann surfaces relevant for computing amplitudes. For the superstring this kind of understanding is largely absent. A canonical formulation of open superstring field theory was provided by Berkovits [41, 42], but it utilises the large Hilbert space which obscures the relation to supermoduli space. Moreover, quantization of the Berkovits theory is not completely understood [111–114]. Motivated by this problem, we seek a different formulation of open superstring field theory satisfying three criteria:

(1) The kinetic term is diagonal in mode number.

(2) Gauge invariance follows from the same algebraic structures which ensure gauge invariance in open bosonic string field theory.

(3) The vertices do not require integration over bosonic moduli.

We assume (1) since we want the theory to have a simple propagator. We assume (2) since we want to be able to quantise the theory in a straightforward manner, following the work of Thorn [115], Zwiebach [17] and others for the bosonic string. Finally we assume (3) for simplicity, but also because we would like to know whether open string field theory can describe closed string physics through its quantum corrections. Once we add stubs to the open string vertices, the nature of the minimal area problem changes and requires separate degrees of freedom for closed strings at the quantum level [26].

Condition (1) rules out the modified cubic theory and its variants [38,39,116–119], and (2) rules out the Berkovits theory. This leaves the original proposal for open superstring field theory at picture -1, described by Witten [34]. The problem is that this theory is singular and incomplete. A picture changing operator in the cubic term leads to a divergence in the four point amplitude which requires subtraction against a divergent quartic vertex [37]. Likely an infinite number of divergent higher vertices are needed to ensure gauge invariance, but have never been constructed.[1]

In this chapter we would like to complete the construction of Witten's open superstring field theory in the NS sector. We achieve this by resolving the singularity in the cubic vertex by spreading the picture changing insertion away from the midpoint. As a result the product is non-associative. But we know how to formulate a gauge invariant action with a non-associative product [25]. The action takes the form

$$S = \frac{1}{2}\omega(\Psi, Q\Psi) + \frac{1}{3}\omega(\Psi, M_2(\Psi,\Psi)) + \frac{1}{4}\omega(\Psi, M_3(\Psi,\Psi,\Psi)) + \dots, \tag{3.1}$$

where ω is the symplectic bilinear form and $Q, M_2, M_3, \dots$ are multi-string products which satisfy the relations of an A_∞ algebra. The fact that one can in principle construct a regularisation of Witten's theory along these lines is well-known. The new ingredient we provide is an exact solution of the A_∞ relations, giving an explicit and computable definition of the vertices to all orders.

The resulting theory is quite simple. However, its explicit form depends on a choice of non-local, BPZ even operator built from the picture changing operator

$$X = \oint \frac{dz}{2\pi \mathrm{i}} f(z) X(z), \tag{3.2}$$

which tells us how to spread the picture changing insertion in the cubic vertex away from the midpoint. As far as we know, there is no canonical way to make this choice. This suggests the result of a partial gauge fixing. In fact, a gauge fixed

[1] There have been some attempts to fix the problems with Witten's theory by changing the nature of the midpoint insertions in the action. These include the modified cubic theory [38, 39] and the theory described in [120].

version of Berkovits' theory resembling our approach has been explored in [51,121]. Our regularisation of the cubic vertex is inspired by this work.

This chapter is organised as follows. In section 3.2 we describe our regularisation of the cubic vertex. The cubic vertex gives rise to a non-associative 2-product and we find the 3-product by requiring that the resulting action satisfies a master equation. The main observation is that the 3-product is Q-exact and leads to the recursive construction of a full solution to the master equation described in section 3.3. As a cross check for our construction we calculate the four-point amplitude in field theory and show that it is identical to the first quantised amplitude in section 3.4. We conclude the chapter with some discussion.

3.2 Witten's theory up to quartic order

The field theory we propose is a cyclic field theory in the sense of section 2.3. We therefore need to construct a cyclic A_∞ algebra on some vector space $\mathcal{H}$. In our case $\mathcal{H}$ is identified with the space of states at picture -1 of some reference boundary superconformal field theory. In order to describe the construction of the higher vertices, it is convenient to think of $\mathcal{H}$, the small Hilbert space, embedded into the large Hilbert space as the kernel of the zero mode of the field η. The grading on $\mathcal{H}$ is given by ghost number and we perform a suspension on $\mathcal{H}$. The shifted grading is called *degree* and is related to ghost number via

$$\deg(a) = \mathrm{gh}(a) - 1. \tag{3.3}$$

The world sheet BRST operator Q enriches $\mathcal{H}$ to a differential graded vector space. This choice guarantees that the cohomology $H^0(Q)$ at degree 0 coincides with the space of physical states. Cyclicity is measured w.r.t. the symplectic form ω of ghost number -3 and picture 2 induced from the BPZ inner product on the small Hilbert space. It carries degree -1, is graded antisymmetric and is non-degenerate. Formally, Witten's star product $*$ gives rise to a picture 0 and ghost number 0 associative product on the state space. Let us call this product m_2. Unfortunately, m_2 does not preserve $\mathcal{H}$ as the product of two -1 states gives a state at picture -2. The main goal of this chapter is to construct a suitable substitute product M_2 that does indeed preserve $\mathcal{H}$.

The original proposal of Witten [34] was

$$M_2 = X(\mathrm{i}) m_2, \tag{3.4}$$

where $X(z)$ is the picture changing operator and carries picture $+1$. This proposal formally preserves $\mathcal{H}$, but gives rise to divergences of the form $X(\mathrm{i})^2$ upon for example calculating the associator or the four-point amplitude [37]. The origin of the divergence can be traced back to the presence of a double pole in the OPE of $X(z)$ with itself. To avoid these problems we make a more general ansatz:

$$M_2(A, B) \equiv \frac{1}{3}\Big[X m_2(A, B) + m_2(XA, B) + m_2(A, XB)\Big], \tag{3.5}$$

where X is a BPZ even non-local operator built out of the picture changing operator[2]:

$$X = \oint \frac{dz}{2\pi \mathrm{i}} f(z) X(z). \tag{3.6}$$

The product M_2 now explicitly depends on a choice of 1-form $f(z)dz$, which describes how the picture changing is spread over the half-string overlaps of the Witten vertex. Provided $f(z)$ is holomorphic in some non-degenerate annulus around the unit circle, products of X with itself are regular, and in particular the 4-point amplitude is finite. Note that the geometry of the cubic vertex (3.5) is the same as in Witten's open bosonic string field theory. This means that the propagator together with the cubic vertex already cover the bosonic moduli space of Riemann surfaces with boundary [122]. Therefore higher vertices must be contact interactions without integration over bosonic moduli.

Since X is BPZ even, the 1-form $f(z)$ satisfies

$$f(z) = -\frac{1}{z^2} f\left(-\frac{1}{z}\right). \tag{3.7}$$

We also assume the normalisation condition

$$\oint \frac{dz}{2\pi \mathrm{i}} f(z) = 1, \tag{3.8}$$

since any other number could be absorbed into a redefinition of the open string coupling constant. Perhaps the simplest choice of X is the zero mode of the picture changing operator:

$$X_0 = \oint \frac{dz}{2\pi \mathrm{i}} \frac{1}{z} X(z). \tag{3.9}$$

If we like, we can also choose X so that it approaches Witten's singular midpoint insertion as a limit. For example we can take

$$f(z) = \frac{1}{z - i\lambda} - \frac{1}{z - \frac{i}{\lambda}}, \tag{3.10}$$

which as $\lambda \to 1^-$ approaches a delta function localising X at the midpoint. Note that the annulus of analyticity,

$$\lambda < |z| < \frac{1}{\lambda}, \tag{3.11}$$

degenerates to zero thickness in the $\lambda \to 1^-$ limit. This is why Witten's original vertex produces contact divergences.

[2]We can choose X to be BPZ even without loss of generality, since if we assume a cyclic vertex any BPZ odd component would cancel out.

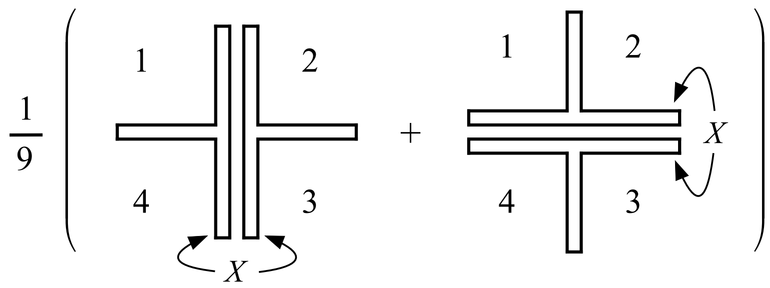


Figure 3.1: Pictorial representation of the associator of M_2. We can take the numbers $1, 2, 3$ to represent the states which are multiplied, and 4 to represent the output of the associator. The "T" shape represents a contour integral of X surrounding the respective Witten vertex, and two factors of $\frac{1}{3}$ comes from the two vertices.

The price we have to pay for the regularisation is now that M_2 is not an associative product anymore so that the action is not gauge-invariant.

$$M_2(M_2 \otimes \mathbb{I} + \mathbb{I} \otimes M_2) \neq 0. \tag{3.12}$$

To restore gauge invariance we search for a 3-product M_3, a 4-product M_4, and so on so that the full set of multilinear maps satisfy the relations of an A_∞ algebra. Using these multilinear maps to define higher vertices, the action

$$S = \frac{1}{2}\omega(\Psi, Q\Psi) + \sum_{n=2}^{\infty} \frac{1}{n+1}\omega(\Psi, M_n(\underbrace{\Psi, ..., \Psi}_{n \text{ times}})) \tag{3.13}$$

is gauge invariant by construction.

As a first step we construct the 3-product M_3 which defines the quartic vertex. The first two A_∞ relations say that Q is nilpotent and a derivation of the 2-product M_2. The third relation characterises the failure of M_2 to associate in terms of the BRST variation of M_3:

$$0 = M_2(M_2 \otimes \mathbb{I} + \mathbb{I} \otimes M_2) + [Q, M_3] \tag{3.14}$$

The last four terms represent the BRST variation of M_3 by placing a Q on each output of the quartic vertex. To visualise how to solve for M_3, consider figure 3.1, which gives a schematic world sheet picture the configuration of X contour integrals in the M_2 associator. To pull a Q off of the X contours, it would clearly help if X were a BRST exact quantity. In the large Hilbert space it is, since we can write

$$X = [Q, \xi], \qquad \xi \equiv \oint \frac{dz}{2\pi \mathrm{i}} f(z)\xi(z), \tag{3.15}$$

where ξ is the mode of the ξ-ghost defined by the 1-form $f(z)$. Now pulling a Q out of the associator simply requires replacing one of the X contours in each term with

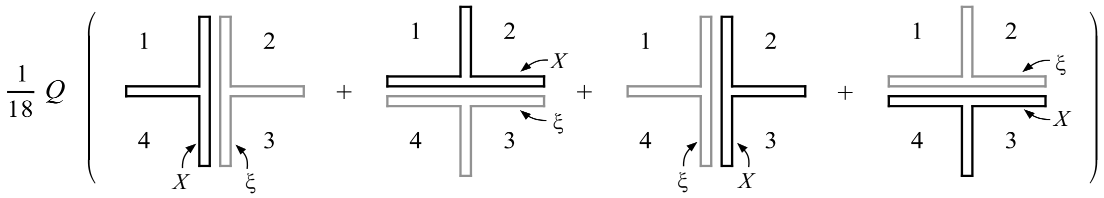


Figure 3.2: Pictorial representation of the associator as a BRST exact quantity. The black "T" shape represents a contour integral of X around the Witten vertex and the grey "T" shape represents the corresponding contour integral of ξ. We have four terms since we require the quartic vertex to be cyclic.

a ξ contour. Since there are two X contours in each term, there are two ways to do this, and by cyclicity we should sum both ways and divide by two. This is shown in figure 3.2. Translating this picture into an equation gives a solution for M_3:

$$M_3 = \frac{1}{2}\Big[M_2(\mu_2 \otimes \mathbb{I} + \mathbb{I} \otimes \mu_2) - \mu_2(M_2 \otimes \mathbb{I} + \mathbb{I} \otimes M_2)\Big] + \ Q\text{-closed}\ , \tag{3.16}$$

where we leave open the possibility of adding a Q-closed piece, which would not contribute to the associator. μ_2 in this equation is a new object that we call the *dressed-2-product*:

$$\mu_2 \equiv \frac{1}{3}\Big[\xi m_2 - m_2(\xi \otimes \mathbb{I} + \mathbb{I} \otimes \xi)\Big]. \tag{3.17}$$

This is essentially the same as M_2, only the X contour has been replaced by a ξ contour. The dressed-2-product has degree 0, and as required its BRST variation is M_2:

$$M_2 = [Q, \mu_2]. \tag{3.18}$$

Acting η on μ_2 gives yet another object which we call the *bare-2-product*:

$$m_2 = [\eta, \mu_2]. \tag{3.19}$$

The bare-2-product has degree 1. As it happens the bare-2-product is the same as Witten's open string star product. Both the dressed-product and the bare-product have nontrivial generalisations to higher number of inputs.

While we can introduce ξ into our calculations as a formal convenience, consistency requires that all multilinear maps defining string vertices must preserve the small Hilbert space. This is already true for M_2, but has to be checked for M_3. For this reason we make use of our freedom to add a BRST closed piece in equation (3.16)

$$Q\text{-closed} = \frac{1}{2}\Big[Q, \mu_3\Big], \tag{3.20}$$

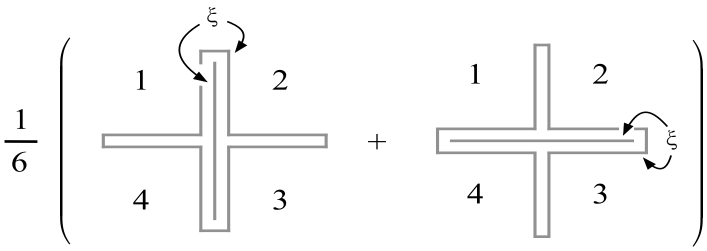


Figure 3.3: Schematic picture of the ξ contours defining the dressed-3-product. The vertical and horizontal lines inside the cross represents an insertion of ξ between open string star products. The cross represents a sum of ξ insertions acting on all external states.

where μ_3 is defined in such a way as to ensure that the total 3-product is in the small Hilbert space. The object μ_3 is called the *dressed-3-product*. Now we require that M_3 preserves the small Hilbert space:

$$0 = [\eta, M_3] = -\frac{1}{2}\Big[M_2, m_2\Big] + \Big[\eta,\ Q\text{-closed}\Big]. \tag{3.21}$$

With some algebra this simplifies to

$$0 = [\eta, M_3] = \frac{1}{2}\Big[Q, -[m_2, \mu_2] + [\eta, \mu_3]\Big]. \tag{3.22}$$

Since $[\eta, M_3]$ should be zero, it is reasonable to assume that the dressed-3-product μ_3 should satisfy

$$[\eta, \mu_3] = [m_2, \mu_2] = \frac{2}{3}m_2(\xi m_2 \otimes \mathbb{I} + \mathbb{I} \otimes \xi m_2) = m_3 \tag{3.23}$$

The right hand side defines what we call the *bare-3-product*, m_3. Using associativity of m_2 it is straightforward to check that m_3 indeed preserves the small Hilbert space so that this equation is consistent. Though equation (3.23) does not uniquely determine μ_3, there is a natural solution: take m_3 and place a ξ on each external state:

$$\mu_3 \equiv \frac{1}{4}\Big[\xi m_3 - m_3(\xi \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \xi \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \xi)\Big]. \tag{3.24}$$

Thus the dressed-3-product is described by a configuration of ξ contours shown in figure 3.3. This gives an explicit definition of the quartic vertex in the small Hilbert space consistent with gauge invariance.

3.3 Solution to all orders

The construction from the previous section can be extended to all orders. We obtain a recursive construction of the higher order vertices in terms of the lower vertices and the lower order bare products. The results is expressed most conveniently in terms of coderivations. The n-th A_∞ relation reads

$$0 = [\mathbf{M}_n, \mathbf{M}_1] + [\mathbf{M}_{n-1}, \mathbf{M}_2] + ... + [\mathbf{M}_2, \mathbf{M}_{n-1}] + [\mathbf{M}_1, \mathbf{M}_n], \tag{3.25}$$

where $\mathbf{M}_1 \equiv \mathbf{Q}$. To express all such relations in a compact form, it is useful to introduce a generating function $\mathbf{M}(t)$:

$$\mathbf{M}(t) \equiv \sum_{n=0}^{\infty} t^n \mathbf{M}_{n+1}, \tag{3.26}$$

where t is some parameter. Then the full set of A_∞ relations is equivalent to the equation

$$[\mathbf{M}(t), \mathbf{M}(t)] = 0. \tag{3.27}$$

The nth relation is found by expanding this equation in a power series and reading off the coefficient of t^{n-1}. The solution we are after takes the form

$$\mathbf{M}_{n+2} = \frac{1}{n+1} \sum_{k=0}^{n} [\mathbf{M}_{n-k+1}, \boldsymbol{\mu}_{k+2}]. \tag{3.28}$$

If we know the products up to $\mathbf{M}_{n+1}$, and the dressed-products up to $\boldsymbol{\mu}_{n+2}$, this equation determines the next product $\mathbf{M}_{n+2}$. The proof is as follows. Define a generating function for the dressed-products:

$$\boldsymbol{\mu}(t) = \sum_{n=0}^{\infty} t^n \boldsymbol{\mu}_{n+2} \tag{3.29}$$

Then the recursive formula (3.28) follows from the t^n component of the differential equation

$$\frac{d}{dt}\mathbf{M}(t) = [\mathbf{M}(t), \boldsymbol{\mu}(t)]. \tag{3.30}$$

This equation implies

$$\frac{d}{dt}[\mathbf{M}(t), \mathbf{M}(t)] = 2[[\mathbf{M}(t), \mathbf{M}(t)], \boldsymbol{\mu}(t)]. \tag{3.31}$$

Let

$$[\mathbf{M}(t), \mathbf{M}(t)]_{n+1} = \sum_{k=0}^{n} [\mathbf{M}_{n-k+1}, \mathbf{M}_{k+1}], \tag{3.32}$$

be the combination of $\mathbf{M}$s appearing in the $n+1$st A_∞ relation, or equivalently the coefficient of t^n in the power series expansion of $[\mathbf{M}(t), \mathbf{M}(t)]$. Then equation (3.31) implies a recursive formula for these coefficients:

$$[\mathbf{M}(t), \mathbf{M}(t)]_{n+2} = \frac{2}{n+1} \sum_{k=0}^{n} [[\mathbf{M}(t), \mathbf{M}(t)]_{n-k+1}, \boldsymbol{\mu}_{k+2}]. \tag{3.33}$$

If $[\mathbf{M}(t), \mathbf{M}(t)]_k$ vanishes for $1 \leq k \leq n+1$, then this formula implies that it must vanish for $k = n+2$. So all we have to do is show that $[\mathbf{M}(t), \mathbf{M}(t)]_k$ vanishes for $k = 1$. It does because

$$[\mathbf{M}(t), \mathbf{M}(t)]_1 = [\mathbf{Q}, \mathbf{Q}] = 0. \tag{3.34}$$

This completes the proof that equation (3.28) implies the A_∞ relations. Next consider the bare-products $\mathbf{m}_n$. For the moment we ignore the possible identification between $\mathbf{m}_n$ and $[\boldsymbol{\eta}, \boldsymbol{\mu}_n]$. Rather, we define the bare-products in terms of the recursive formula

$$\mathbf{m}_{n+3} = \frac{1}{n+1} \sum_{k=0}^{n} [\mathbf{m}_{n-k+2}, \boldsymbol{\mu}_{k+2}]. \tag{3.35}$$

If we know the bare-products up to $\mathbf{m}_{n+2}$ and the dressed-products up to $\boldsymbol{\mu}_{n+2}$, this determines the next bare-product $\mathbf{m}_{n+3}$. We can check that this formula matches our previous calculation of the bare-3-product. Suppose that we define a generating function for the bare-products

$$\mathbf{m}(t) = \sum_{n=0}^{\infty} t^n \mathbf{m}_{n+2}. \tag{3.36}$$

Then equation (3.35) implies the differential equation

$$\frac{d}{dt}\mathbf{m}(t) = [\mathbf{m}(t), \boldsymbol{\mu}(t)]. \tag{3.37}$$

Using a similar argument as just given below equation (3.31), we can prove

$$[\mathbf{m}(t), \mathbf{m}(t)] = 0, \tag{3.38a}$$
$$[\mathbf{m}(t), \mathbf{M}(t)] = 0 \tag{3.38b}$$

recursively from the identities $[\mathbf{m}_2, \mathbf{m}_2] = 0$ and $[\mathbf{m}_2, \mathbf{Q}] = 0$. In components of t^n,

$$\sum_{k=0}^{n} [\mathbf{m}_{n-k+2}, \mathbf{m}_{k+2}] = 0, \tag{3.39a}$$
$$\sum_{k=0}^{n} [\mathbf{m}_{n-k+2}, \mathbf{M}_{k+1}] = 0. \tag{3.39b}$$

This means that the products and bare-products form a pair of mutually commuting A_∞ algebras. This much is true regardless of our choice of dressed-products $\boldsymbol{\mu}_k$. What fixes $\boldsymbol{\mu}_k$ is the additional condition

$$[\boldsymbol{\eta}, \boldsymbol{\mu}_{k+2}] = \mathbf{m}_{k+2}. \tag{3.40}$$

We construct a solution to this condition recursively as follows. First note that $[\boldsymbol{\eta}, \boldsymbol{\mu}_2] = \mathbf{m}_2$ by definition. Second, suppose that we have constructed a solution to equation (3.40) up to $\mathbf{m}_{n+2}$ and $\boldsymbol{\mu}_{n+2}$. Then it follows that the bare-product $\mathbf{m}_{n+3}$ is in the small Hilbert space:

$$[\boldsymbol{\eta}, \mathbf{m}_{n+3}] = -\frac{1}{n+1}\sum_{k=0}^{n}[\mathbf{m}_{n-k+2}, \mathbf{m}_{k+2}] = 0, \tag{3.41}$$

where we used the recursive equation (3.35) and the A_∞ relations (3.39a). Now define the $n+3$rd dressed-product:

$$\boldsymbol{\mu}_{n+3} \equiv \frac{1}{n+4}\left(\xi m_{n+3} - m_{n+3}\sum_{k=0}^{n+2}\mathbb{I}^{\otimes n+2-k}\otimes\xi\otimes\mathbb{I}^{\otimes k}\right). \tag{3.42}$$

Since $\mathbf{m}_{n+3}$ is in the small Hilbert space, this implies

$$[\boldsymbol{\eta}, \boldsymbol{\mu}_{n+3}] = \mathbf{m}_{n+3}. \tag{3.43}$$

Proceeding this way inductively, we find a solution to equation (3.40) for all k.

Next we have to show how this construction implies that all products defining vertices are indeed in the small Hilbert space. Acting $\boldsymbol{\eta}$ on the differential equation (3.30) for $\mathbf{M}$ gives

$$\begin{aligned}\frac{d}{dt}[\boldsymbol{\eta}, \mathbf{M}(t)] &= [[\boldsymbol{\eta}, \mathbf{M}(t)], \boldsymbol{\mu}(t)] - [\mathbf{M}(t), \mathbf{m}(t)],\\ &= [[\boldsymbol{\eta}, \mathbf{M}(t)], \boldsymbol{\mu}(t)],\end{aligned} \tag{3.44}$$

where we used equation (3.40) and the fact that the A_∞ algebras of $\mathbf{M}$ and $\mathbf{m}$ commute. The t^n component of this differential equation implies the recursive formula

$$[\boldsymbol{\eta}, \mathbf{M}_{n+2}] = \frac{1}{n+1}\sum_{k=0}^{n}[[\boldsymbol{\eta}, \mathbf{M}_{n-k+1}], \boldsymbol{\mu}_{k+2}]. \tag{3.45}$$

Note that $\mathbf{M}_1 = \mathbf{Q}$ commutes with η. And this equation implies that if all of the products up to $\mathbf{M}_{n+1}$ are in the small Hilbert space, the next product $\mathbf{M}_{n+2}$ is also in the small Hilbert space. Thus we have a complete solution of the A_∞ relations defining Witten's superstring field theory. The construction we have provided is recursive. Suppose we have determined all products, bare-products, and dressed-products up to $\mathbf{M}_n, \mathbf{m}_n$ and $\boldsymbol{\mu}_n$. To proceed to the next order, first we construct the $n+1$st bare-product $\mathbf{m}_{n+1}$ from equation (3.35). Next we construct the $n+1$st dressed-product $\boldsymbol{\mu}_{n+1}$ from equation (3.42). Finally, using $\boldsymbol{\mu}_{n+1}$ we construct the $n+1$st product $\mathbf{M}_{n+1}$ via equation (3.28), or we can proceed to the next order and compute the $n+2$nd bare-product $\mathbf{m}_{n+2}$, starting the process over.

Our solution to the A_∞ relations depends on the following assumptions:

(1) Q and η are nilpotent and anticommute.

(2) Q and η are derivations of the product m_2.

(3) η has a homotopy ξ satisfying $[\eta, \xi] = 1$.

(4) m_2 is associative.

Within the context of these assumptions we can construct a slightly more general solution by adding an η closed piece to ξ. This can have the effect of replacing X in the cubic vertex with a slightly more general operator. Aside from this, perhaps the most interesting assumption to drop is associativity of m_2, cf. section 4.2, This might be useful, for example, for constructing a theory based on a cubic vertex with world sheet strips attached to each output, as is done in open-closed bosonic string field theory [26]. The solution of the A_∞ relations is not unique. The non-uniqueness can be characterised by our freedom to add an η closed piece to $\boldsymbol{\mu}_n$ at each order. Perhaps the most nontrivial aspect of our construction is that despite this non-uniqueness we were able to find a natural definition of each vertex, without having to make additional choices at each order.

3.4 Four-point amplitudes

It is instructive to see how our regularisation of Witten's theory reproduces the familiar first-quantised scattering amplitudes. Here we focus explicitly on the generic four-point amplitude. The general case is discussed in chapter 6[3].

We start with the color-ordered 4-point amplitude expressed in the form:

$$A_4^{\text{1st}}(\Psi_1,\Psi_2,\Psi_3,\Psi_4) = -\int_0^1 dt \Big\langle \Big(X_0\cdot\Psi_1(0)\Big)\Big(b_{-1}X_0\cdot\Psi_2(t)\Big)\Psi_3(1)\,\Psi_4(\infty)\Big\rangle_{\text{UHP}}. \tag{3.46}$$

Here $\Psi_1,..,\Psi_4$ are on-shell vertex operators in the -1 picture and the correlator is evaluated in the small Hilbert space on the upper half plane. We denote the amplitude with the superscript 1st to indicate that this is the first quantised amplitude, not the string field theory result. As far as bosonic moduli are concerned, this amplitude is structurally the same as in the bosonic string, and following [11] and chapter 2 we can reexpress it using the open string star product and the Siegel gauge propagator in the s- and t-channels:

$$\begin{aligned} A_4^{\text{1st}}(\Psi_1,\Psi_2,\Psi_3,\Psi_4) = &-\omega\left(X_0\Psi_1, m_2\left(X_0\Psi_2, \frac{b_0}{L_0}m_2(\Psi_3,\Psi_4)\right)\right) \\ &-\omega\left(X_0\Psi_1, m_2\left(\frac{b_0}{L_0}m_2(X_0\Psi_2,\Psi_3),\Psi_4\right)\right). \end{aligned} \tag{3.47}$$

The operator X_0 is the operator (3.6) for the special choice $f(z)=\frac{1}{z}$ so that $X_0\Psi_i$ coincides with the picture 0 form of the vertex operator Ψ_i. This is the form of the amplitude we want to compare with Witten's superstring field theory. Now consider

[3]Similar computations of four-point amplitudes in gauge-fixed Berkovits superstring field theory appear in [51].

the 4-point amplitude derived from the Lagrangian:

$$\begin{aligned} A_4(\Psi_1,\Psi_2,\Psi_3,\Psi_4) = -\omega\left(\Psi_1, M_2\left(\Psi_2, \frac{b_0}{L_0}M_2(\Psi_3,\Psi_4)\right)\right) \\ -\,\omega\left(\Psi_1, M_2\left(\frac{b_0}{L_0}M_2(\Psi_2,\Psi_3),\Psi_4\right)\right) \\ +\,\omega\Big(\Psi_1, M_3(\Psi_2,\Psi_3,\Psi_4)\Big). \end{aligned} \tag{3.48}$$

Pulling $\Psi_1,\ldots,\Psi_4$ off to the right we can then express the amplitude

$$\langle A_4| = \langle\omega|\left(\mathbb{I}\otimes M_2\left(-\mathbb{I}\otimes\frac{b_0}{L_0}M_2 - \frac{b_0}{L_0}M_2\otimes\mathbb{I}\right) + \mathbb{I}\otimes M_3\right), \tag{3.49}$$

where $\langle\omega|$ is the symplectic form on the small Hilbert space. We can write this using the coderivations derived from M_2 and M_3:

$$\langle A_4| = \langle\omega|\mathbb{I}\otimes\left(-\mathbf{M}_2\frac{b_0}{L_0}\mathbf{M}_2 + \mathbf{M}_3\right), \tag{3.50}$$

where we use $\frac{b_0}{L_0}\mathbf{M}_2$ to denote the coderivation derived from the map $\frac{b_0}{L_0}M_2$. We can also write the first quantised amplitude (3.47)

$$\langle A_4^{\mathrm{1st}}| = -\langle\omega|\mathbb{I}\otimes\pi_1\left(\mathbf{m}_2\frac{b_0}{L_0}\mathbf{m}_2\right)(X_0\otimes X_0\otimes\mathbb{I}\otimes\mathbb{I}). \tag{3.51}$$

First we check that BRST exact states decouple. Suppose the Ψ_1 is Q-exact. Pulling the Q off Ψ_1 and acting on $\langle A_4|$ gives

$$\begin{aligned} \langle A_4|Q\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} &= -\langle A_4|Q\otimes\left(-\mathbf{M}_2\frac{b_0}{L_0}\mathbf{M}_2 + \mathbf{M}_3\right), \\ &= \langle\omega|\mathbb{I}\otimes\left(-\mathbf{Q}\mathbf{M}_2\frac{b_0}{L_0}\mathbf{M}_2 + \mathbf{Q}\mathbf{M}_3\right), \end{aligned} \tag{3.52}$$

where we used the fact that Q is BPZ odd: $\langle\omega|\mathbb{I}\otimes Q = -\langle\omega|Q\otimes\mathbb{I}$. Since the other three states are Q-closed, we can write the second factor as a commutator with $\mathbf{Q}$:

$$\begin{aligned} \langle A_4|Q\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} &= \langle\omega|\mathbb{I}\otimes\left(\left[\mathbf{Q}, -\mathbf{M}_2\frac{b_0}{L_0}\mathbf{M}_2 + \mathbf{M}_3\right]\right), \\ &= \langle\omega|\mathbb{I}\otimes\Big(\mathbf{M}_2\mathbf{M}_2 + [\mathbf{Q},\mathbf{M}_3]\Big), \\ &= \langle\omega|\mathbb{I}\otimes\left(\frac{1}{2}[\mathbf{M}_2,\mathbf{M}_2] + [\mathbf{Q},\mathbf{M}_3]\right), \\ &= 0. \end{aligned} \tag{3.53}$$

This vanishes as a result of the A_∞ relation for M_2 and M_3. Similarly, Q exact states decouple from the first quantised amplitude (3.51) because of associativity of m_2.

Now we show that the field theory amplitude (3.50) and the first-quantised amplitude (3.51) are identical. For this purpose it is helpful to pass to the large Hilbert space, since this allows us to analyse individual terms which appear in the 3-product M_3 separately. In fact, if we identify the small Hilbert space as the kernel of η in the large Hilbert space, all operators have a natural extension to the large Hilbert space. The BPZ inner product on the large Hilbert space induces another symplectic form ω_L. ω_L carries picture 1 and non-vanishing matrix elements require us to saturate the ξ-zero mode. Hence, the relation of ω_L with the symplectic form ω on the small Hilbert space is not unique but of the form[4],

$$\langle\omega| = \langle\omega_L|(\mathbb{I}\otimes\xi), \tag{3.54}$$

where ξ is arbitrary with $[\eta,\xi]=1$. If b_n is a multilinear map which commutes with η, this implies the relation when restricted to the small Hilbert space

$$\langle\omega|\mathbb{I}\otimes b_n = (-1)^{\deg(b_n)}\langle\omega_L|(\mathbb{I}\otimes b_n)(\mathbb{I}^{\otimes k}\otimes\xi\otimes\mathbb{I}^{n-k}), \tag{3.55}$$

so we can place ξ on any input of the multilinear map as needed.

From the large Hilbert space point of view, physical states are states that are simultaneously annihilated by Q and η. Taking care of the ξ zero mode, the field theory amplitude (3.50) takes the form

$$\langle A_{4,L}| = \langle\omega_L|\mathbb{I}\otimes\xi\pi_1\left(-\mathbf{M}_2\frac{b_0}{L_0}\mathbf{M}_2+\mathbf{M}_3\right), \tag{3.56}$$

where we used (3.54). Since we are in the large Hilbert space, we are free to use our definition of the vertices in terms of dressed and bare products. Write $\mathbf{M}_2=[\mathbf{Q},\boldsymbol{\mu}_2]$ in the first term and pull $[\mathbf{Q},\cdot]$ past the propagator:

$$\begin{aligned}\langle A_{4,L}| &= \langle\omega_L|\mathbb{I}\otimes\xi\left(-\frac{1}{2}\left[\mathbf{Q},\boldsymbol{\mu}_2\frac{b_0}{L_0}\mathbf{M}_2\right]-\frac{1}{2}\left[\mathbf{Q},\mathbf{M}_2\frac{b_0}{L_0}\boldsymbol{\mu}_2\right]-\frac{1}{2}[\mathbf{M}_2,\boldsymbol{\mu}_2]+\mathbf{M}_3\right),\\ &= \langle\omega_L|\mathbb{I}\otimes X\left(-\frac{1}{2}\boldsymbol{\mu}_2\frac{b_0}{L_0}\mathbf{M}_2+\frac{1}{2}\mathbf{M}_2\frac{b_0}{L_0}\boldsymbol{\mu}_2\right)\\ &\qquad + \langle\omega_L|\mathbb{I}\otimes\xi\left(-\frac{1}{2}[\mathbf{M}_2,\boldsymbol{\mu}_2]+\mathbf{M}_3\right).\end{aligned} \tag{3.57}$$

In the second step we moved the $\mathbf{Q}$ commutator past the ξ insertion to act on external states. Note that $-\frac{1}{2}[\mathbf{M}_2,\boldsymbol{\mu}_2]$ already cancels one term in $\mathbf{M}_3$. In the first pair of terms above ξ only appears in the dressed 2-product $\boldsymbol{\mu}_2$. Using equation (3.55) we can move the ξs out of $\boldsymbol{\mu}_2$ onto the second entry of the symplectic form. This leaves the bare 2-product $\mathbf{m}_2$:

$$\begin{aligned}\langle A_{4,L}| &= \langle\omega_L|\mathbb{I}\otimes X\xi\left(-\frac{1}{2}\mathbf{m}_2\frac{b_0}{L_0}\mathbf{M}_2-\frac{1}{2}\mathbf{M}_2\frac{b_0}{L_0}\mathbf{m}_2\right)\\ &\qquad + \langle\omega_L|\mathbb{I}\otimes\xi\left(-\frac{1}{2}[\mathbf{M}_2,\boldsymbol{\mu}_2]+\mathbf{M}_3\right).\end{aligned} \tag{3.58}$$

[4]This identification assumes that the basic ghost correlator in the large Hilbert space is normalised $\langle\xi c\partial c\partial^2 ce^{-2\phi}\rangle = 2$. Note that the sign is opposite from our chosen normalisation of the basic correlator in the small Hilbert space.

Now we repeat this process a second time; Write $\mathbf{M}_2 = [\mathbf{Q}, \boldsymbol{\mu}_2]$ and pull $[\mathbf{Q}, \cdot]$ past the propagator:

$$\langle A_{4,L}| = \langle \omega_L|\mathbb{I} \otimes X\xi \left(\frac{1}{2} \left[\mathbf{Q}, \mathbf{m}_2 \frac{b_0}{L_0} \boldsymbol{\mu}_2 \right] - \frac{1}{2} \left[\mathbf{Q}, \boldsymbol{\mu}_2 \frac{b_0}{L_0} \mathbf{m}_2 \right] - \frac{1}{2} [\mathbf{m}_2, \boldsymbol{\mu}_2] \right) + \langle \omega_L|\mathbb{I} \otimes \xi \left(-\frac{1}{2} [\mathbf{M}_2, \boldsymbol{\mu}_2] + \mathbf{M}_3 \right). \tag{3.59}$$

We pick up a term $[\mathbf{m}_2, \boldsymbol{\mu}_2]$, which happens to be the bare-3-product $\mathbf{m}_3$. Moving Q past the ξ insertion gives

$$\langle A_{4,L}| = \langle \omega_L|\mathbb{I} \otimes X^2 \left(-\frac{1}{2} \mathbf{m}_2 \frac{b_0}{L_0} \boldsymbol{\mu}_2 - \frac{1}{2} \boldsymbol{\mu}_2 \frac{b_0}{L_0} \mathbf{m}_2 \right) - \langle \omega_L|\mathbb{I} \otimes X\xi \left(\frac{1}{2} \mathbf{m}_3 \right) + \langle \omega_L|\mathbb{I} \otimes \xi \left(-\frac{1}{2} [\mathbf{M}_2, \boldsymbol{\mu}_2] + \mathbf{M}_3 \right). \tag{3.60}$$

In the first term, use equation (3.55) to move the ξ out of $\boldsymbol{\mu}_2$ onto the second input of ω_L. In the second term, use equation (3.55) to move the ξ from the second input of ω_L back into the bare-3-product $\mathbf{m}_3$, turning it into the dressed 3-product $\boldsymbol{\mu}_3$:

$$\begin{aligned} \langle A_{4,L}| &= \langle \omega_L|\mathbb{I} \otimes X^2\xi \left(-\mathbf{m}_2 \frac{b_0}{L_0} \mathbf{m}_2 \right) - \langle \omega_L|\mathbb{I} \otimes X \left(\frac{1}{2} \boldsymbol{\mu}_3 \right) \\ &\qquad + \langle \omega_L|\mathbb{I} \otimes \xi \left(-\frac{1}{2} [\mathbf{M}_2, \boldsymbol{\mu}_2] + \mathbf{M}_3 \right), \\ &= \langle \omega_L|\mathbb{I} \otimes X^2\xi \left(-\mathbf{m}_2 \frac{b_0}{L_0} \mathbf{m}_2 \right) \\ &\qquad + \langle \omega_L|\mathbb{I} \otimes \xi \left(-\frac{1}{2} [\mathbf{Q}, \boldsymbol{\mu}_3] - \frac{1}{2} [\mathbf{M}_2, \boldsymbol{\mu}_2] + \mathbf{M}_3 \right). \end{aligned} \tag{3.61}$$

The last three terms cancel by the definition of $\mathbf{M}_3$. Moving back to the small Hilbert space, we have therefore shown

$$\langle A_4| = -\langle \omega|X^2 \otimes \left(\mathbf{m}_2 \frac{b_0}{L_0} \mathbf{m}_2 \right). \tag{3.62}$$

This is almost the first quantised amplitude, except X may be different from the zero mode X_0, and it acts twice on the first input rather than once on the first and once on the second input. But the difference between X and X_0 is a BRST exact, and the change moving X_0 to the second output is also BRST exact. Since external states are on-shell and m_2 is associative, these changes do not effect the amplitude. Therefore

$$\langle A_4| = -\langle \omega|\mathbb{I} \otimes \left(\mathbf{m}_2 \frac{b_0}{L_0} \mathbf{m}_2 \right) (X_0 \otimes X_0 \otimes \mathbb{I} \otimes \mathbb{I}) = \langle A_4^{\mathrm{1st}}|. \tag{3.63}$$

and the string field theory 4-point amplitude agrees with the first quantised result.

3.5 Discussion

We have succeeded in constructing an explicit and non-singular covariant superstring field theory in the small Hilbert space. Virtually by construction, the action satisfies the classical BV master equation,

$$\{S, S\} = 0, \tag{3.64}$$

once we relax the ghost number constraint on the string field. To quantise the theory, we need to incorporate the Ramond sector, cf. chapter 5 and also [64, 65, 75]. There are a couple of different approaches we could take to this problem. One suggested by Berkovits [123] is to distribute the degrees of freedom of the Ramond string field between picture $-\frac{1}{2}$ and picture $-\frac{3}{2}$, which necessarily breaks manifest covariance. One might also try to regulate Witten's original kinetic term for the Ramond string field, which has a midpoint insertion of the inverse picture changing operator Y. Then we would have to see how this extra operator could be incorporated into the A_∞ structure. Once the Ramond sector is included, we would be in good shape to understand the role of closed strings in quantum open string field theory.

Another variation we can consider is adding stubs to the cubic vertex. Then the higher vertices would necessarily require integration over bosonic moduli. It would be interesting to understand the interplay between the picture changing insertions and the A_∞ structure related to integration over bosonic moduli. Once this is understood, closed type II superstring field theory can be constructed in a similar manner. Both problems are solved in chapter 4. Previous formal attempts to construct such a theory have been stymied by the lack of a well-posed minimal area problem on supermoduli space [59]. Another construction of type II closed superstring field theory in the large Hilbert space proposed in [124].

Our construction is purely algebraic. We have not analysed how the vertices and propagators cover the supermoduli space of the disk with NS boundary punctures. Understanding this would undoubtedly provide insight into the foundations of superstring field theory.

Considering that our theory is formulated in the small Hilbert space, the large Hilbert space plays a surprisingly prominent role. This strongly suggests a relation to Berkovits' open superstring field theory. Indeed our formulation can be obtained as a partial gauge-fixing from the Berkovits theory [51, 56–58, 121]. For one thing, there has been recent notable progress in understanding classical solutions in the Berkovits theory [125], and it would be pleasing to incorporate these results in a unified formalism.

Chapter 4

NS-NS sector of closed superstring field theory

The construction open NS-superstring field theory in chapter 3 was crucially based on the associativity of Witten's star product. If the trivalent graphs do not cover the bosonic moduli space, the 2-product becomes non-associative. As the bosonic moduli space of punctured Riemann spheres does not admit a cover through trivalent graphs, extending the previous construction to include non-associative products is necessary for formulating closed superstring field theories. In this chapter we give a construction for a general class of vertices in superstring field theory which include integration over bosonic moduli as well as the required picture changing insertions. We apply this procedure to find a covariant action for the NS-NS sector of type II closed superstring field theory.

This chapter is based on the paper **NS-NS Sector of Closed Superstring Field Theory** by T. Erler, the author and I. Sachs [53].

4.1 Introduction

Though bosonic string field theory has been well-understood since the mid 90's [10, 17, 25, 26], superstring field theory remains largely mysterious. In some cases it is possible to find elegant formulations utilising the large Hilbert space [41, 45, 46, 123, 126], but it seems difficult to push beyond tree level [111–114] and the presumed geometrical underpinning of the theory in terms of the supermoduli space remains obscure. A somewhat old-fashioned alternative [34] is to formulate superstring field theory using fields in the small Hilbert space. A well known complication, however, is that one needs a prescription for inserting picture changing operators into the action. This requires an apparently endless sequence of choices, and while limited work in this direction exists [59, 74, 127], it has not produced a compelling and fully explicit action.

The basic insight of chapter 3 is that the multi-string products of open superstring field theory can be constructed by passing to the large Hilbert space and constructing a particular finite gauge transformation through the space of A_∞ structures. The result is an explicit action for open superstring field theory which automatically satisfies the classical BV master equation. In this paper we generalise these results to define classical actions for the NS sectors of all open and closed superstring field theories. Of particular interest is the NS-NS sector of type II closed superstring field theory. Interestingly, however, picture changing operators still appear to be needed in the action. The main technical obstacle for us will be learning how to accommodate vertices which include integration over bosonic moduli, and for the NS-NS superstring, how to insert additional picture changing operators for the right-moving sector. These results lay the groundwork for serious consideration of the Ramond sector and quantization of superstring field theory. This is of particular interest in the context of recent efforts to obtain a more complete understanding of superstring perturbation theory [49, 92, 95, 128, 129].

This chapter is organised as follows. In section 4.2 we revisit Witten's open superstring field theory in the -1 picture [34], but generalising in chapter 3, we allow vertices which include integration over bosonic moduli as well as the required picture changing insertions. We find that the multi-string products can be derived from a recursion involving a two-dimensional array of products of intermediate picture number. The recursion emerges from the solution to a pair of differential equations which follow uniquely from two assumptions: that the products are derived by gauge transformation through the space of A_∞ structures, and that the gauge transformation is defined in the large Hilbert space. In section 4.3, we explain how this construction generalises to the NS sector of heterotic string field theory. In section 4.4 we consider the NS-NS sector of type II closed superstring field theory. We give one construction which defines the products by applying the open string recursion of section 4.2 twice, first to get the correct picture in the left-moving sector and and again to get the correct picture in the right-moving sector. This construction however treats the left and right-moving sectors asymmetrically. We therefore provide a second, more nontrivial construction which preserves symmetry between left and remover's at every stage in the recursion. Finally, in section 4.5 we give a general discussion about the dependence of the actions on some choices made during their construction.

4.2 Witten's theory with stubs

In this section we revisit the construction of Witten's open superstring field theory. Unlike in chapter 3, where the higher vertices were built from Witten's open string star product, here we consider a more general set of vertices which may include integration over bosonic moduli. Such vertices are at any rate necessary for the closed string [130]. Witten's superstring field theory is based on a string field Ψ in the -1 picture. It has even degree, ghost number 1, and lives in the small Hilbert space. The action is defined by a sequence of multi-string products

$$M_1^{(0)} = Q, \qquad M_2^{(1)}, \qquad M_3^{(2)}, \qquad M_4^{(3)}, \qquad \ldots\ , \tag{4.1}$$

satisfying the relations of a cyclic A_∞ algebra. Since the vertices must have total picture -2, and the string field has picture -1, the $(n+1)$st product $M_{n+1}^{(n)}$ must carry picture n. We keep track of the picture through the upper index of the product. The goal is to construct these products by placing picture changing operators on a set of n-string products defining open bosonic string field theory:

$$M_1^{(0)} = Q, \qquad M_2^{(0)}, \qquad M_3^{(0)}, \qquad M_4^{(0)}, \qquad \ldots\ , \tag{4.2}$$

where the bosonic string products of course carry zero picture. We can choose $M_2^{(0)}$ to be Witten's open string star product, in which case the higher bosonic products $M_3^{(0)}, M_4^{(0)}, \ldots$ can be chosen to vanish. This is the scenario considered in chapter 3. Here we will not assume that $M_3^{(0)}, M_4^{(0)}, \ldots$ vanish. For example, we can consider the open string star product with "stubs" attached to each output:

$$M_2^{(0)}(A,B) = (-1)^{\deg(A)} e^{-\pi L_0}\Big((e^{-\pi L_0}A) * (e^{-\pi L_0}B)\Big). \tag{4.3}$$

The presence of stubs means that the propagators by themselves will not cover the full bosonic moduli space, and the higher products $M_3^0, M_4^0, \ldots$ are needed to cover the missing regions. Though it is natural to think of the $M_n^{(0)}$s as deriving from open bosonic string field theory, this is not strictly necessary. We only require three formal properties:

1) The $M_n^{(0)}$s satisfy the relations of a cyclic A_∞ algebra.

2) The $M_n^{(0)}$s are in the small Hilbert space.

3) The $M_n^{(0)}$s carry vanishing picture number.

Our task is to add picture number to the $M_n^{(0)}$s to define consistent nonzero vertices for Witten's open superstring field theory.

Cubic and quartic vertices

We start with the cubic vertex, defined by a 2-product $M_2^{(1)}$ constructed by placing a picture changing operator X once on each output of $M_2^{(0)}$:

$$M_2^{(1)}(\Psi_1,\Psi_2) \equiv \frac{1}{3}\Big(XM_2^{(0)}(\Psi_1,\Psi_2) + M_2^{(0)}(X\Psi_1,\Psi_2) + M_2^{(0)}(\Psi_1,X\Psi_2)\Big). \tag{4.4}$$

The picture changing operator X takes the following form:

$$X \equiv \oint_{|z|=1}\frac{dz}{2\pi i}f(z)X(z), \qquad\qquad X(z) = Q\xi(z), \tag{4.5}$$

where $f(z)$ a 1-form which is analytic in some non-degenerate annulus around the unit circle, and satisfies

$$f(z) = -\frac{1}{z^2} f\left(-\frac{1}{z}\right), \qquad \oint_{|z|=1} \frac{dz}{2\pi i} f(z) = 1. \tag{4.6}$$

The first relation implies that X is BPZ even, and the second amounts to a choice of the open string coupling constant, which we have set to 1. Since Q and X commute, Q is a derivation of $M_2^{(1)}$:

$$[\mathbf{Q}, \mathbf{M}_2^{(1)}] = 0. \tag{4.7}$$

Together with $[\mathbf{Q}, \mathbf{Q}] = 0$, this means that the first two A_∞ relations are satisfied. However, $M_2^{(1)}$ is not associative, so higher products $M_3^{(2)}, M_4^{(3)}, ...$ are needed to have a consistent A_∞ algebra.

To find the higher products, the key observation is that $M_2^{(1)}$ is BRST exact in the large Hilbert space:[1]

$$\mathbf{M}_2^{(1)} = [\mathbf{Q}, \boldsymbol{\mu}_2^{(1)}]. \tag{4.8}$$

Here we introduce a degree even product

$$\mu_2^{(1)} \equiv \frac{1}{3}\Big(\xi M_2^{(0)} - M_2^{(0)}(\xi \otimes \mathbb{I} + \mathbb{I} \otimes \xi)\Big), \tag{4.9}$$

with $\xi \equiv \oint \frac{dz}{2\pi i} f(z)\xi(z)$, which also satisfies

$$\mathbf{M}_2^{(0)} = [\boldsymbol{\eta}, \boldsymbol{\mu}_2^{(1)}], \tag{4.10}$$

where $\boldsymbol{\eta}$ is the coderivation derived from the η zero mode. The fact that $M_2^{(1)}$ is BRST exact means that it can be generated by a gauge transformation through the space of A_∞ structures, cf. chapter 3. So to find a solution to the A_∞ relations, all we have to do is complete the construction of the gauge transformation so as to ensure that $M_3^{(2)}, M_4^{(3)}, \ldots$ are in the small Hilbert space. The gauge transformation is defined by $\mu_2^{(1)}$ and an array of higher-point products $\mu_l^{(k)}$ of even degree. We call them *gauge products*.

The first nonlinear correction to the gauge transformation determines the 3-product $M_3^{(2)}$, via the formula

$$\mathbf{M}_3^{(2)} = \frac{1}{2}\Big([\mathbf{Q}, \boldsymbol{\mu}_3^{(2)}] + [\mathbf{M}_2^{(1)}, \boldsymbol{\mu}_2^{(1)}]\Big), \tag{4.11}$$

where we introduce a gauge 3-product $\boldsymbol{\mu}_3^{(2)}$ with picture number two. Plugging in and using the Jacobi identity, it is easy to see that the 3rd A_∞ relation is identically satisfied:

$$0 = \frac{1}{2}[\mathbf{M}_2^{(1)}, \mathbf{M}_2^{(1)}] + [\mathbf{Q}, \mathbf{M}_3^{(2)}]. \tag{4.12}$$

[1]Note that the cohomology of Q and η is trivial in the large Hilbert space.

However, the term $[\mathbf{Q}, \boldsymbol{\mu}_3^{(2)}]$ in equation (4.11) does not play a role for this purpose. This term is needed for a different reason: to ensure that $M_3^{(2)}$ lives in the small Hilbert space. Let's define a degree odd 3-product $M_3^{(1)}$ with picture 1, satisfying

$$\mathbf{M}_3^{(1)} = [\boldsymbol{\eta}, \boldsymbol{\mu}_3^{(2)}]. \tag{4.13}$$

Requiring $M_3^{(2)}$ to be in the small Hilbert space implies

$$\begin{aligned}[\boldsymbol{\eta}, \mathbf{M}_3^{(2)}] = 0 &= \frac{1}{2}\Big(- [\mathbf{Q}, \mathbf{M}_3^{(1)}] - [\mathbf{M}_2^{(1)}, \mathbf{M}_2^{(0)}]\Big), \\ &= \frac{1}{2}\left[\mathbf{Q}, -\mathbf{M}_3^{(1)} + [\mathbf{M}_2^{(0)}, \boldsymbol{\mu}_2^{(1)}]\right]. \end{aligned} \tag{4.14}$$

Therefore $M_3^{(1)}$ must satisfy

$$\mathbf{M}_3^{(1)} = [\mathbf{Q}, \boldsymbol{\mu}_3^{(1)}] + [\mathbf{M}_2^{(0)}, \boldsymbol{\mu}_2^{(1)}], \tag{4.15}$$

where we introduce yet another gauge 3-product $\mu_3^{(1)}$ with picture number 1. In chapter 3 it was consistent to set $\mu_3^{(1)} = 0$ because Witten's open string star product is associative. Now we will not assume that $M_2^{(0)}$ is associative, so the term $[\mathbf{Q}, \boldsymbol{\mu}_3^{(1)}]$ is needed to make sure that $M_3^{(1)}$ is in the small Hilbert space, as is required by equation (4.13). We define $\mu_3^{(1)}$ by the relation

$$2\mathbf{M}_3^{(0)} = [\boldsymbol{\eta}, \boldsymbol{\mu}_3^{(1)}], \tag{4.16}$$

where $\mathbf{M}_3^{(0)}$ is the bosonic 3-product. Then taking η of equation (4.14) implies

$$0 = [\mathbf{Q}, \mathbf{M}_3^{(0)}] + \frac{1}{2}[\mathbf{M}_2^{(0)}, \mathbf{M}_2^{(0)}]. \tag{4.17}$$

This is nothing but the 3rd A_∞ relation for the bosonic products. The upshot is that we can determine $M_3^{(2)}$ for Witten's superstring field theory by climbing a "ladder" of products and gauge products starting from $M_3^{(0)}$ as follows:

$$\mathbf{M}_3^{(0)} = \text{given}, \tag{4.18a}$$

$$\mu_3^{(1)} = \frac{1}{2}\Big(\xi M_3^{(0)} - M_3^{(0)}(\xi \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \xi \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \xi)\Big), \tag{4.18b}$$

$$\mathbf{M}_3^{(1)} = [\mathbf{Q}, \boldsymbol{\mu}_3^{(1)}] + [\mathbf{M}_2^{(0)}, \boldsymbol{\mu}_2^{(1)}], \tag{4.18c}$$

$$\mu_3^{(2)} = \frac{1}{4}\Big(\xi M_3^{(1)} - M_3^{(1)}(\xi \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \xi \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \xi)\Big), \tag{4.18d}$$

$$\mathbf{M}_3^{(2)} = \frac{1}{2}\Big([\mathbf{Q}, \boldsymbol{\mu}_3^{(2)}] + [\mathbf{M}_2^{(1)}, \boldsymbol{\mu}_2^{(1)}]\Big). \tag{4.18e}$$

The second and fourth equations invert equations (4.16) and (4.13) by placing a ξ insertion once on each output of the respective 3-product. Incidentally, we construct

$M_2^{(1)}$ by climbing a similar ladder

$$\mathbf{M}_2^{(0)} = \text{given}, \tag{4.19a}$$

$$\mu_2^{(1)} = \frac{1}{3}\Big(\xi M_2^{(0)} - M_2^{(0)}(\xi \otimes \mathbb{I} + \mathbb{I} \otimes \xi)\Big), \tag{4.19b}$$

$$\mathbf{M}_2^{(1)} = [\mathbf{Q}, \boldsymbol{\mu}^{(1)}], \tag{4.19c}$$

but in this case it was easier to postulate the final answer from the beginning. Proceeding in this way, it is not difficult to anticipate that the $(n+1)$-string product $M_{n+1}^{(n)}$ of Witten's superstring field theory can be constructed by ascending a ladder of $n+1$ products

$$M_{n+1}^{(0)}, \qquad M_{n+1}^{(1)}, \qquad \ldots, \qquad M_{n+1}^{(n)}, \tag{4.20a}$$

interspersed with n gauge products

$$\mu_{n+1}^{(1)}, \qquad \mu_{n+1}^{(2)}, \qquad \ldots, \qquad \mu_{n+1}^{(n)}, \tag{4.20b}$$

adding picture number one step at a time. Thus we will have a recursive solution to the A_∞ relations.

All vertices

We now explain how to determine the vertices to to all orders. We start by collecting superstring products into a generating function

$$\mathbf{M}^{[0]}(t) \equiv \sum_{n=0}^{\infty} t^n \mathbf{M}_{n+1}^{(n)}, \tag{4.21}$$

so that the $(n+1)$st superstring product can be extracted by looking at the coefficient of t^n. Here we place an upper index on the generating function (in square brackets) to indicate the "deficit" in picture number of the products relative to what is needed for the superstring. In this case, of course, the deficit is zero. The superstring products must satisfy two properties. First, they must be in the small Hilbert space, and second, they must satisfy the A_∞ relations:

$$[\boldsymbol{\eta}, \mathbf{M}^{[0]}(t)] = 0, \qquad [\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)] = 0. \tag{4.22}$$

Expanding the second equation in powers of t gives the A_∞ relation. To solve the A_∞ relations, we postulate the differential equation

$$\frac{\partial}{\partial t}\mathbf{M}^{[0]}(t) = [\mathbf{M}^{[0]}(t), \boldsymbol{\mu}^{[0]}(t)], \tag{4.23}$$

where

$$\boldsymbol{\mu}^{[0]}(t) = \sum_{n=0}^{\infty} t^n \boldsymbol{\mu}_{n+2}^{(n+1)} \tag{4.24}$$

is a generating function for "deficit-free" gauge products. Expanding (4.23) in powers of t gives previous formulas (4.8) and (4.13) for the 2-product and the 3-product. Note that this differential equation implies

$$\frac{\partial}{\partial t}[\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)] = [[\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)], \boldsymbol{\mu}^{[0]}(t)]. \tag{4.25}$$

Since this is homogeneous in $[\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)]$, the A_∞ relations follow immediately from the fact that $[\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)] = 0$ holds at $t = 0$ (since $\mathbf{Q}$ is nilpotent). Note that the generating function (4.21) can also be interpreted as defining a 1-parameter family of A_∞ algebras, where the parameter t is the open string coupling constant, cf. chapter 3. In this context, the differential equation (4.23) says that changes of the coupling constant are implemented by a gauge transformation through the space of A_∞ structures, and $\boldsymbol{\mu}^{[0]}(t)$ is the infinitesimal gauge parameter.

The statement that the coupling constant is pure gauge normally means that the cubic and higher order vertices can be removed by field redefinition, and the scattering amplitudes vanish [131]. This does not happen here because $\boldsymbol{\mu}^{[0]}(t)$ is in the large Hilbert space, and therefore does not define an admissible gauge parameter. But then the nontrivial condition is that the superstring products are in the small Hilbert space despite the fact that the gauge transformation defining them is not. To see what this condition implies, take $\boldsymbol{\eta}$ of the differential equation (4.23) to find

$$[\mathbf{M}^{[0]}(t), \mathbf{M}^{[1]}(t)] = 0, \tag{4.26}$$

where

$$\mathbf{M}^{[1]}(t) = [\boldsymbol{\eta}, \boldsymbol{\mu}^{[0]}(t)] = \sum_{n=0}^{\infty} t^n \mathbf{M}_{n+2}^{(n)} \tag{4.27}$$

is the generating function for products with a single picture deficit. Now we can solve equation (4.26) by postulating a new differential equation

$$\frac{\partial}{\partial t}\mathbf{M}^{[1]}(t) = [\mathbf{M}^{[0]}(t), \boldsymbol{\mu}^{[1]}(t)] + [\mathbf{M}^{[1]}(t), \boldsymbol{\mu}^{[0]}(t)], \tag{4.28}$$

where

$$\boldsymbol{\mu}^{[1]}(t) = \sum_{n=0}^{\infty} t^n \boldsymbol{\mu}_{n+3}^{(n+1)} \tag{4.29}$$

is a generating function for gauge products with a picture deficit 1.

Now we are beginning to see the outlines of a recursion. Taking $\boldsymbol{\eta}$ of equation (4.28) implies a constraint on the generating function for products with two picture deficits $\mathbf{M}^{[2]}(t)$, which can be solved by postulating yet another differential equation, and so on. The full recursion is most compactly expressed by packaging the generating functions $\mathbf{M}^{[m]}(t)$ and $\boldsymbol{\mu}^{[m]}(t)$ together in a power series in a new parameter

s:

$$\mathbf{M}(s,t) \equiv \sum_{m=0}^{\infty} s^m \mathbf{M}^{[m]}(t) = \sum_{m,n=0}^{\infty} s^m t^n \mathbf{M}^{(n)}_{m+n+1}, \tag{4.30a}$$

$$\boldsymbol{\mu}(s,t) \equiv \sum_{m=0}^{\infty} s^m \boldsymbol{\mu}^{[m]}(t) = \sum_{m,n=0}^{\infty} s^m t^n \boldsymbol{\mu}^{(n+1)}_{m+n+2}. \tag{4.30b}$$

Note that powers of t count the picture number, and powers of s count the deficit in picture number. At $t = 0$ $\mathbf{M}(s,t)$ reduces to a generating function for products of the bosonic string, and at $s = 0$ it reduces to a generating function for products of the superstring:

$$\mathbf{M}(s,0) = \sum_{n=0}^{\infty} s^n \mathbf{M}^{(0)}_{n+1}, \tag{4.31a}$$

$$\mathbf{M}(0,t) = \mathbf{M}^{[0]}(t) = \sum_{n=0}^{\infty} t^n \mathbf{M}^{(n)}_{n+1}. \tag{4.31b}$$

The recursion then emerges from expansion of a *pair* of differential equations

$$\frac{\partial}{\partial t}\mathbf{M}(s,t) = [\mathbf{M}(s,t), \boldsymbol{\mu}(s,t)], \tag{4.32a}$$

$$\frac{\partial}{\partial s}\mathbf{M}(s,t) = [\boldsymbol{\eta}, \boldsymbol{\mu}(s,t)]. \tag{4.32b}$$

Note that these equations imply

$$\frac{\partial}{\partial t}[\mathbf{M}(s,t), \mathbf{M}(s,t)] = [[\mathbf{M}(s,t), \mathbf{M}(s,t)], \boldsymbol{\mu}(s,t)], \tag{4.33a}$$

$$\frac{\partial}{\partial t}[\boldsymbol{\eta}, \mathbf{M}(s,t)] = [[\boldsymbol{\eta}, \mathbf{M}(s,t)], \boldsymbol{\mu}(s,t)] - \frac{1}{2}\frac{\partial}{\partial s}[\mathbf{M}(s,t), \mathbf{M}(s,t)]. \tag{4.33b}$$

Since the first equation is homogeneous in $[\mathbf{M}(s,t), \mathbf{M}(s,t)]$, the A_∞ relations for the bosonic products at $t = 0$ implies $[\mathbf{M}(s,t), \mathbf{M}(s,t)] = 0$ for all s and t. Thus the second equation (4.33b) becomes homogeneous in $[\boldsymbol{\eta}, \mathbf{M}(s,t)]$, and the fact that the bosonic products are in the small Hilbert space at $t = 0$ implies that all products are in the small Hilbert space. Thus

$$[\mathbf{M}(s,t), \mathbf{M}(s,t)] = 0, \qquad [\boldsymbol{\eta}, \mathbf{M}(s,t)] = 0. \tag{4.34}$$

Setting $s = 0$ we recover equation (4.22). Therefore, solving equations (4.32) automatically determines a set of superstring products which live in the small Hilbert space and satisfy the A_∞ relations.

Now all we need to do is solve the differential equations (4.32a) and (4.32b) to determine the products. Expanding equation (4.32a) in s, t and reading off the coefficient of $s^m t^n$ gives the formula:

$$\mathbf{M}^{(n+1)}_{m+n+2} = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{l=0}^{m}[\mathbf{M}^{(k)}_{k+l+1}, \boldsymbol{\mu}^{(n-k+1)}_{m+n-k-l+2}]. \tag{4.35}$$

This determines the product $M^{(n+1)}_{m+n+2}$ if we are given gauge products

$$\mu^{(k)}_l, \qquad 1 \leq k \leq n+1, k+1 \leq l \leq k+m+1, \tag{4.36}$$

and the lower order products

$$M^{(k)}_l, \qquad 0 \leq k \leq n, k+1 \leq l \leq k+m+1. \tag{4.37}$$

The lower order products are either again determined by equation (4.35), or they are products of the bosonic string, which we assume are given. So now we must find the gauge products $\mu^{(k)}_l$. Expanding equation (4.32b) gives

$$[\boldsymbol{\eta}, \boldsymbol{\mu}^{(n+1)}_{m+n+2}] = (m+1)\mathbf{M}^{(n)}_{m+n+2}. \tag{4.38}$$

This equation determines $\mu^{(n+1)}_{m+n+2}$ in terms of $M^{(m)}_{m+n+2}$. The solution is not unique. However there is a natural ansatz preserving cyclicity:

$$\mu^{(m+1)}_{m+n+2} = \frac{n+1}{m+n+3}\left(\xi M^{(m)}_{m+n+2} - M^{(m)}_{m+n+2}\sum_{k=0}^{m+n+1}\mathbb{I}^{\otimes k}\otimes\xi\otimes\mathbb{I}^{\otimes m+n+1-k}\right). \tag{4.39}$$

or, more compactly, we can write $\boldsymbol{\mu}^{(m+1)}_{m+n+2} = (n+1)\xi\circ\mathbf{M}^{(m)}_{m+n+2}$ where $\xi\circ$ denotes the operation of taking the average of ξ acting on the output and on each input of the product. This ansatz works assuming $M^{(m)}_{m+n+2}$ is in the small Hilbert space, but we have to show that the ansatz is consistent with that assumption. To this end, note that if equation (4.32a) is satisfied and the gauge products are defined in (4.39), we have the relation

$$\frac{\partial}{\partial t}[\boldsymbol{\eta}, \mathbf{M}(s,t)] = [[\boldsymbol{\eta}, \mathbf{M}(s,t)], \boldsymbol{\mu}(s,t)] + \left[\mathbf{M}(s,t), \frac{\partial}{\partial s}\xi\circ[\boldsymbol{\eta}, \mathbf{M}(s,t)]\right]. \tag{4.40}$$

Since this equation is homogeneous in $[\boldsymbol{\eta}, \mathbf{M}(s,t)]$, equation (4.39) implies that all products must be in the small Hilbert space.

The construction is recursive. Assume that we have already constructed all products $M^{(k)}_m$ and gauge products $\mu^{(k)}_m$ with $m \leq n$ inputs and with all picture numbers. Then we construct the $(n+1)$st product of Witten's superstring field theory by climbing a ladder of products and gauge products, defined by equations (4.35) and

(4.38):

$$\begin{aligned}
\mathbf{M}^{(0)}_{n+1} &= \text{given},\\
\mu^{(1)}_{n+1} &= \frac{n}{n+2}\left(\xi M^{(0)}_{n+1} - M^{(0)}_{n+1}\sum_{k=0}^{n}\mathbb{I}^{\otimes k}\otimes\xi\otimes\mathbb{I}^{\otimes n-k}\right).\\
\mathbf{M}^{(1)}_{n+1} &= [\mathbf{Q},\boldsymbol{\mu}^{(1)}_{n+1}] + [\mathbf{M}^{(0)}_2,\boldsymbol{\mu}^{(1)}_n] + \ldots + [\mathbf{M}^{(0)}_n,\boldsymbol{\mu}^{(1)}_2],\\
\mu^{(2)}_{n+1} &= \frac{n-1}{n+2}\left(\xi M^{(1)}_{n+1} - M^{(1)}_{n+1}\sum_{k=0}^{n}\mathbb{I}^{\otimes k}\otimes\xi\otimes\mathbb{I}^{\otimes n-k}\right),\\
\mathbf{M}^{(2)}_{n+1} &= \frac{1}{2}\Big([\mathbf{Q},\boldsymbol{\mu}^{(2)}_{n+1}] + [\mathbf{M}^{(0)}_2,\boldsymbol{\mu}^{(2)}_n] + [\mathbf{M}^{(1)}_2,\boldsymbol{\mu}^{(1)}_n] + \ldots\\
&\qquad + [\mathbf{M}^{(0)}_{n-1},\boldsymbol{\mu}^{(2)}_3] + [\mathbf{M}^{(1)}_{n-1},\boldsymbol{\mu}^{(1)}_3] + [\mathbf{M}^{(1)}_n,\boldsymbol{\mu}^{(1)}_2]\Big),\\
&\ \ \vdots\\
\mu^{(n)}_{n+1} &= \frac{1}{n+2}\left(\xi M^{(n)}_{n+1} - M^{(n)}_{n+1}\sum_{k=0}^{n}\mathbb{I}^{\otimes k}\otimes\xi\otimes\mathbb{I}^{\otimes n-k}\right),\\
\mathbf{M}^{(n)}_{n+1} &= \frac{1}{n}\Big([\mathbf{Q},\boldsymbol{\mu}^{(n)}_{n+1}] + [\mathbf{M}^{(1)}_2,\boldsymbol{\mu}^{(n-1)}_n] + \ldots + [\mathbf{M}^{(n-1)}_n,\boldsymbol{\mu}^{(1)}_2]\Big).
\end{aligned}$$

The final step in this ladder is the $n+1$-string product of Witten's open superstring field theory. Incidentally, note that the nature of this construction guarantees that the superstring products will define cyclic vertices if the bosonic products do.

4.3 NS heterotic string

Our analysis of the open superstring almost immediately generalises to a construction of heterotic string field theory in the NS sector. An alternative formulation of this theory, using the large Hilbert space, is described in [45, 46]. The closed string field is a degree even NS state Φ in the superconformal field theory of a heterotic string. Note that the $\beta\gamma$ ghosts and picture only reside in the left-moving sector. The string field has ghost number 2 and picture number -1, and satisfies the level matching constraints (2.62). The symplectic form (2.63) is well-defined only on states whose ghost number adds up to five and whose picture number adds up to -2.

The action is defined by a sequence of degree odd closed string products

$$L^{(0)}_1 = Q, \qquad L^{(1)}_2, \qquad L^{(2)}_3, \qquad L^{(3)}_4, \qquad \ldots\ , \tag{4.41}$$

satisfying the relations of a cyclic L_∞ algebra. Just like in the open string, the nth closed string product must have picture $n-1$ to define a non-vanishing vertex. We construct the products by placing picture changing operators on the products of the closed bosonic string

$$L^{(0)}_1 = Q, \qquad L^{(0)}_2, \qquad L^{(0)}_3, \qquad L^{(0)}_4, \qquad \ldots\ , \tag{4.42}$$

which, or course, have vanishing picture. The explicit definition of the closed bosonic string products is an intricate story [13, 17, 132–134], but for our purposes all we need to know is: 1) they satisfy the relations of a cyclic L_∞ algebra, 2) they are in the small Hilbert space, 3) they carry vanishing picture number, and 4) they are consistent with the level matching constraints.

The problem we need to solve appears completely analogous to the open superstring. Aside from replacing tensor products with wedge products, there is one minor difference. Since the products of the heterotic string must respect the b_0^- and L_0^- constraints, the picture changing operator X in the 2-product

$$L_2^{(1)}(\Phi_1,\Phi_2) = \frac{1}{3}\Big(XL_2^{(0)}(\Phi_1,\Phi_2) + L_2^{(0)}(X\Phi_1,\Phi_2) + L_2^{(0)}(\Phi_1,X\Phi_2)\Big) \tag{4.43}$$

must be identified with the zero mode X_0. This way, we can pull b_0^- and L_0^- past X_0 to act on $L_2^{(0)}$, which vanishes. More generally, we must construct closed superstring products using the ξ zero mode

$$\xi = \xi_0 = \oint_{|z|=1}\frac{dz}{2\pi i}\frac{1}{z}\xi(z), \tag{4.44}$$

rather than a more general charge which would be consistent for the open string.

Following the discussion of the open superstring, we introduce a "triangle" of products

$$L_{n+1}^{(k)}, \qquad\qquad 0\leq n\leq\infty, 0\leq k\leq n, \tag{4.45}$$

and gauge products,

$$\lambda_{n+2}^{(k+1)}, \qquad\qquad 0\leq n\leq\infty, 0\leq k\leq n \tag{4.46}$$

of intermediate picture indicated in the upper index. We build the $(n+1)$-heterotic string product $L_{n+1}^{(n)}$ by climbing a "ladder" of products

$$L_{n+1}^{(0)}, \qquad \lambda_{n+1}^{(1)}, \qquad L_{n+1}^{(1)}, \qquad \ldots, \qquad \lambda_{n+1}^{(n)}, \qquad L_{n+1}^{(n)}, \tag{4.47}$$

adding picture one step at a time. Each step is prescribed by the closed string analogues of equations (4.35) and (4.38):

$$\mathbf{L}_{m+n+2}^{(m+1)} = \frac{1}{m+1}\sum_{k=0}^{m}\sum_{l=0}^{n}[\mathbf{L}_{k+l+1}^{(k)}, \boldsymbol{\lambda}_{m+n-k-l+2}^{(m-k+1)}] \tag{4.48a}$$

$$\lambda_{m+n+2}^{(m+1)} = \frac{n+1}{m+n+3}\Big(\xi_0 L_{m+n+2}^{(m)} - L_{m+n+2}^{(m)}(\xi_0\wedge\mathbb{I}_{m+n+1})\Big). \tag{4.48b}$$

The only differences from the open superstring are that the coderivations act on the symmetrised tensor algebra, and ξ has been replaced by ξ_0.

4.4 NS-NS closed superstring

We are now ready to discuss the NS-NS sector of type II closed superstring field theory. The closed string field is a degree even NS-NS state Φ in the superconformal field theory of a type II superstring. Now $\beta\gamma$ ghosts and picture occupy both the left-moving and right-moving sectors. The string field has ghost number 2, satisfies the level matching constraints (2.62), and has left/right-moving picture number $(-1,-1)$. The symplectic form (2.63) is non-vanishing on states of ghost number 5 and left/right picture $(-2,-2)$.

The theory is defined by a sequence of degree odd closed string products

$$L_1^{(0,0)} = Q, \qquad L_2^{(1,1)}, \qquad L_3^{(2,2)}, \qquad L_4^{(3,3)}, \qquad \dots, \tag{4.49}$$

satisfying the relations of a cyclic L_∞ algebra. The $(n+1)$st closed string product must have left/right picture (n,n). These products should be constructed from the products of the closed bosonic string,

$$L_1^{(0,0)} = Q, \qquad L_2^{(0,0)}, \qquad L_3^{(0,0)}, \qquad L_4^{(0,0)}, \qquad \dots, \tag{4.50}$$

which have vanishing picture. Note that we add an extra index to indicate right-moving picture. Now the situation is somewhat different from the open string, since we need to add twice as much picture and we need to pay attention to how it is distributed between left-moving and right-moving sectors. However, it is not difficult to guess what the 2-product should look like. Starting with $L_2^{(0,0)}$, we surround it once with a left-moving picture changing operator X_0, and again a right-moving picture changing operator $\overline{X}_0$, to produce the expression

$$\begin{aligned} L_2^{(1,1)}(\Phi_1,\Phi_2) = \frac{1}{9}\Big(& X_0\overline{X}_0 L_2^{(0,0)}(\Phi_1,\Phi_2) + X_0 L_2^{(0,0)}(\overline{X}_0\Phi_1,\Phi_2) + X_0 L_2^{(0,0)}(\Phi_1,\overline{X}_0\Phi_2) \\ & + \overline{X_0} L_2^{(0,0)}(X_0\Phi_1,\Phi_2) + L_2^{(0,0)}(X_0\overline{X}_0\Phi_1,\Phi_2) + L_2^{(0,0)}(X_0\Phi_1,\overline{X}_0\Phi_2) \\ & + \overline{X_0} L_2^{(0,0)}(\Phi_1,X_0\Phi_2) + L_2^{(0,0)}(\overline{X}_0\Phi_1,X_0\Phi_2) + L_2^{(0,0)}(\Phi_1,X_0\overline{X}_0\Phi_2)\Big). \end{aligned} \tag{4.51}$$

Consider the 2-product $L_2^{(1,1)}$ written in the form

$$\mathbf{L}_2^{(1,1)} = \frac{1}{2}[\mathbf{Q},\boldsymbol{\lambda}_2^{(1,1)} + \overline{\boldsymbol{\lambda}}_2^{(1,1)}]. \tag{4.52}$$

Now we have introduced *two* gauge products. The first $\boldsymbol{\lambda}_2^{(1,1)}$ is called a left gauge product, and is defined by replacing X_0 in the expression (4.51) for $L_2^{(1,1)}$ with ξ_0. The second $\overline{\boldsymbol{\lambda}}_2^{(1,1)}$ is called a right gauge product, and is defined by replacing $\overline{X}_0$ in $L_2^{(1,1)}$ with $\overline{\xi}_0$. Once we act with $\mathbf{Q}$, $\boldsymbol{\lambda}_2^{(1,1)}$ and $\overline{\boldsymbol{\lambda}}_2^{(1,1)}$ produce the same expression (hence the factor of 1/2), but the advantage of this decomposition is that left/right symmetry is manifest. Denoting the left/right-moving eta zero modes by η and $\overline{\eta}$, we have the relations

$$[\boldsymbol{\eta},\boldsymbol{\lambda}_2^{(1,1)}] = \mathbf{L}_2^{(0,1)}, \qquad [\overline{\boldsymbol{\eta}},\overline{\boldsymbol{\lambda}}_2^{(1,1)}] = \mathbf{L}_2^{(1,0)} \tag{4.53a}$$

$$[\boldsymbol{\eta},\overline{\boldsymbol{\lambda}}_2^{(1,1)}] = 0, \qquad [\overline{\boldsymbol{\eta}},\boldsymbol{\lambda}_2^{(1,1)}] = 0. \tag{4.53b}$$

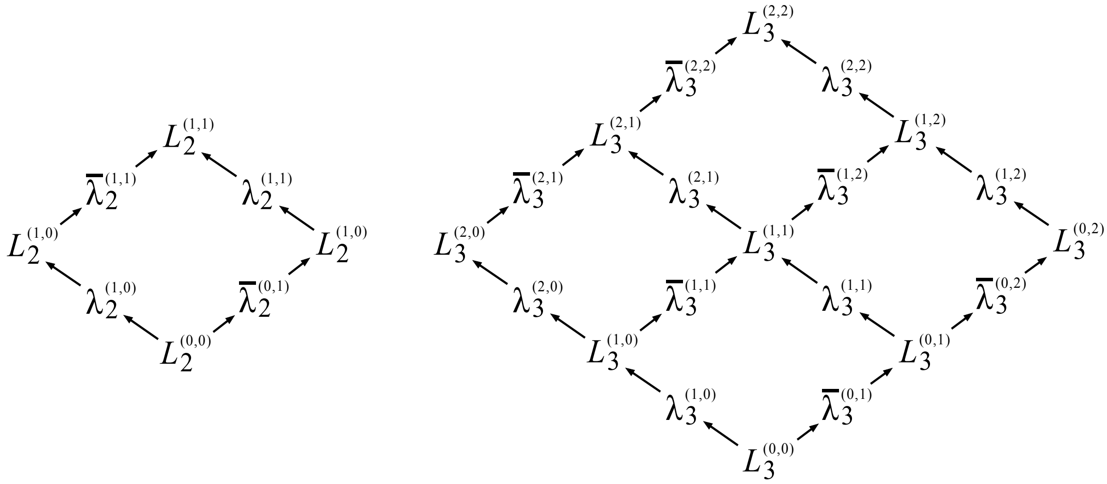


Figure 4.1: Diamond shaped arrangement of products and gauge products needed to construct the 2-product and 3-product of NS-NS closed superstring field theory.

Note that the left gauge product $\boldsymbol{\lambda}_2^{(1,1)}$ is in the right-moving small Hilbert space, while the right gauge product $\overline{\boldsymbol{\lambda}}_2^{(1,1)}$ is in the left-moving small Hilbert space. The products $L_2^{(1,0)}$ and $L_2^{(0,1)}$ now carry a single X_0 or $\overline{X}_0$ insertion, respectively. Pulling $\mathbf{Q}$ out we can write

$$\mathbf{L}_2^{(1,0)} = [\mathbf{Q}, \boldsymbol{\lambda}_2^{(1,0)}], \qquad \mathbf{L}_2^{(0,1)} = [\mathbf{Q}, \overline{\boldsymbol{\lambda}}_2^{(0,1)}], \tag{4.54}$$

where $\boldsymbol{\lambda}_2^{(1,0)}$ and $\overline{\boldsymbol{\lambda}}_2^{(0,1)}$ are left/right gauge products satisfying

$$[\boldsymbol{\eta}, \boldsymbol{\lambda}_2^{(1,0)}] = [\overline{\boldsymbol{\eta}}, \overline{\boldsymbol{\lambda}}_2^{(0,1)}] = \mathbf{L}_2^{(0,0)} \tag{4.55a}$$

$$[\overline{\boldsymbol{\eta}}, \boldsymbol{\lambda}_2^{(1,0)}] = [\boldsymbol{\eta}, \overline{\boldsymbol{\lambda}}_2^{(0,1)}] = 0, \tag{4.55b}$$

and $L_2^{(0,0)}$ is the product of the bosonic string. In this way the superstring product $L_2^{(1,1)}$ is derived by filling a diamond shaped diagram of products and gauge products, as shown in figure 4.1.

Also shown is a diamond illustrating the derivation of the 3-product, which has four cells giving a total of 21 intermediate products. The explicit formulas associated with this diagram are difficult to guess, so we will proceed to motivate the general construction. To find the closed superstring product $L_{n+1}^{(n,n)}$, we need a diamond consisting of $(n+1)^2$ products

$$L_{n+1}^{(p,q)}, \qquad 0 \le p, q \le n, \tag{4.56}$$

$n(n+1)$ left gauge products

$$\boldsymbol{\lambda}_{n+1}^{(p,q)}, \qquad \begin{matrix} 1 \le p \le n, \\ 0 \le q \le n, \end{matrix} \tag{4.57}$$

and $n(n+1)$ right gauge products

$$\overline{\boldsymbol{\lambda}}_{n+1}^{(p,q)}, \qquad \begin{array}{l} 0 \leq p \leq n, \\ 1 \leq q \leq n. \end{array} \tag{4.58}$$

We would like to package the products into three generating functions

$$\mathbf{L}(s, \overline{s}, t), \qquad \boldsymbol{\lambda}(s, \overline{s}, t), \qquad \overline{\boldsymbol{\lambda}}(s, \overline{s}, t), \tag{4.59}$$

which depend on three variables, corresponding to the three indices characterising the products. The variable t counts the total picture number, s the deficit in left-moving picture number, and $\overline{s}$ the deficit in right-moving picture number. Thus we have

$$\mathbf{L}(s, \overline{s}, t) = \sum_{N=0}^{\infty} \sum_{i,j=0}^{N} t^{i+j} s^{N-i}\, \overline{s}^{N-j} \mathbf{L}_{N+1}^{(i,j)}, \tag{4.60a}$$

$$\boldsymbol{\lambda}(s, \overline{s}, t) = \sum_{N=0}^{\infty} \sum_{i=0}^{N} \sum_{j=0}^{N+1} t^{i+j} s^{N-i}\, \overline{s}^{N+1-j} \boldsymbol{\lambda}_{N+2}^{(i+1,j)}, \tag{4.60b}$$

$$\overline{\boldsymbol{\lambda}}(s, \overline{s}, t) = \sum_{N=0}^{\infty} \sum_{i=0}^{N+1} \sum_{j=0}^{N} t^{i+j} s^{N+1-i}\, \overline{s}^{N-j} \overline{\boldsymbol{\lambda}}_{N+2}^{(i,j+1)}. \tag{4.60c}$$

The solution to the L_∞ relations is defined by the system of equations

$$\frac{\partial}{\partial t} \mathbf{L}(s, \overline{s}, t) = \Big[\mathbf{L}(s, \overline{s}, t), \boldsymbol{\lambda}(s, \overline{s}, t) + \overline{\boldsymbol{\lambda}}(s, \overline{s}, t)\Big], \tag{4.61a}$$

$$\frac{\partial}{\partial s} \mathbf{L}(s, \overline{s}, t) = [\boldsymbol{\eta}, \boldsymbol{\lambda}(s, \overline{s}, t)], \qquad [\overline{\boldsymbol{\eta}}, \boldsymbol{\lambda}(s, \overline{s}, t)] = 0, \tag{4.61b}$$

$$\frac{\partial}{\partial \overline{s}} \mathbf{L}(s, \overline{s}, t) = [\overline{\boldsymbol{\eta}}, \overline{\boldsymbol{\lambda}}(s, \overline{s}, t)], \qquad [\boldsymbol{\eta}, \overline{\boldsymbol{\lambda}}(s, \overline{s}, t)] = 0. \tag{4.61c}$$

Note that $\mathbf{L}(s, \overline{s}, t)$ at $t = 0$ reduces to a generating function for bosonic products:

$$\mathbf{L}(s, \bar{s}, 0) = \sum_{n=0}^{\infty} (s\overline{s})^n \mathbf{L}_{n+1}^{(0,0)}. \tag{4.62}$$

Following the argument given in section 4.2, this boundary condition together with the differential equations (4.61) imply, cf. section 4.5,

$$[\mathbf{L}(s, \overline{s}, t), \mathbf{L}(s, \overline{s}, t)] = 0, \qquad [\boldsymbol{\eta}, \mathbf{L}(s, \overline{s}, t)] = 0, \qquad [\overline{\boldsymbol{\eta}}, \mathbf{L}(s, \overline{s}, t)] = 0. \tag{4.63}$$

Evaluating this at $s = \overline{s} = 0$ implies that the closed superstring products are in the small Hilbert space and satisfy the L_∞ relations.

Now we have to solve the system (4.61) to define the products. Expanding (4.61a) in powers gives the formula

$$\mathbf{L}_{n+2}^{(p,q)} = \frac{1}{p+q} \sum_{k=0}^{n} \left(\sum_{r,s} [\mathbf{L}_{n-k+1}^{(r,s)}, \boldsymbol{\lambda}_{k+2}^{(p-r,q-s)}] + \sum_{r,s} [\mathbf{L}_{n-k+1}^{(r,s)}, \overline{\boldsymbol{\lambda}}_{k+2}^{(p-r,q-s)}] \right). \tag{4.64}$$

The sum over r, s include all values such that the product and gauge product in the commutator have admissible picture numbers. Similar to (4.35), this formula determines the products recursively given the products of the bosonic string and the left/right gauge products. The left/right gauge products are defined by solving equations (4.61b) and (4.61c), and following the argument of section 4.2 we find natural solutions

$$\lambda^{(p+1,q)}_{n+2} = \frac{n-p+1}{n+3}\Big(\xi_0 L^{(p,q)}_{n+2} - L^{(p,q)}_{n+2}(\xi_0 \wedge \mathbb{I}_{N+1})\Big), \tag{4.65a}$$

$$\overline{\lambda}^{(p,q+1)}_{n+2} = \frac{n-q+1}{n+3}\Big(\overline{\xi}_0 L^{(p,q)}_{n+2} - L^{(p,q)}_{n+2}(\overline{\xi}_0 \wedge \mathbb{I}_{N+1})\Big). \tag{4.65b}$$

Once we know all products and gauge products with up to $n+1$ inputs, we can determine the $(n+2)$nd superstring product $L^{(n+1,n+1)}_{n+2}$ by filling a "diamond" of products of intermediate picture number, starting from the bosonic product $L^{(0,0)}_{n+2}$ at the bottom. Filling the diamond requires climbing $4(n+1)$ levels, $2(n+1)$ of those require computing gauge products from products using equations (4.65a) and (4.65b), and the other $2(n+1)$ require computing products from gauge products using (4.64).

4.5 General properties

The recursive constructions presented in the previous sections are all derivable from a set of equations similar to (4.32) and (4.61). In this section we discuss a few obvious generalisations thereof and argue that they do not add anything new to the discussion as they produce actions that are related to the described ones by field redefinitions. In the course of this chapter we treat the formal variables t and s on a different footing: The variable s plays no special role and should be thought of as a compact way to write a set of coupled differential equations and algebraic constraints. More generally, we can also add additional formal variables that do not occur in the equations. The variable t on the other hand can be interpreted as a deformation parameter that connects a reference structure at $t = 0$ with the desired final algebraic structure at $t = 1$. Algebraically, the closed string is much richer so that we limit our discussion to that case.

The geometric space underlying the deformation problem is the space of L_∞ structures on a fixed Hilbert space $\mathcal{H}$ with the coefficient functions taking values in a polynomial ring $R = \mathbb{C}[s, \bar{s}]$. In principle we could adjoin more formal variables to the coefficient ring. We can think of $\mathbf{L}$ as a coordinate function on that space that takes values in square 0, degree 1 coderivations on $R \otimes S\mathcal{H}$. A family of L_∞ structures is just a function $\mathbf{L}(t)$ and corresponds to a path in the space of L_∞ structures. Let us consider now an equation of the form

$$\frac{d}{dt}\mathbf{L}(t) = \boldsymbol{\beta}(\mathbf{L}(t)). \tag{4.66}$$

If solutions to this equation should be L_∞ structures for all t, we must have that $[\mathbf{L}(t), \boldsymbol{\beta}(\mathbf{L})]$. Moreover, equations of this form are just the same as the flow equations of a vector field on the space of L_∞ structures. Recall that $[\mathbf{L},\cdot]$ is the *Chevalley-Eilenberg differential* $d_\mathbf{L}$ [135, 136]. For the A_∞ case this differential is called the *Hochschild differential.* Its kernel coincides with tangent vectors to the space of L_∞ algebras.

Among the vector fields there is a special class that corresponds to L_∞ isomorphisms. If $\mathcal{F} = \mathbb{I} + \boldsymbol{\alpha}$ is an infinitesimal L_∞ isomorphism, then $\boldsymbol{\alpha}$ must be a coderivation of degree 0. Under this isomorphism we have $\mathbf{L} \to \mathbf{L} + [\mathbf{L}, \boldsymbol{\alpha}]$. The corresponding vector field is an exact element $\boldsymbol{\beta} = d_\mathbf{L}\boldsymbol{\alpha}$. Usually one would stop here and declare such tangent vectors as gauge transformations and the Chevalley-Eilenberg cohomology is then equal to the tangent space of the moduli space of L_∞ structures, i.e. the space of L_∞ structures modulo isomorphisms. This moduli space can be identified with the space of physical coupling constants of the theory.

However, in the present situation this notion of gauge equivalence fails to capture the impact of the smallness conditions $[\boldsymbol{\eta}, \mathbf{L}] = [\overline{\boldsymbol{\eta}}, \mathbf{L}] = 0$. The appropriate modifications are dictated by requiring that the smallness conditions should be preserved as well. Thus, the tangent space at $\mathbf{L}$ is described by

$$d_\mathbf{L}\boldsymbol{\beta} = 0, \qquad [\boldsymbol{\eta}, \boldsymbol{\beta}] = [\overline{\boldsymbol{\eta}}, \boldsymbol{\beta}] = 0 \tag{4.67a}$$

$$\boldsymbol{\beta} \sim \boldsymbol{\beta} + d_\mathbf{L}\boldsymbol{\alpha} \qquad [\boldsymbol{\eta}, \boldsymbol{\alpha}] = [\overline{\boldsymbol{\eta}}, \boldsymbol{\alpha}] = 0. \tag{4.67b}$$

We now reconsider equation (4.61). The first equation (4.61a) can be interpreted as a flow equation for a vector field $\boldsymbol{\beta} = d_\mathbf{L}\boldsymbol{\lambda} + d_\mathbf{L}\overline{\boldsymbol{\lambda}}$. Although $\boldsymbol{\beta}$ looks like a trivial vector field, the coderivations $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ do not satisfy the smallness constraint required in (4.67b). Let us call the two vector fields $\delta = d_\mathbf{L}\boldsymbol{\lambda}$ and $\bar{\delta} = d_\mathbf{L}\overline{\boldsymbol{\lambda}}$. The coderivation $\boldsymbol{\lambda}$ is not unique, but is determined by

$$\frac{\partial}{\partial s}\mathbf{L} = [\boldsymbol{\eta}, \boldsymbol{\lambda}], \qquad [\overline{\boldsymbol{\eta}}, \boldsymbol{\lambda}] = 0.$$

In section 4.4 we decided to solve for $\boldsymbol{\lambda}$ with the help of the operator $\xi_0 \circ \cdot$. But it is clear that any other solution modifies δ by a term that is exact in the sense (4.67b) and yields isomorphic L_∞ structures. Consequently, there is no loss in generality with the choices made in the previous sections.

The Lie bracket of δ and $\bar{\delta}$ gives an infinitesimal L_∞ isomorphism. In fact,

$$[\delta, \bar{\delta}]\mathbf{L} = [\mathbf{L}, \boldsymbol{\kappa}], \tag{4.68}$$

$$\boldsymbol{\kappa} = [\boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}] + \bar{\xi}_0 \circ \frac{\partial}{\partial \bar{s}}[\mathbf{L}, \boldsymbol{\lambda}] - \xi_0 \circ \frac{\partial}{\partial s}[\mathbf{L}, \bar{\boldsymbol{\lambda}}] - [\mathbf{L}, \xi_0 \circ \bar{\xi}_0 \circ \frac{\partial^2}{\partial \bar{s}\partial s}\mathbf{L}],$$

$$[\boldsymbol{\eta}, \boldsymbol{\kappa}] = [\bar{\boldsymbol{\eta}}, \boldsymbol{\kappa}] = 0.$$

Thus, on the space of L_∞-structures modulo isomorphisms we have $[\delta, \bar{\delta}] = 0$. In particular it follows that the endpoint of the flow of any linear combination $a\delta + b\bar{\delta}$, for possibly t dependent functions a and b, is independent of the particular choice of a and b and depends only on $\int_0^1 dt\, a$ and $\int_0^1 dt\, b$. By redefining s and $\bar{s}$ one can always set these two integrals to 1. The effect is just a rescaling of the coupling constant. It follows that the physical products depend on the choice of integration contour only up to isomorphism.

4.6 Summary and outlook

In this chapter we have constructed explicit actions for all NS superstring field theories in the small Hilbert space. Since these actions share the same algebraic structure as bosonic string field theory, relaxing the ghost number of the string field automatically gives a solution to the classical BV master equation. This is a small, but significant step towards the goal of providing an explicit computational and conceptual understanding of quantum superstring field theory. The next steps of this program include

- Incorporate the Ramond sector(s) so as to maintain a controlled solution to the classical BV master equation. This is done in chapter 5 at the level of the equations of motion and chapter 7 at the level of the action for open superstrings.
- Quantise the theory. Specifically determine the higher genus corrections to the tree-level action needed to ensure a solution to the quantum BV master equation.
- Understand how the vertices and propagators of classical or quantum superstring field theory provide a single cover of the supermoduli space of type II world sheets, cf. chapter 2.
- Understand how this relates to formulations of superstring field theory in the large Hilbert space, which may ultimately be more fundamental. See for example [51, 56–58, 137, 138].

Progress on these questions will not only help to assess whether superstring field theory can be a useful tool beyond tree level, but may provide valuable insights into the systematics of superstring perturbation theory.

Chapter 5

Ramond equations of motion in superstring field theory

In chapter 3 and 4 we constructed actions for the NS or NS-NS subsectors of all superstring theories by a recursive procedure. The inclusion of R fields is non-trivial because of the presence of an odd gluing parameter along R cylinders which causes complications with the construction of cyclic vertices. However, at the level of the equations of motion the recursive construction can be extended to include the Ramond sectors quite straightforwardly. In this chapter we describe such an extension and discuss the realisation of spacetime supersymmetry in open superstring field theory.

This chapter is based on the paper **Ramond Equations of Motion in Superstring Field Theory** by T. Erler, the author and I. Sachs [54].

5.1 Introduction

The recursive constructions of actions for NS string fields, cf. chapters 3 and 4, have raised the prospect of obtaining a second quantised, field theoretic description of all superstring theories. The next step in this programme is to include the Ramond sectors. As is well-known, formulating kinetic terms for the Ramond sector is complicated by the fact that the string field must carry a definite picture, cf. chapter 2, which for a holomorphic Ramond state is naturally chosen to be $-\frac{1}{2}$. This is not the right picture to form the usual string field theory kinetic term,

$$\frac{1}{2}\langle\Psi, Q\Psi\rangle, \tag{5.1}$$

since in the small Hilbert space the BPZ inner product must act on states whose picture adds up to -2. While there are some proposals for circumventing this problem [59,62,123,139], at this time it is not clear what is the most promising way forward.

Therefore it is worth considering a simpler problem first: namely, constructing classical field equations for all superstring theories, including Ramond sectors. This is the goal of the present paper. With the classical equations of motion, we can

- compute of tree level amplitudes including Ramond asymptotic states around the perturbative vacuum or any classical solution;
- investigate the broken and unbroken supersymmetries of classical solutions representing distinct string backgrounds;
- construct classical solutions in type II closed superstring field theory representing nontrivial Ramond-Ramond backgrounds.

The last point is interesting, since Ramond-Ramond backgrounds are quite difficult to describe in the first quantised RNS formalism. While solving the equations of motion of closed string field theory is a tremendously difficult task, it does not appear to be more difficult for Ramond-Ramond backgrounds than other types of background.

The construction of the equations of motion is an extension of the procedure described in chapter 3. The main new ingredient is incorporating additional labels associated with multiplication of Ramond states. A different formulation of the equations of motion using the large Hilbert space has already been provided for the open superstring in [123] and recently the heterotic string in [126, 140]. Our approach has the advantage of describing type II closed superstrings as well, and, once suitable Ramond kinetic terms are formulated, might be generalised to give a classical Batalin-Vilkovisky action.

5.2 Ramond sector of open superstring

In this section we construct the Neveu-Schwarz and Ramond equations of motion for open superstring field theory using Witten's associative star product. We discuss the more general construction based on a non-associative product in the next section. The equations of motion involve two dynamical fields for the NS and R sectors:

$$\Phi_{\mathrm{N}} \in \mathcal{H}_{\mathrm{N}}, \qquad\qquad \Psi_{\mathrm{R}} \in \mathcal{H}_{\mathrm{R}}, \tag{5.2}$$

where $\mathcal{H}_{\mathrm{N}}$ and $\mathcal{H}_{\mathrm{R}}$ are the NS and R open string state spaces, respectively. Both Φ_{N} and Ψ_{R} are Grassmann odd and carry ghost number 1; the NS field Φ_{N} carries picture -1 while the Ramond field Ψ_{R} carries picture $-\frac{1}{2}$.

For clarity, let us explain why the Ramond string fields are spacetime fermions. According to chapter 2 an open string field has always ghost number 1, no internal quantum numbers and positive GSO-parity. Thus, it is always an anticommuting object. On a Minkowski background there are more conserved quantum numbers. For example, world sheet fermion number per complex fermion is conserved modulo 2. Similarly, picture is conserved modulo integers. The fermion number operators form a Cartan subalgebra for the zero mode algebra of the $\mathfrak{so}(9,1)_1$ AKM-algebra realising spacetime Lorentz invariance. Together with picture number the weight lattice is extended from D_5 to a $D_{5,1}$-lattice, the so-called *covariant lattice* [141]. The crucial point is that requiring that all states have positive GSO-parity ensures that all exchange phases between states in the GSO-projected Hilbert space become real. Moreover, the covariant lattice decomposes into four cosets of the covariant root lattice. Two have odd GSO-parity and are projected out. The GSO-even cosets are shifts of the root lattice by the vector weight and by a spinor weight with positive parity. States from the spinor coset are fermions and Grassmann odd, while states from the vector coset are bosons and Grassmann even and the associated coefficients in the string field are even for the vector coset and odd for the spinorial coset, as expected.

Witten's original proposal for open superstring field theory gives the equations of motion [34]

$$0 = Q\Phi_{\mathrm{N}} + X(i)\Phi_{\mathrm{N}} * \Phi_{\mathrm{N}} + \Psi_{\mathrm{R}} * \Psi_{\mathrm{R}}, \tag{5.3a}$$

$$0 = Q\Psi_{\mathrm{R}} + X(i)(\Psi_{\mathrm{R}} * \Phi_{\mathrm{N}} + \Phi_{\mathrm{N}} * \Psi_{\mathrm{R}}), \tag{5.3b}$$

where Q is the BRST operator, $X(z) = Q \cdot \xi(z)$ is a picture changing operator, and $*$ is the open string star product. Since these equations of motion are singular, we regularise them following chapter 3,

$$0 = Q\Phi_{\mathrm{N}} + M_2(\Phi_{\mathrm{N}}, \Phi_{\mathrm{N}}) + m_2(\Psi_{\mathrm{R}}\Psi_{\mathrm{R}}) + \text{higher orders}, \tag{5.4a}$$

$$0 = Q\Psi_{\mathrm{R}} + M_2(\Psi_{\mathrm{R}}, \Phi_{\mathrm{N}}) + M_2(\Phi_{\mathrm{N}}, \Psi_{\mathrm{R}}) + \text{higher orders}, \tag{5.4b}$$

with higher order terms that we construct in a moment. The product M_2 carries picture $+1$ and takes the form (3.5)

$$M_2(A, B) = \frac{1}{3}\Big(Xm_2(A, B) + m_2(XA, B) + m_2(A, XB)\Big), \tag{5.5a}$$

$$\mu_2(A, B) = \frac{1}{3}\Big(\xi m_2(A, B) - m_2(\xi A, B) - (-1)^{\deg(A)} m_2(A, \xi B)\Big). \tag{5.5b}$$

It satisfies $m_2 = [\eta, \mu_2]$. While in chapter 3 the choice of M_2 in equation (5.5a) was dictated by cyclicity, in the current context we are not attempting to construct an action, so cyclicity is not a requirement.

Cubic order

The higher order terms in the equations of motion are defined by a sequence of degree odd multi-string products,

$$\tilde{M}_1 \equiv Q, \qquad \tilde{M}_2, \qquad \tilde{M}_3, \qquad \tilde{M}_4, \qquad \dots, \tag{5.6}$$

which form an A_∞ algebra. We use the tilde over the products to denote a composite object which appropriately multiplies both NS and R sector states. For example, if

N_1, N_2 are NS sector string fields and R_1, R_2 are R sector string fields, the composite 2-product $\tilde{M}_2$ is defined to satisfy

$$\tilde{M}_2(N_1, N_2) \equiv M_2(N_1, N_2), \qquad \tilde{M}_2(N_1, R_1) \equiv M_2(N_1, R_1), \tag{5.7a}$$
$$\tilde{M}_2(R_1, N_1) \equiv M_2(R_1, N_1), \qquad \tilde{M}_2(R_1, R_2) \equiv m_2(R_1, R_2). \tag{5.7b}$$

Introducing a composite string field

$$\tilde{\Phi} = \Phi_{\rm N} + \Psi_{\rm R} \in \tilde{\mathcal{H}} \equiv \mathcal{H}_{\rm N} \oplus \mathcal{H}_{\rm R} \tag{5.8}$$

the equations of motion up to second order can be expressed

$$0 = Q\tilde{\Phi} + \tilde{M}_2(\tilde{\Phi}, \tilde{\Phi}) + \text{higher orders}. \tag{5.9}$$

Projecting onto picture -1 gives the equation of motion (5.4a) and projecting onto picture $-\frac{1}{2}$ gives the equation of motion (5.4b). Up to cubic order the A_∞ relations are

$$Q^2 = 0, \qquad [Q, \tilde{M}_2] = 0, \qquad [Q, \tilde{M}_3] + \frac{1}{2}[\tilde{M}_2, \tilde{M}_2] = 0.$$

The first two A_∞ relations are already satisfied since Q is nilpotent and a derivation of both m_2 and M_2. We use the third A_∞ relation to determine the composite 3-product $\tilde{M}_3$. First, act the third A_∞ relation on three NS states, or two NS states and one R state. In this case, the commutator $[\tilde{M}_2, \tilde{M}_2]$ reduces to $[M_2, M_2]$, and we can take $\tilde{M}_3 = M_3$, where M_3 is the 3-product (3.16) of the NS open superstring field theory. Therefore

$$\tilde{M}_3(N_1, N_2, N_3) = M_3(N_1, N_2, N_3), \quad \tilde{M}_3(N_1, N_2, R_1) = M_3(N_1, N_2, R_1), \tag{5.10a}$$
$$\tilde{M}_3(N_1, R_1, N_2) = M_3(N_1, R_1, N_2), \quad \tilde{M}_3(R_1, N_1, N_2) = M_3(R_1, N_1, N_2). \tag{5.10b}$$

If there is more than one R input, $\tilde{M}_3$ takes a different form. For example, let us act the third A_∞ relation on three Ramond states:

$$\begin{aligned}\Big([Q, \tilde{M}_3] + \frac{1}{2}[\tilde{M}_2, \tilde{M}_2]\Big)\, R_1 \otimes R_2 \otimes R_3 = &\\ = \Big([Q, \tilde{M}_3] + \tilde{M}_2(\tilde{M}_2 \otimes \mathbb{I} + \mathbb{I} \otimes \tilde{M}_2)\Big) R_1 \otimes R_2 \otimes R_3,&\\ = \Big([Q, \tilde{M}_3] + M_2(m_2 \otimes \mathbb{I} + \mathbb{I} \otimes m_2)\Big) R_1 \otimes R_2 \otimes R_3,&\end{aligned} \tag{5.11}$$

where in the second step we acted $\tilde{M}_2$ on the R states to produce M_2 and m_2. Next we use the fact that M_2 is BRST exact in the large Hilbert space:

$$\begin{aligned}\Big([Q, \tilde{M}_3] + \frac{1}{2}[\tilde{M}_2, \tilde{M}_2]\Big)\, R_1 \otimes R_2 \otimes R_3 = &\\ = \Big[Q, \Big(\tilde{M}_3 + \mu_2(m_2 \otimes \mathbb{I} + \mathbb{I} \otimes m_2)\Big)\Big] R_1 \otimes R_2 \otimes R_3.&\end{aligned} \tag{5.12}$$

Since this must be zero, it is natural to identify

$$\tilde{M}_3(R_1, R_2, R_3) = -\mu_2(m_2 \otimes \mathbb{I} + \mathbb{I} \otimes m_2)R_1 \otimes R_2 \otimes R_3. \tag{5.13}$$

Note that this product is in the small Hilbert space,

$$\eta\tilde{M}_3(R_1, R_2, R_3) = 0, \tag{5.14}$$

since η turns μ_2 into m_2, and the result vanishes by associativity of m_2. Similar considerations determine the remaining 3-products between NS and R states:

$$\tilde{M}_3(N_1, R_1, R_2) = m_2(\mu_2(N_1, R_1), R_2) - (-1)^{\deg(N_1)}\mu_2(N_1, m_2(R_1, R_2)), \tag{5.15a}$$
$$\tilde{M}_3(R_1, N_1, R_2) = m_2(\mu_2(R_1, N_1), R_2) + m_2(R_1, \mu_2(N_1, R_2)), \tag{5.15b}$$
$$\tilde{M}_3(R_1, R_2, N_1) = -\mu_2(m_2(R_1, R_2), N_1) + m_2(R_1, \mu_2(R_2, N_1)), \tag{5.15c}$$
$$\tilde{M}_3(R_1, R_2, R_3) = -\mu_2(m_2(R_1, R_2), R_3) - (-1)^{\deg(R_1)}\mu_2(R_1, m_2(R_2, R_3)). \tag{5.15d}$$

In general, when multiplying n strings there will be 2^n formulae representing all ways that NS and R states can multiply. Determining all these formulae seems like a daunting task, but there is a trick to it which we explain in the next subsection.

Before we get to this, however, it is interesting to consider the product of four Ramond states, $\tilde{M}_4(R_1, R_2, R_3, R_4)$. Since this product would contribute to the NS part of the equations of motion (5.4a), its ghost number must be -2 and its picture number must be $+1$. In fact, this is the first product where the ghost number is more negative than the picture number is positive. It is easy to see that any product built from composing Q, m_2 and ξ must satisfy

$$\text{ghost number} + \text{picture number} \geq 0.$$

This inequality is necessarily violated for products of four or more Ramond states. Therefore such products potentially present an obstruction to our solution of the A_∞ relations as they cannot be built from Q, m_2 and ξ. To see how this problem is avoided, consider the fourth A_∞ relation, $[Q, \tilde{M}_4] + [\tilde{M}_3, \tilde{M}_2] = 0$, acting on four Ramond states:

$$\begin{aligned}
0 &= \Big([Q, \tilde{M}_4] + [\tilde{M}_3, \tilde{M}_2]\Big)R_1 \otimes R_2 \otimes R_3 \otimes R_4, \\
&= \Big([Q, \tilde{M}_4] + \tilde{M}_3(\tilde{M}_2 \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \tilde{M}_2 \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \tilde{M}_2) \\
&\qquad\qquad + \tilde{M}_2(\tilde{M}_3 \otimes \mathbb{I} + \mathbb{I} \otimes \tilde{M}_3)\Big)R_1 \otimes R_2 \otimes R_3 \otimes R_4, \\
&= \Big([Q, \tilde{M}_4] + \tilde{M}_3(m_2 \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes m_2 \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes m_2) \\
&\qquad\qquad + m_2(\tilde{M}_3 \otimes \mathbb{I} + \mathbb{I} \otimes \tilde{M}_3)\Big)R_1 \otimes R_2 \otimes R_3 \otimes R_4.
\end{aligned} \tag{5.16}$$

Keeping careful track of the NS and R inputs of $\tilde{M}_3$, this can be further expanded

$$\begin{aligned}0 = \Big(&[Q,\tilde{M}_4] + m_2(\mu_2\otimes\mathbb{I})(m_2\otimes\mathbb{I}\otimes\mathbb{I}) + \mu_2(m_2\otimes m_2) + m_2(\mu_2\otimes\mathbb{I})(\mathbb{I}\otimes m_2\otimes\mathbb{I})\\
&+ m_2(\mathbb{I}\otimes\mu_2)(\mathbb{I}\otimes m_2\otimes\mathbb{I}) - \mu_2(m_2\otimes m_2) + m_2(\mathbb{I}\otimes\mu_2)(\mathbb{I}\otimes\mathbb{I}\otimes m_2)\\
&- m_2(\mu_2\otimes\mathbb{I})(m_2\otimes\mathbb{I}\otimes\mathbb{I}) - m_2(\mu_2\otimes\mathbb{I})(\mathbb{I}\otimes m_2\otimes\mathbb{I})\\
&- m_2(\mathbb{I}\otimes\mu_2)(m_2\otimes\mathbb{I}\otimes\mathbb{I}) - m_2(\mathbb{I}\otimes\mu_2)(\mathbb{I}\otimes m_2\otimes\mathbb{I})\Big)R_1\otimes R_2\otimes R_3\otimes R_4,\\
= [Q,&\tilde{M}_4]R_1\otimes R_2\otimes R_3\otimes R_4. \end{aligned} \tag{5.17}$$

Therefore we can simply choose

$$\tilde{M}_4(R_1,R_2,R_3,R_4) = 0. \tag{5.18}$$

More generally, we claim that all products with four or more Ramond states can be set to zero. Therefore the equations of motion will be cubic in the Ramond string field.

At first this seems somewhat strange. If the equations of motion have terms which are cubic in the Ramond string field, cyclicity would naturally imply that they should have terms that are quartic in the Ramond string field as well. This is a clear indication that the equations of motion cannot be derived from an action. While this was expected, one might still worry that quartic Ramond terms in the equations of motion are needed to get the correct physics. For example, the quadratic Ramond term $m_2(\Psi_\mathrm{R},\Psi_\mathrm{R})$ is not implied by A_∞ relations or gauge invariance, but is required to incorporate the backreaction of the R field on the NS field. The difference at quartic order is that there is no 4-product of Ramond states at the relevant ghost and picture number which is nontrivial in the small Hilbert space BRST cohomology. Therefore, any quartic term in the Ramond string field can be removed by field redefinition. As a cross check on our equations of motion, as a result of chapter 6 we obtain the correct tree level amplitudes.

All orders

A key ingredient in constructing the equations of motion at higher order is to realise that multi-string products can be characterised according to their *Ramond number*. The Ramond number of a product is defined to be the number of Ramond inputs minus the number of Ramond outputs required for the product to be nonzero:

$$\text{Ramond number} = \#(\text{Ramond inputs}) - \#(\text{Ramond outputs})$$

Generally, products do not have well-defined Ramond number. A product of Ramond number N has the specific property that it will be nonzero only when multiplying N or $N+1$ Ramond states (together with possibly other NS states), in which case it respectively produces an NS or R state. When Ramond number is defined, we indicate it by a vertical slash followed by an extra index attached to the product. For example $b_n|_N$ denotes an n-product with Ramond number N. A product can be non-zero only if

$$-1 \leq N \leq n, \tag{5.19}$$

since the number of Ramond inputs cannot exceed the total number of inputs and the number of Ramond outputs cannot exceed one. While generically multi-string products do not possess well-defined Ramond number, it is clear that they can always be decomposed into a sum of products which do,

$$b_n = \sum_{N=-1}^{n} b_n|_N. \tag{5.20}$$

A comment about notation: Generally, we use $b_n|_N$ to denote an n-string product of Ramond number N, but this does not necessarily mean that $b_n|_N$ is derived from a product b_n after projection to Ramond number N. When we do mean this, it should be clear from context. Consider a 1-string product $\mathrm{R}_1 = P_\mathrm{R}$, where P_R denotes the projector onto R states. A product has definite Ramond number N if and only if it satisfies

$$[\mathbf{b}_n|_N, \mathbf{R}_1] = N\,\mathbf{b}_n|_N. \tag{5.21}$$

Using the Jacobi identity, this implies that Ramond number is additive when taking commutators of products:

$$[\mathbf{b}_m|_M, \mathbf{c}_n|_N]\,|_{M+N} = [\mathbf{b}_m|_M, \mathbf{c}_n|_N]\,. \tag{5.22}$$

Finally, let us mention that the products in the equations of motion always carry even Ramond number, since picture changing operators do not mix NS and R sector states. Products of odd Ramond number play a role once we consider supersymmetry in section 5.6.

Now let us revisit the results of the previous subsection. The BRST operator has Ramond number zero:

$$Q|_0 = Q. \tag{5.23}$$

The composite 2-product $\tilde{M}_2$ can be written as the sum of products at Ramond number zero and two. Comparing with equations (5.7), we can apparently write

$$\tilde{M}_2 = M_2|_0 + m_2|_2. \tag{5.24}$$

Note that $M_2|_0$ and $m_2|_0$ can be derived from the Ramond number zero projection of the gauge 2-product μ_2:

$$M_2|_0 = [Q, \mu_2|_0]\,, \tag{5.25a}$$
$$m_2|_0 = [\eta, \mu_2|_0]\,. \tag{5.25b}$$

The composite 3-product $\tilde{M}_3$ can likewise be written as the sum of products at Ramond number zero and two:

$$\tilde{M}_3 = M_3|_0 + m'_3|_2. \tag{5.26}$$

The Ramond number zero piece corresponds to equations (5.10). The Ramond number two piece $m'_3|_2$ is seemingly more complicated, as it must produce four distinct expressions (5.10) depending on how it multiplies two or three Ramond states. To derive the 3-string products, consider the third A_∞ relation:

$$\begin{aligned} 0 &= [Q, \tilde{M}_3] + \frac{1}{2}[\tilde{M}_2, \tilde{M}_2], \\ &= [Q, M_3|_0] + [Q, m'_3|_2] + \frac{1}{2}\left[M_2|_0, M_2|_0\right] + \left[M_2|_0, m_2|_2\right] + \frac{1}{2}\left[m_2|_2, m_2|_2\right], \\ &= [Q, M_3|_0] + [Q, m'_3|_2] + \frac{1}{2}\left[M_2|_0, M_2|_0\right] + \left[M_2|_0, m_2|_2\right]. \end{aligned} \tag{5.27}$$

This is equivalent to two independent equations at Ramond number 0 and 2:

$$0 = [Q, M_3|_0] + \frac{1}{2}\left[M_2|_0, M_2|_0\right], \tag{5.28a}$$

$$0 = [Q, m'_3|_2] + \left[M_2|_0, m_2|_2\right]. \tag{5.28b}$$

The first equation can be solved as in chapter 3 with the gauge 3-product $\mu_3|_0$ and the bare 3-product $m_3|_0$,

$$M_3|_0 = \frac{1}{2}\Big([Q, \mu_3|_0] + [M_2|_0, \mu_2|_0] \Big), \tag{5.29a}$$

$$\mu_3|_0 = \frac{1}{4}\Big(\xi m_3|_0 + m_3|_0(\xi \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \xi \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \xi)\Big), \tag{5.29b}$$

$$m_3|_0 = \left[m_2|_0, \mu_2|_0\right]. \tag{5.29c}$$

With these definitions one can show that $M_3|_0$ preserves the small Hilbert space. To multiply more than one Ramond state we need $m'_3|_2$. By inspection of the Ramond number two component of the A_∞ relation, we can instantly guess the solution

$$m'_3|_2 = \left[m_2|_2, \mu_2|_0\right]. \tag{5.30}$$

This simple formula reproduces all four equations (5.10) for 3-products of two or more Ramond states.

It is not difficult to guess the general form of the products to all orders. We state the answer first and prove it later. The composite $(n+2)$-string product $\tilde{M}_{n+2}$ can be decomposed

$$\tilde{M}_{n+2} = M_{n+2}|_0 + m'_{n+2}|_2. \tag{5.31}$$

As anticipated before, products with four or more Ramond states can be set to zero. In addition we need to introduce supplemental bare products and gauge products. In total we have four kinds of product:

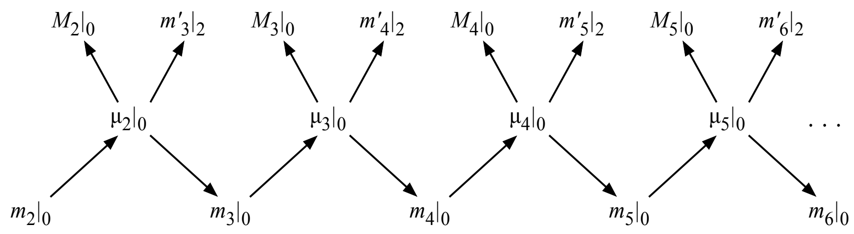


Figure 5.1: Starting from $m_2|_0$ at the lower left corner, this diagram shows the procedure for constructing all products which appear the NS+R equations of motion using intermediate bare products and gauge products.

$$\begin{array}{llllll} \text{gauge products} & \mu_{n+2}|_0: & \text{degree even,} & \text{picture\#} = n+1, & \text{Ramond\#} = 0, \\ \text{products} & \left\{ \begin{array}{l} M_{n+1}|_0: \\ m'_{n+2}|_2: \end{array} \right. & \begin{array}{l} \text{degree odd,} \\ \text{degree odd,} \end{array} & \begin{array}{l} \text{picture\#} = n, \\ \text{picture\#} = n, \end{array} & \begin{array}{l} \text{Ramond\#} = 0 \\ \text{Ramond\#} = 2, \end{array} \\ \text{bare products} & m_{n+2}|_0: & \text{degree odd,} & \text{picture\#} = n, & \text{Ramond\#} = 0, \end{array}$$

which are determined recursively by the equations:

$$\mu_{n+2}|_0 = \frac{1}{n+3}\left(\xi m_{n+2}|_0 - \sum_{k=0}^{n+1} m_{n+2}|_0(\mathbb{I}^{\otimes k} \otimes \xi \otimes \mathbb{I}^{\otimes n+1-k})\right), \tag{5.32a}$$

$$M_{n+2}|_0 = \frac{1}{n+1}\sum_{k=0}^{n} [M_{k+1}|_0, \mu_{n-k+2}|_0], \tag{5.32b}$$

$$m'_{n+3}|_2 = \frac{1}{n+1}\sum_{k=0}^{n} \left[m'_{k+2}|_2, \mu_{n-k+2}|_0\right], \tag{5.32c}$$

$$m_{n+3}|_0 = \frac{1}{n+1}\sum_{k=0}^{n} [m_{k+2}|_0, \mu_{n-k+2}|_0], \tag{5.32d}$$

where

$$M_1|_0 \equiv Q, \qquad m'_2|_2 \equiv m_2|_2.$$

The recursive procedure for constructing the products, gauge products, and bare products is illustrated in figure 5.1. Note that these equations are almost the same as (3.28) determining the NS open superstring field theory. The only major difference is the appearance of a new set of products $m'_{n+2}|_2$ for multiplying 2 or 3 Ramond states. To prove these formulas we introduce generating functions,

$$\mathbf{M}(t) = \sum_{n=0}^{\infty} t^n \mathbf{M}_{n+1}|_0, \tag{5.33a}$$

$$\mathbf{m}'(t) = \sum_{n=0}^{\infty} t^n \mathbf{m}'_{n+2}|_2, \tag{5.33b}$$

$$\mathbf{m}(t) = \sum_{n=0}^{\infty} t^n \mathbf{m}_{n+2}|_0, \tag{5.33c}$$

$$\boldsymbol{\mu}(t) = \sum_{n=0}^{\infty} t^n \boldsymbol{\mu}_{n+2}|_0. \tag{5.33d}$$

Substituting the generating functions and expanding in powers of t, it is straightforward to show that equations (5.32) are equivalent to:

$$\frac{d}{dt}\mathbf{M}(t) = [\mathbf{M}(t), \boldsymbol{\mu}(t)], \tag{5.34a}$$

$$\frac{d}{dt}\mathbf{m}'(t) = [\mathbf{m}'(t), \boldsymbol{\mu}(t)], \tag{5.34b}$$

$$\frac{d}{dt}\mathbf{m}(t) = [\mathbf{m}(t), \boldsymbol{\mu}(t)], \tag{5.34c}$$

$$\boldsymbol{\mu}(t) = \xi \circ \mathbf{m}(t). \tag{5.34d}$$

Let $\mathbf{A}(t)$ or $\mathbf{B}(t)$ stand for $\mathbf{M}(t), \mathbf{m}'(t)$ or $\mathbf{m}(t)$. We have

$$[\mathbf{A}(0), \mathbf{B}(0)] = 0, \tag{5.35}$$

since $\mathbf{Q}, \mathbf{m}_2|_0$ and $\mathbf{m}_2|_2$ mutually anticommute. The differential equations (5.34) imply

$$\frac{d}{dt}[\mathbf{A}(t), \mathbf{B}(t)] = [[\mathbf{A}(t), \mathbf{B}(t)], \boldsymbol{\mu}(t)]. \tag{5.36}$$

Since this equation is homogeneous in $[\mathbf{A}(t), \mathbf{B}(t)]$ and is true at $t = 0$, we conclude

$$[\mathbf{A}(t), \mathbf{B}(t)] = 0. \tag{5.37}$$

In other words, $\mathbf{M}(t), \mathbf{m}'(t)$ and $\mathbf{m}(t)$ are nilpotent and mutually commute. Next note that

$$[\boldsymbol{\eta}, \mathbf{A}(0)] = 0. \tag{5.38}$$

since $\mathbf{Q}, \mathbf{m}_2|_0$ and $\mathbf{m}_2|_2$ are in the small Hilbert space. Equations (5.34) together with equation (5.37) imply

$$\begin{aligned} \frac{d}{dt}[\boldsymbol{\eta}, \mathbf{A}(t)] &= [\boldsymbol{\eta}, [\mathbf{A}(t), \boldsymbol{\mu}(t)]], \\ &= [[\boldsymbol{\eta}, \mathbf{A}(t)], \boldsymbol{\mu}(t)] - [\mathbf{A}(t), \mathbf{m}(t)] + [\mathbf{A}(t), \xi \circ [\boldsymbol{\eta}, \mathbf{m}(t)]] \\ &= [[\boldsymbol{\eta}, \mathbf{A}(t)], \boldsymbol{\mu}(t)] + [\mathbf{A}(t), \xi \circ [\boldsymbol{\eta}, \mathbf{m}(t)]]. \end{aligned} \tag{5.39}$$

Suppose $\mathbf{A}(t) = \mathbf{m}(t)$. Then this equation is homogeneous in $[\boldsymbol{\eta}, \mathbf{m}(t)]$, and since this vanishes at $t = 0$ we conclude $[\boldsymbol{\eta}, \mathbf{m}(t)] = 0$. Therefore

$$\frac{d}{dt}[\boldsymbol{\eta}, \mathbf{A}(t)] = [[\boldsymbol{\eta}, \mathbf{A}(t)], \boldsymbol{\mu}(t)]. \tag{5.40}$$

Since this equation is homogeneous in $[\boldsymbol{\eta}, \mathbf{A}(t)]$ and is true at $t = 0$, we conclude

$$[\boldsymbol{\eta}, \mathbf{A}(t)] = 0. \tag{5.41}$$

In other words, all products and bare products are in the small Hilbert space. Finally, consider the coderivation representing the composite products in the equations of motion

$$\tilde{\mathbf{M}} \equiv \sum_{n=0}^{\infty} \tilde{\mathbf{M}}_{n+1} = \mathbf{M}(1) + \mathbf{m}'(1). \tag{5.42}$$

The above results immediately imply that

$$[\boldsymbol{\eta}, \tilde{\mathbf{M}}] = 0, \qquad\qquad [\tilde{\mathbf{M}}, \tilde{\mathbf{M}}] = 0.$$

The first equation says that the composite products preserve the small Hilbert space, and the second equation says that they satisfy A_∞ relations. This completes the construction of the Neveu-Schwarz and Ramond equations of motion for the open superstring based on Witten's open string star product.

5.3 Ramond sector of open superstring with stubs

In preparation for studying the closed superstring, in this section we provide a more general construction of the open superstring equations of motion which does not require the associativity of Witten's open string star product. Specifically, we build the equations of motion by inserting picture changing operators into a set of elementary products at picture zero:

$$M_1^{(0)} \equiv Q, \qquad M_2^{(0)}, \qquad M_3^{(0)}, \qquad M_4^{(0)}, \qquad \dots . \tag{5.43}$$

We assume that these products have odd degree, live in the small Hilbert space, and satisfy A_∞ relations. For example, we could define the elementary 2-string product $M_2^{(0)}$ by attaching stubs to Witten's open string star product. In the following we need to introduce a multitude of products with different picture and Ramond numbers. The notation $M_{n+1}^{(p)}|_{2r}$ means that the $n+1$-product has total picture p and Ramond number $2r$.

The goal is to construct the NS+R equations of motion,

$$0 = Q\tilde{\Phi} + \tilde{M}_2(\tilde{\Phi}, \tilde{\Phi}) + \tilde{M}_3(\tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi}) + \text{higher order}, \tag{5.44}$$

where $\tilde{\Phi} = \Phi_\mathrm{N} + \Psi_\mathrm{R}$ and $\tilde{M}_{n+1}$ are degree odd composite products which appropriately multiply NS and R states. We require that the composite products preserve the small Hilbert space and satisfy A_∞ relations. The composite products can be decomposed into a sum of products of definite Ramond and picture number,

$$\tilde{M}_{n+1} = M_{n+1}^{(n)}|_0 + M_{n+1}^{(n-1)}|_2 + M_{n+1}^{(n-2)}|_4 + \dots, \tag{5.45}$$

with the sum terminating when the Ramond number exceeds the number of inputs. It is now convenient to count the number of missing picture changing operators in each product instead of its total picture. We modify the definition of the *picture deficit* from section 4.2 in the presence of Ramond states. We say that the product $M_{n+1}^{(p)}|_{2r}$ has picture deficit $d = n - r - p$ and indicate the picture deficit instead of the total picture as $M_{n+1}^{[d]}|_{2r}$. The advantage of the picture deficit is that the maps $\tilde{M}_{n+1}$ from equation (5.45) have homogeneous picture deficit 0. We introduce a list of products and gauge products as follows:

$$\begin{aligned} &\text{products}: && M_{N+1}^{[d]}|_{2r}, && \text{degree odd,} \\ &\text{gauge products}: && \mu_{N+2}^{[d]}|_{2r}, && \text{degree even,} \end{aligned} \tag{5.46}$$

where the integers N, d, r take values in the ranges

$$N \geq 0, \qquad 0 \leq r \leq N, \qquad 0 \leq d \leq N. \tag{5.47}$$

We introduce generating functions $\mathbf{M}(s,t,u)$ and $\boldsymbol{\mu}(s,t,u)$, where the formal variable s counts picture deficit and u counts Ramond number,

$$\mathbf{M}(s,t,u)_N = \sum u^r s^d\, \mathbf{M}_N^{[d]}|_{2r}(t) \tag{5.48a}$$

$$\boldsymbol{\mu}(s,t,u)_N = \sum u^r s^d\, \boldsymbol{\mu}_N^{[d]}|_{2r}(t), \tag{5.48b}$$

where we set $\mathbf{M}_N^{[N-1-r]}|_{2r}(0)$ to the elementary products with no picture changing operators and set all other $\mathbf{M}_N^{[d]}|_{2r}(0) = 0$. The remaining products are determined by a pair of equations analogous to equation (4.32),

$$\frac{\partial}{\partial t}\mathbf{M}(s,t,u) = [\mathbf{M}(s,t,u), \boldsymbol{\mu}(s,t,u)], \tag{5.49a}$$

$$\boldsymbol{\mu}(s,t,u) = \xi \circ \frac{\partial}{\partial s}\mathbf{M}(s,t,u). \tag{5.49b}$$

Since the elementary products at picture zero are assumed to be in the small Hilbert space and form an A_∞ algebra, we have

$$[\mathbf{M}(s,0,u), \mathbf{M}(s,0,u)] = 0, \qquad [\boldsymbol{\eta}, \mathbf{M}(s,0,u)] = 0. \tag{5.50}$$

From the general discussion in section 4.5 we conclude then

$$[\boldsymbol{\eta}, \mathbf{M}(s,t,u)] = 0, \tag{5.51}$$

so that all products are in the small Hilbert space. Finally, consider the coderivation for the composite products which appear in the equations of motion:

$$\tilde{\mathbf{M}} = \sum_{n=0}^{\infty} \tilde{\mathbf{M}}_{n+1} = \mathbf{M}(0,1,1). \tag{5.52}$$

The above results imply that

$$[\boldsymbol{\eta}, \tilde{\mathbf{M}}] = 0, \qquad [\tilde{\mathbf{M}}, \tilde{\mathbf{M}}] = 0, \tag{5.53}$$

so the composite products are in the small Hilbert space and satisfy A_∞ relations. For completeness we give the solution for $\mathbf{M}(s, 1, u)$,

$$\mu_N^{[d]} = \frac{d+1}{N+1}\left(\xi M_N^{[d+1]} - M_N^{[d+1]} \sum_{k=0}^{N-1} \mathbb{I}^{\otimes k} \otimes \xi \otimes \mathbb{I}^{N-k-1}\right), \tag{5.54a}$$

$$M_N^{[d]}|_{2r} = \frac{1}{N-d-r-1} \sum_{\substack{k'+k''=N+1\\ d'+d''=d\\ r'+r''=r}} \left[M_{k'}^{[d']}|_{2r'}, \mu_{k''}^{[d'']}|_{2r''}\right]. \tag{5.54b}$$

This completes the construction of the Neveu-Schwarz and Ramond equations of motion for the open superstring with stubs.

5.4 Ramond sector of heterotic string

The generalisation of the open superstring with stubs to the heterotic string is straightforward. We replace the A_∞ structure with an L_∞ structure and require the level matching constraints (2.62). The string fields are $\Phi_{\rm N} \in \mathcal{H}_{\rm N}$ and $\Psi_{\rm R} \in \mathcal{H}_{\rm R}$, where $\mathcal{H}_{\rm N}$ and $\mathcal{H}_{\rm R}$ are the Neveu-Schwarz and Ramond state spaces of a heterotic string subject to the level matching constraints and GSO projections. For the heterotic string, superconformal ghosts and picture only inhabit the left-moving sector. The NS field $\Phi_{\rm N}$ has ghost number 2 and picture number -1 and the Ramond field $\Psi_{\rm R}$ has ghost number 2 and picture number $-\frac{1}{2}$.

$$\tilde{\Phi} = \Phi_{\rm N} + \Psi_{\rm R} \in \tilde{\mathcal{H}} = \mathcal{H}_{\rm N} \oplus \mathcal{H}_{\rm R}. \tag{5.55}$$

The NS+R equations of motion of the heterotic string take the form

$$0 = Q\tilde{\Phi} + \tilde{L}_2(\tilde{\Phi}, \tilde{\Phi}) + \tilde{L}_3(\tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi}) + \text{higher orders}, \tag{5.56}$$

where $\tilde{L}_1 \equiv Q$ and $\tilde{L}_{n+1}, n = 0, 1, 2, ...$ are composite closed string products with appropriately multiply NS and R states. We require that they are compatible with the level matching constraints, i.e. $b_0^- \tilde{L}_{n+1} = 0, L_0^- \tilde{L}_{n+1} = 0$, that they satisfy L_∞ relation and that they preserve the small Hilbert space. The composite products can be expanded as a sum of products of definite Ramond and picture number

$$\tilde{L}_{n+1} = L_{n+1}^{(n)}|_0 + L_{n+1}^{(n-1)}|_2 + L_{n+1}^{(n-2)}|_4 + \dots. \tag{5.57}$$

Ramond number and picture number are such that the NS part of the equations of motion has picture -1 and the R part of the equations of motion has picture $-\frac{1}{2}$. We want to build the products by inserting picture changing operators on a set of elementary products of odd degree at picture zero:

$$L_1^{(0)} \equiv Q, \qquad L_2^{(0)}, \qquad L_3^{(0)}, \qquad L_4^{(0)}, \qquad \dots, \tag{5.58}$$

which are compatible with the level matching constraints and preserve the small Hilbert space. Moreover, we require that $L^{(0)}_k$ define an L_∞ algebra. The most natural definition of the elementary products would derive from the polyhedral vertices of Saadi and Zwiebach [13], but for our purposes it will not matter how they are chosen.

The procedure for constructing the products is exactly like for the open string with stubs. We introduce a list of products and gauge products:

$$\begin{aligned} &\text{products}: && L^{[d]}_{d+p+r+1}|_{2r}, && \text{degree odd}, \\ &\text{gauge products}: && \lambda^{[d]}_{d+p+r+2}|_{2r}, && \text{degree even}, \end{aligned} \tag{5.59}$$

for $d, p, r \geq 0$. They are defined recursively following equation (5.54) using the equations

$$\lambda^{[d]}_N = \frac{d+1}{N+1}\left(\xi_0 L^{[d+1]}_N - L^{[d+1]}_N(\xi_0 \wedge \mathbb{I}^{\wedge N-1})\right), \tag{5.60a}$$

$$L^{[d]}_N|_{2r} = \frac{1}{N-d-r-1}\sum_{\substack{k'+k''=N+1\\ d'+d''=d\\ r'+r''=r}}\left[L^{[d']}_{k'}|_{2r'}, \lambda^{[d'']}_{k''}|_{2r''}\right]. \tag{5.60b}$$

Note that the first equation uses the ξ zero mode rather than an arbitrary operator built from ξ as is possible for the open string. This guarantees that all products generated in the recursion are compatible with the level matching constraints. The proof that the resulting composite products $\tilde{L}_{n+1}$ are in the small Hilbert space and satisfy L_∞ relations is identical to that of the previous section.

5.5 Ramond sectors of type II closed superstring

For type II closed superstrings, superconformal ghosts appear in both the holomorphic and antiholomorphic sectors. Therefore, string fields and closed string products will have two respective picture numbers, the left- and right-moving picture numbers. To formulate the equations of motion we need four dynamical closed string fields in the small Hilbert space, with respective left-/right-moving pictures:

$$\begin{aligned} &\Phi_{\mathrm{NN}} \in \mathcal{H}_{\mathrm{NN}} \text{ at picture } (-1,-1), && \Psi_{\mathrm{NR}} \in \mathcal{H}_{\mathrm{NR}} \text{ at picture } \left(-1,-\frac{1}{2}\right), \\ &\Psi_{\mathrm{RN}} \in \mathcal{H}_{\mathrm{RN}} \text{ at picture } \left(-\frac{1}{2},-1\right), && \Phi_{\mathrm{RR}} \in \mathcal{H}_{\mathrm{RR}} \text{ at picture } \left(-\frac{1}{2},-\frac{1}{2}\right). \end{aligned}$$

All four string fields have ghost number 2, and satisfy the level matching constraints. The NS-NS and R-R string fields describe spacetime bosons and have commuting coefficient fields. The NS-R and R-NS string fields describe spacetime fermions and their coefficient functions are anticommuting fields. The GSO projections are the same as for type IIA or type IIB superstring theory, i.e. the GSO even states have the same spacetime chirality for type IIB and the opposite chirality for type IIA.

To construct the equations of motion we insert picture changing operators on a set of elementary products of odd degree at left- and right-moving picture zero:

$$L_1^{(0,0)} \equiv Q, \qquad L_2^{(0,0)}, \qquad L_3^{(0,0)}, \qquad L_4^{(0,0)}, \qquad \ldots . \tag{5.61}$$

We assume that these products satisfy L_∞ relations and are compatible with the level matching constraints. Moreover they should preserve the small Hilbert space, i.e. they satisfy $[\eta, L_{n+1}^{(0,0)}] = 0$ and $[\overline{\eta}, L_{n+1}^{(0,0)}] = 0$. As in the heterotic string, we can take these products to be defined by the polyhedral closed string vertices of Saadi and Zwiebach. Contrary to the heterotic string we now have the notion of left- and right-moving Ramond number, which are defined analogously to the open superstring. The construction of the equations of motion requires the definition of many products at intermediate Ramond and picture numbers. We use the notation $L_{N+1}^{(p,\bar{p})}|_{2r,2\bar{r}}$ to indicate that the $N+1$-product has pictures $(p,\bar{p})$ and Ramond numbers $(2r, 2\bar{r})$. Let us introduce the composite string field

$$\tilde{\Phi} = \Phi_{\mathrm{NN}} + \Psi_{\mathrm{NR}} + \Psi_{\mathrm{RN}} + \Phi_{\mathrm{RR}} \in \tilde{\mathcal{H}}. \tag{5.62}$$

We can write the equations of motion in the form

$$0 = Q\tilde{\Phi} + \tilde{L}_2(\tilde{\Phi}, \tilde{\Phi}) + \tilde{L}_3(\tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi}) + \text{higher orders}, \tag{5.63}$$

where $\tilde{L}_{n+1}$ are composite products of odd degree which appropriately multiply the four sectors of states. We require that the composite products satisfy L_∞ relations and are in the small Hilbert space. The composite products can be decomposed into a sum of products with definite left- and right-moving Ramond and picture numbers:

$$\begin{aligned}
\tilde{L}_{n+1} = {} & L_{n+1}^{(n,n)}|_{0,0} + L_{n+1}^{(n,n-1)}|_{0,2} + L_{n+1}^{(n,n-2)}|_{0,4} + \ldots \\
& + L_{n+1}^{(n-1,n)}|_{2,0} + L_{n+1}^{(n-1,n-1)}|_{2,2} + \ldots \\
& + L_{n+1}^{(n-2,n)}|_{4,0} + \ldots \\
& + \ldots .
\end{aligned}$$

It is important that for fixed n the number of terms in the expansion is finite, because increasing the Ramond number by 2 requires one less unit in picture and the total picture is always non-negative. For example we have at lowest orders,

$$\begin{aligned}
\tilde{L}_1 = Q, \qquad & \tilde{L}_2 = L_2^{(1,1)}|_{0,0} + L_2^{(1,0)}|_{0,2} + L_2^{(0,1)}|_{2,0} + L_2^{(0,0)}|_{2,2}, \\
& \tilde{L}_3 = L_3^{(2,2)}|_{0,0} + L_3^{(2,1)}|_{0,2} + L_3^{(1,2)}|_{2,0} + L_3^{(1,1)}|_{2,2}.
\end{aligned}$$

The products with no picture insertions have to agree with the elementary products. The remaining products are constructed recursively with the recursion relation being derived from a set of differential equations for a generating function. To count Ramond number, we introduce formal variables u and $\bar{u}$ with left-moving resp. right-moving Ramond number -2. In addition, we have formal variables s and $\bar{s}$ counting the left-moving resp. right-moving picture deficit. Picture deficit is equal to the number of insertions of picture changing operators needed so that the output has either picture -1 or picture $-\frac{1}{2}$. We define a coderivation from the elementary products, but include the needed powers of the formal variables,

$$\mathbf{L} = \sum_{n,r,\bar{r}\geq 0} u^r \bar{u}^{\bar{r}} s^{n-r} \bar{s}^{n-\bar{r}} \mathbf{L}^{(0,0)}_{n+1}|_{2r,2\bar{r}}. \tag{5.64}$$

Picture deficit and Ramond number are additive under composition and since the elementary products satisfy the L_∞-relations, a straightforward calculation shows that $[\mathbf{L}, \mathbf{L}] = 0$. Similarly, it holds that $[\boldsymbol{\eta}, \mathbf{L}] = [\overline{\boldsymbol{\eta}}, \mathbf{L}] = 0$. Based on the experience with closed NS-NS type II strings, we construct the higher order terms in the equations of motion through the L_∞-structure that solves this set of differential equations with initial conditions given by $\mathbf{L}$,

$$\frac{\partial}{\partial t}\mathbf{L} = [\mathbf{L}, \boldsymbol{\lambda} + \overline{\boldsymbol{\lambda}}], \tag{5.65a}$$

$$\frac{\partial}{\partial s}\mathbf{L} = [\boldsymbol{\eta}, \boldsymbol{\lambda}], \qquad [\overline{\boldsymbol{\eta}}, \boldsymbol{\lambda}] = 0 \tag{5.65b}$$

$$\frac{\partial}{\partial \bar{s}}\mathbf{L} = [\overline{\boldsymbol{\eta}}, \overline{\boldsymbol{\lambda}}], \qquad [\boldsymbol{\eta}, \overline{\boldsymbol{\lambda}}] = 0. \tag{5.65c}$$

Eliminating $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ requires a choice of contracting homotopies for the operators $[\boldsymbol{\eta}, \cdot]$ and $[\overline{\boldsymbol{\eta}}, \cdot]$. We make a particular choice for $\xi_0\circ$ and $\bar{\xi}_0\circ$ based on the zero modes ξ_0 and $\bar{\xi}_0$. For example, the operation $\xi_0\circ$ acts on coderivations and is defined by its action on an n-closed string product

$$\xi_0 \circ b_n = \frac{1}{n+1}\Big(\xi_0 b_n + (-1)^{\deg(b_n)} b_n(\xi_0 \wedge \mathbb{I}_{n-1})\Big). \tag{5.66}$$

With a similar definition for $\bar{\xi}_0\circ$. With these choices the central equations (5.65) can be rewritten as

$$\frac{\partial}{\partial t}\mathbf{L} = [\mathbf{L}, \boldsymbol{\lambda} + \overline{\boldsymbol{\lambda}}], \tag{5.67a}$$

$$\boldsymbol{\lambda} = \xi_0 \circ \frac{\partial}{\partial s}\mathbf{L}, \qquad \overline{\boldsymbol{\lambda}} = \bar{\xi}_0 \circ \frac{\partial}{\partial \bar{s}}\mathbf{L}. \tag{5.67b}$$

Invoking the results of section 4.5, it follows immediately that $\tilde{\mathbf{L}} = \mathbf{L}|_{t=1,s=\bar{s}=0,u=\bar{u}=1}$ contains only products with the correct number of picture changing operators, satisfies the L_∞ relations and preserves the small Hilbert space and the level matching conditions. Thus $\tilde{\mathbf{L}}$ gives a consistent set of equations of motion. It is very instructive to solve equation (5.67) explicitly in terms of the initial data to obtain the recursion formulas. Explicitly we have:

$$\lambda^{(p+1,\overline{p})}_{N+2}|_{2r,2\overline{r}} = \frac{N-p-r+1}{N+3}\Big(\xi_0 L^{(p,\overline{p})}_{N+2}|_{2r,2\overline{r}} - L^{(p,\overline{p})}_{N+2}|_{2r,2\overline{r}}(\xi_0 \wedge \mathbb{I}_{N+1})\Big), \tag{5.68a}$$

$$\overline{\lambda}^{(p,\overline{p}+1)}_{N+2}|_{2r,2\overline{r}} = \frac{N-\overline{p}-\overline{r}+1}{N+3}\Big(\overline{\xi}_0 L^{(p,\overline{p})}_{N+2}|_{2r,2\overline{r}} - L^{(p,\overline{p})}_{N+2}|_{2r,2\overline{r}}(\overline{\xi}_0 \wedge \mathbb{I}_{N+1})\Big), \tag{5.68b}$$

$$L^{(p,\overline{p})}_{N+2}|_{2r,2\overline{r}} = \frac{1}{p+\overline{p}}\left(\sum_{n=0}^{N}\sum_{(p',r')\in A_n}\sum_{(\overline{p}',\overline{r}')\in\overline{A}_n}[L^{(p',\overline{p}')}_{n+1}|_{2r',2\overline{r}'}, \boldsymbol{\lambda}^{(p-p',\overline{p}-\overline{p}')}_{N-n+2}|_{2(r-r'),2(\overline{r}-\overline{r}')}] \right.$$
$$\left. + \sum_{n=0}^{N}\sum_{(p',r')\in B_n}\sum_{(\overline{p}',\overline{r}')\in\overline{B}_n}[L^{(p',\overline{p}')}_{n+1}|_{2r',2\overline{r}'}, \overline{\boldsymbol{\lambda}}^{(p-p',\overline{p}-\overline{p}')}_{N-n+2}|_{2(r-r'),2(\overline{r}-\overline{r}')}]\right). \tag{5.68c}$$

The sums in the last equation are over all values of p', r' and $\overline{p}', \overline{r}'$ such that the products and left/right gauge products in the commutators have admissible left- and right-moving Ramond and picture numbers. Explicitly, (p', r') is in a set A_n or B_n and $(\overline{p}', \overline{r}')$ is in a set $\overline{A}_n$ or $\overline{B}_n$ defined by the conditions:

$$\begin{aligned}
(p',r')\in A_n, \quad &\leftrightarrow \quad 0\le p',\quad 0\le r',\quad p'\le p-1, \quad r'\le r,\\
&\qquad p+r-N+n-1\le p'+r', \quad p'+r'\le N,\\
(\overline{p}',\overline{r}')\in \overline{A}_n, \quad &\leftrightarrow \quad 0\le \overline{p}',\quad 0\le \overline{r}',\quad \overline{p}'\le \overline{p}, \quad \overline{r}'\le \overline{r},\\
&\qquad \overline{p}+\overline{r}-N+n\le \overline{p}'+\overline{r}', \quad \overline{p}'+\overline{r}'\le N,\\
(p',r')\in B_n, \quad &\leftrightarrow \quad 0\le p',\quad 0\le r',\quad p'\le p, \quad r'\le r,\\
&\qquad p+r-N+n\le p'+r', \quad p'+r'\le N,\\
(\overline{p}',\overline{r}')\in \overline{B}_n, \quad &\leftrightarrow \quad 0\le \overline{p}',\quad 0\le \overline{r}',\quad \overline{p}'\le \overline{p}-1, \quad \overline{r}'\le \overline{r},\\
&\qquad \overline{p}+\overline{r}-N+n-1\le \overline{p}'+\overline{r}', \quad \overline{p}'+\overline{r}'\le N.
\end{aligned}$$

This completes the definition of the equations of motion of type II closed superstring field theory. Let us finally mention that it is possible to generalise equation (5.65) by choosing a different contracting homotopy instead of $\xi\circ$ and $\bar{\xi}\circ$ or by modifying the linear combination $\boldsymbol{\lambda} + \overline{\boldsymbol{\lambda}}$ to other, possibly t-dependent combinations. This is the same ambiguity as discussed in section 4.5 and results in theories that are related by proper field redefinitions.

5.6 Supersymmetry

Spacetime supersymmetry is necessary for the consistency of perturbative superstring theory and can thus be regarded as a very remarkable prediction of superstring theory [142]. Albeit this does not imply that supersymmetry is unbroken at low energy scales. After having discussed equations of motion in this chapter, it is quite natural to ask how spacetime supersymmetry is realised in these proposals. For closed superstring theories local supersymmetry transformations are part of the gauge symmetry of the theory. The generators correspond to longitudinal gravitinos and this is a local symmetry as the gravitinos may have a momentum dependence. Taking the soft limit, the symmetry becomes a global symmetry and the gauge-invariance of the S-matrix translates into the Ward identity for the global symmetry. If global supersymmetry is spontaneously broken by the boundary conditions of the vacuum solution at infinity, the Ward identity acquires an anomalous contribution from the change in the asymptotic boundary conditions [61,62,95,142]. Supersymmetry of open superstring theories, on the other hand, is much more subtle, because pure open superstring theories do not contain supergravity. But, since pure open superstring theories violate quantum mechanical unitarity, one could argue that coupling of the open strings to closed strings and therefore also to supergravity, is inevitable. In this case one could construct an *open-closed homotopy algebra (OCHA)* analogously to the construction described in chapters 3, 4 and this chapter and identify the open superstring supersymmetry as the influence of the bulk gauge symmetry on the boundary fields via the OCHA. In this section we take a more conservative route and construct the supersymmetry transformations for open superstrings based on Witten's star product. It would be interesting to see if these results can be recovered from an appropriate bulk-gauge transformation via an OCHA. For completeness, let us mention some previous work in that direction [34,116,123].

Suppose that we work with open superstring field theory formulated around a BPS D-brane with sixteen supersymmetries. From [35,36] we know that the simplest way to realise supersymmetry transformations at the level of the vertex operators is through the action of the integrated fermionic vertex operator s_1 at picture $-\frac{1}{2}$ acting as a 1-string product,

$$s_1 = \sqrt{2} \oint \frac{dz}{2\pi \mathrm{i}}\, \Theta_a e^{-\phi/2} \epsilon^a. \tag{5.69}$$

The parameter ϵ^a is a spacetime spinor with positive chirality and carries internal quantum numbers such that s_1 is a ghost number 0 string product with no internal quantum numbers. Θ_a denotes the spin field creating twisted vacuum for the world sheet fermions. Positive chirality of ϵ^a ensures that s_1 commutes with world sheet fermion number[1][2]. It is natural to identify the linearised supersymmetry transformation of the NS field as

$$\delta\Phi_{\mathrm{N}} = s_1 \Psi_{\mathrm{R}}. \tag{5.70}$$

Likewise, the linearised supersymmetry transformation of the R field should be proportional to the NS field. However, to get the pictures to line up we need a

[1]The factor $\sqrt{2}$ is inserted to obtain the canonical normalisation of the supersymmetry algebra.

[2]Notice that integrating over the position of an additional R vertex operator is very subtle, because the odd moduli of the world sheet are inherently global and that moving a Ramond puncture without changing the other moduli is not possible [95]. Altough a geometric interpretation may be obscured in this way, the constructions here still make sense algebraically.

supersymmetry operator of picture $+\frac{1}{2}$. This can be defined by integrating the zero momentum fermion vertex at picture $+\frac{1}{2}$:

$$S_1 = \sqrt{2}\oint \frac{dz}{2\pi i}\Big(i\partial X_\mu \Theta_{\dot{a}} e^{\phi/2}(z)\,(\Gamma^\mu)^{\dot{a}}{}_a + b\eta\Theta_a\, e^{3\phi/2}(z)\Big)\epsilon^a, \tag{5.71}$$

where Γ^μ denotes a gamma matrix in 9+1 dimensions. Again, this operator has ghost number 0 and no internal quantum numbers. The linearised supersymmetry transformation of the Ramond field should therefore be

$$\delta\Psi_{\mathrm{R}} = S_1\Phi_{\mathrm{N}}. \tag{5.72}$$

We can write both supersymmetry transformations in a single equation using the composite string field $\tilde{\Phi} = \Phi_{\mathrm{N}} + \Psi_{\mathrm{R}}$

$$\delta\tilde{\Phi} = \tilde{S}_1\tilde{\Phi}, \tag{5.73a}$$

$$\tilde{S}_1 = S_1|_{-1} + s_1|_1, \tag{5.73b}$$

with the indicated Ramond numbers of S_1 and s_1. The operators s_1 and S_1 have some important algebraic properties. For example, both preserve the small Hilbert space and commute with the BRST operator,

$$[\eta, s_1] = [\eta, S_1] = 0, \tag{5.74a}$$

$$[Q, s_1] = [Q, S_1] = 0. \tag{5.74b}$$

This is in fact necessary for the supersymmetry transformation to be a symmetry of the linearised equations of motion. Moreover, since both operators are zero modes of weight one primary fields, they are derivations of Witten's star product:

$$[s_1, m_2] = [S_1, m_2] = 0. \tag{5.75}$$

It is also useful to introduce a supersymmetry operator in the large Hilbert space

$$\sigma_1 = \sqrt{2}\oint \frac{dz}{2\pi i}\,\xi\Theta_a\, e^{-\phi/2}(z)\epsilon^a. \tag{5.76}$$

This operator has ghost number -1 and satisfies

$$S_1 = [Q, \sigma_1], \tag{5.77a}$$

$$s_1 = [\eta, \sigma_1]. \tag{5.77b}$$

and is also a derivation of the star product. Note the relationship between the supersymmetry operators S_1, σ_1 and s_1 is somewhat analogous to the relation between the products M_2, μ_2, and m_2.

5.6.1 Perturbative construction of supersymmetry transformation

In this subsection we describe a perturbative completion of the supersymmetry transformation $\tilde{S}_1$. It turns out that the final form of the supersymmetry transformation is easiest to understand in a different set of field variables described in the next subsection. The following derivation, however, has the advantage that it likely generalises to supersymmetry transformations in other forms of superstring field theory.

A symmetry of the equations of motion is a field redefinition that preserves its space of solutions. Infinitesimal field redefinitions are described by degree 0 coderivations $\tilde{\mathbf{S}}$. They preserve the equations of motion encoded in the degree 1 coderivation $\tilde{\mathbf{M}}$ provided

$$[\tilde{\mathbf{M}}, \tilde{\mathbf{S}}] = 0. \tag{5.78}$$

Moreover, we must require that $[\boldsymbol{\eta}, \tilde{\mathbf{S}}] = 0$ so that the small Hilbert space is preserved by the field redefinition, cf. section 2.3. Note that the condition $[\tilde{\mathbf{M}}, \tilde{\mathbf{S}}] = 0$ is somewhat stronger than the statement that the transformation maps solutions into solutions. It places a nontrivial condition on the off-shell form of the supersymmetry transformation. In fact, equation (5.78) is the nearest we can come to the statement that the transformation is a symmetry of the action. All that is missing is a symplectic structure which would allow us to define an action and impose cyclicity.

Equation (5.78) implies that the products $\tilde{S}_{n+1}$ satisfy a hierarchy of identities:

$$[\tilde{M}_1, \tilde{S}_{n+1}] + [\tilde{M}_2, \tilde{S}_n] + ... + [\tilde{M}_{n+1}, \tilde{S}_1] = 0, \qquad n = 0, 1, 2, \dots\,, \tag{5.79}$$

where $\tilde{M}_1 = Q$. We have already discussed $\tilde{S}_1$ (5.73b), and it provides a solution to

$$[Q, \tilde{S}_1] = 0. \tag{5.80}$$

The next step is to derive the 2-product $\tilde{S}_2$ from the identity

$$[Q, \tilde{S}_2] + [\tilde{M}_2, \tilde{S}_1] = 0. \tag{5.81}$$

Since $\tilde{S}_2$ is part of a supersymmetry transformation, it can be split into a sum of products with odd Ramond number:

$$\tilde{S}_2 = S_2|_{-1} + s_2|_1. \tag{5.82}$$

Equation (5.81) breaks up into two independent equations at Ramond number -1 and 1:

$$0 = [Q, S_2|_{-1}] + [M_2|_0, S_1|_{-1}]\,, \tag{5.83a}$$

$$0 = [Q, s_2|_1] + [M_2|_0, s_1|_1] + [m_2|_2, S_1|_{-1}]\,. \tag{5.83b}$$

To solve the first equation, we can pull a Q out of either M_2 or S_1. The solution we prefer is to pull a Q out of M_2, obtaining

$$S_2|_{-1} = [S_1|_{-1}, \mu_2|_0]\,. \tag{5.84}$$

We can check that this is in the small Hilbert space:

$$[\eta, S_2|_{-1}] = [S_1|_{-1}, m_2|_0] = [S_1, m_2]|_{-1} = 0, \tag{5.85}$$

where we used conservation of Ramond number and the fact that S_1 is a derivation of m_2. To solve for the Ramond number 1 component of $\tilde{S}_2$, we pull a Q in the natural way out of the two terms in equation (5.83b)

$$s_2|_1 = [s_1|_1, \mu_2|_0] + [m_2|_2, \sigma_1|_{-1}] \,. \tag{5.86}$$

It turns out to be convenient to introduce separate symbols for these two terms:

$$s_2^{(\mathrm{I})}|_1 \equiv [s_1|_1, \mu_2|_0] \,, \tag{5.87}$$

$$s_2^{(\mathrm{II})}|_1 \equiv [m_2|_2, \sigma_1|_{-1}] \,. \tag{5.88}$$

Check that this is in the small Hilbert space:

$$[\eta, s_2|_1] = [s_1|_1, m_2|_0] - [m_2|_2, s_1|_{-1}] = [s_1, m_2]|_1 = 0,$$

where we used conservation of Ramond number and the fact that s_1 is a derivation of m_2. This completes the definition of the 2-product $\tilde{S}_2$ in the supersymmetry transformation.

To establish a pattern it is helpful to continue on to the 3-product $\tilde{S}_3$, which must satisfy

$$[Q, \tilde{S}_3] + [\tilde{M}_2, \tilde{S}_2] + [\tilde{M}_3, \tilde{S}_1] = 0. \tag{5.89}$$

Let us look at the component of this equation at Ramond number 3:

$$\Big[Q, \tilde{S}_3|_3\Big] + [m_2|_2, s_2|_1] + [m_3'|_2, s_1|_1] = 0. \tag{5.90}$$

Substituting equation (5.86) for $s_2|_1$ and equation (5.30) for $m_3'|_2$,

$$\begin{aligned} 0 &= \Big[Q, \tilde{S}_3|_3\Big] + [m_2|_2, [s_1|_1, \mu_2|_0]] + [m_2|_2, [m_2|_2, \sigma_1|_{-1}]] + [[m_2|_2, \mu_2|_0] \,, s_1|_1] \,, \\ &= \Big[Q, \tilde{S}_3|_3\Big] \,, \end{aligned}$$

where the additional terms either cancel or vanish identically because the Ramond number exceeds the number of inputs. From this we conclude that the Ramond number 3 component of $\tilde{S}_3$ can be set to zero. Therefore $\tilde{S}_3$ must be a sum of products at Ramond number -1 and 1

$$\tilde{S}_3 = S_3|_{-1} + s_3|_1, \tag{5.91}$$

just like $\tilde{S}_1$ and $\tilde{S}_2$. Let's look at the Ramond number -1 component of the identity (5.89):

$$\begin{aligned} 0 &= [Q, S_3|_{-1}] + [M_2|_0, S_2|_{-1}] + [M_3|_0, S_1|_{-1}] \,, \\ &= \left[Q, S_3|_{-1} - \frac{1}{2}\Big([S_1|_{-1}, \mu_3|_0] + [S_2|_{-1}, \mu_2|_0] \Big)\right] . \end{aligned}$$

This suggests we identify

$$S_3|_{-1} = \frac{1}{2}\Big([S_1|_{-1}, \mu_3|_0] + [S_2|_{-1}, \mu_2|_0] \Big). \tag{5.92}$$

In is then straightforward to check that $S_3|_{-1}$ preserves the small Hilbert space. Finally, let's look at the Ramond number 1 component of (5.89):

$$\begin{aligned} 0 &= [Q, s_3|_1] + [M_2|_0, s_2|_1] + [m_2|_2, S_2|_{-1}] + [M_3|_0, s_1|_1] + [m_3'|_2, S_1|_{-1}]\,, \\ &= \Big[Q, s_3|_1 - \frac{1}{2}\Big([s_1|_1, \mu_3|_0] + \big[s_2^{(\mathrm{I})}|_1, \mu_2|_0\big] \Big) - \big[s_2^{(\mathrm{II})}|_1, \mu_2|_0\big]\Big]\,. \end{aligned}$$

Therefore we identify

$$\begin{aligned} s_3|_1 &= s_3^{(\mathrm{I})}|_1 + s_3^{(\mathrm{II})}|_1, \\ s_3^{(\mathrm{I})}|_1 &= \frac{1}{2}\Big([s_1|_1, \mu_3|_0] + \big[s_2^{(\mathrm{I})}|_1, \mu_2|_0\big] \Big), \\ s_3^{(\mathrm{II})}|_1 &= \big[s_2^{(\mathrm{II})}|_1, \mu_2|_0\big]\,. \end{aligned} \tag{5.93}$$

Thus we see a pattern where the supersymmetry product at Ramond number 1 breaks up into a product denoted with (I) and a product denoted with (II), each determined by independent recursions. Again, one can check that $s_3|_1$ is in the small Hilbert space. This completes the definition of the 3-product $\tilde{S}_3$ in the supersymmetry transformation.

Now we can guess the form of the supersymmetry transformation at higher orders. The $(n+1)$st product can be written

$$\tilde{S}_{n+1} = S_{n+1}|_{-1} + s_{n+1}|_1. \tag{5.94}$$

Components with higher Ramond number can be set to zero. In addition, $s_{n+2}|_1$ can be written as a sum

$$s_{n+2}|_1 = s_{n+2}^{(\mathrm{I})}|_1 + s_{n+2}^{(\mathrm{II})}|_1. \tag{5.95}$$

The products are determined recursively by the equations,

$$S_{n+2}|_{-1} = \frac{1}{n+1}\sum_{k=0}^{n} [S_{k+1}|_{-1}, \mu_{n-k+2}|_0]\,, \tag{5.96a}$$

$$s_{n+2}^{(\mathrm{I})}|_1 = \frac{1}{n+1}\sum_{k=0}^{n} \big[s_{k+1}^{(\mathrm{I})}|_1, \mu_{n-k+2}|_0\big]\,, \tag{5.96b}$$

$$s_{n+3}^{(\mathrm{II})}|_1 = \frac{1}{n+1}\sum_{k=0}^{n} \big[s_{k+2}^{(\mathrm{II})}|_1, \mu_{n-k+2}|_0\big]\,, \tag{5.96c}$$

starting from $S_1|_{-1}$ in equation (5.71), $s_1^{(\mathrm{I})}|_1 = s_1|_1$ in equation (5.69), and $s_2^{(\mathrm{II})}|_1 = [m_2|_2, \sigma_1|_{-1}]$. Now let's prove that these products have the required properties. We promote the products to coderivations and define generating functions

$$\mathbf{S}(t) = \sum_{n=0}^{\infty} t^n \mathbf{S}_{n+1}|_{-1},$$
$$\mathbf{s}^{(\mathrm{I})}(t) = \sum_{n=0}^{\infty} t^n \mathbf{s}^{(\mathrm{I})}_{n+1}|_1,$$
$$\mathbf{s}^{(\mathrm{II})}(t) = \sum_{n=0}^{\infty} t^n \mathbf{s}^{(\mathrm{II})}_{n+2}|_1.$$

Using the generating function of the gauge products (5.33d), the recursive equations (5.96) can be reexpressed

$$\frac{\partial}{\partial t}\mathbf{S}(t) = [\mathbf{S}(t), \boldsymbol{\mu}(t)], \tag{5.97a}$$
$$\frac{\partial}{\partial t}\mathbf{s}^{(\mathrm{I})}(t) = [\mathbf{s}^{(\mathrm{I})}(t), \boldsymbol{\mu}(t)], \tag{5.97b}$$
$$\frac{\partial}{\partial t}\mathbf{s}^{(\mathrm{II})}(t) = [\mathbf{s}^{(\mathrm{II})}(t), \boldsymbol{\mu}(t)], \tag{5.97c}$$

The generating functions for the products in the equations of motion and supersymmetry transformation take the form

$$\tilde{\mathbf{M}}(t) = \mathbf{M}(t) + t\mathbf{m}'(t), \qquad \tilde{\mathbf{S}}(t) = \mathbf{S}(t) + \mathbf{s}^{(\mathrm{I})}(t) + t\mathbf{s}^{(\mathrm{II})}(t). \tag{5.98}$$

The differential equations for the generating functions imply a set of equations:

$$\frac{\partial}{\partial t}[\tilde{\mathbf{M}}(t), \tilde{\mathbf{S}}(t)] = [[\tilde{\mathbf{M}}(t), \tilde{\mathbf{S}}(t)], \boldsymbol{\mu}(t)] + [\mathbf{m}'(t), \tilde{\mathbf{S}}(t)] + [\tilde{\mathbf{M}}(t), \mathbf{s}^{(\mathrm{II})}(t)], \tag{5.99a}$$
$$\frac{\partial}{\partial t}\Big([\mathbf{m}'(t), \tilde{\mathbf{S}}(t)] + [\tilde{\mathbf{M}}(t), \mathbf{s}^{(\mathrm{II})}(t)]\Big) = [[\mathbf{m}'(t), \tilde{\mathbf{S}}(t)] + [\tilde{\mathbf{M}}(t), \mathbf{s}^{(\mathrm{II})}(t)], \boldsymbol{\mu}(t)] + 2[\mathbf{m}'(t), \mathbf{s}^{(\mathrm{II})}(t)], \tag{5.99b}$$
$$\frac{\partial}{\partial t}[\mathbf{m}'(t), \mathbf{s}^{(\mathrm{II})}(t)] = [[\mathbf{m}'(t), \mathbf{s}^{(\mathrm{II})}(t)], \boldsymbol{\mu}(t)]. \tag{5.99c}$$

Start with the last equation. Note that $[\mathbf{m}'(0), \mathbf{s}^{(\mathrm{II})}(0)]$ vanishes because

$$[m_2|_2, s_2^{(\mathrm{II})}|_1] = [m_2|_2, [m_2|_2, \sigma_1|_{-1}]] = 0.$$

The last equation implies

$$[\mathbf{m}'(t), \mathbf{s}^{(\mathrm{II})}(t)] = 0.$$

Equation (5.99b) is now homogeneous in $[\mathbf{m}'(t), \tilde{\mathbf{S}}(t)] + [\tilde{\mathbf{M}}(t), \mathbf{s}^{(\mathrm{II})}(t)]$. This vanishes at $t = 0$, because

$$[m_2|_2, S_1|_{-1}] + [Q, [m_2|_2, \sigma_1|_{-1}]] = 0.$$

Therefore equation (5.99b) implies

$$[\mathbf{m}'(t), \tilde{\mathbf{S}}(t)] + [\tilde{\mathbf{M}}(t), \mathbf{s}^{(\mathrm{II})}(t)] = 0.$$

Finally, the first equation (5.99a) is now homogeneous in $[\tilde{\mathbf{M}}(t), \tilde{\mathbf{S}}(t)]$. The commutator $[\tilde{\mathbf{M}}(0), \tilde{\mathbf{S}}(0)]$ vanishes since the supersymmetry operators are BRST invariant. Therefore

$$[\tilde{\mathbf{M}}(t), \tilde{\mathbf{S}}(t)] = 0.$$

Setting $t = 1$, we have in particular

$$[\tilde{\mathbf{M}}, \tilde{\mathbf{S}}] = 0, \tag{5.100}$$

which proves that the products $\tilde{S}_{n+1}$ generate a symmetry. To prove that the transformation preserves the small Hilbert space, consider the following set of equations:

$$\frac{\partial}{\partial t}[\boldsymbol{\eta}, \tilde{\mathbf{S}}(t)] = [[\boldsymbol{\eta}, \tilde{\mathbf{S}}(t)], \boldsymbol{\mu}(t)] + [\tilde{\mathbf{S}}(t), \mathbf{m}(t)] + [\boldsymbol{\eta}, \mathbf{s}^{(\mathrm{II})}(t)], \tag{5.101a}$$

$$\begin{aligned}\frac{\partial}{\partial t}\Big([\tilde{\mathbf{S}}(t), \mathbf{m}(t)] + [\boldsymbol{\eta}, \mathbf{s}^{(\mathrm{II})}(t)]\Big) = [[\tilde{\mathbf{S}}(t), \mathbf{m}(t)] + [\boldsymbol{\eta}, \mathbf{s}^{(\mathrm{II})}(t)], \boldsymbol{\mu}(t)]& \\ + 2[\mathbf{s}^{(\mathrm{II})}(t), \mathbf{m}(t)],&\end{aligned} \tag{5.101b}$$

$$\frac{\partial}{\partial t}[\mathbf{s}^{(\mathrm{II})}(t), \mathbf{m}(t)] = [[\mathbf{s}^{(\mathrm{II})}(t), \mathbf{m}(t)], \boldsymbol{\mu}(t)]. \tag{5.101c}$$

Start with the last equation. Note that $[\mathbf{m}(0), \mathbf{s}^{(\mathrm{II})}(0)]$ vanishes because

$$[m_2|_0, s_2^{(\mathrm{II})}|_1] = [m_2|_0, [m_2|_2, \sigma_1|_{-1}]] = \frac{1}{2}[[m_2, m_2]|_2, \sigma_1|_{-1}] - [m_2|_2, [m_2, \sigma_1]|_{-1}] = 0.$$

The last equation then implies

$$[\mathbf{m}(t), \mathbf{s}^{(\mathrm{II})}(t)] = 0.$$

The next to last equation (5.101b) is now homogeneous in $[\tilde{\mathbf{S}}(t), \mathbf{m}(t)] + [\boldsymbol{\eta}, \mathbf{s}^{(\mathrm{II})}(t)]$. We know that $[\tilde{\mathbf{S}}(0), \mathbf{m}(0)] + [\boldsymbol{\eta}, \mathbf{s}^{(\mathrm{II})}(0)] = 0$ because

$$[S_1|_{-1} + s_1|_1, m_2|_0] + [\eta, [m_2|_2, \sigma_1|_{-1}] = [S_1, m_2]|_0 + [s_1, m_2]|_1 = 0.$$

Therefore equation (5.101b) implies

$$[\tilde{\mathbf{S}}(t), \mathbf{m}(t)] + [\boldsymbol{\eta}, \mathbf{s}^{(\mathrm{II})}(t)] = 0.$$

Finally, consider the first equation (5.101a), which is now homogeneous in $[\boldsymbol{\eta}, \tilde{\mathbf{S}}(t)]$. Since $[\boldsymbol{\eta}, \tilde{\mathbf{S}}(0)] = 0$ we conclude

$$[\boldsymbol{\eta}, \tilde{\mathbf{S}}(t)] = 0, \tag{5.102}$$

so the products $\tilde{S}_{n+1}$ are in the small Hilbert space. This completes the construction of the supersymmetry transformation.

5.6.2 Polynomial form of the supersymmetry transformation

The recursive construction of the supersymmetry transformations given in the previous subsection is quite inconvenient for explicit calculations. Luckily, most of the complexity is due to the field redefinition generated by the coderivation $\boldsymbol{\mu}(t)$. Upon undoing this field redefinition, one obtains polynomial equations of motion. The price one has to play is that the small Hilbert space constraint must be modified into a non-linear, but polynomial constraint. The polynomial form of the equations of motion and the non-linear small Hilbert space constraint is reminiscent of Berkovits' superstring field theory [41, 42, 123]. The finite field redefinition was first analysed in [56, 58] when investigating the relationship between the open superstring field theory from chapter 3 and the NS-sector of Berkovits' superstring field theory.

The finite form of the field redefinition generated by a coderivation $\boldsymbol{\mu}(t)$ is a cohomomorphism $\hat{\mathbf{G}}(t)$ that solves the initial value problem,

$$\frac{\partial}{\partial t}\hat{\mathbf{G}}(t) = \hat{\mathbf{G}}(t)\boldsymbol{\mu}(t), \qquad \hat{\mathbf{G}}(0) = \mathbb{I}. \tag{5.103}$$

We do not need the explicit solution for $\hat{\mathbf{G}}(t)$, but it suffices to know that it exists and is analytic in t and can be constructed order by order in t by integrating the differential equation. The generating function $\mathbf{M}(t)$, $\mathbf{m}'(t)$ and $\mathbf{m}(t)$ can be expressed in terms of $\hat{\mathbf{G}}(t)$ and their initial values,

$$\mathbf{M}(t) = \hat{\mathbf{G}}(t)^{-1}\,\mathbf{Q}\,\hat{\mathbf{G}}(t), \tag{5.104a}$$

$$\mathbf{m}'(t) = \hat{\mathbf{G}}(t)^{-1}\,\mathbf{m}_2|_2\,\hat{\mathbf{G}}(t), \tag{5.104b}$$

$$\mathbf{m}(t) = \hat{\mathbf{G}}(t)^{-1}\,\mathbf{m}_2|_0\,\hat{\mathbf{G}}(t). \tag{5.104c}$$

Moreover, it follows from the computation in [58] that[3]

$$\boldsymbol{\eta} = \hat{\mathbf{G}}(t)^{-1}\,(\boldsymbol{\eta} - t\,\mathbf{m}_2|_0)\,\hat{\mathbf{G}}(t). \tag{5.105}$$

Thus, the small Hilbert space constraint is mapped to a non-linear constraint by the cohomomorphisms $\hat{\mathbf{G}}(t)$. Define a new string field $\tilde{\varphi}$ as $e^{\tilde{\varphi}} = \hat{\mathbf{G}}(1)e^{\tilde{\Phi}}$. This string field lives in the large Hilbert space. Using the above relations it follows immediately, that the small Hilbert space constraint and the equations of motion are equivalently given by the Maurer-Cartan equation for the A_∞-structure $\mathbf{Q} + \mathbf{m}_2 - \boldsymbol{\eta}$,

$$(\mathbf{Q} + \mathbf{m}_2 - \boldsymbol{\eta})\,e^{\tilde{\varphi}} = 0. \tag{5.106}$$

The string field $\tilde{\varphi} = \varphi_{\mathrm{N}} + \psi_{\mathrm{R}}$ can be decomposed into large Hilbert space fields at picture -1 and at picture $-\frac{1}{2}$. In terms of the component fields the Maurer-Cartan

[3] One way to see this identity is as follows: Define $\mathbf{n}(t) = \boldsymbol{\eta} + t\,\mathbf{m}(t)$. This function satisfies the differential equation $\frac{\partial}{\partial t}\mathbf{n}(t) = [\mathbf{n}(t), \boldsymbol{\mu}(t)]$ and $\mathbf{n}(0) = \boldsymbol{\eta}$. Thus in terms of $\hat{\mathbf{G}}(t)$ we have $\mathbf{n}(t) = \hat{\mathbf{G}}(t)^{-1}\,\boldsymbol{\eta}\hat{\mathbf{G}}(t)$, from which together with equation (5.104c) the desired formula follows.

equations read

$$0 = Q\psi_{\mathrm{R}} \tag{5.107a}$$
$$0 = Q\varphi_{\mathrm{N}} + \psi_{\mathrm{R}} * \psi_{\mathrm{R}} \tag{5.107b}$$
$$0 = \eta\psi_{\mathrm{R}} - \psi_{\mathrm{R}} * \varphi_{\mathrm{N}} + \varphi_{\mathrm{N}} * \psi_{\mathrm{R}} \tag{5.107c}$$
$$0 = \eta\varphi_{\mathrm{N}} - \varphi_{\mathrm{N}} * \varphi_{\mathrm{N}}. \tag{5.107d}$$

The first two equations arise from the original equations of motion under the field redefinition and can be interpreted as equations of motion. Moreover, only the first two terms contain spacetime derivatives at the linear level. The last two equations have a purely algebraic linear term and should therefore be regarded as constraints rather than dynamical equations of motion.

Now we consider the new form of the supersymmetry transformation. The differential equations (5.97) imply that the supersymmetry transformation can be written in the form

$$\tilde{\mathbf{S}} = \hat{\mathbf{G}}(1)^{-1}\,\tilde{\mathbf{s}}\,\hat{\mathbf{G}}(1), \tag{5.108a}$$
$$\tilde{\mathbf{s}} = \mathbf{S}_1|_{-1} + \mathbf{s}_1|_1 + [\boldsymbol{\sigma}_1|_{-1}, \mathbf{m}_2|_2]. \tag{5.108b}$$

The coderivation $\tilde{\mathbf{s}}$ is the generator of supersymmetry transformations in the new field variable $\tilde{\varphi}$. Explicitly, in terms of the component fields φ_{N} and ψ_{R} they read

$$\delta\varphi_{\mathrm{N}} = s_1\psi_{\mathrm{R}} + \psi_{\mathrm{R}} * (\sigma_1\varphi_{\mathrm{N}}) + (\sigma_1\varphi_{\mathrm{N}}) * \psi_{\mathrm{R}}, \tag{5.109a}$$
$$\delta\psi_{\mathrm{R}} = S_1\varphi_{\mathrm{N}} + \sigma_1(\psi_{\mathrm{R}} * \psi_{\mathrm{R}}). \tag{5.109b}$$

We can check that $\tilde{\mathbf{s}}$ is a symmetry of the equations of motion:

$$\begin{aligned}[\tilde{\mathbf{s}}, \mathbf{Q} + \mathbf{m}_2|_2] &= [\mathbf{S}_1|_{-1} + \mathbf{s}_1|_1 + [\boldsymbol{\sigma}_1|_{-1}, \mathbf{m}_2|_2], \mathbf{Q} + \mathbf{m}_2|_2]\,,\\ &= -[\mathbf{S}_1|_{-1}, \mathbf{m}_2|_2] + [\mathbf{S}_1|_{-1}, \mathbf{m}_2|_2] + [\mathbf{s}_1|_1, \mathbf{m}_2|_2] + [[\boldsymbol{\sigma}_1|_{-1}, \mathbf{m}_2|_2]\,, \mathbf{m}_2|_2]\,,\\ &= 0.\end{aligned} \tag{5.110}$$

The terms either cancel or vanish because the Ramond number exceeds the number of inputs. We can also check that $\tilde{\mathbf{s}}$ preserves the constraints:

$$\begin{aligned}[\tilde{\mathbf{s}}, \boldsymbol{\eta} - \mathbf{m}|_2|_0] &= [\mathbf{S}_1|_{-1} + \mathbf{s}_1|_1 + [\boldsymbol{\sigma}_1|_{-1}, \mathbf{m}_2|_2]\,, \boldsymbol{\eta} - \mathbf{m}_2|_0]\,,\\ &= -[\mathbf{s}_1|_{-1}, \mathbf{m}_2|_2] - [\mathbf{S}_1|_{-1}, \mathbf{m}_2|_0] - [\mathbf{s}_1|_1, \mathbf{m}_2|_0] - [[\boldsymbol{\sigma}_1|_{-1}, \mathbf{m}_2|_2]\,, \mathbf{m}_2|_0]\,,\\ &= -[\mathbf{s}_1, \mathbf{m}_2]\,|_1 - [\mathbf{S}_1, \mathbf{m}_2]\,|_{-1}\\ &\qquad - \frac{1}{2}\,[\boldsymbol{\sigma}_1|_{-1}, [\mathbf{m}_2, \mathbf{m}_2]\,|_2] + [[\boldsymbol{\sigma}_1, \mathbf{m}_2]\,|_{-1}, \mathbf{m}_2|_2]\,,\\ &= 0.\end{aligned} \tag{5.111}$$

This vanishes since the s_1, S_1 and σ_1 are derivations of the star product and because the star product is associative. Conjugating by $\hat{\mathbf{G}}(1)$, this provides an alternative proof that the supersymmetry transformation $\tilde{\mathbf{S}}$ preserves the equations of motion for $\tilde{\Phi}$ and is consistent with the small Hilbert space constraint.

5.6.3 Supersymmetry algebra

The remaining question is whether the algebra satisfied by the transformations (5.109) is indeed a supersymmetry algebra. This is indeed the case, but it turns out that the algebra closes only on-shell and up to gauge-transformations. Working with coderivations as infinitesimal supersymmetry transformations has the advantage that commutators of field redefinitions are calculated in terms of the ordinary commutator bracket,

$$[\tilde{\mathbf{s}}, \tilde{\mathbf{s}}'],$$

where the prime indicates that the operator is defined by replacing the spinorial parameter ϵ^a with ϵ'^a. The commutator of supersymmetry transformations should produce the momentum operator. We need two different momentum operators, P_μ at picture 0 and p_μ at picture -1, and an auxiliary operator π_μ at picture 0,

$$P_\mu = \oint \frac{dz}{2\pi \mathrm{i}}\, i\partial X_\mu(z). \tag{5.112a}$$

$$p_\mu = \frac{1}{\sqrt{2}} \oint \frac{dz}{2\pi \mathrm{i}}\, \psi_\mu e^{-\phi}(z), \tag{5.112b}$$

$$\pi_\mu = \frac{1}{\sqrt{2}} \oint \frac{dz}{2\pi \mathrm{i}}\, \xi\psi_\mu e^{-\phi}(z). \tag{5.112c}$$

P_μ, p_μ and π_μ are derivations of the star product and satisfy

$$\begin{aligned} P_\mu &= [Q, \pi_\mu], \\ p_\mu &= [\eta, \pi_\mu]. \end{aligned}$$

To compute the supersymmetry algebra we need the following commutators between supersymmetry operators:

$$[s_1, s_1'] = -2\,\bar{\epsilon}\,\Gamma^\mu \epsilon'\, p_\mu \equiv -2p(\epsilon, \epsilon'), \tag{5.113a}$$

$$[S_1, s_1'] = [s_1, S_1'] = -2\,\bar{\epsilon}\,\Gamma^\mu \epsilon'\, P_\mu \equiv -2P(\epsilon, \epsilon') \tag{5.113b}$$

$$\bar{\epsilon}\,\Gamma^\mu \epsilon'\, \pi_\mu \equiv \pi(\epsilon, \epsilon') \tag{5.113c}$$

Now we are ready to compute the supersymmetry algebra. Plugging in equation (5.108b) and expanding cross-terms gives

$$\begin{aligned} [\tilde{\mathbf{s}}, \tilde{\mathbf{s}}'] = {} & [\mathbf{S}_1|_{-1}, \mathbf{s}_1'|_1] + [\mathbf{s}_1|_1, \mathbf{S}_1'|_{-1}] + [\mathbf{S}_1|_{-1}, [\boldsymbol{\sigma}_1'|_{-1}, \mathbf{m}_2|_2]] + [[\boldsymbol{\sigma}_1|_{-1}, \mathbf{m}_2|_2]\,, \mathbf{S}_1'|_{-1}] \\ & + [\mathbf{s}_1|_1, [\boldsymbol{\sigma}_1'|_{-1}, \mathbf{m}_2|_2]] + [[\boldsymbol{\sigma}_1|_{-1}, \mathbf{m}_2|_2]\,, \mathbf{s}_1'|_1] + [[\boldsymbol{\sigma}_1|_{-1}, \mathbf{m}_2|_2]\,, [\boldsymbol{\sigma}_1'|_{-1}, \mathbf{m}_2|_2]]\,. \end{aligned}$$

To extract the momentum operator, rewrite the first two terms

$$\begin{aligned} [\mathbf{S}_1|_{-1}, \mathbf{s}_1'|_1] + [\mathbf{s}_1|_1, \mathbf{S}_1'|_{-1}] &= [\mathbf{S}_1, \mathbf{s}_1'] + [\mathbf{s}_1, \mathbf{S}_1'] - [\mathbf{S}_1|_1, \mathbf{s}_1'|_{-1}] - [\mathbf{s}_1|_{-1}, \mathbf{S}_1'|_1]\,, \\ &= -4\,\mathbf{P}(\epsilon, \epsilon') - [\mathbf{S}_1|_1, \mathbf{s}_1'|_{-1}] - [\mathbf{s}_1|_{-1}, \mathbf{S}_1'|_1]\,, \\ &= -2\,\mathbf{P}(\epsilon, \epsilon') - 2[\mathbf{Q}, \boldsymbol{\pi}(\epsilon, \epsilon')] - [\mathbf{S}_1|_1, \mathbf{s}_1'|_{-1}] - [\mathbf{s}_1|_{-1}, \mathbf{S}_1'|_1]\,, \end{aligned}$$

where $\mathbf{P}(\epsilon,\epsilon')$ is the coderivation corresponding to $P(\epsilon,\epsilon')$ and $\boldsymbol{\pi}(\epsilon,\epsilon')$ is the coderivation corresponding to $\pi(\epsilon,\epsilon')$. In the third step we chose to express part of the translation operator in the form $[Q,\pi_\mu]$, for reasons that will be clear shortly. Substituting we find

$$\begin{aligned}[\tilde{\mathbf{s}},\tilde{\mathbf{s}}'] = -2\,\mathbf{P}(\epsilon,\epsilon') &- 2[\mathbf{Q},\boldsymbol{\pi}(\epsilon,\epsilon')] - [\mathbf{S}_1|_1,\mathbf{s}'_1|_{-1}] - [\mathbf{s}_1|_{-1},\mathbf{S}'_1|_1]\\ &+ [\mathbf{S}_1|_{-1},[\boldsymbol{\sigma}'_1|_{-1},\mathbf{m}_2|_2]] + [[\boldsymbol{\sigma}_1|_{-1},\mathbf{m}_2|_2]\,,\mathbf{S}'_1|_{-1}]\\ &+ [\mathbf{s}_1|_1,[\boldsymbol{\sigma}'_1|_{-1},\mathbf{m}_2|_2]] + [[\boldsymbol{\sigma}_1|_{-1},\mathbf{m}_2|_2]\,,\mathbf{s}'_1|_1]\\ &+ [[\boldsymbol{\sigma}_1|_{-1},\mathbf{m}_2|_2]\,,[\boldsymbol{\sigma}'_1|_{-1},\mathbf{m}_2|_2]]\,. \end{aligned} \tag{5.114}$$

In the simplest supersymmetry algebra, the terms after $-2\,\mathbf{P}(\epsilon,\epsilon')$ would cancel. Unfortunately they do not cancel, and we have to make sense of them. The reason why the extra terms are present is that we are dealing with an on-shell supersymmetry algebra. This is not surprising, since the off-shell fermionic and bosonic degrees of freedom in the string field do not match. For example, at mass level 0 we have 16 fermion fields but only 11 boson fields, including the gauge field, transverse scalars, and an auxiliary field. In the current context, on-shell supersymmetry implies that that the supersymmetry algebra should be expressible as

$$[\tilde{\mathbf{s}},\tilde{\mathbf{s}}'] = -2\,\mathbf{P}(\epsilon,\epsilon') + \left[\mathbf{Q}+\mathbf{m}_2|_2,\tilde{\boldsymbol{\lambda}}\right], \tag{5.115}$$

where $\boldsymbol{\lambda}$ is a coderivation which is consistent with the constraint on the field $\tilde{\varphi}$:

$$\left[\boldsymbol{\eta}-\mathbf{m}_2|_0,\tilde{\boldsymbol{\lambda}}\right] = 0. \tag{5.116}$$

With a little more effort [54] one can show that equations (5.115) and (5.116) are satisfied with

$$\tilde{\boldsymbol{\lambda}} = -2\,\boldsymbol{\pi}(\epsilon,\epsilon') - [\boldsymbol{\sigma}_1|_1,\mathbf{s}'_1|_{-1}] - [\mathbf{s}_1|_{-1},\boldsymbol{\sigma}'_1|_1] + [\boldsymbol{\sigma}_1|_{-1},[\boldsymbol{\sigma}'_1|_{-1},\mathbf{m}_2|_2]]\,. \tag{5.117}$$

In summary we have shown that $\tilde{\mathbf{s}}$ indeed generate an $\mathcal{N}=1$ supersymmetry algebra, but the algebra closes only on-shell and up to gauge transformations. By using the field redefinition in $\hat{\mathbf{G}}$ one can transfer the results to the field equations (5.4) expressed entirely in the small Hilbert space and conclude that the solution space has $\mathcal{N}=1$ supersymmetry.

5.7 Summary

In this chapter we constructed consistent classical field equations for all superstring theories and gave for the open superstring an explicit analysis of supersymmetry. A proof that our field equations imply the correct tree-level amplitudes can be found in chapter 6. Let us conclude by discussing future directions.

Though we don't know how to write a fully satisfactory action for the Ramond sector, it should be possible to formulate a tree-level action which includes two Ramond string fields (typically, at picture $-\frac{1}{2}$ and $-\frac{3}{2}$), which are afterwards related by imposing a self-dual constraint on classical solution space [139]. See [140, 143] for recent discussion. One version of this idea was recently suggested in [62], and its applicability to open superstrings is discussed in chapter 7. However, the required products in the equations of motion are slightly different from those introduced here. In a sense they are more complicated, since even at a given Ramond number the products differ depending on the number of Ramond states being multiplied. But this is not an insurmountable complication. It would be particularly nice if an action with constraint could be realised for type II closed superstring field theory, as it would give a potentially interesting gauge invariant observable for Ramond-Ramond backgrounds. However, it remains to be seen whether an action with constraint helps in defining the quantum theory.

Chapter 6

The S-matrix in superstring field theory

Any open superstring field theory should reproduce the traditional perturbative superstring scattering amplitudes through its associated Feynman perturbation series. In this chapter we establish this property for the classical superstring field theories described in chapters 3, 4 and 5. In the proof we focus on open superstring field theory and we exploit the fact that the vertices are obtained by a field redefinition in the large Hilbert space. The result extends to include the NS-NS subsector of type II superstring field theory and the equations of motions for the Ramond fields. In addition, our proof implies that the S-matrix obtained from Berkovits' WZW-like string field theory then agrees with the perturbative S-matrix to all orders.

This chapter is based on the paper **The S-matrix in superstring field theory** by the author [55].

6.1 Introduction

A field theoretical formulation of string theory can give valuable insight into a possible non-perturbative description of the moduli space of quantum string vacua. For the open bosonic string such a formulation was first described in light-cone gauge and later reformulated in covariant form [10]. The algebraic structures contained in the latter are still at the heart of any covariant string field theory in use today. Almost at the same time an analogous formalism for superstring field theory was proposed [34]. However, its construction was highly formal and turned out to give divergent results due to collisions of local operators on the world-sheet and therefore required regularisation [37]. The modified string field theory was proposed in [38, 39] and dealt with the problem by using a modified kinetic term. But, it is not clear whether this field theory reproduces the correct particle spectrum. A new regularisation in terms of small Hilbert space fields and smeared picture changing operators was described in chapter 3. If the latter formulation defines a valid open superstring field theory, its S-matrix must necessarily coincide with the usual perturbative string S-matrix calculated in the formalism of picture changing operators [35, 36] or in terms of integrals over supermoduli space [95, 144]. In this paper we prove such equivalence to the former formalism at tree-level or genus 0.

Let us now outline the main ingredients of this proof for open string field theory. At the perturbative level bosonic string field theory provides a definition of the Polyakov path-integral for arbitrary matter part with $c = 26$. This means that its tree-level perturbation series gives rise to a regularised version of integrals over the whole moduli space of punctured discs. The Feynman perturbation series of planar tree-level diagrams in Siegel gauge coincides with the usual description of the color-ordered amplitude as an integral over the positions of all but three punctures. On the other hand the vertices of open string field theory satisfy the axioms of a cyclic A_∞ algebra. At the algebraic level, the connection between the S-matrix and the A_∞ algebraic structure is established through the so-called *minimal model*. For any A_∞ structure there exists an A_∞ structure on the cohomology $H^\bullet(Q)$ in such a way that this induced structure is A_∞-quasi-isomorphic to the original one. An explicit formula for the minimal model and the A_∞-quasi-isomorphisms is formulated in terms of sums over all planar tree diagrams [22] and we argue that the matrix elements of the induced maps coincide with the color-ordered S-matrix.

The open superstring field theory action for the NS-sector was found by requiring the vertices to be in the small Hilbert space and that they constitute a cyclic A_∞ algebra, cf. chapter 3. The solution was eventually obtained through a field redefinition in the large Hilbert space from a free theory. The field redefinition was constructed by integration of a pair of differential equations (4.32),

$$\delta\mathbf{M} = \frac{\partial}{\partial t}\mathbf{M} = [\mathbf{M}, \boldsymbol{\mu}] \tag{6.1a}$$

$$\frac{\partial}{\partial s}\mathbf{M} = [\boldsymbol{\eta}, \boldsymbol{\mu}], \tag{6.1b}$$

where t was a deformation parameters, s a formal parameter counting the so-called *picture deficit* and $\boldsymbol{\mu}$ was an arbitrary function $\mathbf{M}(s, t)$. In this paper, we show that at the level of the S-matrix this field redefinition leads to the needed insertions of picture changing operators (PCO) at the external legs. One important feature of the proof is that it only requires the above two equations and can thus be applied to any family of A_∞ algebras satisfying equations (6.1) and its validity is independent of the choice of contracting homotopy for $[\boldsymbol{\eta}, \cdot]$. The proof itself is divided into three steps. In the first step we find an explicit expression of the minimal model using homological perturbation theory. For technical reasons we need to consider a slight modification of the minimal model that we call the *almost minimal model*. Next, we argue that the products of the (almost) minimal model are identical to the perturbative, color-ordered S-matrix elements. We do this by showing that they satisfy a recursion relation that generates all planar tree diagrams. Finally, we evaluate the minimal model of open superstring field theory and relate it to the minimal model of the underlying bosonic string products. From which the postulated equivalence of the

S-matrix of superstring field theory with the perturbative S-matrix calculated in the PCO formalism follows.

The outline of this chapter is as follows: In section 6.2, we discuss some mathematical properties of the minimal model and find an explicit expression through an application of the homological perturbation lemma. Moreover we discuss the connection of the minimal model with the perturbative S-matrix. Section 6.3 contains the key result of this paper. We apply the previously described techniques to evaluate the (almost) minimal model of open NS-superstring field theory. Quite interestingly, the proof can be adapted to all other superstring field theories based on homotopy algebraic methods. This includes the extension to the classical closed NS-NS superstring, the heterotic NS string, cf. chapter 4, the equations of motion for the complete classical open superstring, closed superstring and heterotic string, cf. chapter5. As the arguments are very similar, we only discuss the extension to the closed NS-NS superstring in section 6.4.1 and the extension to the equations of motion of the complete open superstring in section 6.4.2. From there it should be clear that the extension to the remaining cases is straightforward. We also comment on the implications of our results on the S-matrix of Berkovits' WZW-like superstring field theory in section 6.4.3.

6.2 The minimal model

6.2.1 The minimal model of an A_∞-algebra

The essential idea behind cohomology theories is that certain quantities interest can be represented in many different ways. Such quantities could be geometrical, topological invariants in mathematics or scattering matrix elements in physics. The various representations of that data are called models for the cohomology theory. When modelled with the help of dg-chain complexes, they typically include lots of auxiliary data and encode the physical information in the cohomology of some differential Q together with some additional algebraic structure on that cohomology, like the gauge-invariant "S-matrix". The calculation of the gauge-invariant data living on the cohomology can be done in various models. Some of them lead to nice interpretations, while some of them allow for easy calculations.

A_∞ algebras are special cases of this idea: every A_∞ algebra $\mathbf{M}$ induces an A_∞-algebra structure $\tilde{\mathbf{M}}$ on the cohomology $H^\bullet(Q)$, the so-called *minimal model*. We need a little more terminology. Given two A_∞ algebras on $\mathcal{H}$ and $\mathcal{H}'$ described by coderivations $\mathbf{M}$ and $\mathbf{M}'$, we can define an A_∞-*morphism* $\mathcal{F} : (\mathcal{H}, \mathbf{M}) \to (\mathcal{H}', \mathbf{M}')$ as a cohomomorphism $\mathcal{F} : T\mathcal{H} \to T\mathcal{H}'$ that intertwines both structures, $\mathcal{F}\mathbf{M} = \mathbf{M}'\mathcal{F}$. $\mathcal{F}$ is called an A_∞-*isomorphism* if it is invertible as a cohomomorphism. Let us denote by $f_k = \pi_1 \mathcal{F} \iota_k$ the component maps of $\mathcal{F}$. The component f_1 must the satisfy

$$f_1 Q = Q' f_1.$$

Consequently f_1 is a chain map and, therefore, gives rise to a map $H^\bullet(Q) \to H^\bullet(Q')$.

If the latter map is invertible, $\mathcal{F}$ is called an A_∞-quasi-isomorphism. An A_∞-algebra is called minimal if $Q = 0$. The important *minimal model theorem* states that any A_∞-algebra is isomorphic to a minimal A_∞-algebra and that this minimal model characterises the A_∞ algebra $\mathbf{M}$ completely, i.e. any two A_∞ algebras with A_∞-isomorphic minimal models are quasi-isomorphic [22, 103]. A nice review is [24].

In the following we want to motivate the construction of the minimal model structure $\tilde{\mathbf{M}}$ via homological perturbation theory. In general there are two ways to define an algebraic structure on $H^\bullet(Q)$. The first approach takes arbitrary representatives for each cohomology class and defines a cohomology class by specifying some Q-closed vector. Then, one has to show that by changing the representative by a Q-exact piece modifies the answer only by a Q-exact piece. Alternatively, one can make a fixed choice of representative for each cohomology class and define the structure on them. The drawback of the latter construction is that it depends on the particular choice of representative and is not manifestly independent of it. Since one does not expect that the algebraic structures are independent of the choice of representative, one should at least require that two different choices give rise to isomorphic algebraic structures. In our case at hand, the first method only works for 2-product so that we need to resort to the second method for the higher products.

Let us now see how the first two products of the minimal model structure are constructed explicitly. In the first approach the induced 2-product $\tilde{M}_2$ is obviously defined through the product M_2:

$$\tilde{M}_2 = M_2.$$

This is a good choice because Q is a derivation of this product so that the product of two Q-closed vectors is again Q-closed and shifting the representative by a Q-exact piece shifts the product by a Q-exact term. Thus, we have defined a binary product on cohomology $\tilde{M}_2 : H^\bullet(Q) \otimes H^\bullet(Q) \to H^\bullet(Q)$. Since M_2 is associative up to homotopy, the induced product is completely associative. At this point we already have obtained an A_∞-algebraic structure on the cohomology. Unfortunately, this new structure is not the minimal model because it is not A_∞-quasi-isomorphic to the original structure in general. Finding the 3-product is a little bit more complicated. The naive guess $\tilde{M}_3 = M_3$ does not work, since it does not map Q-closed states into Q-closed states. Unless the induced 2-product on cohomology is trivial, there is no way to define the 3-product such that all Q-exact states decouple, so that the first method fails and we have to resort the the second. Choosing a representative for each cohomology class means that we select a section $i : H^\bullet(Q) \hookrightarrow \mathcal{H}$ of the canonical projection $p : \ker(Q) \subset \mathcal{H} \twoheadrightarrow H^\bullet(Q)$. Consequently we can find a homotopy $Q^\dagger$ such that we have

$$\mathbb{I} = p\, i, \tag{6.2a}$$

$$\mathbb{I} - Q^\dagger Q - QQ^\dagger = i\, p \equiv P. \tag{6.2b}$$

Our choice of binary product can then be expressed as

$$\tilde{M}_2 = p\, M_2\, i^{\otimes 2}.$$

Using the A_∞-relations together with (6.2) one can show that this is indeed an associative product. Playing around a little bit, one discovers that the following product

$$\tilde{M}_3 = p\left(M_3 + M_2((-Q^\dagger M_2)\otimes \mathbb{I} + \mathbb{I}\otimes(-Q^\dagger M_2))\right) i^{\otimes 3}$$

satisfies $[\tilde{\mathbf{M}}_3, \tilde{\mathbf{M}}_2] = 0$. Hence, $\tilde{M}_2$ and $\tilde{M}_3$ satisfy the first two non-trivial A_∞ relations.

As an example consider the DGA of differential forms on a compact manifold X, the so-called *de Rham complex*. The cohomology theory that it models is purely topological and is known as the cohomology of X. The cohomology classes are in one-to-one correspondence with the homology $H_\bullet(X)$ through Poincaré duality. The induced associative product $\tilde{M}_2$ is then the cup product. If X is orientable, we can also endow the DGA with an invariant symplectic form so that we obtain a cyclic DGA. The symplectic form is given by integration over X. In this case the product $\tilde{M}_2$ also calculates intersection numbers between cycles. The higher products $\tilde{M}_k, k \geq 3$ correspond to the Massey products and give refined topological information about X [24].

Guessing the higher order products is quite cumbersome and we want a systematic way to construct them. The answer is given in terms of the homological perturbation lemma [145]. Our goal is to construct $\tilde{\mathbf{M}}$ together with a pair of mutually inverse A_∞-quasi-isomorphisms from/to the original A_∞-structure. This means that we look for A_∞-morphisms $\mathfrak{p} : (\mathcal{H}, \mathbf{M}) \to (H^\bullet(Q), \tilde{\mathbf{M}})$ and $\mathfrak{i} : (H^\bullet(Q), \tilde{\mathbf{M}}) \to (\mathcal{H}, \mathbf{M})$ such that $\mathfrak{p}\mathfrak{i} = \mathbb{I}_{T\mathcal{H}}$ and $\mathfrak{i}\mathfrak{p} \cong \mathbb{I}_{T\mathcal{H}}$, where the last requirement means that $\mathfrak{i}\mathfrak{p}$ is homotopic to $\mathbb{I}_{T\mathcal{H}}$, i.e. there is a homotopy $H : T\mathcal{H} \to T\mathcal{H}$ such that

$$\begin{aligned}
\mathfrak{i}\mathfrak{p} &= \mathbb{I}_{T\mathcal{H}} + \mathbf{M}H + H\mathbf{M} \\
\mathfrak{p}\mathbf{M} &= \tilde{\mathbf{M}}\mathfrak{p} \\
\mathfrak{i}\tilde{\mathbf{M}} &= \mathbf{M}\mathfrak{i}.
\end{aligned}$$

If $\mathbf{M} = \mathbf{Q}$ there are no induced products and the problem is easily solved in terms of $Q^\dagger$, p and i from (6.2). The appropriate choices are

$$\begin{aligned}
H &= \sum_{r,s\geq 0} \mathbb{I}^{\otimes r} \otimes (-Q^\dagger) \otimes P^{\otimes s} \\
\mathfrak{p} &= \sum_{r\geq 0} p^{\otimes r}, \\
\mathfrak{i} &= \sum_{r\geq 0} i^{\otimes r} \\
\tilde{\mathbf{M}} &= 0.
\end{aligned}$$

They satisfy important compatibility conditions with the coalgebra structures Δ on $T\mathcal{H}$ and $T\mathcal{H}_p$,

$$\Delta\mathbf{Q} = (\mathbf{Q} \otimes' \mathbb{I}_{T\mathcal{H}} + \mathbb{I}_{T\mathcal{H}} \otimes' \mathbf{Q})\Delta \tag{6.3a}$$

$$\Delta H = (\mathbb{I}_{T\mathcal{H}} \otimes' H + H \otimes' \hat{P})\Delta \tag{6.3b}$$

$$\Delta\hat{P} = (\hat{P} \otimes' \hat{P})\Delta. \tag{6.3c}$$

The homological perturbation lemma allows us to modify this solution of the "free" problem into a solution of the complete problem.

Let V be a graded vector space and d some differential on V. The homological perturbation lemma [145] gives then a collection of formulæ that allow for calculating the cohomology of a perturbed differential $d+\delta$, where δ is small in an appropriate sense. For our purposes one may think of $V = T\mathcal{H}$ and of d as the codifferential Q representing the free theory. We treat then the difference $\mathbf{M}_{\text{int}} = \mathbf{M} - \mathbf{Q}$ as a perturbation in the sense of HPT. In order to state the perturbation lemma, we need a little bit more terminology. Let (V, d) and (W, D) be two chain complexes and let $\mathfrak{p} : V \to W$ and $\mathfrak{i} : W \to V$ be chain maps, s.t. $\mathfrak{p}\mathfrak{i} = \mathbb{I}$ and $\hat{P} = \mathfrak{i}\mathfrak{p} = \mathbb{I} + hd + dh$ for some linear map $h : V \to V$ called the *homotopy*. This collection of data $(V, W, d, D, \mathfrak{p}, \mathfrak{i}, h)$ is called *homotopy equivalence data* and (W, D) is said to be a *deformation retract* of (V, d). Let now δ be a perturbation of the chain complex (V, d), i.e. $(d+\delta)^2 = 0$. δ should be small in the sense that $(1-\delta\ h)^{-1}$ exists. In our context $\delta = \mathbf{M}_{\text{int}}$ represents the interaction part of the theory and is proportional to some coupling constant, so that the inverse exists at least perturbatively in the coupling constant. The main statement of the perturbation lemma is now that it is possible to deform the rest of the homotopy equivalence data in such a way that one retains valid homotopy equivalence data. More precisely,

$$d' = d + \delta \tag{6.4a}$$
$$\mathfrak{i}' = \mathfrak{i} + h\delta(1 - h\delta)^{-1}\mathfrak{i} \tag{6.4b}$$
$$\mathfrak{p}' = \mathfrak{p} + \mathfrak{p}\delta(1 - h\delta)^{-1}h \tag{6.4c}$$
$$D' = D + \mathfrak{p}\delta(1 - h\delta)^{-1}\mathfrak{i} \tag{6.4d}$$
$$h' = h + h\delta(1 - h\delta)^{-1}h\,. \tag{6.4e}$$

If the homotopy h satisfies additional properties,

$$h\mathfrak{i} = \mathfrak{p}h = h^2 = 0, \tag{6.5}$$

then (W, D) is called a *strong deformation retract* of (V, d). Applying the homological perturbation lemma to the case at hand means replacing

$$\begin{aligned} V &\to T\mathcal{H} & W &\to TH^\bullet(Q) \\ d &\to \mathbf{Q} & D &\to 0 \\ \delta &\to \mathbf{M}_{\text{int}} & D' &\to \tilde{\mathbf{M}}. \end{aligned}$$

In particular, the new differential on $\tilde{\mathbf{M}}$ on $TH^\bullet(Q)$ reads as

$$\tilde{\mathbf{M}} = \mathfrak{p}(1 - \mathbf{M}_{\text{int}}H)^{-1}\mathbf{M}_{\text{int}}\mathfrak{i}. \tag{6.6}$$

Since $(TH^\bullet(Q), 0)$ is a strong deformation retract of $(T\mathcal{H}, \mathbf{Q})$, one can show that $\tilde{\mathbf{M}}$ is a coderivation and so defines a minimal A_∞ structure on $H^\bullet(Q)$ and that $\mathfrak{p}$ and $\mathfrak{i}$ A_∞-quasi-isomorphisms and are homotopy inverses of each other. This result is known as the *minimal model theorem* or *Kadeishvili's theorem* [23, 146].

6.2.2 The minimal model and Siegel gauge

In string field theory the S-matrix is usually calculated in Siegel gauge. The various terms in the S-matrix calculated with the Siegel gauge propagator have a nice geometric interpretation in terms of disks obtained by gluing strips between the vertices and integrating over their lengths. A standard result in bosonic string field theory implies that one obtains the correct perturbative S-matrix [17, 105]. In this section we assume that the string background is flat Minkowski space or contains some uncompactified directions. The construction of the minimal model requires a choice of contracting homotopy $Q^\dagger$ of the Hilbert space onto the cohomology. While such an operator always exists, it is not necessarily equal to the Siegel gauge propagator. To see this, we make the choice $Q^\dagger = \frac{b_0}{L_0}(1 - e^{-\infty L_0})$. From this it follows that the physical projector P is given by

$$P = 1 - [Q, Q^\dagger] = e^{-\infty L_0}.$$

Although P is a projection operator, its image also contains unphysical states as $QP \neq 0$ and P is not a projection operator onto $H^\bullet(Q)$ but onto a larger vector space. We obtain a deformation retract of the original Hilbert space onto the image of P. At the end of section 6.2.1 we argued that if we start with a strong deformation retract, we obtain an A_∞ algebraic structure on $\mathcal{H}_p = P\mathcal{H}$ together with a pair of A_∞-quasi-isomorphisms. Actually, we can relax these assumptions a little bit and require only that $Q^\dagger i = pQ^\dagger = (Q^\dagger)^2 = 0$. The differential on $\mathcal{H}_p$ is QP. Application of the homological perturbation lemma means that we set

$$\begin{aligned} V &\to T\mathcal{H} & W &\to T\mathcal{H}_p \\ d &\to \mathbf{Q} & D &\to \mathbf{Q}\hat{P} \\ \mathfrak{i} &\to \hat{P} & \mathfrak{p} &\to \hat{P} \\ \delta &\to \mathbf{M}_{\text{int}} & D' &\to \mathcal{S}(\mathbf{M}). \end{aligned}$$

This gives then the induced A_∞-structure $\mathcal{S}(\mathbf{M})$ as

$$\mathcal{S}(\mathbf{M}) = \mathbf{Q}\hat{P} + \hat{P}\mathbf{M}_{\text{int}}(1 - H\mathbf{M}_{\text{int}})^{-1}\hat{P}. \tag{6.7}$$

We call $\mathcal{S}(\mathbf{M})$ the *almost minimal model* and call the actual minimal model $\tilde{\mathbf{M}}$ the *algebraic minimal model* whenever these distinctions are relevant. Since $\mathcal{S}(\mathbf{M})$ and $\mathbf{M}$ are A_∞-quasi-isomorphic to each other, by the minimal model theorem the minimal model will be $\tilde{\mathbf{M}}$ in both cases. As discussed in the next section, the maps in $\mathcal{S}(\mathbf{M})$ are given by sums over planar tree-level diagrams with propagators $-\frac{b_0}{L_0}$. Calculation of the minimal model for $\mathcal{S}(\mathbf{M})$ requires us to choose a contracting homotopy for QP. The projector P puts the states onto the mass-shell. This means that for operator $\mathcal{O}_1$ and $\mathcal{O}_2$ that are obtained by restricting space-time momentum preserving operators on $\mathcal{H}$ to operators on $\mathcal{H}_p$ we have the identity

$$\mathcal{O}_1 P \mathcal{O}_2 = 0 \tag{6.8}$$

for states with generic momentum. Thus, only diagrams with no internal lines contribute generically to the minimal model maps, but the only such diagrams are the vertices of $\mathcal{S}(\mathbf{M})$, which are identical to the perturbative S-matrix and coincide with the minimal model generically. Consequently, we can calculate the S-matrix either in Siegel gauge or using a complete gauge-fixing, but obtain the same answers.

6.2.3 The minimal model and the S-matrix

From a physical point of view the relevant information contained in the equations of motion are the observables and their expectation values. Observables are functions of the fields ϕ, but we are not interested in arbitrary such functions, but only those that are gauge invariant. Moreover, we identify two gauge invariant functions if their difference vanishes on-shell, i.e. is proportional to the equations of motion. The equivalence classes are the observables and can be thought of as functions on the moduli space of solutions modulo gauge-equivalence. The S-matrix measures then the obstruction for this moduli space to be smooth at $\phi = 0$ [89]. The most popular method for calculating the S-matrix perturbatively is through the use of Feynman diagrams. This approach, however, is not necessarily the best method for our purposes because the combinatorics for large n-point amplitudes is rather involved. Instead we use homological perturbation theory (HPT) which hides this difficulty and gives easy to manipulate formulæ for the S-matrix. Using HPT to generate the full Feynman perturbation series is not new, see for example [147–149].

In the previous section we constructed the almost minimal model $\mathcal{S}(\mathbf{M})$ of $\mathbf{M}$. The claim is that its products are identical to the color-ordered S-matrix elements of a string field theory with vertices encoded in the codifferential $\mathbf{M}$. More precisely, we claim that the color-ordered S-matrix S for Q-closed states Φ_i can be written as

$$S(\Phi_1, \Phi_2, \dots, \Phi_{n+1}) = (-1)^{\Phi_1} \omega(\Phi_1 \otimes \pi_1 \mathcal{S}(\mathbf{M})_n(\Phi_2, \Phi_3, \dots, \Phi_{n+1})),$$

or more abstractly as

$$S = \omega(\mathbb{I} \otimes \pi_1 \mathcal{S}(\mathbf{M})),$$

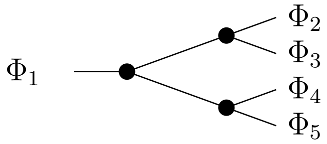
where $\hat{P}$ is an extension of the physical projector to the tensor algebra, ω the symplectic form. The almost minimal model $\mathcal{S}(\mathbf{M})$ is gauge-invariant in the sense that

$$[Q, \mathcal{S}(\mathbf{M})] = 0. \tag{6.9}$$

Let us now justify these formulas.

The products in $\mathcal{S}(\mathbf{M})$ are identical to the color-ordered S-matrix elements of the underlying field theory $d + \delta$ since they are A_∞-quasi isomorphic and thus have identical moduli spaces [22]. In the L_∞-case this has been shown by Kontsevich in [103] in his proof of deformation quantization. However, there is a more elementary way to verify this claim. The essential contribution to $\mathcal{S}(\mathbf{M})$ is given by

$$\hat{P}\mathbf{M}_{\text{int}}(1 - H\mathbf{M}_{\text{int}})^{-1}\hat{P}. \tag{6.10}$$



$$= (-1)^{\Phi_1}\omega(\Phi_1 \otimes M_2(-Q^\dagger M_2(\Phi_2,\Phi_3) \otimes -Q^\dagger M_2(\Phi_4,\Phi_5)))$$

Figure 6.1: Illustration of the Feynman rules for five states.

We want to interpret this expression as a sum over planar rooted tree diagrams. To this end, we need to introduce a set of Feynman rules. A *planar rooted tree diagram* is a planar graph of genus 0 with a distinguished external line that we call its root. A Feynman diagram for states $\Phi_1, \Phi_2, \ldots, \Phi_n$ is obtained from a planar rooted tree with n external lines as follows:

1. Assign the state $(-1)^{\Phi_1}\omega(\Phi_1 \otimes \mathbb{I})$ to the root, Φ_2 to the first leg next to the root in clockwise order and so on.

2. To the vertex connected to the root assign the multilinear map M_k, where $k+1$ is the number of edges connected to it.

3. To each other vertex of valence $k+1$ assign the operator $-Q^\dagger M_k$.

4. Compose these multilinear maps according to the shape of the diagram.

Figure 6.1 illustrates this procedure using a particular example. With this new terminology we consider equation (6.10) and show that it is equal to the sum over all Feynman diagrams as just defined. Let us define two maps on the tensor algebra[1],

$$\begin{aligned} A &= (1 - H\mathbf{M}_{\text{int}})^{-1}\hat{P} \\ \Sigma &= \mathbf{M}_{\text{int}}A. \end{aligned}$$

Notice that $\hat{P}\Sigma$ agrees with (6.10). The map A is a cohomomorphism and, consequently, is determined by its component map $\pi_1 A : T\mathcal{H} \to \mathcal{H}$. Using the explicit form (2.53) of a cohomomorphism and the definition of the maps M_k in equation (2.52), one easily deduces the following pair of equations,

$$\pi_1 A = P + (-Q^\dagger)\pi_1\Sigma \tag{6.11a}$$

$$\pi_1\Sigma = \sum_{k\geq 2} M_k(\pi_1 A)^{\otimes k} = \sum_{k\geq 2} M_k(P + (-Q^\dagger)\pi_1\Sigma)^{\otimes k}. \tag{6.11b}$$

Equation (6.11) provides us with a recursive algorithm for the restrictions of $\pi_1\Sigma$ to n inputs. The reason being that the sum on the right hand side starts at $k=2$ and

[1] At first one might think that one could replace $H\mathbf{M}_{\text{int}}$ with $-Q^\dagger\mathbf{M}_{\text{int}}$ in the next formula, but this leads to the wrong combinatorics. For example for the 5-point function it generates too many tree diagrams with two 3-vertices. The projections P play an important role in establishing the identification with the S-matrix.

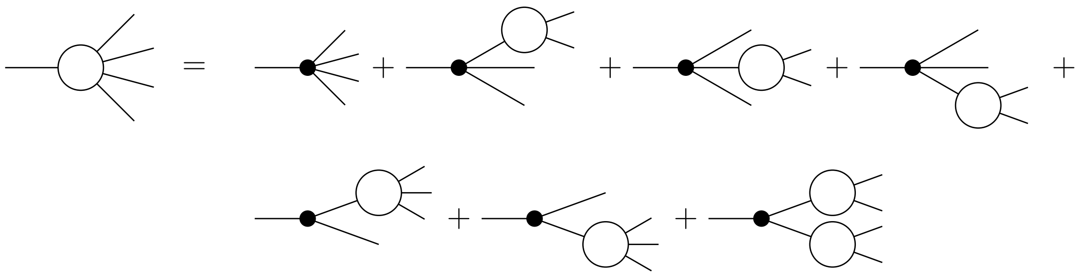


Figure 6.2: Graphical representation of equation (6.11) for four inputs. Big white circles represent components of $\pi_1\Sigma$ and small solid circles elementary vertices M_k. Internal lines correspond to propagators $-Q^\dagger$. External lines pointing to the right represent physical states.

so involves restrictions of $\pi_1\Sigma$ to at most $n-1$ inputs. A graphical representation of equation (6.11) can be found in figure (6.2). From the graphical representation it follows that $\pi_1\Sigma$ is constructed from a sum over all planar tree-level Feynman diagrams. Equation (6.11) is recognised as the classical Dyson-Schwinger equation once one identifies $-Q^\dagger$ with the propagator and M_k as interaction vertices of an action.

The important property (6.9) that encodes the gauge-invariance of the S-matrix follows straightforwardly,

$$\begin{aligned}[\mathbf{Q},\mathcal{S}(\mathbf{M})] &= (1-\mathbf{M}_{\text{int}}H)^{-1}\Big([\mathbf{Q},\mathbf{M}_{\text{int}}H]\mathbf{M}_{\text{int}}(1-H\mathbf{M}_{\text{int}})^{-1}+[\mathbf{Q},\mathbf{M}_{\text{int}}]\Big)\\ &= -\mathcal{S}(\mathbf{M})\hat{P}\mathcal{S}(\mathbf{M})=0,\end{aligned}$$

where in the last step we used that internal lines are generically off-shell.

Finally, we discuss the cyclicity properties of the S-matrix obtained from the almost minimal model for a cyclic A_∞-structure. Since the operator Q is BPZ-odd, any contracting homotopy $Q^\dagger$ can be written as the sum of a BPZ-even operator α and a BPZ-odd and Q-closed operator β, $[Q,\beta]=0$. In order to calculate the deviation from cyclicity, we consider the sum,

$$\begin{aligned}&\omega(\pi_1\Sigma\otimes P+P\otimes\pi_1\Sigma)=\\ &\quad\sum_{k\geq 2}\omega\Big(M_k(P+(-Q^\dagger)\pi_1\Sigma)^{\otimes k}\otimes P\\ &\qquad\qquad -M_k(P\otimes(P+(-Q^\dagger)\pi_1\Sigma)^{\otimes(k-1)})\otimes(P+(-Q^\dagger)\pi_1\Sigma)\Big)\\ &\quad=-\sum_{k\geq 2}\omega\Big((-Q^\dagger)\pi_1\Sigma\otimes\pi_1\Sigma+\pi_1\Sigma\otimes(-Q^\dagger)\pi_1\Sigma)\Big)\\ &\quad=2\omega(\beta(\pi_1\Sigma)\otimes\pi_1\Sigma).\end{aligned}\tag{6.12}$$

The S-matrix is manifestly cyclic provided $\beta=0$. Moreover, if $\beta=[Q,R]$ for some operator R, the gauge-invariance (6.9) tells us that the S-matrix is still cyclic, so that one can relax the requirement that $Q^\dagger$ has to be BPZ-even.

6.3 Evaluation of the minimal model

In this section we apply the formalism for tree-level perturbation theory explained in the previous section to open superstring field theory. We review the relevant main ingredients of the construction and express its S-matrix in terms of the usual perturbative S-matrix. Equations (6.1) are the only two ingredients used in the evaluation of the almost minimal model along with a choice of propagator $Q^\dagger$ and any choice of contracting homotopy $\xi \circ \cdot$ for $[\boldsymbol{\eta}, \cdot]$, not necessarily related to the homotopy used to construct $\mathbf{M}$. We claim that the following equation holds

$$X \circ \frac{\partial}{\partial s}\mathcal{S}(\mathbf{M}) - \frac{\partial}{\partial t}\mathcal{S}(\mathbf{M}) = [\boldsymbol{\eta}, [\mathbf{Q}, \mathbf{T}]]\,, \tag{6.13}$$

where $\mathbf{T}$ denotes some coderivation whose particular form is not relevant for the S-matrix. Before proving formula (6.13), we deduce the announced equivalence to the usual superstring S-matrix. To this end we recall that the S-matrix elements of our theory are calculated from the physical vertices $\mathbf{M}^{[0]}$, that is from the string products that are proportional to s^0. As the formula for the almost minimal model (6.7) does not involve any operations on s, we conclude that the coefficient of s^0 in $\mathcal{S}(\mathbf{M})$, $\mathcal{S}(\mathbf{M})^{[0]}$, must be identical to the S-matrix of our field theory. Let us consider the n-product $\mathcal{S}(\mathbf{M})_n$. Since picture deficit is additive when composing multilinear maps, the highest power in s of $\mathcal{S}(\mathbf{M})_n$ must be s^{n-1}. This is precisely the case when each vertex has maximal possible picture deficit. But these vertices are identical to the bosonic vertices, so that $\mathcal{S}(\mathbf{M})_n^{[n-1]}$ must calculate the bosonic S-matrix elements. In order to convert the $\mathcal{S}(\mathbf{M})$ into a real S-matrix, we need to convert its output into an input using the symplectic form on the small Hilbert space $\omega_S = \omega(\mathbb{I} \otimes \xi)$, where the right-hand side is expressed in terms of the large Hilbert space symplectic form ω. The S-matrix now reads as

$$S = \omega(\mathbb{I} \otimes \xi\pi_1\mathcal{S}(\mathbf{M})) = \sum_{k\geq 0} s^k S^{[k]}. \tag{6.14}$$

This linear functional has to be evaluated on vectors in the small Hilbert space that are generalised solutions to the equation $Q\phi = 0$. Moreover, this functional also has an expansion in terms of the variable s. Let us now look at the various coefficients of s in (6.13). The right-hand side is a Q-exact and η-exact coderivation, so that we find on $H^\bullet(Q|\eta)$,

$$\omega(\mathbb{I} \otimes \xi[\boldsymbol{\eta}, [\mathbf{Q}, \mathbf{T}]]) = \omega(\mathbb{I} \otimes [\mathbf{Q}, \mathbf{T}]) = 0,$$

where we used the compatibility of ω with both Q and η. Thus, we can deduce from equation (6.13) the recursion relation, $k \geq 0$,

$$(k+1)S^{[k+1]}X = \omega(\mathbb{I} \otimes \xi\,\pi_1\frac{\partial}{\partial t}\mathcal{S}(\mathbf{M})^{[k]}) = \frac{\partial}{\partial t}S^{[k]}. \tag{6.15}$$

In order to solve this hierarchy of differential equations, we fix the number of external states to $n+1$ and consider $S_{n+1} = S\iota_{n+1}$ and we find that it satisfies the following differential equations,

$$\begin{aligned}
\frac{\partial}{\partial t} S_{n+1}^{[0]} &= S_{n+1}^{[1]} X = \frac{1}{n+1} \sum_{r+s=n} S_{n+1}^{[1]} \left(\mathbf{1}^{\otimes r} \otimes X \otimes \mathbf{1}^{\otimes s} \right), \\
\frac{\partial}{\partial t} S_{n+1}^{[1]} &= 2 S_{n+1}^{[2]} X, \\
&\vdots \\
\frac{\partial}{\partial t} S_{n+1}^{[n-1]} &= n S_{n+1}^{[n]} X, \\
\frac{\partial}{\partial t} S_{n+1}^{[n]} &= 0.
\end{aligned} \tag{6.16}$$

In the last equation we obtain zero, because all vertices have maximal possible picture deficit. These equations can be integrated, if we use the initial conditions $S_{n+1}^{[n]}(0) = S_{n+1}^{\text{bos}}$ and $S_{n+1}^{[k]}(0) = 0$ for $k < n$, where S_{n+1}^{bos} denotes the bosonic $(n+1)$-S-matrix element. The result is,

$$S_{n+1}^{[0]}(t) = t^n S_{n+1}^{\text{bos}} X^{n-1}.$$

Moving around the picture changing operators X does not change the S-matrix because of equation (6.9). We can therefore distribute the PCOs such that each external leg has at most one X. Hence, the S-matrix $S^{[0]}$ can be calculated by taking all but two vertex operators in the 0-picture and the remaining two in the -1-picture. The functional S^{bos} then inserts these vertex operators at the boundary of a disc and integrates over the possible positions, essentially by the validity of the usual bosonic string field theory construction. Consequently, $S^{[0]}$ is identical to the perturbative string S-matrix, as claimed.

It remains to prove (6.13). Before we consider the completely general case, we concentrate on the 3-product in (6.13) for which the proof can be carried out by hand. In this case $\mathcal{S}(\mathbf{M})_3$ reads as

$$\mathcal{S}(\mathbf{M})_3 = \hat{P}(\mathbf{M}_3 + \mathbf{M}_2(-Q^\dagger \mathbf{M}_2 \otimes \mathbf{1} + \mathbf{1} \otimes -Q^\dagger \mathbf{M}_2))\hat{P}^{\otimes 3}.$$

Note that is equation is between polynomials in s, so that it actually represents multiple equations. We can apply two operators, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ to this equality,

$$\begin{aligned}
\frac{\partial}{\partial t} \mathcal{S}(\mathbf{M})_3 = \hat{P} \Bigg(& \frac{\partial}{\partial t} \mathbf{M}_3 + \frac{\partial}{\partial t} \mathbf{M}_2 (-Q^\dagger \mathbf{M}_2 \otimes \mathbb{I} + \mathbb{I} \otimes -Q^\dagger \mathbf{M}_2) \\
& + \mathbf{M}_2 (-Q^\dagger \frac{\partial}{\partial t} \mathbf{M}_2 \otimes \mathbb{I} + \mathbb{I} \otimes -Q^\dagger \frac{\partial}{\partial t} \mathbf{M}_2) \Bigg) \hat{P} \\
\frac{\partial}{\partial s} \mathcal{S}(\mathbf{M})_3 = \hat{P} \Big(& [\boldsymbol{\eta}, \boldsymbol{\mu}_3] + [\boldsymbol{\eta}, \boldsymbol{\mu}_2] (-Q^\dagger \mathbf{M}_2 \otimes \mathbb{I} + \mathbb{I} \otimes -Q^\dagger \mathbf{M}_2) \\
& + \mathbf{M}_2 (-Q^\dagger [\boldsymbol{\eta}, \boldsymbol{\mu}_2] \otimes \mathbb{I} + \mathbb{I} \otimes -Q^\dagger [\boldsymbol{\eta}, \boldsymbol{\mu}_2]) \Big) \hat{P} \\
= \hat{P} \Big(& [\boldsymbol{\eta}, \boldsymbol{\mu}_3 + \boldsymbol{\mu}_2 (-Q^\dagger \mathbf{M}_2 \otimes \mathbb{I} + \mathbb{I} \otimes -Q^\dagger \mathbf{M}_2) \\
& + \mathbf{M}_2 (-Q^\dagger \boldsymbol{\mu}_2 \otimes \mathbb{I} + \mathbb{I} \otimes -Q^\dagger \boldsymbol{\mu}_2)] \Big) \hat{P}.
\end{aligned}$$

In order to proceed, we apply $[\mathbf{Q}, \xi \circ \cdot]$ to the second equality and use the property $[\boldsymbol{\eta}, \xi \circ \cdot] + \xi \circ [\boldsymbol{\eta}, \cdot] = \mathbb{I}$ to find,

$$\begin{aligned}[\mathbf{Q}, \xi \circ \frac{\partial}{\partial s}\mathcal{S}(\mathbf{M})]_3 &= [\mathbf{Q}, [\boldsymbol{\eta}, \cdots]] + \hat{P}[\mathbf{Q}, \boldsymbol{\mu}_3 + \boldsymbol{\mu}_2(-Q^\dagger \mathbf{M}_2 \otimes \mathbb{I} + \mathbb{I} \otimes -Q^\dagger \mathbf{M}_2) \\ &\qquad + \mathbf{M}_2(-Q^\dagger \boldsymbol{\mu}_2 \otimes \mathbb{I} + \mathbb{I} \otimes -Q^\dagger \boldsymbol{\mu}_2)]\hat{P} \\ &= [\mathbf{Q}, [\boldsymbol{\eta}, \cdots]] + \frac{\partial}{\partial t}\mathcal{S}(\mathbf{M})_3,\end{aligned}$$

where the ellipsis corresponds to some irrelevant terms contributing to $\mathbf{T}$. Rearranging the latter equation a little bit yields,

$$X \circ \frac{\partial}{\partial s}\mathcal{S}(\mathbf{M})_3 - \frac{\partial}{\partial t}\mathcal{S}(\mathbf{M})_3 = [\mathbf{Q}, [\boldsymbol{\eta}, \cdots]].$$

During the calculation we made use of equation (6.8), which allows us to drop terms involving the physical projector P between operators when evaluating this expression on physical states, because generically the internal lines will be off-shell. Thus, for $n = 3$, equation (6.13) follows.

The general case can be derived analogously. The starting point is equation (6.7). The homotopy H is constructed from the propagator $-Q^\dagger$ and from the physical projector P via formula (6.3c). It follows then straightforwardly that

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{S}(\mathbf{M}) &= \hat{P}(1 - \mathbf{M}_{\text{int}}H)^{-1}\frac{\partial}{\partial t}\mathbf{M}H(1 - \mathbf{M}_{\text{int}}H)^{-1}\mathbf{M}_{\text{int}}\hat{P} + \hat{P}(1 - \mathbf{M}_{\text{int}}H)^{-1}\frac{\partial}{\partial t}\mathbf{M}\hat{P} \\ &= \hat{P}(1 - \mathbf{M}_{\text{int}}H)^{-1}[\mathbf{M}, \boldsymbol{\mu}]H(1 - \mathbf{M}_{\text{int}}H)^{-1}\mathbf{M}_{\text{int}}\hat{P} \\ &\qquad + \hat{P}(1 - \mathbf{M}_{\text{int}}H)^{-1}[\mathbf{M}, \boldsymbol{\mu}]\hat{P} \\ &= \hat{P}(1 - \mathbf{M}_{\text{int}}H)^{-1}[\mathbf{M}, \boldsymbol{\mu}](1 - H\mathbf{M}_{\text{int}})^{-1}\hat{P} \end{aligned} \tag{6.17}$$

$$\begin{aligned}\frac{\partial}{\partial s}\mathcal{S}(\mathbf{M}) &= \hat{P}(1 - \mathbf{M}_{\text{int}}H)^{-1}\frac{\partial}{\partial s}\mathbf{M}(1 - H\mathbf{M}_{\text{int}})^{-1}\hat{P} \\ &= \hat{P}(1 - \mathbf{M}_{\text{int}}H)^{-1}[\boldsymbol{\eta}, \boldsymbol{\mu}](1 - H\mathbf{M}_{\text{int}})^{-1}\hat{P} \\ &= [\boldsymbol{\eta}, \hat{P}(1 - \mathbf{M}_{\text{int}}H)^{-1}\boldsymbol{\mu}(1 - H\mathbf{M}_{\text{int}})^{-1}\hat{P}] \equiv [\boldsymbol{\eta}, \boldsymbol{\rho}],\end{aligned} \tag{6.18}$$

where in the last step we used the fact that the interaction term, the physical projector and the homotopy H commute with the coderivation $\boldsymbol{\eta}$. Note that $\boldsymbol{\rho}$ is a coderivation. Now, we solve (6.18) for $\boldsymbol{\rho}$ modulo η-exact terms using the contracting homotopy $\xi\circ$. We find that

$$\boldsymbol{\rho} = \xi \circ \frac{\partial}{\partial s}\mathcal{S}(\mathbf{M}) + [\boldsymbol{\eta}, \xi \circ \boldsymbol{\rho}]. \tag{6.19}$$

In order to produce a PCO instead of a ξ on the right-hand side, we calculate the commutator with the coderivation $\mathbf{Q}$. Calculating the commutator of $\mathbf{Q}$ with operators of the form $(1 - A)^{-1}$ is easy, once one recognises that the Leibniz rule for

$[\mathbf{Q},\cdot]$ implies that $[\mathbf{Q},(1-A)^{-1}] = (1-A)^{-1}[\mathbf{Q},A](1-A)^{-1}$. In our case we have $A = \mathbf{M}_{\text{int}}H$ and, hence,

$$[\mathbf{Q},\boldsymbol{\rho}] = \hat{P}(1-\mathbf{M}_{\text{int}}H)^{-1}[\mathbf{Q},\,\mathbf{M}_{\text{int}}H]\boldsymbol{\rho}\hat{P} + \hat{P}\boldsymbol{\rho}[\mathbf{Q},H\mathbf{M}_{\text{int}}](1-H\mathbf{M}_{\text{int}})^{-1}\hat{P}$$
$$+ \hat{P}(1-\mathbf{M}_{\text{int}}H)^{-1}[\mathbf{Q},\boldsymbol{\mu}](1-H\mathbf{M}_{\text{int}})^{-1}\hat{P}.$$

Note that $\mathbf{M}^2 = 0$ implies that $[\mathbf{Q},\mathbf{M}_{\text{int}}] = -\mathbf{M}_{\text{int}}^2$ and that H is a homotopy from $\mathbb{I}$ to $\hat{P}$, see (6.4). Therefore, the first and the second commutators yield

$$[\mathbf{Q},\mathbf{M}_{\text{int}}H] = \mathbf{M}_{\text{int}}\,(1-\mathbf{M}_{\text{int}}H) - \mathbf{M}_{\text{int}}\hat{P}, \tag{6.20a}$$
$$[\mathbf{Q},H\mathbf{M}_{\text{int}}] = -\,(1-H\mathbf{M}_{\text{int}})\,\mathbf{M}_{\text{int}} + \hat{P}\mathbf{M}_{\text{int}}. \tag{6.20b}$$

Using these results, we can simplify (6.20a) further and arrive at the identity

$$[\mathbf{Q},\boldsymbol{\rho}] = \hat{P}(1-\mathbf{M}_{\text{int}}H)^{-1}[\mathbf{M},\boldsymbol{\mu}](1-H\mathbf{M}_{\text{int}})^{-1}\hat{P} = \frac{\partial}{\partial t}\mathcal{S}(\mathbf{M}).$$

Using equation (6.19) together with the gauge-invariance of the S-matrix (6.9), we finally deduce a relation of the form

$$X \circ \frac{\partial}{\partial s}\mathcal{S}(\mathbf{M}) - \frac{\partial}{\partial t}\mathcal{S}(\mathbf{M}) = -[\mathbf{Q},[\boldsymbol{\eta},\xi\circ\boldsymbol{\rho}]], \tag{6.21}$$

from which the main equation (6.13) follows. This concludes the proof of equivalence of open superstring field theory from chapter 3 with the ordinary perturbative string S-matrix for open superstrings in the NS-sector.

6.4 Variations

In the previous section we presented a proof of the equivalence of open superstring field theory to usual perturbative string theory in the NS-sector. However, the homological perturbation theoretical proof is applicable to some other, closely related physical systems: The action of NS-NS sector of closed type II-superstring theory, cf. chapter 4, and the extension to the R-sectors at the level of the equations of motion, cf. chapter 5. In both cases the construction is obtained by integrating the flow generated by an exact homological vector field on the formal manifold of homotopy algebraic structures. Now, in both cases the fundamental equation (6.13) still holds true,

$$X \circ \frac{\partial}{\partial s}\mathcal{S}(\mathbf{M}) - \frac{\partial}{\partial t}\mathcal{S}(\mathbf{M}) = [\boldsymbol{\eta},[\mathbf{Q},\cdots]].$$

From the proof in section 6.3 this follows quite trivially, because we only assumed that $\frac{\partial}{\partial t}\mathbf{M} = [\mathbf{M},\mathbf{R}]$ and that $\frac{\partial}{\partial s}\mathbf{M} = [\boldsymbol{\eta},\mathbf{R}]$ for some $\mathbf{R}$.

6.4.1 Closed type II-superstring

On the world-sheet of a closed type II superstring we have holomorphic and anti-holomorphic degrees of freedom. The type II world sheet has both a holomorphic and antiholomorphic super Riemann surface structure and the world-sheet theory now comes with a holomorphic and an antiholomorphic picture number, both of which have to add up to -2 individually in order to obtain a well-defined correlator. In chapter 4 a holomorphic and an antiholomorphic picture deficit together with formal variables s and $\bar{s}$ were introduced. A generic coderivation $\mathbf{L}$ can then be expanded as

$$\mathbf{L} = \sum_{k,l\geq 0} s^k \bar{s}^l \mathbf{L}^{[k,l]},$$

where $\mathbf{L}^{[k,l]}$ has holomorphic picture deficit k and antiholomorphic picture deficit l.

Closed string products are graded-symmetric, hence, the underlying homotopy algebraic structure is an L_∞ algebra instead of an A_∞-algebra. However, it is possible to take the universal envelope of an L_∞-algebra [102] and obtain an A_∞-algebra to which the usual construction can be applied. Alternatively, one can think of the construction in the dual geometric picture and skip the universal enveloping algebra completely. Eventually, two vector fields δ and $\bar{\delta}$ were introduced,

$$\delta\mathbf{L} = [\mathbf{L}, \boldsymbol{\lambda}] \qquad \bar{\delta}\mathbf{L} = [\mathbf{L}, \bar{\boldsymbol{\lambda}}] \tag{6.22a}$$

$$[\boldsymbol{\eta}, \boldsymbol{\lambda}] = \frac{\partial}{\partial s}\mathbf{L} \qquad [\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{\lambda}}] = \frac{\partial}{\partial \bar{s}}\mathbf{L}. \tag{6.22b}$$

The equations (6.22b) were then solved using the special contracting homotopy for η or $\bar{\eta}$ that was built using the zero-modes of the ξ- or $\bar{\xi}$-fields. This was required to preserve the level matching constraints on the closed string state space. However, in the following we do not require this choice for $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$.

The closed string products are defined as a solution of equation (4.61) with the initial conditions $\mathbf{L} = \mathbf{L}_{\text{bos}}$ given by the closed string vertices of closed bosonic string field theory [17]. The main equation (6.22a) now tells us that

$$X \circ \mathcal{S}(\mathbf{L}) + \bar{X} \circ \mathcal{S}(\mathbf{L}) = \frac{\partial}{\partial t}\mathcal{S}(\mathbf{L}) + [\boldsymbol{\eta}, [\bar{\boldsymbol{\eta}}, [\mathbf{Q}, \cdots]]]. \tag{6.23}$$

The rest of the argument is very similar to the one given in section 6.3. We only work out the details for the four-point S-matrix elements here. The closed string S-matrix elements are calculated from $\mathcal{S}(\mathbf{L})$ using the symplectic form $\omega_S = \omega(\mathbb{I} \otimes \xi_0\bar{\xi}_0 c_0^-)$, where ω denotes the BPZ-inner product for the world-sheet theory formulated in the large Hilbert space. The S-matrix is then the restriction of the functional S

$$S = \omega_S(\mathbb{I} \otimes \mathcal{S}(\mathbf{L}))$$

to the relative cohomology $H^\bullet(Q|\eta, \bar{\eta})$. Equation (6.23) decomposes into a system of differential equations in the deformation parameter t by reading off the coefficients of terms homogeneous in s and $\bar{s}$:

$$\frac{\partial}{\partial t}S_4^{[0,0]}(t) = S_4^{[1,0]}(t)X + S_4^{[0,1]}(t)\bar{X}, \qquad \frac{\partial}{\partial t}S_4^{[1,0]}(t) = 2S_4^{[2,0]}(t)X + S_4^{[1,1]}(t)\bar{X},$$
$$\frac{\partial}{\partial t}S_4^{[0,1]}(t) = S_4^{[1,1]}(t)X + 2S_4^{[0,2]}(t)\bar{X}, \qquad \frac{\partial}{\partial t}S_4^{[1,1]}(t) = 2S_4^{[2,1]}(t)X + 2S_4^{[1,2]}(t)\bar{X},$$
$$\frac{\partial}{\partial t}S_4^{[2,0]}(t) = S_4^{[2,1]}(t)\bar{X}, \qquad \frac{\partial}{\partial t}S_4^{[0,2]}(t) = S_4^{[1,2]}(t)X,$$
$$\frac{\partial}{\partial t}S_4^{[2,1]}(t) = 2S_4^{[2,2]}(t)\bar{X}, \qquad \frac{\partial}{\partial t}S_4^{[1,2]}(t) = 2S_4^{[2,2]}(t)X,$$
$$\frac{\partial}{\partial t}S_4^{[2,2]}(t) = 0.$$

In the last equation we used the fact that the highest picture deficit for 3-products in this construction is $[2,2]$ so that there are no source terms of the last differential equation. Indeed, the functional $S_4^{[2,2]}$ is identical to the S-matrix calculated from bosonic CSFT described by the initial vertices. It is clear that this system of equations can be integrated directly and we can express the S-matrix $S_4^{[0,0]}$ in terms of the bosonic CSFT-S-matrix $S_{\text{bos},4} = S_4^{[2,2]}(0)$ and picture changing operators X and $\bar{X}$ located at the punctures,

$$S_4^{[0,0]} = S_{\text{bos},4}X^2\bar{X}^2.$$

Moreover, if the external states are on-shell, we can move the PCOs arbitrarily and may adjust them such that all external states are in the $(0,0)$ picture except for two that are in the $(-1,-1)$ picture.

6.4.2 Equations of motion for the Ramond fields

Formulating the dynamics of the Ramond string fields in the small Hilbert space using an action principle is still an open problem. Finding covariant equations of motion is a somewhat simpler problem and was solved recently using homotopy algebraic methods in chapter 5. In this subsection we only discuss the validity of the resulting equations of motion for the open superstring obtained from the stubified bosonic open string products from section 5.2. The extension to the closed type II superstring and the heterotic string contains no new conceptual ideas and we leave the details to the enthusiastic reader.

The string field $\phi = \phi_{\text{NS}} + \phi_{\text{R}}$ now takes values in the CFT state space $\mathcal{H}_{\text{NS}} \oplus \mathcal{H}_{\text{R}}$, where the NS field is at picture -1 and the R field is at picture $-\frac{1}{2}$. The final result of the construction of chapter 5 is given as the coderivation $\tilde{\mathbf{M}}$ in equation (5.52). The equations of motion are the Maurer-Cartan equations associated to $\tilde{\mathbf{M}}$. If these equations of motion came from an action S, it would correspond to the Euler-Lagrange vector field Q obtained as the Hamiltonian vector field associated to the action S by some degree -1 symplectic form, i.e. $dS = \iota_Q\omega$. However, even if the equations of motion do not derive from an action, it makes sense to discuss the structure of the space of solutions modulo gauge transformations. In the presence of an action principle the smoothness of the solution space is characterised by the classical S-matrix. More generally, the deviation from smoothness is measured by the minimal model of the homological vector field Q [89]. We therefore mimic the classical S-matrix by a symplectic form on the solution space. The minimal model is an A_∞ structure on $H^\bullet(Q|\eta)$ that is obtained by restricting $\mathcal{S}(\mathbf{M})$ to the cohomology. We may contract the minimal model structure with a non-degenerate symplectic form on $H^\bullet(Q|\eta)$ to obtain a linear functional S, which is the classical S-matrix of the equations of motion (5.44). In order to define said symplectic structure, we need to introduce an inverse picture changing operators Y that is required to be BPZ-even and a homotopy inverse of X. We now introduce an operator $\mathcal{O}$ by

$$\mathcal{O}\phi = \phi_{\rm NS} + Y\phi_{\rm R}.$$

The sought for symplectic form $\tilde{\omega}$ is now in terms of the large Hilbert space BPZ-inner product ω,

$$\tilde{\omega} = \omega(\mathbb{I} \otimes \xi\mathcal{O}).$$

It is readily checked that $\tilde{\omega}$ is Q-closed and, hence, descends to a non-degenerate pairing on $H^\bullet(Q|\eta)$. The S-matrix for a homological vector field $\mathbf{M}$ is then

$$S = \tilde{\omega}(\mathbb{I} \otimes \pi_1\mathcal{S}(\mathbf{M})). \tag{6.24}$$

The main difference to the construction for the pure NS-subsector is that we now have two component fields $\phi_{\rm NS}$ and $\phi_{\rm R}$ which carry different picture number. Thus, the required number of PCO insertions will depend on the sector of the inputs to a vertex. The problem was solved in section 5.2 and the final coderivation $\tilde{\mathbf{M}} = \mathbf{M}^{[0]}|_{t=1,u=1}$ was obtained through integrating a set of differential equations akin to equations (6.1) in formula (5.52). We are now ready to evaluate the S-matrix (6.24) in the same way as in section 6.3. The functional S is then equal to the bosonic S-matrix with vertex operators inserted in the correct picture if the output of $\mathcal{S}(\tilde{\mathbf{M}})$ is an NS-state. If it is an R-state, we can use one of the PCOs to remove the Y operator at the output and we still obtain the perturbative string S-matrix. Let us see how this works for the four-point amplitude of two R-states R_1 and R_2 with two NS-states NS_1 and NS_2. The relevant component of $S_4^{[0]}$ has Ramond number 0 and is given in terms of the bosonic S-matrix $\mathcal{S}(\mathbf{M}_{\rm bos})$ as

$$\begin{aligned}
S_4^{[0]}(R_1, R_2, NS_1, NS_2) &= \omega(R_1 \otimes \xi Y(X \circ X \circ \mathcal{S}(\mathbf{M}_{\rm bos})(R_2, NS_1, NS_2)) \\
&= \omega(XYR_1 \otimes \xi\mathcal{S}(\mathbf{M}_{\rm bos})(XR_2, NS_1, NS_2)) \\
&= S_{\rm bos}(R_1, XR_2, NS_1, NS_2).
\end{aligned}$$

This concludes our discussion of the validity of the construction of chapter 5 as valid Ramond equations of motion.

6.4.3 Relation to Berkovits' WZW-like theory

Our result has further implications. In [58] it was shown that the CS-like formulation of open super string field theory from chapter 3 is related to a gauge-fixed version of Berkovits' WZW-type super string field theory through a field redefinition. Now, since the S-matrix is invariant under field redefinitions up to a similarity transformation, our result states that the S-matrix of Berkovits' WZW-type formulation agrees with the usual perturbative super string S-matrix. Previously [51, 143, 150] some checks in this direction were performed, but remained restricted to the four-point and five-point S-matrix elements. Equivalence of CS-like heterotic string field theory and its WZW-like formulation has been studied recently in [137].

6.5 Summary

We showed the equivalence of perturbative string theory with the superstring field theories based on the small Hilbert space. This equivalence requires that the solution space of the linearised equations of motion coincides with the physical string spectrum and that the S-matrix around a given vacuum agrees with that of perturbative string theory. The first requirement was true by construction and we only had to show the second. In doing so, the special form of the cohomological vector field encoding the equations of motion was crucial: It allowed us to relate the S-matrix of the underlying bosonic string field theory to the real S-matrix by a sequence of descent equations (6.16) without employing complicated combinatorial arguments involving Feynman diagrams or world sheet diagrams.

Despite the progress at the algebraic level, there are still some open questions to address. For example, it would be interesting to see if and how the algebraic construction and properties arise from the world-sheet point of view. Since the formulation is entirely in terms of the small Hilbert space expressing the interaction vertices in terms of integrals over the moduli space of super Riemann surfaces should be easier than in the large Hilbert space formulations. However, even though formulated in terms of small Hilbert space fields, we still use the bosonised β-γ-ghosts. A first step towards a geometric formulation would be to reformulate the construction in terms of operators manifestly built from modes of the β and γ-ghosts and to find a geometric interpretation of the descent equations. Quantization of the theory necessitates an action principle. However, since except for the open superstring only equations of motion are known for the Ramond string fields, the first step must be to reformulate them in terms of an action principle. In turn we would need to find a suitable symplectic form of picture number -1 on the space of (off-shell) Ramond fields. Most likely such a construction would require a constraint on the Hilbert space, similar to the construction of closed string field theory. We propose a complete classical action principle for open superstrings in chapter 7.

Chapter 7

Open superstring field theory on the restricted Hilbert space

In this chapter we improve on the results from chapter 5, providing a construction of a classical gauge-invariant action for open superstring field theory including both NS- and R-sectors. In order to formulate a kinetic term, we either need to impose a restriction on the Ramond field or introduce an auxiliary field at picture $-\frac{3}{2}$.

This chapter is based on the paper **Open Superstring Field Theory on the Restricted Hilbert Space** by the author and I. Sachs [64].

7.1 Introduction

The problem of formulating an action for interacting covariant open superstring field theory has a long history, starting with Witten's cubic action [34].

This cubic theory has two short comings: One problem is the presence of singularities in the Neveu-Schwarz (NS) sector due to collisions of picture changing operators. Another issue is that the kinetic term (more precisely the inner product) is degenerate in the Ramond (R) sector. The first problem can be remedied by smearing out the picture changing operator, cf. chapter 3, see also [51] for earlier work in this direction. This results in a consistent, although non-polynomial BV-action for the NS sector of open superstring field theory on the small Hilbert space. An action for the NS sector on the other hand in the large Hilbert space has been formulated long time ago by Berkovits [41]. This theory is attractive due its simple form and is well suited for explicit calculations, e.g. [125], but its BV-quantization is less clear. However, recently it has been shown that Berkovits' theory is related to the BV-action on the small Hilbert space by a field redefinition [56, 58]. Furthermore, it was shown in chapter 6 that the non-polynomial BV-action (3.13) and thus the Berkovits action do reproduce the perturbative tree-level S-matrix to all orders. Hence the latter does indeed realise a decomposition of the supermoduli space.

For the combined theory of NS- and R-sectors gauge invariant field equations have been formulated in chapter 5 and shown to produce the correct tree-level S-matrix elements in chapter 6. But, due to the lack of cyclicity, of the multistring products these field equations cannot derive from an action. Moreover, the above-mentioned issue with the kinetic term in the Ramond (R) sector was not addressed in chapter 5. In [59] and [75] the degeneracy of the Ramond kinetic term was avoided with the help of a suitable restriction of the Ramond Hilbert space. Indeed, it was noticed [151] in the early days of string field theory that Witten's theory propagates only a subset of constrained string fields [152–157]. This was subsequently related to the presence of an extra gauge symmetry, not generated by the BRST charge. It can be fixed to remove all fields that do not satisfy the constraint [158], see also [159].

A gauge invariant action for the interacting theory was proposed in [75], see also [160], with smeared picture changing operators and Ramond fields in the restricted Hilbert space. Problems with cyclicity of the vertices were avoided by taking the the NS field to live in the large Hilbert space akin to the Berkovits formulation. On another front, in [59] a geometric approach, based on a decomposition of the supermoduli space was outlined, which is formulated in the small Hilbert space with a constrained Ramond sector. Furthermore, in [73] another geometric construction was proposed where the restriction on the Ramond fields is substituted by the introduction of auxiliary fields[1].

In this chapter we clarify first the relation between the restricted and unrestricted Ramond Hilbert spaces. In particular, we show explicitly that the restrictions used in [75] and [59] are the same and argue that the cohomology of the restricted Hilbert space is the same as that of the unrestricted space. The latter result was previously obtained in [161]. In the second part we carefully choose the contracting homotopy $\xi\circ$ used to construct the R-NS vertices (5.32) so that the vertices become cyclic on the small, restricted Hilbert space. Provided the picture changing operators used in [59, 75] can be defined in a way that is compatible with the interaction vertices, our construction immediately gives a classical action for the open superstring in the small, restricted Hilbert space. More generally, the vertices can be regarded as an algebraic construction of the interaction vertices of the auxiliary field construction of [73]. Then, invoking the results of chapter 6 one concludes that this action reproduces the correct tree-level S-matrix.

7.2 The restricted Hilbert space

Let us start with the restricted Ramond Hilbert space spanned by vectors of the form [75, 151–159]

$$\psi = \phi_1|{\downarrow}\rangle + \gamma_0\phi_2|{\downarrow}\rangle - (-1)^{|\phi_1|}G_0\phi_2|{\uparrow}\rangle\,, \tag{7.1}$$

where $|{\downarrow}\rangle = b_0|{\uparrow}\rangle$, $|\phi|$ denotes the Grassmann parity of ϕ, γ_0 is the zero mode of the commuting superconformal ghost and G_0 the (matter plus ghost) supercharge with

[1] In fact, the proposals [59] and [73] were worked out for the closed type II superstring but the idea is easily adapted to the open string, cf. chapter 2.

the $\gamma_0 b_0$ contribution subtracted. More concretely, we decompose the BRST charge Q as

$$Q = c_0 L_0 + b_0 M + \gamma_0 G_0 + \beta_0 K - \gamma_0^2 b_0 + \tilde{Q} \tag{7.2}$$

where $L_0, M, G_0, K, \tilde{Q}$ have no dependence on the ghost zero modes, see e.g. [159] for details. Using that $\{\tilde{Q}, G_0\} = 0$ and $G_0^2 = L_0$ it is not hard to see that

$$\begin{aligned} Q\psi = &\Big(M(G_0\phi_2) + K(\phi_2) + \tilde{Q}(\phi_1)\Big)\,|\downarrow\rangle + \gamma_0\left(G_0(\phi_1) + \tilde{Q}(\phi_2)\right)|\downarrow\rangle \\ &+ (-1)^{|\phi_1|}G_0\left(G_0(\phi_1) + \tilde{Q}(\phi_2)\right)|\uparrow\rangle\,. \end{aligned} \tag{7.3}$$

According to [159], ϕ_2 can be gauged away completely[2]. The closedness condition reduces to

$$\tilde{Q}\phi_1 = G_0\phi_1 = 0\,, \tag{7.4}$$

with a residual gauge freedom

$$\delta_\lambda\phi_1 = \tilde{Q}\lambda\,, \quad G_0\lambda = 0\,. \tag{7.5}$$

Let us now compare this with the cohomology of the unrestricted Ramond sector. Because the cohomology of Q is known to be isomorphic to the relative cohomology $H^\bullet_{rel}(Q)$ calculated on on the subspace defined by $b_0\psi = \beta_0\psi = 0$ [161, 162] we consider this case. A generic vector in this subspace is given by $\psi = \phi\,|\downarrow\rangle$ with ϕ independent of γ_0 and c_0. Then, $Q\psi = 0$ reduces to

$$\tilde{Q}\phi = G_0\phi = 0\,, \tag{7.6}$$

with the same residual gauge freedom as above. Thus the cohomology of the restricted Ramond sector (7.1) agrees with that of the unrestricted Ramond Hilbert space as previously shown in [161].

Next, we compare the restriction (7.1) with the approach of [59]. The constraint, originally formulated in [74], arose from the need to have a right-inverse Y_0 for the picture-changing operator

$$X_0 = (G_0 - 2\gamma_0 b_0)\delta(\beta_0) + b_0\delta'(\beta_0). \tag{7.7}$$

This operator acts on picture $-\frac{3}{2}$ states and existence of Y_0 implies that X_0 cannot have a cokernel[3]. This leads to the condition on picture $-\frac{1}{2}$ states ψ,

$$\beta_0^2\psi = 0 \tag{7.8}$$

[2]Notice however, that there are some subtleties when $G_0\phi_2 = 0$.

[3]Note that there is no well-established algebraic characterisation of the picture $-\frac{1}{2}$ states in terms of the modes of β and γ. For (7.7), one possible choice is to require that $\beta_k^{n_k}|\psi\rangle = \gamma_l^{m_l}|\psi\rangle = 0$ for $l > 0$ and $k \geq 0$ and natural numbers n_k and m_l. This is not a problem for free string field theory but becomes an issue in the presence of interaction vertices which generically do not preserve this definition.

with general solution,

$$\psi = \phi_1^{(0)}|\downarrow\rangle + \gamma_0\phi_1^{(1)}|\downarrow\rangle + \phi_2^{(0)}|\uparrow\rangle + \gamma_0\phi_2^{(1)}|\uparrow\rangle \tag{7.9}$$

where $\phi_i^{(j)}$ are independent of γ_0 and c_0. Now requiring that the condition (7.8) is preserved by Q implies that $\phi_2^{(1)} = 0$ and $\phi_2^{(0)} = -(-1)^{|\phi_1^{(1)}|}G_0\phi_1^{(1)}$ and thus (7.8) and (7.1) define the same invariant subspace. Finally we note that X_0 is indeed no cokernel, i.e. every vector in this subspace can be written as $\psi = X_0\tilde{\psi}$, where $\tilde{\psi}$ is an arbitrary string field with picture $-\frac{3}{2}$. This follows from the identities [74]

$$\delta(\gamma_0) = |0,-\frac{3}{2}\rangle\langle 0,-\frac{3}{2}| \tag{7.10a}$$

$$\delta(\beta_0) = |0,-\frac{1}{2}\rangle\langle 0,-\frac{1}{2}|, \tag{7.10b}$$

$$\delta'(\beta_0) = -|0,-\frac{1}{2}\rangle\langle 1,-\frac{1}{2}| + |1,-\frac{1}{2}\rangle\langle 0,-\frac{1}{2}| \tag{7.10c}$$

where the index $-\frac{1}{2}$ resp. $-\frac{3}{2}$ denotes the picture and $|n,-\frac{1}{2}\rangle = \gamma_0^n|0,-\frac{1}{2}\rangle$. Then, for $\tilde{\psi} = \phi_1|\downarrow\rangle + \phi_2|\uparrow\rangle$ with $\phi_i = \sum_{n=0}^{\infty}\beta_0^n\phi_i^{(n)}\delta(\gamma_0)$ we find

$$X_0\tilde{\psi} = \left(G_0(\phi_1^{(0)}) - (-1)^{|\phi_2|}\phi_2^{(1)}\right)|\downarrow\rangle - (-1)^{|\phi_2|}\gamma_0\phi_2^{(0)}|\downarrow\rangle + G_0\phi_2^{(0)}|\uparrow\rangle \tag{7.11}$$

where we have used that $\delta(\gamma_0)\delta(\beta_0) = |0,-\frac{3}{2}\rangle\langle 0,-\frac{1}{2}|$. We then see that $X_0\tilde{\psi}$ is indeed of the form (7.1) with

$$\phi_1 = G_0(\phi_1^{(0)}) - (-1)^{|\phi_2|}\phi_2^{(1)} \qquad \text{and} \qquad \phi_2 = (-1)^{|\phi_2|}\phi_2^{(0)}\,. \tag{7.12}$$

7.3 Open superstring field theory

The vertices of open superstring field theory can be written as

$$C_n(\Psi_1,\cdots,\Psi_n) = \omega(\Psi_1, M_{n-1}(\Psi_2,\cdots,\Psi_{n-1})), \tag{7.13}$$

where Ψ denotes a combined string field in the R- and NS-sector and M_n are string n-products. These products are constructed through a gauge transformation of the free theory defined by a hierarchy of gauge products on the large Hilbert space with each gauge product obtained from lower order products by means of a contracting homotopy ξ for the nilpotent operator η_0. More precisely, we require the existence of an operator ξ such that $[\eta_0,\xi] = 1$. Upon changing ξ, the construction produces actions that are related by field redefinitions, so that any choice for ξ is equally good. One additional condition on ξ is that the resulting vertices should be non-singular. In chapter 3 a class of such good homotopies built out of

$$\xi = \oint\frac{dz}{2\pi i}f(z)\xi(z) \tag{7.14}$$

was proposed, where $f(z)$ is required to be holomorphic in some annulus that contains the unit circle.

In chapter 5 the homotopy for $[\eta_0, \cdot]$ was taken to be the same irrespective of whether the string products defining the string products have zero or one Ramond input. To illustrate this we consider the string product

$$M_2 = \frac{1}{3}\{X, m_2\}P_2^{<0>} + Xm_2P_2^{<1>} + m_2P_2^{<2>} \tag{7.15}$$

where $P_2^{<n>}$ is the projector on n Ramond inputs among the two inputs of m_2 and $m_2 = *$ is Witten's star product. The picture changing operator, X is related to ξ through the graded commutator, $X = [Q, \xi]$. Finally, $\{X, m_2\}$ is the graded anti-commutator of X and m_2. For zero Ramond inputs M_2 is cyclic with respect to the standard symplectic form by construction since the combination $\{X, m_2\}$ sums over all possible insertions of a picture changing operator. For vertices involving two Ramond fields we have

$$\omega(N, M_2(R, R)) \ = \ \omega(N, m_2(R, R)) = \omega(R, m_2(R, N)) \tag{7.16}$$

where N and R denote NS- and R- string fields respectively. At first sight it looks as if M_2 were not cyclic since there is an X missing in front of m_2 on the right hand side of (7.16). However, we will see in the end that this is exactly what we need, because of subtleties in defining a symplectic form on the R-string fields.

Next, let us consider the 4-vertex. First, we have from (7.15)

$$[M_2, M_2](R, R, R) = 2Xm_2 \circ m_2(R, R, R) = 0 \tag{7.17}$$

due to associativity of the star product ($m_2 \circ m_2 = 0$). Thus, to this order the A_∞ consistency condition, or equivalently the BV-equation, allows us to set $M_3(R, R, R) = 0$. For two Ramond inputs we have

$$\frac{1}{2}[M_2, M_2](R, N, R) = m_2 \circ Xm_2(R, N, R) = -[Q, [m_2, \mu_2]](R, N, R)\,,$$

where

$$\mu_2 = \xi m_2 P_2^{<1>} + \frac{1}{3}\{\xi, m_2\}P_2^{<0>}\,. \tag{7.18}$$

Since the gauge products μ_n never have more than one Ramond input, the A_∞ consistency condition, $\frac{1}{2}[M_2, M_2] + [Q, M_3] = 0$, then fixes M_3 completely as

$$M_3(R, N, R) = m_3(R, N, R)\,, \tag{7.19}$$

where $m_3 = [m_2, \mu_2]$ and we have used associativity of m_2. Associativity then also implies that $\eta M_3(R, N, R) = -\eta[m_2, \mu_2](R, N, R) = 0$ and thus M_3 is in the small Hilbert space.

Similarly, for one Ramond input

$$\begin{aligned}\frac{1}{2}[M_2, M_2](N, R, N) &= Xm_2 \circ Xm_2(N, R, N)\\ &= -\frac{1}{2}[Q, [Xm_2 P_2^{<1>}, \mu_2 P_2^{<1>}]](N, R, N) = -\frac{1}{2}[Q, [M_2, \mu_2 P_2^{<1>}](N, R, N)\\ &= -\frac{1}{2}[Q, [M_2, \mu_2]](N, R, N)\,. \end{aligned} \tag{7.20}$$

To continue we choose the homotopy for η defining the gauge product μ_3 as

$$\mu_3 = \frac{1}{4}\{\xi, m_3\} P_3^{<0>} + \xi m_3 P_3^{<1>}\,. \tag{7.21}$$

Then,

$$\mu_3(N, R, N) = \xi m_3(N, R, N) = \xi m_2 \circ \xi m_2(N, R, N)\,. \tag{7.22}$$

Using, associativity of m_2 again we then find

$$\begin{aligned} M_3(N, R, N) &= \frac{1}{2}\left([M_2, \mu_2] + [Q, \mu_3]\right)(N, R, N)\\ &= M_2^{<1>}\mu_2(N, R, N) = Xm_2^{<1>}\mu_2(N, R, N)\\ &= Xm_3 P_3^{<1>}(N, R, N) \end{aligned} \tag{7.23}$$

which is in the small Hilbert space. More generally, for a generic permutation of the R- and NS inputs

$$M_3 P_3^{<1>} = Xm_2^{<1>}\mu_2 P_3^{<1>} = Xm_3 P_3^{<1>} \tag{7.24}$$

holds. Thus, modulo the factor X that will be dealt with below, proving cyclicity of M_3 is reduced to show cyclicity of m_3. Explicitly, we have

$$\begin{aligned} \omega(N_1, M_3(R_1, N_2, R_2)) &= \omega(N_1, m_3(R_1, N_2, R_2))\\ &= \omega_L(N_1, \xi_0 m_2(\xi m_2(R_1, N_2), R_2))\\ &\quad + \omega_L(N_1, \xi_0 m_2(R_1, \xi m_2(N_2, R_2)))\,, \end{aligned} \tag{7.25}$$

where ω_L is the symplectic form evaluated in the large Hilbert space and which reproduces the symplectic form, ω, on the small Hilbert space upon insertion of the zero mode ξ_0. Now, commuting ξ_0 through to R_1 and using cyclicity of m_2 we get

$$\begin{aligned} \omega(N_1, M_3(R_1, N_2, R_2)) &= \omega_L(\xi m_2(\xi_0 R_1, N_2), m_2(R_2, N_1))\\ &\quad + \omega_L(\xi_0 R_1, m_2(\xi m_2(N_2, R_2), N_1))\,. \end{aligned} \tag{7.26}$$

Since ξ is BPZ-even we then have

$$\begin{aligned} \omega(N_1, M_3(R_1, N_2, R_2)) &= \omega_L(m_2(\xi_0 R_1, N_2), \xi m_2(R_2, N_1))\\ &\quad + \omega_L(\xi_0 R_1, m_2(\xi m_2(N_2, R_2), N_1))\\ &= \omega_L(\xi_0 R_1, m_2(N_2, \xi m_2(R_2, N_1)))\\ &\quad + \omega_L(\xi_0 R_1, m_2(\xi m_2(N_2, R_2), N_1))\\ &= \omega(R_1, m_3(N_2, R_2, N_1))\,. \end{aligned} \tag{7.27}$$

Similarly, for two adjacent Ramond inputs,

$$\begin{aligned}\omega(N_1, M_3(R_1,R_2,N_2)) &= \omega(N_1, m_2(R_1, \mu_2(R_2,N_2))) \\ &\quad - \omega(N_1, \mu_2(m_2(R_1,R_2),N_2)) \\ &= -\omega_L(N_1, m_2(\xi_0 R_1, \mu_2(N_2,R_2))) \\ &\quad - \omega_L(N_1, \mu_2(m_2(\xi_0 R_1,R_2),N_2)) \,. \end{aligned} \tag{7.28}$$

Now, for the first term we use cyclicity of m_2 while for the second we use cyclicity of μ_2 for two R-inputs which gives

$$\begin{aligned}\omega(N_1, M_3(R_1,R_2,N_2)) &= \omega_L(\xi_0 R_1, m_2(\mu_2(R_2,N_2),N_1)) \\ &\quad + \omega_L(m_2(\xi_0 R_1,R_2), \mu_2(N_2,N_1)) \\ &= \omega_L(R_1, \xi_0 m_2(\mu_2(R_2,N_2),N_1)) \\ &\quad + \omega_L(R_1, \xi_0 m_2(R_2, \mu_2(N_2,N_1))) \\ &= \omega(R_1, m_3(R_2,N_2,N_1)) \,. \end{aligned} \tag{7.29}$$

Thus, m_3 is cyclic with respect to the symplectic form $\omega(\cdot,\cdot)$. In order to prove cyclicity to arbitrary order we first recall the recursion relations defining the higher order products (5.32). For zero or one Ramond input we have

$$M^{<0/1>}_{n+2} = \frac{1}{n+1}\sum_{k=0}^{n}[M_{k+1},\mu_{n-k+2}]P^{<0/1>}_{n+2} \,, \qquad M_1 = Q \tag{7.30}$$

and for two Ramond inputs

$$M^{<2>}_{n+3} = m_{n+3}P^{<2>}_{n+3} = \frac{1}{n+1}\sum_{k=0}^{n}[m_{k+2},\mu_{n-k+2}]P^{<2>}_{n+3} \tag{7.31}$$

where

$$m_{n+3} = \frac{1}{n+1}\sum_{k=0}^{n}[m_{k+2},\mu_{n-k+2}] \tag{7.32}$$

with $m_2 = *$. Finally, the gauge products μ_n are given by

$$\mu_{n+2} = \frac{1}{n+3}\{\xi, m_{n+2}\}P^{<0>}_{n+2} + \xi m_{n+2}P^{<1>}_{n+2} \,. \tag{7.33}$$

Mathematical induction shows that from $M_3(R,R,R) = 0$ it immediately follows the vanishing of $M_{n+3}(\cdots,R,\cdots R,\cdots,R,\cdots)$ for all n. Indeed, upon inspection of equations (7.32) and (7.33), it is apparent that such a term would have to be of the form $\xi\sum_{k=0}^{n} m_{n-k+2}m_{k+2}$ which vanishes due to the A_∞ condition $[m,m]=0$. Furthermore, it holds that

$$(n-1)M^{<1>}_{n+1} = X\left(m^{<1>}_{n}\mu_2 + m^{<1>}_{n-1}\mu_3 + \cdots\right) = (n-1)Xm_{n+1}P^{<1>}_{n+1} \,. \tag{7.34}$$

To show this identity we proceed by induction. We have from (7.30)

$$\begin{aligned} nM_{n+1}^{<1>} = [M_n^{<1>}, \mu_2^{<1>}] + [M_{n-1}^{<1>}, \mu_3^{<1>}] + \cdots + [Q, \mu_{n+1}^{<1>}] \\ + M_n^{<1>}\mu_2^{<0>} + M_{n-1}^{<1>}\mu_3^{<0>} + \cdots \\ - \mu_2^{<1>}M_n^{<0>} - \mu_3^{<1>}M_{n-1}^{<0>} + \cdots . \end{aligned} \tag{7.35}$$

Now, we use $[Q, \mu_p^{<1>}] = Xm_p^{<1>} - \xi[Q, m_p^{<1>}]$ together with the identity, $[m, M] = 0$, that is,

$$\begin{aligned} [Q, \mu_{n+1}^{<1>}] = Xm_{n+1}^{<1>} + \xi \Big([m_n^{<1>}, M_2^{<1>}] + [m_{n-1}^{<1>}, M_3^{<1>}] + \cdots \\ + M_2^{<1>}m_n^{<0>} + M_3^{<1>}m_{n-1}^{<0>} + \cdots \\ + m_n^{<1>}M_2^{<0>} + m_{n-1}^{<1>}M_3^{<0>} + \cdots \Big) . \end{aligned} \tag{7.36}$$

Upon substitution of (7.36) into (7.35) and using (7.33) as well as $[m, m] = 0$ the result follows.

Thanks to equations (7.31) and (7.34) the problem of proving cyclicity of M_n is again reduced to show cyclicity of m_n. To prove cyclicity of m_{n+3}, $n \geq 1$, one proceeds exactly as in (7.25)-(7.29) expressing m_{n+3} in terms of $[m_{k+2}, \mu_{n-k+2}]$ and then using cyclicity of m_q, $q \leq n+2$ as well as cyclicity of μ_p, $p \leq n+2$ for p NS-inputs.

Let us now explain how these vertices lead to a gauge-invariant action for the open superstring in the small Hilbert space. Following [73] we write

$$\begin{aligned} S = \frac{1}{2}\omega(\phi, Q\phi) - \frac{1}{2}\omega(\tilde{\psi}, XQ\tilde{\psi}) + \omega(\tilde{\psi}, Q\psi) \\ + \frac{1}{3}\omega(\Psi, \mathcal{M}_2(\Psi, \Psi)) + \frac{1}{4}\omega(\Psi, \mathcal{M}_3(\Psi, \Psi, \Psi)) + \cdots \end{aligned} \tag{7.37}$$

where, $\Psi = \phi + \psi$ and $\tilde{\psi}$ is an auxiliary Ramond string field with picture $-\frac{3}{2}$. The higher string products $\mathcal{M}_n$ are given by

$$\mathcal{M}_n = M_n P^{<0>} + m_n(P^{<1>} + P^{<2>}) \tag{7.38}$$

which differs from (7.15) by the ubiquitous factor X. To prove gauge invariance we use that $\mathcal{M}_n$ is cyclic w.r.t. ω. The standard proof of gauge-invariance has to be modified as $\mathcal{M}$ is not an A_∞-algebra. However, M is an A_∞-algebra and differs from $\mathcal{M}$ in that it contains an additional X-insertion on Ramond outputs and contains no BRST operator Q. There are three different types of gauge-transformations with odd parameters Λ, λ and $\tilde{\lambda}$ having picture -1, $-\frac{1}{2}$ and $-\frac{3}{2}$.

Using antisymmetry of ω and cyclicity of $\mathcal{M}_n$ one arrives at the identities, for $n, k \geq 2$,

$$\begin{aligned} \omega(\Lambda, M_n \circ M_k) &= \omega(\Lambda, \mathcal{M}_n \circ M_k) \\ &= \omega(\mathcal{M}_n\boldsymbol{\Lambda}, P_1^{<0>}\mathcal{M}_k + XP_1^{<1>}\mathcal{M}_k) = \omega(M_n\boldsymbol{\Lambda}, \mathcal{M}_k), \end{aligned} \tag{7.39a}$$

$$\omega(\Lambda, QM_k) = \omega(Q\Lambda, M_k) = \omega(Q\boldsymbol{\Lambda}, \mathcal{M}_k), \tag{7.39b}$$

$$\omega(\Lambda, M_n \circ Q) = \omega(\mathcal{M}_n\boldsymbol{\Lambda}, Q). \tag{7.39c}$$

where $\mathbf{\Lambda}$ denotes the coderivation built from Λ as its 0-string map and we suppressed the string field Ψ. Explicitly, (7.39c) reads as

$$\omega(\Lambda, M_n(Q\Psi, \ldots, \Psi) + M_n(\Psi, Q\Psi, \ldots, \Psi) + \cdots) \\ = \omega(\mathcal{M}_n(\Lambda, \Psi, \ldots, \Psi) + \mathcal{M}_n(\Psi, \Lambda, \ldots, \Psi) + \ldots, Q\Psi).$$

Define the transformation $\delta\phi$, $\delta\psi$, $\delta\tilde{\psi}$ as

$$\delta\phi + \delta\tilde{\psi} = Q\Lambda + \sum_{n\geq 2} \mathcal{M}_n \mathbf{\Lambda}(e^{\Psi}), \tag{7.40a}$$

$$\delta\psi = X\delta\tilde{\psi}. \tag{7.40b}$$

Summing equations (7.39) we obtain zero on the left-hand side due to the A_∞ relations, while on the right-hand side we find,

$$\begin{aligned} 0 &= \omega(\delta\phi, Q\phi) + \omega(\delta\tilde{\psi}, Q\psi) + \sum_{k\geq 2} \omega((\delta\phi + \delta\psi), \mathcal{M}_k(\Psi, \Psi, \ldots, \Psi)) \\ &= \delta\left(\frac{1}{2}\omega(\phi, Q\phi) + \omega(\tilde{\psi}, Q\psi) + \sum_{k\geq 2} \frac{1}{k+1}\omega(\Psi, \mathcal{M}_k(\Psi, \Psi, \ldots, \Psi))\right) - \omega(\tilde{\psi}, Q\delta\psi) \\ &= \delta S, \end{aligned} \tag{7.41}$$

where we used $\omega(\tilde{\psi}, Q\delta\psi) = \delta\left(\frac{1}{2}\omega(\tilde{\psi}, QX\tilde{\psi})\right)$ in the last step. Consequently, the transformations (7.40) are a bosonic gauge symmetry of the action. By replacing Λ with $\tilde{\lambda}$ in (7.39) one verifies that the following transformation is a fermionic gauge symmetry,

$$\delta\phi + \delta\tilde{\psi} = Q\tilde{\lambda} + \sum_{n\geq 2} \mathcal{M}_n \mathbf{X}\tilde{\boldsymbol{\lambda}}(e^{\Psi}), \tag{7.42a}$$

$$\delta\psi = X\delta\tilde{\psi}, \tag{7.42b}$$

where $\mathbf{X}\tilde{\boldsymbol{\lambda}}$ denotes the coderivation with 0-string product $X\lambda$.

In order to derive the gauge transformations corresponding to the parameter λ, let us recall that M_n and $m_n(P^{<0>} + P^{<1>})$ give two commuting A_∞ structures, cf. chapter 5. Together with cyclicity of $m_n(P^{<0>} + P^{<1>})$ w.r.t. ω one can then deduce that the following transformations are a gauge symmetry of S, by imitating the previous derivation,

$$\delta\phi + \delta\tilde{\psi} = \sum_{n\geq 2} \mathcal{M}_n \boldsymbol{\lambda}(e^{\Psi}), \tag{7.43a}$$

$$\delta\psi = Q\lambda + X\delta\tilde{\psi}. \tag{7.43b}$$

Notice that all gauge transformations preserve the constraint $\psi = X\tilde{\psi}$ up to states of the form $Q\lambda$ with λ not expressible in the form $\lambda = X\rho$ for some picture $-\frac{3}{2}$ state ρ.

Let us now comment on the applicability of our formalism to writing the proposal for the superstring action [75] in the small Hilbert space. Assuming the constraint (7.8), we can rewrite (7.37) without the need for the auxiliary field $\tilde{\psi}$ as

$$S = \frac{1}{2}\omega(\phi, Q\phi) + \frac{1}{2}\omega(\psi, YQ\psi) + \frac{1}{3}\omega(\Psi, \mathcal{M}_2(\Psi, \Psi)) + \frac{1}{4}\omega(\Psi, \mathcal{M}_3(\Psi, \Psi, \Psi)) + \cdots \tag{7.44}$$

where $Y = c_0\delta'(\gamma_0)$ is the inverse picture changing operator in the restricted Hilbert space. The gauge transformation of this action agrees with that of (7.37) up to the contribution coming from the kinetic term that is

$$\delta S \propto \omega((X - X_0)(m_2(\Psi, \Lambda) + m_2(\Lambda, \Psi) + m_3(\Psi, \Lambda, \Psi + \cdots)), YQ\psi) \tag{7.45}$$

Formally this term can be removed by replacing X by X_0 (as well as ξ by $\Theta(\beta_0)$) in the definition of the higher string products M_n and the gauge products μ_n when applied to states containing one or two Ramond states, e.g. instead of (7.15) we take

$$M_2 = \frac{1}{3}\{X, m_2\}P_2^{<0>} + X_0 m_2 P_2^{<1>} + m_2 P_2^{<2>} \tag{7.46}$$

and instead of (7.18) we take

$$\mu_2 = \Theta(\beta_0) m_2 P_2^{<1>} + \frac{1}{3}\{\xi, m_2\}P_2^{<0>}\,. \tag{7.47}$$

However, for this choice of homotopy to be well defined, one needs that the m_ns are compatible with the particular realisation of the picture $-\frac{1}{2}$ states in terms of the zero modes β_0 and γ_0 described in section 7.2.

7.4 Summary

In this chapter we constructed an action principle (7.37) for classical open superstrings. The string field Ψ contains bosonic fields at picture -1 and fermionic fields at picture $-\frac{1}{2}$. Moreover, it involves an auxiliary field $\tilde{\psi}$ at picture $-\frac{3}{2}$. The purpose of $\tilde{\psi}$ is to perform the integration over the odd modulus present when sewing two R punctures at picture $-\frac{1}{2}$ as it effectively inserts an additional picture changing operator at every internal R line. The results from chapter 6 imply that this action principle reproduces the correct open superstring scattering amplitudes.

Alternatively, one can remove $\tilde{\psi}$ by taking the picture $-\frac{1}{2}$ components of Ψ to live in the restricted Hilbert space of section 7.2. In this formulation one works with a minimal number of fields. Gauge invariance dictates now that one should formally take $X = X_0$. But it is not clear whether this choice makes sense beyond the formal level. On the other hand, the restricted Hilbert space is well motivated by geometrical considerations. Thus it would be interesting to find a precise geometric interpretation of (7.37) in terms of type II world sheet structures.

Conclusions

The main objective of this thesis has been the construction of classical superstring field theories. String field theories provide a connection between field theories and string theories. String theories are defined in the first quantised picture, in which a Hilbert space of free asymptotic string states and an S-matrix is defined. The definition of the S-matrix as an integral over supermoduli space gives rise to power series in the string coupling constant and is intrinsically perturbative. In this first quantised picture string dynamics can thus be visualised by strings propagating through spacetime and interacting by joining and splitting. Perturbative field theories have a similar interpretation in terms of Feynman diagrams. One can visualise such diagrams as evolution of point particles through spacetime and interactions are described by a predefined set of vertices encoded in the action. The advantage of field theories is the fewer amount of parameters. Instead of defining all scattering amplitudes individually, specification of the action automatically gives a consistent set of scattering amplitudes. Moreover, field theories allow one to replace the sum over Feynman diagrams by a proper path integral and give rise to a non-perturbative definition of the theory. String field theory formulates conventional string perturbation theory as a sum over Feynman diagrams of a field theory action. The key observation is that the moduli space of world sheets is non-compact and the way world-sheets degenerate at infinity is reminiscent of the factorisation of Feynman diagrams through the Deligne-Mumford compactification of the moduli space. Since the world-sheet path integral respects these factorisation properties, one can translate gauge-invariance of the S-matrix into a set of algebraic vertices satisfying a classical BV-master equation. With this observation the construction of string field theories becomes entirely algebraic: find a solution to the BV master equation and show that the associated perturbative S-matrix reproduces the conventional string S-matrix.

This program was subsequently executed for open superstring theory, heterotic string theory and type II superstring theory. The solutions were found recursively. The procedure starts with string vertices for the underlying bosonic string field theory and dresses them interatively with picture changing operators and ξ-ghosts. For the pure Neveu-Schwarz fields the prescription is relatively straightforward and gives rise to solutions to BV master equations. The appearance of picture changing operators on internal Ramond lines complicated the inclusion of the Ramond fields, but it was still possible to find a solution to the A_∞-/L_∞-consistency conditions.

Thus gauge-invariant equations of motion exist for all types of superstring theories. We discussed the realisation of spacetime supersymmetry for open superstrings. The validity of the solutions were checked by evaluating the classical S-matrix. Algebraically, the S-matrix is closely related to the minimal model of the homotopy algebras. We showed that this S-matrix coincides with the traditional string S-matrix. For open superstrings it was possible to beyond equations of motions for the Neveu-Schwarz fields and formulate an action principle. In doing so we either had to impose an algebraic constraint on the Ramond string field or add an auxiliary string field. In both cases we formulated a gauge-invariant action principle.

Despite this progress some questions still remain open. The recursive solution to the consistency conditions is not very satisfactory on the conceptual level. In string theory and bosonic string field theory, the origin of the algebraic structures can be traced back to properties of geometric or topological structures on the world sheet. We have no such explanation for the superstring vertices. Although we argued in favour of such a geometric origin, the actual construction makes no use of such. In bosonic string field theory the vertices, i.e. disjoint regions of the moduli space equipped with coordinate discs, are found by the unique solution of a minimal area metric problem. It would therefore be very interesting to see if the algebraic construction can be lifted to the geometrical level by an analogous problem. Open bosonic string field theory action is polynomial in the string field, but the recursive construction gives rise to a non-polynomial action for open superstring field theory. Because there is a field redefinition that makes the equations of motion polynomial and turns the small Hilbert space constraint into a non-linear constraint, the non-polynomiality appears to be unnatural and seems to be an artefact of the usage of the small Hilbert space. It would therefore be very interesting to ascertain the role of the large Hilbert space and its connection with a possible geometric construction of the vertices. Progress on this question may also give precious insight on the quantization of the theory. A solution to the quantization problem would require finding a solution to the quantum BV-master equation or, equivalently, construct a loop-homotopy algebra and then define the full string path-integral through a gauge-fixing of this master equation. Without spurious singularities it is conceivable that the construction could be generalised to quantum superstrings. Unfortunately, spurious singularities may give rise to an obstruction. If successful, this program gives rise to a non-perturbative definition of superstring theory.

On the mathematical side we observe that the classical superstring field theories seem to carry more structure than just homotopy algebras. For example, the recursive construction finds the vertices starting from the bosonic products with the highest picture deficit and reduces it subsequently. Very interestingly, the formal sum over all picture deficits $\mathbf{M}(s)$ satisfies the A_∞-relations itself. If we set $\mathbf{N}^{[p]}(v) = (-1)^{\mathrm{pic}(v)+l}\mathbf{M}^{[p]}(v), v \in \mathcal{H}^{\otimes l}$, the A_∞-relations are equivalent to $\sum_{p,q}(-1)^p\mathbf{N}^{[p]}\mathbf{N}^{[q]} = 0$. Note that $\mathbf{N}^{[p]}$ is not a coderivation. The latter equation tells us that $\mathbf{N}$ defines a *derived A_∞ algebra* [163]. Such a structure seems to be unique to the superstring and related to the picture changing operation and it would be interesting to investigate the significance of this structure for superstring field theory and find interpretation of this and other structures within the first quantised picture.

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