Forest-Fire Models and Their Critical Limits

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Abstract

Forest-fire processes were first introduced in the physics literature as a toy model for self-organized criticality. The term self-organized criticality describes interacting particle systems which are governed by local interactions and are inherently driven towards a perpetual critical state. As in equilibrium statistical physics, the critical state is characterized by long-range correlations, power laws, fractal structures and self-similarity.

We study several different forest-fire models, whose common features are the following: All models are continuous-time processes on the vertices of some graph. Every vertex can be “vacant” or “occupied by a tree”. We start with some initial configuration. Then the process is governed by two competing random mechanisms: On the one hand, vertices become occupied according to rate 1 Poisson processes, independently of one another. On the other hand, occupied clusters are “set on fire” according to some predefined rule. In this case the entire cluster is instantaneously destroyed, i.e. all of its vertices become vacant.

The self-organized critical behaviour of forest-fire models can only occur on infinite graphs such as planar lattices or infinite trees. However, in all relevant versions of forest-fire models, the destruction mechanism is a priori only well-defined for finite graphs. For this reason, one starts with a forest-fire model on finite subsets of an infinite graph and then takes the limit along increasing sequences of finite subsets to obtain a new forest-fire model on the infinite graph.

In this thesis, we perform this kind of limit for two classes of forest-fire models and investigate the resulting limit processes. The text is structured as follows:

In **Chapter 1**, we give a brief introduction to self-organized criticality and present some previous results on forest-fire models.

In **Chapter 2**, we consider the following forest-fire model on the upper half-plane of the two-dimensional square lattice $\mathbb{Z}^2$: At time 0 all vertices are vacant. Then trees grow at rate 1, independently for all vertices. If an occupied cluster reaches the boundary of the upper half-plane or if it is about to become infinite, the cluster is instantaneously destroyed. Additionally, we demand that the model is invariant under translations along the x-axis.

We prove that such a model exists and arises naturally as a subsequential limit of forest-fire processes in finite boxes when the box size tends to infinity.

Moreover, the model exhibits a phase transition in the following sense: There exists a critical time $t_c = t_c(\mathbb{Z}^2)$ (which corresponds with the critical probability $p_c = p_c(\mathbb{Z}^2)$ in ordinary site percolation by $1 - e^{-t_c} = p_c$) such that *before* t_c, only vertices close to

the boundary have been affected by destruction, whereas *after* t_c, vertices on the entire half-plane have been affected by destruction.

In **Chapter 3**, we study the forest-fire model of Chapter 2 on the upper half-plane of the triangular lattice $\mathbb{T}$ rather than the square lattice. Mutatis mutandis, the statements of Chapter 2 are also valid for the triangular lattice. In addition, the existence of critical exponents for site percolation on $\mathbb{T}$ allows us to show that even *at* the critical time $t_c = t_c(\mathbb{T})$, the effect of destruction in the forest-fire model is only felt near the boundary of the half-plane. More precisely, we prove the following: Choose an arbitrary infinite cone in the half-plane whose apex lies on the boundary of the half-plane and whose boundary lines are non-horizontal. Then a.s. only finitely many sites in the cone have been affected by destruction up to time t_c.

Chapter 4 is about the convergence of certain forest-fire models on increasing finite subsets of a regular rooted tree T. For every natural number n, let B_n be the finite subtree of vertices with graph distance at most n from the root. Consider the following model on B_n: At time 0 all vertices are vacant. Then trees grow at rate 1, independently for all vertices. Independently thereof and independently for all vertices, "lightning" hits vertices at rate $\lambda(n) > 0$. When a vertex is hit by lightning, its occupied cluster is instantaneously destroyed.

Now suppose that $\lambda(n)$ decays exponentially in n but much more slowly than $1/|B_n|$. We show that then there exist a supercritical time τ and $\epsilon > 0$ such that the forest-fire model on B_n between time 0 and time $\tau + \epsilon$ tends to the following process on T as n goes to infinity: At time 0 all vertices are vacant. Between time 0 and time τ vertices become occupied at rate 1, independently for all vertices. At time τ all infinite occupied clusters become vacant. Between time τ and time $\tau + \epsilon$ vertices again become occupied at rate 1, independently for all vertices. At time $\tau + \epsilon$ all occupied clusters are finite. This process is a dynamic version of self-destructive percolation.

Zusammenfassung

Forest-Fire-Prozesse wurden erstmals in der Physik-Literatur als Toy-Modell für selbst-organisierte Kritikalität eingeführt. Der Begriff selbst-organisierte Kritikalität beschreibt Vielteilchensysteme, die von lokalen Wechselwirkungen bestimmt werden und aus eigenem Antrieb in einen immer währenden kritischen Zustand geraten. Wie in Gleichgewichtssystemen der statistischen Physik zeichnet sich der kritische Zustand durch langreichweitige Korrelationen, Potenzgesetze, fraktale Strukturen und Selbstähnlichkeit aus.

Wir untersuchen mehrere unterschiedliche Forest-Fire-Modelle, die folgende gemeinsame Eigenschaften haben: Alle Modelle sind zeitstetige Prozesse auf den Knoten eines Graphen. Jeder Knoten kann „frei“ oder „mit einem Baum belegt“ sein. Wir starten mit irgendeiner Anfangskonfiguration. Dann wird der Prozess von zwei konkurrierenden Zufallsmechanismen bestimmt: Einerseits werden Knoten unabhängig voneinander gemäß Poissonprozessen mit Rate 1 belegt. Andererseits werden belegte Cluster gemäß einer festgelegten Regel „in Brand gesetzt“. In diesem Fall wird das gesamte Cluster instantan zerstört, d.h. alle seine Knoten werden frei.

Das selbst-organisiert kritische Verhalten von Forest-Fire-Modellen kann nur auf unendlichen Graphen wie zum Beispiel ebenen Gittern oder unendlichen Bäumen auftreten. Jedoch ist der Zerstörungsmechanismus in allen relevanten Varianten von Forest-Fire-Modellen a priori nur für endliche Graphen wohldefiniert. Aus diesem Grund beginnt man mit einem Forest-Fire-Modell auf endlichen Teilmengen eines unendlichen Graphen und bildet dann den Limes entlang aufsteigender Folgen endlicher Teilmengen, um so ein neues Forest-Fire-Modell auf dem unendlichen Graphen zu erhalten.

In dieser Arbeit führen wir diese Art von Limes für zwei Klassen von Forest-Fire-Modellen aus und untersuchen den resultierenden Limes-Prozess. Der Text hat folgende Gliederung:

In **Kapitel 1** geben wir eine kurze Einführung in selbst-organisierte Kritikalität und stellen einige frühere Ergebnisse zu Forest-Fire-Modellen vor.

In **Kapitel 2** betrachten wir folgendes Forest-Fire-Modell auf der oberen Halbebene des zweidimensionalen Quadratgitters $\mathbb{Z}^2$: Zur Zeit 0 sind alle Knoten frei. Dann wachsen Bäume mit Rate 1, unabhängig für alle Knoten. Wenn ein belegtes Cluster den Rand der oberen Halbebene erreicht oder im Begriff ist unendlich zu werden, wird das Cluster instantan zerstört. Zusätzlich fordern wir, dass das Modell invariant unter Translationen entlang der x-Achse ist.

Wir beweisen, dass ein derartiges Modell existiert und auf natürliche Weise als Teilfol-

genlimes von Forest-Fire-Prozessen in endlichen Boxen entsteht, wenn die Boxgröße nach unendlich strebt.

Außerdem weist das Modell einen Phasenübergang im folgenden Sinne auf: Es existiert eine kritische Zeit $t_c = t_c(\mathbb{Z}^2)$ (die der kritischen Wahrscheinlichkeit $p_c = p_c(\mathbb{Z}^2)$ in gewöhnlicher Knoten-Perkolation vermöge $1 - e^{-t_c} = p_c$ entspricht), so dass *vor* t_c nur Knoten in der Nähe des Randes von Zerstörung betroffen sind, während *nach* t_c Knoten auf der gesamten Halbebene von Zerstörung betroffen sind.

In **Kapitel 3** untersuchen wir das Forest-Fire-Modell aus Kapitel 2 auf der oberen Halbebene des Dreiecksgitters $\mathbb{T}$ anstatt auf dem Quadratgitter. Mutatis mutandis gelten die Aussagen von Kapitel 2 auch für das Dreiecksgitter. Zusätzlich ermöglicht uns die Existenz kritischer Exponenten für Knoten-Perkolation auf $\mathbb{T}$ zu zeigen, dass der Effekt der Zerstörung sogar *bei* der kritischen Zeit $t_c = t_c(\mathbb{T})$ nur in der Nähe des Randes wahrnehmbar ist. Genauer gesagt beweisen wir Folgendes: Man wähle einen beliebigen unendlichen Kegel in der Halbebene, dessen Spitze auf dem Rand der Halbebene liegt und dessen Randlinien nicht-horizontal sind. Dann sind bis zur Zeit t_c f.s. nur endlich viele Knoten im Kegel von Zerstörung betroffen.

Kapitel 4 befasst sich mit der Konvergenz bestimmter Forest-Fire-Modelle auf wachsenden endlichen Teilmengen eines regulären gewurzelten Baums T. Für jede natürliche Zahl n sei B_n der endliche Teilbaum der Knoten, deren Graphendistanz von der Wurzel höchstens n ist. Betrachte folgendes Modell auf B_n: Zur Zeit 0 sind alle Knoten frei. Dann wachsen Bäume mit Rate 1, unabhängig für alle Knoten. Unabhängig davon und unabhängig für alle Knoten schlagen „Blitze“ mit Rate $\lambda(n) > 0$ ein. Wenn ein Knoten vom Blitz getroffen wird, wird sein belegtes Cluster instantan zerstört.

Nehmen wir nun an, dass $\lambda(n)$ exponentiell in n abfällt, aber viel langsamer als $1/|B_n|$. Wir zeigen, dass dann eine superkritische Zeit τ und $\epsilon > 0$ existieren, so dass das Forest-Fire-Modell auf B_n zwischen der Zeit 0 und der Zeit $\tau + \epsilon$ für n nach unendlich gegen den folgenden Prozess auf T strebt: Zur Zeit 0 sind alle Knoten frei. Zwischen der Zeit 0 und der Zeit τ werden Knoten mit Rate 1 belegt, unabhängig für alle Knoten. Zur Zeit τ werden alle unendlichen belegten Cluster frei. Zwischen der Zeit τ und der Zeit $\tau + \epsilon$ werden Knoten wieder mit Rate 1 belegt, unabhängig für alle Knoten. Zur Zeit $\tau + \epsilon$ sind alle belegten Cluster endlich. Dieser Prozess ist eine dynamische Version von Self-Destructive Percolation.

Chapter 1

Introduction to forest-fire models

> *"To poke a wood fire is more solid enjoyment than almost anything else in the world."*
>
> Charles Dudley Warner [War73]

1.1 Self-organized criticality

The concept of self-organized criticality was introduced in the seminal paper [BTW87] by P. Bak, C. Tang and K. Wiesenfeld as an explanation for the ubiquity of scale-free structures in nature. We first address the meaning of "criticality" before we specify the supplement "self-organized".

Criticality is a well-known phenomenon in lattice models of equilibrium statistical physics such as independent site percolation or the Ising model. These systems depend on a model parameter which greatly influences their behaviour (the density p of open sites in the case of percolation and the inverse temperature β in the case of the Ising model). At a certain "critical" value of the parameter these systems experience a phase transition. Formally, the critical value can be defined as the threshold between the regime with no infinite cluster and the regime with at least one infinite cluster. The critical state is typically characterized by the following closely related features:

(C1) long-range correlations;

(C2) power laws;

(C3) scale-invariance (or even conformal invariance) and fractal structures.

Taking independent site percolation on the two-dimensional triangular lattice as an example, these properties become manifest in the following way:

(1) The correlation length diverges at criticality.

(2) The distributions of various observables (e.g. the cluster size) have asymptotic power laws at or near criticality (see [SW01]).

(3) In the scaling limit, the critical percolation exploration paths converge to the Schramm-Loewner evolution with parameter $\kappa = 6$, which is conformally invariant (see [Smi01] and [CN07]). Its trace has Hausdorff dimension 7/4 and its boundary has Hausdorff dimension 4/3 (see [Bef04]). Moreover, in the scaling limit, the full percolation configuration converges to the Continuum Nonsimple Loop process, which is conformally invariant, too (see [CN06]).

Critical site percolation on other two-dimensional lattices is currently not as well understood from a mathematically rigorous point of view but it is widely believed that the essential properties of critical systems are universal in the sense that they do not depend on the underlying lattice.

The characteristic features of criticality (items (C1), (C2), (C3) above) are also frequently observed in nature. However, the equilibrium systems considered so far can hardly serve as an explanation for the widespread occurrence of critical phenomena because this would require the model parameter to be tuned at exactly the right value.

To explain this paradoxon, the authors of [BTW87] claimed that a large class of dynamical systems inherently evolve into a critical state - a phenomenon which they called "self-organized criticality". This concept can be described as follows: Consider an interacting particle system with an arbitrary initial configuration which is governed by local interactions and slowly driven by some external force. The adverb "slowly" means that there should be a separation of time scales between the external driving process and the internal relaxation mechanism. Then such a system may evolve into a critical state without the tuning of any external parameters. The critical state is an unstable yet stationary state which has the same characteristic features (C1), (C2), (C3) as in the equilibrium theory. In fact, (C1) can be interpreted in a stronger sense by requiring that both temporal and spatial long-range correlations should be present. In applications of this theory to real systems, the features (C1) and (C2) often appear in the form of large catastrophic events.

As a first example of such a self-organized critical system, the following sandpile model[1] was proposed in [BTW87]: Sand trickles down randomly onto a plane. (This is the external drive.) After a while, heaps of sand develop. If a sand grain falls down at a site where the slope is too steep, it causes the sand grains at this site to topple to neighbouring sites. (This corresponds to the internal relaxation.) The toppled sand grains can induce further topplings at their new positions and thus induce avalanches. The "size" of these avalanches can be measured in different ways, e.g. by their lifetime or by the total number of affected sites. Both quantities are believed to have a power law distribution (see [Jen98], Section 4.2.3). Despite its name, the sandpile model was not primarily intended to describe real sandpiles but on a more abstract level was meant to explain the emergence of self-organized critical behaviour. In particular, the authors of [BTW87] envisaged their model as an abstract justification of the ubiquity of $1/f$ noise, which we briefly discuss in the next paragraph.

In the physics literature (e.g. [Jen98], Sections 2.2 and 2.3), $1/f$ noise is defined as

[1]In [BTW87], the model was pictorially described in terms of twisted coupled pendulums rather than sandpiles but both descriptions boil down to the same dynamics.

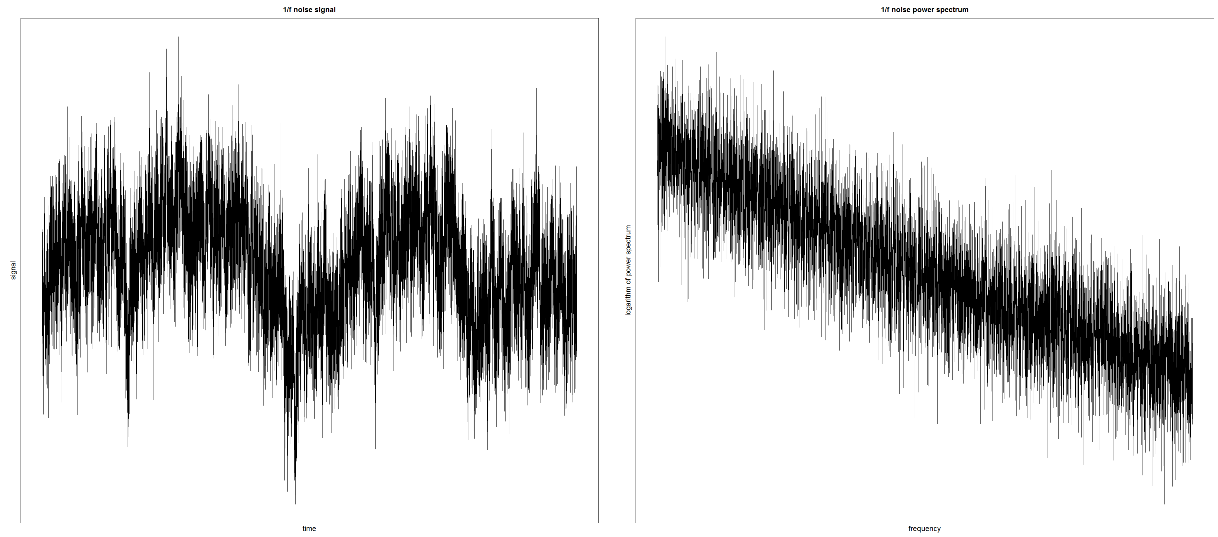


Figure 1.1: A $1/f$ noise signal (left) and a log-lin plot of the corresponding power spectrum (right)

a random time-dependent signal $N(t)$ which is stationary and scale-invariant and whose power spectral density

$$S(f) := \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} N(t) e^{2i\pi f t}\, dt \right|^2 \tag{1.1}$$

decays like $1/f$ for large $|f|$. Such behaviour is observed in various systems, for instance as flicker noise in electronic devices, in sunspot activity, in highway traffic or in the flow of the river Nile. Figure 1.1 shows a simulation of a $1/f$ noise signal and the corresponding power spectrum (calculated from a discrete and finite approximation of equation (1.1)). The power spectrum is plotted as a log-lin graph, i.e. with a logarithmic scale on the y-axis and a linear scale on the x-axis. The fact that the values in this plot are clustered around a decreasing straight line confirms that the spectral density has a power law decay.

Apart from $1/f$ noise, self-organized criticality has been conjectured to be at the origin of numerous other phenomena such as earthquakes, landscape formation, economic bubbles and biological evolution. Comprehensive accounts of self-organized criticality with details to these and many more examples are given in [Bak96], [Jen98], [Pru04] and [Pru12].

1.2 Forest-fire models in the physics literature

In addition to the sandpile model, numerous other toy models of self-organized criticality have been developed, with one of the most prominent examples being the forest-fire model. Since its first appearance in the paper [BCT90] by P. Bak, K. Chen and C. Tang different variants of this model have been studied. The most common version in the physics literature goes back to the paper [DS92] by B. Drossel and F. Schwabl and is described in the sequel. Let d be a natural number and let B be a finite box in the d-dimensional lattice $\mathbb{Z}^d$. The Drossel-Schwabl forest-fire model is a discrete-time Markov process on B in which each site can be either "occupied by a tree", "burning" or "vacant". The process starts with an arbitrary initial configuration. At each time step, the configuration is changed according to the following rules:

[GROWTH] If a site is vacant, it becomes occupied with probability p (independently of all other sites).

[IGNITION] If a site is occupied, it becomes burning with probability f (independently of [GROWTH] and all other sites).

[SPREAD] If a site is occupied and has a burning neighbour, it becomes burning in the next time step.

[INCINERATION] If a site is burning, it becomes vacant in the next time step.

This model is believed to become self-organized critical if the following conditions are satisfied:

(DS)
- There is a double separation of time scales in the sense that $1 \gg p \gg f$. In other words, the spread of fire is much faster than the growth of trees, and the growth of trees is much faster than ignition.
- The size of the box B is much larger than the number of sites which are typically burnt down as a consequence of a single ignition event.

In the physics literature, these conditions are interpreted as characterizations of the order of magnitude so that p and f/p are assumed to be small but positive and B is taken to be large but finite. Sometimes the model is simplified by making the spread of fire instantaneous. Then the model only has the two states "occupied" and "vacant", and [IGNITION], [SPREAD] and [INCINERATION] are replaced by the following single rule:

[DESTRUCTION] If a site is occupied, it is hit by "lightning" with probability f (independently of [GROWTH] and all other sites). In this case its entire occupied cluster becomes vacant.

The first condition of (DS) is accordingly reduced to $p \gg f$.

The original paper [DS92] claimed that in the regime where (DS) is satisfied, the cluster size distribution and various other observables of the forest-fire model exhibit power laws. The corresponding critical exponents were calculated by mean field arguments and their agreement with the spatial model was verified by simulations. However, based on more extensive simulations, several follow-up papers (e.g. [Gra93], [Hen93] and [CDS94]) suggested more complicated values for these exponents and proposed corrections to the scaling ansatz in [DS92]. Some results for the one-dimensional case which were obtained non-rigorously in [DCS93] were later proven in [vdBJ05] and [BP06] under a slightly different setting (see Section 1.3.4), while other predictions of [DCS93] turned out to be incorrect. More recent and larger simulations (e.g. [Gra02], [PJ04]) have cast some doubt whether the condition (DS) really results in critical behaviour in two dimensions.

The previous paragraph only covers a small fraction of the vast physics literature on the Drossel-Schwabl forest-fire model. A good overview of the current state of research can be found in Section 5.2 of the book [Pru12] by G. Pruessner.

1.3 Forest-fire models in the mathematics literature

1.3.1 Existence and uniqueness results

Soon after their introduction in the physics literature, forest-fire models also caught the interest of mathematicians. Most of the mathematics literature on this subject deals with a continuous-time version of the simplified Drossel-Schwabl model (where clusters are destroyed instantaneously). Additionally, the model has been generalized from finite boxes to more general graphs. Important existence and uniqueness results for infinite graphs were achieved by M. Dürre in [Dür06a], [Dür06b] and [Dür09]. So let V be the vertex set of a connected graph[2] and let $\lambda > 0$. The Dürre forest-fire model[3] on V with ignition rate λ is a continuous-time process on V in which each site can be either "occupied" or "vacant". We start with an arbitrary initial configuration which does not contain infinite occupied clusters. Then the process develops according to the following rules:

[GROWTH] Vertices turn from "vacant" to "occupied" according to independent rate 1 Poisson processes (the so-called growth processes).

[DESTRUCTION] Independently of [GROWTH], vertices are hit by "lightning" according to independent rate λ Poisson processes (the so-called ignition processes). If a vertex is hit by lightning, its occupied cluster instantaneously becomes vacant.

If the vertex set V is finite, then the existence and uniqueness of such a forest-fire model is clear: Given independent growth and ignition processes, we can order the jump events chronologically and thus obtain a unique corresponding forest-fire process by a graphical construction. A similar statement is true for the one-dimensional case $V = \mathbb{Z}$ (endowed with the standard edges): Given independent growth and ignition processes and an arbitrary time T, there are infinitely many sites which are neither occupied in the initial configuration nor experience a growth event up to time T. We can therefore partition $\mathbb{Z}$ into a random collection of finite subsets which do not interact up to time T and perform a graphical construction on each of these sets to obtain a unique forest-fire process up to time T. Since T is arbitrary, this yields a unique forest-fire process for all times.

For general infinite graphs V, the approach above does not work any more so that the existence and uniqueness of forest-fire processes on V require more sophisticated methods.

[2] In slight abuse of notation we will denote the graph by V, too.

[3] For simplicity, we name the model after M. Dürre for all graphs V even though for finite V, the model was studied by other authors before.

Henceforth, we assume that the graph V has bounded vertex degree. Then a forest-fire process on V exists for all initial configurations which do not contain infinite occupied clusters ([Dür06a], Theorem 1). Now suppose that the initial configuration fulfills a slightly stronger condition, the so-called "conditional cluster size bound" ([Dür09], Definition 7). This condition is for instance satisfied by the vacant configuration and by independent site percolation on V with no infinite occupied clusters. Then the corresponding forest-fire process on V is unique and has the Markov property ([Dür09], Theorem 3). Moreover, if 0 is some vertex in V and V_n denotes the set of vertices whose graph distance from 0 is at most n, then for $n \to \infty$, the forest-fire process on V_n converges a.s. to the forest-fire process on V, where these processes are coupled in the natural way, i.e. through their growth and ignition processes ([Dür09], Theorem 1). In fact, a closer look at the proof of the a.s. convergence reveals that the forest-fire process on V can even be perfectly simulated. This is explained in more detail in Appendix A.

For Dürre forest-fire models on $\mathbb{Z}^d$ with $d \in \mathbb{N}$, it is also known that there exists a stationary, translation-invariant distribution. In the one-dimensional case (on which we will elaborate in Section 1.3.4) this goes back to R. Brouwer and J. Pennanen ([BP06], Proposition 5.1), while for $d \geq 2$ this was proven more recently by A. Stahl ([Sta12], Theorem 1).

1.3.2 The critical limit

Let us now come back to the presumed self-organized critical behaviour of forest-fire models. In the following informal discussion, let V again be an infinite graph with bounded vertex degree, where we primarily have "classical" graphs like d-dimensional lattices for $d \geq 2$ or regular trees in mind. Transferring the conditions (DS) imposed on the Drossel-Schwabl model in Section 1.2 to the Dürre model and interpreting them as limits rather than just orders of magnitude, we expect the situations (D1) and (D2) to be particularly important:

(D1) The limit $\lambda \downarrow 0$ of the Dürre forest-fire model on V with ignition rate λ.

(D2) The limit $n \to \infty$ of the Dürre forest-fire model on V_n with ignition rate $\lambda(n)$, where $\lambda(n) \to 0$ for $n \to \infty$ but the decay of $\lambda(n)$ is sufficiently slow compared to the size of V_n.

We call both (D1) and (D2) "critical limits". If such a limit exists (in a weak sense, a subsequential limit can always be obtained by Prokhorov's theorem - compare Lemmas 1 and 2 in Chapter 2), we thus get a new limit process on V. The interesting question, of course, is what its dynamics are like, i.e. how the growth and destruction mechanism are transferred to the limit process. It seems natural that [GROWTH] should remain unchanged in the critical limit. As for [DESTRUCTION], it is obvious that *finite* clusters cannot be destroyed any more in the limit process because the ignition rate tends to zero in both (D1) and (D2). On the other hand, in case (D1) one can heuristically argue that in the limit process *infinite* clusters should be destroyed as soon as they appear because for arbitrarily small $\lambda > 0$, they would be hit instantaneously if there was lightning according to rate λ ignition processes. In case (D2), the condition that $\lambda(n)$ decays sufficiently slowly is meant in the same spirit: The intersection of infinite clusters in the limit process with V_n should have a high probability of being hit by lightning within short time if there was lightning according to rate $\lambda(n)$ ignition processes. One might therefore conjecture that the limit process has the following dynamics:

[GROWTH]	Sites turn from "vacant" to "occupied" according to independent rate 1 Poisson processes.
[DESTRUCTION]	If an occupied cluster becomes infinite, it is instantaneously destroyed, i.e. all of its sites turn vacant.

A process with these dynamics is called a permanent self-destructive percolation process on V (regardless of whether it is obtained as the critical limit of forest-fire processes or otherwise).

However, at least for the two-dimensional lattice $V = \mathbb{Z}^2$ the heuristic arguments above are not correct, for it has recently been proven by D. Kiss, I. Manolescu and V. Sidoravicius that a permanent self-destructive percolation process with vacant initial configuration does not exist on $\mathbb{Z}^2$ ([KMS13], Theorem 6). A first step towards this proof was already achieved by J. van den Berg and R. Brouwer ([vdBB04], Theorem 4.1), who reduced the non-existence statement to an unproven but numerically confirmed technical conjecture about independent site percolation on $\mathbb{Z}^2$. There is no clear-cut alternative conjecture concerning the true dynamics of limit forest-fire processes on $\mathbb{Z}^2$ but some insight is given in another paper by J. van den Berg and R. Brouwer ([vdBB06], Theorem 2.2): Let $t_c = t_c(\mathbb{Z}^2)$ denote the critical time after which an infinite occupied cluster first appears on the square lattice $\mathbb{Z}^2$ if we start with a vacant initial configuration and then only have [GROWTH] but no destruction mechanism. For $n \in \mathbb{N}$ and $(x, y) \in \mathbb{Z}^2$, let $B_n(x, y) := (x, y) + [-n, n]^2 \cap \mathbb{Z}^2$ denote the box with centre (x, y) and radius n, and for $\lambda > 0$, let η^λ be a Dürre forest-fire process on $\mathbb{Z}^2$ with vacant initial configuration and ignition rate λ. Then there exists $t > t_c$ such that for all $n \in \mathbb{N}$

$$\liminf_{\lambda \downarrow 0} \mathbf{P}\left[\eta^\lambda \text{ has destruction in } B_n(0,0) \text{ before time } t\right] \leq \frac{1}{2}$$

holds. The original result in [vdBB06] was phrased slightly differently and stated conditionally on a then unproven conjecture but thanks to more recent results in [Dür06a], [Dür09] and [KMS13], it can now be reformulated in this way. Having discussed the intricacy of the critical limit for the square lattice $\mathbb{Z}^2$, we should emphasize that the situation is expected to be different for other graphs such as lattices in high dimensions and transitive unimodular non-amenable graphs (see [ADCKS13], [AST14]). In particular, the permanent self-destructive percolation process might exist on these graphs. We will come back to these references in Section 1.4.

1.3.3 On the contents of this thesis

The focus of this thesis lies on two problems which are closely related with the critical limit and which we describe next. In Chapter 4, we study a situation like (D2) for the case where V is an infinite regular rooted tree and the forest-fire process starts with a vacant initial configuration. However, we let $\lambda(n)$ tend to zero slightly faster than required for true critical behaviour. As a consequence, the limit process on V first becomes supercritical in the sense that infinite clusters appear. But at a deterministic time (which depends on the rate of decay of $\lambda(n)$), all infinite clusters are destroyed so that the configuration becomes subcritical again. These dynamics can be interpreted as a version of self-destructive percolation, which is introduced in Section 1.4.

The starting point of Chapters 2 and 3 is a forest-fire model on the boxes $B_n(0,0)$ with centre $(0,0)$ and radius $n \in \mathbb{N}$, in which clusters are not destroyed by lightning but instead are destroyed when they reach the boundary of the box. More precisely, the process starts with a vacant initial configuration and is then determined by the following rules:

[GROWTH] Sites turn from "vacant" to "occupied" according to independent rate 1 Poisson processes.

[DESTRUCTION] If an occupied cluster reaches the boundary of the box, it is instantaneously destroyed, i.e. all of its sites turn vacant.

Since $B_n(0,0)$ is finite, the existence and uniqueness of such a process is immediate. For this model, the critical limit is obtained for $n \to \infty$, and again the question arises what the dynamics of the corresponding limit process on $\mathbb{Z}^2$ are (provided that the limit exists in a suitable sense). Employing similar heuristic arguments as above, one could analogously come to the flawed conclusion that the limit process is equal to the permanent self-destructive percolation process on $\mathbb{Z}^2$, whose existence is disproved in [KMS13]. But apart from this negative result not much is known about the critical limit of this model. One of the reasons why the problem is so difficult to tackle is the fact that in the limit $n \to \infty$, the boundary disappears so that the effect of destruction - if there is any at all - must come from infinitely far away.

With this in mind, we now modify the original setting in the following way: Instead of fixing the centre of the box and letting the box tend to infinity in all four directions, we fix the bottom side and let the box tend to infinity in the remaining three directions. In other words, we consider an analogous forest-fire model on the boxes $B_n(0,n)$ with centre $(0,n)$ and radius $n \in \mathbb{N}$. In addition, we make two further changes of technical nature.[4] In Chapter 2, we show that for $n \to \infty$, the forest-fire processes on $B_n(0,n)$ subsequentially converge to limit processes on the upper half-plane $\overline{\mathbb{H}} := \{(x,y) \in \mathbb{Z}^2 : y \geq 0\}$ of the square lattice. Any such limit process starts with a vacant initial configuration and then has the following dynamics:

[GROWTH] Sites turn from "vacant" to "occupied" according to independent rate 1 Poisson processes.

[DESTRUCTION] If an occupied cluster reaches the boundary of the half-plane or if it

[4]Namely, we restrict the destruction mechanism to the fixed bottom side and use periodic boundary conditions in the x-direction.

becomes infinite, it is instantaneously destroyed, i.e. all of its sites turn vacant.

It is worth noting that on a formal level, this destruction mechanism is a hybrid between the destruction at the boundary in the finite-size model and the destruction of infinite clusters in the hypothetical permanent self-destructive percolation process on $\mathbb{Z}^2$. Since the limit process starts with a vacant initial configuration, one can easily verify that *before* the critical time $t_c = t_c(\mathbb{Z}^2)$, the effect of destruction is limited to areas near the boundary. In Chapter 2, we additionally prove the less straightforward fact that *after* t_c, the effect of destruction extends to the entire half-plane. However, it remains unknown whether this is caused by the destruction of infinite clusters or by the destruction of increasingly large but finite clusters. In fact, it is unclear if the destruction of infinite clusters ever occurs with positive probability.

Assuming the existence of two critical exponents for independent site percolation on $\mathbb{Z}^2$, one can prove that even *at* t_c, the effect of destruction is still limited to areas near the boundary. At the moment, these exponents are rigorously confirmed for the triangular lattice $\mathbb{T}$ (see [SW01] or [Nol08]) but not for the square lattice $\mathbb{Z}^2$. Yet all the above forest-fire results can be transferred to the upper half-plane of the triangular lattice if the appropriate changes are made (e.g. replacing the critical time $t_c(\mathbb{Z}^2)$ of the square lattice by the critical time $t_c(\mathbb{T})$ of the triangular lattice). Thus, the statement about the extent of destruction *at* the critical time becomes fully rigorous on the triangular lattice. This is the content of Chapter 3.

1.3.4 One-dimensional and mean field models

While this thesis is mainly concerned with forest-fire models on two-dimensional lattices and regular rooted trees, it should be pointed out that models of this kind have also been studied quite extensively on other graphs. In the following, we review some of these results.

In the first part of this section, we summarize some results about Dürre forest-fire models on $\mathbb{Z}$ with ignition rate λ. In [vdBJ05], J. van den Berg and A. Járai show that regardless of the initial configuration, after time of order $\log(1/\lambda)$ the density of vacant sites in the forest-fire process is of order $1/\log(1/\lambda)$. Additionally, they derive certain bounds for the cluster size distribution. The subsequent paper [BP06] by R. Brouwer and J. Pennanen is concerned with the cluster size distribution of a stationary, translation-invariant forest-fire distribution (whose existence is proved, as well). The authors define a threshold $s_{\max}$ by $s_{\max} \log s_{\max} = 1/\lambda$ and show that for $\alpha < 1/3$ and $s \leq s_{\max}^{\alpha}$, the probability that the cluster at a fixed site has size s is of order $1/(s\log(1/\lambda))$, uniformly in λ and s. On the other hand, it follows from results in [vdBJ05] that this uniform decay does not hold any more for s of order $s_{\max}$.

The steady state of forest-fire processes on $\mathbb{Z}$ with ignition rate $\lambda = 1$ has been analysed more closely by X. Bressaud and N. Fournier in [BF09]: For $\lambda = 1$, there is a unique stationary distribution, which is exponentially mixing and can be perfectly simulated. Moreover, for any initial distribution, the forest-fire process tends to equilibrium exponentially fast.

In [BF10], the same authors investigate the critical limit $\lambda \downarrow 0$ of the one-dimensional forest-fire model. In this regard, the one-dimensional case differs from higher dimensions because without rescaling, as $\lambda \downarrow 0$, the forest-fire process on $\mathbb{Z}$ trivially converges to the pure growth process (i.e. the process with [GROWTH] only and no destruction mechanism). However, if time is accelerated by a factor $\log(1/\lambda)$ and space is compressed by a factor $\lambda \log(1/\lambda)$, then a non-trivial continuous process is obtained for $\lambda \downarrow 0$; the limit process admits a graphical construction and can be perfectly simulated. In the consecutive paper [BF11], this result is extended to generalized forest-fire processes on $\mathbb{Z}$, where the growth and ignition processes need not be Poisson processes any more but are only required to be stationary renewal processes.

In the second part of this section, we describe a mean field forest-fire model which is studied by B. Ráth and B. Tóth in [RT09] and for which self-organized critical features have been rigorously established. The model lives on the complete graph K_n with n vertices and resembles the Dürre forest-fire model but exhibits two major differences: Firstly, trees grow on the edges of K_n, while lightning strikes at the vertices of K_n. Secondly, the growth rate of trees is $1/n$ rather than 1. An explicit characterization of the Ráth-Tóth model reads as follows: Each edge can be "occupied" or "vacant". The process starts with a vacant initial configuration[5] and then develops according to the following rules:

[GROWTH] Edges turn from "vacant" to "occupied" according to independent rate $1/n$ Poisson processes.

[DESTRUCTION] Independently of [GROWTH], vertices are hit by "lightning" according to independent rate $\lambda(n)$ Poisson processes. If a vertex is hit by lightning, its occupied cluster instantaneously becomes vacant.

If [DESTRUCTION] is omitted, then the resulting process is simply a dynamical formulation of the Erdős-Rényi random graph model. For $k \in \mathbb{N}$ and $t \geq 0$, let $v_{n,k}^{\mathrm{ER}}(t)$ denote the random fraction of vertices which are in a cluster of size k at time t in the Erdős-Rényi model. It is well-known that for $n \to \infty$, $v_{n,k}^{\mathrm{ER}}(t)$ converges to a deterministic function $v_k^{\mathrm{ER}}(t)$ and that this function has a phase transition at the gelation time $T_{\mathrm{gel}} = 1$ in the sense that

$$\sum_{k \in \mathbb{N}} v_k^{\mathrm{ER}}(t) \begin{cases} = 1 & \text{if } 0 \leq t \leq T_{\mathrm{gel}}, \\ < 1 & \text{if } t > T_{\mathrm{gel}}. \end{cases}$$

The mass defect for $t > T_{\mathrm{gel}}$ is caused by the appearance of the giant component, whose size is of order n. At the gelation time, the tail $\sum_{l=k}^{\infty} v_l^{\mathrm{ER}}(T_{\mathrm{gel}})$ decays like $k^{-1/2}$, while for $t \neq T_{\mathrm{gel}}$, $v_k^{\mathrm{ER}}(t)$ decays exponentially in k. So the Erdős-Rényi model is subcritical for $t < T_{\mathrm{gel}}$, critical for $t = T_{\mathrm{gel}}$ and supercritical for $t > T_{\mathrm{gel}}$.

Now let us return to the Ráth-Tóth forest-fire model on K_n with ignition rate $\lambda(n)$. Loosely speaking, the main result in [RT09] says that if $\lambda(n)$ tends to zero sufficiently slowly (in the sense of (D2) in Section 1.3.2), then in the limit $n \to \infty$ the model behaves

[5] The results in [RT09] also cover more general initial configurations.

critically for all times $t \geq T_{\text{gel}}$. More precisely, the following holds true: Suppose that $\lambda(n)$ satisfies $1/n \ll \lambda(n) \ll 1$. For $k \in \mathbb{N}$ and $t \geq 0$, let $v_{n,k}^{\text{RT}}(t)$ denote the random fraction of vertices which are in a cluster of size k at time t in the Ráth-Tóth model. Then for $n \to \infty$, $v_{n,k}^{\text{RT}}(t)$ converges to a deterministic function $v_k^{\text{RT}}(t)$. For $0 \leq t \leq T_{\text{gel}}$ we have $v_k^{\text{RT}}(t) = v_k^{\text{ER}}(t)$. For $t \geq T_{\text{gel}}$, the tail $\sum_{l=k}^{\infty} v_l^{\text{RT}}(t)$ decays like $k^{-1/2}$. For all $t \geq 0$, $v_k^{\text{RT}}(t)$ satisfies $\sum_{k \in \mathbb{N}} v_k^{\text{RT}}(t) = 1$.

1.4 Self-destructive percolation

Introduced by J. van den Berg and R. Brouwer in [vdBB04], self-destructive percolation is a modification of ordinary Bernoulli percolation and is closely related with the (hypothetical) permanent self-destructive percolation process of Section 1.3.2. Let G be an infinite connected graph, let 0 be a distinguished site in G and let p_c be the critical probability for independent site percolation on G (where we assume $p_c < 1$). Then self-destructive site percolation on G with parameters $p \in (p_c, 1)$ and $\delta \in (0, 1)$ is defined in three steps:

1. First, every site becomes occupied with probability p (independently of all other sites).
2. Then the sites of all infinite occupied clusters become vacant.
3. Finally, every vacant site becomes occupied with probability δ (independently of the first step and of all other sites).

Let $\theta(p, \delta)$ denote the probability that the site 0 is part of an infinite occupied cluster in the final configuration. Furthermore, for fixed p, let

$$\delta_c(p) := \inf \{\delta > 0 : \theta(p, \delta) > 0\}$$

be the minimal enhancement needed in the third step to have an infinite occupied cluster at the site 0 with positive probability. It is not difficult to see that $\theta(p, \delta)$ is zero if and only if the final configuration a.s. contains no infinite cluster, and $\theta(p, \delta)$ is positive if and only if the final configuration a.s. contains at least one infinite cluster. Thus, $\delta_c(p)$ can also be interpreted as the threshold between the regime with no infinite cluster and the regime with at least one infinite cluster. The quantities $\theta(p, \delta)$ and $\delta_c(p)$ can be analogously defined for self-destructive *edge* percolation. In either case, the most interesting question about the model is how $\delta_c(p)$ behaves as p approaches the critical value p_c.

As it turns out, the answer to this question crucially depends on the underlying graph G. For self-destructive site percolation on the binary tree and on transitive unimodular non-amenable graphs and for self-destructive edge percolation on $\mathbb{Z}^d$ with d sufficiently large it is known that

$$\lim_{p \downarrow p_c} \delta_c(p) = 0$$

holds (see [vdBB04], Theorem 5.1; [AST14], Theorem 2.2; [ADCKS13], Theorem 1). Theorem 5.1 in [vdBB04] also establishes the fact that for self-destructive site percolation on the binary tree $\delta_c(p) > 0$ holds for all $p \in (p_c, 1)$. This result is extended to regular rooted trees in Section 4.4.

By contrast, for self-destructive site percolation on the square lattice $\mathbb{Z}^2$ (and other two-dimensional lattices) D. Kiss, I. Manolescu and V. Sidoravicius ([KMS13], Theorem 2) have recently proven that

$$\inf_{p>p_c} \delta_c(p) > 0. \tag{1.2}$$

Equation (1.2) is strongly connected with the fact that the permanent self-destructive percolation process with vacant initial configuration does not exist on $\mathbb{Z}^2$. Indeed, both statements follow from a more general theorem about critical percolation on finite-size rectangles ([KMS13], Theorem 4). Equation (1.2) had already been conjectured for the square lattice in the original paper [vdBB04] on self-destructive percolation. Some weaker linear lower bounds on $\delta_c(p)$ for various two-dimensional lattices have also been obtained in [vdBdL09].

The definition of self-destructive percolation can be trivially extended from $p \in (p_c, 1)$ to arbitrary $p \in (0, 1)$ (in which case the destruction of all infinite clusters in the second step may become an empty condition). Now consider again self-destructive site percolation on $\mathbb{Z}^2$: Then on the one hand, we have $\theta(p_c, \delta) > 0$ for all $\delta \in (0, 1)$. On the other hand, (1.2) implies the existence of $\delta_0 \in (0, 1)$ such that $\theta(p, \delta) = 0$ holds for all $p \in (p_c, 1), \delta \in (0, \delta_0)$. Consequently, the function $\theta(\,\cdot\,, \cdot\,)$ is discontinuous on the segment $\{p_c\} \times (0, \delta_0)$. In [vdBBV08], J. van den Berg, R. Brouwer and B. Vágvölgyi show that this is essentially the only region with discontinuity. More precisely, there exists $\delta_1 \in (0, 1)$ such that $\theta(\,\cdot\,, \cdot\,)$ is continuous outside the segment $\{p_c\} \times (0, \delta_1)$.

Chapter 2

A forest-fire model on the upper half-plane

An article which closely follows this chapter has been published in the *Electronic Journal of Probability* [Gra14b].

2.1 Introduction and statement of the main results

In this chapter we consider forest-fire models which are defined on subsets of the square lattice $\mathbb{Z}^2$. We assume the vertex set $\mathbb{Z}^2$ to be equipped with the standard lattice edge set, where two sites in $\mathbb{Z}^2$ are connected by an edge if and only if they have Euclidean distance 1. For practical purposes we will identify $\mathbb{Z}^2 \subset \mathbb{R}^2$ with $\mathbb{Z}+i\mathbb{Z} \subset \mathbb{C}$ (where $i := \sqrt{-1} = (0,1)$) and mostly use the complex number notation even though we do not use the multiplicative structure of $\mathbb{C}$. The finite volume versions of the model will be defined on boxes

$$B_n(w) := w + [-n,n]^2 \cap \mathbb{Z}^2 \tag{2.1}$$

with centre $w \in \mathbb{Z}^2$ and radius $n \in \mathbb{N}$. To begin with, we endow the vertex set $B_n(w)$ with the standard edges inherited from the square lattice $\mathbb{Z}^2$ and we denote this by writing $B_n^{\mathrm{s}}(w)$ instead of $B_n(w)$. Later on, for each $k \in \{-n, -n+1, \ldots, n\}$, we will insert an additional edge between the vertex $w-n+ik$ on the left and the vertex $w+n+ik$ on the right in order to make the setup periodic in the x-direction; in this case we write $B_n^{\mathrm{p}}(w)$ instead of $B_n(w)$. The graph $B_n^{\mathrm{p}}(w)$ is best visualized as a cylinder. The infinite volume version of the forest-fire model will be defined on the "closed" upper half-plane

$$\overline{\mathbb{H}} := \{x+iy \in \mathbb{Z}+i\mathbb{Z} : y \geq 0\},$$

which we endow with the edges inherited from the square lattice $\mathbb{Z}^2$. We will also denote by

$$\mathbb{H} := \{x+iy \in \mathbb{Z}+i\mathbb{Z} : y > 0\}$$

the "open" upper half-plane.

In order to explain some more notation, let us for a moment consider an arbitrary connected graph with vertex set V. (In practice, this will usually be one of the graphs $B_n^{\mathrm{s}}(w)$, $\mathbb{Z}^2$, $B_n^{\mathrm{p}}(w)$ or $\overline{\mathbb{H}}$.) For a subset $S \subset V$, we write

$$\partial S := \{v \in V \setminus S : (\exists w \in S : v \text{ and } w \text{ are neighbours})\}$$

for the **(outer) boundary** of S in V. For the subset $\mathbb{H} \subset \overline{\mathbb{H}}$, for instance, we simply have $\partial\mathbb{H} = \mathbb{Z}$. At any given time, the forest-fire model will be described by a random configuration $(\alpha_v)_{v\in V} \in \{0,1\}^V$, which induces a subgraph of V on the vertex set $\{v \in V : \alpha_v = 1\}$. For $z \in V$ the maximal connected component of this subgraph containing z is called the **cluster** of z in the configuration $(\alpha_v)_{v\in V}$. If $\alpha_z = 0$, then the cluster of z is just the empty set.

We are now ready to describe the forest-fire model on the box $B_n^{\mathrm{s}}(0)$ ($n \in \mathbb{N}$) which is the starting point of this chapter. It is a continuous-time Markov process on the state space $\{0,1\}^{B_n^{\mathrm{s}}(0)}$, where a site with "1" is said to be "occupied by a tree" and a site with "0" is said to be "vacant". At the starting time all sites are vacant. Then the process is governed by the following two conflicting mechanisms:

[GROWTH] Sites turn from "vacant" to "occupied" according to independent rate 1 Poisson processes.

[DESTRUCTION] If an occupied cluster reaches the inner boundary $B_n(0) \setminus B_{n-1}(0)$ of the box[1], it is instantaneously destroyed, i.e. all of its sites turn vacant.

The most interesting aspect about this model is the question of what happens in the limit $n \to \infty$ (provided that it exists in a suitable sense). It is in this limit that the model is expected to exhibit self-organized criticality, and the intuitive reasoning goes as follows: For large n, small clusters are unlikely to get destroyed but sufficiently large clusters are still vulnerable to destruction. So a hypothetical limit process on $\mathbb{Z}^2$ might have the following dynamics: At the starting time all sites are vacant. Then the process is governed by the following two conflicting mechanisms:

[GROWTH] Sites turn from "vacant" to "occupied" according to independent rate 1 Poisson processes.

[DESTRUCTION] If an occupied cluster becomes infinite, it is instantaneously destroyed, i.e. all of its sites turn vacant.

However, these heuristics cannot be true because it has recently been proven in [KMS13] that a process with these dynamics does not exist on $\mathbb{Z}^2$. A mathematically rigorous treatment of the question of convergence for $n \to \infty$ currently seems hard to achieve.

A first step towards a better understanding of the $n \to \infty$ limit probably lies in the analysis of the behaviour of the sites close to the inner boundary when n is large. We therefore change our perspective in the following way:

[1] where we set $B_0(0) := \{0\}$

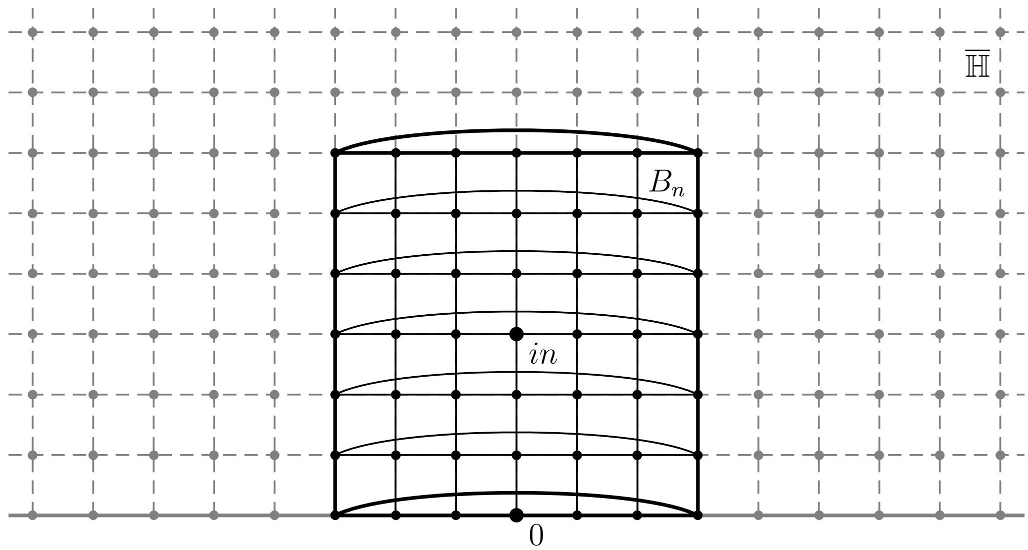


Figure 2.1: The box $B_n := B_n^{\mathrm{p}}(in)$ for $n = 3$ (black) and the upper half-plane $\overline{\mathbb{H}}$ (grey)

- Instead of keeping the centre of the box fixed and letting the box tend to infinity in all four directions, we keep the bottom side fixed and let the box tend to infinity in the remaining three directions. In other words, we consider the process on the box $B_n(in)$ instead of the box $B_n(0)$. In the (subsequential) limit $n \to \infty$ we thus get a process on the upper half-plane $\overline{\mathbb{H}}$.

Additionally, we make the following changes, which are natural for the new setting:

- We restrict the destruction mechanism [DESTRUCTION] to clusters which reach the fixed bottom side instead of destroying clusters at all four sides.
- We use periodic boundary conditions in the x-direction, i.e. we work on $B_n^{\mathrm{p}}(in)$ instead of $B_n^{\mathrm{s}}(in)$.

Let us define this new process more formally, in a fashion similar to the definition of the Dürre forest-fire model in [Dür06a]. We include the underlying Poisson growth processes into our notation and thus obtain a continuous-time process on the state space $(\{0,1\} \times \mathbb{N}_0)^{B_n^{\mathrm{p}}(in)}$. For convenience we henceforth abbreviate $B_n := B_n^{\mathrm{p}}(in)$. Figure 2.1 depicts the box B_n and its edges, embedded into the upper half-plane $\overline{\mathbb{H}}$. In accordance with the periodic boundary conditions of B_n, for $z \in B_n$ and $x \in \mathbb{Z}$ we define "periodic addition" by

$$z \oplus x := [((\operatorname{Re} z + x) + n) \bmod (2n+1)] - n + i \operatorname{Im} z \in B_n.$$

Moreover, for a function $[0,\infty) \ni t \mapsto f_t \in \mathbb{R}$ we write $f_{t^-} := \lim_{s \uparrow t} f_s$ for the left-sided limit at $t > 0$, provided the limit exists.

Definition 1. Let $n \in \mathbb{N}$. Let $(\eta_{t,z}^n, G_{t,z}^n)_{t \geq 0, z \in B_n}$ be a process[2] with values in $(\{0,1\} \times$

[2] A more precise but more cumbersome notation would be $\big((\eta_{t,z}^n, G_{t,z}^n)_{z \in B_n}\big)_{t \geq 0}$.

$\mathbb{N}_0)^{[0,\infty)\times B_n}$, initial condition $\eta^n_{0,z} = 0$ for $z \in B_n$ and boundary condition $\eta^n_{t,x} = 0$ for $t \geq 0, x \in \partial\mathbb{H} \cap B_n$. Suppose that for all $z \in B_n$ the process $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0}$ is càdlàg, i.e. right-continuous with left limits. For $z \in B_n$ and $t > 0$, let $C^n_{t^-,z}$ denote the cluster of z in the configuration $(\eta^n_{t^-,w})_{w\in B_n}$.

Then $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in B_n}$ is called a B_n**-forest-fire process** if the following conditions are satisfied:

[POISSON] The processes $(G^n_{t,z})_{t\geq 0}$, $z \in B_n$, are independent Poisson processes with rate 1.

[ROT-INV] The distribution of $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in B_n}$ is invariant under rotations of the cylinder B_n, i.e. the processes $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in B_n}$ and $(\eta^n_{t,z\oplus 1}, G^n_{t,z\oplus 1})_{t\geq 0, z\in B_n}$ have the same distribution.

[GROWTH] For all $t > 0$ and all $z \in \mathbb{H} \cap B_n$ the following implications hold:

(i) $G^n_{t^-,z} < G^n_{t,z} \Rightarrow \eta^n_{t,z} = 1$,
i.e. the growth of a tree at the site z at time t implies that the site z is occupied at time t;

(ii) $\eta^n_{t^-,z} < \eta^n_{t,z} \Rightarrow G^n_{t^-,z} < G^n_{t,z}$,
i.e. if the site z gets occupied at time t, there must have been the growth of a tree at the site z at time t.

[DESTRUCTION] For all $t > 0$ and all $x \in \partial\mathbb{H}\cap B_n$, $z \in \mathbb{H}\cap B_n$ the following implications hold:

(i) $G^n_{t^-,x} < G^n_{t,x} \Rightarrow \forall w \in C^n_{t^-,x+i} : \eta^n_{t,w} = 0$,
i.e. if the cluster at $x + i$ grows to the boundary $\partial\mathbb{H} \cap B_n$ at time t, it is destroyed at time t;

(ii) $\eta^n_{t^-,z} > \eta^n_{t,z} \Rightarrow \exists u \in \partial C^n_{t^-,z} \cap \partial\mathbb{H} : G^n_{t^-,u} < G^n_{t,u}$,
i.e. if the site z is destroyed at time t, its cluster must have grown to the boundary $\partial\mathbb{H} \cap B_n$ at time t.

Due to the finiteness of the box B_n, the existence and uniqueness (in distribution) of a B_n-forest-fire process is clear: Given independent rate 1 Poisson processes $(G^n_{t,z})_{t\geq 0}$, $z \in B_n$, a unique corresponding càdlàg process $(\eta^n_{t,z})_{t\geq 0}$, $z \in B_n$, which has the required initial and boundary conditions and satisfies [GROWTH] and [DESTRUCTION] can be obtained by a so-called graphical construction, and [ROT-INV] then follows automatically by the rotation-invariance of the cylinder B_n. For more details on graphical constructions, the reader is referred to [Lig85].

Above, we raised the question of what happens with forest-fire processes on boxes of size n when $n \to \infty$. As far as the dynamics are concerned, this question is partially answered for B_n-forest-fire processes by the following result, where $\mathbb{Q}^+_0 := \mathbb{Q} \cap [0, \infty)$ denotes the set of non-negative rational numbers:

Theorem 1. *For $n \in \mathbb{N}$ let $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in B_n}$ be a B_n-forest-fire process. Embed this process into the upper half-plane $\overline{\mathbb{H}}$ by setting $(\eta^n_{t,z}, G^n_{t,z}) := (0,0)$ for $z \in \overline{\mathbb{H}} \setminus B_n$ and all $t \geq 0$. Then for any strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers, there exists a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ such that $(\eta^{n_{k_l}}_{t,z}, G^{n_{k_l}}_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$ converges weakly to some random variable $(\eta^{\mathbb{Q}}_{t,z}, G^{\mathbb{Q}}_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, where convergence is understood in the space $(\{0,1\}\times\mathbb{N}_0)^{\mathbb{Q}_0^+\times\overline{\mathbb{H}}}$ endowed with the product topology. Moreover, the right-sided limit*

$$(\eta_{t,z}, G_{t,z}) := \lim_{s\downarrow t, s\in\mathbb{Q}_0^+} (\eta^{\mathbb{Q}}_{s,z}, G^{\mathbb{Q}}_{s,z}), \qquad t \geq 0, z \in \overline{\mathbb{H}},$$

exists a.s., and restricted to the complement of a null set, the resulting process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ is an $\overline{\mathbb{H}}$-forest-fire process in the sense of Definition 2 below.

Definition 2. Let $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ be a process[3] with values in $(\{0,1\}\times\mathbb{N}_0)^{[0,\infty)\times\overline{\mathbb{H}}}$, initial condition $\eta_{0,z} = 0$ for $z \in \overline{\mathbb{H}}$ and boundary condition $\eta_{t,x} = 0$ for $t \geq 0, x \in \partial\mathbb{H}$. Suppose that for all $z \in \overline{\mathbb{H}}$ the process $(\eta_{t,z}, G_{t,z})_{t\geq 0}$ is càdlàg. For $z \in \overline{\mathbb{H}}$ and $t > 0$, let $C_{t^-,z}$ denote the cluster of z in the configuration $(\eta_{t^-,w})_{w\in\overline{\mathbb{H}}}$.

Then $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ is called an $\overline{\mathbb{H}}$-**forest-fire process** if the following conditions are satisfied:

[POISSON] The processes $(G_{t,z})_{t\geq 0}$, $z \in \overline{\mathbb{H}}$, are independent Poisson processes with rate 1.

[TRANSL-INV] The distribution of $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ is invariant under translations along the real line, i.e. the processes $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ and $(\eta_{t,z+1}, G_{t,z+1})_{t\geq 0, z\in\overline{\mathbb{H}}}$ have the same distribution.

[GROWTH] For all $t > 0$ and all $z \in \mathbb{H}$ the following implications hold:

(i) $G_{t^-,z} < G_{t,z} \Rightarrow \eta_{t,z} = 1$,
i.e. the growth of a tree at the site z at time t implies that the site z is occupied at time t;

(ii) $\eta_{t^-,z} < \eta_{t,z} \Rightarrow G_{t^-,z} < G_{t,z}$,
i.e. if the site z gets occupied at time t, there must have been the growth of a tree at the site z at time t.

[DESTRUCTION] For all $t > 0$ and all $x \in \partial\mathbb{H}$, $z \in \mathbb{H}$ the following implications hold:

(i) $(G_{t^-,x} < G_{t,x} \Rightarrow \forall w \in C_{t^-,x+i} : \eta_{t,w} = 0) \wedge$
$(|C_{t^-,z}| = \infty \Rightarrow \forall w \in C_{t^-,z} : \eta_{t,w} = 0)$,

[3] Again, a more precise but more cumbersome notation would be $\big((\eta_{t,z}, G_{t,z})_{z\in\overline{\mathbb{H}}}\big)_{t\geq 0}$.

i.e. if the cluster at $x+i$ grows to the boundary $\partial\mathbb{H}$ at time t, it is destroyed at time t, and if the cluster at z is about to become infinite at time t, it is destroyed at time t;

(ii) $\eta_{t^-,z} > \eta_{t,z} \Rightarrow ((\exists u \in \partial C_{t^-,z} \cap \partial\mathbb{H} : G_{t^-,u} < G_{t,u}) \vee |C_{t^-,z}| = \infty)$, i.e. if the site z is destroyed at time t, its cluster either must have grown to the boundary $\partial\mathbb{H}$ at time t or it must have been about to become infinite at time t.

For the remainder of this section, let $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ be any $\overline{\mathbb{H}}$-forest-fire process (not necessarily the specific process constructed in Theorem 1). A closely related auxiliary process is the **pure growth process** $(\sigma_{t,z})_{t\geq 0,z\in\overline{\mathbb{H}}}$, which is obtained when the destruction mechanism [DESTRUCTION] in Definition 2 is omitted, and which is formally defined by

$$\sigma_{t,z} := 1_{\{G_{t,z}>0\}}, \qquad t \geq 0, z \in \overline{\mathbb{H}}, \tag{2.2}$$

where we write 1_A for the indicator function of an event A. Obviously, $(\sigma_{t,z})_{t\geq 0,z\in\overline{\mathbb{H}}}$ is monotone increasing in t and dominates $(\eta_{t,z})_{t\geq 0,z\in\overline{\mathbb{H}}}$ in the sense that

$$\sigma_{t,z} \geq \sigma_{s,z} \geq \eta_{s,z}, \qquad 0 \leq s \leq t, z \in \overline{\mathbb{H}}, \tag{2.3}$$

holds. For a fixed time t, the configuration $(\sigma_{t,z})_{z\in\overline{\mathbb{H}}}$ is simply independent site percolation on $\overline{\mathbb{H}}$, where each site is open with probability $1 - e^{-t}$. In particular, if p_c denotes the critical probability of independent site percolation on $\overline{\mathbb{H}}$ (or equivalently $\mathbb{Z}^2$), then the critical time t_c, defined by $1 - e^{-t_c} = p_c$, has the property that a.s. for $t \leq t_c$, there exists no infinite cluster in the configuration $(\sigma_{t,z})_{z\in\overline{\mathbb{H}}}$, while for $t > t_c$, there exists exactly one infinite cluster in the configuration $(\sigma_{t,z})_{z\in\overline{\mathbb{H}}}$.

However, [DESTRUCTION] in Definition 2 and the fact that the paths of $(\eta_{t,z}, G_{t,z})_{t\geq 0,z\in\overline{\mathbb{H}}}$ are càdlàg imply that for all $t \geq 0$ there exists no infinite cluster in the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$. This gives rise to the question to what extent the processes $(\eta_{t,z})_{t\geq 0,z\in\overline{\mathbb{H}}}$ and $(\sigma_{t,z})_{t\geq 0,z\in\overline{\mathbb{H}}}$ differ and motivates the following definition:

Definition 3. For $t \geq 0$, $x \in \partial\mathbb{H}$ let

$$Y_{t,x} := \sup\{y \in \mathbb{N} : (\exists 0 < t' < t'' \leq t : \eta_{t',x+iy} = 1, \eta_{t'',x+iy} = 0)\} \vee 0 \;\in\; \mathbb{N}_0 \cup \{\infty\}$$

be the height up to which points with real part x have been destroyed up to time t. We call $Y_{t,x}$ the **height of destruction** at the point x up to time t.

Note that for $t \geq 0$ and $x \in \partial\mathbb{H}$

$$\{Y_{t,x} < \infty\} \subset \{\forall 0 \leq s \leq t \,\forall y \geq Y_{t,x} + 1 : \sigma_{s,x+iy} = \eta_{s,x+iy}\} \tag{2.4}$$

holds. It turns out that as a function of time, the height of destruction experiences a phase transition:

Theorem 2. *Let* $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ *be an* $\overline{\mathbb{H}}$*-forest-fire process and let* $(Y_{t,x})_{t\geq 0, x\in\partial\mathbb{H}}$ *be the corresponding heights of destruction. Then for all* $x \in \partial\mathbb{H}$*, the following holds a.s.:* $Y_{t,x} < \infty$ *for* $t < t_c$ *and* $Y_{t,x} = \infty$ *for* $t > t_c$*.*

Informally speaking, this means that after the critical time t_c, the influence of the destruction mechanism [DESTRUCTION] in Definition 2 is not just confined to areas close to the boundary $\partial\mathbb{H}$ but is global on all of $\mathbb{H}$.

We will prove Theorems 1 and 2 in Sections 2.3 and 2.4, respectively. In Section 2.2 we draw the reader's attention to some obvious but open questions about $\overline{\mathbb{H}}$-forest-fire processes. In Appendix B we present simulations of the forest-fire processes on the boxes $B_n^{\mathrm{s}}(w)$ and $B_n^{\mathrm{p}}(w)$ with $n = 200$ and $w = 0$.

2.2 Open problems

The following natural questions about $\overline{\mathbb{H}}$-forest-fire processes $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ and the corresponding heights of destruction $(Y_{t,x})_{t\geq 0, x\in\partial\mathbb{H}}$ remain open:

- Are $\overline{\mathbb{H}}$-forest-fire processes unique in distribution? Is $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ adapted to the filtration generated by the growth processes $(G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$?
- Does there exist $z \in \mathbb{H}$ such that the event $\{\exists t > 0 : |C_{t-,z}| = \infty\}$ has positive probability (where $C_{t-,z}$ is defined as in Definition 2), i.e. do infinite clusters in the left-sided limit occur with positive probability?
- How does the height of destruction behave at the critical time t_c? For instance, does $Y_{t_c,x} < \infty$ a.s. hold for $x \in \partial\mathbb{H}$? We will come back to this question in Chapter 3.

2.3 Proof of Theorem 1

The construction of the limit process in Theorem 1 is partly analogous to the construction of the infinite volume Dürre forest-fire model in [Dür06a]. However, a new strategy is needed when it comes to infinite clusters in the process. This is where we will make use of the translation-invariance property [TRANSL-INV] of the process. We will only give a brief sketch of the parts that are similar to [Dür06a] in Sections 2.3.1, 2.3.2 and 2.3.3 and then focus on the issue of infinite clusters in Sections 2.3.4 and 2.3.5.

For the remainder of this section, consider the following setup: For $n \in \mathbb{N}$ let $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in B_n}$ be a B_n-forest-fire process. Embed this process into the upper half-plane $\overline{\mathbb{H}}$ by setting $(\eta^n_{t,z}, G^n_{t,z}) := (0, 0)$ for $z \in \overline{\mathbb{H}} \setminus B_n$ and all $t \geq 0$. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers.

2.3.1 Construction of the limit process and easy properties

Lemma 1. *(i) The sequence* $(\eta^n_{t,z}, G^n_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, *is tight in the space* $(\{0,1\} \times \mathbb{N}_0)^{\mathbb{Q}_0^+ \times \overline{\mathbb{H}}}$ *endowed with the product topology.*

(ii) There exists a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ *of natural numbers such that* $(\eta^{n_{k_l}}_{t,z}, G^{n_{k_l}}_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$ *converges weakly to some random variable* $(\eta^{\mathbb{Q}}_{t,z}, G^{\mathbb{Q}}_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$.

Proof. First note that since the index set $\mathbb{Q}_0^+ \times \overline{\mathbb{H}}$ is countable, the product spaces $\{0,1\}^{\mathbb{Q}_0^+ \times \overline{\mathbb{H}}}$, $\mathbb{N}_0^{\mathbb{Q}_0^+ \times \overline{\mathbb{H}}}$ and $(\{0,1\} \times \mathbb{N}_0)^{\mathbb{Q}_0^+ \times \overline{\mathbb{H}}}$ are metrizable and, in fact, are Polish spaces. By Tychonoff's theorem, the space $\{0,1\}^{\mathbb{Q}_0^+ \times \overline{\mathbb{H}}}$ is compact and hence the sequence $(\eta^n_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, is trivially tight. Moreover, the sequence $(G^n_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, is clearly convergent and therefore tight by Prokhorov's theorem. As we work in the product topology, we conclude that the joint sequence $(\eta^n_{t,z}, G^n_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, is tight, as well. This proves (i). Part (ii) then follows from (i) by another application of Prokhorov's theorem (in the opposite direction). $\square$

It is easy to see that the limit random variable $(\eta^{\mathbb{Q}}_{t,z}, G^{\mathbb{Q}}_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$ can be extended to a process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, which we henceforth call the limit process:

Lemma 2. *A.s. the right-sided limit*

$$(\eta_{t,z}, G_{t,z}) := \lim_{s\downarrow t, s\in\mathbb{Q}_0^+} (\eta^{\mathbb{Q}}_{s,z}, G^{\mathbb{Q}}_{s,z}), \qquad t \geq 0, z \in \overline{\mathbb{H}},$$

exists.

Proof. This is proved analogously to Lemma 7 in [Dür06a]. $\square$

We now realize the processes $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, and $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ on a joint probability space $(\Omega, \mathcal{A}, \mathbf{P})$, where $\mathcal{A}$ is the completion of the σ-field

$$\sigma\left(\eta^n_{t,z}, G^n_{t,z}; \eta_{t,z}, G_{t,z} : t \geq 0, z \in \overline{\mathbb{H}}, n \in \mathbb{N}\right).$$

There is a very useful relation between the limit process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ and the B_n-forest-fire processes $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, which allows to transfer properties from the B_n-forest-fire processes to the limit process:

Lemma 3. *Let A be an event which is described by the configuration of finitely many sites and finitely many points in time, i.e. there exist $h \in \mathbb{N}$ and a finite set $S \subset \overline{\mathbb{H}}$ such that $A \in \mathcal{P}((\{0,1\} \times \mathbb{N}_0)^{[h]\times S})$, where $\mathcal{P}(X)$ denotes the power set of a set X and $[h] := \{1, 2, \ldots, h\}$. If there exists $N \in \mathbb{N}$ such that for all $0 \leq t_1 < t_2 < \ldots < t_h$ and all $n \geq N$*

$$\mathbf{P}\left[(\eta^n_{t_j,z}, G^n_{t_j,z})_{j\in[h], z\in S} \in A\right] = 0$$

holds, then

$$\mathbf{P}\left[\exists 0 \leq t_1 < t_2 < \ldots < t_h : (\eta_{t_j,z}, G_{t_j,z})_{j\in[h], z\in S} \in A\right] = 0$$

also holds.

Proof. This is proved analogously to Lemma 9 in [Dür06a]. □

The construction of the limit process in Lemma 2 immediately implies that a.s. for all $z \in \overline{\mathbb{H}}$ the process $(\eta_{t,z}, G_{t,z})_{t\geq 0}$ is càdlàg. For $z \in \overline{\mathbb{H}}$ and $t \geq 0$, let $C_{t,z}$ denote the cluster of z in the configuration $(\eta_{t,w})_{w\in\overline{\mathbb{H}}}$, and for $z \in \overline{\mathbb{H}}$ and $t > 0$, let $C_{t^-,z}$ denote the cluster of z in the configuration $(\eta_{t^-,w})_{w\in\overline{\mathbb{H}}}$. Then the following properties of the limit process are straightforward:

Lemma 4. *A.s. the process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ satisfies the initial condition $\eta_{0,z} = 0$ for $z \in \overline{\mathbb{H}}$ and the boundary condition $\eta_{t,x} = 0$ for $t \geq 0, x \in \partial\mathbb{H}$. Moreover, it a.s. has the properties [POISSON] and [GROWTH] (ii) of Definition 2 and satisfies [TRANSL-INV].*

Proof. The proofs for the initial condition and the property [GROWTH] (ii) are easy consequences of Lemma 3 above and are analogous to the proofs of Lemmas 26 and 10 in [Dür06a]. The proof of the property [POISSON] is identical to the proof of Lemma 5 in [Dür06a]. The zero boundary condition for the limit process is trivial since the same boundary condition is satisfied by the B_n-forest-fire processes for all n. Finally, the translation-invariance [TRANSL-INV] of the limit process is a consequence of the rotation-invariance [ROT-INV] of the B_n-forest-fire processes for all n. □

2.3.2 Some auxiliary lemmas

It thus remains to show that the process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ also a.s. has the properties [GROWTH] (i) and [DESTRUCTION] (i), (ii) of Definition 2. In this section we state some auxiliary lemmas, which are in a sense weaker versions of these properties.

We first introduce some further notation: For $0 \leq s \leq t$, $z \in \overline{\mathbb{H}}$ and $n \in \mathbb{N}$, let

$$\mathrm{G}_{s,t,z} := \{G_{s,z} < G_{t,z}\}, \qquad \mathrm{G}^n_{s,t,z} := \left\{G^n_{s,z} < G^n_{t,z}\right\}$$

be the events that the growth of a tree occurs at the site z in the time interval $(s, t]$, and for $0 < s \leq t$, $z \in \overline{\mathbb{H}}$ and $n \in \mathbb{N}$, let

$$\mathrm{G}_{s^-,t,z} := \{G_{s^-,z} < G_{t,z}\}, \qquad \mathrm{G}^n_{s^-,t,z} := \left\{G^n_{s^-,z} < G^n_{t,z}\right\}$$

be the events that the growth of a tree occurs at the site z in the time interval $[s, t]$. Moreover, if $X \ni x \mapsto f_x \in U$ is any function from a set X to a set U, then for $X' \subset X$, $u \in U$ we abbreviate the expression $\forall x \in X' : f_x = u$ by $f_{X'} = u$. Finally, if A, B are two events, we will denote the complement of A by $\complement A$, and (in slight abuse of notation) we will write $\{A, B\}$ instead of $A \cap B$.

Lemma 5 is a weaker version of [GROWTH] (i):

Lemma 5. *Suppose that $w, z \in \mathbb{H}$ are neighbouring sites. Then*

$$\mathbf{P}\left[\exists t > 0 : \eta_{t,w} = 1, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0\right] = 0$$

holds; in other words: A.s. if there is the growth of a tree at the site z at some time t and a neighbouring site w is occupied at time t, then the site z is also occupied at time t.

Proof. Let $w, z \in \mathbb{H}$ be neighbouring sites. Since $(G_{t,w})_{t\geq 0}$ and $(G_{t,z})_{t\geq 0}$ are independent Poisson processes (see Lemma 4), a.s. they do not have jumps at the same time. Using this and the fact that Poisson process paths are a.s. piecewise constant and càdlàg, we obtain

$$\begin{aligned}\{\exists t>0 : \eta_{t,w}=1, \mathrm{G}_{t^-,t,z}, \eta_{t,z}=0\} &\overset{\text{a.s.}}{\subset} \left\{\exists t>0 : \complement\, \mathrm{G}_{t^-,t,w}, \eta_{t,w}=1, \mathrm{G}_{t^-,t,z}, \eta_{t,z}=0\right\}\\ &\overset{\text{a.s.}}{\subset} \left\{\exists 0\leq s<t : \complement\, \mathrm{G}_{s,t,w}, \eta_{t,w}=1, \mathrm{G}_{t^-,t,z}, \eta_{t,z}=0\right\}\\ &\overset{\text{a.s.}}{\subset} \left\{\exists 0\leq s<t : \complement\, \mathrm{G}_{s,t,w}, \eta_{t,w}=1, \mathrm{G}_{s,t,z}, \eta_{t,z}=0\right\}.\end{aligned}$$

Now for all sufficiently large n (such that $w, z \in B_n$) and arbitrary $0 \leq s < t$, it is easy to deduce from [GROWTH] and [DESTRUCTION] in Definition 1 that B_n-forest-fire processes satisfy

$$\mathbf{P}\left[\complement\, \mathrm{G}^n_{s,t,w}, \eta^n_{t,w}=1, \mathrm{G}^n_{s,t,z}, \eta^n_{t,z}=0\right]=0.$$

The result therefore follows from Lemma 3. □

Lemmas 6 and 7 are about the destruction of occupied clusters:

Lemma 6. *For all $w, z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists 0\leq s<t : w\in C_{s,z}, \eta_{t,w}=0, \eta_{t,z}=1, \complement\, \mathrm{G}_{s,t,z}\right]=0;$$

in other words: A.s. if a site w was occupied at some time s but is vacant at some later time $t > s$, then any other site z which was in the cluster of w at time s must be vacant at time t unless there is the growth of a tree at that site in the time interval $(s, t]$.

Proof. This is a consequence of Lemma 3 above and is proved analogously to Lemma 12 in [Dür06a]. □

Lemma 7 is the first half of [DESTRUCTION] (i) in Definition 2:

Lemma 7. *For all $x \in \partial\mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t>0 : \mathrm{G}_{t^-,t,x}, \exists w\in C_{t^-,x+i} : \eta_{t,w}=1\right]=0;$$

in other words: A.s. if the cluster at $x + i$ grows to the boundary $\partial\mathbb{H}$ at some time t, it is destroyed at time t.

Proof. Let $x \in \partial\mathbb{H}$. For $z \in \mathbb{H}$, let C^{fin}_z denote the (countable) set of all finite connected subsets of $\mathbb{H}$ which contain the site z. Due to the equality

$$\{\exists t>0 : \mathrm{G}_{t^-,t,x}, \exists w\in C_{t^-,x+i} : \eta_{t,w}=1\} = \bigcup_{S\in C^{\mathrm{fin}}_{x+iy}} \bigcup_{w\in S} \{\exists t>0 : \mathrm{G}_{t^-,t,x}, \eta_{t^-,S}=1, \eta_{t,w}=1\}$$

it suffices to show that for all $S \in C^{\mathrm{fin}}_{x+iy}$ and $w \in S$

$$\mathbf{P}\left[\exists t>0 : \mathrm{G}_{t^-,t,x}, \eta_{t^-,S}=1, \eta_{t,w}=1\right]=0$$

holds. So let $S \in C^{\mathrm{fin}}_{x+iy}$ and $w \in S$ be fixed. Since $(G_{t,x})_{t\geq 0}$ and $(G_{t,w})_{t\geq 0}$ are independent Poisson processes (see Lemma 4), a.s. they do not have jumps at the same time. Using this and the fact that the paths of the limit process are a.s. piecewise constant and càdlàg, we obtain

$$\{\exists t > 0 : \mathrm{G}_{t-,t,x}, \eta_{t-,S} = 1, \eta_{t,w} = 1\} \overset{\text{a.s.}}{\subset} \left\{\exists 0 \leq s < t : \mathrm{G}_{s,t,x}, \eta_{s,S} = 1, \eta_{t,w} = 1, \complement\, \mathrm{G}_{s,t,w}\right\}.$$

Now for all sufficiently large n (such that $\{x\} \cup S \subset B_n$) and arbitrary $0 \leq s < t$, it is easy to deduce from [GROWTH] and [DESTRUCTION] in Definition 1 that B_n-forest-fire processes satisfy

$$\mathbf{P}\left[\mathrm{G}^n_{s,t,x}, \eta^n_{s,S} = 1, \eta^n_{t,w} = 1, \complement\, \mathrm{G}^n_{s,t,w}\right] = 0.$$

The result therefore follows from Lemma 3. □

2.3.3 A Markov-type property of the limit process

Let $(\mathcal{F}_t)_{t\geq 0}$ be the completion of the canonical filtration of the limit process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, i.e. $\mathcal{F}_t$ is the completion of the σ-field

$$\sigma\left((\eta_{s,z}, G_{s,z}) : 0 \leq s \leq t, z \in \overline{\mathbb{H}}\right)$$

generated up to time $t \geq 0$. As is customary, if T is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$, we define the σ-field up to time T by

$$\mathcal{F}_T := \{A \in \mathcal{A} : (\forall t \geq 0 : A \cap \{T \leq t\} \in \mathcal{F}_t)\},$$

where $\mathcal{A}$ is the full σ-field introduced in the paragraph below Lemma 2. Then the limit process satisfies the following Markov-type property:

Lemma 8. *Let T be a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. Then for all $A \in \mathcal{F}_T$*

$$\mathbf{P}\left[(G_{T+t,z} - G_{T,z})_{t\geq 0, z\in\overline{\mathbb{H}}} \in \cdot\,, T < \infty, A\right] = \mathbf{P}\left[(G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}} \in \cdot\,\right] \mathbf{P}\left[T < \infty, A\right]$$

holds; in other words: On the event $\{T < \infty\}$, the increments $(G_{T+t,z} - G_{T,z})_{t\geq 0}$, $z \in \overline{\mathbb{H}}$, of the Poisson processes after time T are independent of the σ-field $\mathcal{F}_T$ and are again independent Poisson processes.

Proof. This is proved analogously to Lemma 19 in [Dür06a]. □

2.3.4 Non-existence of infinite clusters

The aim of this section is to prove Lemma 11, which states that a.s. there do not exist infinite clusters in the process $(\eta_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$.

Lemma 9. *For all $t \geq 0$, $z \in \mathbb{H}$ and $R \in \mathbb{N}$ we have*

$$\mathbf{P}\left[|\{w \in C_{t,z} : \operatorname{Im} w = R\}| = \infty\right] = 0;$$

in other words: For any fixed time t there a.s. does not exist a cluster which contains infinitely many sites with the same distance R from $\partial\mathbb{H}$.

Intuitively, the reason why Lemma 9 should hold is the following: Suppose that the cluster $C_{t,z}$ contains infinitely many sites w with distance $\operatorname{Im} w = R$ from $\partial\mathbb{H}$. Lemma 6 and the fact that the paths of the limit process are a.s. piecewise constant and càdlàg imply that a.s. the cluster $C_{t,z}$ persists during time $[t, t+\epsilon]$ for some $\epsilon > 0$. However, for any $\epsilon > 0$ there a.s. is a growth sequence within $[t, t+\epsilon]$ from one of the sites w with $\operatorname{Im} w = R$ down to the boundary $\partial\mathbb{H}$, which causes the cluster at z to be destroyed before time $t + \epsilon$ - a contradiction. We now make this argument rigorous.

Proof. Let $t \geq 0$, $z \in \mathbb{H}$ and $R \in \mathbb{N}$. We abbreviate

$$E_{t,z} := \{|\{w \in C_{t,z} : \operatorname{Im} w = R\}| = \infty\}.$$

On the event $E_{t,z}$, let $(W_k)_{k\in\mathbb{Z}}$ be a disjoint enumeration of the sites $w \in C_{t,z}$ with $\operatorname{Im} w = R$. Moreover, for $w \in \mathbb{H}$ and $s \geq 0$, $\gamma > 0$ let

$$\text{V-GROWTH-SEQ}(w, s, \gamma) := \left\{\forall j \in \{1, \ldots, \operatorname{Im} w\} : \mathrm{G}_{s+\frac{j-1}{\operatorname{Im} w}\gamma, s+\frac{j}{\operatorname{Im} w}\gamma, w-ji}\right\}$$

denote the event that there is a vertical growth sequence from the site $w - i$ to the boundary $\partial\mathbb{H}$ between times s and $s + \gamma$ (with the jth growth event between times $s + \frac{j-1}{\operatorname{Im} w}\gamma$ and $s + \frac{j}{\operatorname{Im} w}\gamma$ for $j = 1, \ldots, \operatorname{Im} w$).

Since the paths of the limit process are a.s. piecewise constant and càdlàg, we have

$$E_{t,z} \overset{\text{a.s.}}{\subset} \left\{E_{t,z}, \exists \epsilon \in \mathbb{Q} \cap (0, \infty) : \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}\right\}.$$

It therefore suffices to show

$$\mathbf{P}\left[E_{t,z}, \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}\right] = 0 \tag{2.5}$$

for arbitrary $\epsilon > 0$.

So pick $\epsilon > 0$. Lemma 8 implies that conditional on $E_{t,z}$ (we can assume $\mathbf{P}[E_{t,z}] > 0$ without loss of generality), the events V-GROWTH-SEQ(W_k, t, ϵ), $k \in \mathbb{Z}$, are independent with

$$\mathbf{P}\left[\text{V-GROWTH-SEQ}(W_k, t, \epsilon) |\, E_{t,z}\right] = \mathbf{P}\left[\text{V-GROWTH-SEQ}(iR, 0, \epsilon)\right] > 0$$

for all $k \in \mathbb{Z}$. We therefore conclude from the Borel-Cantelli lemma that

$$E_{t,z} \overset{\text{a.s.}}{\subset} \{E_{t,z}, \text{V-GROWTH-SEQ}(W_k, t, \epsilon) \text{ for infinitely many } k\} \tag{2.6}$$

holds.

For the moment, let $k \in \mathbb{Z}$ be fixed. Considering the first $R-1$ growth events (in $\mathbb{H}$) of the event V-GROWTH-SEQ(W_k, t, ϵ) and applying Lemmas 5 and 6 repeatedly, we see that

$$\begin{aligned}&\left\{E_{t,z}, \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}, \text{V-GROWTH-SEQ}(W_k, t, \epsilon)\right\}\\ &\qquad\overset{\text{a.s.}}{\subset} \left\{E_{t,z}, \forall s \in [t + \frac{R-1}{R}\epsilon, t+\epsilon] : \underbrace{W_k - (R-1)i}_{=\operatorname{Re} W_k + i} \in C_{s,z}\right\}. \end{aligned} \tag{2.7}$$

However, considering the last growth event (in $\partial\mathbb{H}$) of the event V-GROWTH-SEQ(W_k, t, ϵ) and using (2.7) and Lemma 7, it follows that

$$\begin{aligned}&\left\{E_{t,z}, \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}, \text{V-GROWTH-SEQ}(W_k, t, \epsilon)\right\}\\ &\qquad\overset{\text{a.s.}}{\subset} \left\{E_{t,z}, \exists s \in [t + \frac{R-1}{R}\epsilon, t+\epsilon] : \eta_{s,z} = 0\right\}. \end{aligned} \tag{2.8}$$

Since the condition $\exists s \in [t + \frac{R-1}{R}\epsilon, t+\epsilon] : \eta_{s,z} = 0$ on the right side of (2.8) contradicts the condition $\eta_{[t,t+\epsilon],z} = 1$ on its left side, we conclude that

$$\mathbf{P}\left[E_{t,z}, \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}, \text{V-GROWTH-SEQ}(W_k, t, \epsilon)\right] = 0 \tag{2.9}$$

for all $k \in \mathbb{Z}$.

Equation (2.5) is now a direct consequence of (2.6) and (2.9). □

Definition 4. For $t \geq 0$ let $N_t \in \mathbb{N}_0 \cup \{\infty\}$ denote the number of infinite clusters in the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$.

Lemma 10. *For all $t \geq 0$ we have $\mathbf{P}[N_t = 0] = 1$; in other words: For any fixed time t there a.s. does not exist an infinite cluster in the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$.*

Intuitively, the reason why Lemma 10 should hold is the following: Due to the translation-invariance [TRANSL-INV] of the limit process we expect $N_t \in \{0, 1, \infty\}$ a.s. If $N_t = 1$, then the translation-invariance implies that a.s. there exists $R \in \mathbb{N}$ such that there are infinitely many sites w with $\operatorname{Im} w = R$ in the unique infinite cluster at time t, but this is ruled out by Lemma 9. On the other hand, if $N_t = \infty$, then the translation-invariance implies that a.s. there exists $R \in \mathbb{N}$ such that there are infinitely many infinite clusters with distance R from $\partial\mathbb{H}$ at time t. But due to the translation-invariance and the limited amount of space these clusters must be very close to one another. Using this observation and the fact that the paths of the limit process are a.s. piecewise constant and càdlàg, we find that a.s. there exists $\epsilon > 0$ such that by time $t + \epsilon$, the above-mentioned clusters have grown together to form one infinite cluster containing infinitely many sites with distance R from $\partial\mathbb{H}$. Yet once more, this is ruled out by Lemma 9. It should be noted that the classical Burton-Keane argument to rule out the case $N_t = \infty$ cannot be applied here because we work on the half-plane $\overline{\mathbb{H}}$ and not on $\mathbb{Z}^2$, and because the translation-invariance [TRANSL-INV] only holds in the x-direction. We now make the above heuristics rigorous.

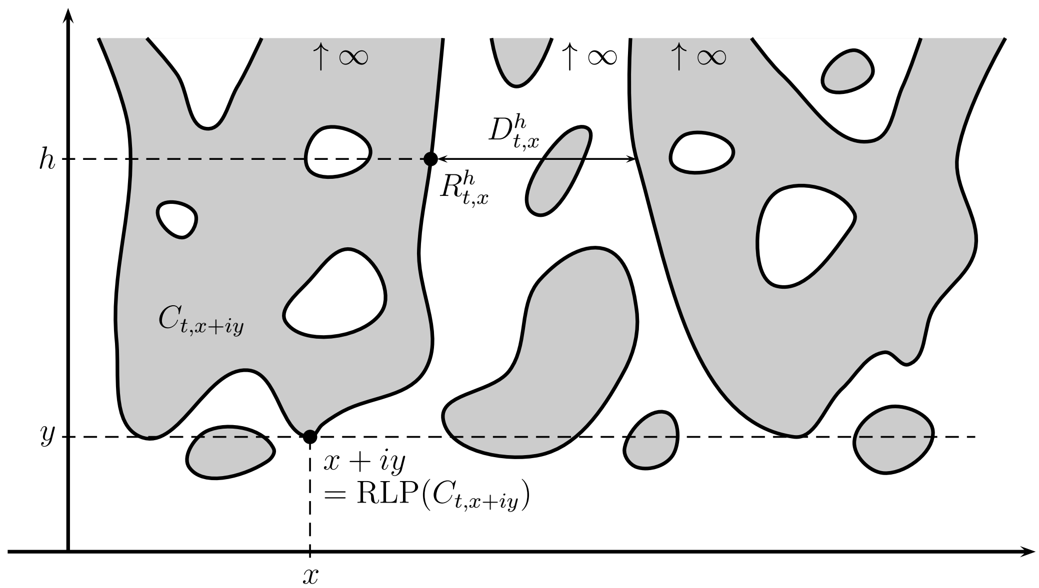


Figure 2.2: A visualization of the event $A_{t,x}$ and the associated random variables $R^h_{t,x}$, $D^h_{t,x}$

Proof. Let $t \geq 0$. In the following, for a subset $S \subset \overline{\mathbb{H}}$ we write

$$\mathrm{dist}(S, \partial\mathbb{H}) := \min\{\mathrm{Im}\, w : w \in S\}$$

for its vertical distance from $\partial\mathbb{H}$. Let us call a site $z \in \mathbb{H}$ the rightmost lowest point of its cluster $C_{t,z}$ (hereinafter abbreviated by $z = \mathrm{RLP}(C_{t,z})$) if

- $\mathrm{Im}\, z$ is minimal in $C_{t,z}$, i.e. $\mathrm{Im}\, z = \mathrm{dist}(C_{t,z}, \partial\mathbb{H})$, and
- $\mathrm{Re}\, z$ is maximal among all $w \in C_{t,z}$ with $\mathrm{Im}\, w = \mathrm{dist}(C_{t,z}, \partial\mathbb{H})$.

Lemma 9 implies that a.s. every non-empty cluster in the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$ has a rightmost lowest point, so that

$$\{N_t \geq 1\} \overset{\text{a.s.}}{\subset} \{\exists x \in \mathbb{Z}\, \exists y \in \mathbb{N} : x + iy = \mathrm{RLP}(C_{t,x+iy}), |C_{t,x+iy}| = \infty\}.$$

Let $y \in \mathbb{N}$ be fixed, and set

$$A_{t,x} := \{x + iy = \mathrm{RLP}(C_{t,x+iy}), |C_{t,x+iy}| = \infty\}$$

for all $x \in \mathbb{Z}$. Due to the translation-invariance [TRANSL-INV] of the limit process we have $\mathbf{P}[A_{t,x}] = \mathbf{P}[A_{t,0}]$ for all $x \in \mathbb{Z}$, and it thus suffices to prove $\mathbf{P}[A_{t,0}] = 0$.

Step 1: Using the translation-invariance [TRANSL-INV] of the limit process again, we see that for all $x \in \mathbb{Z}$

$$A_{t,x} \overset{\text{a.s.}}{\subset} \{A_{t,u} \text{ for infinitely many } u \in \mathbb{N}_0\} \tag{2.10}$$

holds by the Poincaré recurrence theorem (see e.g. [Shi95], Section V.1, Theorem 1). Since the rightmost lowest point of a cluster is unique (if it exists), (2.10) in particular implies that on the event $A_{t,x}$, there a.s. exist infinitely many infinite clusters at time t which are to the right of the cluster $C_{t,x+iy}$ and have vertical distance y from $\partial\mathbb{H}$. For $x \in \mathbb{Z}$ and integer $h \geq y$, on the event $A_{t,x}$, let

$$R^h_{t,x} := \max\{r \in \mathbb{Z} : r + ih \in C_{t,x+iy}\} + ih$$

be the rightmost point of the cluster $C_{t,x+iy}$ at height h, and let

$$D^h_{t,x} := \min\Big\{d \in \mathbb{N} : |C_{t,R^h_{t,x}+d}| = \infty, \operatorname{dist}(C_{t,R^h_{t,x}+d}, \partial\mathbb{H}) = y\Big\}$$

be the horizontal distance from $R^h_{t,x}$ to the "next" infinite cluster with vertical distance y from $\partial\mathbb{H}$. On the event $A_{t,x}$, $R^h_{t,x}$ and $D^h_{t,x}$ are a.s. well-defined because obviously

$$A_{t,x} \overset{\text{a.s.}}{\subset} \{A_{t,x}, \forall h \geq y\, \exists r \in \mathbb{Z} : r + ih \in C_{t,x+iy}\}$$

holds, and because of Lemma 9 and the observation below equation (2.10). See Figure 2.2 for a visualization of the event $A_{t,x}$ and the associated random variables $R^h_{t,x}$, $D^h_{t,x}$. The aim of Step 1 is to prove that

$$A_{t,x} \overset{\text{a.s.}}{\subset} \Big\{A_{t,x}, \liminf_{h\to\infty} D^h_{t,x} < \infty\Big\} \tag{2.11}$$

holds for all $x \in \mathbb{Z}$.

Suppose that (2.11) is not true. Then there exist sequences $(c_h)_{h\geq y}$ and $(d_h)_{h\geq y}$ of natural numbers with $d_h \uparrow \infty$ as $h \to \infty$ such that the events

$$B_{t,x} := \big\{A_{t,x}, \forall h \geq y : \big|\operatorname{Re} R^h_{t,x} - x\big| \leq c_h, D^h_{t,x} \geq d_h\big\}$$

have positive probability for all $x \in \mathbb{Z}$. (Of course, the translation-invariance [TRANSL-INV] of the limit process implies $\mathbf{P}[B_{t,x}] = \mathbf{P}[B_{t,0}]$ for all $x \in \mathbb{Z}$.) From the translation-invariance [TRANSL-INV] of the limit process and the Birkhoff ergodic theorem (see e.g. [Shi95], Section V.3, Theorem 1) we thus deduce that there exists $\beta > 0$ such that

$$\mathbf{P}\left[\sum_{x=0}^{n-1} 1_{B_{t,x}} > \beta n \text{ eventually as } n \to \infty\right] > 0.$$

On the event $\big\{\sum_{x=0}^{n-1} 1_{B_{t,x}} > \beta n \text{ eventually as } n \to \infty\big\}$, for large n the sites $iy, 1 + iy, \ldots, (n-1) + iy$ are part of at least $\lceil \beta n \rceil$ different infinite clusters for which the following holds: For $h \geq y$ their rightmost points at height h are all contained in the interval $[-c_h, (n-1) + c_h] + ih$ and have at least horizontal distance d_h from one another. Hence the horizontal distance between the right-most points at height h of the first and the $\lceil \beta n \rceil$th cluster is less than $n + 2c_h$ but greater than or equal to $(\lceil \beta n \rceil - 1)d_h$. In particular, it holds that

$$\frac{\lceil \beta n \rceil - 1}{n} d_h \leq 1 + \frac{2c_h}{n}.$$

Letting $n \to \infty$, we obtain $\beta d_h \leq 1$ for all $h \geq y$. But since $\beta > 0$, this contradicts the condition that $d_h \uparrow \infty$ for $h \to \infty$. We have thus proven (2.11).

Step 2: We now prove $\mathbf{P}[A_{t,0}] = 0$. Let $\epsilon > 0$ be arbitrary; since the paths of the limit process are a.s. piecewise constant and càdlàg, it suffices to show

$$\mathbf{P}\left[A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}\right] = 0.$$

In fact, we will prove

$$\left\{A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}\right\} \overset{\text{a.s.}}{\subset} \{|\{w \in C_{t+\epsilon,iy} : \operatorname{Im} w = y\}| = \infty\}, \tag{2.12}$$

and the latter is a null set by Lemma 9. Let $K \in \mathbb{N}$ be arbitrary; the inclusion (2.12) then follows if we can show

$$\left\{A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}\right\} \overset{\text{a.s.}}{\subset} \{|\{w \in C_{t+\epsilon,iy} : \operatorname{Im} w = y\}| > K\}. \tag{2.13}$$

On the event $A_{t,0}$, we recursively define

$$X_1 := 0,\ Z_1 := iy$$

and for $k \geq 2$

$$X_k := \min\left\{x > X_{k-1} : 1_{A_{t,x}} = 1\right\},\ Z_k := X_k + iy,$$

which is a.s. well-defined by (2.10). Informally speaking, for $k \in \mathbb{N}$, C_{t,Z_k} is "the kth infinite cluster with distance y from $\partial\mathbb{H}$", where we count clusters from left to right, starting with the cluster at iy. Since the paths of the limit process are a.s. piecewise constant and càdlàg, and because of (2.11), we have

$$\begin{aligned} A_{t,0} \overset{\text{a.s.}}{\subset} \Big\{ & A_{t,0}, \exists \tilde{\epsilon} \in \mathbb{Q} \cap (0,\epsilon), \exists d \in \mathbb{N}\, \forall k \in \{1,\ldots,K\} : \eta_{[t,t+\tilde{\epsilon}],Z_{k+1}} = 1, \complement\, \mathrm{G}_{t,t+\tilde{\epsilon},Z_{k+1}}, \\ & \liminf_{h\to\infty} D^h_{t,X_k} \leq d \Big\}. \end{aligned}$$

So let $0 < \tilde{\epsilon} < \epsilon$ and $d \in \mathbb{N}$ be arbitrary. For the proof of (2.13) it then suffices to show

$$\begin{aligned} & \Big\{ A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}, \forall k \in \{1,\ldots,K\} : \eta_{[t,t+\tilde{\epsilon}],Z_{k+1}} = 1, \complement\, \mathrm{G}_{t,t+\tilde{\epsilon},Z_{k+1}}, \\ & \liminf_{h\to\infty} D^h_{t,X_k} \leq d \Big\} \overset{\text{a.s.}}{\subset} \{|\{w \in C_{t+\epsilon,iy} : \operatorname{Im} w = y\}| > K\}. \end{aligned} \tag{2.14}$$

During the next two paragraphs, let $k \in \{1, \dots, K\}$ be fixed. On the event $\left\{A_{t,0}, \liminf_{h\to\infty} D^h_{t,X_k} \le d\right\}$, we recursively define

$$H_{k,1} := \min\left\{h \ge y : D^h_{t,X_k} \le d\right\}$$

and for $l \ge 2$

$$H_{k,l} := \min\left\{h > H_{k,l-1} : D^h_{t,X_k} \le d\right\}.$$

(This is well-defined since D^h_{t,X_k} is integer-valued.) Then for $l \in \mathbb{N}$, $R^{H_{k,l}}_{t,X_k}$ is the "lth rightmost point of the cluster C_{t,Z_k} whose horizontal distance to the cluster $C_{t,Z_{k+1}}$ is less than or equal to d", where we count these points from bottom to top. Moreover, for $w \in \mathbb{H}$, $c \in \mathbb{N}$ and $s \ge 0$, $\gamma > 0$ let

$$\text{H-GROWTH-SEQ}(w, c, s, \gamma) := \left\{\forall j \in \{1, \dots, c\} : \mathrm{G}_{s+\frac{j-1}{c}\gamma, s+\frac{j}{c}\gamma, w+j}\right\}$$

denote the event that there is a horizontal growth sequence from the site $w+1$ to the site $w+c$ between times s and $s+\gamma$ (with the jth growth event between times $s+\frac{j-1}{c}\gamma$ and $s+\frac{j}{c}\gamma$ for $j = 1, \dots, c$).

Lemma 8 implies that conditional on $A_{t,0}$ (we can assume $\mathbf{P}[A_{t,0}] > 0$ without loss of generality), the events H-GROWTH-SEQ$(R^{H_{k,l}}_{t,X_k}, d, t, \tilde{\epsilon})$, $l \in \mathbb{N}$, are independent with

$$\mathbf{P}\left[\text{H-GROWTH-SEQ}(R^{H_{k,l}}_{t,X_k}, d, t, \tilde{\epsilon}) \middle| A_{t,0}\right] = \mathbf{P}\left[\text{H-GROWTH-SEQ}(i, d, 0, \tilde{\epsilon})\right] > 0$$

for all $l \in \mathbb{N}$. We therefore conclude from the Borel-Cantelli lemma that

$$A_{t,0} \overset{\text{a.s.}}{\subset} \left\{A_{t,0}, \text{H-GROWTH-SEQ}(R^{H_{k,l}}_{t,X_k}, d, t, \tilde{\epsilon}) \text{ for infinitely many } l\right\} \tag{2.15}$$

holds.

But for any fixed numbers $l_1, \dots, l_K \in \mathbb{N}$ repeated applications of Lemmas 5 and 6 yield

$$\begin{aligned}
&\Big\{A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}, \forall k \in \{1, \dots, K\} : \eta_{[t,t+\tilde{\epsilon}],Z_{k+1}} = 1, \complement\, \mathrm{G}_{t,t+\tilde{\epsilon},Z_{k+1}},\\
&\liminf_{h\to\infty} D^h_{t,X_k} \le d, \text{H-GROWTH-SEQ}(R^{H_{k,l_k}}_{t,X_k}, d, t, \tilde{\epsilon})\Big\}\\
&\qquad\overset{\text{a.s.}}{\subset} \left\{A_{t,0}, C_{t+\epsilon,iy} \supset \bigcup_{k=1}^{K+1} C_{t,Z_k}\right\}\\
&\qquad\overset{\text{a.s.}}{\subset} \left\{\left|\{w \in C_{t+\epsilon,iy} : \operatorname{Im} w = y\}\right| \ge K+1\right\}.
\end{aligned} \tag{2.16}$$

Equation (2.14) is now a direct consequence of (2.15) and (2.16). □

Lemma 11. *We have* $\mathbf{P}[\forall t \ge 0 : N_t = 0] = 1$*; in other words: A.s. there does not exist an infinite cluster in the configuration* $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$ *for any time* $t \ge 0$.

Proof. Using the fact that the paths of the limit process are a.s. piecewise constant and càdlàg, and then applying Lemma 6, we obtain

$$\begin{aligned}\{\exists t \geq 0 : N_t \geq 1\} &\overset{\text{a.s.}}{\subset} \left\{\exists t \geq 0\, \exists z \in \mathbb{H}\, \exists \epsilon > 0 : |C_{t,z}| = \infty, \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}\right\} \\ &\overset{\text{a.s.}}{\subset} \left\{\exists t \geq 0\, \exists z \in \mathbb{H}\, \exists \epsilon > 0 : |C_{t,z}| = \infty, \forall w \in C_{t,z} : \eta_{[t,t+\epsilon],w} = 1\right\} \\ &\subset \left\{\exists t \in \mathbb{Q}_0^+ : N_t \geq 1\right\}.\end{aligned}$$

Since the set $\mathbb{Q}_0^+$ is countable, the last event is a null set by Lemma 10. □

2.3.5 Infinite clusters in the left-sided limit

The aim of this section is to prove Lemma 16, which states that a.s. clusters in the process $(\eta_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ are destroyed if they are about to become infinite. We start with the following weaker version of this statement:

Lemma 12. *For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 1\right] = 0;$$

in other words: A.s. if the left-sided limit of the cluster at z is infinite at some time t, then the site z gets destroyed at time t unless there is the growth of a tree at z at time t.

Proof. Let $z \in \mathbb{H}$. Since a.s. $|C_{t,z}| < \infty$ holds for any time $t \geq 0$ (Lemma 11) and since the paths of the limit process are a.s. piecewise constant and càdlàg, it follows that

$$\begin{aligned}&\left\{\exists t > 0 : |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 1\right\} \\ &\qquad \overset{\text{a.s.}}{\subset} \left\{\exists t > 0 : |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 1, \exists w \in C_{t^-,z} : \eta_{t,w} = 0\right\} \\ &\qquad \overset{\text{a.s.}}{\subset} \left\{\exists 0 \leq s < t : \complement\, \mathrm{G}_{s,t,z}, \eta_{t,z} = 1, \exists w \in C_{s,z} : \eta_{t,w} = 0\right\}.\end{aligned}$$

But the latter is a null set by Lemma 6. □

Lemma 13. *(i) For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists 0 < s < t : |C_{s^-,z}| = \infty, |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{s^-,t,z}\right] = 0; \tag{2.17}$$

in other words: A.s. if the left-sided limit of the cluster at z is infinite at some time s, the left-sided limit of the cluster cannot be infinite at some later time $t > s$ unless there is the growth of a tree at z in the time interval $[s, t]$.

(ii) For all $z \in \mathbb{H}$ the set $\{t > 0 : |C_{t^-,z}| = \infty\}$ of times at which the left-sided limit of the cluster at z is infinite a.s. has no accumulation points.

Proof. Let $z \in \mathbb{H}$. Lemma 12 and the property [GROWTH] (ii) of the limit process (see Lemma 4) imply

$$\begin{aligned}\left\{\exists 0 < s < t : |C_{s^-,z}| = \infty, |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{s^-,t,z}\right\} \\ \overset{\text{a.s.}}{\subset} \left\{\exists 0 < s < t : \eta_{s,z} = 0, \complement\, \mathrm{G}_{s,t,z}, |C_{t^-,z}| = \infty\right\} \\ \overset{\text{a.s.}}{\subset} \left\{\exists 0 < s < t : \eta_{[s,t],z} = 0, |C_{t^-,z}| = \infty\right\}.\end{aligned}$$

But since the conditions $\eta_{[s,t],z} = 0$ and $|C_{t^-,z}| = \infty$ in the last event obviously contradict each other, we conclude that (2.17) holds indeed.

It now follows from (2.17) that a.s. if the set $\{t > 0 : |C_{t^-,z}| = \infty\}$ has an accumulation point, then the set $\{t > 0 : \mathrm{G}_{t^-,t,z}\}$ of times at which a tree grows at the site z also has an accumulation point. But since $(G_{t,z})_{t\geq 0}$ is a Poisson process, this happens with probability zero. □

For $z \in \mathbb{H}$, we recursively define $T_{0,z} := 0$ and for $k \in \mathbb{N}$

$$T_{k,z} := \inf \{t > T_{k-1,z} : |C_{t^-,z}| = \infty\} \in (0, \infty].$$

Lemma 13 (ii) implies that a.s. the inclusion

$$\{t > 0 : |C_{t^-,z}| = \infty\} \subset \{T_{k,z} : k \in \mathbb{N}\} \tag{2.18}$$

holds. This allows us to treat the issue of infinite left-sided clusters at z by considering the countable sequence of random times $T_{k,z}$, $k \in \mathbb{N}$. In fact, these random times are predictable stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ introduced in Section 2.3.3, where predictability is defined as follows:

Definition 5. A stopping time T with respect to $(\mathcal{F}_t)_{t\geq 0}$ is called **predictable** if there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times with respect to $(\mathcal{F}_t)_{t\geq 0}$ which a.s. satisfy $T_n \uparrow T$ for $n \to \infty$ and $T_n < T$ for all $n \in \mathbb{N}$. In this case, the sequence $(T_n)_{n\in\mathbb{N}}$ is said to announce the stopping time T.

Lemma 14. *For all $z \in \mathbb{H}$ and $k \in \mathbb{N}$, $T_{k,z}$ is a predictable stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Proof. Let $z \in \mathbb{H}$ and $k \in \mathbb{N}$. Then $T_{k,z}$ is obviously a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. We now prove that $T_{k,z}$ is announced by the sequence

$$T_{k,z,n} := \inf \{t > T_{k-1,z} : |C_{t,z}| \geq n\} \wedge n, \qquad n \in \mathbb{N}.$$

Clearly, for each $n \in \mathbb{N}$, $T_{k,z,n}$ is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$, which a.s. satisfies $T_{k,z,n} < T_{k,z}$ and $T_{k,z,n} \leq T_{k,z,n+1}$. Consequently, the limit $\tilde{T}_{k,z} := \lim_{n\to\infty} T_{k,z,n}$ exists a.s. and satisfies $\tilde{T}_{k,z} \leq T_{k,z}$ a.s. From the latter we deduce that

$$\tilde{T}_{k,z} = T_{k,z} \text{ a.s. on the event } \left\{\tilde{T}_{k,z} = \infty\right\}$$

holds. On the other hand, the definition of $T_{k,z,n}$, Lemma 11 and the fact that the paths of the limit process are a.s. piecewise constant and càdlàg imply that for all $n \in \mathbb{N}$

$$\left\{\tilde{T}_{k,z} < \infty\right\} \overset{\text{a.s.}}{\subset} \left\{\tilde{T}_{k,z} < \infty, \exists s \in [0, \tilde{T}_{k,z}) : |C_{s,z}| \geq n, \forall t \in [s, \tilde{T}_{k,z}) \, \forall w \in H_n(z) : \eta_{t,w} = \eta_{s,w}\right\}$$

holds, where $H_n(z) := z + [-n, n]^2 \cap \mathbb{H}$. Since n is arbitrary, this yields

$$\left\{\tilde{T}_{k,z} < \infty\right\} \overset{\text{a.s.}}{\subset} \left\{\tilde{T}_{k,z} < \infty, |C_{\tilde{T}_{k,z}^-,z}| = \infty\right\}. \tag{2.19}$$

Moreover, since a.s. no two growth events occur at the same time, Lemma 12 in particular shows that on the event $\left\{\tilde{T}_{k,z} < \infty\right\}$, we a.s. have $|C_{T_{k-1,z},z}| \leq 1$ and hence $\tilde{T}_{k,z} > T_{k-1,z}$. We can therefore conclude from (2.19) that

$$\tilde{T}_{k,z} = T_{k,z} \text{ a.s. on the event } \left\{\tilde{T}_{k,z} < \infty\right\}$$

holds, which completes the proof of the lemma. $\square$

The Markov-type property stated in Lemma 8 now implies the following:

Lemma 15. *(i) Let T be a predictable stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. Then for all $z \in \overline{\mathbb{H}}$ we have*

$$\mathbf{P}\left[T < \infty, \mathrm{G}_{T^-,T,z}\right] = 0. \tag{2.20}$$

(ii) For any $w \in \mathbb{H}$, $z \in \overline{\mathbb{H}}$ it holds that

$$\mathbf{P}\left[\exists t > 0 : |C_{t^-,w}| = \infty, \mathrm{G}_{t^-,t,z}\right] = 0.$$

Proof. Part (i): Let T be a predictable stopping time which is announced by some sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times. Let $z \in \overline{\mathbb{H}}$. Pick $\epsilon > 0$ arbitrary. Then the definition of predictability yields

$$\mathbf{P}\left[T < \infty, \mathrm{G}_{T^-,T,z}\right] = \lim_{n\to\infty} \mathbf{P}\left[T < \infty, T - T_n < \epsilon, \mathrm{G}_{T^-,T,z}\right].$$

Fixing $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\mathbf{P}\left[T < \infty, T - T_n < \epsilon, \mathrm{G}_{T^-,T,z}\right] &\leq \mathbf{P}\left[T < \infty, T - T_n < \epsilon, \mathrm{G}_{T_n,T_n+\epsilon,z}\right] \\
&\leq \mathbf{P}\left[\mathrm{G}_{T_n,T_n+\epsilon,z}\right] \\
&= \mathbf{P}\left[\mathrm{G}_{0,\epsilon,z}\right] = 1 - e^{-\epsilon},
\end{aligned}$$

where we used Lemma 8 for the penultimate equality. It thus follows that

$$\mathbf{P}\left[T < \infty, \mathrm{G}_{T^-,T,z}\right] \leq 1 - e^{-\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, this proves equation (2.20).

Part (ii): Let $w \in \mathbb{H}$, $z \in \overline{\mathbb{H}}$. Equation (2.18) implies

$$\{\exists t > 0 : |C_{t^-,w}| = \infty, \mathrm{G}_{t^-,t,z}\} \overset{\text{a.s.}}{\subset} \left\{\exists k \in \mathbb{N} : T_{k,w} < \infty, \mathrm{G}_{T_{k,w}^-,T_{k,w},z}\right\}.$$

But the latter is a null set by part (i) because $T_{k,w}$ is a predictable stopping time for all $k \in \mathbb{N}$ by Lemma 14. □

We have thus proved that the limit process a.s. satisfies the second half of [DESTRUCTION] (i) in Definition 2:

Lemma 16. *For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : |C_{t^-,z}| = \infty, \eta_{t,z} = 1\right] = 0;$$

in other words: A.s. if the left-sided limit of the cluster at z is infinite at some time t, the site z becomes vacant at time t.

Proof. This is an immediate consequence of Lemma 12 and Lemma 15 (ii). □

2.3.6 Completion of the proof of Theorem 1

We next prove that the limit process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ a.s. satisfies [DESTRUCTION] (ii) in Definition 2:

Lemma 17. *For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : \eta_{t^-,z} > \eta_{t,z}, |C_{t^-,z}| < \infty, \forall u \in \partial C_{t^-,z} \cap \partial\mathbb{H} : \complement\, \mathrm{G}_{t^-,t,u}\right] = 0;$$

in other words: A.s. if the site z becomes vacant at some time t and its cluster was not about to become infinite at time t, its cluster must have grown to the boundary at time t.

Proof. The following argument is similar to the proof of Lemma 23 in [Dür06a]. Let $z \in \mathbb{H}$. As in Lemma 7, let C_z^{fin} denote the (countable) set of all finite connected subsets of $\mathbb{H}$ which contain the site z. Then the relation

$$\begin{aligned}
&\left\{\exists t > 0 : \eta_{t^-,z} > \eta_{t,z}, |C_{t^-,z}| < \infty, \forall u \in \partial C_{t^-,z} \cap \partial\mathbb{H} : \complement\, \mathrm{G}_{t^-,t,u}\right\} \\
&\qquad = \bigcup_{S \in C_z^{\text{fin}}} \left\{\exists t > 0 : \eta_{t^-,z} > \eta_{t,z}, C_{t^-,z} = S, \forall u \in \partial S \cap \partial\mathbb{H} : \complement\, \mathrm{G}_{t^-,t,u}\right\} \\
&\qquad \subset \bigcup_{S \in C_z^{\text{fin}}} \{\exists t > 0 : D_{S,t,z}\}
\end{aligned}$$

holds, where we abbreviate

$$D_{S,t,z} := \left\{\eta_{t^-,z} > \eta_{t,z}, \forall w \in \partial S : \eta_{t^-,w} = 0, \forall u \in \partial S \cap \partial\mathbb{H} : \complement\, \mathrm{G}_{t^-,t,u}\right\}.$$

So let $S \in C_z^{\text{fin}}$; it then suffices to show $\mathbf{P}\left[\exists t > 0 : D_{S,t,z}\right] = 0$. We distinguish whether or not at time t there is the growth of a tree at some site in $\partial S \cap \mathbb{H}$ and thus obtain

$$\{\exists t > 0 : D_{S,t,z}\} = A_{S,z} \cup B_{S,z}$$

with

$$\begin{aligned} A_{S,z} &:= \left\{\exists t > 0 : D_{S,t,z}, \forall v \in \partial S \cap \mathbb{H} : \complement\, \mathrm{G}_{t^-,t,v}\right\}, \\ B_{S,z} &:= \left\{\exists t > 0 : D_{S,t,z}, \exists v \in \partial S \cap \mathbb{H} : \mathrm{G}_{t^-,t,v}\right\}. \end{aligned}$$

We first consider the event $A_{S,z}$: Since the paths of the limit process are a.s. piecewise constant and càdlàg, and since the set ∂S is finite, it follows that

$$A_{S,z} \overset{\text{a.s.}}{\subset} \left\{\exists 0 \le s < t : \eta_{s,z} > \eta_{t,z}, \forall w \in \partial S : \eta_{s,w} = 0, \complement\, \mathrm{G}_{s,t,w}\right\}.$$

Now for all sufficiently large n (such that $S \cup \partial S \subset B_n$, where the boundary ∂S is taken in $\overline{\mathbb{H}}$) and arbitrary $0 \le s < t$, it is easy to deduce from [GROWTH] and [DESTRUCTION] in Definition 1 that B_n-forest-fire processes satisfy

$$\mathbf{P}\left[\eta^n_{s,z} > \eta^n_{t,z}, \forall w \in \partial S : \eta^n_{s,w} = 0, \complement\, \mathrm{G}^n_{s,t,w}\right] = 0.$$

Hence Lemma 3 yields $\mathbf{P}[A_{S,z}] = 0$.

We now consider the event $B_{S,z}$: Resorting to Lemma 15 (ii), we obtain

$$\begin{aligned} B_{S,z} &\overset{\text{a.s.}}{\subset} \{\exists t > 0 : D_{S,t,z}, \exists v \in \partial S \cap \mathbb{H} : \mathrm{G}_{t^-,t,v}, \forall x \in \mathbb{H} : |C_{t^-,x}| < \infty\} \\ &\subset \Big\{\exists t > 0 : D_{S,t,z}, \exists v \in \partial S \cap \mathbb{H} : \mathrm{G}_{t^-,t,v}, \exists S' \in C_z^{\text{fin}} : S' = S \cup \{v\} \cup \bigcup_{x \in \partial\{v\}} C_{t^-,x}\Big\} \\ &\subset \bigcup_{S' \in C_z^{\text{fin}}} \{\exists t > 0 : \eta_{t^-,z} > \eta_{t,z}, \forall w \in \partial S' : \eta_{t^-,w} = 0, \exists v \in S' : \mathrm{G}_{t^-,t,v}\} \\ &\overset{\text{a.s.}}{\subset} \bigcup_{S' \in C_z^{\text{fin}}} \left\{\exists t > 0 : \eta_{t^-,z} > \eta_{t,z}, \forall w \in \partial S' : \eta_{t^-,w} = 0, \complement\, \mathrm{G}_{t^-,t,w}\, \exists v \in S' : \mathrm{G}_{t^-,t,v}\right\} \\ &\subset \bigcup_{S' \in C_z^{\text{fin}}} A_{S',z}, \end{aligned}$$

where in the penultimate step we used that a.s. no two growth events occur at the same time. So the above implies $\mathbf{P}[B_{S,z}] = 0$. □

Finally, we show that the limit process also a.s. satisfies [GROWTH] (i) in Definition 2:

Lemma 18. *For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0\right] = 0;$$

in other words: A.s. if a tree grows at the site z at some time t, then the site z is occupied at time t.

Proof. The argument to come is similar to the proof of Lemma 24 in [Dür06a]. Let $z \in \mathbb{H}$. Then the following inclusions hold:

$$
\begin{aligned}
&\{\exists t > 0 : \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0\} \\
&\qquad \overset{\text{a.s.}}{\subset} \left\{\exists t > 0 : \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0, \forall w \in \overline{\mathbb{H}} \setminus \{z\} : \complement\, \mathrm{G}_{t^-,t,w}, |C_{t^-,w}| < \infty\right\} \\
&\qquad \overset{\text{a.s.}}{\subset} \left\{\exists t > 0 : \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0, \forall w \in \partial\{z\} : \eta_{t^-,w} = \eta_{t,w}, \complement\, \mathrm{G}_{t^-,t,w}\right\} \\
&\qquad \overset{\text{a.s.}}{\subset} \left\{\exists 0 \le s < t : \mathrm{G}_{s,t,z}, \eta_{t,z} = 0, \forall w \in \partial\{z\} : \eta_{s,w} = \eta_{t,w}, \complement\, \mathrm{G}_{s,t,w}\right\}.
\end{aligned}
$$

Indeed, the first inclusion follows from Lemma 15 (ii) and the fact that a.s. no two growth events occur at the same time, the second inclusion is a consequence of the properties [GROWTH] (ii) and [DESTRUCTION] (ii) in Definition 2 (which have already been proved for the limit process in Lemmas 4 and 17), and the third inclusion is due to the fact that the paths of the limit process are a.s. piecewise constant and càdlàg. (The case $w \in \partial\mathbb{H}$ in these events is somewhat separate but trivial due to the zero boundary condition proved in Lemma 4.) Now for all sufficiently large n (such that $\{z\} \cup \partial\{z\} \subset B_n$, where the boundary $\partial\{z\}$ is taken in $\overline{\mathbb{H}}$) and arbitrary $0 \le s < t$, it is easy to deduce from [GROWTH] and [DESTRUCTION] in Definition 1 that B_n-forest-fire processes satisfy

$$
\mathbf{P}\left[\mathrm{G}^n_{s,t,z}, \eta^n_{t,z} = 0, \forall w \in \partial\{z\} : \eta^n_{s,w} = \eta^n_{t,w}, \complement\, \mathrm{G}^n_{s,t,w}\right] = 0.
$$

The result therefore follows from Lemma 3. □

Lemmas 1, 2, 4, 7, 16, 17 and 18 combined thus provide the proof of Theorem 1.

2.4 Proof of Theorem 2

Throughout this section, let $(\eta_{t,z}, G_{t,z})_{t \ge 0, z \in \overline{\mathbb{H}}}$ be an $\overline{\mathbb{H}}$-forest-fire process (see Definition 2), let $(\sigma_{t,z})_{t \ge 0, z \in \overline{\mathbb{H}}}$ be the associated pure growth process defined by equation (2.2), and let $(Y_{t,x})_{t \ge 0, x \in \partial\mathbb{H}}$ be the heights of destruction of the process $(\eta_{t,z}, G_{t,z})_{t \ge 0, z \in \overline{\mathbb{H}}}$ (see Definition 3). As we already noted in Section 2.1, for fixed $t \ge 0$ the distribution of $\sigma_t := (\sigma_{t,z})_{z \in \overline{\mathbb{H}}}$ is independent site percolation on $\overline{\mathbb{H}}$, where each site is open with probability $1 - e^{-t}$.

In the following, it will also be convenient to consider independent site percolation on the whole lattice $\mathbb{Z}^2$. So for $t \ge 0$, let $\xi_t := (\xi_{t,z})_{z \in \mathbb{Z}^2}$ be distributed according to independent site percolation on $\mathbb{Z}^2$, where each site is open with probability $1 - e^{-t}$. We realize both $(\eta_{t,z}, G_{t,z})_{t \ge 0, z \in \overline{\mathbb{H}}}$ and ξ_t, $t \ge 0$, on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

A key concept for the treatment of site percolation on the square lattice $\mathbb{Z}^2$ is the so-called matching lattice $\mathbb{Z}^{2*}$, which is obtained from the square lattice $\mathbb{Z}^2$ by adding diagonal edges to all faces in $\mathbb{Z}^2$. In this way, certain statements about open sites on the square lattice $\mathbb{Z}^2$ can be reformulated as statements about closed sites on the matching lattice $\mathbb{Z}^{2*}$; see [Gri99], Section 3.1, for more details. We therefore extend our terminology as follows: Let W be a subset of $\mathbb{Z}^2$ and let $\alpha := (\alpha_w)_{w \in W} \in \{0,1\}^W$ be any configuration on W. Let $\mathbb{Z}^2|_{\alpha,1}$ denote the subgraph of the square lattice $\mathbb{Z}^2$ induced by the vertex set

$\{w \in W : \alpha_w = 1\}$. Then by a 1-**path** in the configuration α we simply mean any path on the graph $\mathbb{Z}^2|_{\alpha,1}$. Similarly, let $\mathbb{Z}^{2*}|_{\alpha,0}$ denote the subgraph of the matching lattice $\mathbb{Z}^{2*}$ induced by the vertex set $\{w \in W : \alpha_w = 0\}$. Then a $0*$-**path** in the configuration α is simply any path on the graph $\mathbb{Z}^{2*}|_{\alpha,0}$.

For $w \in \mathbb{Z}^2$ and $n \in \mathbb{N}$, let

$$B_n(w) := w + [-n,n]^2 \cap \mathbb{Z}^2 = \left\{ z \in \mathbb{Z}^2 : |\operatorname{Re}(z-w)| \le n, |\operatorname{Im}(z-w)| \le n \right\} \tag{2.21}$$

denote the box with centre w and radius n, and let

$$\begin{aligned} S_n(w) := &\left\{ z \in \mathbb{Z}^2 : |\operatorname{Re}(z-w)| = n, |\operatorname{Im}(z-w)| \le n \right\} \\ &\cup \left\{ z \in \mathbb{Z}^2 : |\operatorname{Re}(z-w)| \le n, |\operatorname{Im}(z-w)| = n \right\} \end{aligned}$$

denote the inner boundary of that box. For later reference we also define the left side

$$L_n(w) := \left\{ z \in \mathbb{Z}^2 : \operatorname{Re}(z-w) = -n, |\operatorname{Im}(z-w)| \le n \right\} \tag{2.22}$$

and the right side

$$R_n(w) := \left\{ z \in \mathbb{Z}^2 : \operatorname{Re}(z-w) = n, |\operatorname{Im}(z-w)| \le n \right\} \tag{2.23}$$

of the box $B_n(w)$.

We will need the following two well-known results from percolation theory:

Correlation length. For all $t > t_c$ the "inverse correlation length"

$$c(t) := \lim_{n\to\infty} \frac{\log \mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } 0 \text{ to } S_n(0)\right]}{-n}$$

is well-defined and positive, and there exist universal constants $\rho, \sigma > 0$ such that

$$\rho n^{-1} e^{-c(t)n} \le \mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } 0 \text{ to } S_n(0)\right] \le \sigma n e^{-c(t)n} \tag{2.24}$$

holds for all $t > t_c$ and all $n \in \mathbb{N}$ (see [Gri99], Section 6.1, for instance[4]). We will only use the left inequality in (2.24).

Percolation on subsets of the half-plane. Let $t > t_c$. Define the bijective function $h_t : [e, \infty) \to [\frac{1}{c(t)}, \infty)$ by

$$h_t(y) := \frac{1}{c(t)} \left(\log y + 3 \log\log y \right), \qquad y \ge e, \tag{2.25}$$

[4] In this reference the statement is proved for independent bond percolation on $\mathbb{Z}^2$ but the proof is identical for independent site percolation on $\mathbb{Z}^2$.

and let $g_t : [\frac{1}{c(t)}, \infty) \to [e, \infty)$ be its inverse function. Extend g_t continuously to $[0, \infty)$ by setting

$$g_t(x) := e, \qquad 0 \le x < \frac{1}{c(t)}.$$

(The specific way of the extension is immaterial.) Then

$$\mathbf{P}\left[(\sigma_{t,x+iy})_{x\ge 0, y \ge g_t(x)} \text{ contains an infinite cluster}\right] = 1 \tag{2.26}$$

holds; in other words: A.s. the restriction of σ_t to the area $\left\{x + iy \in \overline{\mathbb{H}} : x \ge 0, y \ge g_t(x)\right\}$ (endowed with the edges inherited from $\overline{\mathbb{H}}$) contains an infinite cluster. A more detailed account of this topic can be found in [Gri99], Section 11.5, or in the original papers [Gri83], [CC86][5].

Remark. A closer look shows that the core of the proof of Theorem 2 only relies on the following weaker versions of equations (2.24) and (2.26): For all $t > t_c$ there exists $a(t) > 0$ such that for all $n \in \mathbb{N}$

$$\mathbf{P}\left[\xi_t \text{ contains a } 0{*}\text{-path from } 0 \text{ to } S_n(0)\right] \ge e^{-a(t)n}$$

holds and there exists $b(t) > 0$ such that

$$\mathbf{P}\left[(\sigma_{t,x+iy})_{x\ge 0, y \ge e^{b(t)x}} \text{ contains an infinite cluster}\right] = 1$$

holds. However, if we used only these equations, the statements of some of the lemmas to come would have to be weakened accordingly, e.g. the width of the semi-infinite tube in Lemma 20 would then also depend on t.

As a direct consequence of (2.26), we deduce the following lemma:

Lemma 19. *For $t > t_c$ define the function $f_t : (0, \infty) \to (0, \infty)$ by*

$$f_t(x) := \frac{1}{(c(t)x)^3} e^{c(t)x}, \qquad x > 0.$$

Then for all $t > t_c$ we have

$$\mathbf{P}\left[Y_{t,x} \ge f_t(x) \textit{ for infinitely many } x \in \mathbb{N}\right] = 1. \tag{2.27}$$

Proof. Let $t > t_c$. From equations (2.4) and (2.26), together with the fact that the configuration $(\eta_{t,z})_{z \in \overline{\mathbb{H}}}$ a.s. does not contain an infinite cluster, we conclude

$$\mathbf{P}\left[Y_{t,x} \ge g_t(x) \text{ for infinitely many } x \in \mathbb{N}\right] = 1.$$

[5] Again, in these references the statement is proved for independent bond percolation on $\mathbb{Z}^2$ but the proof carries over to independent site percolation on $\mathbb{Z}^2$ when duality of lattices is replaced by matching of lattices.

It is therefore enough to show that $g_t(x) \geq f_t(x)$ holds for all sufficiently large $x \in \mathbb{N}$. Indeed, the definition of g_t (below (2.25)) yields

$$x = \frac{1}{c(t)} \left(\log g_t(x) + 3 \log\log g_t(x)\right), \qquad x \geq \frac{1}{c(t)},$$

and applying f_t on both sides of this equation gives

$$f_t(x) = \left(\frac{\log g_t(x)}{\log g_t(x) + 3\log\log g_t(x)}\right)^3 g_t(x), \qquad x \geq \frac{1}{c(t)}.$$

Since $g_t(x) \geq e$ for $x \geq \frac{1}{c(t)}$, we have

$$\left(\frac{\log g_t(x)}{\log g_t(x) + 3\log\log g_t(x)}\right)^3 \leq 1, \qquad x \geq \frac{1}{c(t)},$$

which completes the proof. □

The first inequality in (2.24) also implies the following:

Lemma 20. *For $t > t_c$ and $x \in \mathbb{N}$ let*

$$T_{t,x} := \left[\tfrac{3}{4}x, \tfrac{5}{4}x\right] \times \left[\tfrac{1}{2}f_t(x), \infty\right) \cap \mathbb{H}$$

be the semi-infinite tube with vertical midline at x, width $2\lfloor \frac{x}{4} \rfloor$ and starting height $\lceil \frac{1}{2} f_t(x) \rceil$, and let

$$D_{t,x} := \left[\tfrac{3}{4}x, \tfrac{5}{4}x\right] \times \left\{\lceil \tfrac{1}{2} f_t(x) \rceil\right\} \cap \mathbb{H}$$

be its baseline. Additionally, let

$$\text{PATH-IN-TUBE}_{t,x} := \{\exists y \geq f_t(x) : \sigma_t \textit{ contains a 1-path from } x + iy \textit{ to } D_{t,x} \textit{ within } T_{t,x}\}$$

be the event that in the configuration σ_t there exists a site with real part x and imaginary part at least $f_t(x)$ which is connected by a 1-path to the baseline $D_{t,x}$ within the tube $T_{t,x}$. Then for all $t > t_c$ we have

$$\mathbf{P}\left[\text{PATH-IN-TUBE}_{t,x} \textit{ for infinitely many } x \in \mathbb{N}\right] = 0. \tag{2.28}$$

Proof. Let $t > t_c$ and $x \in \mathbb{N}$. As depicted in Figure 2.3, we partition the tube $T_{t,x}$ up to height $f_t(x)$ into disjoint boxes of radius $\lfloor \frac{x}{4} \rfloor$ such that adjacent boxes have vertical distance 1. Let $K_{t,x} := \left\lfloor \frac{f_t(x) - \lceil \frac{1}{2} f_t(x) \rceil + 1}{2\lfloor \frac{x}{4} \rfloor + 1} \right\rfloor$ be the number of these boxes, and for $k \in \{1, \ldots, K_{t,x}\}$ let

$$z_{t,x,k} := x + i\left(\lceil \tfrac{1}{2} f_t(x) \rceil + (2k-1)\lfloor \tfrac{x}{4} \rfloor + (k-1)\right)$$

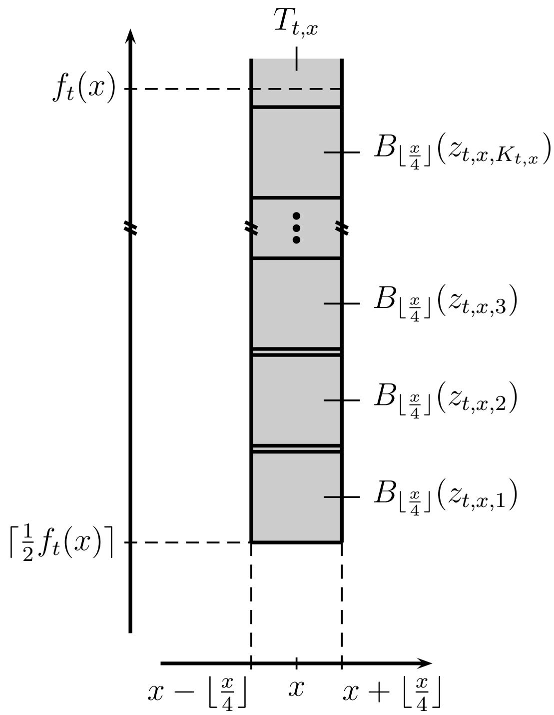


Figure 2.3: Partition of the tube $T_{t,x}$ into $K_{t,x}$ boxes

be the centre of the kth such box. Recalling the notation introduced in equations (2.21), (2.22) and (2.23), and passing from the lattice $\mathbb{Z}^2$ to the matching lattice $\mathbb{Z}^{2*}$, we obtain

$$\begin{aligned}
&\mathbf{P}\left[\text{PATH-IN-TUBE}_{t,x}\right] \\
&\qquad \leq \mathbf{P}\Big[\forall k \in \{1, \ldots, K_{t,x}\} : \sigma_t \text{ contains no } 0*\text{-path from } L_{\lfloor \frac{x}{4} \rfloor}(z_{t,x,k}) \\
&\qquad\qquad \text{to } R_{\lfloor \frac{x}{4} \rfloor}(z_{t,x,k}) \text{ within } B_{\lfloor \frac{x}{4} \rfloor}(z_{t,x,k})\Big] \\
&\qquad = \left(1 - \mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } L_{\lfloor \frac{x}{4} \rfloor}(0) \text{ to } R_{\lfloor \frac{x}{4} \rfloor}(0) \text{ within } B_{\lfloor \frac{x}{4} \rfloor}(0)\right]\right)^{K_{t,x}}.
\end{aligned}$$

Now an argument similar to the proof of Theorem 11.55 in [Gri99] gives

$$\begin{aligned}
&\mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } L_{\lfloor \frac{x}{4} \rfloor}(0) \text{ to } R_{\lfloor \frac{x}{4} \rfloor}(0) \text{ within } B_{\lfloor \frac{x}{4} \rfloor}(0)\right] \\
&\qquad \geq \mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } L_{\lfloor \frac{x}{4} \rfloor}(0) \text{ through } 0 \text{ to } R_{\lfloor \frac{x}{4} \rfloor}(0) \text{ within } B_{\lfloor \frac{x}{4} \rfloor}(0)\right] \\
&\qquad \geq \left(\frac{1}{4}\mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } 0 \text{ to } S_{\lfloor \frac{x}{4} \rfloor}(0)\right]\right)^2 \\
&\qquad \geq \frac{1}{16}\rho^2 \frac{1}{\lfloor \frac{x}{4} \rfloor^2} e^{-2c(t)\lfloor \frac{x}{4} \rfloor} \\
&\qquad \geq \Omega\left(e^{-\frac{4c(t)}{6}x}\right) \qquad \text{for } x \to \infty.
\end{aligned}$$

Here the second inequality is obtained by an application of the FKG inequality, the third inequality is a consequence of (2.24), and in the last inequality we use Landau notation. In addition, it is evident from the definition of $K_{t,x}$ that

$$K_{t,x} \geq \Omega\left(e^{\frac{5c(t)}{6}x}\right) \qquad \text{for } x \to \infty$$

holds. From all this we conclude that $\mathbf{P}\left[\text{PATH-IN-TUBE}_{t,x}\right]$ decays at least exponentially for $x \to \infty$, in particular

$$\sum_{x=1}^{\infty} \mathbf{P}\left[\text{PATH-IN-TUBE}_{t,x}\right] < \infty$$

holds. Equation (2.28) therefore follows from the Borel-Cantelli lemma. □

In Lemma 19 we saw that for any time t there are a.s. infinitely many $x \in \mathbb{N}$ with $Y_{t,x} \geq f_t(x)$. Very roughly speaking, we now want to prove that if $Y_{t,x} \geq f_t(x)$ holds for some $x \in \mathbb{N}$, then for all $\tilde{x}$ of order x the corresponding height of destruction $Y_{t,\tilde{x}}$ is also of order at least $f_t(x)$, i.e. there cannot be "large fluctuations" in the heights of destruction at time t. The precise statement is as follows:

Lemma 21. *For $t > t_c$ and $x \in \mathbb{N}$ let*

$$\begin{aligned}\text{LARGE-FLUCT}_{t,x} := \big\{ & Y_{t,x} \geq f_t(x), \exists x_1, x_2 \in \mathbb{N} : \tfrac{3}{4}x \leq x_1 < x < x_2 \leq \tfrac{5}{4}x, \\ & Y_{t,x_1} < \tfrac{1}{2} f_t(x), Y_{t,x_2} < \tfrac{1}{2} f_t(x) \big\}\end{aligned}$$

denote the event that the height of destruction at x up to time t is at least $f_t(x)$ but there exist $\frac{3}{4}x \leq x_1 < x < x_2 \leq \frac{5}{4}x$ such that the height of destruction at x_1 and x_2 up to time t is less than $\frac{1}{2}f_t(x)$ (see Figure 2.4). Then for all $t > t_c$ we have

$$\mathbf{P}\left[\text{LARGE-FLUCT}_{t,x} \text{ for infinitely many } x \in \mathbb{N}\right] = 0. \tag{2.29}$$

Proof. Let $t > t_c$ and $x \in \mathbb{N}$. We are going to prove

$$\text{LARGE-FLUCT}_{t,x} \overset{\text{a.s.}}{\subset} \text{PATH-IN-TUBE}_{t,x}\,, \tag{2.30}$$

from which equation (2.29) follows by Lemma 20. So assume that the event $\text{LARGE-FLUCT}_{t,x}$ occurs. Then by the definition of the height of destruction, there exist $y \geq f_t(x)$ and $0 < s \leq t$ such that

$$\eta_{s^-,x+iy} = 1, \eta_{s,x+iy} = 0$$

holds. According to the property [DESTRUCTION] in Definition 2, this means that one of the following two cases occurs:
Case 1: $|C_{s^-,x+iy}| = \infty$.
Case 2: $C_{s^-,x+iy}$ contains a site in $\partial\mathbb{H} + i$.

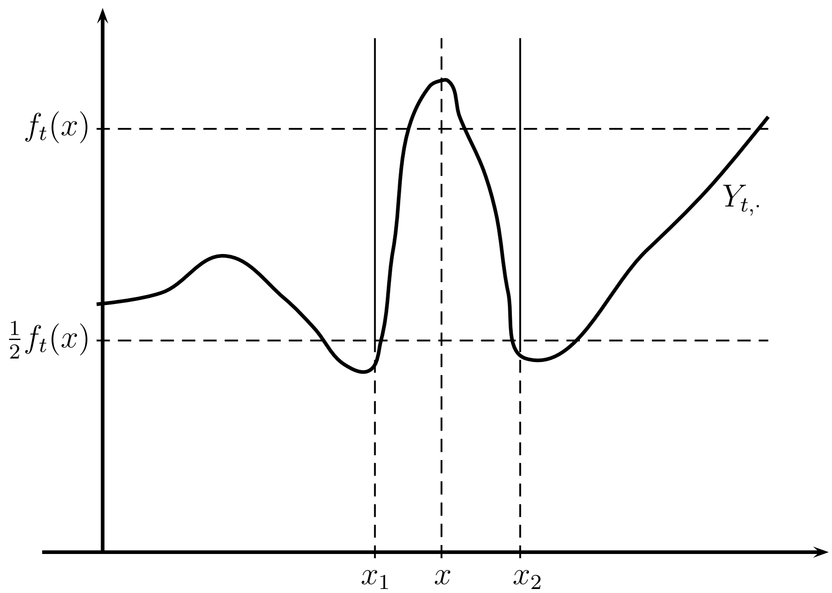


Figure 2.4: A visualization of the event LARGE-FLUCT$_{t,x}$

However, the condition $Y_{t,x_1} < \frac{1}{2}f_t(x), Y_{t,x_2} < \frac{1}{2}f_t(x)$ in the event LARGE-FLUCT$_{t,x}$ implies that all sites of the form x_1+iy_1, x_2+iy_2 with $y_1, y_2 \geq \frac{1}{2}f_t(x)$ cannot be part of $C_{s^-,x+iy}$. It is easy to see that a.s. in both cases this implies that the configuration $(\eta_{s^-,z})_{z\in\overline{\mathbb{H}}}$ contains a 1-path which runs from $x+iy$ to the baseline

$$(x_1, x_2) \times \left\{\lceil \tfrac{1}{2}f_t(x)\rceil\right\} \cap \mathbb{H}$$

within the half-infinite tube

$$(x_1, x_2) \times \left[\tfrac{1}{2}f_t(x), \infty\right) \cap \mathbb{H}. \tag{2.31}$$

(For case 1 observe that a.s. there exists $v \geq y$ with $\eta_{s^-,u+iv} = 0$ for all $u \in \{x_1, x_1+1, \ldots, x_2\}$ so that the cluster $C_{s^-,x+iy}$ cannot stretch to infinity within the tube (2.31).) Since the tube (2.31) is a subset of the tube $T_{t,x}$ and because of the basic inequality (2.3), this proves the inclusion (2.30). $\square$

Lemmas 19 and 21 enable us to prove the following lemma, which is only slightly weaker than Theorem 2:

Lemma 22. *For all $t > t_c$ we have* $\mathbf{P}[Y_{t,0} = \infty] = 1$.

Proof. Let $t > t_c$. Suppose that the lemma is not true; then there exists $y \in \mathbb{N}_0$ with $\mathbf{P}[Y_{t,0} = y] > 0$. The translation-invariance of $\overline{\mathbb{H}}$-forest-fire processes ([TRANSL-INV] in Definition 2) and the Birkhoff ergodic theorem (see e.g. [Shi95], Section V.3, Theorem 1) imply that the sequence $\frac{1}{n}\sum_{x=0}^{n-1} 1_{\{Y_{t,x}=y\}}$, $n \in \mathbb{N}$, is a Cauchy sequence a.s. and that there exists $\epsilon > 0$ such that the event

$$A := \left\{ \frac{1}{n}\sum_{x=0}^{n-1} 1_{\{Y_{t,x}=y\}} > \epsilon \text{ eventually as } n \to \infty \right\}$$

satisfies $\mathbf{P}[A] > 0$. Consequently, on the event A there a.s. exists $n_0 \in \mathbb{N}$ such that for all $n_1, n_2 \geq n_0$

$$\left| \frac{1}{n_1}\sum_{x=0}^{n_1-1} 1_{\{Y_{t,x}=y\}} - \frac{1}{n_2}\sum_{x=0}^{n_2-1} 1_{\{Y_{t,x}=y\}} \right| < \frac{1}{9}\epsilon \tag{2.32}$$

and

$$\frac{1}{n_1}\sum_{x=0}^{n_1-1} 1_{\{Y_{t,x}=y\}} > \epsilon$$

hold.

However, given n_0 on the event A, it follows from Lemmas 19 and 21 that there a.s. exists $n_1 \geq \max\{n_0, 8\}$ such that for all $x \in \left\{n_1, \ldots, n_1 + \lfloor \frac{n_1}{4} \rfloor\right\}$

$$1_{\{Y_{t,x}=y\}} = 0$$

holds. With this n_1 and $n_2 := n_1 + \lfloor \frac{n_1}{4} \rfloor$ we obtain

$$\begin{aligned} \frac{1}{n_1}\sum_{x=0}^{n_1-1} 1_{\{Y_{t,x}=h\}} - \frac{1}{n_2}\sum_{x=0}^{n_2-1} 1_{\{Y_{t,x}=h\}} &= \frac{1}{n_1}\sum_{x=0}^{n_1-1} 1_{\{Y_{t,x}=h\}} \left(1 - \frac{n_1}{n_1 + \lfloor \frac{n_1}{4} \rfloor}\right) \\ &> \epsilon \cdot \left(1 - \frac{1}{\frac{5}{4} - \frac{1}{n_1}}\right) \geq \frac{1}{9}\epsilon, \end{aligned}$$

which is opposed to (2.32). Hence $\mathbf{P}[A] > 0$ cannot hold - a contradiction. □

Theorem 2 is now an immediate consequence of Lemma 22: The translation-invariance [TRANSL-INV] implies that we only need to consider the case $x = 0$ in Theorem 2. Since $Y_{t,0}$ is obviously monotone increasing in t, we have

$$\{\forall t > t_c : Y_{t,0} = \infty\} = \{\forall t \in (t_c, \infty) \cap \mathbb{Q} : Y_{t,0} = \infty\}$$

so that

$$\mathbf{P}\left[\forall t > t_c : Y_{t,0} = \infty\right] = 1 \tag{2.33}$$

follows from Lemma 22. Moreover, if $0 \leq t < t_c$ and $y \in \mathbb{N}_0$, then the definition of the height of destruction, the condition [DESTRUCTION] in Definition 2 and equation (2.3) yield

$$\{Y_{t,0} \geq y\} \subset \{\exists v \geq y : \sigma_t \text{ contains a 1-path from } iv \text{ to } \partial\mathbb{H}\}.$$

As a consequence of the exponential decay of the radius for subcritical independent site percolation on $\mathbb{Z}^2$, the probability of the latter event decays to zero as $y \to \infty$ so that

$$\mathbf{P}\left[Y_{t,0} = \infty\right] = \lim_{y\to\infty} \mathbf{P}\left[Y_{t,0} \geq y\right] = 0$$

holds for $0 \leq t < t_c$. Herefrom we readily deduce

$$\mathbf{P}\left[\exists 0 \leq t < t_c : Y_{t,0} = \infty\right] = 0 \tag{2.34}$$

by a similar monotonicity argument as above. Equations (2.33) and (2.34) complete the proof of Theorem 2.

Chapter 3

Critical heights of destruction for a forest-fire model on the half-plane

An article which closely follows this chapter has been uploaded on *arXiv* [Gra14a] and submitted to a journal.

3.1 Introduction and statement of the main result

In Chapter 2 we obtained a forest-fire model on the half-plane as a subsequential limit of forest-fire models on finite-size boxes and analysed a corresponding collection of time- and space-dependent random variables, the so-called heights of destruction. We proved that the heights of destruction in semi-infinite tubes show a phase transition in the sense that they are a.s. finite *before* a certain critical time and infinite *after* the critical time. In this chapter we show that the heights of destruction in semi-infinite tubes and even in infinite cones[1] are a.s. finite *at* the critical time - provided that two critical exponents of site percolation exist. Since these exponents (equations (3.11) and (3.12)) are currently only known for the triangular lattice, we formulate the model on the triangular lattice, whereas Chapter 2 uses the square lattice. In the present section, we give a self-contained account of our result (Theorem 3), in Section 3.2 we put this result in the context of Chapter 2, and in Section 3.3 we give the proof of Theorem 3.

Let $i = \sqrt{-1}$ denote the imaginary unit, let

$$\mathbb{T} := \left\{ k + l e^{i\pi/3} : k, l \in \mathbb{Z} \right\}$$

be the set of sites of the triangular lattice, let

$$\mathbb{C}^{\mathrm{u}} := \{ z \in \mathbb{C} : \operatorname{Im} z \geq 0 \}$$

be the upper half-plane and let $\mathbb{T}^{\mathrm{u}} := \mathbb{T} \cap \mathbb{C}^{\mathrm{u}}$ be the set of sites of the half-plane triangular lattice (see Figure 3.1). Note that according to our definition the relation $\mathbb{Z} \subset \mathbb{T}^{\mathrm{u}} \subset \mathbb{C}^{\mathrm{u}}$

[1]In this chapter we generalize the concept of the height of destruction from vertical semi-infinite tubes to arbitrary sets, see Definition 7.

holds, where $\mathbb{Z}$ can be interpreted as the inner boundary of $\mathbb{T}^{\mathrm{u}}$ in $\mathbb{T}$. Two sites $v, w \in \mathbb{T}$ of the triangular lattice are said to be **neighbours** if their Euclidean distance is 1. For a subset $S \subset \mathbb{T}^{\mathrm{u}}$ of the half-plane triangular lattice, we write

$$\partial S := \{v \in \mathbb{T}^{\mathrm{u}} \setminus S : (\exists w \in S : v \text{ and } w \text{ are neighbours})\}$$

for the **outer boundary** of S in $\mathbb{T}^{\mathrm{u}}$. For a site $x \in \mathbb{Z}$, for example, we have $\partial\{x\} = \{x+1, x+e^{i\pi/3}, x+e^{2i\pi/3}, x-1\}$.

In order to introduce some further notation, let $V \in \{\mathbb{T}^{\mathrm{u}}, \mathbb{T}\}$, let $(\alpha_v)_{v\in V} \in \{0,1\}^V$ and let $j \in \{0,1\}$. A j**-path** in $(\alpha_v)_{v\in V}$ from a site $y \in V$ to a site $z \in V$ is a sequence $v_0, v_1, \ldots, v_l$ of distinct sites in V (where $l \in \mathbb{N}_0$) such that the following holds:

- $v_0 = y$, $v_l = z$;
- v_{k-1} is a neighbour of v_k for all $k \in \{1, \ldots, l\}$;
- $\alpha_{v_k} = j$ for all $k \in \{0, \ldots, l\}$.

If $Y, Z \subset V$ are subsets, then a j-path in $(\alpha_v)_{v\in V}$ from Y to Z is simply any j-path in $(\alpha_v)_{v\in V}$ from a site $y \in Y$ to a site $z \in Z$. Moreover, the **cluster** of a site $y \in V$ in $(\alpha_v)_{v\in V}$ is the set of all sites z in V such that there exists a 1-path in $(\alpha_v)_{v\in V}$ from y to z. If $\alpha_y = 0$, then the cluster of y in $(\alpha_v)_{v\in V}$ is just the empty set.

Informally, the forest-fire model may be described as follows: Each site can be "vacant" (denoted by 0) or "occupied by a tree" (denoted by 1). At time 0 all sites are vacant. Then the process is governed by two competing random mechanisms: On the one hand, trees grow according to rate 1 Poisson processes, independently for all sites. On the other hand, if an occupied cluster reaches the boundary of the upper half-plane, the cluster is instantaneously destroyed, i.e. all of its sites turn vacant. At the critical time $t_c := \log 2$ the process is stopped.

We now give the formal definition of the forest-fire model, which is similar to the definitions in Chapter 2 and [Dür06a]. Here, if $I \subset \mathbb{R}$ is a left-open interval and $I \ni t \mapsto f_t \in \mathbb{R}$ is a function, we write $f_{t^-} := \lim_{s\uparrow t} f_s$ for the left-sided limit at t, provided the limit exists.

Definition 6. Let $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c], z\in\mathbb{T}^{\mathrm{u}}}$ be a process with values in $(\{0,1\} \times \mathbb{N}_0)^{[0,t_c]\times\mathbb{T}^{\mathrm{u}}}$, initial condition $\eta_{0,z} = 0$ for $z \in \mathbb{T}^{\mathrm{u}}$ and boundary condition $\eta_{t,x} = 0$ for $t \in [0,t_c], x \in \mathbb{Z}$. Suppose that for all $z \in \mathbb{T}^{\mathrm{u}}$ the process $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c]}$ is càdlàg. For $z \in \mathbb{T}^{\mathrm{u}}$ and $t \in (0, t_c]$, let $C_{t^-,z}$ denote the cluster of z in the configuration $(\eta_{t^-,w})_{w\in\mathbb{T}^{\mathrm{u}}}$.

Then $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c], z\in\mathbb{T}^{\mathrm{u}}}$ is called a $\mathbb{T}^{\mathrm{u}}$**-forest-fire process** if the following conditions are satisfied:

[POISSON] The processes $(G_{t,z})_{t\in[0,t_c]}$, $z \in \mathbb{T}^{\mathrm{u}}$, are independent Poisson processes with rate 1.

[GROWTH] For all $t \in (0, t_c]$ and all $z \in \mathbb{T}^{\mathrm{u}} \setminus \mathbb{Z}$ the following implications hold:

(i) $G_{t^-,z} < G_{t,z} \Rightarrow \eta_{t,z} = 1$,
i.e. the growth of a tree at the site z at time t implies that the site z is occupied at time t;

(ii) $\eta_{t^-,z} < \eta_{t,z} \Rightarrow G_{t^-,z} < G_{t,z}$,
i.e. if the site z gets occupied at time t, there must have been the growth of a tree at the site z at time t.

[DESTRUCTION] For all $t \in (0, t_c]$ and all $x \in \mathbb{Z}$, $z \in \mathbb{T}^{\mathrm{u}} \setminus \mathbb{Z}$ the following implications hold:

(i) $G_{t^-,x} < G_{t,x} \Rightarrow \forall v \in \partial\{x\}\, \forall w \in C_{t^-,v} : \eta_{t,w} = 0$,
i.e. if a cluster grows to the boundary $\mathbb{Z}$ at time t, it is destroyed at time t;

(ii) $\eta_{t^-,z} > \eta_{t,z} \Rightarrow \exists u \in \partial C_{t^-,z} \cap \mathbb{Z} : G_{t^-,u} < G_{t,u}$,
i.e. if a site is destroyed at time t, its cluster must have grown to the boundary $\mathbb{Z}$ at time t.

In order to construct a $\mathbb{T}^{\mathrm{u}}$-forest-fire process, we start with independent rate 1 Poisson processes $(G_{t,z})_{t\in[0,t_c]}$, $z \in \mathbb{T}^{\mathrm{u}}$, on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathcal{F}$ is the completion of the σ-field generated by $(G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$. We first consider the corresponding **pure growth process**

$$\sigma_{t,z} := 1_{\{G_{t,z}>0\}}, \qquad t \in [0, t_c], z \in \mathbb{T}^{\mathrm{u}},$$

on $\mathbb{T}^{\mathrm{u}}$, where 1_A denotes the indicator function of an event A. For a fixed time $t \in [0, t_c]$, the configuration $\sigma_t^{\mathrm{u}} := (\sigma_{t,z})_{z\in\mathbb{T}^{\mathrm{u}}}$ is simply independent site percolation on $\mathbb{T}^{\mathrm{u}}$, where each site is occupied with probability $1-e^{-t}$. From the RSW theory we know that at the critical time t_c (where sites are occupied with probability 1/2), for all $x \in \mathbb{R}$ we have

$$\mathbf{P}\left[\sigma_{t_c}^{\mathrm{u}} \text{ contains infinitely many disjoint 0-paths from } \mathbb{Z}_{<x} \text{ to } \mathbb{Z}_{>x}\right] = 1, \tag{3.1}$$

$$\mathbf{P}\left[\sigma_{t_c}^{\mathrm{u}} \text{ contains infinitely many disjoint 1-paths from } \mathbb{Z}_{<x} \text{ to } \mathbb{Z}_{>x}\right] = 1, \tag{3.2}$$

where $\mathbb{Z}_{<x} := \{x' \in \mathbb{Z} : x' < x\}$ and $\mathbb{Z}_{>x} := \{x' \in \mathbb{Z} : x' > x\}$. Moreover, it is clear that *if* a $\mathbb{T}^{\mathrm{u}}$-forest-fire process $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ can be constructed from $(G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$, then $(\sigma_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ dominates $(\eta_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ in the sense that

$$\eta_{s,z} \le \sigma_{s,z} \le \sigma_{t,z}, \qquad 0 \le s \le t \le t_c, z \in \mathbb{T}^{\mathrm{u}}. \tag{3.3}$$

Equations (3.1) and (3.3) imply that given $(G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$, there exists a unique corresponding $\mathbb{T}^{\mathrm{u}}$-forest-fire process $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$, which can be obtained by partitioning $\mathbb{T}^{\mathrm{u}}$ into a random collection of finite sets separated by 0-paths in $\sigma_{t_c}^{\mathrm{u}}$ and performing a graphical construction on each of these sets. (Since (3.1) is only an a.s.-property, we may have to change $(G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ on a null set to enable the described partitioning of $\mathbb{T}^{\mathrm{u}}$ everywhere on Ω.) More details on graphical constructions of interacting particle systems can be found in the book [Lig85] by T. Liggett or the paper [Har72] by T. Harris, who was the first to apply this method.

In this chapter we analyse the total effect of destruction in the $\mathbb{T}^{\mathrm{u}}$-forest-fire process up to the final time t_c, which is quantified by the so-called heights of destruction:

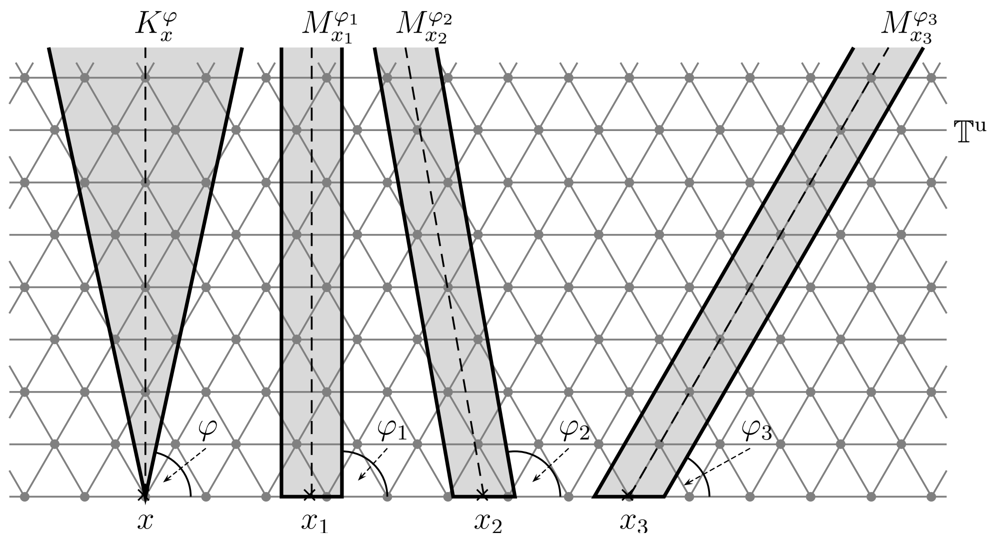


Figure 3.1: The half-plane triangular lattice $\mathbb{T}^{\mathrm{u}}$, a cone K_x^{φ} and three semi-infinite tubes $M_{x_j}^{\varphi_j}$ $(j = 1, 2, 3)$.

Definition 7. For $t \in [0, t_c]$ and $S \subset \mathbb{C}^{\mathrm{u}}$, let

$$Y_t(S) := \sup \{\operatorname{Im} z : z \in S \cap \mathbb{T}^{\mathrm{u}}, \exists s \in (0, t] : \eta_{s^-,z} > \eta_{s,z}\} \vee 0 \tag{3.4}$$

be the height up to which sites in S have been destroyed up to time t (where $Y_t(S)$ can take values in $[0, \infty]$). We call $Y_t(S)$ the **height of destruction**[2] in S up to time t.

Note that $Y_t(S)$ is monotone increasing in t and S in the sense that for $t_1, t_2 \in [0, t_c]$ and $S_1, S_2 \subset \mathbb{C}^{\mathrm{u}}$ the implication

$$(t_1 \leq t_2 \;\wedge\; S_1 \subset S_2) \Rightarrow Y_{t_1}(S_1) \leq Y_{t_2}(S_2) \tag{3.5}$$

holds. We will study the height of destruction in cones of the following kind:

Definition 8. For $x \in \mathbb{R}$ and $\varphi \in (0, \pi/2)$, let

$$K_x^{\varphi} := \left\{x + ae^{i\varphi} + be^{i(\pi-\varphi)} : a, b \geq 0\right\}$$

denote the infinite cone whose apex is x and whose boundary lines have angular directions φ and $\pi - \varphi$, respectively (see Figure 3.1).

Equation (3.2) indicates that $Y_{t_c}(K_x^{\varphi})$ could potentially be equal to ∞. We prove that this case a.s. does not occur:

[2]This definition is more general than Definition 3 in Chapter 2. If for $x \in \mathbb{Z}$ we define $M_x^{\pi/2}$ as in Definition 10 below and ignore the difference in the underlying lattices, then the height of destruction $Y_{t,x}$ of Definition 3 morally corresponds to the height of destruction $Y_t(M_x^{\pi/2})$ of Definition 7.

Theorem 3. *For all $x \in \mathbb{R}$ and $\varphi \in (0, \pi/2)$ we have* $\mathbf{P}\left[Y_{t_c}(K_x^{\varphi}) < \infty\right] = 1$.

Roughly speaking, Theorem 3 means that up to the final time t_c, the influence of the destruction mechanism [DESTRUCTION] in Definition 6 is confined to areas close to the inner boundary $\mathbb{Z}$ of the half-plane lattice $\mathbb{T}^{\mathrm{u}}$.

3.2 Extension of the model beyond the critical time

It is a natural question to ask how the forest-fire model and the corresponding heights of destruction behave when we let the process run beyond the critical time t_c. In this case the local graphical construction above does not work any more so that we must first give thought to the existence of such an extended process. In fact, it is not known whether a process $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ satisfying Definition 6 for all $t \in [0, \infty)$ exists. However, if we additionally demand that clusters are also destroyed when they are about to become infinite, then the extension does exist. This motivates the following definition:

Definition 9. Let $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ be a process with values in $(\{0,1\} \times \mathbb{N}_0)^{[0,\infty)\times\mathbb{T}^{\mathrm{u}}}$, initial condition $\eta_{0,z} = 0$ for $z \in \mathbb{T}^{\mathrm{u}}$ and boundary condition $\eta_{t,x} = 0$ for $t \in [0, \infty), x \in \mathbb{Z}$. Suppose that for all $z \in \mathbb{T}^{\mathrm{u}}$ the process $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty)}$ is càdlàg. For $z \in \mathbb{T}^{\mathrm{u}}$ and $t \in (0, \infty)$, let $C_{t^-,z}$ denote the cluster of z in the configuration $(\eta_{t^-,w})_{w\in\mathbb{T}^{\mathrm{u}}}$.

Then $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ is called an **extended $\mathbb{T}^{\mathrm{u}}$-forest-fire process** if the following conditions are satisfied:

[POISSON] The processes $(G_{t,z})_{t\in[0,\infty)}$, $z \in \mathbb{T}^{\mathrm{u}}$, are independent Poisson processes with rate 1.

[TRANSL-INV] The distribution of $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ is invariant under translations along the real line, i.e. the processes $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ and $(\eta_{t,z+1}, G_{t,z+1})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ have the same distribution.

[GROWTH] For all $t \in (0, \infty)$ and all $z \in \mathbb{T}^{\mathrm{u}} \setminus \mathbb{Z}$ the following implications hold:

(i) $G_{t^-,z} < G_{t,z} \Rightarrow \eta_{t,z} = 1$,
i.e. the growth of a tree at the site z at time t implies that the site z is occupied at time t;

(ii) $\eta_{t^-,z} < \eta_{t,z} \Rightarrow G_{t^-,z} < G_{t,z}$,
i.e. if the site z gets occupied at time t, there must have been the growth of a tree at the site z at time t.

[DESTRUCTION] For all $t \in (0, \infty)$ and all $x \in \mathbb{Z}$, $z \in \mathbb{T}^{\mathrm{u}} \setminus \mathbb{Z}$ the following implications hold:

(i) $(G_{t-,x} < G_{t,x} \Rightarrow \forall v \in \partial\{x\}\, \forall w \in C_{t-,v} : \eta_{t,w} = 0) \wedge$
$(|C_{t-,z}| = \infty \Rightarrow \forall w \in C_{t-,z} : \eta_{t,w} = 0)$,
i.e. if a cluster grows to the boundary $\mathbb{Z}$ at time t, it is destroyed at time t, and if a cluster is about to become infinite at time t, it is destroyed at time t;

(ii) $\eta_{t-,z} > \eta_{t,z} \Rightarrow ((\exists u \in \partial C_{t-,z} \cap \mathbb{Z} : G_{t-,u} < G_{t,u}) \vee |C_{t-,z}| = \infty)$,
i.e. if a site is destroyed at time t, its cluster either must have grown to the boundary $\mathbb{Z}$ at time t or it must have been about to become infinite at time t.

The existence of an extended $\mathbb{T}^{\mathrm{u}}$-forest-fire process follows from Theorem 1 in Chapter 2: There we showed that an analogous process on the upper half-plane of the square lattice $\mathbb{Z}^2$ exists, and the proof can be directly transferred to the triangular lattice. Conversely, it is currently not known whether extended $\mathbb{T}^{\mathrm{u}}$-forest-fire processes are unique in distribution. This is the reason why we have included the translation-invariance property [TRANSL-INV] in Definition 9, whereas for the unextended $\mathbb{T}^{\mathrm{u}}$-forest-fire process, the translation-invariance is already implied by the uniqueness of this process. Since extended $\mathbb{T}^{\mathrm{u}}$-forest-fire processes are also dominated by the corresponding pure growth process, in which there are a.s. no infinite clusters until the critical time t_c, the destruction of infinite clusters in extended $\mathbb{T}^{\mathrm{u}}$-forest-fire processes a.s. does not occur until t_c, i.e. [DESTRUCTION] in Definition 9 and [DESTRUCTION] in Definition 6 a.s. coincide until t_c. (In fact, it is unclear whether the destruction of infinite clusters in extended $\mathbb{T}^{\mathrm{u}}$-forest-fire processes ever occurs with positive probability.) Hence, if $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ is an extended $\mathbb{T}^{\mathrm{u}}$-forest-fire process, then restricted to the complement of a null set, $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ is a $\mathbb{T}^{\mathrm{u}}$-forest-fire process.

For the remainder of this section, let $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ be an extended $\mathbb{T}^{\mathrm{u}}$-forest-fire process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathcal{F}$ is the completion of the σ-field generated by $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$. For $t \in [0,\infty)$ and $S \subset \mathbb{C}^{\mathrm{u}}$, we define the corresponding height of destruction $Y_t(S)$ in S up to time t as in equation (3.4). Moreover, for $z \in \mathbb{C}$ and $S \subset \mathbb{C}$, we define the distance between z and S by

$$\operatorname{dist}(z, S) := \inf\{|z - z'| : z' \in S\}. \tag{3.6}$$

We now look at the height of destruction in semi-infinite tubes of the following kind:

Definition 10. For $x \in \mathbb{R}$ and $\varphi \in (0, \pi)$, let

$$L_x^\varphi := \left\{x + ye^{i\varphi} : y \geq 0\right\}$$

denote the half-line with starting point x and angular direction φ and let

$$M_x^\varphi := \left\{z \in \mathbb{C}^{\mathrm{u}} : \operatorname{dist}(z, L_x^\varphi) \leq \frac{1}{2}\right\}$$

denote the semi-infinite tube with centre line L_x^φ and width 1 (see Figure 3.1).

Combining Theorem 3 with results in Chapter 2, we obtain the following statement:

Corollary 1. *For $x \in \mathbb{R}$ and $\varphi \in (0, \pi)$ we have*

$$\mathbf{P}\left[\forall t \in [0, t_c] : Y_t(M_x^\varphi) < \infty\right] = 1, \tag{3.7}$$
$$\mathbf{P}\left[\forall t \in (t_c, \infty) : Y_t(M_x^\varphi) = \infty\right] = 1. \tag{3.8}$$

In other words, the height of destruction in the semi-infinite tube M_x^φ shows a phase transition in the sense that it is finite until the critical time t_c and becomes infinite immediately after t_c.

Proof of Corollary 1. Let $x \in \mathbb{R}$ and $\varphi \in (0, \pi)$. Pick $\alpha \in (0, \pi/2)$ such that $\alpha < \min\{\varphi, \pi - \varphi\}$ holds. Since $(M_x^\varphi \setminus K_x^\alpha) \cap \mathbb{T}^{\mathrm{u}}$ contains only finitely many sites and since the height of destruction is monotone increasing in the sense of (3.5), equation (3.7) is an immediate consequence of Theorem 3.

Equation (3.8) can be proved along the lines of Theorem 2 in Chapter 2: There we proved a corresponding statement for a slightly different setting, namely for an analogous forest-fire model on the upper half-plane of the square lattice $\mathbb{Z}^2$ and for $x \in \mathbb{Z}$, $\varphi = \pi/2$. (The associated height of destruction up to time t is denoted by $Y_{t,x}$ in Chapter 2.) However, the backbone of the proof in Chapter 2 does not depend on these particular assumptions. A crucial property of M_x^φ in the course of the proof is the fact that any 1-path which crosses from the left of M_x^φ to the right of M_x^φ has at least one site in M_x^φ; this is the reason why we have defined M_x^φ to have width 1. □

3.3 Proof of Theorem 3

Let $x \in \mathbb{R}$ and $\varphi \in (0, \pi/2)$. Throughout this section, we consider the following setting: Let $(G_{t,z})_{t\in[0,t_c]}$, $z \in \mathbb{T}$, be independent Poisson processes on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathcal{F}$ is the completion of the σ-field generated by $(G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}}$, and let

$$\sigma_{t,z} := 1_{\{G_{t,z}>0\}}, \qquad t \in [0, t_c], z \in \mathbb{T},$$

be the corresponding pure growth process on $\mathbb{T}$. (It will be convenient to have these processes on the whole triangular lattice $\mathbb{T}$ and not just on $\mathbb{T}^{\mathrm{u}}$.) Moreover, let $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ be the corresponding $\mathbb{T}^{\mathrm{u}}$-forest-fire process (for the construction of which $(G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ may have to be changed on a null set) and let $Y_{t_c}(K_x^\varphi)$ be the associated height of destruction in the cone K_x^φ up to the critical time t_c. For $t \in [0, t_c]$, we henceforth abbreviate $\eta_t := (\eta_{t,z})_{z\in\mathbb{T}^{\mathrm{u}}}$, $\sigma_t^{\mathrm{u}} := (\sigma_{t,z})_{z\in\mathbb{T}^{\mathrm{u}}}$ and $\sigma_t := (\sigma_{t,z})_{z\in\mathbb{T}}$.

We will frequently use the following terminology: Let $V \in \{\mathbb{T}^{\mathrm{u}}, \mathbb{T}\}$, let $(\alpha_v)_{v\in V} \in \{0,1\}^V$ be a random configuration and let $w \in V$, $S \subset \mathbb{C}$. Then we write $\{w \leftrightarrow S \text{ in } (\alpha_v)_{v\in V}\}$ (in words: w is connected to S in $(\alpha_v)_{v\in V})$ for the event that there exists a 1-path in $(\alpha_v)_{v\in V}$ from a site $y \in V$ to a site $z \in V$ such that y is a neighbour of w and $\operatorname{dist}(z, S) \le 1$ holds, where $\operatorname{dist}(z, S)$ is defined as in (3.6). Note that our definition of $\{w \leftrightarrow S \text{ in } (\alpha_v)_{v\in V}\}$ does not impose any condition on the site w itself.

3.3.1 Tools from percolation theory

We will need the following results from percolation theory:

Exponential decay in the subcritical regime. For $z \in \mathbb{T}$ and $n \in \mathbb{N}$, let

$$\begin{aligned} S_n^\varphi(z) := &\left\{z + u + ve^{i\varphi} : u, v \in \mathbb{R}, |u| = n, |v| \le n\right\} \\ &\cup \left\{z + u + ve^{i\varphi} : u, v \in \mathbb{R}, |u| \le n, |v| = n\right\} \end{aligned}$$

denote the surface of the rhombus with centre z, side length $2n$ and sides parallel to the $\mathbb{R}$-basis $\{1, e^{i\varphi}\}$ of $\mathbb{C}$. There exists a function $\xi_\varphi : (0, t_c) \to (0, \infty)$ such that for all $t \in (0, t_c)$ the full-plane one-arm event $\{0 \leftrightarrow S_n^\varphi(0) \text{ in } \sigma_t\}$ satisfies

$$\lim_{n\to\infty} -\frac{\log \mathbf{P}\left[0 \leftrightarrow S_n^\varphi(0) \text{ in } \sigma_t\right]}{n} = \frac{1}{\xi_\varphi(t)}; \tag{3.9}$$

$\xi_\varphi(t)$ is called the correlation length of the configuration σ_t. Moreover, there exists a universal constant $c \in (0, \infty)$ such that for all $t \in (0, t_c)$ and $n \in \mathbb{N}$

$$\mathbf{P}\left[0 \leftrightarrow S_n^\varphi(0) \text{ in } \sigma_t\right] \le cn \exp\left(-\frac{n}{\xi_\varphi(t)}\right) \tag{3.10}$$

holds. For the proof of (3.9) and (3.10) the reader is referred to [Gri99], Section 6.1. (In this reference, analogous statements are proven for bond percolation on the square lattice $\mathbb{Z}^2$ and $\varphi = \pi/2$ but the proofs can be transferred one-to-one to our setting.)

Critical exponents. Near the critical time t_c, the correlation length behaves like

$$\xi_\varphi(t) = (t_c - t)^{-4/3+o(1)} \qquad \text{for } t \uparrow t_c. \tag{3.11}$$

At the critical time t_c, the probability of the half-plane one-arm event $\{0 \leftrightarrow S_n^\varphi(0) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma_{t_c}^{\mathrm{u}}\}$ decays like

$$\mathbf{P}\left[0 \leftrightarrow S_n^\varphi(0) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma_{t_c}^{\mathrm{u}}\right] = n^{-1/3+o(1)} \qquad \text{for } n \to \infty. \tag{3.12}$$

Equations (3.11) and (3.12) were first proven by S. Smirnov and W. Werner in [SW01] (Theorems 1(iv) and 3) and are also discussed in the survey article [Nol08] (Theorems 33(i) and 22). (In these references, the statements are not based on the rhombus $S_n^\varphi(0)$ used here but on the circle with centre 0 and radius n and the rhombus $S_n^{\pi/3}(0)$ with angle $\pi/3$, respectively. In fact, the exact shape of the boundary line is irrelevant. However, the current proof of (3.11) and (3.12) only works for the triangular lattice.)

3.3.2 The core of the proof of Theorem 3

We now prove Theorem 3, i.e. we show that $\mathbf{P}[Y_{t_c}(K_x^\varphi) = \infty] = 0$. Since the $\mathbb{T}^{\mathrm{u}}$-forest-fire process $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ is dominated by the corresponding pure growth process $(\sigma_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ in the sense of equation (3.3), a.s. all destroyed clusters in $(\eta_{t,z}, G_{t,z})_{t\in[0,t_c],z\in\mathbb{T}^{\mathrm{u}}}$ are finite. Hence, if $Y_{t_c}(K_x^\varphi) = \infty$ holds, then a.s. infinitely many clusters which reach from K_x^φ to the inner boundary $\mathbb{Z}$ must have been destroyed up to the critical time t_c. Moreover, since there are only finitely many jumps in a rate 1 Poisson process up to time t_c, every site on the inner boundary $\mathbb{Z}$ can only be the origin of finitely many destruction events up to time t_c. This implies the inclusion

$$\{Y_{t_c}(K_x^\varphi) = \infty\} \overset{\text{a.s.}}{\subset} \limsup_{n\to\infty} \mathcal{A}_{x,n}^\varphi \cup \limsup_{n\to\infty} \mathcal{A}_{x,-n}^\varphi,$$

where we define

$$\begin{aligned} \mathcal{A}_{x,n}^\varphi &:= \left\{\exists t \in [0,t_c) : \lceil x\rceil + n \leftrightarrow K_x^\varphi \text{ in } \eta_t, \mathrm{G}_{t,t_c,\lceil x\rceil+n}\right\}, \\ \mathcal{A}_{x,-n}^\varphi &:= \left\{\exists t \in [0,t_c) : \lfloor x\rfloor - n \leftrightarrow K_x^\varphi \text{ in } \eta_t, \mathrm{G}_{t,t_c,\lfloor x\rfloor-n}\right\} \end{aligned}$$

for $n \in \mathbb{N}$ and use the abbreviation

$$\mathrm{G}_{s,t,z} := \{G_{s,z} < G_{t,z}\}$$

for $0 \le s < t \le t_c$ and $z \in \mathbb{T}^{\mathrm{u}}$. By symmetry, we have $\mathbf{P}[\mathcal{A}_{x,-n}^\varphi] = \mathbf{P}[\mathcal{A}_{-x,n}^\varphi]$ for all $n \in \mathbb{N}$; consequently, it suffices to prove

$$\mathbf{P}\left[\limsup_{n\to\infty} \mathcal{A}_{x,n}^\varphi\right] = 0. \tag{3.13}$$

Applying equation (3.3) once more and using the topological fact that any connection $\lceil x\rceil + n \leftrightarrow K_x^\varphi$ necessarily contains a connection $\lceil x\rceil + n \leftrightarrow S_n^\varphi(\lceil x\rceil + n) \cap \mathbb{C}^{\mathrm{u}}$, we obtain the inclusions

$$\begin{aligned} \mathcal{A}_{x,n}^\varphi &\subset \left\{\exists t \in [0,t_c) : \lceil x\rceil + n \leftrightarrow K_x^\varphi \text{ in } \sigma_t^{\mathrm{u}}, \mathrm{G}_{t,t_c,\lceil x\rceil+n}\right\} \\ &\subset \left\{\exists t \in [0,t_c) : \lceil x\rceil + n \leftrightarrow S_n^\varphi(\lceil x\rceil + n) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma_t^{\mathrm{u}}, \mathrm{G}_{t,t_c,\lceil x\rceil+n}\right\} =: \mathcal{B}_{x,n}^\varphi. \end{aligned} \tag{3.14}$$

Now choose an arbitrary $\delta \in (0, 1/12)$ and consider the event

$$\mathcal{C}_{x,n}^{\varphi,\delta} := \left\{\exists t \in [0, t_c - n^{-3/4+\delta}) : \lceil x\rceil + n \leftrightarrow S_n^\varphi(\lceil x\rceil + n) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma_t^{\mathrm{u}}\right\}$$

that the connection $\lceil x\rceil + n \leftrightarrow S_n^\varphi(\lceil x\rceil + n) \cap \mathbb{C}^{\mathrm{u}}$ in the pure growth process already occurs before time $t_c - n^{-3/4+\delta}$ (where $n \in \mathbb{N}$ is assumed to be large enough to guarantee

$t_c - n^{-3/4+\delta} > 0$). We can estimate the probability of this event from above as follows:

$$\begin{aligned}
\mathbf{P}\left[\mathcal{C}^{\varphi,\delta}_{x,n}\right] &\leq \mathbf{P}\left[\exists t \in [0, t_c - n^{-3/4+\delta}) : 0 \leftrightarrow S^{\varphi}_n(0) \text{ in } \sigma_t\right] \\
&= \mathbf{P}\left[0 \leftrightarrow S^{\varphi}_n(0) \text{ in } \sigma_{t_c - n^{-3/4+\delta}}\right] \\
&\leq cn \exp\left(-\frac{n}{\xi_\varphi\left(t_c - n^{-3/4+\delta}\right)}\right) \\
&= cn \exp\left(-\frac{n}{\left(n^{-3/4+\delta}\right)^{-4/3+o(1)}}\right) \qquad \text{for } n \to \infty \\
&= cn \exp\left(-n^{(4/3)\delta + o(1)}\right) \qquad \text{for } n \to \infty.
\end{aligned}$$

Here we first drop the condition that the connection occurs in the upper half-plane $\mathbb{C}^{\mathrm{u}}$ and use the translation-invariance of the pure growth process; then we employ the fact that σ_t is monotone increasing in t; finally we successively apply equations (3.10) and (3.11). In particular, this estimate implies

$$\sum_{n=1}^{\infty} \mathbf{P}\left[\mathcal{C}^{\varphi,\delta}_{x,n}\right] < \infty$$

and hence

$$\mathbf{P}\left[\limsup_{n\to\infty} \mathcal{C}^{\varphi,\delta}_{x,n}\right] = 0$$

by the Borel-Cantelli lemma. Regarding the limes superior of the events $\mathcal{B}^{\varphi}_{x,n}$ (defined in (3.14)), we thus conclude

$$\limsup_{n\to\infty} \mathcal{B}^{\varphi}_{x,n} \overset{\text{a.s.}}{\subset} \limsup_{n\to\infty} \left(\mathcal{B}^{\varphi}_{x,n} \setminus \mathcal{C}^{\varphi,\delta}_{x,n}\right) \subset \limsup_{n\to\infty} \mathcal{D}^{\varphi,\delta}_{x,n}, \tag{3.15}$$

where we abbreviate

$$\mathcal{D}^{\varphi,\delta}_{x,n} := \left\{\exists t \in [t_c - n^{-3/4+\delta}, t_c) : \lceil x \rceil + n \leftrightarrow S^{\varphi}_n(\lceil x \rceil + n) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma^{\mathrm{u}}_t, \mathrm{G}_{t, t_c, \lceil x \rceil + n}\right\}$$

for $n \in \mathbb{N}$ satisfying $t_c - n^{-3/4+\delta} > 0$. The probability of the event $\mathcal{D}^{\varphi,\delta}_{x,n}$ can be bounded from above as follows:

$$\begin{aligned}
\mathbf{P}\left[\mathcal{D}^{\varphi,\delta}_{x,n}\right] &\leq \mathbf{P}\left[\lceil x \rceil + n \leftrightarrow S^{\varphi}_n(\lceil x \rceil + n) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma^{\mathrm{u}}_{t_c}, \mathrm{G}_{t_c - n^{-3/4+\delta}, t_c, \lceil x \rceil + n}\right] \\
&= \mathbf{P}\left[0 \leftrightarrow S^{\varphi}_n(0) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma^{\mathrm{u}}_{t_c}\right] \mathbf{P}\left[\mathrm{G}_{0, n^{-3/4+\delta}, 0}\right] \\
&= n^{-1/3+o(1)} \cdot \left(1 - \exp\left(-n^{-3/4+\delta}\right)\right) \qquad \text{for } n \to \infty \\
&\leq n^{-1/3+o(1)} \cdot n^{-3/4+\delta} \qquad \text{for } n \to \infty \\
&= n^{-13/12+\delta+o(1)} \qquad \text{for } n \to \infty.
\end{aligned}$$

Here we first relax the condition on the times at which the connection and the growth event occur, resorting to the fact that σ_t is monotone increasing in t; then we use the independence and translation-invariance of the events $\{\lceil x\rceil + n \leftrightarrow S_n^\varphi(\lceil x\rceil + n) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma_{t_c}^{\mathrm{u}}\}$ and $\mathrm{G}_{t_c - n^{-3/4+\delta}, t_c, \lceil x\rceil + n}$; in the next step we apply equation (3.12); finally we use the inequality $1 - e^{-y} \leq y$ which is valid for all $y \in \mathbb{R}$. Since $-13/12 + \delta < -1$ holds by our choice of δ, the previous estimate shows

$$\sum_{n=1}^{\infty} \mathbf{P}\left[\mathcal{D}_{x,n}^{\varphi,\delta}\right] < \infty.$$

Invoking the Borel-Cantelli lemma again, we get

$$\mathbf{P}\left[\limsup_{n\to\infty} \mathcal{D}_{x,n}^{\varphi,\delta}\right] = 0. \tag{3.16}$$

Together with (3.14) and (3.15), equation (3.16) yields the proof of (3.13) and hence of Theorem 3.

Chapter 4

Self-destructive percolation as a limit of forest-fire models on regular rooted trees

An article which closely follows this chapter has been uploaded on *arXiv* [Gra14c] and submitted to a journal.

4.1 Introduction and statement of the main results

4.1.1 The forest-fire model

In this chapter we study the Dürre forest-fire model (see Section 1.3.1) on regular rooted trees or, more precisely, large finite subtrees thereof. Let us start by introducing some notation about regular rooted trees. For the remainder of this chapter, let $r \in \{2, 3, \dots\}$ be fixed. The r**-regular rooted tree** is the unique tree (up to graph isomorphisms) in which one vertex, called the **root** of the tree, has degree r and every other vertex has degree $r+1$. We denote the r-regular tree by T and the root of T by $\emptyset$. In slight abuse of notation, we will use the term T both for the r-regular tree as a graph and for its vertex set. Let $|u|$ denote the graph distance of a vertex $u \in T$ from the root $\emptyset$. For two vertices $u, v \in T$, we say that u is the **parent** of v (or equivalently that v is a **child** of u) if u and v are neighbours and $|u| = |v| - 1$ holds. Moreover, for $u, v \in T$, we say that u is an **ancestor** of v (abbreviated by $u \preceq v$) if there exist $k \in \mathbb{N}_0$ and a sequence of vertices $z_0, z_1, \dots, z_k \in T$ such that $z_0 = u$, $z_k = v$ and z_{i-1} is the parent of z_i for all $i \in \{1, 2, \dots, k\}$. For $n \in \mathbb{N}_0$, we say that $u \in T$ is in the n**th generation** of T if $|u| = n$, and we define

$$T_n := \{z \in T : |z| = n\}$$

to be the set of all vertices in the nth generation and

$$B_n := \{z \in T : |z| \leq n\} = \bigcup_{i=0}^{n} T_i$$

to be the set of all vertices with graph distance at most n from the root $\emptyset$.

In order to explain some further terminology, let V be a subset of T and let $\alpha = (\alpha_v)_{v\in V} \in \{0,1\}^V$. We say that a vertex $v \in V$ is occupied in α if $\alpha_v = 1$, and we say that v is vacant in α if $\alpha_v = 0$. The set

$$T|_{\alpha,1} := \{v \in V : \alpha_v = 1\} \subset T$$

of occupied vertices in α induces a subgraph of T, which (in slight abuse of notation) we denote by $T|_{\alpha,1}$, too. For any vertex $z \in V$ the maximal connected component of $T|_{\alpha,1}$ containing z is called the **(occupied) cluster** of z in α. Moreover, if W is a connected subset of T, we say that a vertex $z \in T$ is the **root** of W if $z \in W$ holds and z is in the lowest generation among all vertices contained in W.

Let $n \in \mathbb{N}$. We now define the forest-fire model on B_n. Informally, the model can be described as follows: Each vertex in B_n can be vacant or occupied. At time 0 all vertices are vacant. Then the process is governed by two opposing mechanisms: Vertices become occupied according to independent rate 1 Poisson processes, the so-called growth processes. Independently, vertices are hit by "lightning" according to independent rate $\lambda(n)$ Poisson processes (where $\lambda(n) > 0$), the so-called ignition processes. When a vertex is hit by lightning, its occupied cluster is instantaneously destroyed, i.e. it becomes vacant. Occupied vertices are usually pictured to be vegetated by a tree, so occupied clusters correspond to pieces of woodland and the destruction of clusters corresponds to the burning of forests by fires, which are caused by strokes of lightning. However, we avoid this terminology here because we already use the term tree in the graph-theoretic sense. A more formal definition of the forest-fire model goes as follows (where for a function $[0,\infty) \ni t \mapsto f_t \in \mathbb{R}$, we write $f_{t^-} := \lim_{s\uparrow t} f_s$ for the left-sided limit at $t > 0$, provided the limit exists):

Definition 11. Let $n \in \mathbb{N}$ and $\lambda(n) > 0$. Let $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$ be a process with values in $(\{0,1\} \times \mathbb{N}_0 \times \mathbb{N}_0)^{[0,\infty)\times B_n}$ and initial condition $\eta^n_{0,z} = 0$ for $z \in B_n$. Suppose that for all $z \in B_n$ the process $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0}$ is càdlàg, i.e. right-continuous with left limits. For $z \in B_n$ and $t > 0$, let $C^n_{t^-,z}$ denote the cluster of z in the configuration $(\eta^n_{t^-,w})_{w\in B_n}$.

Then $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$ is called a **forest-fire process** on B_n with parameter $\lambda(n)$ if the following conditions are satisfied:

[POISSON] The processes $(G_{t,z})_{t\geq 0}$ and $(I^n_{t,z})_{t\geq 0}$, $z \in B_n$, are independent Poisson processes with rates 1 and $\lambda(n)$, respectively.

[GROWTH] For all $t > 0$ and all $z \in B_n$ the following implications hold:

(i) $G_{t^-,z} < G_{t,z} \Rightarrow \eta^n_{t,z} = 1$,
i.e. the growth of a tree at the site z at time t implies that the site z is occupied at time t;

(ii) $\eta^n_{t^-,z} < \eta^n_{t,z} \Rightarrow G_{t^-,z} < G_{t,z}$,
i.e. if the site z gets occupied at time t, there must have been the growth of a tree at the site z at time t.

[DESTRUCTION] For all $t > 0$ and all $z \in B_n$ the following implications hold:

(i) $I^n_{t^-,z} < I^n_{t,z} \Rightarrow \forall w \in C^n_{t^-,z} : \eta^n_{t,w} = 0$,
i.e. if the cluster at z is hit by lightning at time t, it is destroyed at time t;

(ii) $\eta^n_{t^-,z} > \eta^n_{t,z} \Rightarrow \exists v \in C^n_{t^-,z} : I^n_{t^-,v} < I^n_{t,v}$,
i.e. if the site z is destroyed at time t, its cluster must have been hit by lightning at time t.

Given independent Poisson processes $(G_{t,z})_{t\geq 0}$ and $(I^n_{t,z})_{t\geq 0}$, $z \in B_n$, with rates 1 and $\lambda(n)$, respectively, a unique corresponding forest-fire process $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$ on B_n can be obtained by a graphical construction (see [Lig85]). For this construction it is crucial that B_n is finite. Using different methods, M. Dürre obtained results on existence and uniqueness of forest-fire models for all connected infinite graphs with bounded vertex degree (see [Dür06a], [Dür06b], [Dür09]).

One of the most interesting aspects about the forest-fire process on B_n is the question of what happens when n tends to infinity. Assuming that the limit $n \to \infty$ exists in a suitable sense, we obtain a process on the infinite tree T, and the question thus concerns the dynamics of this limit process. It is intuitively clear that the growth mechanism carries over to the limit process but it is in general highly non-trivial what becomes of the destruction mechanism. Of course, the answer will depend strongly on the asymptotic behaviour of $\lambda(n)$. If a, b are functions from $\mathbb{N}$ to $(0, \infty)$, we write

(i) $a(n) \ll b(n)$ for $n \to \infty$ if $a(n)/b(n) \to 0$ for $n \to \infty$;

(ii) $a(n) \approx b(n)$ for $n \to \infty$ if $\log a(n)/\log b(n) \to 1$ for $n \to \infty$.

Heuristically, one expects four regimes of $\lambda(n)$ with qualitatively different asymptotics, which we now describe informally.

1. If $\lambda(n) \ll r^n$, then the number of lightnings in B_n tends to 0 for $n \to \infty$. Therefore, in the limit $n \to \infty$ no clusters can ever be destroyed so that the resulting process on T is simply a dynamical formulation of Bernoulli percolation.

2. If $\lambda(n) \approx 1/m^n$ for some $1 < m < r$, then in the limit $n \to \infty$ no finite clusters and no "thin" infinite clusters (i.e. those in which on average every vertex has fewer than m occupied child vertices) can be destroyed but "fat" infinite clusters (i.e. those in which on average every vertex has more than m occupied child vertices) should still be hit by lightning as soon as they appear. The resulting process on T should therefore have the following dynamics: Vertices become occupied at rate 1, independently for all vertices. If an infinite cluster becomes "fat", it is instantaneously destroyed.

3. If $1/m^n \ll \lambda(n) \ll 1$ for every $m > 1$, then in the limit $n \to \infty$ no finite clusters can be destroyed but one would expect any infinite cluster to be dense enough that it is hit by lightning as soon as it appears. The resulting process on T should therefore

have the following dynamics: Vertices become occupied at rate 1, independently for all vertices. If a cluster becomes infinite, it is instantaneously destroyed.

4. If $\lambda(n) = \lambda$ for some constant $\lambda > 0$, then the limit $n \to \infty$ should yield a forest-fire model on T with the following dynamics: Vertices become occupied at rate 1, independently for all vertices. Independently thereof and independently for all vertices, vertices are hit by lightning at rate λ. If a vertex is hit by lightning, its cluster is instantaneously destroyed.

In this chapter, we give a partial result for regime 2 in the sense that we prove the conjectured asymptotics between time 0 and a deterministic time shortly *after* the first destruction of infinite clusters in the limit process on T. Before we proceed to the precise statement, we briefly comment on the other regimes and give a short overview of related results.

Regime 1 is the simplest case and the above statement on this regime can easily be made rigorous. The statement on regime 4 follows from work by M. Dürre in [Dür09]. In fact, the results of [Dür09] are much more general in the sense that they are not restricted to regular rooted trees but hold for all connected infinite graphs with bounded vertex degree. Regime 3 is undoubtedly the most difficult case with few rigorous results yet. It is even unknown whether the hypothetical limit process described in 3 exists at all. For the square lattice $\mathbb{Z}^2$, the corresponding process does not exist (conjectured by J. van den Berg and R. Brouwer in [vdBB04] and recently proven by D. Kiss, I. Manolescu and V. Sidoravicius in [KMS13]). Regime 3 is expected to behave similarly to the case where we first set $\lambda(n) = \lambda$ for some $\lambda > 0$ and then take the double limit $\lim_{\lambda\downarrow 0} \lim_{n\to\infty}$ (assuming that it exists in a suitable sense). In [vdBB06] this case was investigated for forest-fire models on the directed binary tree and on the square lattice. For forest-fire models on the square lattice $\mathbb{Z}^2$, an analogous heuristic description of four different regimes of the lightning rate can be found in the paper [RT09] by B. Ráth and B. Tóth. The main content of [RT09], however, is the analysis of a forest-fire model which arises as a modification of the Erdős-Rényi evolution and which also shows four regimes of the lightning rate with different asymptotic behaviour (compare the summary in Section 1.3.4).

4.1.2 The pure growth process

In the following, if A is an event, we write 1_A for its indicator function, and if B is any set, we write $|B|$ for the number of elements in B (where $|B|$ can take values in $\mathbb{N}_0 \cup \{\infty\}$).

Definition 12. Let $(G_{t,z})_{t\geq 0}$, $z \in T$, be independent rate 1 Poisson processes and let

$$\sigma_{t,z} := 1_{\{G_{t,z}>0\}}, \qquad t \geq 0, z \in T.$$

Then $(\sigma_{t,z}, G_{t,z})_{t\geq 0, z\in T}$ is called a **pure growth process**[1] on T. Moreover, for $x \in T$ and $t \geq 0$, we denote by $S_{t,x}$ the cluster of x in the configuration $(\sigma_{t,z})_{z\in T}$, and for $t \geq 0$, we

[1] Unlike in Chapters 2 and 3, we now include the underlying Poisson processes into the notation of the pure growth process.

denote by

$$O_t := \{z \in T : |S_{t,z}| = \infty\}$$

the set of all vertices which are in an infinite cluster in the configuration $(\sigma_{t,z})_{z\in T}$.

Above we claimed that as $n \to \infty$ in regime 2, the forest-fire process on B_n should initially behave like the pure growth process on T until "fat" infinite clusters appear for the first time. We now want to make this statement more precise.

We first observe that for $t \geq 0$, the configuration $(\sigma_{t,z})_{z\in T}$ is identical with Bernoulli percolation on T, where each vertex is occupied with probability $1 - e^{-t}$ and vacant with probability e^{-t}. From percolation theory it is well-known that there is a critical time $t_c := \log \frac{r}{r-1}$ such that a.s. for $t \leq t_c$ there is no infinite cluster in $(\sigma_{t,z})_{z\in T}$ and for $t > t_c$ there are infinitely many infinite clusters in $(\sigma_{t,z})_{z\in T}$. For $z \in T$ and $t \geq 0$, conditionally on the event $\{z \text{ is the root of } S_{t,z}\}$, the cluster $S_{t,z}$ can also be identified with a Galton-Watson process whose offspring distribution is binomially distributed with parameters r and $1 - e^{-t}$. In particular, the offspring distribution at time $t \geq 0$ has mean

$$m(t) := r(1 - e^{-t}) \tag{4.1}$$

and variance

$$\sigma^2(t) := r(1 - e^{-t})e^{-t}. \tag{4.2}$$

It is a consequence of the Kesten-Stigum theorem for Galton-Watson processes (see [KS66]) that for $z \in T$ and $t \geq 0$, there exists a random variable $W_{t,z}$ with values in $[0, \infty)$ such that

$$\lim_{n\to\infty} \frac{|S_{t,z} \cap B_n|}{m(t)^n} = W_{t,z} \text{ a.s.} \tag{4.3}$$

and

$$W_{t,z} > 0 \text{ a.s. on the event } \{|S_{t,z}| = \infty\} \tag{4.4}$$

hold. (We will prove a different version later, see Proposition 2.) This suggests that if the lightning rate in the forest-fire process on B_n satisfies $\lambda(n) \approx 1/m^n$ for some $1 < m < r$, then the time threshold between "thin" and "fat" infinite clusters in the pure growth process should be the unique $\tau \in (t_c, \infty)$ with $m(\tau) = m$. In other words, in the limit $n \to \infty$, we expect to obtain a process on T which is equal to the pure growth process between time 0 and time τ and in which all infinite clusters are destroyed at time τ.

4.1.3 Statement of the main results

We will make the heuristics of the previous paragraph rigorous in the following way:

Definition 13. Let $n \in \mathbb{N}$ and let $\lambda(n) > 0$. We say that a forest-fire process $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$ on B_n with parameter $\lambda(n)$ and a pure growth process $(\tilde{\sigma}_{t,z}, \tilde{G}_{t,z})_{t\geq 0, z\in T}$ on T are **coupled in the canonical way** if they are realized on the same probability space and $(G_{t,z})_{t\geq 0, z\in B_n} = (\tilde{G}_{t,z})_{t\geq 0, z\in B_n}$ holds.

Theorem 4. *Let $\tau \in (t_c, \infty)$ and suppose that $\lambda : \mathbb{N} \to (0,\infty)$ satisfies $\lambda(n) \approx 1/m(\tau)^n$ for $n \to \infty$. For $n \in \mathbb{N}$, let $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$ be a forest-fire process on B_n with parameter $\lambda(n)$ and let $(\sigma_{t,z}, G_{t,z})_{t\geq 0, z\in T}$ be a pure growth process on T, coupled in the canonical way under some probability measure $\mathbf{P}$. For $t \geq 0$, let O_t be defined as in Definition 12. Then for all finite subsets $E \subset T$ and for all $\delta > 0$,*

$$\lim_{n\to\infty} \mathbf{P}\left[\sup_{z\in E, 0\leq t\leq \tau-\delta} \left|\eta^n_{t,z} - \sigma_{t,z}\right| = 0, \forall z \in O_\tau \cap E\, \exists t \in (\tau-\delta, \tau+\delta) : \eta^n_{t^-,z} > \eta^n_{t,z}\right] = 1$$

holds.

The condition on λ in Theorem 4 can be written in a different way: Given $\tau \in (t_c, \infty)$ and a function $\lambda : \mathbb{N} \to (0,\infty)$, define the function $g : \mathbb{N} \to (0,\infty)$ by

$$g(n) := \lambda(n) m(\tau)^n, \qquad n \in \mathbb{N}. \tag{4.5}$$

Then it is easy to see that the following are equivalent:

(i) $\lambda(n) \approx 1/m(\tau)^n$ for $n \to \infty$;

(ii) $\sqrt[n]{g(n)} \to 1$ for $n \to \infty$.

Under additional assumptions on g we can determine whether the destruction of the infinite clusters asymptotically occurs immediately *before* or *after* time τ:

Theorem 5. *Consider the situation of Theorem 4. In particular, suppose that the function g defined by (4.5) satisfies $\sqrt[n]{g(n)} \to 1$ for $n \to \infty$.*

(i) If g satisfies $g(n) \ll n/\log n$ for $n \to \infty$, then for all finite subsets $E \subset T$ and for all $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left[\sup_{z\in E, 0\leq t\leq \tau} \left|\eta^n_{t,z} - \sigma_{t,z}\right| = 0, \forall z \in O_\tau \cap E\, \exists t \in (\tau, \tau+\delta) : \eta^n_{t^-,z} > \eta^n_{t,z}\right] = 1$$

holds, i.e. the infinite clusters are asymptotically destroyed immediately after *time τ.*

(ii) If there exists $\alpha \in (0,1)$ such that g satisfies $g(n) \gg \exp(n^\alpha)$ for $n \to \infty$, then for all finite subsets $E \subset T$ and for all $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left[\sup_{z\in E, 0\leq t\leq \tau-\delta} \left|\eta^n_{t,z} - \sigma_{t,z}\right| = 0, \forall z \in O_\tau \cap E\, \exists t \in (\tau-\delta, \tau) : \eta^n_{t^-,z} > \eta^n_{t,z}\right] = 1$$

holds, i.e. the infinite clusters are asymptotically destroyed immediately before *time* τ.

Theorems 4 and 5 will be proved in Sections 4.2 and 4.3. Before, we give an interpretation of Theorem 4 in terms of self-destructive percolation.

4.1.4 Interpretation in terms of self-destructive percolation

Definition 14. Let $\tau \in (t_c, \infty)$, let $\epsilon > 0$ and let $(\sigma_{t,z}, G_{t,z})_{t \geq 0, z \in T}$ be a pure growth process on T. For $t \geq 0$, let O_t be defined as in Definition 12. We define $\rho_{t,z}$ for $0 \leq t \leq \tau + \epsilon, z \in T$ in three steps:
Firstly,

$$\rho_{t,z} := \sigma_{t,z}, \qquad 0 \leq t < \tau, z \in T,$$

i.e. at time 0 all vertices are vacant and between time 0 and time τ vertices become occupied at rate 1, independently for all vertices. Secondly,

$$\rho_{\tau,z} := \sigma_{\tau,z} \, 1_{\{z \notin O_\tau\}}, \qquad z \in T,$$

i.e. at time τ all infinite occupied clusters are destroyed. Thirdly,

$$\rho_{t,z} := \rho_{\tau,z} \vee 1_{\{G_{t,z} - G_{\tau,z} > 0\}}, \qquad \tau < t \leq \tau + \epsilon, z \in T,$$

i.e. between time τ and time $\tau + \epsilon$ vertices become occupied at rate 1, independently for all vertices and independently of what happened between time 0 and time τ. Then $(\rho_{t,z}, G_{t,z})_{0 \leq t \leq \tau + \epsilon, z \in T}$ is called a **self-destructive percolation process**[2] on T with parameters τ and ϵ.

Self-destructive percolation was first introduced by J. van den Berg and R. Brouwer in [vdBB04] and has subsequently also been studied in [vdBBV08], [vdBdL09], [AST14], [ADCKS13] and [KMS13]. For our purposes, the following property of self-destructive percolation is of particular importance:

Proposition 1. *For all* $\tau \in (t_c, \infty)$ *there exists* $\epsilon > 0$ *such that a.s. there is no infinite cluster in the final configuration* $(\rho_{\tau+\epsilon,z})_{z \in T}$ *of a self-destructive percolation process* $(\rho_{t,z}, G_{t,z})_{0 \leq t \leq \tau+\epsilon, z \in T}$ *on* T *with parameters* τ *and* ϵ.

[2] not to be confused with the hypothetical *permanent* self-destructive percolation process of Section 1.3.2

For the case where T is the binary tree (i.e. $r = 2$), this has already been proved by J. van den Berg and R. Brouwer ([vdBB04], Theorem 5.1). The proof of Proposition 1 for general r is based on an extension of the ideas in [vdBB04] and will be given in Section 4.4.

Theorem 4 and Proposition 1 imply that given $\tau \in (t_c, \infty)$, we can choose $\epsilon > 0$ such that between time 0 and time $\tau + \epsilon$ every forest-fire process on B_n with parameter $\lambda(n) \approx 1/m(\tau)^n$ converges to the self-destructive percolation process on T with parameters τ and ϵ. The formal statement is as follows:

Definition 15. Let $n \in \mathbb{N}$ and let $\lambda(n) > 0$. Moreover, let $\tau \in (t_c, \infty)$ and $\epsilon > 0$. We say that a forest-fire process $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$ on B_n with parameter $\lambda(n)$ and a self-destructive percolation process $(\tilde{\rho}_{t,z}, \tilde{G}_{t,z})_{0\leq t\leq \tau+\epsilon, z\in T}$ on T with parameters τ and ϵ are **coupled in the canonical way** if they are realized on the same probability space and $(G_{t,z})_{0\leq t\leq\tau+\epsilon, z\in B_n} = (\tilde{G}_{t,z})_{0\leq t\leq\tau+\epsilon, z\in B_n}$ holds.

Corollary 2. *Let $\tau \in (t_c, \infty)$, let $\epsilon > 0$ be as in Proposition 1 and suppose that $\lambda : \mathbb{N} \to (0, \infty)$ satisfies $\lambda(n) \approx 1/m(\tau)^n$ for $n \to \infty$. For $n \in \mathbb{N}$, let $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$ be a forest-fire process on B_n with parameter $\lambda(n)$ and let $(\rho_{t,z}, G_{t,z})_{0\leq t\leq\tau+\epsilon, z\in T}$ be a self-destructive percolation process on T, coupled in the canonical way under some probability measure $\mathbf{P}$. Then for all finite subsets $E \subset T$ and for all $\delta \in (0, \epsilon)$,*

$$\lim_{n\to\infty} \mathbf{P}\left[\sup_{z\in E, 0\leq t\leq \tau-\delta} \left|\eta^n_{t,z} - \rho_{t,z}\right| = 0, \sup_{z\in E, \tau+\delta\leq t\leq\tau+\epsilon} \left|\eta^n_{t,z} - \rho_{t,z}\right| = 0\right] = 1 \tag{4.6}$$

holds.

Proof of Corollary 2 given Theorem 4 and Proposition 1. Let τ, ϵ, λ be as in Corollary 2. Likewise, for $n \in \mathbb{N}$, let $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$, $(\rho_{t,z}, G_{t,z})_{0\leq t\leq\tau+\epsilon, z\in T}$ be as in Corollary 2. Moreover, let $E \subset T$ be a finite subset and let $\delta \in (0, \epsilon)$. For the proof of (4.6) we may assume without loss of generality that E is a singleton, i.e. $E = \{x\}$ for some $x \in T$. In view of Theorem 4 it then suffices to prove

$$\lim_{n\to\infty} \mathbf{P}\left[\sup_{\tau+\delta\leq t\leq\tau+\epsilon} \left|\eta^n_{t,x} - \rho_{t,x}\right| = 0\right] = 1. \tag{4.7}$$

Before we continue with the proof, let us introduce some notation: For a non-empty subset $S \subset T$, let

$$\partial S := \{z \in T \setminus S : (\exists w \in S : z \text{ and } w \text{ are neighbours})\}$$

be the boundary of S in T. For $t \in [0, \tau + \epsilon]$ and $z \in T$, let $R_{t,z}$ denote the cluster of z in the configuration $(\rho_{t,w})_{w\in T}$ and let

$$\overline{R}_{t,z} := \begin{cases} R_{t,z} \cup \partial R_{t,z} & \text{if } R_{t,z} \neq \emptyset, \\ \{z\} & \text{if } R_{t,z} = \emptyset, \end{cases}$$

be its "closure". For $t \in [0, \tau + \epsilon]$, $z \in T$ and $n \in \mathbb{N}$ we similarly write $C^n_{t,z}$ for the cluster of z in the configuration $(\eta^n_{t,w})_{w \in B_n}$ and define its closure by

$$\overline{C}^n_{t,z} := \begin{cases} C^n_{t,z} \cup \partial C^n_{t,z} & \text{if } C^n_{t,z} \neq \emptyset, \\ \{z\} & \text{if } C^n_{t,z} = \emptyset. \end{cases}$$

Finally, we denote by C^{fin}_x the (countable) set of all finite connected subsets of T which contain the site x.

Since $R_{\tau+\epsilon,x}$ (and hence $\overline{R}_{\tau+\epsilon,x}$) is a.s. finite by Proposition 1, we have the equality

$$\mathbf{P}\left[\sup_{\tau+\delta \le t \le \tau+\epsilon} \left|\eta^n_{t,x} - \rho_{t,x}\right| = 0\right] = \sum_{A \in C^{\text{fin}}_x} \mathbf{P}\left[\sup_{\tau+\delta \le t \le \tau+\epsilon} \left|\eta^n_{t,x} - \rho_{t,x}\right| = 0, \overline{R}_{\tau+\epsilon,x} = A\right]$$

for all $n \in \mathbb{N}$ with $x \in B_n$. So pick $A \in C^{\text{fin}}_x$ such that $\mathbb{A} := \left\{\overline{R}_{\tau+\epsilon,x} = A\right\}$ satisfies $\mathbf{P}[\mathbb{A}] > 0$. By the dominated convergence theorem, (4.7) holds once we know

$$\lim_{n\to\infty} \mathbf{P}\left[\sup_{\tau+\delta \le t \le \tau+\epsilon} \left|\eta^n_{t,x} - \rho_{t,x}\right| = 0 \middle| \mathbb{A}\right] = 1. \tag{4.8}$$

It is thus enough to show (4.8).

Given the set A, by Theorem 4 we can choose a sequence $(\alpha(n))_{n\in\mathbb{N}}$ with $\alpha(n) > 0$ and $\lim_{n\to\infty} \alpha(n) = 0$ such that the event

$$\begin{aligned} \mathbb{C}_n := \Big\{ &\forall z \in O_\tau \cap A \, \exists t \in (\tau - \alpha(n), \tau + \alpha(n)) : \eta^n_{t^-,z} > \eta^n_{t,z}, \\ &\forall z \in A : G_{\tau-\alpha(n),z} = G_{\tau+\alpha(n),z}, I^n_{\tau+\alpha(n),z} = 0 \Big\} \end{aligned}$$

(where O_τ is defined as in Definition 14 and $n \in \mathbb{N}$ is assumed to be large enough to ensure $A \subset B_n$) satisfies $\lim_{n\to\infty} \mathbf{P}[\mathbb{C}_n] = 1$. As an auxiliary step towards (4.8), we prove that for all $n \in \mathbb{N}$ with $A \subset B_n$ the inclusion

$$\mathbb{A} \cap \mathbb{C}_n \subset \left\{\forall z \in A : \eta^n_{\tau+\alpha(n),z} = \rho_{\tau+\alpha(n),z}\right\} \tag{4.9}$$

holds. So let $z \in A$, let $n \in \mathbb{N}$ be large enough to ensure $A \subset B_n$ and suppose that the event $\mathbb{A} \cap \mathbb{C}_n$ occurs. We distinguish two cases:

Case 1: $z \in O_\tau$. Then there exists $t \in (\tau - \alpha(n), \tau + \alpha(n))$ such that $\eta^n_{t,z} = 0$ holds. Since $G_{\tau-\alpha(n),z} = G_{\tau+\alpha(n),z}$, it follows that we also have $\eta^n_{\tau+\alpha(n),z} = 0$. On the other hand, the assumption $z \in O_\tau$ implies $\rho_{\tau,z} = 0$, and from $G_{\tau-\alpha(n),z} = G_{\tau+\alpha(n),z}$ we again deduce $\rho_{\tau+\alpha(n),z} = 0$. Hence we conclude $\eta^n_{\tau+\alpha(n),z} = 0 = \rho_{\tau+\alpha(n),z}$.

Case 2: $z \notin O_\tau$. By construction $z \notin O_\tau$ implies $\overline{R}_{t,z} \subset \overline{R}_{\tau+\epsilon,z}$ for all $t \in [0, \tau + \epsilon]$. Since we assume $\overline{R}_{\tau+\epsilon,x} = A$ and $z \in A$, we also have $\overline{R}_{\tau+\epsilon,z} \subset A$. In particular we see that $\overline{R}_{\tau-\alpha(n),z} \subset A$ holds. Together with the fact that $I^n_{\tau-\alpha(n),w} = 0$ for all $w \in A$ this yields $\overline{C}^n_{\tau-\alpha(n),z} = \overline{R}_{\tau-\alpha(n),z} \subset A$. If we now use that $G_{\tau-\alpha(n),w} = G_{\tau+\alpha(n),w}$ and

$I^n_{\tau-\alpha(n),w} = I^n_{\tau+\alpha(n),w}$ hold for all $w \in A$, it follows that $\overline{C}^n_{\tau+\alpha(n),z} = \overline{R}_{\tau+\alpha(n),z}$, which shows $\eta^n_{\tau+\alpha(n),z} = \rho_{\tau+\alpha(n),z}$.

Having proved (4.9), we now observe that the event

$$\mathbb{D}_n := \left\{\forall z \in A : I^n_{\tau+\alpha(n),z} = I^n_{\tau+\epsilon,z}\right\}$$

also satisfies $\lim_{n\to\infty} \mathbf{P}[\mathbb{D}_n] = 1$ and that

$$\mathbb{A} \cap \mathbb{D}_n \cap \left\{\forall z \in A : \eta^n_{\tau+\alpha(n),z} = \rho_{\tau+\alpha(n),z}\right\} \subset \left\{\sup_{z\in A, \tau+\alpha(n)\le t\le \tau+\epsilon} \left|\eta^n_{t,z} - \rho_{t,z}\right| = 0\right\} \tag{4.10}$$

holds for all $n \in \mathbb{N}$ with $A \subset B_n$. Since we have $\lim_{n\to\infty} \mathbf{P}\left[\mathbb{C}_n \cap \mathbb{D}_n \middle| \mathbb{A}\right] = 1$ and $\alpha(n) < \delta$ for n large enough, equation (4.8) follows from (4.9) and (4.10). □

4.2 Proof of Theorem 4

We first prove some general properties of the pure growth process in Section 4.2.1 before we come to the core of the proof of Theorem 4 in Section 4.2.2.

4.2.1 Properties of the pure growth process

Let $(\sigma_{t,z}, G_{t,z})_{t\ge 0, z\in T}$ be a pure growth process on T under some probability measure $\mathbf{P}$. For $x \in T$, $t \ge 0$ and $n \in \mathbb{N}_0$, let $S_{t,x}$ denote the cluster of x in the configuration $(\sigma_{t,z})_{z\in T}$ and let

$$S^n_{t,x} := S_{t,x} \cap B_n$$

be the set of vertices in $S_{t,x}$ whose graph distance from the root $\emptyset$ is at most n. Recall the definition of $m(t)$ and $\sigma^2(t)$ in equations (4.1) and (4.2). We start with some estimates for the first and second moment of $|S^n_{t,\emptyset}|$ in the supercritical case $t > t_c$:

Lemma 23. *Let $t > t_c$ and $n \in \mathbb{N}_0$. Then we have*

$$1 - e^{-t} \le \mathbf{E_P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n}\right] \le \frac{m(t)}{m(t)-1}, \tag{4.11}$$

$$\mathbf{E_P}\left[\frac{|S^n_{t,\emptyset}|^2}{m(t)^{2n}}\right] \le \left(\frac{\sigma^2(t)}{m(t)(m(t)-1)} + 1\right)\left(\frac{m(t)}{m(t)-1}\right)^2. \tag{4.12}$$

Proof. Let $t > t_c$ and abbreviate $m := m(t)$, $\sigma^2 := \sigma^2(t)$. We will prove (4.11) and (4.12) by means of Galton-Watson theory. So let $X_{n,i}$, $n, i \in \mathbb{N}$, be i.i.d. $\{0, 1, \ldots, r\}$-valued random variables under some probability measure $\tilde{\mathbf{P}}$ such that $X_{n,i}$ is binomially distributed with parameters r and $1 - e^{-t}$. (In particular, $X_{n,i}$ has mean m and variance σ^2.) Define Z_n,

$n \in \mathbb{N}_0$, recursively by $Z_0 := 1$ and $Z_n := \sum_{i=1}^{Z_{n-1}} X_{n,i}$, $n \in \mathbb{N}$, and set $S_n := \sum_{i=0}^{n} Z_i$, $n \in \mathbb{N}_0$. Then Z_n, $n \in \mathbb{N}_0$, is a supercritical Galton-Watson process, and Z_n has mean

$$\mathbf{E}_{\tilde{\mathbf{P}}}[Z_n] = m^n \tag{4.13}$$

and variance

$$\mathbf{Var}_{\tilde{\mathbf{P}}}[Z_n] = \sigma^2 m^{n-1}\frac{m^n - 1}{m-1} \tag{4.14}$$

(see e.g. [Har63], Section I.5). Moreover, let U be a $\{0,1\}$-valued random variable on the same probability space which is independent of $X_{n,i}$, $n, i \in \mathbb{N}$, and Bernoulli distributed with parameter $1 - e^{-t}$. Then the distribution of $|S^n_{t,\emptyset}|$ under $\mathbf{P}$ and the distribution of US_n under $\tilde{\mathbf{P}}$ coincide, and $\mathbf{E}_{\tilde{\mathbf{P}}}[U] = 1 - e^{-t} \leq 1$. For the proof of (4.11) and (4.12), it therefore suffices to show the following inequalities for $n \in \mathbb{N}_0$:

$$m^n \leq \mathbf{E}_{\tilde{\mathbf{P}}}[S_n] \leq \frac{m}{m-1} m^n, \tag{4.15}$$

$$\mathbf{E}_{\tilde{\mathbf{P}}}\left[S_n^2\right] \leq \left(\frac{\sigma^2}{m(m-1)} + 1\right)\left(\frac{m}{m-1}\right)^2 m^{2n}. \tag{4.16}$$

Proof of (4.15): Using equation (4.13), we obtain

$$\mathbf{E}_{\tilde{\mathbf{P}}}[S_n] = \sum_{i=0}^{n} m^i \leq \frac{m}{m-1} m^n$$

for all $n \in \mathbb{N}_0$, which proves both sides of (4.15).

Proof of (4.16): For $i \in \mathbb{N}_0$, we easily deduce from equations (4.13) and (4.14)

$$\mathbf{E}_{\tilde{\mathbf{P}}}\left[Z_i^2\right] = \sigma^2 m^{i-1}\frac{m^i - 1}{m-1} + m^{2i} \leq \left(\frac{\sigma^2}{m(m-1)} + 1\right) m^{2i}.$$

Furthermore, for $i, j \in \mathbb{N}_0$ with $i < j$, we have

$$\mathbf{E}_{\tilde{\mathbf{P}}}[Z_i Z_j] = \mathbf{E}_{\tilde{\mathbf{P}}}\left[Z_i^2\right] m^{j-i} \leq \left(\frac{\sigma^2}{m(m-1)} + 1\right) m^{i+j}.$$

We thus obtain

$$\mathbf{E}_{\tilde{\mathbf{P}}}\left[S_n^2\right] = \sum_{i,j=0}^{n} \mathbf{E}_{\tilde{\mathbf{P}}}[Z_i Z_j] \leq \left(\frac{\sigma^2}{m(m-1)} + 1\right) \sum_{i,j=0}^{n} m^{i+j}$$

for all $n \in \mathbb{N}_0$. The last sum can be bounded from above by

$$\sum_{i,j=0}^{n} m^{i+j} = \left(\sum_{i=0}^{n} m^i\right)^2 \leq \left(\frac{m}{m-1}\right)^2 m^{2n},$$

which completes the proof of (4.16). $\square$

Recall equations (4.3) and (4.4). We now want to prove similar statements which are uniform in t. The price we pay for this kind of uniformity is that in contrast to (4.3) and (4.4), our statements are in probability rather than almost surely. The precise formulation is as follows:

Proposition 2. *Let $x \in T$ and $a > t_c$. Then we have*

$$\lim_{C\to\infty} \sup_{n\in\mathbb{N}:\, n>|x|} \sup_{t\in[a,\infty)} \mathbf{P}\left[|S^n_{t,x}| > Cm(t)^n\right] = 0 \tag{4.17}$$

and

$$\lim_{c\downarrow 0} \sup_{n\in\mathbb{N}:\, n>|x|} \sup_{t\in[a,\infty)} \mathbf{P}\left[|S^n_{t,x}| < cm(t)^n \,||S_{t,x}| = \infty\right] = 0. \tag{4.18}$$

Proof. Let $a > t_c$.

Step 1: We first prove (4.17) and (4.18) for $x = \emptyset$.

For $C > 0$ and $n \in \mathbb{N}$, $t \in [a, \infty)$, the Markov inequality and equation (4.11) yield

$$\mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq C\right] \leq \frac{1}{C}\,\mathbf{E_P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n}\right] \leq \frac{1}{C}\frac{m(t)}{m(t)-1}. \tag{4.19}$$

Since $m(t)$ is bounded away from 1 for $t \in [a, \infty)$, this implies (4.17) for $x = \emptyset$.

As preparatory work for the proof of (4.18) we next show that there exist $c, \delta > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{t\in[a,\infty)} \mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq c\right] \geq \delta \tag{4.20}$$

holds. For arbitrary $0 < c < C$ and $n \in \mathbb{N}$, $t \in [a, \infty)$, we have

$$\begin{aligned} 1 - e^{-t} &\leq \mathbf{E_P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n}\right] \\ &\leq c + C\,\mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq c\right] + \mathbf{E_P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n}\,1_{\{\frac{|S^n_{t,\emptyset}|}{m(t)^n}\geq C\}}\right], \end{aligned}$$

where the first inequality is due to (4.11) and the second inequality is obtained by distinguishing in which of the intervals $[0, c)$, $[c, C)$, $[C, \infty)$ the rescaled cluster size $|S^n_{t,\emptyset}|/m(t)^n$ lies. The last summand can be bounded from above by

$$\begin{aligned} \mathbf{E_P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n}\,1_{\{\frac{|S^n_{t,\emptyset}|}{m(t)^n}\geq C\}}\right] &\leq \left(\mathbf{E_P}\left[\frac{|S^n_{t,\emptyset}|^2}{m(t)^{2n}}\right]\right)^{1/2}\left(\mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq C\right]\right)^{1/2} \\ &\leq \frac{1}{C^{1/2}}\left(\frac{\sigma^2(t)}{m(t)(m(t)-1)} + 1\right)^{1/2}\left(\frac{m(t)}{m(t)-1}\right)^{3/2}, \end{aligned}$$

where we first use the Cauchy-Schwarz inequality and then apply equations (4.12) and (4.19). We thus obtain

$$\mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq c\right] \geq \frac{1}{C}\left(1 - e^{-t} - c - \frac{1}{C^{1/2}}\left(\frac{\sigma^2(t)}{m(t)(m(t)-1)} + 1\right)^{1/2}\left(\frac{m(t)}{m(t)-1}\right)^{3/2}\right).$$

Since $\sigma^2(t)/m(t) = e^{-t}$ and $m(t)$ is bounded away from 1 for $t \in [a, \infty)$, this proves the existence of $c, \delta > 0$ satisfying (4.20).

We now prove (4.18). Intuitively, (4.18) follows from (4.20) because conditionally on $\{|S_{t,\emptyset}| = \infty\}$, the cluster $S_{t,\emptyset}$ contains arbitrarily many independent subtrees in which an asymptotic growth of the form (4.20) can occur. The formal proof goes as follows:

Let $\epsilon > 0$. We first construct a finite set $U \subset [a, \infty)$ such that

$$\forall t \in [a, \infty)\, \exists u \in U : u \leq t, \mathbf{P}\left[|S_{u,\emptyset}| = \infty\right] \geq \mathbf{P}\left[|S_{t,\emptyset}| = \infty\right] - \frac{\epsilon}{4} \tag{4.21}$$

holds: Define $f : [a, \infty) \to [0, 1], f(t) := \mathbf{P}\left[|S_{t,\emptyset}| = \infty\right]$, and

$$R := \left\{f(a) + i\frac{\epsilon}{4} : i \in \mathbb{N}_0\right\} \cap [0, 1).$$

Then R is clearly finite. Since f is continuous, strictly monotone increasing and maps $[a, \infty)$ onto $[f(a), 1)$, it follows that $U := f^{-1}(R)$ is finite and satisfies (4.21).

Let $\delta, c > 0$ be as in equation (4.20). Given ϵ, δ, c, we choose constants $k, l \in \mathbb{N}$ in the following way: First, we take $k \in \mathbb{N}$ such that $(1 - \delta)^k \leq \epsilon/4$ holds. Then we choose $l \in \mathbb{N}$ such that

$$\forall u \in U : \mathbf{P}\left[\{|S_{u,\emptyset} \cap T_l| \geq k\} \triangle \{|S_{u,\emptyset}| = \infty\}\right] \leq \frac{\epsilon}{4} \tag{4.22}$$

holds, where $\triangle$ denotes the symmetric difference: For each individual $u \in U$ such an l exists because of the Kesten-Stigum theorem (whose full statement is of course much stronger), and since the set U is finite, we can choose l uniformly for all $u \in U$. Finally, we set $\tilde{c} := c/r^l$. Now let $n \in \{l+1, l+2, \ldots\}$ and $t \in [a, \infty)$ be arbitrary. Given t, choose $u \in U$ as in (4.21). Then we can make the estimates

$$\begin{aligned}
\mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq \tilde{c}, |S_{t,\emptyset}| = \infty\right] &\geq \mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq \tilde{c}, |S_{u,\emptyset}| = \infty\right] \\
&\geq \mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq \tilde{c}, |S_{u,\emptyset} \cap T_l| \geq k\right] - \frac{\epsilon}{4},
\end{aligned} \tag{4.23}$$

where the first inequality holds because of $u \leq t$ and the second inequality follows from (4.22). On the event $\{|S_{u,\emptyset} \cap T_l| \geq k\}$, let $Z_{u,1}, \ldots, Z_{u,k}$ be an enumeration of the "first" k vertices in $S_{u,\emptyset} \cap T_l$. For $z \in T$ let

$$\hat{S}_{t,z} := \{v \in S_{t,z} : z \preceq v\} \tag{4.24}$$

be the part of the cluster of $S_{t,z}$ which lies in the r-regular rooted subtree of T originating from z and let

$$\hat{S}_{t,z}^n := \hat{S}_{t,z} \cap B_n. \tag{4.25}$$

Since $u \le t$, on the event $\{|S_{u,\emptyset} \cap T_l| \ge k\}$, we have $Z_{u,i} \in S_{t,\emptyset}$ and hence $\hat{S}_{t,Z_{u,i}} \subset S_{t,\emptyset}$ for all $i \in \{1, \dots, k\}$. This gives

$$\begin{aligned}
&\mathbf{P}\left[\frac{|S_{t,\emptyset}^n|}{m(t)^n} \ge \tilde{c}, |S_{u,\emptyset} \cap T_l| \ge k\right] \\
&\qquad \ge \mathbf{P}\left[\exists i \in \{1, \dots k\} : \frac{|\hat{S}_{t,Z_{u,i}}^n|}{m(t)^n} \ge \tilde{c}, |S_{u,\emptyset} \cap T_l| \ge k\right] \\
&\qquad = \left(1 - \left(1 - \mathbf{P}\left[\frac{|S_{t,\emptyset}^{n-l}|}{m(t)^n} \ge \tilde{c} \,\middle|\, \sigma_{t,\emptyset} = 1\right]\right)^k\right) \mathbf{P}\left[|S_{u,\emptyset} \cap T_l| \ge k\right],
\end{aligned} \tag{4.26}$$

where the last equality follows from the following observations about the pure growth process:

- The configuration on B_l at time u and the configuration on $T \setminus B_l$ at time t are independent.
- The configurations at time t on the r-regular rooted subtrees originating from the vertices in T_l are independent and identically distributed as the configuration at time t on the entire tree T.

Using the inequality $\tilde{c}m(t)^l \le c$ and the defining equations for c, δ and k, we can estimate the first factor in (4.26) by

$$\begin{aligned}
1 - \left(1 - \mathbf{P}\left[\frac{|S_{t,\emptyset}^{n-l}|}{m(t)^n} \ge \tilde{c} \,\middle|\, \sigma_{t,\emptyset} = 1\right]\right)^k &\ge 1 - \left(1 - \mathbf{P}\left[\frac{|S_{t,\emptyset}^{n-l}|}{m(t)^n} \ge \tilde{c}\right]\right)^k \\
&\ge 1 - \left(1 - \mathbf{P}\left[\frac{|S_{t,\emptyset}^{n-l}|}{m(t)^{n-l}} \ge c\right]\right)^k \\
&\ge 1 - (1-\delta)^k \ge 1 - \frac{\epsilon}{4}.
\end{aligned} \tag{4.27}$$

The second factor in (4.26) is bounded from below by

$$\begin{aligned}
\mathbf{P}\left[|S_{u,\emptyset} \cap T_l| \ge k\right] &\ge \mathbf{P}\left[|S_{u,\emptyset}| = \infty\right] - \frac{\epsilon}{4} \\
&\ge \mathbf{P}\left[|S_{t,\emptyset}| = \infty\right] - \frac{\epsilon}{2}
\end{aligned} \tag{4.28}$$

because of (4.22) and (4.21). Putting equations (4.23), (4.26), (4.27) and (4.28) together, we obtain

$$\mathbf{P}\left[\frac{|S_{t,\emptyset}^n|}{m(t)^n} \ge \tilde{c}, |S_{t,\emptyset}| = \infty\right] \ge \mathbf{P}\left[|S_{t,\emptyset}| = \infty\right] - \epsilon.$$

Since this holds uniformly for $n \in \{l+1, l+2, \ldots\}$ and $t \in [a, \infty)$ and since $\mathbf{P}\left[|S_{t,\emptyset}| = \infty\right] \geq \mathbf{P}\left[|S_{a,\emptyset}| = \infty\right] > 0$ for all $t \in [a, \infty)$, we conclude

$$\sup_{n\in\{l+1,l+2,\ldots\}} \sup_{t\in[a,\infty)} \mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq \tilde{c} \,\middle|\, |S_{t,\emptyset}| = \infty \right] \geq 1 - \frac{\epsilon}{\mathbf{P}\left[|S_{a,\emptyset}| = \infty\right]}. \tag{4.29}$$

Additionally, we also have the trivial estimate

$$\sup_{n\in\{1,\ldots,l\}} \sup_{t\in[a,\infty)} \mathbf{P}\left[\frac{|S^n_{t,\emptyset}|}{m(t)^n} \geq \frac{1}{r^l} \,\middle|\, |S_{t,\emptyset}| = \infty \right] = 1$$

for $n \in \{1, \ldots, l\}$. Together with (4.29) this proves (4.18) for $x = \emptyset$.

Step 2: We now prove equations (4.17) and (4.18) for general $x \in T$. So let $x \in T$ and let $n \in \{|x|+1, |x|+2, \ldots\}$, $t \in [a, \infty)$. For both equations we distinguish which vertex z of the finitely many ancestors of x is the root of the cluster $S_{t,x}$ (the case $S_{t,x} = \emptyset$ being irrelevant) and then use the fact that the r-regular rooted subtree originating from z is isomorphic to T. Let $\hat{S}_{t,z}$ and $\hat{S}^n_{t,z}$ be defined as in (4.24) and (4.25) respectively. Regarding (4.17) we then obtain for all $C > 0$

$$\begin{aligned}
\mathbf{P}\left[|S^n_{t,x}| > Cm(t)^n\right] &= \sum_{z\in T:\, z\preceq x} \mathbf{P}\left[|S^n_{t,x}| > Cm(t)^n, z \text{ is the root of } S_{t,x}\right] \\
&\leq \sum_{z\in T:\, z\preceq x} \mathbf{P}\left[|\hat{S}^n_{t,z}| > Cm(t)^n\right] \\
&= \sum_{z\in T:\, z\preceq x} \mathbf{P}\left[|S^{n-|z|}_{t,\emptyset}| > Cm(t)^n\right] \\
&\leq \sum_{z\in T:\, z\preceq x} \mathbf{P}\left[|S^{n-|z|}_{t,\emptyset}| > Cm(a)^{|z|} \cdot m(t)^{n-|z|}\right],
\end{aligned}$$

and regarding (4.18) we similarly obtain for all $c > 0$

$$\begin{aligned}
&\mathbf{P}\left[|S^n_{t,x}| < cm(t)^n \,\middle|\, |S_{t,x}| = \infty\right] \\
&\quad = \sum_{z\in T:\, z\preceq x} \frac{\mathbf{P}\left[|S^n_{t,x}| < cm(t)^n, z \text{ is the root of } S_{t,x}, |S_{t,x}| = \infty\right]}{\mathbf{P}\left[|S_{t,x}| = \infty\right]} \\
&\quad \leq \sum_{z\in T:\, z\preceq x} \frac{\mathbf{P}\left[|\hat{S}^n_{t,z}| < cm(t)^n, |\hat{S}_{t,z}| = \infty\right]}{\mathbf{P}\left[|S_{t,\emptyset}| = \infty\right]} \\
&\quad = \sum_{z\in T:\, z\preceq x} \frac{\mathbf{P}\left[|S^{n-|z|}_{t,\emptyset}| < cm(t)^n, |S_{t,\emptyset}| = \infty\right]}{\mathbf{P}\left[|S_{t,\emptyset}| = \infty\right]} \\
&\quad \leq \sum_{z\in T:\, z\preceq x} \mathbf{P}\left[|S^{n-|z|}_{t,\emptyset}| < cr^{|z|} \cdot m(t)^{n-|z|} \,\middle|\, |S_{t,\emptyset}| = \infty\right].
\end{aligned}$$

Together with Step 1 this completes the proof of (4.17) and (4.18). □

4.2.2 The core of the proof of Theorem 4

Throughout this section, consider the setup of Theorem 4: Let $\tau \in (t_c, \infty)$ and suppose that $\lambda : \mathbb{N} \to (0, \infty)$ satisfies $\lambda(n) \approx 1/m(\tau)^n$ for $n \to \infty$. For $n \in \mathbb{N}$, let $(\eta^n_{t,z}, G_{t,z}, I^n_{t,z})_{t\geq 0, z\in B_n}$ be a forest-fire process on B_n with parameter $\lambda(n)$ and let $(\sigma_{t,z}, G_{t,z})_{t\geq 0, z\in T}$ be a pure growth process on T, coupled in the canonical way under some probability measure $\mathbf{P}$. As before, we use the following notation: For $x \in T$, $t \geq 0$ and $n \in \mathbb{N}$, let $S_{t,x}$ denote the cluster of x in the configuration $(\sigma_{t,z})_{z\in T}$ of the pure growth process at time t, let $S^n_{t,x} := S_{t,x} \cap B_n$ and let $O_t := \{z \in T : |S_{t,z}| = \infty\}$. Similarly, for $x \in B_n$, let $C^n_{t,x}$ denote the cluster of x in the configuration $(\eta^n_{t,z})_{z\in B_n}$ of the forest-fire process at time t.

Choose an arbitrary function $f : \mathbb{N} \to (0, \infty)$ which satisfies

$$1 \ll f(n) \ll \frac{n}{\log n} \qquad \text{for } n \to \infty \tag{4.30}$$

and $\lambda(n) < f(n) < r^n \lambda(n)$ for all $n \in \mathbb{N}$. Define a corresponding sequence $(\tau_n)_{n\in\mathbb{N}}$ of time points in such a way that $\lambda(n) = f(n)/m(\tau_n)^n$ holds, i.e.

$$\tau_n := m^{-1}\left(\sqrt[n]{\frac{f(n)}{\lambda(n)}}\right),$$

where

$$m^{-1}(y) = \log \frac{r}{r-y}, \qquad y \in [0, r),$$

denotes the inverse function of $m(t)$, $t \geq 0$. Since $\sqrt[n]{\lambda(n)} \to 1/m(\tau)$ and $\sqrt[n]{f(n)} \to 1$ for $n \to \infty$ (the first limit follows from $\lambda(n) \approx 1/m(\tau)^n$ for $n \to \infty$, the second limit is a consequence of (4.30)), we then have

$$\lim_{n\to\infty} \tau_n = \tau. \tag{4.31}$$

In particular, for all $x \in T$ it is true that

$$\lim_{n\to\infty} \mathbf{P}\left[|S_{\tau_n,x}| = |S_{\tau,x}|\right] = 1. \tag{4.32}$$

Equations (4.31) and (4.32) imply that for the proof of Theorem 4, it is enough to verify the following statement: For all finite subsets $E \subset T$ and for all $\delta > 0$

$$\lim_{n\to\infty} \mathbf{P}\left[\sup_{z\in E, 0\leq t\leq \tau_n} \left|\eta^n_{t,z} - \sigma_{t,z}\right| = 0, \forall z \in O_{\tau_n} \cap E\, \exists t \in (\tau_n, \tau_n + \delta) : \eta^n_{t-,z} > \eta^n_{t,z}\right] = 1 \tag{4.33}$$

holds.

Since E is finite, we may assume without loss of generality that E is a singleton, i.e. $E = \{x\}$ for some $x \in T$. So let $x \in T$ and $\delta > 0$ be fixed and define

$$\mathbf{Q}_n\left[\,\cdot\,\right] := \mathbf{P}\left[\,\cdot\; \middle|\, |S_{\tau_n,x}| = \infty\right]$$

for all $n \in \mathbb{N}$. (Due to $f(n) > \lambda(n)$ we have $\tau_n > t_c$ for all $n \in \mathbb{N}$.) It then suffices to prove

$$\lim_{n\to\infty} \mathbf{P}\left[\sup_{0\le t\le\tau_n} \left|\eta^n_{t,x} - \sigma_{t,x}\right| = 0\right] = 1 \tag{4.34}$$

and

$$\lim_{n\to\infty} \mathbf{Q}_n\left[\exists t \in (\tau_n, \tau_n + \delta) : \eta^n_{t^-,x} > \eta^n_{t,x}\right] = 1. \tag{4.35}$$

Before we go into the details, let us briefly outline the strategy for the proof of (4.34) and (4.35): We investigate how the vertices in the cluster $S^n_{\tau_n,x}$ of the pure growth process on B_n at time τ_n behave in the forest-fire process on B_n between time 0 and time τ_n. We will see that typically destruction only occurs in high generations of $S^n_{\tau_n,x}$ and only few vertices in $S^n_{\tau_n,x}$ are affected by destruction. This has two consequences: Firstly, it shows that (4.34) holds indeed. Secondly, it implies that if $S_{\tau_n,x}$ is infinite, then the cluster $C^n_{\tau_n,x}$ has the same order of magnitude as $S^n_{\tau_n,x}$, namely $m(\tau_n)^n$. But since $\lambda(n)m(\tau_n)^n = f(n)$ and $f(n) \to \infty$ for $n \to \infty$, it follows that $C^n_{\tau_n,x}$ is typically hit by ignition soon after time τ_n, which proves (4.35).

We now make these arguments rigorous. In doing so, we will use the following Landau-type notation: If X^n, $n \in \mathbb{N}$, is a sequence of real-valued random variables under the probability measure $\mathbf{P}$ and $h : \mathbb{N} \to [0,\infty)$ is a non-negative function, we write

$$X^n \overset{\mathbf{P}}{=} O(h(n)) \text{ for } n\to\infty :\Leftrightarrow \lim_{c\to\infty} \liminf_{n\to\infty} \mathbf{P}\left[|X^n| \le ch(n)\right] = 1;$$
$$X^n \overset{\mathbf{P}}{=} \Omega(h(n)) \text{ for } n\to\infty :\Leftrightarrow \lim_{c\to\infty} \liminf_{n\to\infty} \mathbf{P}\left[|X^n| \ge \frac{1}{c}h(n)\right] = 1.$$

Lemma 24. *Let*

$$\iota^n := \inf\left\{t \in [0,\tau_n) : \left(\exists z \in S^n_{\tau_n,x} : I^n_{(\tau_n - t)^-,z} < I^n_{\tau_n - t,z}\right)\right\} \wedge \tau_n$$

be the amount of time between τ_n and the last time of lightning in $S^n_{\tau_n,x}$ before τ_n. (On the event $\{\forall z \in S^n_{\tau_n,x} : I^n_{\tau_n,z} = 0\}$ we have $\iota^n = \tau_n$ by definition.) Then we have

$$\iota^n \overset{\mathbf{P}}{=} \Omega\left(\frac{1}{f(n)}\right) \qquad \textit{for } n \to \infty.$$

Proof. Let $c, \tilde{c} > 0$, $n \in \mathbb{N}$ with $\tau_n \ge 1/(cf(n))$ and let

$$\mathbb{E}_{n,\tilde{c}} := \left\{|S^n_{\tau_n,x}| \le \tilde{c}m(\tau_n)^n\right\}. \tag{4.36}$$

By Proposition 2, equation (4.17), it suffices to show

$$\forall \tilde{c} > 0 : \lim_{c\to\infty} \limsup_{n\to\infty} \mathbf{P}\left[\iota^n < \frac{1}{cf(n)}, \mathbb{E}_{n,\tilde{c}}\right] = 0. \tag{4.37}$$

Indeed, we have

$$\begin{aligned}\mathbf{P}\left[\iota^n < \frac{1}{cf(n)}, \mathbb{E}_{n,\tilde{c}}\right] &= \mathbf{E_P}\left[\mathbf{P}\left[\iota^n < \frac{1}{cf(n)} \middle| S^n_{\tau_n,x}\right] 1_{\mathbb{E}_{n,\tilde{c}}}\right] \\ &= \mathbf{E_P}\left[\left(1 - \exp\left(-\frac{1}{cf(n)}\lambda(n)|S^n_{\tau_n,x}|\right)\right) 1_{\mathbb{E}_{n,\tilde{c}}}\right] \\ &\le 1 - \exp\left(-\frac{\tilde{c}}{c}\right) \xrightarrow[c\to\infty]{} 0,\end{aligned}$$

which proves (4.37). □

Lemma 25. *Let*

$$N^n := \sum_{z \in S^n_{\tau_n,x}} I^n_{\tau_n,z}$$

be the number of lightnings in $S^n_{\tau_n,x}$ up to time τ_n. Then we have

$$N^n \overset{\mathbf{P}}{=} \mathcal{O}\left(f(n)\right) \qquad \text{for } n \to \infty.$$

Proof. Let $c, \tilde{c} > 0$, $n \in \mathbb{N}$ and let $\mathbb{E}_{n,\tilde{c}}$ be defined as in (4.36). By Proposition 2, equation (4.17), it suffices to show

$$\forall \tilde{c} > 0 : \lim_{c\to\infty} \limsup_{n\to\infty} \mathbf{P}\left[N^n > cf(n), \mathbb{E}_{n,\tilde{c}}\right] = 0. \tag{4.38}$$

Indeed, we have

$$\begin{aligned}\mathbf{P}\left[N^n > cf(n), \mathbb{E}_{n,\tilde{c}}\right] &\le \frac{1}{cf(n)} \mathbf{E_P}\left[N^n 1_{\mathbb{E}_{n,\tilde{c}}}\right] \\ &= \frac{1}{cf(n)} \mathbf{E_P}\left[\mathbf{E_P}\left[N^n \middle| S^n_{\tau_n,x}\right] 1_{\mathbb{E}_{n,\tilde{c}}}\right] \\ &= \frac{1}{cf(n)} \mathbf{E_P}\left[\tau_n \lambda(n) |S^n_{\tau_n,x}| 1_{\mathbb{E}_{n,\tilde{c}}}\right] \\ &\le \frac{\tau_n \tilde{c}}{c}\end{aligned}$$

and (by (4.31))

$$\lim_{c\to\infty} \lim_{n\to\infty} \frac{\tau_n \tilde{c}}{c} = \lim_{c\to\infty} \frac{\tau \tilde{c}}{c} = 0,$$

which proves (4.38). □

Lemma 26. *Let*

$$K^n := \max\left\{k \in \{0, \ldots, n\} : \left(\exists z \in S^n_{\tau_n,x} \cap T_{n-k} : I^n_{\tau_n,z} > 0\right)\right\} \vee (-1)$$

be the "depth" of lightning in $S^n_{\tau_n,x}$ up to time τ_n. (On the event $\{\forall z \in S^n_{\tau_n,x} : I^n_{\tau_n,z} = 0\}$ we have $K^n = -1$ by definition.) Then we have

$$K^n \overset{\mathbf{P}}{=} \mathcal{O}\left(\log n\right) \qquad \text{for } n \to \infty.$$

Proof. Let $c, \tilde{c} > 0$, $n \in \mathbb{N}$ with $n \geq \lfloor c \log n \rfloor + 2$ and let

$$\mathbb{E}_{n,c,\tilde{c}} := \left\{ |S_{\tau_n,x}^{n-\lfloor c\log n\rfloor -1}| \leq \tilde{c} m(\tau_n)^{n-\lfloor c\log n\rfloor -1} \right\}.$$

By Proposition 2, equation (4.17), it suffices to show

$$\forall \tilde{c} > 0 : \lim_{c\to\infty} \limsup_{n\to\infty} \mathbf{P}\left[K^n > c\log n, \mathbb{E}_{n,c,\tilde{c}}\right] = 0. \tag{4.39}$$

Indeed, we have

$$\begin{aligned}
\mathbf{P}\left[K^n > c\log n, \mathbb{E}_{n,c,\tilde{c}}\right] &= \mathbf{E}_{\mathbf{P}}\left[\mathbf{P}\left[\exists z \in S_{\tau_n,x}^{n-\lfloor c\log n\rfloor -1} : I_{\tau_n,z}^n > 0 \middle| S_{\tau_n,x}^{n-\lfloor c\log n\rfloor -1}\right] 1_{\mathbb{E}_{n,c,\tilde{c}}}\right] \\
&= \mathbf{E}_{\mathbf{P}}\left[\left(1 - \exp\left(-\tau_n \lambda(n) |S_{\tau_n,x}^{n-\lfloor c\log n\rfloor -1}|\right)\right) 1_{\mathbb{E}_{n,c,\tilde{c}}}\right] \\
&\leq 1 - \exp\left(-\tilde{c}\tau_n \frac{f(n)}{m(\tau_n)^{\lfloor c\log n\rfloor +1}}\right).
\end{aligned}$$

Equations (4.30) and (4.31) imply that for n large enough $f(n) \leq n$ and $m(\tau_n) > 1$ hold and hence

$$\tau_n \frac{f(n)}{m(\tau_n)^{\lfloor c\log n\rfloor +1}} \leq \tau_n \frac{n}{m(\tau_n)^{c\log n}} = \tau_n \exp\left((\log n)(1 - c\log m(\tau_n))\right).$$

By (4.31), for $c > 1/\log m(\tau)$ we thus obtain

$$\lim_{n\to\infty} \tau_n \frac{f(n)}{m(\tau_n)^{\lfloor c\log n\rfloor +1}} = 0,$$

which proves (4.39). □

Lemma 27. *Let*

$$J^n := \max\left\{ j \in \{0,\ldots,n\} : \left(\exists z \in S_{\tau_n,x}^n \cap T_{n-j}\, \exists t \in (0,\tau_n] : \eta_{t^-,z}^n > \eta_{t,z}^n\right)\right\} \vee (-1)$$

be the "depth" of destruction in $S_{\tau_n,x}^n$ up to time τ_n. (On the event $\{\forall z \in S_{\tau_n,x}^n\, \forall t \in (0,\tau_n] : \eta_{t^-,z}^n \leq \eta_{t,z}^n\}$ we have $J^n = -1$ by definition.) Then we have

$$J^n \stackrel{\mathbf{P}}{=} \mathcal{O}\left(f(n)\log n\right) \qquad \textit{for } n\to\infty.$$

Proof. Let $c, \tilde{c} > 0$, $n \in \mathbb{N}$ and let

$$\mathbb{F}_{n,\tilde{c}} := \left\{ \iota^n \geq \frac{1}{\tilde{c} f(n)}, N^n \leq \tilde{c} f(n), K^n \leq \tilde{c}\log n \right\}.$$

By Lemmas 24, 25 and 26, it suffices to show

$$\forall \tilde{c} > 0 : \lim_{c\to\infty} \limsup_{n\to\infty} \mathbf{P}\left[J^n > c f(n)\log n, \mathbb{F}_{n,\tilde{c}}\right] = 0. \tag{4.40}$$

Let $(Z_i^n)_{i=1,\dots,N^n}$ be an enumeration of the sites in $S^n_{\tau_n,x}$ which are hit by ignition up to time τ_n, where we count these sites with multiplicity, i.e. for each $z \in S^n_{\tau_n,x}$ the relation $|\{i \in \{1,\dots,N^n\} : Z_i^n = z\}| = I^n_{\tau_n,z}$ holds. (On the event $\{N^n = 0\}$ the sequence $(Z_i^n)_{i=1,\dots,N^n}$ is empty.) For $t \geq 0$ and $z \in T$ let

$$A_{t,z} := |z| - \min\{|w| : w \in S_{t,z}\}$$

be the difference between the generation of z and the lowest generation which is contained in the cluster of z in the pure growth process at time t. (On the event $\{S_{t,z} = \emptyset\}$ we have $A_{t,z} = -\infty$.) Now suppose that $0 \leq t_1 \leq t_2$, $z \in T$ and $k \in \mathbb{N}$ are given: Using the inclusion $\{A_{t_1,z} \geq k\} \subset \{A_{t_2,z} \geq k\}$ and the fact that the growth processes at different sites are independent, one can show that

$$\mathbf{P}\left[A_{t_1,z} \geq k \,|(\sigma_{t_2,w})_{w\in T}\right] = \left(\frac{1-e^{-t_1}}{1-e^{-t_2}}\right)^{k+1} 1_{\{A_{t_2,z}\geq k\}}.$$

Additionally, let $\mathcal{S}_{t_2}\mathcal{I}^n := \sigma\left((\sigma_{t_2,w})_{w\in T}, (I^n_{t,w})_{t\geq 0, w\in B_n}\right)$ denote the σ-field generated by the configuration of the pure growth process at time t_2 and all ignition processes. Since the growth processes and the ignition processes are independent and since $A_{t_1,z}$ only depends on the growth processes, it follows from the previous equation that

$$\mathbf{P}\left[A_{t_1,z} \geq k \,|\mathcal{S}_{t_2}\mathcal{I}^n\right] = \left(\frac{1-e^{-t_1}}{1-e^{-t_2}}\right)^{k+1} 1_{\{A_{t_2,z}\geq k\}} \leq \left(\frac{1-e^{-t_1}}{1-e^{-t_2}}\right)^{k+1}. \tag{4.41}$$

We now relate these preliminaries with the proof of (4.40): Assume that n is large enough so that $cf(n)\log n \geq \tilde{c}\log n$ and $\tau_n \geq 1/(\tilde{c}f(n))$ hold. Then

$$\begin{aligned}
&\{J^n > cf(n)\log n, \mathbb{F}_{n,\tilde{c}}\} \\
&\qquad \subset \left\{\exists i \in \{1,\dots,N^n\} : (n - |Z_i^n|) + A_{\tau_n - 1/(\tilde{c}f(n)), Z_i^n} > cf(n)\log n, \mathbb{F}_{n,\tilde{c}}\right\} \\
&\qquad \subset \left\{\exists i \in \{1,\dots,N^n\} : A_{\tau_n - 1/(\tilde{c}f(n)), Z_i^n} > cf(n)\log n - \tilde{c}\log n, \mathbb{F}_{n,\tilde{c}}\right\}
\end{aligned} \tag{4.42}$$

holds, where the first inclusion uses $\mathbb{F}_{n,\tilde{c}} \subset \{\iota^n \geq 1/(\tilde{c}f(n))\}$ and the second inclusion is due to the fact that $\mathbb{F}_{n,\tilde{c}} \subset \{\forall i \in \{1,\dots,N^n\} : (n - |Z_i^n|) \leq \tilde{c}\log n\}$. Furthermore, we deduce from (4.42) and (4.41) that

$$\begin{aligned}
&\mathbf{P}\left[J^n > cf(n)\log n, \mathbb{F}_{n,\tilde{c}}\right] \\
&\quad \leq \mathbf{E_P}\left[\sum_{i=1}^{N^n} 1_{\{A_{\tau_n-1/(\tilde{c}f(n)),Z_i^n} > cf(n)\log n - \tilde{c}\log n\}} 1_{\mathbb{F}_{n,\tilde{c}}}\right] \\
&\quad = \mathbf{E_P}\left[\sum_{i=1}^{N^n} \sum_{z\in S^n_{\tau_n,x}} \mathbf{P}\left[A_{\tau_n-1/(\tilde{c}f(n)),z} \geq \lfloor cf(n)\log n - \tilde{c}\log n\rfloor + 1 \,\middle|\, \mathcal{S}_{\tau_n}\mathcal{I}^n\right] 1_{\{Z_i^n = z\}} 1_{\mathbb{F}_{n,\tilde{c}}}\right] \\
&\quad \leq \tilde{c}f(n)\left(\frac{1-e^{-\tau_n+1/(\tilde{c}f(n))}}{1-e^{-\tau_n}}\right)^{\lfloor cf(n)\log n - \tilde{c}\log n\rfloor + 2}
\end{aligned}$$

holds. Now let n be large enough so that $f(n) \leq n$ holds (which is possible by (4.30)). Then

$$\tilde{c}f(n)\left(\frac{1-e^{-\tau_n+1/(\tilde{c}f(n))}}{1-e^{-\tau_n}}\right)^{\lfloor cf(n)\log n-\tilde{c}\log n\rfloor+2} \leq \tilde{c}n\left(\frac{1-e^{-\tau_n+1/(\tilde{c}f(n))}}{1-e^{-\tau_n}}\right)^{cf(n)\log n-\tilde{c}\log n}.$$

In order to determine the behaviour of the last term for $n \to \infty$, we rewrite it as

$$\begin{aligned} &n\left(\frac{1-e^{-\tau_n+1/(\tilde{c}f(n))}}{1-e^{-\tau_n}}\right)^{cf(n)\log n-\tilde{c}\log n} \\ &\quad= \exp\left(\log n + (cf(n)\log n - \tilde{c}\log n)\log\left(1-\frac{e^{1/(\tilde{c}f(n))}-1}{e^{\tau_n}-1}\right)\right) \\ &\quad= \exp\left((\log n)\left(1+\left(c-\frac{\tilde{c}}{f(n)}\right)f(n)\log\left(1-\frac{e^{1/(\tilde{c}f(n))}-1}{e^{\tau_n}-1}\right)\right)\right). \end{aligned} \tag{4.43}$$

Since $f(n) \to \infty$ and $\tau_n \to \tau$ for $n \to \infty$ (see (4.30) and (4.31)), we calculate

$$\begin{aligned} \lim_{n\to\infty}\frac{e^{\tau_n}-1}{e^{1/(\tilde{c}f(n))}-1}\log\left(1-\frac{e^{1/(\tilde{c}f(n))}-1}{e^{\tau_n}-1}\right) &= \lim_{y\downarrow 0}\frac{\log(1-y)}{y} = -1, \\ \lim_{n\to\infty}\tilde{c}f(n)\left(e^{1/(\tilde{c}f(n))}-1\right) &= \lim_{y\downarrow 0}\frac{e^y-1}{y} = 1, \\ \lim_{n\to\infty}\frac{1}{\tilde{c}\,(e^{\tau_n}-1)} &= \frac{1}{\tilde{c}\,(e^{\tau}-1)}. \end{aligned}$$

Multiplying these equations yields

$$\lim_{n\to\infty} f(n)\log\left(1-\frac{e^{1/(\tilde{c}f(n))}-1}{e^{\tau_n}-1}\right) = \frac{-1}{\tilde{c}\,(e^{\tau}-1)}. \tag{4.44}$$

From (4.43) and (4.44) we conclude that for $c > \tilde{c}\,(e^{\tau}-1)$ we have

$$\lim_{n\to\infty}\tilde{c}n\left(\frac{1-e^{-\tau_n+1/(\tilde{c}f(n))}}{1-e^{-\tau_n}}\right)^{cf(n)\log n-\tilde{c}\log n} = 0,$$

which proves (4.40). □

Proof of (4.34) and (4.35) (and hence of Theorem 4). Equation (4.34) is an immediate consequence of Lemma 27 and the fact that $f(n)\log n \ll n$ for $n \to \infty$ by (4.30).

Proof of (4.35): Let $c > 0$, $n \in \mathbb{N}$ with $x \in B_n$ and let

$$\mathbb{G}_{n,c} := \left\{\frac{|S^n_{\tau_n,x}|}{m(\tau_n)^n} \geq \frac{1}{c}, N^n \leq cf(n), J^n \leq cf(n)\log n\right\}.$$

By Proposition 2, equation (4.18), Lemma 25 and Lemma 27, it suffices to show

$$\forall c > 0: \lim_{n\to\infty}\mathbf{Q}_n\left[\exists t \in (\tau_n, \tau_n+\delta): \eta^n_{t^-,x} > \eta^n_{t,x}\,\middle|\,\mathbb{G}_{n,c}\right] = 1. \tag{4.45}$$

We first observe that

$$\left|S^n_{\tau_n,x} \setminus C^n_{\tau_n,x}\right| \leq N^n \frac{r^{J^n+1}-1}{r-1} \tag{4.46}$$

holds: In the case where $J^n = -1$ we have $C^n_{\tau_n,x} = S^n_{\tau_n,x}$ so that (4.46) holds indeed. In the case where $J^n \geq 0$ the cluster $C^n_{\tau_n,x}$ can only differ from $S^n_{\tau_n,x}$ in the maximal subtrees of $S^n_{\tau_n,x}$ whose roots are in T_{n-J^n}. Each of these maximal subtrees can have at most $\sum_{j=0}^{J^n} r^j = \frac{r^{J^n+1}-1}{r-1}$ vertices. Moreover, since these subtrees are disconnected, at most N^n of them can have been affected by destruction up to time τ_n. This proves (4.46) in the second case. On the event $\mathbb{G}_{n,c}$, we hence have

$$\begin{aligned}\left|S^n_{\tau_n,x} \setminus C^n_{\tau_n,x}\right| &\leq cf(n)\frac{r^{cf(n)\log n+1}-1}{r-1} \\ &\leq cf(n) r^{cf(n)\log n+1}\end{aligned}$$

and

$$\begin{aligned}\left|C^n_{\tau_n,x}\right| &= \left|S^n_{\tau_n,x}\right| - \left|S^n_{\tau_n,x} \setminus C^n_{\tau_n,x}\right| \\ &\geq \frac{1}{c} m(\tau_n)^n - cf(n) r^{cf(n)\log n+1}.\end{aligned} \tag{4.47}$$

For $t \geq 0$, let $\mathcal{F}^n_t := \sigma((G_{s,w})_{0\leq s\leq t, w\in T}, (I^n_{s,w})_{0\leq s\leq t, w\in B_n})$ denote the σ-field generated by the growth and ignition processes up to time t. We then deduce

$$\begin{aligned}&\mathbf{Q}_n\left[\exists t \in (\tau_n, \tau_n+\delta) : \eta^n_{t^-,x} > \eta^n_{t,x}, \mathbb{G}_{n,c}\right] \\ &\quad\geq \mathbf{Q}_n\left[\exists z \in C^n_{\tau_n,x} : I^n_{\tau_n+\delta,z} > I^n_{\tau_n,z}, \mathbb{G}_{n,c}\right] \\ &\quad= \mathbf{E}_{\mathbf{Q}_n}\left[\mathbf{Q}_n\left[\exists z \in C^n_{\tau_n,x} : I^n_{\tau_n+\delta,z} > I^n_{\tau_n,z} \middle| \mathcal{F}^n_{\tau_n}\right] 1_{\mathbb{G}_{n,c}}\right] \\ &\quad= \mathbf{E}_{\mathbf{Q}_n}\left[\left(1-\exp\left(-\delta\lambda(n)\left|C^n_{\tau_n,x}\right|\right)\right) 1_{\mathbb{G}_{n,c}}\right] \\ &\quad\geq \left(1-\exp\left(-\delta\lambda(n)\left(\frac{1}{c}m(\tau_n)^n - cf(n)r^{cf(n)\log n+1}\right)\right)\right)\mathbf{Q}_n\left[\mathbb{G}_{n,c}\right],\end{aligned} \tag{4.48}$$

where the last inequality follows from (4.47). In order to determine the behaviour of the exponential argument for $n \to \infty$, we consider the two summands separately: For the first summand we clearly have

$$\lim_{n\to\infty} \lambda(n) m(\tau_n)^n = \lim_{n\to\infty} f(n) = \infty \tag{4.49}$$

(see (4.30)). The second summand can be rewritten as

$$\begin{aligned}\lambda(n) f(n) r^{cf(n)\log n} &= f(n)^2 \frac{r^{cf(n)\log n}}{m(\tau_n)^n} \\ &= \exp\left(2\log f(n) + (c\log r) f(n)\log n - (\log m(\tau_n))n\right) \\ &= \exp\left(n\left(2\frac{\log f(n)}{n} + (c\log r)\frac{f(n)\log n}{n} - \log m(\tau_n)\right)\right).\end{aligned}$$

By (4.30), the function f satisfies $f(n) \ll n/\log n$ for $n \to \infty$, and this also implies $\log f(n) \ll n$ for $n \to \infty$. Using these asymptotics and recalling (4.31), we thus conclude

$$\lim_{n\to\infty} \left(2\frac{\log f(n)}{n} + (c\log r)\frac{f(n)\log n}{n} - \log m(\tau_n) \right) = -\log m(\tau)$$

and

$$\lim_{n\to\infty} \lambda(n) f(n) r^{cf(n)\log n} = 0. \tag{4.50}$$

Putting (4.48), (4.49) and (4.50) together yields the proof of (4.45). □

4.3 Proof of Theorem 5

Consider the same setup as in Section 4.2.2; additionally, let $g : \mathbb{N} \to (0, \infty)$ be defined as in (4.5). By assumption, g satisfies $\sqrt[n]{g(n)} \to 1$ for $n \to \infty$.

Part (i): Suppose that g also satisfies $g(n) \ll n/\log n$ for $n \to \infty$. Then the function f of Section 4.2.2 can be chosen in such a way that for n large enough $f(n) \geq g(n)$ holds. Since m^{-1} is monotone increasing, we conclude that for n large enough

$$\tau_n = m^{-1}\left(m(\tau) \sqrt[n]{\frac{f(n)}{g(n)}} \right) \geq \tau$$

holds. By (4.31) we also have $\tau_n \to \tau$ for $n \to \infty$. Theorem 5 (i) therefore follows from (4.33).

Part (ii): Suppose that there exists $\alpha \in (0,1)$ such that g satisfies $g(n) \gg \exp(n^\alpha)$ for $n \to \infty$. Choose $\beta, \gamma \in (0,1)$ such that $0 < \beta < \alpha$ and $0 < 1 - \beta < \gamma$ hold. Take the function f of Section 4.2.2 to be $f(n) := n^\gamma$ for large n. Clearly, (4.30) is satisfied for this choice of f, and for n large enough we have $f(n) \leq g(n)$. Hence, similar arguments as above show that for n large enough $\tau_n \leq \tau$ holds. Again, by (4.31) we also have $\tau_n \to \tau$ for $n \to \infty$. Using these facts and arguing analogously to Section 4.2.2, we conclude that for Theorem 5 (ii) it suffices to prove

$$\lim_{n\to\infty} \mathbf{Q}_n \left[\exists t \in (\tau_n, \tau) : \eta^n_{t^-,x} > \eta^n_{t,x} \right] = 1 \tag{4.51}$$

for $x \in T$, where $\mathbf{Q}_n$ is defined as in Section 4.2.2. In Section 4.2.2, we deduced that equation (4.35) follows from (4.49) and (4.50); in exactly the same way it can be shown that equation (4.51) follows from

$$\lim_{n\to\infty} (\tau - \tau_n) f(n) = \infty \tag{4.52}$$

and

$$\lim_{n\to\infty} (\tau - \tau_n) \lambda(n) f(n) r^{cf(n)\log n} = 0. \tag{4.53}$$

Now (4.53) is an immediate consequence of (4.50). It thus remains to prove (4.52).

To this end we first rewrite $\tau - \tau_n$ as

$$\tau - \tau_n = \log\left(\frac{r - m(\tau)\sqrt[n]{f(n)/g(n)}}{r - m(\tau)}\right) = \log\left(1 + \frac{m(\tau)}{r - m(\tau)}\left(1 - \sqrt[n]{\frac{f(n)}{g(n)}}\right)\right).$$

Since $\sqrt[n]{f(n)} \to 1$ and $\sqrt[n]{g(n)} \to 1$ for $n \to \infty$ (the first limit follows from (4.30), the second limit holds by assumption), we conclude that

$$\lim_{n\to\infty}\left(1 - \sqrt[n]{\frac{f(n)}{g(n)}}\right)^{-1}(\tau - \tau_n) = \lim_{y\downarrow 0} y^{-1}\log\left(1 + \frac{m(\tau)}{r - m(\tau)}y\right) = \frac{m(\tau)}{r - m(\tau)} > 0$$

holds. For n large enough, we also have $\frac{f(n)}{g(n)} \leq \frac{n^\gamma}{\exp(n^\alpha)} \leq \frac{1}{\exp(n^\beta)}$ and hence

$$f(n)\left(1 - \sqrt[n]{\frac{f(n)}{g(n)}}\right) \geq n^\gamma\left(1 - \exp\left(-n^{\beta-1}\right)\right).$$

Moreover, the limit $n \to \infty$ of the last term is given by

$$\lim_{n\to\infty} n^\gamma\left(1 - \exp\left(-n^{\beta-1}\right)\right) = \lim_{n\to\infty} n^{\gamma+\beta-1} \cdot \lim_{n\to\infty} n^{-(\beta-1)}\left(1 - \exp\left(-n^{\beta-1}\right)\right) = \infty.$$

This yields the proof of (4.52).

4.4 Proof of Proposition 1

Let $\tau \in (t_c, \infty)$, let $\epsilon > 0$ and let $(\rho_{t,z}, G_{t,z})_{0\leq t\leq\tau+\epsilon, z\in T}$ be a self-destructive percolation process on T with parameters τ and ϵ under some probability measure $\mathbf{P}$. So far we have parametrized self-destructive percolation in terms of the length of the time intervals $[0, \tau)$ and $[\tau, \tau + \epsilon]$. For the proof of Proposition 1, however, it will be more convenient to parametrize the final configuration $\rho_{\tau+\epsilon} := (\rho_{\tau+\epsilon,z})_{z\in T}$ in terms of the Bernoulli probabilities $p := 1 - e^{-\tau}$ and $\delta := 1 - e^{-\epsilon}$ for growth at a fixed vertex in the time intervals $[0, \tau)$ and $[\tau, \tau + \epsilon]$, respectively. We therefore use the following alternative notation (which follows along the lines of [vdBB04]):

Let X_v, $v \in T$, and Y_v, $v \in T$, be independent $\{0, 1\}$-valued random variables under some probability measure $\mathbf{P}_{p,\delta}$ such that

$$\begin{aligned} \mathbf{P}_{p,\delta}[X_v = 1] &= p, & \mathbf{P}_{p,\delta}[X_v = 0] &= 1 - p, \\ \mathbf{P}_{p,\delta}[Y_v = 1] &= \delta, & \mathbf{P}_{p,\delta}[Y_v = 0] &= 1 - \delta \end{aligned}$$

for all $v \in T$. Let $X := (X_v)_{v\in T}$, $Y := (Y_v)_{v\in T}$ and define $X^* = (X^*_v)_{v\in T}$, $Z = (Z_v)_{v\in T}$ by

$$X^*_v := \begin{cases} 1 & \text{if } X_v = 1 \text{ and the cluster of } v \text{ in } X \text{ is finite,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Z_v := X_v^* \vee Y_v.$$

Then the distribution of the final configuration $\rho_{\tau+\epsilon}$ under $\mathbf{P}$ and the distribution of Z under $\mathbf{P}_{p,\delta}$ are clearly identical.

Let

$$\theta(p) := \mathbf{P}_{p,\delta}\,[\text{the cluster of } \emptyset \text{ in } X \text{ is infinite}]$$

be the probability that the root $\emptyset$ is in an infinite cluster after the first step of self-destructive percolation (i.e. in independent site percolation on T with parameter p), and let

$$\theta(p,\delta) := \mathbf{P}_{p,\delta}\,[\text{the cluster of } \emptyset \text{ in } Z \text{ is infinite}]$$

be the probability that the root $\emptyset$ is in an infinite cluster in the final configuration of self-destructive percolation. Using the fact that the final configuration Z is positively associated ([vdBB04], Sections 2.2 and 2.3), it is easy to see that the equivalence

$$\mathbf{P}_{p,\delta}\,[Z \text{ contains an infinite cluster}] = 0 \Leftrightarrow \theta(p,\delta) = 0$$

holds. For the proof of Proposition 1 it therefore suffices to prove the following proposition, where $p_c := \frac{1}{r} = 1 - e^{-t_c}$ denotes the critical probability of independent site percolation on T:

Proposition 3. *For all $p \in (p_c, 1)$ there exists $\delta \in (0,1)$ such that $\theta(p,\delta) = 0$.*

Proposition 3 is a generalization of a result by J. van den Berg and R. Brouwer ([vdBB04], Theorem 5.1), who proved the following statement for the case where T is the binary tree (i.e. $r = 2$): If $p \in (p_c, 1)$ and $\delta > 0$ satisfies

$$p\,(1-\delta) \geq p_c, \tag{4.54}$$

then $\theta(p,\delta) = 0$. Our proof of Proposition 3 for general r is based on the same principal ideas as [vdBB04] but eventually takes a different route due to the occurrence of higher order terms for $r \geq 3$. Although these terms turn out to be asymptotically negligible, they are the reason why for $r \geq 3$ we do not obtain an explicit condition on δ like (4.54).

We first prove a weaker version of Proposition 3:

Lemma 28. *For all $p \in (p_c, 1)$ we have $\lim_{\delta \downarrow 0} \theta(p,\delta) = 0$.*

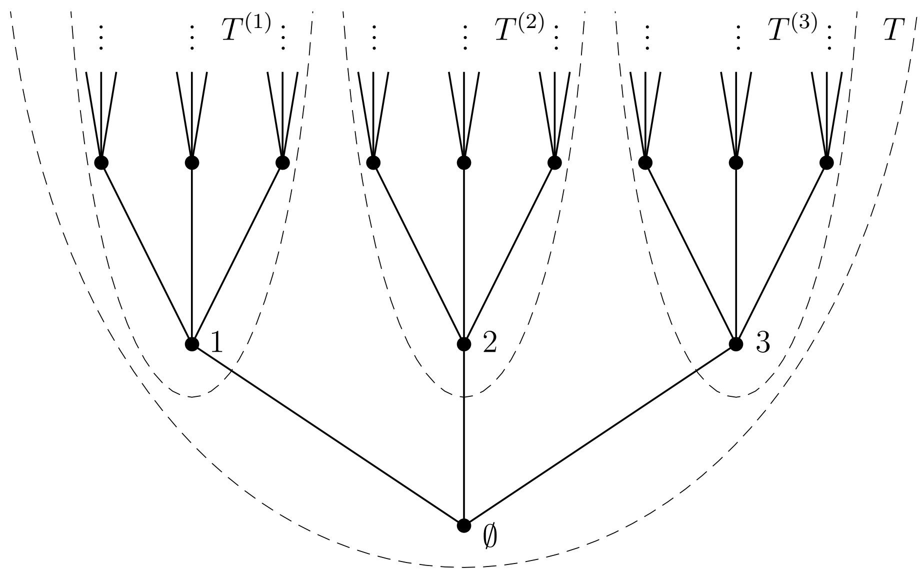


Figure 4.1: Illustration of the case $r = 3$ - the 3-regular rooted tree T with the root $\emptyset$, its children $1, 2, 3$ and the 3-regular rooted subtrees $T^{(1)}, T^{(2)}, T^{(3)}$

Proof of Lemma 28. Let $p \in (p_c, 1)$ and $\delta \in (0, 1)$. By distinguishing whether or not the root $\emptyset$ is in an infinite cluster after the first step of self-destructive percolation we obtain the inequality

$$\begin{aligned}\theta(p,\delta) \le\ & \mathbf{P}_{p,\delta}\left[\text{the cluster of } \emptyset \text{ in } X \text{ is infinite}, Y_\emptyset = 1\right] \\ & + \mathbf{P}_{p,\delta}\left[\text{the cluster of } \emptyset \text{ in } X \text{ is finite}, \text{the cluster of } \emptyset \text{ in } X \vee Y \text{ is infinite}\right] \\ =\ & \theta(p)\delta + \left(\theta(p + (1-p)\delta) - \theta(p)\right).\end{aligned}$$

Since $\theta(\,\cdot\,)$ is continuous, the last expression tends to zero for $\delta \downarrow 0$, which proves the lemma. $\square$

Proof of Proposition 3. Suppose that Proposition 3 is not true. Then there exists $p_0 \in (p_c, 1)$ such that for all $\delta \in (0, 1)$ we have $\theta(p_0, \delta) > 0$. In fact, even the stronger statement

$$\forall p \in (p_c, p_0]\, \forall \delta \in (0, 1) : \theta(p, \delta) > 0 \tag{4.55}$$

is true. This is due to the fact that if $p_1, p_2 \in (p_c, 1)$ and $\delta_1, \delta_2 \in (0, 1)$ satisfy $p_1 \ge p_2$ and $p_1 + (1-p_1)\delta_1 = p_2 + (1-p_2)\delta_2$, then $\theta(p_1, \delta_1) \le \theta(p_2, \delta_2)$ holds (see [vdBB04], Lemma 2.3). We will show that (4.55) leads to a contradiction.

Let $p \in (p_c, p_0]$, $\delta \in (0, 1)$ and define the probability measure $\mathbf{P}_{p,\delta}$ and the random configurations X, Y, X^*, Z as above (at the beginning of Section 4.4). We will derive an inequality for $\theta(p, \delta)$ by exploiting the recursive structure of the tree T. So let us denote the r children of the root $\emptyset$ by $1, \ldots, r$. For $i = 1, \ldots, r$, let $T^{(i)}$ be the r-regular rooted subtree of T which has i as its root (see Figure 4.1 for an illustration of the case $r = 3$). As before, we will use the term $T^{(i)}$ both for the graph and its vertex set. Let $X^{(i)} := (X_v)_{v \in T^{(i)}}$

and $Y^{(i)} := (Y_v)_{v \in T^{(i)}}$ be the configurations we obtain when we restrict X and Y to the subtree $T^{(i)}$. Moreover, let $X^{*(i)} = (X_v^{*(i)})_{v \in T^{(i)}}$ and $Z^{(i)} = (Z_v^{(i)})_{v \in T^{(i)}}$ be the corresponding configurations for self-destructive percolation on $T^{(i)}$, i.e.

$$X_v^{*(i)} := \begin{cases} 1 & \text{if } X_v = 1 \text{ and the cluster of } v \text{ in } X^{(i)} \text{ is finite,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Z_v^{(i)} := X_v^{*(i)} \vee Y_v.$$

Then the quadruples of configurations $(X^{(i)}, Y^{(i)}, X^{*(i)}, Z^{(i)})$, $i = 1, \ldots, r$, are independent and have the same distribution as (X, Y, X^*, Z).

Now consider the events

$$A := \{X_\emptyset \vee Y_\emptyset = 1, \exists i \in \{1, \ldots, r\} : \text{the cluster of } i \text{ in } Z^{(i)} \text{ is infinite}\}$$

and

$$B := \Big\{X_\emptyset = 1, Y_\emptyset = 0, \exists i, j \in \{1, \ldots, r\} : i \neq j, \text{the cluster of } i \text{ in } Z^{(i)} \text{ is infinite,} \\ \text{the cluster of } j \text{ in } X^{(j)} \text{ is infinite}\Big\}.$$

Since these events satisfy the inclusions

$$\begin{aligned} \{\text{the cluster of } \emptyset \text{ in } Z \text{ is infinite}\} &\subset A, \\ \{\text{the cluster of } \emptyset \text{ in } Z \text{ is finite}\} &\supset B, \\ B &\subset A, \end{aligned}$$

we have

$$\theta(p, \delta) \leq \mathbf{P}_{p,\delta}[A] - \mathbf{P}_{p,\delta}[B]. \tag{4.56}$$

From the definition of A we readily deduce

$$\begin{aligned} \mathbf{P}_{p,\delta}[A] &= (p + (1-p)\delta)\left(1 - (1 - \theta(p, \delta))^r\right) \\ &= (p + (1-p)\delta) \cdot r\theta(p, \delta) + \mathcal{O}(\theta(p, \delta)^2) \\ &\quad \text{for } \delta \downarrow 0, \text{ uniformly for } p \in (p_c, p_0]. \end{aligned} \tag{4.57}$$

In order to calculate $\mathbf{P}_{p,\delta}[B]$, we define

$$D_i := \Big\{\text{the cluster of } i \text{ in } Z^{(i)} \text{ is infinite,} \\ \exists j \in \{1, \ldots, r\} \setminus \{i\} : \text{the cluster of } j \text{ in } X^{(j)} \text{ is infinite}\Big\}$$

for $i = 1, \ldots, r$ and rewrite B as

$$B = \{X_\emptyset = 1, Y_\emptyset = 0\} \cap \bigcup_{i=1}^{r} D_i. \tag{4.58}$$

For $i \in \{1, \ldots, r\}$ the definition of D_i implies

$$\mathbf{P}_{p,\delta}[D_i] = \theta(p,\delta)\left(1 - (1-\theta(p))^{r-1}\right),$$

and for $k \in \{2, \ldots, r\}$ and $1 \le i_1 < \ldots < i_k \le r$ we have the upper bound

$$\mathbf{P}_{p,\delta}[D_{i_1} \cap \ldots \cap D_{i_k}] \le \theta(p,\delta)^k.$$

Hence equation (4.58) yields

$$\begin{aligned}
\mathbf{P}_{p,\delta}[B] &= p(1-\delta)\left(\sum_{k=1}^{r} \sum_{1 \le i_1 < \ldots < i_k \le r} (-1)^{k+1}\, \mathbf{P}_{p,\delta}[D_{i_1} \cap \ldots \cap D_{i_k}]\right) \\
&= p(1-\delta) \cdot r\theta(p,\delta)\left(1 - (1-\theta(p))^{r-1}\right) + \mathcal{O}(\theta(p,\delta)^2) \qquad (4.59)\\
&\quad \text{for } \delta \downarrow 0, \text{ uniformly for } p \in (p_c, p_0].
\end{aligned}$$

Inserting (4.57) and (4.59) into the inequality (4.56) and dividing both sides by $\theta(p,\delta)$ (which is possible because of our assumption (4.55)), we obtain

$$\begin{aligned}
1 &\le (p + (1-p)\delta) \cdot r - p(1-\delta) \cdot r\left(1 - (1-\theta(p))^{r-1}\right) + \mathcal{O}(\theta(p,\delta)) \\
&\quad \text{for } \delta \downarrow 0, \text{ uniformly for } p \in (p_c, p_0].
\end{aligned}$$

Finally, letting δ tend to zero and using Lemma 28, we get

$$1 \le pr\,(1-\theta(p))^{r-1}.$$

In the remainder of the proof we show that this inequality leads to a contradiction when p tends to p_c. Expanding the right side of the inequality in powers of $\theta(p)$, we obtain

$$1 \le pr\,(1 - (r-1)\theta(p)) + \mathcal{O}(\theta(p)^2) \qquad \text{for } p \downarrow p_c. \tag{4.60}$$

On the other hand, the recursive structure of the tree T implies

$$\begin{aligned}
\theta(p) &= p\left(1 - (1-\theta(p))^r\right) \\
&= p\left(r\theta(p) - \frac{1}{2}r(r-1)\theta(p)^2\right) + \mathcal{O}\left(\theta(p)^3\right) \qquad \text{for } p \downarrow p_c.
\end{aligned}$$

Dividing both sides by $\theta(p)$ (which is positive for $p \in (p_c, p_0]$) gives

$$1 = pr\left(1 - \frac{1}{2}(r-1)\theta(p)\right) + \mathcal{O}\left(\theta(p)^2\right) \qquad \text{for } p \downarrow p_c. \tag{4.61}$$

Subtracting (4.61) from (4.60) and dividing by $\theta(p)$ again then leads to the inequality

$$0 \le -\frac{1}{2}pr(r-1) + \mathcal{O}(\theta(p)) \qquad \text{for } p \downarrow p_c.$$

But since $\theta(p) \to 0$ for $p \downarrow p_c$, this produces a contradiction. $\square$

Appendix A

Perfect simulations of Dürre forest-fire models on infinite graphs

In Section 1.3.1 we mentioned that the results in [Dür09] can be used to perfectly simulate Dürre forest-fire models on infinite graphs. It is the purpose of this appendix to make this statement more precise and to sketch how it can be derived from the proofs in [Dür09]. Since this is only a side aspect of the present thesis, we will keep the presentation short and refer the reader to [Dür09] for more details, including some basic definitions.

Consider an infinite connected graph with vertex set V whose vertex degree is bounded by some natural number $d \in \mathbb{N}$. Let $0 \in V$ be a distinguished vertex and for $n \in \mathbb{N}$, let V_n be the set of vertices whose graph distance from 0 is at most n. Fix some ignition rate $\lambda > 0$. Let $(\eta_{0,z})_{z\in V}$ be a (possibly random) 0-1-configuration on V such that the conditional cluster size bound $\mathrm{CCSB}(0, \lambda/(4d^2), m)$ (see [Dür09], Definition 7) is satisfied for some $m \in \mathbb{N}$. Moreover, let $(G_{t,z})_{t\geq 0}$ and $(I_{t,z})_{t\geq 0}$, $z \in V$, be independent Poisson processes with rates 1 and λ respectively, independently of $(\eta_{0,z})_{z\in V}$. Now let $\bar{\eta} := (\eta_{t,z}, G_{t,z}, I_{t,z})_{t\geq 0, z\in V}$ be the almost surely uniquely defined Dürre forest-fire model on V (see [Dür09], Definition 3) with initial configuration $(\eta_{0,z})_{z\in V}$, growth processes $(G_{t,z})_{t\geq 0, z\in V}$ and ignition processes $(I_{t,z})_{t\geq 0, z\in V}$. (The feasibility of this construction follows from [Dür09], Theorems 1 and 3.) Similarly, for $n \in \mathbb{N}$, let $\bar{\eta}^n := (\eta^n_{t,z}, G_{t,z}, I_{t,z})_{t\geq 0, z\in V_n}$ be the unique Dürre forest-fire model on V_n with initial configuration $(\eta_{0,z})_{z\in V_n}$, growth processes $(G_{t,z})_{t\geq 0, z\in V_n}$ and ignition processes $(I_{t,z})_{t\geq 0, z\in V_n}$. In this way, $\bar{\eta}$ and $\bar{\eta}^n$ are naturally coupled through their growth and ignition processes. This coupling can be used to perfectly simulate $(\eta_{t,z})_{t\geq 0, z\in V}$ in the following sense:

Proposition 4. *Let $T > 0$ and let $E \subset V$ be finite. For $n \in \mathbb{N}$ let*

$$\mathcal{F}_{T,n} := \sigma\left((\eta_{0,z})_{z\in V_n}, (G_{t,z}, I_{t,z})_{0\leq t\leq T, z\in V_n}\right)$$

denote the σ-field generated by the initial condition on V_n and the growth and ignition processes on V_n between times 0 and T. Then there exists a stopping time $\nu_{T,E}$ with respect *to the filtration* $(\mathcal{F}_{T,n})_{n\in\mathbb{N}}$ *such that* $\mathbf{P}\left[\nu_{T,E}<\infty\right]=1$ *and*

$$\mathbf{P}\left[\sup_{n\geq\nu_{T,E}}\sup_{0\leq t\leq T}\sup_{x\in E}\left|\eta_{t,x}-\eta^n_{t,x}\right|=0\right]=1$$

hold.

An explicit choice for $\nu_{T,E}$ is given in the proof of the proposition (see equation (A.6)).

Proposition 4 is an extension of Theorem 1 in [Dür09], which states that for $n\to\infty$, the forest-fire process on V_n converges a.s. to the forest-fire process on V. As in [Dür09], the so-called blur process is an important ingredient in the proof of Proposition 4. For $S\subset V$, let

$$\partial S:=\{v\in V\setminus S:(\exists w\in S:v\text{ and }w\text{ are neighbours})\}$$

denote the (outer) boundary of S, and let $\overline{S}:=S\cup\partial S$ be the closure of S. Then for $t\geq 0$ and $S\subset V$, the (t,S)-blur process $(\beta^{t,S}_{T,z})_{T\geq t,z\in\overline{S}}$ with respect to the forest-fire process $\bar{\eta}$ can informally be described as follows: Each vertex is either "blurred" (denoted by "2") or "not blurred" (denoted by "0"). At the starting time t a vertex in $\overline{S}$ is blurred if and only if it lies on the boundary ∂S or its cluster is connected to the boundary ∂S. After time t a vertex in $\overline{S}$ becomes blurred if and only if its cluster gets connected to a vertex which has been blurred before. If a vertex is blurred once, it remains blurred forever. A more formal definition of the blur process is given in [Dür09], Definition 9.

The (t,S)-blur process at time $T\geq t$ indicates for which vertices in S the configuration of the infinite volume forest-fire process at time T can be determined only by the configuration $(\eta_{t,z})_{z\in S}$ on S at time t and the increments $(G_{t',z}-G_{t,z})_{t\leq t'\leq T,z\in S}$, $(I_{t',z}-I_{t,z})_{t\leq t'\leq T,z\in S}$ of the growth and ignition processes in S between times t and T. Moreover, the blur process yields a sufficient criterion to decide whether the infinite volume forest-fire process and the corresponding finite volume process agree:

Lemma 29. *Let* $0\leq t<T$*, let* $k,l\in\mathbb{N}$ *with* $k\geq l$ *and let* $x\in V_l$*. Then the inclusion*

$$\left\{\sup_{n\geq k}\sup_{z\in V_l}\left|\eta_{t,z}-\eta^n_{t,z}\right|=0,\sup_{n\geq k}\sup_{t\leq t'\leq T}\left|\eta_{t',x}-\eta^n_{t',x}\right|>0\right\}\subset\left\{\beta^{t,V_l}_{T,x}=2\right\}$$

holds.

Proof. This is exactly Proposition 1 in [Dür09]. □

For the purpose of Proposition 4 the following aspect is important, too:

Lemma 30. *Let* $0\leq t\leq T$*,* $S\subset V$ *and* $x\in\overline{S}$*. Then* $\beta^{t,S}_{T,x}$ *is measurable with respect to the* σ*-field*

$$\sigma\left((\eta_{t,z})_{z\in S},(G_{t',z}-G_{t,z},I_{t',z}-I_{t,z})_{t\leq t'\leq T,z\in S}\right)$$

generated by the configuration on S of the infinite volume forest-fire process at time t and the increments of the growth and ignition processes in S between times t and T.

Proof. This can be seen from the explicit construction of the blur process, which is given in [Dür09], Lemma 1. □

In the next lemma, we compare a (t_1, S_1)-blur process with a (t_2, S_2)-blur process, where $t_1 \leq t_2$ and $S_2 \subset S_1$.

Lemma 31. *Let $0 \leq t_1 \leq t_2$ and $S_2 \subset S_1 \subset V$. Then the inclusion*

$$\left\{\forall x \in S_2 : \beta^{t_1,S_1}_{t_2,x} \leq \beta^{t_2,S_2}_{t_2,x}\right\} \subset \left\{\forall T \geq t_2 \, \forall x \in S_2 : \beta^{t_1,S_1}_{T,x} \leq \beta^{t_2,S_2}_{T,x}\right\} \tag{A.1}$$

holds. In particular, if $t_1 = t_2 = t$ for some $t \geq 0$, then

$$\forall T \geq t \, \forall x \in S_2 : \beta^{t,S_1}_{T,x} \leq \beta^{t,S_2}_{T,x} \tag{A.2}$$

holds pointwise.

Proof. We first observe that for $t_1 = t_2$, the condition on the left side of (A.1) is always satisfied so that (A.2) follows indeed from (A.1). Equation (A.2) is proved in [Dür09], Lemma 2. The more general version (A.1) can be proved analogously. □

Together with results in [Dür09], Lemma 31 can be used to prove the following:

Lemma 32. *For $T \geq 0$ and $x \in V$ we have*

$$\lim_{l\to\infty} \mathbf{P}\left[\beta^{0,V_l}_{T,x} = 2\right] = 0. \tag{A.3}$$

Proof. Combining Theorem 1, Proposition 2 and Proposition 3 in [Dür09], one can deduce the existence of constants $\gamma > 0$ and $\epsilon > 0$ (depending on the ignition rate λ, the bound d on the vertex degree and the constant m from the conditional cluster size bound) with the following property: Let $t_0 := 0$, $t_1 := \gamma$ and recursively define $t_i := t_{i-1} + \epsilon$ for $i \geq 2$. Then for $i \in \mathbb{N}$ and $x \in V$

$$\lim_{l\to\infty} \mathbf{P}\left[\beta^{t_{i-1},V_l}_{t_i,x} = 2\right] = 0 \tag{A.4}$$

holds. By induction on i we now show that for $i \in \mathbb{N}$ and $x \in V$

$$\lim_{l\to\infty} \mathbf{P}\left[\beta^{0,V_l}_{t_i,x} = 2\right] = 0 \tag{A.5}$$

is true, as well. For $i = 1$ equations (A.5) and (A.4) coincide. So let $i \geq 2$ and pick $\delta > 0$. By (A.4) we can first choose $k \in \mathbb{N}$ such that $V_k \ni x$ and

$$\mathbf{P}\left[\beta^{t_{i-1},V_k}_{t_i,x} = 2\right] < \frac{\delta}{2}$$

hold. By the induction hypothesis we can next choose $l_0 \geq k$ such that for all $l \geq l_0$ and all $z \in V_k$

$$\mathbf{P}\left[\beta^{0,V_l}_{t_{i-1},z} = 2\right] < \frac{\delta}{2|V_k|}$$

holds, where $|V_k|$ denotes the number of elements of V_k. For $l \geq l_0$ we then obtain

$$\begin{aligned}\mathbf{P}\left[\beta^{0,V_l}_{t_i,x} = 2\right] &< \mathbf{P}\left[\beta^{0,V_l}_{t_i,x} = 2, \beta^{t_{i-1},V_k}_{t_i,x} = 0, \forall z \in V_k : \beta^{0,V_l}_{t_{i-1},z} = 0\right] + \delta \\ &\leq \mathbf{P}\left[\beta^{0,V_l}_{t_i,x} = 2, \beta^{t_{i-1},V_k}_{t_i,x} = 0, \forall z \in V_k : \beta^{0,V_l}_{t_{i-1},z} \leq \beta^{t_{i-1},V_k}_{t_{i-1},z}\right] + \delta \\ &= \delta,\end{aligned}$$

where the first inequality follows from the choice of k and l and the probability in the second line is zero because of Lemma 31, equation (A.1). This completes the induction step and thus the proof of (A.5). Since $\beta^{0,V_l}_{T,x}$ is monotone increasing in T and $t_i \to \infty$ for $i \to \infty$, equation (A.5) implies (A.3). $\square$

We now have all the necessary ingredients for the proof of Proposition 4.

Proof of Proposition 4. Let $T > 0$ and let $E \subset V$ be finite. Lemma 32 and Lemma 31, equation (A.2), imply

$$\mathbf{P}\left[\exists l \in \mathbb{N} : V_l \ni x, \beta^{0,V_l}_{T,x} = 0\right] = 1$$

for all $x \in V$ and

$$\left\{\beta^{0,V_l}_{T,x} = 0\right\} \subset \left\{\forall n \geq l : \beta^{0,V_n}_{T,x} = 0\right\}$$

for all $x \in V$ and $l \in \mathbb{N}$ with $V_l \ni x$. Consequently

$$\nu_{T,E} := \inf\left\{n \in \mathbb{N} : \left(V_n \supset E, \forall x \in E : \beta^{0,V_n}_{T,x} = 0\right)\right\} \tag{A.6}$$

satisfies $\mathbf{P}\left[\nu_{T,E} < \infty\right] = 1$, and on the event $\{\nu_{T,E} < \infty\}$ we have

$$\forall n \geq \nu_{T,E}\, \forall x \in E : \beta^{0,V_n}_{T,x} = 0$$

and hence

$$\sup_{n \geq \nu_{T,E}} \sup_{0 \leq t \leq T} \sup_{x \in E} \left|\eta_{t,x} - \eta^n_{t,x}\right| = 0$$

by Lemma 29. Moreover, it follows from Lemma 30 that $\nu_{T,E}$ is a stopping time with respect to $(\mathcal{F}_{T,n})_{n \in \mathbb{N}}$. $\square$

Appendix B

Simulations of forest-fire models on finite boxes

In this appendix to Chapter 2, we look at simulations of the two forest-fire models on finite boxes which are the starting point of Chapter 2 and discuss how the simulation results reflect conjectured and proven properties of $\overline{\mathbb{H}}$-forest-fire processes (see Definition 2).

B.1 General remarks and observations

Let $n \in \mathbb{N}$, $w \in \mathbb{Z}^2$ and define the box $B_n(w) \subset \mathbb{Z}^2$ with centre w and radius n by (2.1). As in Chapter 2, we write $B_n^{\mathrm{s}}(w)$ if the box $B_n(w)$ is endowed with the standard edges of the square lattice, and $B_n^{\mathrm{p}}(w)$ if additional edges from left to right are inserted to make the box periodic in the x-direction.

In Chapter 2, we initially considered the following forest-fire model on the box $B_n^{\mathrm{s}}(w)$:

> At time 0 all sites are vacant. Then trees grow according to independent rate 1 Poisson processes, but if an occupied cluster (with respect to the edges of $B_n^{\mathrm{s}}(w)$) reaches any of the four sides of the box, it is instantaneously destroyed.

Let us call this process a $B_n^{\mathrm{s}}(w)$**-forest-fire process**. (See page 14 for a more detailed description.) We then introduced the following forest-fire model on the box $B_n^{\mathrm{p}}(w)$:

> At time 0 all sites are vacant. Then trees grow according to independent rate 1 Poisson processes, but if an occupied cluster (with respect to the edges of $B_n^{\mathrm{p}}(w)$) reaches the bottom side of the box, it is instantaneously destroyed.

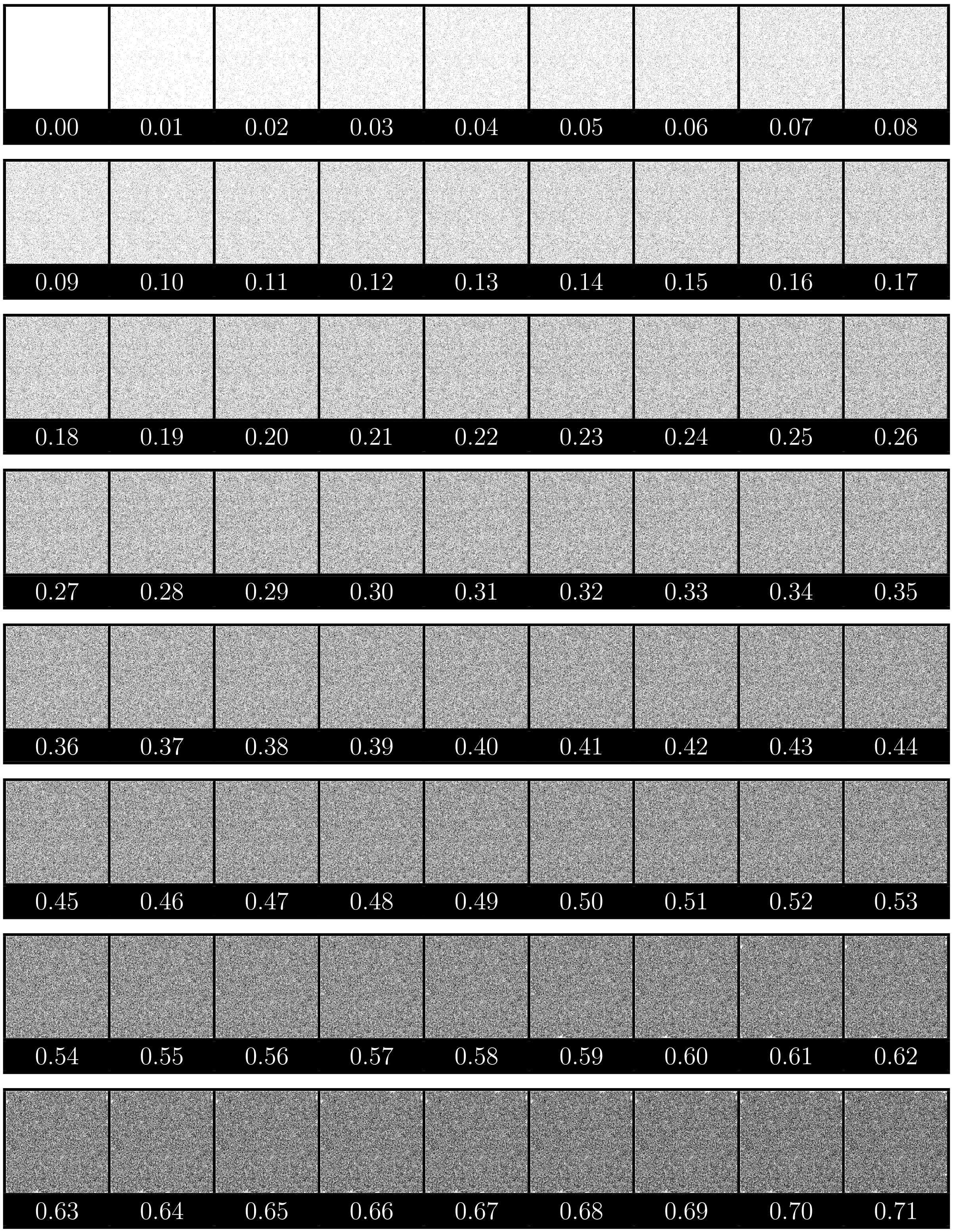
Let us call this process a $B_n^{\mathrm{p}}(w)$**-forest-fire process**. (In Definition 1 we defined this process under the additional assumption that $w = in$, where $i := \sqrt{-1} = (0, 1)$ denotes the imaginary unit.)

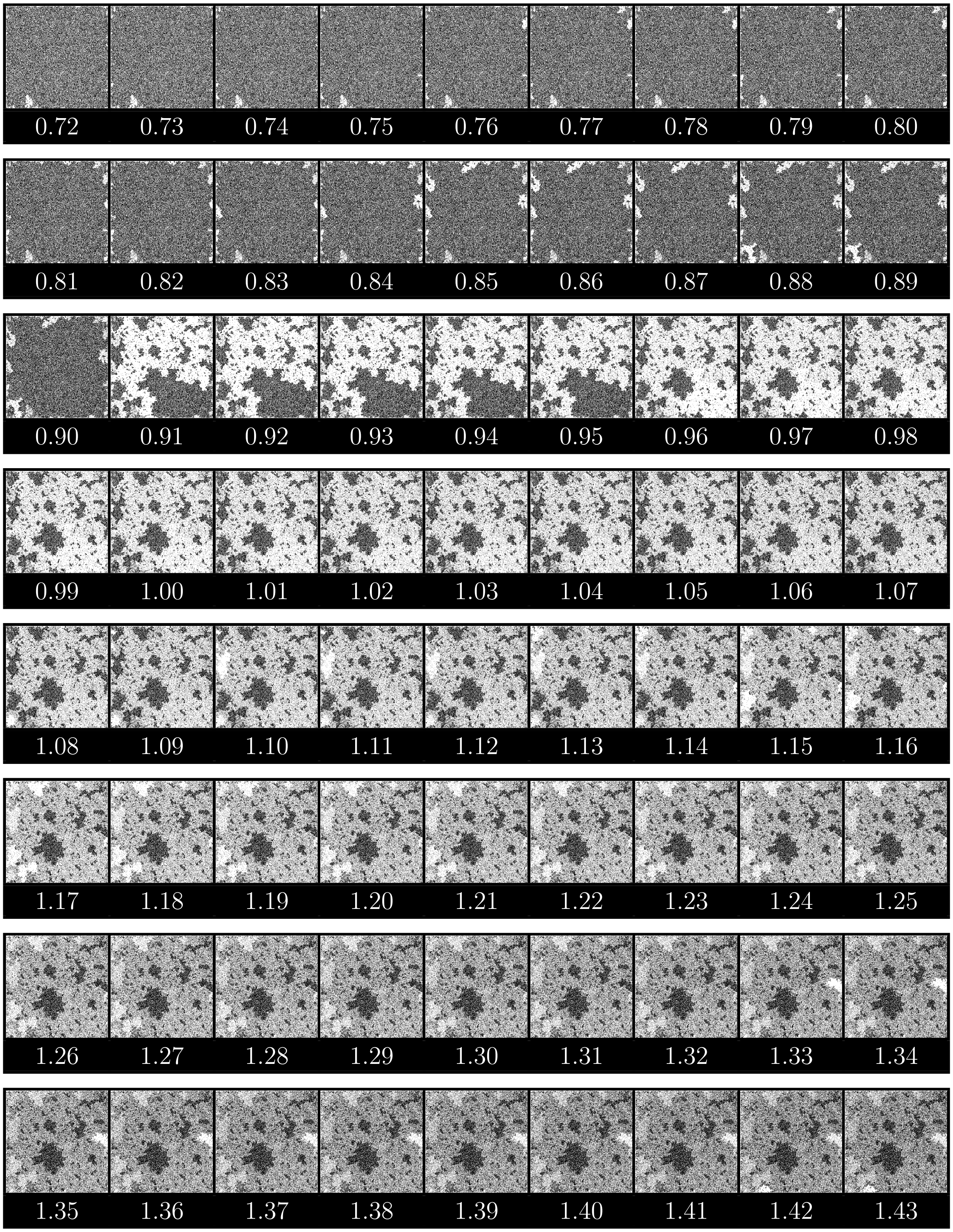
Sections B.2 and B.3 show simulations of these processes on the box $B_{200}(0)$ of radius 200 (i.e. a box with $401 \cdot 401 = 160801$ sites) between time 0 and time 2. For ease of notation, we choose the centre of the box to be the origin 0 but this is obviously irrelevant with regard to the simulations. For instance, we may as well consider the simulations to be performed on the box $B_{200}(200i)$, whose bottom side is on the x-axis. While Section B.2 shows a $B^{\mathrm{s}}_{200}(0)$-forest-fire process, Section B.3 shows a $B^{\mathrm{p}}_{200}(0)$-forest-fire process; however, both simulations are based on the same realization of the underlying growth processes. The configurations of the forest-fire processes are depicted in time steps of 0.1, where each site is represented by a tiny square which is white if the site is vacant and black if it is occupied by a tree.

Let us give some heuristic interpretations of the simulation results in tabular form:

Observation about simulation results	**Corresponding conjecture or theorem for $\overline{\mathbb{H}}$-forest-fire processes**
The $B^{\mathrm{s}}_{200}(0)$-forest-fire process and the $B^{\mathrm{p}}_{200}(0)$-forest-fire process qualitatively show the same behaviour.	We conjecture that in the limit $n \to \infty$, the $B^{\mathrm{s}}_{n}(ni)$-forest-fire process and the $B^{\mathrm{p}}_{n}(ni)$-forest-fire process converge to the same limit process (in a suitable sense). The limit process is expected to be an $\overline{\mathbb{H}}$-forest-fire process. Theorem 1 proves this kind of convergence for subsequences of $B^{\mathrm{p}}_{n}(ni)$-forest-fire processes.
In both simulations, the effect of destruction is first only felt locally (i.e. close to the side(s) which cause destruction) but is then suddenly felt globally (i.e. in the whole box). In the $B^{\mathrm{s}}_{200}(0)$-forest-fire process, this transition is observed between times 0.90 and 0.91, and in the $B^{\mathrm{p}}_{200}(0)$-forest-fire process, it is observed between times 0.92 and 0.93.	According to Theorem 2, the heights of destruction for the $\overline{\mathbb{H}}$-forest-fire process (see Definition 3) are a.s. finite before the critical time t_c and a.s. infinite after t_c. Numerical estimates indicate that t_c lies between 0.89 and 0.90. See [Hug96], Section 3.4.3, for an overview of numerical results on critical probabilities for Bernoulli percolation on various lattices.
The observed transition times in the two finite-size models are slightly larger than the critical time t_c.	We conjecture that the heights of destruction for the $\overline{\mathbb{H}}$-forest-fire process are a.s. finite at the critical time t_c. For the triangular lattice, a similar statement is proved in Chapter 3, Theorem 3.
The transition between local and global destruction occurs again at a later time. In the $B^{\mathrm{s}}_{200}(0)$-forest-fire process, this transition is observed between times 1.84 and 1.85, and in the $B^{\mathrm{p}}_{200}(0)$-forest-fire process, it is observed between times 1.96 and 1.97.	There are no well-founded conjectures let alone rigorous results which match this observation.

B.2 Simulation of a $B^{\mathrm{s}}_{200}(0)$-forest-fire process




0.72 0.73 0.74 0.75 0.76 0.77 0.78 0.79 0.80
0.81 0.82 0.83 0.84 0.85 0.86 0.87 0.88 0.89
0.90 0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98
0.99 1.00 1.01 1.02 1.03 1.04 1.05 1.06 1.07
1.08 1.09 1.10 1.11 1.12 1.13 1.14 1.15 1.16
1.17 1.18 1.19 1.20 1.21 1.22 1.23 1.24 1.25
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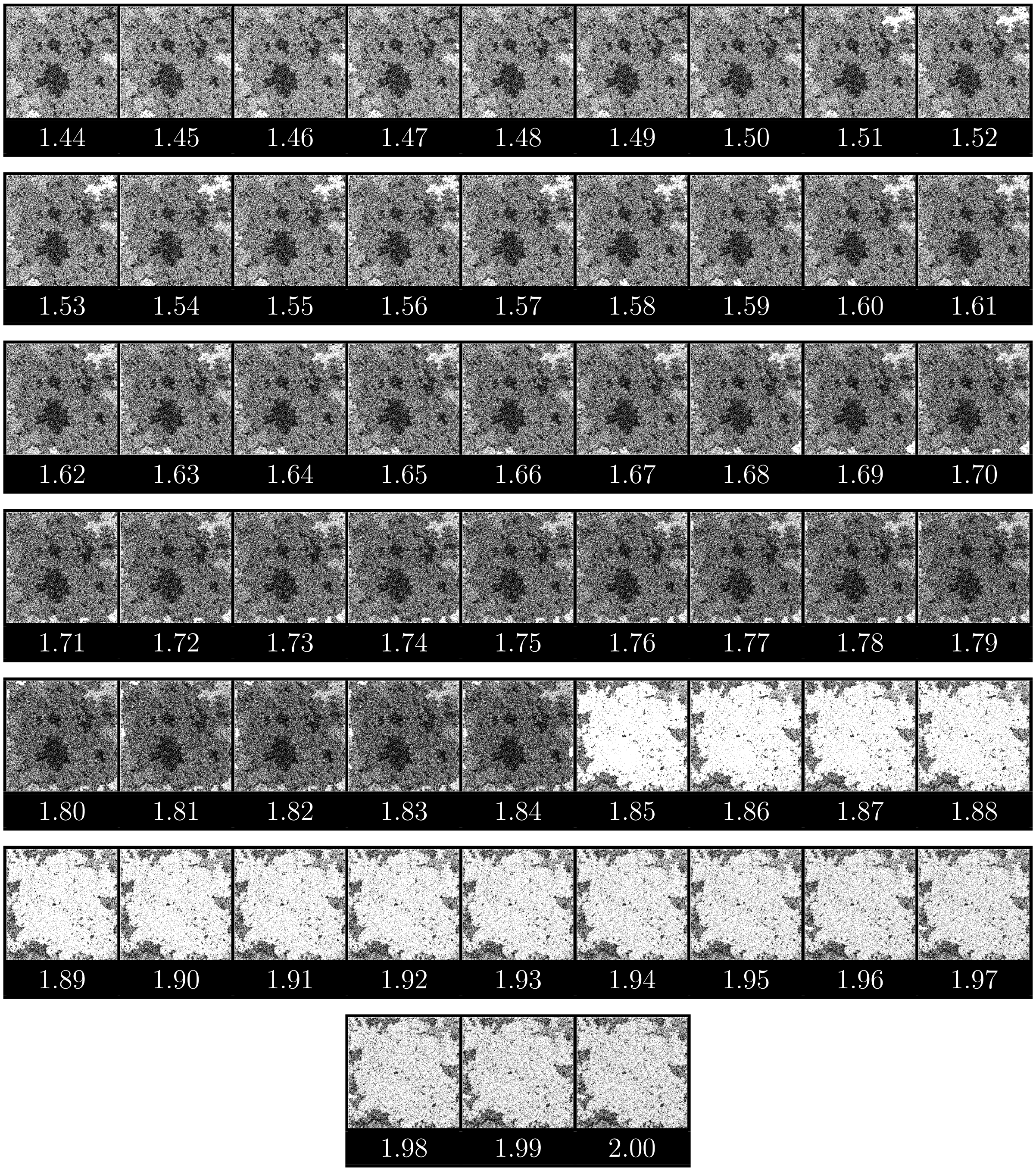
1.35 1.36 1.37 1.38 1.39 1.40 1.41 1.42 1.43


1.44 1.45 1.46 1.47 1.48 1.49 1.50 1.51 1.52
1.53 1.54 1.55 1.56 1.57 1.58 1.59 1.60 1.61
1.62 1.63 1.64 1.65 1.66 1.67 1.68 1.69 1.70
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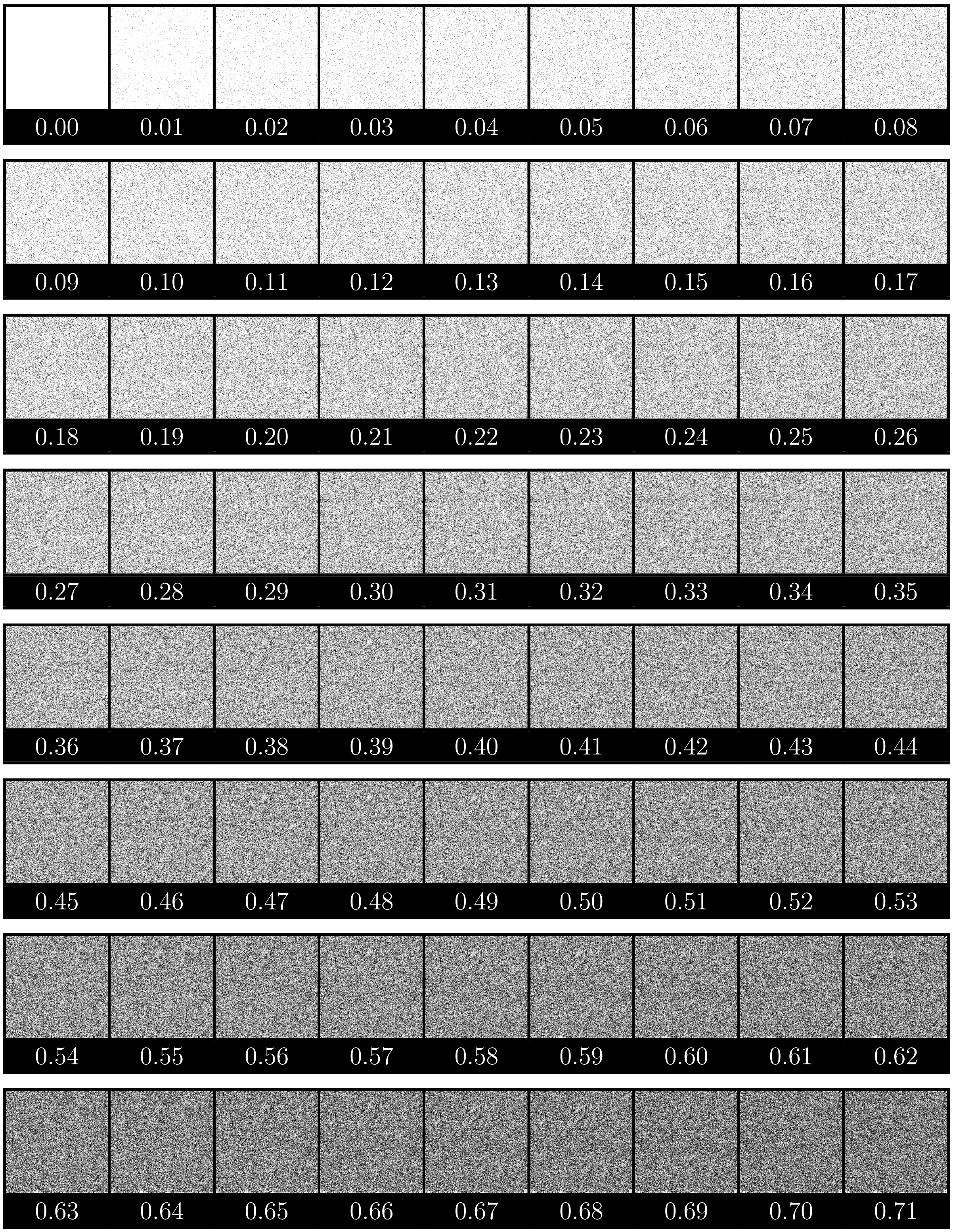
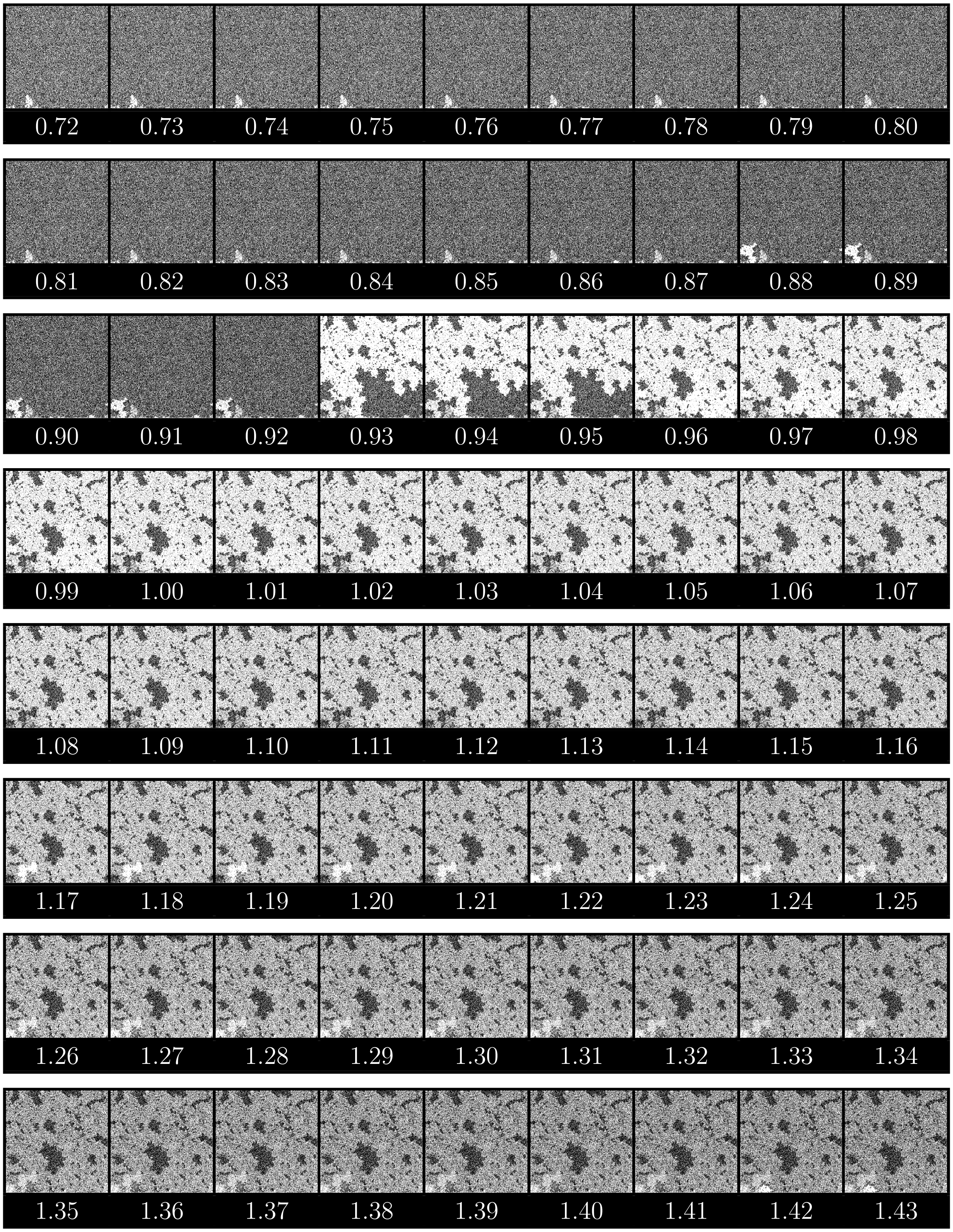
1.89 1.90 1.91 1.92 1.93 1.94 1.95 1.96 1.97
1.98 1.99 2.00

B.3 Simulation of a $B^{\mathrm{p}}_{200}(0)$-forest-fire process

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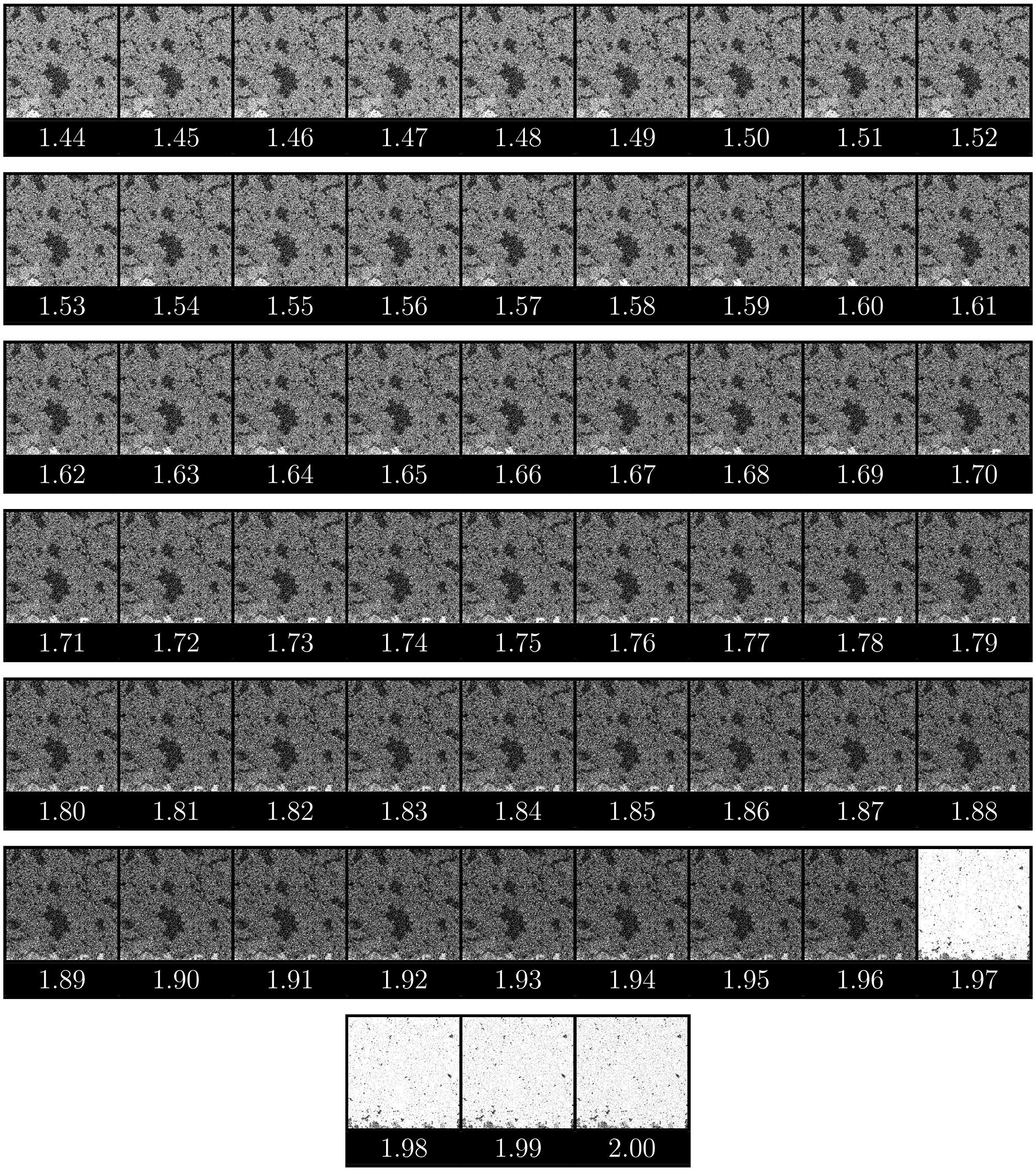
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1.71 1.72 1.73 1.74 1.75 1.76 1.77 1.78 1.79
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1.89 1.90 1.91 1.92 1.93 1.94 1.95 1.96 1.97
1.98 1.99 2.00

Bibliography

[ADCKS13] D. Ahlberg, H. Duminil-Copin, G. Kozma, and V. Sidoravicius. Seven-dimensional forest fires. *arXiv:1302.6872*, 2013.

[AST14] D. Ahlberg, V. Sidoravicius, and J. Tykesson. Bernoulli and self-destructive percolation on non-amenable graphs. *Electron. Commun. Probab.*, 19(40):1–6, 2014.

[Bak96] P. Bak. *How nature works: the science of self-organized criticality*. Copernicus Series. Springer, 1996.

[BCT90] P. Bak, K. Chen, and C. Tang. A forest-fire model and some thoughts on turbulence. *Phys. Lett. A*, 147(5-6):297–300, 1990.

[Bef04] V. Beffara. Hausdorff dimensions for SLE6. *Ann. Probab.*, 32(3B):2606–2629, 2004.

[BF09] X. Bressaud and N. Fournier. On the invariant distribution of a one-dimensional avalanche process. *Ann. Probab.*, 37(1):48–77, 2009.

[BF10] X. Bressaud and N. Fournier. Asymptotics of one-dimensional forest fire processes. *Ann. Probab.*, 38(5):1783–1816, 2010.

[BF11] X. Bressaud and N. Fournier. One-dimensional general forest fire processes. *arXiv:1101.0480*, 2011.

[BP06] R. Brouwer and J. Pennanen. The cluster size distribution for a forest-fire process on $\mathbb{Z}$. *Electron. J. Probab.*, 11(43):1133–1143, 2006.

[BTW87] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality: an explanation of $1/f$ noise. *Phys. Rev. Lett.*, 59(4):381–384, 1987.

[CC86] J. Chayes and L. Chayes. Critical points and intermediate phases on wedges of $\mathbb{Z}^d$. *J. Phys. A: Math. Gen.*, 19(15):3033–3048, 1986.

[CDS94] S. Clar, B. Drossel, and F. Schwabl. Scaling laws and simulation results for the self-organized critical forest-fire model. *Phys. Rev. E*, 50(2):1009–1018, 1994.

[CN06] F. Camia and C. M. Newman. Two-dimensional critical percolation: the full scaling limit. *Commun. Math. Physics*, 268(1):1–38, 2006.

[CN07] F. Camia and C. M. Newman. Critical percolation exploration path and SLE6: a proof of convergence. *Probab. Theory Related Fields*, 139(3-4):473–519, 2007.

[DCS93] B. Drossel, S. Clar, and F. Schwabl. Exact results for the one-dimensional self-organized critical forest-fire model. *Phys. Rev. Lett.*, 71(23):3739–3742, 1993.

[DS92] B. Drossel and F. Schwabl. Self-organized critical forest-fire model. *Phys. Rev. Lett.*, 69(11):1629–1632, 1992.

[Dür06a] M. Dürre. Existence of multi-dimensional infinite volume self-organized critical forest-fire models. *Electron. J. Probab.*, 11(21):513–539, 2006.

[Dür06b] M. Dürre. Uniqueness of multi-dimensional infinite volume self-organized critical forest-fire models. *Electron. Commun. Probab.*, 11(31):304–315, 2006.

[Dür09] M. Dürre. *Self-organized critical phenomena: forest fire and sandpile models.* PhD thesis, LMU München, 2009.

[Gra93] P. Grassberger. On a self-organized critical forest-fire model. *J. Phys. A: Math. Gen.*, 26(9):2081–2089, 1993.

[Gra02] P. Grassberger. Critical behaviour of the Drossel-Schwabl forest fire model. *New Journal of Physics*, 4(17):1–15, 2002.

[Gra14a] R. Graf. Critical heights of destruction for a forest-fire model on the half-plane. *arXiv:1406.1992*, 2014.

[Gra14b] R. Graf. A forest-fire model on the upper half-plane. *Electron. J. Probab.*, 19(8):1–27, 2014.

[Gra14c] R. Graf. Self-destructive percolation as a limit of forest-fire models on regular rooted trees. *arXiv:1404.0325*, 2014.

[Gri83] G. Grimmett. Bond percolation on subsets of the square lattice, and the threshold between one-dimensional and two-dimensional behaviour. *J. Phys. A: Math. Gen.*, 16(3):599–604, 1983.

[Gri99] G. Grimmett. *Percolation.* Grundlehren der mathematischen Wissenschaften. Springer, 2nd edition, 1999.

[Har63] T. E. Harris. *The theory of branching processes.* Grundlehren der mathematischen Wissenschaften. Springer, 1963.

[Har72] T. E. Harris. Nearest-neighbor Markov interaction processes on multidimensional lattices. *Advances in Math.*, 9(1):66–89, 1972.

[Hen93] C. Henley. Statics of a "self-organized" percolation model. *Phys. Rev. Lett.*, 71(17):2741–2744, 1993.

[Hug96] B. D. Hughes. *Random walks and random environments: random environments*. Oxford Science Publications. Clarendon Press, 1996.

[Jen98] H. Jensen. *Self-organized criticality: emergent complex behavior in physical and biological systems*. Cambridge lecture notes in physics. Cambridge University Press, 1998.

[KMS13] D. Kiss, I. Manolescu, and V. Sidoravicius. Planar lattices do not recover from forest fires. *arXiv:1312.7004*, 2013.

[KS66] H. Kesten and B. P. Stigum. A limit theorem for multidimensional Galton-Watson processes. *Ann. Math. Statist.*, 37(5):1211–1223, 1966.

[Lig85] T. Liggett. *Interacting particle systems*. Classics in Mathematics. Springer, 1985.

[Nol08] P. Nolin. Near-critical percolation in two dimensions. *Electron. J. Probab.*, 13(55):1562–1623, 2008.

[PJ04] G. Pruessner and H. Jensen. Efficient algorithm for the forest fire model. *Phys. Rev. E*, 70(6):1–25, 2004.

[Pru04] G. Pruessner. *Studies in self-organised criticality*. PhD thesis, Imperial College London, 2004.

[Pru12] G. Pruessner. *Self-organised criticality: theory, models and characterisation*. Cambridge University Press, 2012.

[RT09] B. Ráth and B. Tóth. Erdős-Rényi random graphs + forest fires = self-organized criticality. *Electron. J. Probab.*, 14(45):1290–1327, 2009.

[Shi95] A. N. Shiryayev. *Probability*. Graduate Texts in Mathematics. Springer, 2nd edition, 1995.

[Smi01] S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Math. Acad. Sci. Paris*, 333(3):239–244, 2001.

[Sta12] A. Stahl. Existence of a stationary distribution for multi-dimensional infinite volume forest-fire processes. *arXiv:1203.5533*, 2012.

[SW01] S. Smirnov and W. Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8(5-6):729–744, 2001.

[vdBB04] J. van den Berg and R. Brouwer. Self-destructive percolation. *Random Struct. Algorithms*, 24(4):480–501, 2004.

[vdBB06] J. van den Berg and R. Brouwer. Self-organized forest-fires near the critical time. *Commun. Math. Physics*, 267(1):265–277, 2006.

[vdBBV08] J. van den Berg, R. Brouwer, and B. Vágvölgyi. Box-crossings and continuity results for self-destructive percolation in the plane. In *In and Out of Equilibrium 2*, volume 60 of *Progress in Probability*, pages 117–135. Birkhäuser Basel, 2008.

[vdBdL09] J. van den Berg and B. N. B. de Lima. Linear lower bounds for $\delta_c(p)$ for a class of 2d self-destructive percolation models. *Random Struct. Algorithms*, 34(4):520–526, 2009.

[vdBJ05] J. van den Berg and A. A. Járai. On the asymptotic density in a one-dimensional self-organized critical forest-fire model. *Commun. Math. Physics*, 253(3):633–644, 2005.

[War73] C. D. Warner. *Backlog Studies.* J. R. Osgood, 1873.

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Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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