Non-perturbative Effects in field theory and gravity

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Zusammenfassung

Nichtperturbative Effekte spielen eine zentrale Rolle für das Verständnis der Dynamik von Quantenfeldtheorien, wie z.B. für Confinement oder den Zerfall schwarzer Löcher. In dieser Arbeit betrachten wir zwei Systeme in denen nichtperturbative Effekte eine zentrale Rolle spielen. Im ersten Teil beschäftigen wir uns mit der Dynamik nichtabelscher Eichtheorien; im zweiten Teil versuchen wir bestimmte mysteriöse Eigenschaften schwarzer Löcher in einem Modell von Dvali und Gomez aufzuklären.
Nichtabelsche Eichtheorien sind ein zentrales Element des Standardmodells der Teilchenphysik. Viele dynamische Aspekte dieser Theorien sind immer noch unklar. $\mathcal{N} = 1$ supersymmetrische Yang-Mills Theorien mit Eichgruppe $SU(N_C)$ haben Domänenwälle mit besonderen Eigenschaften. Man erwartet, dass Eichfelder mit Chern-Simons(CS) Term auf ihrem Weltvolumen leben. Im 't Hooft Limes von großem N_C verhalten sich diese sehr ähnlich zu D-Branen in Stringtheorie. Ähnliche Domänenwälle werden auch in nicht supersymmetrischen Yang-Mills Theorien vermutet. Wir konstruieren ein einfaches Modell in dem ein Eichfeld mit CS Term auf einem Domänenwall lokalisiert ist. Hierzu erweitern wir ein vorheriges Modell von Dvali und Shifman. Daraufhin leiten wir die Eigenschaften des CS Terms aus Effekten der darunterliegenden mikroskopischen Theorie ab. Dann schauen wir uns die eigentlich interessante Theorie an. Der neue Teil unserer Untersuchung ist hierbei der Fokus auf den topologischen Teil der Yang-Mills Theorie. Dieser erlaubt es, robuste Aussagen zu machen, obwohl die Theorie stark gekoppelt ist. Wir konstruieren die effektive Wirkung bei niedrigen Energien für die superymmetrische und die nicht supersymmetrische Yang-Mills Theorie. Wegen der Massenlücke ist dies eine topologische Feldtheorie. Diese topologische Feldtheorie enkodiert sowohl die Aharanov-Bohm Phasen in der Theorie als auch die Phasen durch Schnittpunkte von Flussschläuchen. In dieser topologischen Feldtheorie sehen wir, dass die Weltvolumentheorie

der Domänenwälle einen CS Term mit Niveau N_C enthält. Dieser CS Term wurde bereits in vorherigen Arbeiten auf Basis stringtheoretischer Konstruktionen vermutet. Hier geben wir die erste feldtheoretische Konstruktion. Wir nutzen diese Konstruktion auch um die Unterschiede zwischen supersymmetrischen und nicht supersymmetrischen Domänenwällen zu zeigen.
Danach zeigen wir Verbindungen der beobachtete Effekte zu ähnlichen Effekten in kritischer Stringtheorie auf und geben einige Spekulationen inwieweit das Verhalten dieser Domänenwälle eine Analogie zum fraktionellen Quanten Hall Effekt habe.
Im zweiten Teil der Arbeit untersuchen wir nichtperturbative Aspekte der Physik schwarzer Löcher. Hierzu betrachten wir ein Modell von Dvali und Gomez in dem schwarze Löcher als Bose-Einstein-Kondensate (BEC) schwach wechselwirkender Gravitonen in der Nähe eines quantenkritischen Punktes beschrieben werden. Hier konzentrieren wir uns auf nichtpertubative Eigenschaften eines systems attraktiv selbstwechselwirkender nicht relativistischer Bosonen das als ein vereinfachtes Modell für Graviton BECs von Dvali und Gomez vorgeschlagen wurde. In dieser Arbeit betrachten wir dieses System primär mit einer vollständig nichtperturbativen Methode die exakte Diagonalisierung genannt wird. Zunächst untersuchen wir Verschränkungseigenschaften des Grundzustandes des Systems. Hierbei zeigen wir, dass der Grundzustand in der Nähe des quantenkritischen Punktes stark verschränkt ist. Um dies präzise zu machen führen wir die sogenannte Fluktuationsverschränkung ein. Daraufhin berechnen wir diese Grösse zunächst in einer Bogoliubov-Analyse und extrahieren sie dann auch aus der exakten Diagonalisierung. Wir betrachten dann die Zeitevolution des Systems. Hier sind wir interessiert daran ein Analog der vermuteten schnellen Chiffrierung von schwarzen Löchern zu finden die von Hayden und Preskill vorgeschlagen wurde. Hier beschränken wir uns auf eine schwächere Eigenschaft, sogenanntes schnelles Quantenbrechen. Wir zeigen, dass in unserem vereinfachten Modell die Quantenbrechzeit konsistent mit der Zeitskala, die für das schnelle Chiffrieren im Kontext schwarzer Löcher vermutet wird, ist. Abschließend zeigen wir auf, wie diese Resultate in verschiedene Richtungen erweitert werden können.

Abstract

Nonperturbative effects are crucial to fully understand the dynamics of quantum field theories including important topics such as confinement or black hole evaporation. In this thesis we investigate two systems where nonperturbative effects are of paramount importance. In the first part we study the dynamics of non-abelian gauge theories, while in the second part we try to shed light on mysterious properties of black holes using a model proposed earlier by Dvali and Gomez.
Non-abelian gauge theories are the central element in the standard model of particle physics and many dynamical aspects remain elusive. $\mathcal{N} = 1$ supersymmetric Yang-Mills theories with $SU(N_C)$ allows for domain walls with several curious properties. They are expected to have gauge fields with a Chern-Simons (CS) term living on their worldvolume, while in the 't Hooft limit of a large number of colors many of their properties seem reminiscent of string theoretic D-Branes. Similar domain walls were also conjectured to be present in non supersymmetric Yang Mills theories. In our work, we investigate this problem from several points of view. We construct a toy model of how to localize a gauge field with a CS term on a domain wall extending earlier work by Dvali and Shifman. We then derive the peculiar properties of CS terms in terms of effects of the underlying microscopic dynamics. Then we look at the actual theory of interest. Here the main novelty is the focus on the topological part of the Yang-Mills theory allowing us to make robust statements despite working in a strongly coupled theory. We construct the low energy effective action of both the non-supersymmetric as well as the supersymmetric Yang Mills theory, which due to the presence of a mass gap is a topological field theory. This topological field theory encodes the Aharanov-Bohm phases in the theory as well as phases due to intersection of flux tubes. In this topological field theory we see that the worldvolume theory of domain walls contains a level N_C CS term. The presence of this term was already

conjectured in ealier works based on string theoretic constructions. Here we give its first purely field theoretical construction. Within this construction we also illuminate differences between domain walls in the supersymmetric and non-supersymmetric case.
Lastly we try to relate the effects observed to similar effects in critical string theories and we also speculate on whether the behaviour of these domain walls is due to an analog of the fractional quantum hall effect.
In the second part of this thesis we investigate non-perturbative aspects of black hole physics. Here we consider a model for a low energy description of black holes due to Dvali and Gomez, where black holes are described in terms of a Bose-Einstein condensate (BEC) of weakly interacting gravitons near a quantum critical point. We focus on nonperturbative properties of a system of attractively self-interacting non-relativistic bosons, which was proposed as a toy model for graviton BECs by Dvali and Gomez. In this thesis we investigate this system mostly relying on a fully non-perturbative approach called exact diagonalization. We first investigate entanglement properties of the ground state of the system, showing that the ground state becomes strongly entangled as one approaches the quantum critical point. In order to make this notion precise we introduce the notion of fluctuation entanglement. We then compute it in a Bogoliubov analysis and extract it from the exact diagonlization procedure as well. We also consider the real time evolution of the system. Here we are interested in finding an analog of the conjectured fast scrambling property of black holes originally introduced by Hayden and Preskill. We only consider the weaker notion of quantum breaking and show that the toy model has a quantum break time consistent with the fast scrambling time scale conjectured in the black hole context. We then conclude by pointing out several possible extensions of these results.

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Chapter 1

Introduction and Outline

General relativity and quantum field theory are two of the most important advances in physics in the 20th century forming the central pillars of our current description of reality. On the other hand both can be traced back to the early beginning of natural sciences. The concept of space, which is the key player in general relativity was first discussed in the celebrated Elements by Euclid over two centuries ago, while the idea that objects consist out of particles dates back to Demokrit. From there it took mankind over two centuries of experimentation and thinking to arrive at our current description of nature, which is able to describe effects ranging from the dynamics of the universe as a whole to the behaviour of quarks inside a proton.
The commonly used description of quantum field theories in terms of Feynman diagrams rests on a perturbative description. For quantum electrodynamics this description works extremely well, for example the measured electric dipole moment of the electron matches the value predicted by quantum electrodynamics to 12 significant digits. The need for non-perturbative effects in quantum field theories was discovered early after the inception of quantum field theory and dates back to a seminal paper by Freeman Dyson from 1952 [1]. Dyson argued convicingly that the perturbation series for almost any physically relevant quantum field theory has to have zero radius of convergence and therefore should be understood as an asymptotic series. This implies that in order to render the perturbative expansion meaningful there should be additional effects which are not captured by the perturbative expansion. Over 60 years after the discovery of non-perturbative effects there exists now a plethora of methods which are able to quantify them. One approach is to start from a perspective which does not take into account per-

turbation theory at all, but instead tries to reduce the problem to a simpler problem with a finite configuration space using some type of coarse graining and then doing numerical computations, numerical studies of lattices gauge theories as well as numerical exact diagonalization would be examples of this. Another approach is to use topological and symmetry properties of the theories in order to gain a certain type nonperturbative information which is protected by topology or symmetry. A good example for this is the computation of the ground state degeneracy of a quantum hall system, which depends only on the topology of the sample.

1.1 Nonabelian Gauge Theories

Gauge theories provide a very interesting playground for nonperturbative effects. The possible interactions of massless spin-1 particles are much more constrained than the interactions for scalar fields, however they lack the notorious difficulties of theories containing spins higher than one, like absence of local operators in theories of quantum gravity. Furthermore nonperturbative effects that are still poorly understood are very important for the problem of confinement in quantum chromodynamics, which has obvious implications for our real world. Yang-Mills theories typically contain various objects with quantum numbers protected by topological considerations. Due to the robustness of topological properties against perturbation those provide a clear window into nonperturbative physics. The well known objects of this type include domain walls, magnetic monopoles and instantons. Out of these, magnetic monopoles are conjectured to play a pivotal role in the description of confinement, which is thought to be due condensation of magnetic monopoles(see [2] and references therein). In this picture confinement is the analog of Meissner effect, which in this case squeezes chromoelectric flux into flux tubes. In this thesis we will work out some further consequences of this magnetic Meissner effects for the low energy dynamics of supersymmetric as well as non-supersymmetric Yang-Mills theories. In particular we study Domain Walls encountered in pure Yang-Mills theories and show that they contain topological degrees of freedom on their worldvolume. Furthermore we show how the non-perturbative objects encountered in Yang-Mills theories can be naturally interpreted in terms of analogs of D-Branes in noncritical string theory.

1.2 Black Holes and Information Paradox

After discussing non perturbative physics in the context of nonabelian gauge theories we will now discuss aspects of gravity. The most puzzling object in gravity is certainly a black hole with two of the big open questions being the origin of black hole entropy and the so called information paradox. According to an argument by Bekenstein black holes should have an entropy [3–5], which is proportional to the horizon area of the black hole measured in planck units. A clear microscopic interpretation of this entropy has not been found so far. There have been microscopic computations of the degeneracy of extremal black holes in the context of string theory, which provides an ultraviolet completion to general relativity. This approach has been pioneered in a seminal paper by Strominger and Vafa [6], where they compute the degeneracy of BPS black holes using an intricate series of very indirect arguments. However due to their indirect nature, these computations do not give a clear interpretation of the degeneracy, however they provide a clear check as to the correctness of associating an entropy to black holes. It is not clear whether these arguments can be extended to Schwarzschild black holes. The information paradox [7] arises when quantizing a field theory on a black hole background. Hawking has shown [8], that due to quantum fluctuations a black hole will emit particles with a thermal spectrum with a temperature proportional to the inverse Schwarzschild radius $T_H \sim R^{-1}$. The information paradox arises when taking this result literally. It would imply that time evolution in a theory of quantum gravity should not be unitary. The black hole can be formed from a pure state, however if the black hole completely evaporates we are left with a thermal state, which is mixed. A unitary time evolution leaves all eigenvalues of the density matrix invariant, this implies that the corresponding time evolution can not be unitary. In order to resolve this issue there are two approaches one can take. We could take this as an indication that general relativity is incomplete and needs to be extended to a more complete theory like string theory. The other approach would be to conclude that the semi classical analysis of Hawking misses parts of the relevant physics. Since Hawking's calculation is strictly valid only in the case of an infinite black hole mass and therefore infinite black hole life time going beyond a semi classical treatment might resolve the paradox.

Dvali and Gomez [9–14] proposed to describe black holes as Bose-Einstein condensates of many gravitons with a wavelength given by the Schwarzschild radius. The individual gravitons in the condensate are weakly interacting, however due to their large number there are strong collective interactions. Hawking evaporation is thought to be due to so called quantum depletion of the condensate. It was argued that the collective interactions are in fact so strong that the condensate should be close to the critical point of a quantum phase transition. The proximity to a quantum phase transition also explains why quantum effects are more relevant than naively thought and why the semiclassical treatment might miss relevant effects.

In order to describe the graviton condensate one would need to study Bose Condensation in a derivatively interacting quantum field theory of massless spin-2 particles, so far this proved to be an insurmountable task. In this thesis we will focus on a non-relativistic toy model proposed by Dvali and Gomez, which captures certain aspects of the black hole condensate picture. To be precise we find that the ground state close to the critical point is very strongly entangled. Furthermore we show that after preparing the system in a classical state the system ceases to be well described by a classical description on a very short timescale, i.e. the so called quantum break time is very short. In fact the quantum break time for the toy model is consistent with the so called scrambling time scale in the black hole context.

1.3 Outline

This work is split into three main parts, the first parts discusses non-perturbative aspects of the dynamics of Yang-Mills theories, the second part contains a discussion of aspects of black hole dynamics in the context of the graviton condensate picture proposed by Dvali and Gomez while the last part of the thesis consists of reprints of the peer reviewed publications of the author.

The first part starts with a review of the dynamics of Yang-Mills theories and their behaviour in the so called 't Hooft limit of a large number of colors. We also review the relation of the 't Hooft expansion to the genus expansion known from string theory. From there it proceeds with a short review of non local operators and topological field theories. Following this there is a review of how to localize massless gauge fields on topological defects as well as a short description of the mechanism for localizing topologically massive gauge fields on domain walls invented by the author. The corresponding more detailed explanation can be found in the third part of this thesis. The next section deals with a topological field theory description of the low energy dynamics of ordinary Yang-Mills theories as well as discussing the behaviour of domain walls appearing in these theories extending previous results by Seiberg and collaborators. Here we will also be able to shed light on the appearance of topological degrees of freedom on the world volume of these walls. The subsequent two sections are devoted to an extension of these results to supersymmetric gauge theories. We first review the properties of supersymmetric Yang-Mills theories and the existing computations of the tension of domain walls in supersymmetric Yang-Mills theories. Then we follow up with an extension of our work in the previous chapter to show that there are topological degrees of freedom living on these domain walls as well. We conclude this part with a summary of the results achieved in this thesis with respect to domain walls in Yang-Mills theory and furthermore show how several puzzling properties of Yang-Mills theories can be seen to have natural analogs in critical string theories. We also point out a striking analogy to fractional quantum hall systems. Parts of the second part will be basis for an upcoming publication together with Markus Dierigl.

In the second part of the thesis we start by reviewing well known properties of black holes encountered in a classical as well as a semiclassical treatment of general relativity. Here we also discuss the well known information paradox as well the so called fast scrambling conjecture. We also discuss the importance of black holes for the high energy behaviour of theories of quantum gravity. Then we follow with an introduction of a model for black holes by Dvali and Gomez. Here black holes are described as Bose-Einstein condensates of gravitons at a quantum critical point. We then discuss a toy model consisting of nonrelativistic bosons that captures aspects of the relevant physics. Then we sketch the results of the author's two publications which study entanglement properties and the physics of fast scrambling in this toy model. We conclude with a discussion of possible interesting extensions of this work.

The final part consists of ad verbatim reprints of three of the author's published publications, where one [15] is concerned with localization of gauge fields with a Chern Simons interaction. The second [16] studies entanglement properties of the ground state of a toy model for black holes described as a graviton condensate. The final publication [17] is focused on dynamics of this toy model, here we study how the system evolves from an initial classical state to an entangled non classical state and quantify the corresponding time scale.

Chapter 2

Branes in Field Theory

2.1 Yang-Mills Theory

In this section we are going to describe some of the features of non-abelian gauge theories, non-abelian gauge theories date back to the 1950's and were introduced by Yang and Mills. [18] However they were largely ignored, since the classical lagrangian describes a set of massless interacting gauge fields, as there is no evidence for any long range forces besides electromagnetism and gravity. The great sucess of the parton model [19] for the description of meson as well as hadron physics led to the development of QCD, which is a particular non-abelian gauge theory. Here quantum mechanics gives the way out of the paradox of the absence of a long range force: due to quantum fluctuations the strength of the gauge coupling depends on the length scale at which the theory is probed. In the IR the coupling is strong therefore the conclusion that there should be a long range force turns out to be wrong. Since the coupling is strong the theory can not be treated semiclassically and any conclusion derived from semiclassical treatment shouldn't be trusted. This implies, that the absence of massless states in the spectrum of Yang Mills theories is in principle consistent. The absence of massless gauge bosons in the spectrum of the theory goes under the name of confinement and has not been understood analytically even today.

2.1.1 Basics of YM theory

In this section we will review a few of the basics about nonabelian gauge theories and also review some non-perturbative issues, which give the conventional picture of confinement. Furthermore we will explain how an additional term, the so called θ-term can be added to the action of Yang Mills and how it affects the physics. Then finally we will review how Yang-Mills simplifies in a particular double scaling limit in which the number of colors is sent to infinity.

The Lagrangian of Pure YM

Now we want to describe how one can construct Yang-Mills theory as the most general action describing massless interacting spin-1 particles at sufficiently low energies, the following discussion sketches the main ideas, a more detailed discussion can be found in Weinberg's book. As a warmup we are going to start by looking at a single massless spin-1. First we should choose the representation of the corresponding field under the Lorentz group, since we want to describe a spin-1 particle the simplest possibility to consider would be a vector field A_μ. In order to appreciate the constraints imposed by masslessness one should recall that massless particles are described by helicity using Wigner's approach of the little group. Here we have to remember that this forces part of the Lorentz group to act trivially on the states, for example if we have a particle with a three-momentum in the z direction, then we can boost in the y direction and subsequently rotate in the yz plane such that the 4-momentum stays invariant, Wigner tells us that these kind of transformation should act trivially on the corresponding Hilbertspace. However these transformations necessarily change the vector field, the only way to keep a Lorentz covariant description of our system is now to declare that the original and the transformed field lead to the same quantum state, i.e. we have to introduce a redudancy into our description. A more precise derivation shows that this redundancy is given by

$$A^\mu \to A^\mu + \partial_\mu f, \tag{2.1}$$

for a general function f, that decays sufficiently quickly at infinity. Demanding that the Lagrangian respects this redundancy, contains no more than two derivatives and no operators of dimension larger than 4 leads to

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.2}$$

where we defined the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the factor of $1/4$ is a normalization factor. Now we can try to couple our spin-1 field to additional degrees of freedom, enforcing again, that the relevant part of the Lorentz group acts correctly, strongly constrains the interactions to be gauge invariant, where now the additional fields, that interact with the spin-1 field also transform under the gauge redundancy. If the additional fields have spin-1 as well we see that now all the spin-1's transform under the redundancies of the other spin-1's. It turns out that the only consistent way to couple them is, when the individual redundancies combine into a non-abelian Lie group G. This leads to Yang Mills theory.

In order to conveniently describe the Yang Mills theory we combine all the individual spin-1 gauge fields into a non-abelian gauge field A, which takes values in the adjoint of the Lie algebra g of G. We can relate this to the individual gauge fields via

$$A_\mu = A_\mu^a T^a \quad , \tag{2.3}$$

where T^a denote the generators of the Lie group in some convenient representation R, for the case of $SU(N)$ we take the fundamental representation, here the index a runs from 1 to $\dim(G)$. The normalization of the generators is determined by the index $T(R)$ of R

$$\mathrm{Tr}\left(T^a T^b\right) = T(R)\delta^{ab} \ , \tag{2.4}$$

it is 1/2 for the fundamental representation of $SU(N)$. In order for the transformation generated by the T^a to form a representation of G, they have to satisfy the commutation relations

$$\left[T^a, T^b\right] = i f^{abc} T^c \ , \tag{2.5}$$

wheref^{abc} are the structure constants of the Lie algebra g.
In all the discussion that follows we will choose the fundamental representation as our representation R. Under gauge redundancies given by the group element U the gauge field changes as

$$A_\mu \to U^{-1} A_\mu U + U^{-1}\partial_\mu U. \tag{2.6}$$

The antisymmetric field strength tensor of a non-Abelian gauge field is given via

$$F_{\mu\nu} = F^a_{\mu\nu} T^a = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu\right) T^a. \tag{2.7}$$

The dual field strength tensor is defined as

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \ . \tag{2.8}$$

This leads to the Lagrangian for pure Yang-Mills

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} \ , \tag{2.9}$$

with the gauge coupling constant g. This differs from the form that is presented in most textbooks, by a rescaling of the gauge field by the coupling. It leads to non-canonical kinetic terms, which has to be taken into account carefully when deriving Feynman rules for example. One should note that after writing the action in this form, g^2 plays the same role as $\hbar$, making it clear why semiclassical methods are not appropiate to use at strong coupling. The classical equations of motion are given by

$$D_\mu F^{\mu\nu} = \left(\partial_\mu F^{a\mu\nu} + f^{abc} A^b_\mu F^{c\mu\nu}\right) T^a = 0 \ , \tag{2.10}$$

with the covariant derivative

$$D_\mu = \partial_\mu - i A^a_\mu T^a \ , \tag{2.11}$$

in the case of a single spin-1 field this simply reduces to two of the Maxwell equations. Furthermore, the field strength tensor fulfills the Bianchi identity

$$D_\mu \tilde{F}^{\mu\nu} = 0 \ , \tag{2.12}$$

which for a single spin-1 gives the remaining Maxwell equations. Let us mention one more important point, in the previous section we were careful to point out, that the gauge redundancies should decay sufficiently quickly at infinity. The transformations, that don't decay at infinity are actually global symmetry transformations of the theory. So in a gauge theory, gauge transformations always contain a part that is just a redundancy as well as a part, that is really a bona fide global symmetry. A particularly important feature is the running of the coupling constant with energy, which leads to asymptotic freedom [20]. For high energies the coupling constant decreases and one can describe the theory using standard perturbative methods, asymptotic freedom is the main pillar in the justification of the parton model. After solving the renormalization group equations for the running coupling in pure Yang-Mills we get

$$\frac{g^2(M)}{8\pi^2} \approx \frac{1}{\beta_0 \log\left(M/\Lambda\right)} \ , \quad \text{with} \ \ \beta_0 = \frac{11N}{3} \ . \tag{2.13}$$

Λ is a dynamically generated scale, which signals the breakdown of the weak coupling approximation, which is the energy scale where confinement should take place, it is given by

$$\Lambda \approx M \exp\left(-\frac{24\pi^2}{11Ng^2}\right) \ . \tag{2.14}$$

We see, that even though we started with a theory without any mass scale, we end up with a mass scale generated in the full quantum theory, i.e. the classical scale symmetry is anomalous. At high energies the pure gauge theory is well described by weakly interacting gluons, at low energies composite objects called glueballs, which are made from gluons will be the physical low energy degrees of freedom. These bound states will always be singlets under the global SU(N) symmetry, the lightest of these with a mass $M = \mathcal{O}(1)\Lambda$.

This phenomenon is called confinement and there is no analytical explanation for it, however a sucessful idea is monopole condensation, which can also be seen to be at work in various supersymmetric cousins of YM.

The θ-Term

There is an additional operator of dimension four, which is both gauge and Lorentz invariant, the so-called θ-term

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \quad . \tag{2.15}$$

θ turns out to be an angle, i.e. if we add this term to the Yang-Mills action, we should get the same physics if we change it by 2π. This term doesn't influence the equations of motion and it also cannot be seen in any finite order in perturbation theory, since it can be written as a total derivative,

$$F^a_{\mu\nu} \tilde{F}^{a\mu\nu} = \partial_\mu K^\mu \quad , \tag{2.16}$$

with the topological Chern-Simons current

$$K^\mu = 2\epsilon^{\mu\nu\alpha\beta} \left(A^a_\nu \partial_\alpha A^a_\beta + \frac{1}{3} f^{abc} A^a_\nu A^b_\alpha A^c_\beta \right) \quad . \tag{2.17}$$

We can use this current to define a corresponding topological charge Q_{top}, which measures the winding or instanton number of the field configuration, e.g. [21]

$$Q_{\text{top}} = \frac{1}{32\pi^2} \int \mathrm{d}^3 x K_0 \quad . \tag{2.18}$$

For an $SU(N)$ gauge theory charge turns out to be an integer for any field configuration, that is pure gauge at infinity, i.e. that can be written as $A_\mu = U^{-1}\partial_\mu U$. If we are only interested in finite action configurations, this seems to be a necessary condition and therefore we would conclude, that θ should be periodic. It can also easily be checked that this term violates both P and CP.

Quarks

Now we will consider how the presence of massless fermions changes the previous discussion, we will call the fermions quarks. We introduce N_f flavours of quarks in some representation R. These are described by a Dirac spinor ψ_f and labeled by a flavor index f, in all that follows we supress the gauge indices. The fermionic part of the Lagrangian is just the usual Dirac Lagrangian with the partial derivative replaced by a gauge covariant derivative

$$\mathcal{L}_f = i\bar{\psi}_f \not{D} \psi_f \ . \tag{2.19}$$

In the presence of Dirac fermions the beta functions changes to

$$\beta_0 = \frac{11}{3} T(A) - \frac{4}{3} T(R) n_f \ . \tag{2.20}$$

For $SU(N)$ and N_f fundamental quarks this becomes:

$$\beta_0 = \frac{11}{3} N - \frac{2}{3} n_f \ , \tag{2.21}$$

which reduces to the usual result for QCD. A Dirac spinor can be decomposed into two Weyl spinors and since our fermions are massless there is no mixing between them, i.e. we have a

$$U(n_f)_L \times U(n_f)_R \tag{2.22}$$

global flavour symmetry. However in QCD we know that there is a quark condensate, which breaks the chiral symmetry to the diagonal subgroup, i.e. we have the breaking pattern

$$U(n_f)_L \times U(n_f)_R \to U(n_f)_{\text{diag}} \ . \tag{2.23}$$

Since we have spontaneous symmetry breaking, we expect massless Goldstone bosons, in QCD these will be pions and kaons, one should note that in QCD these are only pseudo Goldstone bosons, since the the quarks have a bare mass, which gives a small explicit breaking of chiral symmetry. There is one important caveat here, the naively expected Goldstone boson corresponding to the central $U(1)$, the η' is heavier than expected, in order to understand this we have to turn to the chiral anomaly. The transformation corresponding to the broken central $U(1)$ is

$$\psi_f \to e^{i\gamma_5 \alpha} \psi_f \ . \tag{2.24}$$

Classically the theory is invariant under this transformation, however quantum corrections do not conserve the chiral current

$$j_5^\mu = \bar{\psi}_f \gamma^\mu \gamma_5 \psi_f \ . \tag{2.25}$$

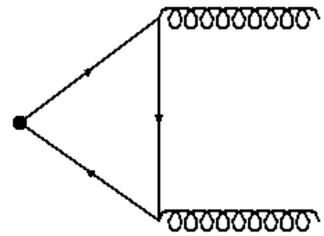


Figure 2.1: Triangle diagram causing the chiral anomaly, the black dot marks a divergence of the chiral current, $\partial_\mu j_5^\mu$

The effect is due to the triangle diagram depicted in figure 2.1. Its contribution can be calculated as, see e.g. [22]

$$\frac{n_f}{16\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \quad . \tag{2.26}$$

So we see, that at the quantum level there is an additional explicit breaking, which we can expect to give additional contributions to the mass of the η'. Therefore we can potentially still interpret the η' as a pseudo goldstone boson, despite it's surprisingly large mass. In order to make sense of this idea however we have to find a way to vary some parameter, such that we can turn off the anomaly. The large N expansion we will look at later provides precisely that. This also implies, that the θ angle in QCD is not observable, only a linear combination of the phase of the quark condensate and the bare θ angle is observable.

2.1.2 Axions

We saw that the θ term in a YM theory violates the CP symmetry, however we don't observe this kind of CP violation in nature and there is a strong bound on the size of θ_{eff} $\mathcal{O}(10^{-9})$, see [23].
A particular solution to this problem is to introduce a new pseudoscalar field, a, the axion. The axion couples to the same combination of gauge fields as the θ-angle and dynamically sets the relevant parameter $\theta + a$ to zero. It was first suggested by Weinberg [24] and Wilczek [25] following up on the work of Peccei and Quinn [26, 27]. The idea is to make θ into a field and then to explain why the potential of this field should have a minimum at a CP conserving value.
We will also use the axion to discuss the so called Witten effect [28], i.e. that

magnetically charged particles pick up electric charges if θ is varied. This can be easily understood in terms of axionic electrodynamics, this explanation was discovered by Wilczek [29] and we will follow his discussion closely.

The Axion and its Consequnces

The axion transforms as a pseudoscalar field, that couples to the Pontryagin density

$$\frac{a}{32\pi^2 f_a} F^a_{\mu\nu} F^{a\mu\nu} \ , \tag{2.27}$$

where f_a is the axion decay constant.
The kinetic term of the axion is

$$\mathcal{L}_{\text{ax}} \propto \partial_\mu a \partial^\mu a \ , \tag{2.28}$$

which is invariant under the shift symmetry

$$a \to a + c \ , \text{with} \ c \in \mathbb{R} \ . \tag{2.29}$$

We see, that the axion acts as an effective θ term and the observed effective θ_{eff} will be given by a combination of the vev of the axion and the actual θ parameter. In general quantum effects will generate a potential for the axion, this potential however will be symmetric under shifts of the effective θ angle by 2π. The observed θ will be small if it can be guaranteed, that the minimum of the axion is at a CP conserving value, this can be guaranteed by a theorem by Vafa and Witten [30].

The Witten Effect

In 1979 Witten showed that a non-zero value of the θ angle induces an electic charge for magnetic monopoles, turning them into dyons. Here we want to give a simple explanation for this using axionic electrodynamics, which was first discussed by Wilczek [29], here the axion plays the same role as the θ-angle.
We take the Maxwell Langrangian and add a topological term coupled to the axion

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{32\pi^2} a \tilde{F}_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{\text{ax}} \ , \tag{2.30}$$

where $\mathcal{L}_{\rm ax}$ contain the potential and kinetic term for the axion. The analog of Maxwell's equations in the absence of a source are now given by

$$-\frac{1}{e^2}\partial_\mu F^{\mu\nu} + \frac{1}{8\pi^2}\partial_\mu\left(a\tilde{F}^{\mu\nu}\right) = 0 \qquad (2.31)$$

and the usual Bianchi identity. Using Bianchi identity and writing the equations explicitly in electric and magnetic fields leads to the following expression for the analog of Gauss' law:

$$\vec{\nabla}\cdot\vec{E} = \frac{e^2}{8\pi^2}(\vec{\nabla}a)\cdot\vec{B}\ , \qquad (2.32)$$

where $\vec{E}$ and $\vec{B}$ represent the electric and magnetic field respectively.
Let us consider a spherical axionic domain wall with $\langle a\rangle = 0$ inside and $\langle a\rangle = 2\pi$ outside of the wall, depicted in figure 2.2. We then put a magnetic

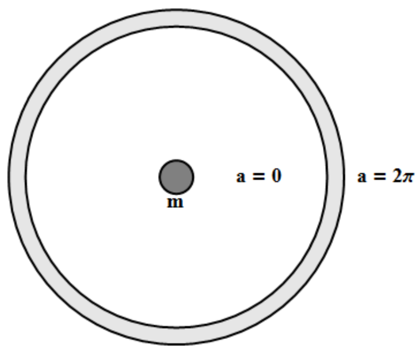


Figure 2.2: Setup for the illustration of the Witten effect

monopole of magnetic charge m in the center of the sphere, in the region $\langle a\rangle = 0$ there is a radial magnetic field. 2.32 now implies that a magnetic field, which is parallel to a gradient in the axion effectively acts as a charge for the electric field. This however is precisely what is happening here, therefore if we measure the electric field at ∞ we conclude that the monopole picked up a charge. Looking more closely we see, that the monopole picked up a charge of

$$\begin{aligned} q &= \int_V \mathrm{d}^3x\left[\vec{\nabla}\cdot\vec{E}\right] = \int_V \mathrm{d}^3x\left[\frac{e^2}{8\pi^2}(\vec{\nabla}a)\cdot\vec{B}\right] \\ &= \frac{e^2}{8\pi^2}\int_V \mathrm{d}^3x\left[2\pi\delta(r-R)\frac{m}{4\pi}\frac{\hat{e}_r\cdot\hat{e}_r}{r^2}\right] = \frac{1}{4\pi}me^2\ . \end{aligned} \qquad (2.33)$$

As in [28] we assume the fundamental magnetic charge to be that of a 't Hooft-Polyakov monopole, which we can get by breaking an $SU(2)$ gauge group to $U(1)$ and is $m = \frac{4\pi}{e}$. Hence for $\langle a \rangle \to \langle a \rangle + 2\pi$ it generates a fundamental unit of electric charge

$$q = \frac{1}{4\pi} m e^2 = e \ . \tag{2.34}$$

This can be generalized to non-abelian groups in a straightforward way.

2.1.3 Electric and Magnetic Charges in YM Theory

The local degrees of freedom of pure YM theories, i.e. the gauge fields A, take values in the Lie algebra g and hence are insensitive to the global structure of the gauge group. If the universal cover of the Lie algebra g is denoted by $\tilde{G}$, the theories giving rise to the same local degrees of freedom can be written as quotient

$$G = \tilde{G}/H \ . \tag{2.35}$$

H denotes a subgroup of the center, $\mathcal{Z}$, of the universal cover. The center is the subgroup of the elements in G that commute with all group elements.
In order not to confuse the important implications by staying too abstract we will restrict now to the Lie algebra $su(N)$. The universal cover is the Lie group $SU(N)$ the center is

$$\mathcal{Z}_{SU(N)} = \mathbb{Z}_N \ . \tag{2.36}$$

We will only be interested now in two relevant groups

$$G = SU(N), \quad \text{and} \quad G = SU(N)/\mathbb{Z}_N \ , \tag{2.37}$$

however one should note, that one could in principle mod out $SU(N)$ by a Z_q where q could be any divisor of N.
The center of $SU(N)$ consists of the elements

$$\mathcal{Z}_{SU(N)} = \left\{ \mathbb{1}_N \exp\left(2\pi i \frac{p}{N}\right), \ p = 0, \ldots, N-1 \right\} \ , \tag{2.38}$$

where $\mathbb{1}_N$ is the $N \times N$ identity matrix.
As mentioned, the local degrees of freedom of both theories are the same but we will be able to distinguish the theories by non-local operators discussed

later.
If we restrict our attention to confining theories (or theories completely higgsed by adjoint fields) we can label the physical charges and fluxes seen in the IR by elements of the center $\mathcal{Z}_G$. The fact that a confining theory sees only the transformation properties of the corresponding representation R under the center and is not sensitive to the precise representation can be easily seen as follows. The following is a summary of the arguments presented in [2], for the original papers relevant see references therein. Imagine a flux tube streched between two infinitely heavy probe charges in an arbitrary representation R of the gauge group G, see figure 2.3. The flux tube has a tension

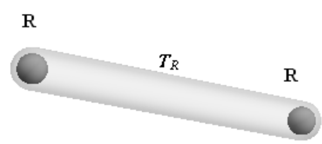


Figure 2.3: Flux tube spanned between heavy probe particles in representation R

T_R that depends on the representation R. By binding gluons, which are in the adjoint representation to the probe paricles we can construct any other representation with the same center charge. If the flux tube is very long, i.e. we are sufficiently deep in the IR these gluons will be subleading, when calculating the energy of the whole configuration which is dominated by the string tension T_R times the string length. Consequently, the string tension for long flux tubes does not depend on the representation of the probe particles, but just on the center charge, as we can just pick the representation leading to the lowest tension by adding gluons. Since adding adjoint fields does not change the charge under the center of the gauge group the only value characterizing the flux tube is its center charge.
Another subtle point is that physical flux tubes (e.g. chromoelectric fluxes in QCD) must not carry colorflux, because they should be gauge independent quantities. This implies for example that it is not correct to think of a meson in QCD as for example consisting of a blue quark and an antiblue antiquark, the correct description is as a quantum state, which is a linear superposition of all colors.
This implies that the relevant probe particles can be characterized by the center.

The magnetic charges and fluxes can be constructed similarly from the weights of the Langland-dual Lie algebra modulo their root lattice, see [31]. This is isomorphic to center $\mathcal{Z}_G$. It can also be understood as the first homotopy class of the gauge group, $\pi_1(G)$.
For the two interesting cases we find

$$\mathcal{Z}_{SU(N)} = \mathbb{Z}_N, \qquad \pi_1(SU(N)) = \mathbb{1} \ , \tag{2.39}$$

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_N} = \mathbb{1}, \qquad \pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N \ . \tag{2.40}$$

We therefore can specify the relevant charges and fluxes with a lattice $\mathbb{Z}_N \times \mathbb{Z}_N$. This is in turn relevant for confinement via Dyon/Monopole condensation as different condensates lead to different subsets of $\mathbb{Z}_N \times \mathbb{Z}_N$ being confined.

2.2 Gauge Theories at large N_c

In this section we are going to review the relevant basics of gauge theory in the limit of a large number of colours, the main idea is that the dynamics of the gauge theory will become a lot simpler to treat if we make the gauge coupling small and at the same time take the number of fields to infinity in order to keep the dynamics non-trivial. The more non-trivial point is that this procedure still manages to catch the relevant physics like for example confinement and chiral symmetry breaking. The idea that theories can become simpler in the limit when the number of some internal degrees of freedom becomes infinite goes back a long time and was introduced in the context of spin models in statistical physics in 1968 by Stanley [32], this was then applied to gauge theories by 't Hooft in 1973 [33]. Technically the point is that in a theory of matrix valued fields the number of fields will enter into the diagrams via combinatoric factors simply because we have to sum over intermediate states whose number grows with N. Here different diagrams will have different combinatoric factors associated to them and the idea of the large N expansion is to keep only those diagrams that give the largest combinatoric factors and hence should dominate the physics. This formed the basis of many important advances in field theory afterwards. In order to study the large N limit we write the Yang Mills action

$$S = \int d^4x - \frac{1}{4g^2}\mathrm{T}r F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}i\not{D}\psi. \tag{2.41}$$

In order to have a useful large N limit we should try to keep the QCD scale Λ_{QCD} fixed, to do this we look at the one-loop renormalization group equation for the coupling

$$\mu\frac{dg}{d\mu} = -b_0\frac{g^3}{16\pi^2} + \mathcal{O}(g^5),\; b_0 = \frac{11}{3}N - \frac{2}{3}N_F, \tag{2.42}$$

this equation will not have a sensible large N limit if we keep g fixed as we vary N since b_0 is $\mathcal{O}(N)$, this implies that masses of particles would be N dependent. From the RG equation we see that the natural quantity to keep fixed in the large N limit is $\lambda = Ng^2$, the RG equation for λ is given by

$$\mu\frac{d\lambda}{d\mu} = -\left(\frac{11}{3} - \frac{2}{3}\frac{N_F}{N}\right)\frac{\lambda^2}{32\pi^2}. \tag{2.43}$$

The position of the Landau pole for λ has a well defined large N limit, implying that this is a way to take the large N limit that has a chance to catch some of the non-trivial dynamics related to confinement.

In order to understand what this limit means diagramatically we can use a trick due to 't Hooft, where we track the color flow in the Feynman diagrams. The quark propagator is given by

$$\langle \psi^a(x)\bar{\psi}^b(y)\rangle = \delta^{ab} S(x-y), \tag{2.44}$$

this is represented by a single line, where the color at the beginning is the same as at the end due to δ^{ab}. For gluons it is more useful to think of them as $N \times N$ matrices with two indices in the N and $\bar{N}$ representation instead of treating them as fields with a single adjoint index., i.e. we define

$$A^a_{b\mu} = A^A_\mu (T^A)^a_b. \tag{2.45}$$

The gluon propagator can then be written as

$$\langle A^a_{b\mu}(x) A^c_{d\nu}(y)\rangle = D_{\mu\nu}(x-y)\left(\frac{1}{2}\delta^a_d\delta^c_b - \frac{1}{2N}\delta^a_b\delta^c_d\right). \tag{2.46}$$

For large N we can neglect the second term in the former relation, neglecting this term corresponds to replacing $SU(N)$ by $U(N)$. Now we are going to denote the gluon propagator by a double line, we see, that in terms of the flow of color in the diagrams we can think of a gluon as being composed out of a quark and an antiquark. The vertices can also be written using the double line notation. This allows to rewrite any Feynman graph as a sum of double line graphs. Let us illustrate that in a simple example. The diagram depicted in figure 2.4 has two vertices ($\propto g^2$) and one loop. The ingoing

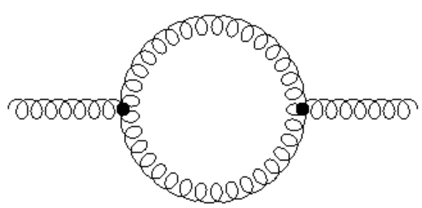


Figure 2.4: Basic gluon loop

and outgoing gluons must posses the same color decomposition. This leaves one free fundamental color index for the loop that has to be summed over.

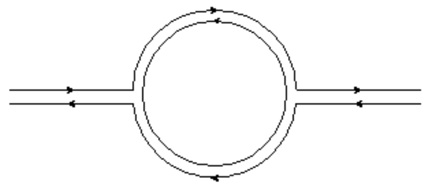


Figure 2.5: Double line representation of the diagram in 2.4, the directions of the arrows differ for quark and antiquark

Therefore, the overall contribution is of order $g^2N = \lambda$. In the double line notation the diagram transforms to figure 2.5 and the summation over the single color index in the inner loop becomes apparent. In general, we can think of each double line graph as a surface obtained by gluing together polygons at the double lines. Because each line has an arrow on it in $SU(N)$ and double lines have opposing arrows we can only construct orientable surfaces in an $SU(N)$ theory. For $SO(N)$ the fundamental representation is real, therefore the lines do not have arrows and we can construct non-orientable surfaces. The preceding discussion allows us to formulate counting rules for the order in N for a given graph. Each vertex gets a factor of N, while every propagator has a factor of $1/N$, furthermore every loop gives a factor of N since it represents a sum over N colors. In terms of the surfaces constructed by gluing polygons each closed loop corresponds to the edge of a polygon and to a face of the surface, using this one finds for a connected vacuum graph, i.e. no external legs

$$N^{V-E+F} = N^{\chi}. \tag{2.47}$$

Here V is the number of vertices, E is the number of edges, F is the number of faces and χ is a topological invariant known as the euler character. For a connected orientable surface this is given by

$$\chi = 2 - 2h - b, \tag{2.48}$$

where h is the number of handles and b is the number of boundaries. We see that we can use this result if we identify the edges with gluon propagators, the faces with loops and the vertices with the actual interaction vertices, we see that our large N counting exactly reproduces the euler characteristic. Since a quark is presented by a single line a closed quark loop is a boundary, therefore every quark loop brings a supression of $1/N$, for a surface without holes we have $\chi = 2 - 2g$, where g is the genus of the surface. So the leading contribution is coming from surfaces with the topology of a sphere. Let us translate this result back into the language of Feynman graphs. First we note that since this doesn't have any holes there will be no quark contributions. Furthermore we can do the following trick: Let us remove one face of the corresponding spherical surface and project the rest onto a plane, in this way we will obtain a planar graph. Furthermore it is geometrically easy to see, that this will only work for spherical surfaces, i.e. the leading contribution in the large N limit is given by planar graphs. In order to illustrate the planar expansion we would like to consider the standard diagrams for the correction of the gluon propagator and calculate their N and λ dependence, see e.g. [34] or [35]. The 1-loop and 2-loop planar diagrams are depicted in figure 2.6. The left diagram contains two vertices, two inner propagators, and one loop,

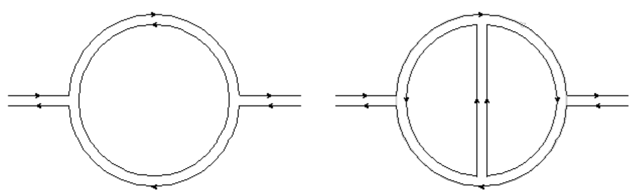


Figure 2.6: 1-loop and 2-loop contributions to gluon propagator in doubleline notation

thus its contribution is $g^2N = \lambda$. The diagram on the right has four vertices, four inner propagators, and two loops and is of order $g^4N^2 = \lambda^2$. Both of the diagrams are not $1/N$ suppressed since they are planar. In figure 2.7 the simplest non-planar diagram is presented. It cannot be drawn in a plane

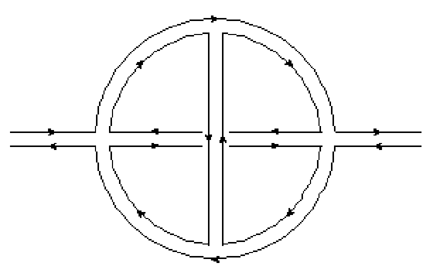


Figure 2.7: Simplest non-planar diagram in doubleline notation

without intersections and there are six vertices, six inner gluon propagators, but only one single loop, which determines the contribution $g^6N = \frac{1}{N^2}\lambda^3$. It is indeed suppressed by a factor $1/N^2$.

The vacuum energy density is calculated to first order by the contribution of vacuum-vacuum amplitudes. The dominant term is due to genus $g = 0$ surfaces and leads to

$$N^{\chi} = N^{2-2g} = N^2 \ . \tag{2.49}$$

This coincides with our expectation that the vacuum energy density is proportional to the degrees of freedom of the theories. Since we can understand the $1/N$ as an expansion in the genus it is natural to try to interpret this as some kind of string theory, where we try to interpret the surfaces as string world sheets. The genus expansion in string theory corresponds to an expansion in the string coupling g_{st}. This leads to the conjecture that QCD in the large N expansion is related to a string theory with $g_{st} \sim N^{-2}$. [33] Furthermore we can see that quark diagrams come with factors of $1/N$ as compared to purely gluonic diagrams, where the expansion is purely in terms of $1/N^2$, i.e. with quarks we see a coupling of $\sqrt{g_{st}}$ emerge. This is also natural in terms of string theory, where the coupling of open strings g_{open} is related to the closed string coupling via

$$g_{\mathrm{st}} = g_{\mathrm{open}}^2. \tag{2.50}$$

This implies that in the string theory language the difference between QCD and pure gluodynamics should be, that one contains open strings, while the other contains only closed strings.

2.2.1 Mesons and Glueballs

In order to get a further understanding of the stringy interpretation of large N gauge theories we will now investigate the phenomenology of gluons and mesons. Since in the limit we took kept Λ_{QCD} fixed we can expect confinement to survive in the large N limit and furthermore that typical meson and glueball states will have masses of $\mathcal{O}(1)$. First of all let us note that the N counting rules alluded to earlier also apply to Green's functions. We can use this to get the effective interactions of glueballs and mesons by looking at Green's functions for operators which have an $\mathcal{O}(1)$ probability of creating glueballs and mesons respectively. This leads to a scattering amplitude for glueballs which is $\mathcal{O}(1/N^2)$, while it is $\mathcal{O}(1/N)$ for mesons. This suggests to identify glueballs with closed strings and mesons with open strings.

2.2.2 Baryons

Now we are going to consider baryons, a baryon is a state of N quarks, that is completely antisymmetric in color. Baryons in the large N expansion were first studied by Witten [35] and here we will closely follow his exposition. There are two big differences to mesons, the first is that for mesons we always consider the same diagrams but with different combinatoric factors associated to them, however for baryons we have to consider different diagrams for different values of N. Second in gluon exchange amplitudes we don't get an enhancement due to index sums, since the quarks have to remain in a color singlet state, however we get a combinatoric enhancement due to the N quarks in the Baryon. Let us for example look at the one gluon exchange between a pair of quarks in the baryon, the coupling constant for this is of order $1/\sqrt{N}$, however there is an enhancement, since the exchange could have happened between any two quarks in the baryon. Since there are $1/2N(N-1)$ such pairs the total contribution of one gluon exchange diagrams will be of order

$$N^2 \frac{1}{N} \sim N, \tag{2.51}$$

i.e. these grow linearly in N, so it seems perturbation theory does not work. The same will hold in higher orders, for example if we look at two gluon exchange this will also grow like N. This will hold for all the connected n-gluon tree exchanges.[1] This implies that at large N all the $k - body$ interactions are important, and give contributions $\mathcal{O}(N)$, therefore the baryon mass can have a smooth large N limit if it is $\mathcal{O}(N)$. The baryon is completely antisymmetric in the color indices, this implies that their wavefunctions in space, spin and flavour are symmetric, i.e. after stripping of the trivial colour dependence they behave effectively as bosons. So we are led to a system of a large number of weakly interacting bosons which can be naturally treated using Hartree approximation, i.e. we should be able to regard each quark as moving in a self-consistent potential generated by the other quarks.

In order to simplify the problem we are going to take heavy quarks, since in this case we can do a non-relativistic Hartree treatment which is considerably simpler than the relativistic version, however it can be checked in lower dimensions that this gives the same answer on a qualitative level. Furthermore it will be evident that the same logic can be applied in the relativistic case. The baryon we are considering is a many body bound state

[1] Here n should be small compared to N

with a binding energy that small compared to Nm, where m is the mass of the quark. The total Hamiltonian for this system has the form

$$H = Nm + \sum_a \frac{-\Delta_a}{2m} + \frac{1}{2N} \sum_{a \neq b} V^{(2)}(x_a, x_b) + \frac{1}{6N^2} \sum_{a \neq b \neq c} V^{(3)}(x_a, x_b, x_c) + \dots \tag{2.52}$$

where $V^{(n)}$ denotes the n-body potential. In the Hartree approximation[2] the wavefunction for the baryon factorizes into a product of one-body wavefunctions, for simplicity we will consider the baryon completely symmetric in flavour and spin (this would be the analog of the Δ^{++} in QCD). In this case the spatial wavefunction will be totally symmetric, i.e.

$$\psi_b(x_1, \dots, x_N) = \Pi_i \psi_0(x_i). \tag{2.53}$$

Now the one body wavefunction ψ_0 is usually determined variationally, the answer will be

$$\langle \text{Baryon} | H | \text{Baryon} \rangle = N \left(m + \mathcal{O}(1) \right), \tag{2.54}$$

since each m-body interaction gives a contribution of order N. Since N factors out the one body wave function will be independent of N in the large N limit, this means that while the mass of the baryon is $\mathcal{O}(N)$ it's size is $\mathcal{O}(1)$. This also explains why Hartree approximation is valid, since as we have seen as N gets larger the density of quarks grows, meaning that each quarks interacts with many others. This again has an interesting interpretation in terms of the string picture. That the mass of Baryon goes as N suggests that it behaves like a soliton of mesons, which in our stringy interpretation would correspond to a D0 brane. Actually in this case this can be made more precise by considering the chiral Lagrangian describing the interactions of pions in which the Baryon appears as a topological defect called skyrmion. [36]

2.2.3 Large N as a Classical Limit

In any gauge theory the natural observables are expectation values of (products of) gauge invariant operators. The large N limit can be understood as a peculiar kind of classical limit in a beautiful argument due to Witten [37].

[2] Here we will only sketch the Hartree approximation, since we only want to draw qualitative conclusions, in the second part of the thesis we review it in more detail.

For some gauge invariant operator X let us look at the expectation value of its square, i.e. $\langle X^2 \rangle$ and compare this to the square of its expectation value $\langle X \rangle^2$, for an operator which has an expectation value in leading order in large N we expect $\langle X \rangle \sim N^2$. This implies $\langle X \rangle^2 \sim N^4$, which in diagrammatic language can be understood easily since $\langle X \rangle^2 \sim N^4$ corresponds to two disconnected diagrams. The difference between these two quantities will in general be down by a factor of $1/N^2$:

$$\langle X^2 - \langle X \rangle^2 \rangle = 0 \, \mathrm{a}s \, N \to \infty. \tag{2.55}$$

This can be easily understood, since the difference is given in diagrammatic language only by connected diagrams, which only give one factor of N^2 instead of two, thereby implying that it vanishes in $N \to \infty$. The same argument can also be extended to products of gauge invariant operators, implying that the expecation value of a product of gauge invariant operators is equal to the product of the expectation values in large N. For example let X,Y,Z be gauge invariant operators, then

$$\langle XYZ \rangle = \langle X \rangle \langle Y \rangle \langle Z \rangle. \tag{2.56}$$

(2.55) and (2.56) show that gauge invariant operators in large N can be described as c-numbers instead of as quantum mechanical operators, precisely characteristic of a classical limit.

In terms of a path integral treatment this implies that only a single gauge field configuration should contribute to the path integral at $N = \infty$, furthermore using translation invariance we can choose a gauge where this field configuration is constant, i.e. Yang Mills theory at large N can be described by a set of 4 $N \times N$ matrices transforming as a Lorentz vector. This implies that any observable in $N = \infty$ Yang Mills should be described by knowledge of a single classical gauge field configuration, which precisely characterizes this as a classical limit, which however seems to be distinct from the usual classical limit related to weak coupling. However finding these matrices[3] has not been possible so far. This field configuration was originally introduced by Witten in [37] and has been dubbed Master field.

[3]actually since $N = \infty$ those should be thought of as linear operators on a Hilbert space instead of as matrices

2.2.4 Large N and θ

Next we are going to investigate how θ enters in the large N limit, the appearance of θ leads to surprising consequences in confining theories originally pointed out by Witten in [38]. In discussing how the θ term influences gauge theories at large N we have to distinguish several different cases. If our theory is weakly coupled the effect θ has on the theory is determined via instantons. Instantons or Anti-instantons give contributions proportional to $\exp(-8\pi^2/g^2)\exp(\pm i\theta)$. In the case of a spontaneously broken gauge theory for example we have a weak effective coupling and instantons have characteristic size given by the relevant inverse mass gap, in this case instanton effects can be reliably computed in an instanton expansion. However in the case of an asymptotically free gauge theory this can not be done, since there is no natural size for the instanton and instantons of all sizes contribute. This means especially that it is not possible to treat the instantons as a dilute gas so it is not clear what kind of effective description should be used. Moreover the instanton computation would suggest that in the large N limit the theta dependence should be exponentially supressed as $N \to \infty$. To be more precise they should be supressed as $\exp(-8\pi^2 N/\lambda)$, here again we see a similarity to D-Branes, since naively we would expect the action to be supressed with $\exp(-N^2)$, since the effective glueball coupling is $1/N^2$, so it is natural to interpret the instanton in the string theory analog as a $D-1$ brane. However due to the aforementioned difficulties we can expect that this conclusion is too naive, in fact both the way how the $U(1)$-problem is resolved in large N as well as the fact that the θ dependence can be seen in large N at leading order using current algebra techniques [39] can be seen as evidence that θ dependence should enter in leading order in $1/N$. This gains further support from certain two dimensional models with many features in common with QCD where θ dependence can also be seen in leading order in $1/N$ [38]. However this poses a serious problem, which can be seen as follows: In the large N limit we have to scale out an overall factor of N from the action, this suggests that in the large N limit we should keep θ/N fixed. To be more specific consider the vacuum energy which we naively expect to be proportional to N^2, i.e. we expect $E(\theta)$ to have the form

$$E(\theta) = N^2 h(\theta/N), \tag{2.57}$$

for some function h. Furthermore $E(\theta) = E(\theta + 2\pi)$, these two conditions however seem to be incompatible, since a smooth function of θ/N can only be invariant under $\theta \to \theta + 2\pi$ if it is constant. The most straightforward way to resolve this issue is that $E(\theta)$ is a multibranched function due to many candidate vacua, that all become stable for $N = \infty$, where the question which of these has lowest energy will depend on θ. In vacuum k the energy will be given by

$$E_k(\theta) = N^2 h((\theta + 2\pi k)/N). \tag{2.58}$$

In order to get the true ground state we have to minimize E_k with respect to k for a given value of θ, the actual vacuum energy will therefore be given by

$$E(\theta) = N^2 \min_k h((\theta + 2\pi k)/N). \tag{2.59}$$

Under CP we have $\theta \to -\theta$, since CP is a symmetry iff $\theta = 0, \pi$. Furthermore $E(\theta)$ will have its absolute minimum at $\theta = 0$, since exactly then the integral of the euclidean path integral is real and positive. Therefore this function will be periodic but not smooth since as every time θ/π is an odd integer two branches will cross. This picture actually implies that at these values of θ there are quantum phase transitions. Furthermore the solution of the $U(1)$-problem in the presence of light quarks suggests that $d^2h/d\theta^2 \neq 0$ at $\theta = 0$, if this is the case we can write $h(\theta) = C\theta + \ldots$, where the higher order terms don't contribute to the vacuum energy in leading order in $1/N$, this leads to the following vacuum energy

$$E(\theta) = \text{const} \cdot \min_k (\theta + 2\pi k)^2 + \mathcal{O}(1/N). \tag{2.60}$$

In order to study the stability of those different vacua, Witten tried to answer this question in [40] using methods from M-Theory. In general we are looking for a bubble nucleation process, so essentially we are looking for domain walls between different of these would be vacua and we are interested in their tension. In Witten's argument those correspond to D-Branes and the tension between neighboring vacua should scale as N for large N, while the difference in energy density is $\mathcal{O}(\infty)$, therefore their lifetimes go to infinity for $N \to \infty$. This implies that for $N = \infty$ we have an infinite number of stable "vacua". A further quite non-trivial point is that chromoelectric flux tubes can end on these domain walls. Since their tension scales like a D-Brane tension would, flux tubes can end on them it is very natural to suggest that these are a close field theoretic analog to D-Branes. One should note that the argument presented in for this relies on an embedding into string theory and an identification of the domain walls with actual D-Branes, therefore an understanding of this D-Brane analog from a field theory viewpoint would be highly desirable.

A field theoretic argument for the leading low energy part of the world-volume theory of these D-Brane analogs, as well as more qualitive arguments for other parts of their dynamics will be the main part of the following chapters.

2.2.5 Summary of large N

What we have seen in this chapter is that Yang Mills theory simplifies in the large N limit. In particular we have seen that at leading order in $1/N$ only planar diagrams contribute and expectation values of gauge invariant operators factorize. Furthermore we have seen, that it is natural to think of Yang Mills theory at large N as some kind of noncritical string theory, where glueballs behave like closed strings, while mesons behave like open strings. Furthermore we have seen that in this picture the mass of baryons is consistent with interpreting them as $D0$-branes, although here one should note, that the more natural interpretation is in terms of a soliton of mesons, which can be seen in chiral peturbation theory as a skyrmion. A similar situation arises in Type I string theory, here we have a 5-brane, as well as open strings in the bulk. The type-I 5-brane can be either seen as an open string soliton or as a D-Brane. So in string theories with bulk open string there is no sharp distinction between D-Branes and open string solitons. [41]

The open string sector gives a SO(32) gauge theory. We can take the instanton solution of 4 dimensional Yang-Mills theory and lift it to 10 dimensions, by simply making it extend into the other 6 directions, this gives us a six dimensional object. In this language this object is just a soliton of the open string sector, however the instanton has a scale modulus. If we shrink the instanton to zero size it can actually be understood as the 5-brane, so we see, that the question of whether a given object should be understood as a brane or a soliton is somewhat ambiguous if there is an open string sector and which description should be used depends on the problem.
After this we have seen that in pure Yang Mills we have several metastable vacua with domain walls interpolating in between them. These domain walls can be naturally interpreted as D-Branes in the string picture. It should be noted that in this case we are in a situation similar to type-II string theories, which are string theories only described in terms of closed strings, however

once D-Branes are taken into account open strings appear as well, whose ends are bound to the world volumes of the D-Branes. From a field theory perspective this seems particularly confusing, since we started with a theory just containing adjoint matter, however in order to describe the ends of flux tubes there should be matter in the fundamental representation. Another interesting point is that from D-Brane dynamics we would expect a $U(1)$ gauge field to be localized on a wall interpolating between neighboring vacua, however if this would be a massless gauge field and the relationship to string theory holds, then open-closed duality implies that there should also be a massless spin-2 glueball in the bulk, which is most certainly not the case. Furthermore if we interpolate between k vacua we would expect the theory on such a wall to be described by a $U(k)$ gauge theory. This is fairly surprising and suggests that there is a different mechanism at work here than the gauge field localization mechanism due to Dvali and Shifman [42], which we will discuss later. On top of this, there is also no clear candidate for condensing on the wall and thereby breaking the $SU(N)$.

In the end we are left with a very interesting analogy with string theory which if better understood might lead to a more detailed understanding of the microscopic dynamics of string theory. As we have seen many of the interesting features of D-Branes also arise in the field theory context, however here they seem deeply mysterious, in the remainder we will try to point out how field theoretic mechanisms might explain those. For this we will study these walls both in supersymmetric as well as in nonsupersymmetric Yang-Mills theory.

2.3 Non-Local Operators and TFT

In order to understand the structures appearing on the world volume theory of the walls as well as in the bulk of Yang-Mills theory we have to understand some basic properties of non-local operators and topological field theories. An important tool to investigate gauge theories are non-local operators. The line and surface operators are of special importance in four dimensions and are briefly reviewed in the following.
Subsequently, we state the relevant facts and mechanisms of topological field theories in various dimensions. Starting from $2 + 1$ dimensional Chern-Simons-theories, after some generalizations we describe the $3+1$ dimensional BF-theories that will be the foundation of our construction later.
To simplify the notation and avoid confusion caused by the vast amount of indices we switch to differential notation in this chapter.

2.3.1 Non-Local Operators

The classification of gauge theories concerning their different phases is an interesting topic by itself, e.g. [34]. In the present case our theories are believed to be in the confining phase, which dynamically generates a mass gap.
Nevertheless, there are further possibilities to destiguish various realization of this confining phase. A powerful tool to investigate the fundamental structure and properties of the theory are non-local operators, i.e. line and surface operators, [43] and [44]. We first review the electric line and surface operators connected to Wilson loops, introduced in [45] and their magnetic analogs, called 't Hooft loops. These allow us to destiguish between different global gauge groups that exhibit the same local degrees of freedom.
The differential geometry notation proves to be very useful, since p-form are the natural objects to be integrated over p-dimensional manifolds. Our notation will follow that in [46] and [43].

Electric Line and Surface Operators

For a gauge group G the Wilson line operator $\mathcal{W}_R(C)$ is constructed from a curve C and a representation R as

$$\mathcal{W}_R(C) = \mathrm{Tr}\left[\mathcal{P}\exp\left(iq\oint_C A_R\right)\right] \,, \tag{2.61}$$

where R is the representation of the gauge group and $\mathcal{P}$ denotes the path ordering operator. In everything that follows path ordering will be left implicit. In the case when C is a closed curve the operator will be called Wilson loop for obvious reasons. We can give a simple spacetime interpretation of the Wilson loop as follows: We create an particle antiparticle pair in representation R at some point of C, move the particle along half the loop in one direction, move the antiparticle along half the loop in the other direction and annihilate them again as in figure 2.8. In most cases we will be interested in

Figure 2.8: Illustration of a Wilson loop with infinitely heavy probe particles

Wilson loops in the fundamental representation.
Now let us see, why this is a relevant quantity. In order to do that we go to euclidean space and take a rectangle in spacetime with two of the sides aligned with the euclidean time direction. This can be interpreted as keeping the probe particles at a distance d for time T , the expectation value of the Wilson loop can then be expressed in terms of the potential energy, $V(d)$ between the two particles as:(see e.g. [47])

$$\langle \mathcal{W}_R \rangle = \langle \exp\left(-V(d)T\right) \rangle \ . \tag{2.62}$$

There are two behaviours relevant for the subsequent discussion, the exponent can be proportional to the perimeter of the loop C, which is $2(T+d)$ (perimeter law), or it can be proportional to the area enclosed by C, roughly RT (area law). In the second case the potential energy is proportional to the distance d, which is the expected behaviour for a flux tube with a finite tension, which connects the two charges (e.g. [34]), this means the charges are confined.

There is a problem with using this criterion for confinement in Yang-Mills theories with light fundamental matter, since once the probe charges are connected by a flux tube containing enough energy to create a pair of these light particles the flux tube will break and we end up again with a perimeter law at large distances. This actually means, that in the presence of light charged particles the terminology confinement is actually a bit misleading and should better be replaced by e.g. cloaking of color (see e.g. [2]), however in order to stay consistent with common literature we will nevertheless refer to the absence of color charged states in the physical spectrum of Yang-Mills as confinement even in the presence of light charged particles. Furthermore for most of our discussion this will be irrelevant since we are only considering pure yang-mills theories. For our later discussion it will be sufficient to consider Abelian theories, therefore we will from now on focus on the case $G = U(1)$. In this case the Wilson loop of charge q and can be expressed as

$$\mathcal{W}(C,q) = \exp\left(iq\oint_C A\right) \ . \tag{2.63}$$

Since we are only interested in the case of compact $U(1)$ we can assume q to be an integer. If the flux corresponding to the charges traversing C is confined to a two dimensional surface Σ, i.e. $C = \partial\Sigma$, we can be rewrite the Wilson loop into

$$\mathcal{W}(C,q) = \exp\left(iq\int_\Sigma \mathrm{d}A\right) = \exp\left(iq\int_\Sigma F\right) \ , \tag{2.64}$$

using a generalized Stoke's theorem for p-forms Ω and p+1 dimensional surfaces Σ_{p+1}

$$\int_{\Sigma_{p+1}} \mathrm{d}\Omega = \oint_{\partial\Sigma_{p+1}} \Omega \ . \tag{2.65}$$

We will now introduce electric surface operators. In order to measure the electric charge in a spacelike region $\mathcal{V}$ bounded by the 2-surface $\partial\mathcal{V}$ we can simply integrate the dual field strength, $\tilde{F} = *F$, over $\partial\mathcal{V}$, see [48]

$$Q \propto \oint_{\partial\mathcal{V}} *F \ , \tag{2.66}$$

this process can be thought of as encircling the charge with a magnetic flux tube and measuring the charge with the corresponing Aharanov-Bohm phase.

We can define an analog of the electric line operators of a connection 1-form A for 2-forms B,this is a surface operator corresponding to a 2-manifold Σ, which is defined analogously via

$$\mathcal{W}_S(\Sigma, \eta) = \exp\left(i\eta \int_\Sigma B\right) \ . \tag{2.67}$$

In order to construct the analog of a Wilson loop Σ should be a closed 2-surface (i.e. $\partial\Sigma = 0$). Instead of a particle this can be pictured by a flux tube sweeping out a two dimensional worldsheet embedded in the spacetime manifold (note the analogy to stings), see figure 2.9.

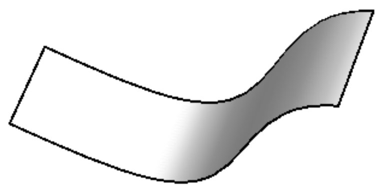


Figure 2.9: 2-dimensional worldsheet of a flux tube embedded in higher dimensional spacetime

Magnetic Line and Surface Operators

We consider instead of a pair of electric charges a pair of magnetic charges (monopole-antimonopole pair) and construct line operators in an analogous manner. These are called 't Hooft operators and were first considered in [49]. We do this by considering the dual, magnetic gauge field $\tilde{A}$ ($\tilde{F} = \mathrm{d}\tilde{A}$, where magnetic and electric field are interchanged compared to $F = \mathrm{d}A$). The 't Hooft loop is now given by

$$\mathcal{H}(C, m) = \exp\left(im \oint_C \tilde{A}\right) \ . \tag{2.68}$$

There can picture a 't Hooft operator in the same way as a Wilson loop. The magnetic charge in a spacelike region $\mathcal{V}$ can again be evaluated via a similar construction as in 2.66, [48]

$$Q_m = \frac{1}{2\pi} \oint_{\partial\mathcal{V}} F \ . \tag{2.69}$$

This implies that we can alternatively construct the 't Hooft operator by removing a cylindrical neighborhood from spacetime,this looks locally like $\mathbb{R}$ along the worldline times S^2 enclosing the monopole. We have to enforce (see [46])

$$\frac{1}{2\pi}\oint_{S^2} F = m \ , \tag{2.70}$$

on the boundary of the removed region. The familiar Dirac Quantization condition becomes for any closed 2-surface $\partial\mathcal{V}$, [50]:

$$\frac{1}{2\pi}\oint_{\partial\mathcal{V}} F \in \mathbb{Z} \ . \tag{2.71}$$

Equation 2.70 immediately implies, that $F = \mathrm{d}A$ can not be exactly true in the presence of magnetic charges, but should contain singularities along the world lines of magnetic charges.
Similar singularities are present for magnetic surface operators,here they are on the worldsheets of magnetic flux tubes and lead to a singularity of the field strength on the surface Σ, [43]

$$F = 2\pi\alpha\delta_\Sigma + \text{nonsingular} \ . \tag{2.72}$$

α is subject to a Dirac quantization condition. The δ_Σ is a 2-form, which is orthogonal to the world-sheet Σ,i.e. for an arbitrary two form Ω we get

$$\int \delta_\Sigma \wedge \Omega = \int_\Sigma \Omega \ . \tag{2.73}$$

Dyonic Operators

We can combine 't Hooft and Wilson line operators into a new class of non-local operators. These are similar in construction however instead of using probe electric charge we use dyons, i.e. particles with magnetic as well as electric charges.
The dyonic operator is simply a linear combination of a 't Hooft line and a Wilson line along the same curve C with electric charge q and magnetic charge m

$$\mathcal{D}(C, q, m) = \exp\left(iq\oint_C A + im\oint_C \tilde{A}\right) \ . \tag{2.74}$$

There is an obvious generalization of operators describing flux tubes carrying both electric as well as magnetic flux.

For line and surface operators there are certain properties that don't depend on the precise shape of the curve or surface in question, but only on topological properties like e.g. whether the curve describes a knot or whether the surface has intersections, these can naturally be described by topological field theory, which will be what we are largely interested in in what follows.

More general nonlocal operators

We can easily generalize this discussion, by considering more general fields and going to general dimensions. The objects that naturally couple to extended objects are form fields, therefore it seems natural to consider form fields. If we have a p+1 form field we can define an operator $\mathcal{W}_p$ corresponding to a $(p+1)d$ worldvolume $\mathcal{V}$ via

$$\mathcal{W}_p = \exp\left(i\eta_p \int_{\mathcal{V}} C\right), \tag{2.75}$$

in the case of p=0 this would just correspond to the Wilson line operator. Furthermore we can define a flux operator $\mathcal{F}_p$ corresponding to a $(p+2)d$ volume $\mathcal{V}$ via

$$\mathcal{F}_p = \exp\left(i\eta_p \int_{\mathcal{V}} dC\right), \tag{2.76}$$

this would be the analog of electric surface operators. By introducing singularities into C or dC we can also define the analogs of our 't Hooft line and surface operators. These will again be constrained by an analogous quantization condition.

2.3.2 Duality Transformations

Here we are going to generalize the well known electric magnetic duality of Maxwell theory to the case of general p-form fields in both the massless as well as the massive case. Physically the idea is to rewrite a given theory in terms of different fields, where topological and electrical charges are interchanged when switching between the two theories.

Massless Case

Let us start with the Lagrangian for a free massless (p+1)-Form field C_p given by

$$\mathcal{L} = F_p \wedge \star F_p, \tag{2.77}$$

where $F_p = dC_p$. Now the procedure is as follows, magnetic charge density corresponds to dF_p, we introduce a new field C_q, which couples to magnetic charge density. This leads to the Lagrangian

$$\mathcal{L} = F_p \wedge \star F_p + C_q dF_p, \tag{2.78}$$

we note that this action is equivalent to the original one, as integrating out C_q immediately shows. Next we integrate out F_p in order to arrive at an action for C_q, this gives us

$$\mathcal{L} = dC_q \wedge \star dC_q, \tag{2.79}$$

so we see, that this gives us a free $(q+1)$-form, where $q = D - 4 - p$. In the end what we see here is, that an $(p+1)$-form has an equivalent description in terms of a $(q+1)$-form, with $p+q = D-4$, the electric object is here a $(p+1)$ dimensional membrane, while the magnetic object is $(q+1)$ dimensional. Furthermore we should note that the electric field strength is a $p+2$ form, while the "magnetic" field strength is a $q+2$ form is simply the hodge dual of the former exactly as happens in electromagnetism.

Massive Case

Now let us proceed to studying an analogous procedure in the massive case, the Lagrangian in this case has the form

$$\mathcal{L} = F_p \wedge \star F_p + C_p \wedge \star C_p. \tag{2.80}$$

Now it looks like we are in trouble, since the procedure we used in the previous section relied on the Lagrangian depending only on the field strength. However there is a way out, we can decompose a massive form as $C_p + dB_{p-1}$, in the case of massive electrodynamics, B would simply correspond to the phase of the Higgs field. Notice that by doing this we have introduced an additional p-form gauge redundancy into our description, B shifts under this redundancy, while C shift by a derivative. Now we can dualize B, after doing this we see, that the Lagrangian now only depends on F_p and not on C_p itself, which allows us in turn to dualize C, after doing this we end up with

$$\mathcal{L} = F_q \wedge \star F_q + (B_q + dC_{q-1}) \wedge \star (B_q + dC_{q-1}), \tag{2.81}$$

where $p + q = D - 3$. We can now easily see, that this is just the Lagrangian for a massive q-form, therefore the behaviour is very similar to the massless case, except that the dimensions are off by one. We should also note, that the magnetic objects in the massless case are now confined and have to be boundaries for objects of one dimension higher, which explains why it is to be expected, that our counting is off by one.

2.3.3 Topological Field Theories

Pure topological field theories (TFTs) are non-dynamical, i.e. they don't contain any propagating degrees of freedom. Furthermore computing the Hamiltonian density we see, that it is zero, i.e. there is no non-trivial time evolution. This is both a blessing and a curse. Topological theories can be treated much more easily on the other hand the available observables are quite limited, however they can describe the vacuum structure in gapped theories as well as Aharanov Bohm type phases.
In this chapter we review some properties of topological field theories. We start by discussing Chern-Simons (CS)-theory in $2+1$ dimensions introduced in [51] (for a good review see [52]). Then we proceed to the more general BF theories [53].

Chern-Simons Theory

Pure CS-theory is the most prominent example of a topological field theory and has a plethora of applications in various areas of physics, the most striking example being probably fractional quantum hall effect. It contains a 1-form gauge connection A (of a general gauge group G) with a non-standard kinetic term, which is only allowed in three dimensions. It can be easily seen, that the theory does not depend on the metric of the manifold, but only on topological properties, therefore observables can only depend on topological invariants of the base manifold. Despite of this there are some nontrivial properties which we will explain in the following.

Abelian CS-Theory

For gauge group $U(1)$ the pure CS-action with source is given by

$$S_{\mathrm{CS}} = \int \left[\frac{k}{4\pi} A \wedge \mathrm{d}A - A \wedge *j \right] \, . \tag{2.82}$$

The equations of motion can be derived as

$$*j = \frac{k}{2\pi}\,\mathrm{d}A \ . \tag{2.83}$$

In the absence of a source we get an equation of motion $\mathrm{d}A = F = 0$, which means the solutions are simply all flat gauge connections. This implies that in order to get non-trivial results we have to go to manifolds with nontrivial topology. Moreover, as the action is only first order in spacetime derivatives, there are no propagating degrees of freedom in the pure CS-theory
For a point charge q, i.e. $j = q\delta^2(\vec{x})\mathrm{d}t$, we see that it induces a magnetic field

$$B = \frac{2\pi q}{k}\delta^2(\vec{x}) \ , \tag{2.84}$$

which is attached to it. Note that in $2+1$ dimensions the magnetic field is a pseudoscaler rather than a vector field. A useful way to picture this is to think of three dimensions, then electric charges will acquire a magnetic field perpendicular to the plane, see figure 2.10. This leads to an additional phase

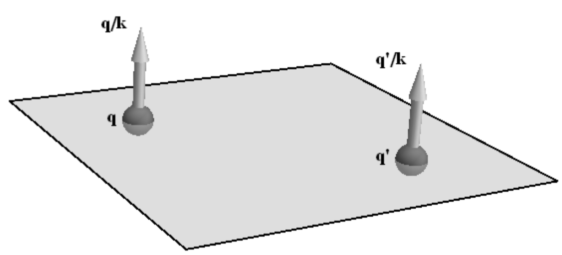


Figure 2.10: Magnetic field attached to electric charges due to Chern-Simons term

if two particles are interchanged, see [54], which in three dimensional terms is just a manifestation of the Aharonov-Bohm effect in a 2+1 dimensional setup. We see that this effect changes the exchange statistics. By the spin-statistics theorem this should also change the spin, which it does, since a composite of a charge and a magnetic field has an angular momentum. It can be checked that the induced angular momentum matches the exchange phase. In contrast to theories in $3+1$ dimensions this phase shift does not have to add up to an integral multiple of π, since spin can be an arbitrary real number in two spatial dimensions and therefore the statistics is not constrained to lead to fermions or bosons. This shows, that in 2+1 dimensions there is an exchange phase $e^{i\alpha}$ determined by the prefactor k of the CS-term (anyons, [55]). α can be determined explicitly for two particles of charge q and q':

$$\alpha = \frac{\pi}{k} q q' \ . \tag{2.85}$$

For an alternative derivation, we can calculate the regulated expectation value of two line operators (section 2.3.1), see [56]

$$\langle \exp\left(iq \oint_{C_1} A \right) \exp\left(iq' \oint_{C_2} A \right) \rangle = \exp\left(\frac{2\pi i}{k} q q' L(C_1, C_2) \right) \ , \tag{2.86}$$

with $L(C_1, C_2)$ the linking number of the two lines, see figure 2.11 for illustration. The case where we move one charge around another, is equivalent to

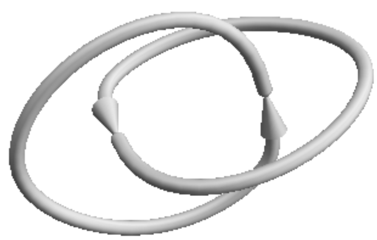


Figure 2.11: Example of two paths with linking number 1

a linking number of 1. The CS-theory therefore measures the linking number of line operators and multiplies the wavefunction by the correct phase. Since only the linking number enters equation 2.86 the line operators can be deformed, as long as they don't cross and we still get the same result.

In the presence of a Maxwell kinetic term for the gauge field the CS-term leads to a topological mass term. This, however, does not change the number of propagating degrees of freedom as the Higgs-mechanism would, as the resulting massive photon only propagates a given spin (e.g. positive spin OR negative spin). This is akin to the reason why we have two helicities for massless particles in 4 dimensions, where the negative helicity is enforced by CPT. In the massive case in 3 dimensions, CPT acts trivially, however P would enforce us to have both spins. Note that this is only allowed since the CS term breakes parity. The photon mass is given by

$$m_{\rm CS} = k e^2 \ , \tag{2.87}$$

where e^2 the coupling constant appearing in front of the kinetic term, see [52]. Under gauge transformations

$$A \to A + \mathrm{d}f \ , \tag{2.88}$$

with f a 2π-periodic function (0-form), the CS-action changes by a total derivative

$$\Delta\mathcal{L}_{\mathrm{CS}} = \frac{ik}{2}\mathrm{d}\left(f \wedge \mathrm{d}A\right) \ . \tag{2.89}$$

Therefore, the equations of motion will not change. The action itself is not gauge invariant in the presence of a boundary of the spacetime. In this case we have to include boundary degrees of freedom in order for the action to be gauge invariant, see e.g. [52]. In the Quantum Hall effect these boundary degrees of freedom are the famous edge currents.

Non-Abelian CS-Theory

For CS-theory with a non-Abelian gauge group there is a cubic coupling of the gauge fields, to be specific we get

$$\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\,\mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right) \ . \tag{2.90}$$

Under a gauge transformations U the gauge field A transforms as

$$A = A_\mu \mathrm{d}x^\mu \to \left(U^{-1}A_\mu U + U^{-1}\partial_\mu U\right)\mathrm{d}x^\mu = U^{-1}AU + U^{-1}\mathrm{d}U \ . \tag{2.91}$$

The CS-Lagrangian now changes by a term that cannot be written as a total derivative (see [52])

$$\Delta\mathcal{L}_{\mathrm{CS}} = -\frac{k}{12\pi}\,\mathrm{Tr}\left(U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U\right) \ . \tag{2.92}$$

For a non-Abelian gauge group the integral of this quantity is the winding number of the gauge transformation as discussed in e.g. [57]

$$\frac{1}{24\pi^2}\int \mathrm{Tr}\left(U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U \wedge U^{-1}\mathrm{d}U\right) \in \mathbb{Z} \ . \tag{2.93}$$

This implies, that the coefficient of the nonabelian CS-theory has to be quantized

$$k \in \mathbb{Z} \ , \tag{2.94}$$

where k is called the level of the CS theory. Similar arguments can be made in the abelian case if the gauge group is compact, see [58].

Two Field Topological Theories (BF-Theories)

In Chern-Simons theories the topological action is constructed with a single gauge field A. We can generalize this to actions with multiple fields and more general p-forms.
Let us start with the so called BF action, see e.g. [59]

$$S_{\rm BF} = \frac{k}{2\pi}\int B\wedge \mathrm{d}A = \frac{k}{2\pi}\int B\wedge F \ . \tag{2.95}$$

Again, the action does not depend on the spacetime metric and therefore is again topological. Here the Hilbert space is given by the flat connections of both B and A modulo gauge redundancies. This can again be seen from the equations of motion without sources

$$B = F = 0 \ . \tag{2.96}$$

These theories were originally considered in [53].
Since A and B are two distinct fields, these models are not restricted to three spacetime dimensions as the CS-theory. If A is a 1-form gauge field we see that in D dimensions B has to be a (D-2)-form, we will call this case BF theory and we will call the other cases generalized BF theories.
In the following, we will discuss the cases $D = 3$ and $D = 4$ in some detail and sketch the general case.

BF-Theory in D=3

In three spacetime dimensions A and B are 1-form fields and transform under 0-form gauge transformations

$$\begin{aligned} A &\to A + \mathrm{d}f \\ B &\to B + \mathrm{d}g \ , \end{aligned} \tag{2.97}$$

with two 2π-periodic 0-forms f and g, since we are only considering BF theories with compact gauge group. Introducing currents for the two fields we see

$$S_{\rm BF} = \int \left[\frac{k}{2\pi}B\wedge \mathrm{d}A - A\wedge *j - B\wedge *J\right] \ . \tag{2.98}$$

The equations of motion are given by

$$\begin{aligned} \mathrm{d}A &= \frac{2\pi}{k} * J \ , \\ \mathrm{d}B &= \frac{2\pi}{k} * j \ . \end{aligned} \tag{2.99}$$

The field configuration is again determined by the currents.
The magnetic flux of A is attached to the charges of B and vice versa. This leads to a non-trivial exchange phase between A and B charges

$$\alpha = \frac{\pi q m}{k} \ , \tag{2.100}$$

where m and q are the charges. As in the CS case we can express this in terms of the linking number via

$$\langle \exp\left(iq\oint_{C_1} A\right) \exp\left(im\oint_{C_2} B\right)\rangle = \exp\left(\frac{2\pi i}{k} qmL(C_1, C_2)\right) \ . \tag{2.101}$$

This action actually has a very concrete physical realization given by the type II superconductor in two spatial dimensions, i.e. a superconductor with magnetic vortices. Instead of describing a real type II superconductor we will use an Abelian Higgs Model, see e.g. [34]. An explicit derivation of the emergent BF-theory for the dual superconductor will be given later, an alternative approach via path integrals is given in [59]. In this description j couples to electric charges and J to magnetic vortices, which shows that the phases we observed is just the familiar Aharonov-Bohm effect.

BF-Theory in D=4

For $D = 4$ B is a 2-form field and transforms under 1-form gauge transformations λ

$$B \to B + \mathrm{d}\lambda \ . \tag{2.102}$$

In our superconductor analogy, B now couples to the worldsheet of magentic flux tubes rather than the worldline of vortices, see [59] and [60]. This is the natural extension for topological field theories in $D = 4$, because there are no Aharonov-Bohm phases between two point particles in four spacetime dimensions; there is no analog of a linking number of two line operators in as two loops can always be unwound. In the case of a single curve this is just the familiar statement, that we can unknot any knot without cutting it in $D \geq 4$. Instead, the phases are present for a point paricle moving in the field of an infinitly long solenoid, which is just a fluxtube. This can also be seen mathematically as we can define a linking number for a line and a surface operator. This can be understood pictorially by compatifying one dimension and winding the surface operator one time around the compact dimension,

this leads to the $D = 3$ case.
[53] introduced another term into the action of the form

$$\Delta S_{\mathrm{BF}} \propto \int B \wedge B \ , \tag{2.103}$$

which is allowed for spacetime dimensions divisible by 4. This term counts the number of intersections [61], which is a topological invariant. In four spacetime dimensions this is the familiar statement, that two dimensional worldsheets intersect in points.
If we reconsider the superconductor case we see, that this term leads to a phase, when flux tubes cross. This can be understood physically as follows [56], when we want one of the vortices to cross we have to move it, i.e. we have to boost it in the direction of the other vortex. This however leads to an electric field, just by usual rules of Lorentz transformation, this field will have a component parallel to the second flux tube. It can be easily checked, that the integral of $\vec{B} \cdot \vec{E}$, doesn't depend on any details of the event like crossing angle, width of the vortex or speed of the vortex, i.e. we see that it is again topological. One should furthermore note, that $\vec{B} \cdot \vec{E}$ is just the familiar θ term, i.e. we see, that the θ term can in principle be observed in electromagnetism, which is usually disregarded. The point is, that putting the vortices puts nontrivial boundary conditions, which means that θ can change the physics even though it is a boundary term.

Groundstate Degeneracy in Topological Theories

On compact, topologically non-trivial space manifolds there is an additional observable in TFTs, which is the groundstate degeneracy. For simplicity we will consider only the $2+1$ dimensional CS-theory of level k in this section, where space is the torus T^2.
On the torus there are two fundamental non-contractible loops γ_1 and γ_2. We can add Wilson line operators of the field A along these directions

$$\mathcal{W}_i = \exp\left(i \oint_{\gamma_i} A \right) \ . \tag{2.104}$$

With $\mathcal{W}_i^{-1}$ we denote the line operator with reversed path $-\gamma_i$. Adiabatically switching on a unit of flux, $\frac{2\pi}{k}$ $(q = 1)$, through the hole of the torus, see figure 2.12, leads to a phase of $\mathcal{W}_1$, e.g. [62]. The Hamiltonian of the theory does not

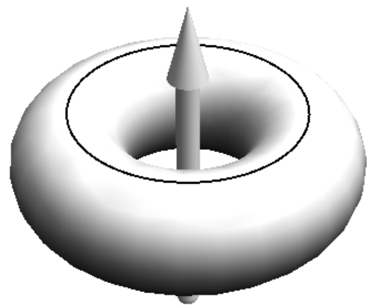


Figure 2.12: Torus with additional flux through the hole and fundamental line operator around it

change by adding the flux adiabatically and we see that this produces another groundstate. These two states can be distinguished by the expectation value of the Wilson loops. After inserting k elemetary fluxes the phase will be a multiple of 2π and hence is unobservable, i.e. we expect to get back to the original groundstate, therefore we find k different groundstates on the torus. In general, one could argue that by adding a flux through the tube of the torus we get even more degenerate ground states, altering the expectation values of $\mathcal{W}_2$. But this is not the case since we find

$$\langle \mathcal{W}_2^{-1} \mathcal{W}_1^{-1} \mathcal{W}_2 \mathcal{W}_1 \rangle = \exp\left(\frac{2\pi i}{k}\right) , \tag{2.105}$$

since it can be deformed to two loops of linking number 1. This immediately implies

$$\langle \mathcal{W}_2 \mathcal{W}_1 \rangle = \langle \mathcal{W}_1 \mathcal{W}_2 \rangle \exp\left(\frac{2\pi i}{k}\right) , \tag{2.106}$$

i.e. the Wilson lines form a Heisenberg Algebra. Thus, inserting fluxes in the tube of the torus and probing with $\mathcal{W}_2$ is the same as inserting flux trough the hole of the torus and probing with the operator $\mathcal{W}_1$.

For an Abelian CS-theory at level k we find a ground state degeneracy of k, which can be pictured by putting a flux of $q \in \{0, \ldots, k-1\}$ through the hole of the torus.

In general, if we the manifold has genus g surface the groundstate degeneracy will be k^g for the level k CS-theory.

For the BF-theory in the previous chapter the situation changes slightly.

Instead of two different line operators we get four

$$\mathcal{W}_i(\phi) = \exp\left(i\oint_{\gamma_i}\phi\right), \quad \text{with } \phi = A, B \ . \tag{2.107}$$

There are two different Heisenberg algebras, similar to (2.106), and we find a number of ground states of $(2k)^g$. These correspond to adding $q \in \{0, \dots, k-1\}$ fluxes of A and B through the holes of the genus g surface, see [59].

Continuum Description of a TFT

Now we are going to derive a TFT as a low energy effective description of an abelian Higgs model. First of all we should notice that the Higgs mechanism leads to mass gap in the theory, therefore the effective theory at energies way below the mass gap shouldn't contain any propagating degrees of freedom. Using Poincare invariance immediately implies, that the Hamiltonian of the low energy theory should be trivial and therefore that we can at most hope for a topological field theory as a low energy theory. In this section we are going to derive this theory explicitly. This has been known in the condensed matter literature for quite some time [59]. Here we are mostly following the derivation of [46], which uses an index free notation. Our model is the standard abelian Higgs model, with a Higgs field of charge k in 4 dimensions, for notational simplicity we will work in euclidean space. The Abelian Higgs model is a theory of a complex scalar field ϕ of charge k and a $U(1)$ gauge field A. The gauge transformations are given as

$$\begin{aligned} \phi &\to e^{ikf}\phi \\ A &\to A + \mathrm{d}f \ . \end{aligned} \tag{2.108}$$

After splitting ϕ into modulus part and phase

$$\phi = \rho e^{i\varphi} \ , \tag{2.109}$$

the gauge tranformation acts as a shift of the phase φ

$$\varphi \to \varphi + kf \ . \tag{2.110}$$

The kinetic term for ϕ and A is given by

$$S_{\text{kin}} = -\frac{1}{2g^2}F\wedge *F + D\phi\wedge(*D\phi)^* \equiv -\frac{1}{2g^2}F\wedge *F + |D\phi|^2 \ , \tag{2.111}$$

where * denotes complex conjugation. We take the standard Higgs potential for the scalar field

$$V(\phi) = \frac{\lambda}{4}\left(|\phi|^2 - v^2\right)^2 \quad , \tag{2.112}$$

with self-coupling λ and $v \geq 0$. This potential vanishes for $\rho = v$, which gives the groundstate of the theory. Here we should note that we can still fix the phase of ϕ to an arbitrary number. Therefore we would naively conclude, that the ground state has a continuous degeneracy labeled by the phase of ϕ. This however turns out not to be true, since the corresponding field configurations are related by gauge transformations, i.e. they describe the same state. This is also how the Higgs mechanism evades Goldstone's theorem, Goldstone's theorem strongly relies on the non-uniqueness of a vacuum state. Here however there is no continuous number of vacuum states, therefore we should not expect to see a gapless particle in the spectrum, for a more detailed discussion see [63]. This vacuum expectation value describes the condensation of the charge k particles and the gauge field acquires a mass given by

$$m_A^2 = 2v^2k^2 \quad . \tag{2.113}$$

The excitations of the Higgs field $H = \rho - v$ have a mass

$$m_H^2 = 2\lambda v^2 \quad . \tag{2.114}$$

At very low energies, i.e. for $v \gg E$, we can set $\rho = v$ everywhere except at the locations of possible flux tubes that are characterized by

$$\oint_C A \neq 0 \quad , \tag{2.115}$$

if the path C encloses the worldsheet of the flux. They can be equivalently described as singularities in φ. In this limit the magnetic vortices are infinitely thin, since their radius scales as the Compton wavelength, $\propto \frac{1}{m}$, of the fundamental particles, [64]

$$m_H, m_A \overset{v\to\infty}{\to} \infty \quad . \tag{2.116}$$

This shows, that the width of the vortices are much smaller then the characteristic length scale we are probing, making an approximation as line like object reasonable. A particle of charge k should not develop a non-trivial phase in the presence of flux tubes as otherwise the fluxtubes would destroy superconductivity in the bulk. This implies, that the flux, Φ, has to be quantized as

$$k \oint_C A \in 2\pi\mathbb{Z} \Rightarrow \Phi \in \frac{2\pi}{k}\mathbb{Z} \ . \tag{2.117}$$

k fluxtubes lead to a trivial phase even for unit charges and cannot be detected.This shows, that the flux takes values in $\mathbb{Z}_k$. It should be emphasized again, that this conclusion only applies to the topological properties of the theory.

Since all the local degrees of freedom are gapped and weakly coupled we can construct the low energy effective action, which will encode the Aharonov-Bohm phases and will be a topological field theory.

The procedure for obtaining a topological theory encoding Aharanov-Bohm type phases and discrete ground state degeneracies can also be applied to theories without a mass gap, i.e. gauge theories in the Coulomb phase. In this case however the corresponding action can not be obtained as a simple low energy approximation. Formally this can be done by introducing a topological scaling, which differs from the Wilsonian scaling. In the Wilsonian scaling as we vary the scaling parameter (which is the renormalization scale) we vary the couplings as well, such that the correlation length stays fixed. In the topological scaling we look at the system on scale R. Now we keep the couplings fixed as we send $R \to \infty$ and only keep the parts of correlation functions, which scale as R^0. The Wilsonian scaling keeps all the information except for the information related to short distance effects, while the topological scaling only keeps topological properties. For more details see [65]. However in the case of a gapped system, this is equivalent to Wilsonian scaling, as the only effects that survive at low energies will be of topological nature and since topological features can be detected at all scales, we are guaranteed to be able to capture them at low energies. After integrating out the radial mode, the leading term at low energies is

$$S_{\text{top}} = v^2 \int (\mathrm{d}\varphi - kA) \wedge *(\mathrm{d}\varphi - kA) \ . \tag{2.118}$$

For large v the Euclidean action is dominated by the classical contributions and in fact we can rewrite 2.118 as a contraint imposed by a Lagrange multiplier 3-form h, see [43]

$$S_{\text{top}} = \int h \wedge (\mathrm{d}\varphi - kA) \ . \tag{2.119}$$

This constraint ensures that away from vortex lines the gauge field is pure gauge. At the vortex line positions the phase φ as well as the gauge field A are singular. After dualizing φ into a two form B we get again a BF theory, where electric surface operators of B described the worldsheets of vortices, while Wilson lines of A describe the worldlines of charges. Here we can again describe the Aharanov Bohm phases as before in terms of linking numbers.

Generalized BF Theories

In this section we will sketch how to generalize the discussion of the previous sections to generalized BF theories. We will follow closely the paper of Kapustin and Seiberg [66]. We are in D (Euclidean) dimensions, our fields are a $(q+1)$ form $C_{(q+1)}$ and a $(D-q-2)$ form $C_{(D-q-2)}$. The action is a straightforward generalization of the BF action considered previously

$$S_{BF} = \frac{in}{2\pi} \int C_{(q+1)} \wedge dC_{(D-q-2)}. \tag{2.120}$$

The action is invariant under two abelian $U(1)$ gauge transformations

$$C_{(q+1)} \quad \rightarrow C_{(q+1)} + d\lambda_{(q)} \tag{2.121}$$

$$C_{(D-q-2)} \quad \rightarrow C_{(D-q-2)} + d\lambda_{(D-q-3)} \tag{2.122}$$

We can define gauge invariant field strengths in the usual way via $F_{(q+2)} = dC_{(q+1)}$ and $F_{(D-q-1)} = dC_{(D-q-2)}$. In the absence of sources the equation of motion imply that the two field strengths vanish, i.e.

$$F_{(D-q-1)} = F_{(q+2)} = 0. \tag{2.123}$$

We can again introduce electric type operators for both gauge fields, for a p-form C_p these are defined as

$$W^{(p)}(\Sigma) = e^{i\int_\Sigma C_p}, \tag{2.124}$$

where Σ is a closed p-dimensional submanifold of spacetime. We can understand the effect of these by noting, that for example introducing the Wilson operator $W_{(D-2-q)}(\Sigma)$ leads to a modified equation of motion $nF_{(q+2)} = 2\pi\delta_\Sigma$, where delta is a form, that behaves like a delta function and that is orthogonal to Σ. This delta function implies, that the holonomy of $W_{(q+1)}$ around Σ is $e^{2\pi i/n}$, in the case of $D=4$ and $q=0$ this holonomy just describes the familiar Aharanov-Bohm phases encountered earlier. These are the central observables of the generalized BF theory, in the case when $D = 0 \bmod 4$, there is an additional term for $q = D/2 - 1$, which is given by

$$C_{(q+1)} \wedge C_{(q+1)}. \tag{2.125}$$

In the special case of $D = 4$ and $q = 1$ this term is the BB term discussed earlier.

2.4 Localization of Gauge Fields

In this section we are going to review the gauge field localization mechanism of Dvali and Shifman and then introduce an extension due to the author, which introduces a Chern-Simons term in this setup.

In order to describe localization of fields we are looking at a theory, where we have some higher dimensional bulk theory, which has some effectively lower dimensional objects like D-Branes or topological solitons. At sufficiently low energies for probes localized on the lower dimensional object we expect to see degrees of freedom that are localized to the wall. A typical example for this would be sound waves along a guitar string for example, which can be thought of as Goldstone bosons corresponding to broken translations. It turns out that it is fairly easy to localize massless particles of spin-0 and spin-1/2, since spin-0 particles are guaranteed to be there due to translation breaking, while spin-1/2 particles can shown to be generically localized using index theorem techniques.[4] For higher spin particles the situation is significantly more complicated. For example there is no known way to localize a massless spin-2 particle, this can be expected to be hard, since gravity couples universally and furthermore there are strong restrictions on the way a massless spin-2 particle could possibly emerge as some kind of bound state due to a theorem by Weinberg and Witten. [67] However there is a mechanism due to Dvali, Gabadadze and Porrati that achieves a sort of partial localization of gravity with interesting phenomenological implications. [68]

In order to localize a gauge field there seem to be two possible directions one could take, first the gauge field could appear as an emergent quasiparticle due to some complicated dynamics, but again there are very strong restrictions in how this can be done since one needs to evade the Weinberg-Witten theorem. The second possibility which Dvali and Shifman chose is to take a gauge field that already exists in the bulk theory and arrange that it only propagates on the wall. Apart from theoretical interest localization of gauge field is also relevant from a more phenomenological perspective as one area of research in the extension of the standard model lies in extra dimensions, here it would be natural to localize the Standard Model we see to such a wall. [69–71]

[4] here we are mostly interested in localizing massless particles, massive particles can be straightforwardly localized.

2.4.1 Dvali-Shifman Mechanism

We will begin by describing a very naive idea to localize a $U(1)$ gauge field, which will turn out to fail. Then we will exploit the way in which this approach fails to construct an improved mechanism which truly localizes a massless gauge field. In order to localize a gauge field we will simply take electromagnetism and give the photon a mass everywhere except in some region of space. It turns out however that this does not localize a massless gauge field on the wall. In order to see this we should realize that a massive $U(1)$ field corresponds to a superconductor.

Now we imagine the setup of a superconductor in the bulk, while the wall is the vacuum; this is precisely the well known Josephson junction. [72] Now it is physically clear why we don't localize the gauge field: In the superconductor there is a large number of free charges. This implies that a test charge in the wall will polarize the free charges in the superconductor and this will in turn screen the electric field.
This can be seen technically as follows, in the limit where the photon is infinitely heavy outside of the wall, the gauge field is present only inside the wall, i.e. the corresponding action is given

$$S = \int d^3x \int_{-d/2}^{d/2} F_{\mu\nu} F^{\mu\nu}, \tag{2.126}$$

where d is the width of the wall. This is akin to a four dimensional theory compactified to three dimensions. However the spectrum of the low-energy theory depends crucially on the boundary conditions imposed at $|z| = \frac{d}{2}$. The would-be massless photon corresponds to the zero mode resulting from the Kaluza Klein decomposition. If the boundary conditions project that mode out from the spectrum we don't localize the vector field.
In our case the boundary conditions are as follows:

- The electric field has to be perpendicular to the boundary, since we have a conductor
- In the case of a superconductor the magnetic field should be parallel to the boundary.

This means $E_1 = E_2 = 0$ and $B_3 = 0$ on the boundary, or written covariantly as $F_{ab} = 0$ where $a, b = 0, 1, 2$. The zero mode however has to be constant along the z direction. This implies that the zero mode of F_{ab} is not in the spectrum of the low energy theory. Therefore it is clear that we do not have a massless photon. The lowest mode of spin-1 we see is a massive photon of mass $m \propto d^{-1}$. What remains in the low energy spectrum is the massless KK scalar F^{3a}, which does not couple to electric charge. By rewriting the boundary conditions as $\tilde{F}^{a3} = 0$, we see that the zero mode of the dual field strength tensor $\tilde{F}^{ab}$ remains massless. Therefore the massless scalar in the spectrum behaves as a magnetic photon.[5]

An alternative way to understand this is to consider two charges inside the wall, separated by a distance $l \ll d$. Here we will see three dimensional electromagnetism. Now we move the charges apart, in a superconductor electric flux lines decay exponentially fast, while magnetic flux lines are expelled by the superconductor due to the Meissner effect and can only penetrate in the form of flux tubes[6]. In the limit $l \gg d$ most electric flux lines end on the superconductor instead of going through the wall, i.e. flux decays exponentially fast as we move the charges further from each other. This explains why we do not see a massless photon in the large distance limit. On the other hand, the magnetic flux lines show the behaviour we would like electric flux lines to have. They are confined to the wall and give the right behaviour expected of two dimensional Maxwellian electromagnetism.

This gives us a clear way out: we can simply take the electric-magnetic dual of the setup. More concretely, we want to have dual superconductivity in the bulk, which as reviewed earlier is expected to be realized in the confining phase of Yang-Mills theories. We will take a non-abelian confining gauge theory in the bulk which gets Higgsed to a $U(1)$ inside the wall. To be concrete we start with a $SU(2)$ gauge theory with a scalar field in the adjoint representation. For the scalar we take a potential

$$V(\phi) = \frac{\phi^a\phi^a}{M^2}(\phi^a\phi^a - M^2)^2. \tag{2.127}$$

$V(\phi)$ has minima at $\phi^a = 0$ and $\phi^a = Mn^a$, where n^a is an arbitrary unit vector.[7] The potential is tuned in such a way that all the minima lie at $V = 0$. This theory has two phases: One is the broken phase, where just the

[5] This works because in 2+1 dimensions a massless vector field is dual to a scalar field.

[6] In the limit of infinite mass, magnetic fluxes can not penetrate the superconductor at all, while electric fluxes decay infinitely fast on the boundary.

[7] Here we will ignore subtle quantum effects like Coleman Weinberg type corrections,

$U(1)$ symmetry is manifest and which is free in the infrared. In the other phase we have the confining SU(2) with confinement scale Λ. In our model we replace the wall by the U(1) vacuum and the bulk by the SU(2) vacuum. This gives the right picture: electric flux lines can only penetrate into the bulk in terms of flux tubes, while magnetic flux lines end on the boundary. This shows that we really localize a $U(1)$ gauge field. Another interesting point to note is that flux tubes in the bulk can end on the wall, since the $U(1)$ is in an unconfined phase. Let us look into this in a little more detail:

In order to get a better understanding we will do a thought experiment, where we move a charge from the wall into the bulk. The electric field will be expelled, but since the flux has to be conserved by Gauss' law, it will form a string of electric flux between the wall and the charge. This implies that our branes have a common feature with D-Branes from string theory, i.e. strings can end on them.

As a last point let us return to the question of confinement in $2+1$d QED, which has been first discussed by Polyakov [73] and is due to a dilute gas of instantons. Here the euclidean instantons have the field configurations of monopoles in 3 spatial dimensions, therefore we can expect them to lead to confinement. We can also understand this either in the microscopic bulk theory or as done before in terms of instantons in the low energy world volume theory. We can simply note that what we are considering is a literal magnetic analog of a Josephson junction, i.e. we expect a current of tunneling monpoles across the wall. These tunneling monopoles will be precisely the instantons in $2+1$d and the dilute gas of these instantons will lead to debye screening of the magnetic photon, i.e. to confinement of the electric field on the wall. For a more detailed discussion see [74].

2.4.2 Localization of CS term

This section is going to give a concise summary of how to extend the Dvali Shifman mechanism in order to localize a Chern-Simons term on the wall worldvolume. This part was the subject of a publication of the author which is reproduced ad verbatim in the final part of the thesis. As we have described before, the θ term is the derivative of the Chern-Simons form, this leads to the conclusion, that a gradient in θ across the domain wall in Dvali-Shifman

since this will be unimportant for the considerations, at most we will need a more complicated model.

will lead to a Chern-Simons term in the $2+1$ dimensional effective theory on the world-volume of the wall. After integration by parts we get have

$$\begin{aligned}&\frac{\theta(z)}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,(F_{\mu\nu}F_{\rho\sigma})\\=&\big(\partial_\mu\theta(z)\big)\frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[A_\nu\partial_\rho A_\sigma-\frac{2}{3}ieA_\nu A_\rho A_\sigma\right].\end{aligned}\tag{2.128}$$

This leads us to conclude, that we will have a CS term of level

$$k=\frac{\Delta\theta}{8\pi^2}\tag{2.129}$$

This situation can be achieved by having the domain wall behaving as an axionic wall as well as having a Higgs vev. An explicit potential that achieves this is

$$\begin{aligned}V=&m^2\,\mathrm{Tr}[\phi^2]+\frac{\lambda}{2}\,\mathrm{Tr}[\phi^2]^2+\\&A\cos(\theta)+\\&C\,\mathrm{Tr}[\phi^2]\cos(\theta)+\cdots\end{aligned}\tag{2.130}$$

Here θ is the axion, while ϕ is the Higgs field. For a certain parameter range, this model contains precisely the domain walls we are looking for. The parameter range is given by $A<0$ and

$$0<C-m^2<2\sqrt{\lambda|A|}.\tag{2.131}$$

We immediately see, that the vacua are at $\theta=2\pi\mathbb{Z}$ and $\phi=0$, so we can have domain walls interpolating between different values of θ. Furthermore we see, that within the wall $\cos(\theta)$ will become sufficiently negative to dominate over the mass term and leading to a condensation of the Higgs field. This is qualitatively shown in fig. 2.13. Let us remark, that since the potential contains a tree level potential for the axion the model as given would not solve the strong CP problem, however the model is only supposed to be a proof of principle for localization of a CS term using Dvali-Shifman mechanism. In our publication it is explicitly shown how to see the CS term using image charge techniques.

Let us now try to understand some of the features of abelian CS theories in terms of bulk physics.

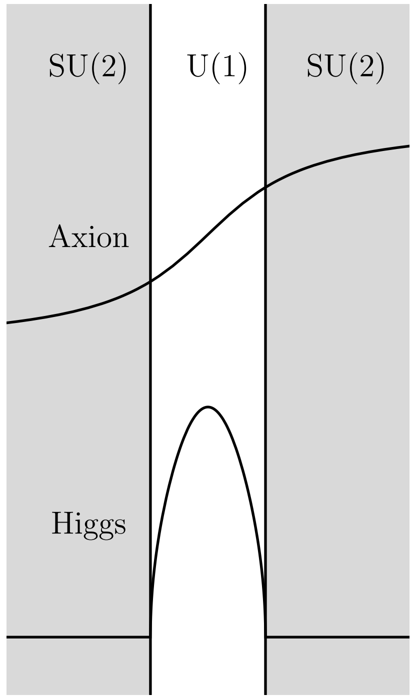


Figure 2.13: Schematic plot of the (classical) field profile across an axionic domain wall in our model. Note: the profile was determined in the approximation $A \gg C$.

Magnetic Events

Let us consider the process, where a magnetic monopole crosses the domain wall. The effective 2+1d theory sees a magnetic charge for a short time, while the monopole is on the wall, this charge will disappear afterwards once it leaves the wall, since its magnetic charge is screened by the condensate. Furthermore the magnetic flux across the wall should change by g. These features lead us to identify this process as a so called magnetic event [75] in the effective Chern-Simons theory.

When the magnetic monopole flies across the brane, it experiences an "adiabatic" change in θ, thereby acquiring electric charge via the Witten effect reviewed earlier. Thereby it leaves the brane on the other side as an electrically charged dyon. This rather interesting interplay between magnetic monopoles and axionic domain walls was already pointed out by Sikivie in

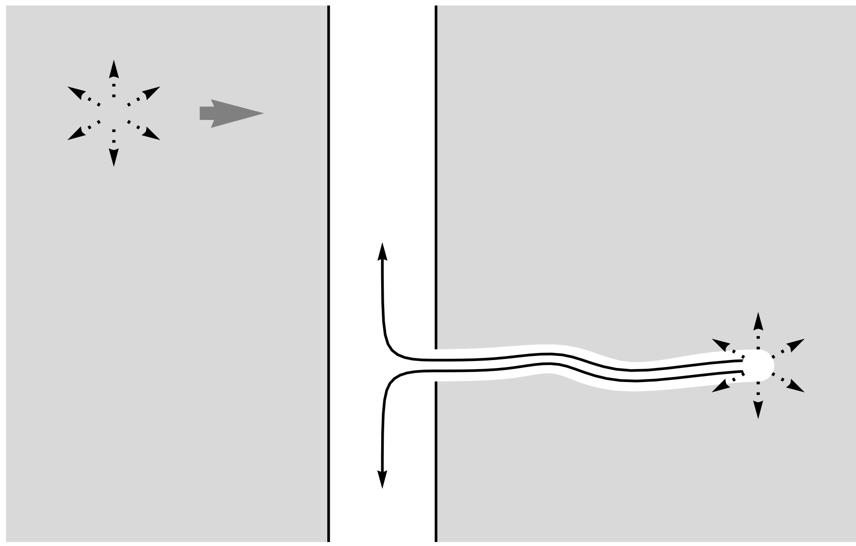


Figure 2.14: Qualitative sketch of a monopole crossing the axionic domain wall. As described in the text, the monopole picks up an electric charge. Because electric flux is conserved in the bulk, an electric flux tube connects the monopole to the wall. The end of the flux tube appears as an electric charge in the effective wall-world volume theory.

[76]. In our setup, the bulk however is confining and electric flux conserved. Thus, when the monopole leaves the brane, it trails an electric flux tube which connects it with the brane. A sketch of this process is shown in figure 2.14.

From the the 2+1d theory, the end of the electric flux tube looks like an electric charge. Like any electric charge in the Chern-Simons theory, it is screened and produces a magnetic flux, as we expected.

Furthermore it can be easily seen, that the confinement of compact QED will disappear once we turn on the CS term. As reviewed earlier the confinement can be understood in the Dvali-Shifman setup as being due to tunelling monopoles, however in our setup the objects condensing on both sides are different objects, therefore the tunnelling monopole can not be absorbed by the condensate, but instead is still conected to the wall by a flux tube, this clearly explains, why in the CS theory there is no confinement.

In our model there are domain walls with $\Delta\theta \in 2\pi\mathbb{Z}$ appear, this leads

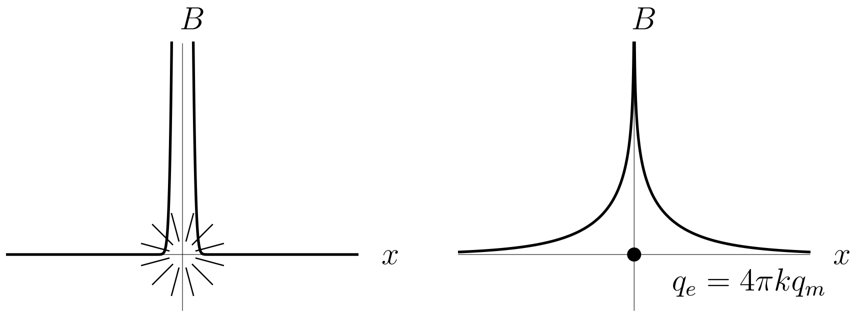


Figure 2.15: Sketch of a magnetic event in Maxwell-Chern-Simons theory. (Left) The magnetic event occurs and creates a localized magnetic flux. (Right) The event leaves an electric remnant charge and — after some electromagnetic waves have dispersed — the magnetic field profile shown here.

to a CS term, whose coefficient is quantized as

$$k \in \frac{1}{4\pi}\mathbb{Z}. \tag{2.132}$$

This agrees perfectly well with the fact that the Chern-Simons term appears as a term in the microscopic SU(2) theory and the quantization condition, that appears in the nonabelian CS theory.

Thus we have seen how to localize a CS term in the Dvali-Shifman setup, in the rest of the thesis we will show how to show, that on certain domain walls in pure YM theory and supersymmetric YM theory we see a CS term on the worldvolume of these walls, here however a different mechanism than Dvali-Shifman seems to be at work.

2.5 TFT for YM-Theories

After having reviewed the set of tools we are going to use we now come to the derivation of the low energy theory of pure Yang-Mills theory, this will also lead to an interpretation of the quasivacua in large N YM we discussed earlier. The central assumption here is, that confinement occurs by monopole condensation. Now we have to identify what kinds of monopoles condense, in order to do that we should remember, that in a superconductor the fluxes of vortices, which are present are very directly related to the charges which are condensing. An alternative useful way of viewing the quantization of vortex flux is as follows, let us imagine ending the flux running through the vortex on a magnetic monopole, then vortex literally becomes the Dirac string of the monopole, now the quantization of the flux can be understood as a Dirac quantization between the confined monopole and the charge of the condensate. Now let us look at what sort of flux tubes we expect to see in Yang-Mills theory: We certainly expect to see flux tubes, with a flux corresponding to a quark charge, however we don't expect to see flux tubes corresponding to fractional quark charges. This immediately implies, that the condensing monopoles should have charge N in the $Z_N \times Z_N$ classifcation of charges explained earlier, i.e. they have to correspond to adjoint monopoles, one should also note, that this naturally explains the appearance of a Z_N topological theory at low energies. In the following we are going to explain how to arrive at this topological theory.

2.5.1 Classification by the Choice of Line Operators

Before we saw that line operators in Yang Mills theories can be characterised by their electric and magnetic charges (q, m) taking values in $\mathbb{Z}_N \times \mathbb{Z}_N$. In [44] the authors introduce the notion of genuine line operators to identify the allowed spectrum of probe particles and classify the gauge theory, [66]. Genuine line operators are operators, that do not need a surface operator attached to them for gauge invariance.
This means that using the generalization of the Dirac quantization for non-local operators, [77]

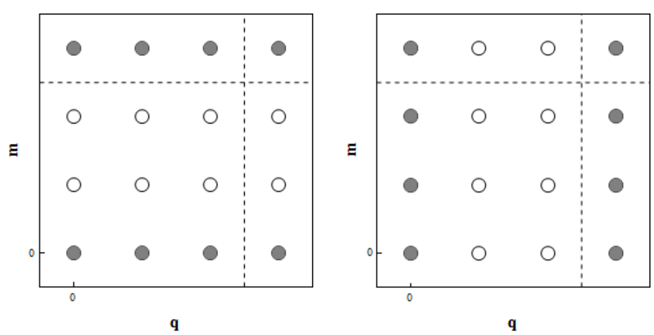
$$qm' - mq' \in N\mathbb{Z} \ , \tag{2.133}$$

we only allow for operators that generate a trivial phase, i.e. a multiple of 2π. In other words we admit line operators in the spectrum for which we

are not able to detect the Dirac string produced by other valid probes. For example a monopole in the fundamental of the magnetic group and a particle in the fundamental of the electric group would pick up a phase of $1/N$, which means we cannot have both in the theory.
For later convenience we rescale our electric charges by a factor of $1/N$, i.e. fundamental quarks have charge $1/N$. The Dirac quantization now is in the more familiar form

$$qm' - mq' \in \mathbb{Z} \ . \tag{2.134}$$

Let us consider two examples. If we want the fundamental Wilson line $\mathcal{W}$,with charge $(1/N, 0)$, of the $SU(N)$ theory to be a genuine line operators, the Dirac quantization (2.133) directly tells us that only magnetic charges are allowed, which are a multiple of N. For $SU(3)$ this is pictured in the left panel of figure 2.16, for a more detailed discussion see [44]. If we take



(a) $SU(3)$, genuine $\mathcal{W}$ (b) $SU(3)$, genuine $\mathcal{H}$

Figure 2.16: Spectrum of non-local operators in $SU(3)$ depending on the choice of genuine line operators, denoted by solid points in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ lattice, the dashed lines mark the periodicity of $\mathbb{Z}_3$

fundamental 't Hooft lines $\mathcal{H}$ to be genuine, only integer electric charges, i.e. N multiples of the fundamental charge, are allowed (right panel of figure 2.16).
All other operators have to be attached to surface operators in order to account for the non-trivial exchange phases. The set of allowed operators gives a more fine grained classification of the theory.
We can get further insights into the theory by considering Witten effect, Witten effect implies that under a change of θ by 2π a dyonic line operator

of charge $(p/N, m)$ will become a dyonic line operator of charge

$$\left(\frac{p}{N}, m\right) \overset{\theta\to\theta+2\pi}{\to} \left(\frac{1}{N}\left[(p+m) \bmod N\right], m\right) \quad . \tag{2.135}$$

The different choices of genuine line operators lead to different theories labelled by the gauge group and an additional parameter p, we can label the different theories as $(SU(N)/Z_m)_p$, where m should be a divisor of N and p is between 1 and m.
Furthermore, we should also notice, that Witten effect implies, that under a shift of θ the monopole condensate picks up an adjoint electric charge, i.e. we end up with a dyon condensate. However just naively changing θ should not produce the ground state of YM at $\theta = 2\pi$, but the first excited quasivacuum. This implies, that the different quasivacua correspond to condensates of dyons with different electric charges.

Let us consider this effect for the spectra of figure 2.16 for $\theta = 0$. Previously we saw that a charge $(\frac{p}{N}, m)$ in fundamental units in the $k = 0$ theory corresponds to a charge

$$\left(\frac{1}{N}\left[(p-m) \bmod N\right], m\right) \quad , \tag{2.136}$$

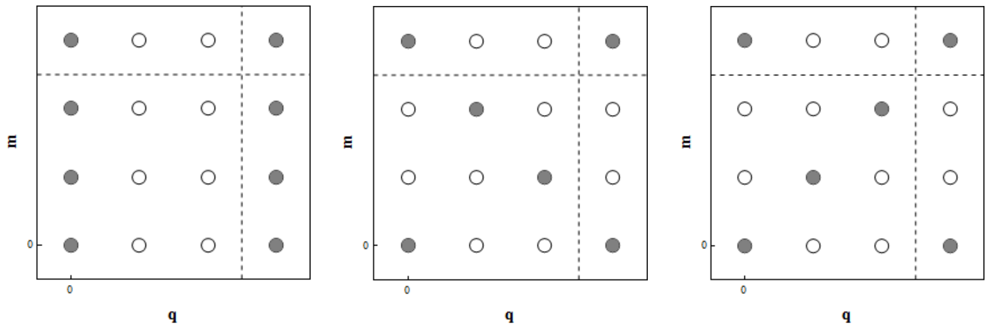
in the theory with $k = 1$.
For genuine line operators $\mathcal{W}$ at $k = 0$ the genuine line operators at $k = 1$ are unchanged, because only monopoles with magnetic charge $\in N\mathbb{Z}$ are allowed whose contribution is canceled by $\bmod N$.
For genuine line operators $\mathcal{H}$ at $k = 0$ their analogs on the branch $k = 1$ are

$$(0, m) \text{ at } k = 0 \leftrightarrow \left(\frac{1}{N}(-m \bmod N), 0\right) \text{ at } k = 1 \quad . \tag{2.137}$$

In figure 2.17 the situation is depicted for $N = 3$. We see that the spectrum indeed does change. Its peridicity in k is N as suggested by [40]. The allowed line operators become dyonic under the change of k as can be suspected by the Witten effect. For a more detailed discussion of this see [44].

2.5.2 Construction of the TFT

With the choice made above and all tools at hand we now construct a topological field theory of $SU(N)$ YM theories. These models are described by a



(a) $k = 0$ (b) $k = 1$ (c) $k = 2$

Figure 2.17: Change of the spectrum of genuine line operators for $SU(3)$ in different quasi-stable vacua labeled by k

$\mathbb{Z}_N \times \mathbb{Z}_N$ charge lattice for the line and surface operators.
We further consistently implement the Witten effect and the different condensate charges in the quasi-stable vacua. Introducing YM domain walls, interpolating between the branches for fixed θ, we find that we have to include a level N CS-action on the wall in order to preserve gauge invariance. This mechanism supports the string theory consideration of [78] and yields a new field theoretical interpretation.
In the following we work in four dimensional Euclidean spacetime. Since our theory only contains the phases, all terms are purely imaginary resulting in the phase $-iS_E$.

The Topological Action for the Dual Superconductor

The fact that all our charges and fluxes are described by $\mathbb{Z}_N$ allows us to construct the topological theory using an Abelian construction, which simplifies the task tremendously, see [43]. Previously we derived the topological action for a $\mathbb{Z}_k$ theory by condensing a charge k field and starting from the Abelian-Higgs model. We will use the same action in the following but with one crucial difference. In order to create confinement of electric charges we do not want to condense electrically but magnetically charged fields. To preserve the discrete symmetry $\mathbb{Z}_N$ we condense charge N monopoles. We should note, that once we say we are condensing charge N monopoles we are implicitly working in $SU(N)/Z_N$, we will comment on the interpretation of this later.

The topological action we start with is just the magnetic version of (2.119) and was already considered in [43]

$$S = \int h \wedge (\mathrm{d}\varphi - N\tilde{A}) \ , \tag{2.138}$$

with Lagrange multiplier 3-form h. The gauge transformations are given by

$$\begin{aligned} \varphi &\to \varphi + Nf \ , \\ A &\to A + \mathrm{d}f \ , \end{aligned} \tag{2.139}$$

with a 0-form gauge function f which is 2π periodic, $f \sim f + 2\pi$. We dualize the magnetic scalar field φ into a 2-form field B and integrate out the Lagrange multiplier leading to

$$S = \int \left[h \wedge (\mathrm{d}\varphi - N\tilde{A}) + \frac{i}{2\pi} \mathrm{d}\varphi \wedge \mathrm{d}B \right] \ . \tag{2.140}$$

Now we can integrate out $d\varphi$, this leads to

$$S = \frac{iN}{2\pi} \int \tilde{A} \wedge \mathrm{d}B \ . \tag{2.141}$$

Here B couples to the vortices of φ, i.e. the electric flux tubes of the confining theory. We assume that the spacetime manifold does not have a boundary and integrate the above action by parts

$$S = -\frac{iN}{2\pi} \int \mathrm{d}\tilde{A} \wedge B = -\frac{iN}{2\pi} \int \tilde{F} \wedge B \ . \tag{2.142}$$

This is the BF-action we had earlier, which described the topological properties of a superconductor in four dimensional spacetime.
In order to recover the more familiar gauge field A we dualize $\tilde{A}$

$$S = \int \left[-\frac{iN}{2\pi} \tilde{F} \wedge B + \frac{i}{2\pi} \mathrm{d}\tilde{A} \wedge \mathrm{d}A \right] = \frac{i}{2\pi} \int \tilde{F} \wedge (F - NB) \ . \tag{2.143}$$

F and B are propotional to each other on the equations of motion. $\tilde{F}$ is a Lagrange multiplier, just as we would expect for a field B that couples to electric flux tubes. The gauge field A has its usual 0-form gauge transformation properties, since it only appears as $\mathrm{d}A$. However we have an additional 1-form gauge transformation λ

$$\begin{aligned} B &\to B + \mathrm{d}\lambda \ , \\ A &\to A + N\lambda \ , \\ \tilde{A} &\to \tilde{A} \ . \end{aligned} \tag{2.144}$$

The field strength F transforms to $F + N\mathrm{d}\lambda$, therefore a surface operator of F on a closed submanifold Σ ($\partial\Sigma = 0$) changes as

$$\frac{N}{2\pi} \oint_\Sigma \mathrm{d}\lambda \tag{2.145}$$

Gauge invariance demands this to be a multiple of N. This leads to a quantization condition for the 1-form transformations λ

$$\frac{1}{2\pi} \oint \mathrm{d}\lambda \in \mathbb{Z} \ . \tag{2.146}$$

This action together with the gauge transformations describes the dual superconductor and its spectrum of non-local operators. In the next section we will perform some nontrivial checks on the consistency of this description.

Non-Local Operators for the Dual Superconductor

In order to see the allowed electric charges for Wilson lines we look at the electric surface operators integrated over a closed 2-surface Σ. All the allowed surface operators should be gauge invariant under 1-form gauge transformations

$$\begin{aligned} \exp\left(i\eta \oint_\Sigma F\right) &\to \exp\left(i\eta \oint_\Sigma [F + N\mathrm{d}\lambda]\right) \\ &= \exp\left(i\eta \oint_\Sigma F + 2\pi i\eta N k\right) \stackrel{!}{=} \exp\left(i\eta \oint_\Sigma F\right), \quad \text{for } k \in \mathbb{Z} \ . \end{aligned} \tag{2.147}$$

Therefore, the electric fluxes and consequently the charges producing them have to be quantized

$$\eta, q \in \frac{1}{N}\mathbb{Z}_N \ . \tag{2.148}$$

This is the same as in the Abelian-Higgs model considered before and just tells us that we reproduce usual charge quantization correctly. The valid Wilson loops can be parametrized as before ($p \in \{0, \ldots, N-1\}$)

$$\exp\left(i\frac{p}{N}\oint_C A\right) \to \exp\left(i\frac{p}{N}\oint_C [A + N\lambda]\right) \ . \tag{2.149}$$

As expected, in general they are not gauge invariant. However if we attach a surface operator with corresponding surface Σ, where $\partial\Sigma = C$ the Wilson loop becomes invariant, since

$$\Delta\left(i\frac{k}{N}\oint_C A - ik\int_\Sigma B\right) = i\frac{k}{N}\oint_C N\lambda - ik\int_\Sigma \mathrm{d}\lambda = 0 \ , \tag{2.150}$$

where we used Stokes' theorem. We see, that in the presence of fundamental monopoles Wilson loops with fundamental quark charges are not gauge invariant. They are only gauge invariant if a surface operator is attached to them, i.e. we have to attach an observable string to them. In the $(SU(N)/Z_N)_0$ theory this is exactly what we expect. The 't Hooft loops on the other hand are gauge invariant on their own exactly reproducing our choice of genuine line operators.

Inclusion of the Witten Effect

In the action, equation (2.143), we describe a dual superconductor, however we neither have a parameter, that we can identify with θ nor do we have a parameter that we can identify with the p of $(SU(N)/Z_N)_p$. In order to correctly identify these effects we make use of Witten effect.
Starting with a condensate of charge $(0, N)$, the Witten effect suggests that under $\theta \to \theta + 2\pi$ its chromoelectric charge changes by N times the fundamental charge and now carries the charge of $(1, N)$. Pure 't Hooft loops should not be gauge invariant anymore but should also require a surface operator. This can be achieved by modifying the 1-form gauge transformations of the dual gauge field $\tilde{A}$, (for a similar discussion in a different context see [66])

$$\tilde{A} \to \tilde{A} - \frac{\theta}{2\pi}\lambda \ . \tag{2.151}$$

For $\theta = 0$ 't Hooft loops are invariant. For $\theta = 2\pi$ the following combination is

$$\exp\left(im\oint_{\partial\Sigma} \tilde{A} + im\int_\Sigma B\right) \ . \tag{2.152}$$

As expected now a dyonic line is gauge invariant. Moreover, we want loop operators, that are powers of the gauge invariant line operators to be gauge invariant as well. For $\theta = 2\pi$ this implies, that any loop operator with $(\frac{m}{N}, m)$ should be gauge invariant, which is exactly what happens.
As we changed the gauge transformations we also have to change the action. The action 2.143 is not anymore gauge invariant with the modified transformations for $\tilde{A}$

$$\Delta S = -\frac{i\theta}{4\pi^2} \int \mathrm{d}\lambda \wedge (F - NB) \ . \tag{2.153}$$

The term $\propto \mathrm{d}\lambda \wedge B$ has to be canceled, i.e. we have to change the action. Earlier we saw that for spacetime dimensions divisible by four an additional term of the form $B \wedge B$ is allowed, see [53]. It can be checked that this has the correct transformation properties. The combination

$$\frac{i\theta N}{8\pi^2} B \wedge B \to \frac{i\theta N}{8\pi^2} B \wedge B + \frac{i\theta N}{4\pi^2} \mathrm{d}\lambda \wedge B + \frac{i\theta N}{8\pi^2} \mathrm{d}(\lambda \wedge \mathrm{d}\lambda) \ , \tag{2.154}$$

is able to cancel the dangerous term. The total derivative does not pose a problem for spacetime manifolds without boundary.
Thus we arrive at a gauge invariant action that takes Witten effect into account

$$S = \frac{i}{2\pi} \int \left[\tilde{F} \wedge (F - NB) - \frac{N\theta}{4\pi} B \wedge B \right] \ , \tag{2.155}$$

with 1-form gauge transformations

$$\begin{aligned} B &\to B + \mathrm{d}\lambda \ , \\ A &\to A + N\lambda \ , \\ \tilde{A} &\to \tilde{A} - \frac{\theta}{2\pi}\lambda \ . \end{aligned} \tag{2.156}$$

In order to understand the physical effect of the $B \wedge B$-term we use the description of 't Hooft loops given earlier, we remove a cylinder around the worldline of the monopole, which is locally $\mathbb{R} \times S^2$ and fix the magnetic flux through the S^2

$$\frac{1}{2\pi} \oint_{S^2} B = \frac{m}{N} \ . \tag{2.157}$$

Now we split the 2-form B into a singular part, corresponding to magnetic charges, and smooth part for electric configurations

$$B = B_{\text{sing}} + B_{\text{sm}} \ , \tag{2.158}$$

the part of the $B\wedge B$-term containing the singular contribution reads ($B_{\text{sing}}\wedge B_{\text{sing}}$ vanishes due to properties of the wedge product)

$$\frac{iN\theta}{8\pi^2}\int 2B_{\text{sing}}\wedge B_{\text{sm}} = \frac{im\theta}{2\pi}\int_{S^2_\perp} B_{\text{sm}} \ . \tag{2.159}$$

It transforms as

$$\frac{im\theta}{2\pi}\int_{S^2_\perp} B_{\text{sm}} \to \frac{im\theta}{2\pi}\int_{S^2_\perp}(B_{\text{sm}}+\mathrm{d}\lambda) \ . \tag{2.160}$$

For $\theta \neq 0$ we see, that it needs to be the boundary of a surface operator. This surface carries the appropriate electric flux $-m/N$ as discovered before. This shows that the new term reproduces the expected behaviour. Now we should note several things, first of all introducing the θ term leads to a phase, when two flux tubes intersect. Furthermore when $\theta = 2\pi$ this phase is $2\pi/N$.This means we only see a periodicity of $2\pi N$ in θ, which for the $SU(N)/Z_N$ theory is perfectly fine however as discussed in [44, 79]. As discussed before this phase however has a very natural interpretation in the case of superconductors and especially doesn't rely on any subtleties regarding the spectrum of allowed line operators, but just on the presence of non-trivial topological vortices. This implies that the BB term we observed should be there even in the $SU(N)$ case. However in the pure $SU(N)$ case we expect to see a 2π periodicity and not a $2\pi N$ periodicity, therefore this result seems rather odd. We should note however that we haven't taken into account that we expect the $SU(N)$ to have N different quasistable vacua.
Now, we want to incorporate the presence of these N different quasi-stable vacua labeled by k. For this purpose we make use of the analogy

$$k \to k+1 \leftrightarrow \theta \to \theta - 2\pi \ . \tag{2.161}$$

The change of the vacuum energy cannot be quantified in the topological framework. The Aharanov Bohm type phases as well as the phases that fluxes acquire upon crossing are however well described by the topological theory. The action describing a dual superconductor with θ term and (quasi) vacua labeled by k is then given by

$$S = \frac{i}{2\pi}\int\left[\tilde{F}\wedge(F-NB) - \frac{N\theta}{4\pi}B\wedge B + \frac{Nk}{2}B\wedge B\right] \ . \tag{2.162}$$

The effect of the change in the condensate charge for different k alters the gauge transformations for $\tilde{A}$ as well

$$\tilde{A} \to \tilde{A} - \frac{\theta}{2\pi}\lambda + k\lambda \ . \tag{2.163}$$

Almost the same action and transformations were presented without derivation in [66], but with a different application. There the authors coupled this type of TFT to a dynamical $SU(N)$ theory in order to obtain a $SU(N)/\mathbb{Z}_N$ theory. Our interpretation is different, we regard the action as a topological field theory for the confining phase of $SU(N)$ YM theory itself, where the label k labels the different (quasi) vacua of the YM theory and the TFT should only be used to compute non trivial AB type phases. Similar actions are found for theories with oblique confinement in [43] and for lattice models of Yang-Mills theories in [80], however neither paper tried to identify the quasivacua in YM or considered domain walls between them.
The effect of the $B \wedge B$-term in the gauge theory can be seen by integrating out $\tilde{F}$. Remember that by dualizing $\tilde{A}$, $\tilde{F}$ is regarded as independent field fulfilling the Bianchi identity induced by the dualization term. Therefore, we use the equations of motions of $\tilde{F}$ rather than $\tilde{A}$ and obtain the constraint equation

$$B = \frac{1}{N}F \ . \tag{2.164}$$

Plugging this constraint back into the action 2.155 we find

$$S = -\frac{i\theta}{8\pi^2 N}\int F \wedge F + \frac{ik}{4\pi N}\int F \wedge F \ . \tag{2.165}$$

This exactly reproduced the θ-term in YM gauge theory up to the factor of $1/N$.[8] Taking into account, that k labels N different (quasi) vacua restores the 2π symmetry which is needed in the $SU(N)$ theory. In the following discussion we keep the vacuum angle θ fixed at 0. In pure YM theory without axions θ is a free parameter and can be fixed to some constant. The action therefore is given by

$$S = \frac{i}{2\pi}\int \left[\tilde{F} \wedge (F - NB) + \frac{Nk}{2} B \wedge B\right] \ , \tag{2.166}$$

[8] here one should also note, that our fluxes are normalized such that fundamental charges are 1/N

with gauge transformations

$$\begin{aligned} B &\to B + \mathrm{d}\lambda \ , \\ A &\to A + N\lambda \ , \\ \tilde{A} &\to \tilde{A} + k\lambda \ . \end{aligned} \tag{2.167}$$

YM Domain Walls

With the action (2.166) and the corresponding gauge transformations we can now investigate properties of YM domain walls in this topological setting. For that we add a jump in k on a codimension one surface, i.e. a domain wall. We only consider fundamental walls for which $\Delta k = 1$. The three dimensional worldvolume of the wall is denoted $\mathcal{V}$.
This jump in k has consequences for the properties under gauge transformations of the action. The first term in (2.166) changes under 1-form transformations 2.167 to

$$\frac{i}{2\pi}\tilde{F}\wedge(F-NB) \to \frac{i}{2\pi}\tilde{F}\wedge(F-NB)+\frac{i}{2\pi}(\mathrm{d}k\wedge\lambda+k\mathrm{d}\lambda)\wedge(F-NB) \ . \tag{2.168}$$

The term $\mathrm{d}k$ can be regarded as a δ-function on the worldvolume $\mathcal{V}$ in the following sense. For an arbitrary 3-from Ω

$$\int \mathrm{d}k \wedge \Omega = \int_{\mathcal{V}} \Omega \ . \tag{2.169}$$

The second term in (2.166) changes as well

$$\frac{iNk}{4\pi}B \wedge B \to \frac{iNk}{4\pi}B \wedge B + \frac{iNk}{2\pi}\mathrm{d}\lambda \wedge B + \frac{iNk}{4\pi}\mathrm{d}(\lambda \wedge \mathrm{d}\lambda) \ . \tag{2.170}$$

The total change of the action reads

$$\Delta S = \int \left[\frac{i}{2\pi}\mathrm{d}k \wedge \lambda \wedge (F - NB) + \frac{ik}{2\pi}\mathrm{d}\lambda \wedge F + \frac{iNk}{4\pi}\mathrm{d}(\lambda \wedge \mathrm{d}\lambda)\right] \tag{2.171}$$

Splitting the terms into a total derivative and $\mathrm{d}k$ contribution

$$\begin{aligned} \frac{iNk}{4\pi}\mathrm{d}(\lambda \wedge \mathrm{d}\lambda) &= \mathrm{d}\left(\frac{iNk}{4\pi}\lambda \wedge \mathrm{d}\lambda\right) - \frac{iN}{4\pi}\mathrm{d}k \wedge \lambda \wedge \mathrm{d}\lambda \ , \\ \frac{ik}{2\pi}\mathrm{d}\lambda \wedge F &\overset{\mathrm{d}F=0}{=} \mathrm{d}\left(\frac{ik}{2\pi}\lambda \wedge F\right) - \frac{i}{2\pi}\mathrm{d}k \wedge \lambda \wedge F \ , \end{aligned} \tag{2.172}$$

the total change in the action becomes

$$\Delta S = \int \left[\frac{i}{2\pi} \mathrm{d} \left(k\lambda \wedge F + \frac{Nk}{2} \lambda \wedge \mathrm{d}\lambda \right) - \frac{iN}{4\pi} \mathrm{d}k \wedge (2\lambda \wedge B + \lambda \wedge \mathrm{d}\lambda) \right] . \tag{2.173}$$

The first term is a total derivative and hence only relevant if the spacetime manifold has a boundary. The second term is more intersesting. It implies that YM domain walls generate a contribution to the action on the worldvolume of the wall ($\Delta k = 1$)

$$\Delta S_{\mathrm{wall}} = -\frac{iN}{4\pi} \int_{\mathcal{V}} [2\lambda \wedge B + \lambda \wedge \mathrm{d}\lambda] \ . \tag{2.174}$$

To allow for domain walls in the topological theory and simultaneously retain the gauge symmetry of the action we consequently have to introduce degrees of freedom on the worldvolume of the domain wall. These new degrees of freedom have to transform under the 1-form gauge transformation. The natural choice is a 1-form field $\mathcal{A}$ transforming under shift symmetry

$$\mathcal{A} \to \mathcal{A} - \lambda \ . \tag{2.175}$$

This is very similar to statistical gauge fields often used in condensed matter physics, e.g. for the fractional quantum Hall effect, see [81].
In order to cancel the contribution of the fundamental domain wall the appropriate worldvolume action for $\mathcal{A}$ is

$$S_{\mathcal{V}} = -\frac{iN}{4\pi} \int_{\mathcal{V}} [2\mathcal{A} \wedge B + \mathcal{A} \wedge \mathrm{d}\mathcal{A}] \ . \tag{2.176}$$

This action contains a coupling of the 2-form field B to $\mathcal{A}$ and a $U(1)$ CS-term at level N. An equivalent action arises for boundaries of the spacetime manifold, see [66].
The full action in the presence of domain walls is the sum of both contributions, $S + S_{\mathcal{V}}$.
Gauge invariance in the presence of a YM domain wall is thus only possible if one includes a level N CS-term on the domain wall. Precisely this term was predicted by string theory investigations for SYM theories in [78] and by breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry in [82]. Our construction now shows that such a term should as well be present on YM domain walls, even without supersymmetry. The construction with some modifications also

works for SYM domain walls, we will will discuss this in the next two chapters. This is to our knowledge the first direct construction of the level N CS-theory for YM domain walls and because it only uses topological characteristics of the theory it should be unaltered under various deformations of the dynamical theory.

Fluxe Tubes Ending on Domain Walls

The string theory constructions of [83] for SYM and [40] for YM theories suggest that electric flux tubes can end on these domain walls, just like fundamental strings end on D-branes. In this section we would like to investigate whether this is also true in our formalism.

The criterion for the existence of certain operators in the topological theory is the gauge invariance under 1-form gauge transformations. The 2-form field B couples to the electric fluxes, but the pure surface operator of B is not gauge invariant on its own and has to be extended by a Wilson loop, $\mathcal{W}$, of the gauge field A. With the domain wall on the other hand there is yet another field transforming under the 1-form transformations, i.e. $\mathcal{A}$.

This opens the following possibility. Consider a open electric surface operator of B over the 2-surface Σ, where the boundary $\partial\Sigma$ is located in the worldvolume of the domain wall $\mathcal{V}$, $(\partial\Sigma \subset \mathcal{V})$. Then, the operator

$$\exp\left(iN\eta\int_{\Sigma} B + iN\eta\oint_{\partial\Sigma}\mathcal{A}\right) \ , \tag{2.177}$$

is gauge invariant. This mechnism allows the electric flux tubes, coupling to B, to end on a domain wall, see figure 2.18. The topological theory successfully comprises the occurrence of the level N CS-term as well as the possibility for electric flux tubes to end on the domain wall. These properties are very hard to reconstruct in dynamical models but become rather simple in this topological framework, both describing the properties of a statistical gauge field $\mathcal{A}$.[9]

The downside of the topological theory, however, is that all dynamical phenomena can not be investigated. Values like the string tension, the domain

[9] The terminology statistical gauge field is borrowed from the quantum hall literature, the Chern-Simons gauge field we described only encodes statistics changing Aharanov Bohm type phases and no dynamics, hence the name statistical

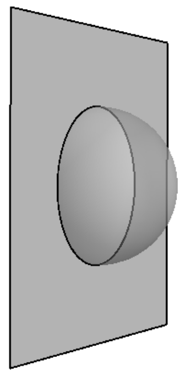


Figure 2.18: An electric surface operator ending on a YM domain wall

wall tension, or dynamical interactions are invisible in the TFT approach.

2.6 SYM Theories

Next we will look at supersymmetric $\mathcal{N} = 1$ SYM theories. Supersymmetry allows us to gain some insights and exact results for the strong coupling regime by analytic continuation from weak coupling results, where supersymmetry guarantees, that certain quantities are analytic. Therefore understanding these supersymmetric theories might yield non-trivial checks of some of the mechanisms considered in the non-supersymmetric case. It also differs in some non-trivial properties making it in the certain sense closer to QCD.

There is strong evidence that SYM shares some of the most important properties of the non-supersymmetric theory. First, there should be only colorless asymptotic states in the spectrum. Second, fundamental electric charges should be confined, represented by the area law of Wilson loops in the fundamental representation. And finally, the theory should dynamically generate a mass gap, so there are no massless degrees of freedom in the spectrum [84].This can be summarized by saying, that confinement qualitatively works in the same way as before.

Again we restrict our discussion to the gauge group $SU(N)$. The notation concerning supersymmetry is taken from [34]. In this chapter we will again use an index notation.

2.6.1 Lagrangian density of SYM

For the supersymmetric extension of pure gauge theories we need a gauge field in the adjoint representation A^a_μ, and its fermionic superpartners λ^a, called gauginos, which are majorana fermions transforming in the adjoint representation.

We can construct a vector superfield with these component fields and use the Wess-Zumino gauge (see e.g. [85]) to get ($A^a_{\alpha\dot{\alpha}} = A^a_\mu \left(\sigma^\mu\right)_{\alpha\dot{\alpha}}$)

$$\begin{aligned} V &= V^a T^a \quad , \quad \text{with} \\ V^a &= -2\theta^\alpha \bar{\theta}^{\dot{\alpha}} A^a_{\alpha\dot{\alpha}} - 2i\bar{\theta}^2(\theta\lambda^a) + 2i\theta^2(\bar{\theta}\bar{\lambda}^a) + \theta^2\bar{\theta}^2 D^a \quad . \end{aligned} \tag{2.178}$$

The scalar field D^a is an auxiliary field and non-dynamical. It should not be confused with D_μ or $D_{\alpha\dot{\alpha}}$ which denotes the covariant derivative in Lorentz vector or spinorial notation respectively and does not carry a color index.

The non-abelian field strength tensor superfield is given by

$$W^a_\alpha = i\left(\lambda^a_\alpha + i\theta_\alpha D^a - \theta^\beta F^a_{\alpha\beta} - i\theta^2 D_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha a}\right) \quad , \tag{2.179}$$

where

$$\begin{aligned} F^a_{\alpha\beta} &= -\frac{1}{2}F^a_{\mu\nu}\left(\sigma^\mu\right)_{\alpha\dot\alpha}\left(\sigma^\nu\right)_\beta{}^{\dot\alpha} \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu \\ D_\mu X^a &= \partial_\mu X^a + f^{abc}A^b_\mu X^c \quad , \end{aligned} \tag{2.180}$$

where X can stand for any field in the adjoint.
The kinetic term is generated by the following term in the superpotential

$$\begin{aligned} W^{a\alpha}W^a_\alpha = &- \lambda^a\lambda^a - 2i(\lambda^a\theta)D^a + 2\lambda^{a\alpha}F^a_{\alpha\beta}\theta^\beta + \\ &+ \theta^2\left(D^aD^a - \frac{1}{2}F^{a\alpha\beta}F^a_{\alpha\beta}\right) + 2i\theta^2\bar\lambda^a_{\dot\alpha}D^{\dot\alpha\alpha}\lambda^a_\alpha \quad . \end{aligned} \tag{2.181}$$

In the absence of matter fields, this is the only term in the SYM Lagrangian. The θ-term of the gauge theory occurs naturally in supersymmetric theories by the complexification of the coupling constant

$$\frac{1}{g^2} \to \frac{1}{g^2} - i\frac{\theta}{8\pi^2} \quad . \tag{2.182}$$

The full Lagrangian density of the pure gluodynamics is

$$\begin{aligned} \mathcal{L} &= \frac{1}{4g^2}\int \mathrm{d}^2\theta W^{a\alpha}W^a_\alpha + h.c. \\ &= -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{i}{g^2}\lambda^{a\alpha}D_{\alpha\dot\beta}\bar\lambda^{a\dot\beta} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde F^{a\mu\nu} \quad , \end{aligned} \tag{2.183}$$

with dual field strength tensor $\tilde F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The only difference to the non-supersymmetric term is the kinetic terms for the gauginos.
The supersymmetric action is just the usual QCD action with one Majorana flavor in the adjoint representation.
One important difference to normal YM is the occurence of the gluino condensate , which has a close analogue in QCD, the observed quark condensate.

2.6.2 Gluino Condensation

In this section we are going to explain how to determine the gluino condensate exactly thanks to supersymmetry and what properties this predicts.

Chiral Anomaly

We are looking for a spontaneous breaking of chiral symmetry. First we should notice that as in QCD the chiral symmetry

$$\lambda^a \to e^{i\alpha}\lambda^a, \quad \bar{\lambda}^a \to e^{-i\alpha}\bar{\lambda}^a \ , \tag{2.184}$$

is explicitly broken via the chiral anomaly. In pure SYM the chiral current is just the R-symmetry current and can be written [34] as

$$R^\mu = \frac{1}{g^2}\bar{\lambda}^a \bar{\sigma}^\mu \lambda^a \ . \tag{2.185}$$

The same triangle diagram as in massless QCD, where the gauginos run in the loop leads to an anomaly of the chiral current. Relative to quarks in QCD there is an additional group theoretic factor of N, since gauginos transform in the adjoint

$$\partial_\mu R^\mu = \frac{N}{16\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \ . \tag{2.186}$$

The factor N implies that there is an unbroken $\mathbb{Z}_{2N}$ subgroup of the chiral $U(1)$ under which the gauginos transform as

$$\lambda^a \to \exp\left(i\pi \frac{j}{N}\right)\lambda^a, \ \ j \in \{0, \dots 2N-1\} \ . \tag{2.187}$$

Here we also see, that part of the Z_{2N} just corresponds to spacetime rotations by 2π. Now we can try to understand the spontaneous breaking of this Z_{2N} via gaugino condensation. The gaugino bilinear acquires a non-vanishing vacuum expectation value, which breaks the $\mathbb{Z}_{2N}$ down to $\mathbb{Z}_2$. Thus, there are N equivalent vacua for $\mathcal{N}=1$ SYM theories, which are parametrized by the phase of the gaugino condensate

$$\langle \lambda^a \lambda^a \rangle \propto \exp\left(2\pi i \frac{k}{N}\right), \quad k \in \{0, ..., N-1\} \ . \tag{2.188}$$

Computing the Witten index also predicts N vacua [86]. In contrast to pure YM theory these are degenerate vacua which due to supersymmetry have a zero energy density.

The anomaly can be used to remove the phase of the gaugino condensate at the expense of introducing a θ term, this was first pointed out in [87]. The θ term is exactly like in YM

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \quad . \tag{2.189}$$

Using a chiral transformation of the gaugino fields with parameter α we shift the divergence of the R-current by

$$\partial_\mu R^\mu = \frac{N}{16\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \rightarrow (1+\alpha) \frac{N}{16\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \quad . \tag{2.190}$$

So we can move the phase of the gaugino condensate into a nontrivial θ term. What we observe in the end is

$$\langle \lambda^a \lambda^a \rangle_\theta = \langle \lambda^a \lambda^a \rangle_0 \exp\left(i \frac{\theta}{N} \right) \quad . \tag{2.191}$$

For $\theta \rightarrow \theta + 2\pi$ we see that the vacua are exchanged cyclically as shown in figure 2.19. One should note however , that the phase of the gaugino con-

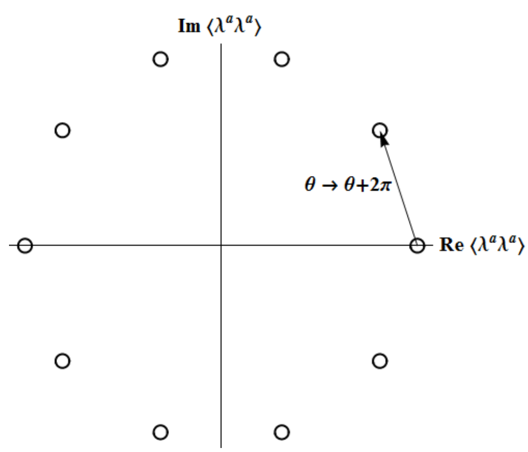


Figure 2.19: Schematic illustration of the vacua of $SU(N=10)$ SYM and the action of a shift of θ

densate can vary in space, i.e. it should be considered a field, which implies that we have an analogue of the axion or η'.

Derivation of the Gaugino Condensate

In 1987 Shifman and Vainshtein explicitly calculated the gaugino condensate, see [87].
For simplicity we will only review their derivation for the $SU(2)$ case and just state the general result, the presentation follows [34]
We introduce two chiral matter fields in the spinor representation Φ_i^c

$$\Phi_i^c = \phi_i^c + \sqrt{2}\theta\psi_i^c + \theta^2 F_i^c \ . \tag{2.192}$$

$i \in \{1,2\}$ represents is flavor index and $a \in \{1,2\}$ the color index. Both are $SU(2)$ indices. We also add a mass term to the superpotential

$$\mathcal{W} = \frac{m}{2}\Phi_i^c\Phi_c^i \ . \tag{2.193}$$

This means our Lagrangian is given by

$$\mathcal{L} = \frac{1}{2g^2}\int \mathrm{d}\theta^2 W^{a\alpha}W_\alpha^a + \frac{1}{4}\int \mathrm{d}\theta^2\mathrm{d}\bar{\theta}^2\,\bar{\Phi}^i e^V \Phi^i + \frac{1}{4}m\left[\int \mathrm{d}^2\theta\,\Phi_i^c\Phi_c^i + h.c.\right] \tag{2.194}$$

For zero mass m the scalar potential comes from the D-terms and is given by

$$V(\phi_i) = \frac{g^2}{8}\left(\sum_i \bar{\phi}_i T^a \phi_i\right)^2 , \quad \text{with} \quad T^a = \frac{1}{2}\sigma^a \ , \tag{2.195}$$

where ϕ_i are the squark fields. This shows, that there is a modulus which can be parametrized by one complex variable v

$$\phi_1 = \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ v \end{pmatrix} \ . \tag{2.196}$$

This degeneracy is exact to all orders in perturbation theory due to non-renormalization theorems; it might however be lifted by non-perturbative effects.
For $vg \gg \Lambda$ (Λ denotes the analog of the QCD scale) the gauge group is completely broken. By the usual Higgs mechanism all gauge bosons acquire a mass $m_v \propto gv$. The fields can be rearranged into three massive vector superfields and one light chiral superfield, this is called super-Higgs mechanism. The light chiral field describes the flat direction $\Phi^2 = \Phi_i^c\Phi_c^i$. In the low energy limit we can integrate out the vector fields and we get an effective

theory for the light superfield Φ^2. Since the vacuum structure of the theory is an IR property of the theory we should be able to understand it from the effective theory. As long as $vg \gg \Lambda$ integrating out the vector fields is a well defined procedure.
Once we switch on the mass term the flat direction is lifted, as we get an additional term in the potential

$$V_m \propto |mv|^2 \ . \tag{2.197}$$

This would lead us to the conclusion, that v should vanish, i.e. the Higgs mechanism should be undone. However once this happens we are not allowed to ignore non-perturbative effects, as they might become relevant.There is an additional contribution from instantons, this contribution is strongly constrained by an anomaly free R-symmetry [88, 89], this contribution is given by

$$\mathcal{W}_{inst} = C\frac{\Lambda^5}{\Phi^2} \ , \tag{2.198}$$

where C is a finite constant.This calculation can only be done as long as $gv \gg \Lambda$, since otherwise one encounters the well known problems of large instantons, which implies a breakdown of the dilute gas approximation. This gives an additional effect due to the F-Term

$$\bar{F} = -\frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} = -\frac{m}{2}\Phi + 2C\frac{\Lambda^5}{(\Phi^2)^2}\Phi \ . \tag{2.199}$$

This has to vanish in a vacuum state, i.e.

$$v^2 = \pm 2\left(\frac{C\Lambda^5}{m}\right)^{\frac{1}{2}} \ . \tag{2.200}$$

As long as the mass is small we see that this approximation is justified, i.e. $gv \gg \Lambda$.
Now we can use the Konishi anomaly to relate this to the value of the gaugino condensate (see [90])

$$\frac{1}{8}\bar{D}^2\left(\bar{\Phi}^i e^V \Phi^i\right) = \frac{1}{2}m\Phi^2 + \frac{1}{16\pi^2}W^{a\alpha}W^a_\alpha \ . \tag{2.201}$$

By looking at the lowest component we easily see

$$\frac{1}{16\pi^2}\langle\lambda^a\lambda^a\rangle = \frac{1}{2}m\langle\phi^2\rangle = \pm\left(C\Lambda^5 m\right)^{\frac{1}{2}} \ . \tag{2.202}$$

This is the gaugino condensate with one light flavor.
In order to take the $m \to \infty$ in which the above theory is simply pure SYM we use holomorphicity.
In order to make use of holomorphicity we promote the mass parameter m to a chiral spurion superfield M, where m is the vev of lowest component of M. This new superfield leads to an extended non-anomalous R-symmetry, which survives in the strong coupling regime

$$W_\alpha \to e^{i\gamma} W_\alpha\,, \;\; \Phi_i \to e^{-i\gamma}\Phi_i\,, \;\; M \to e^{4i\gamma} M\,, \;\; \theta_\alpha \to e^{i\gamma}\theta_\alpha \;\; . \tag{2.203}$$

Using this we get

$$\langle W^{a\alpha} W^a_\alpha \rangle \propto M^{\frac{1}{2}} \;\; \Rightarrow \;\; \langle \lambda^a \lambda^a \rangle \propto m^{\frac{1}{2}} \;\; . \tag{2.204}$$

This is an exact result and valid at weak and at strong coupling. A similar analysis gives

$$\langle \Phi^2 \rangle \propto M^{-\frac{1}{2}} \;\; \Rightarrow \;\; \langle \phi^2 \rangle \propto m^{-\frac{1}{2}} \;\; . \tag{2.205}$$

This allows us to extend the previous results to strong coupling and we can take the limit $m \to \infty$. In order to do this we should match the dynamically generated scales in both theories. The β-function with N_i flavors and N colors is given by [34, 91]

$$\beta_0 = 3N - \frac{1}{2} N_i \;\; . \tag{2.206}$$

The dynamically generated scale Λ can be deduced from (2.14) and is given by

$$\frac{\alpha(M)}{2\pi} = \frac{g^2(M)}{8\pi^2} \approx \frac{1}{\beta_0 \ln\left(\frac{M}{\Lambda}\right)} \Rightarrow \Lambda = M \exp\left(-\frac{2\pi}{\beta_0 \alpha}\right) \;\; . \tag{2.207}$$

In our case we get $\beta_0 = 5$ for small m and $\beta_0 = 6$ for large m. This leads to two different mass scales Λ' and Λ, which for $M = m$ are given by

$$\Lambda' = m \exp\left(-\frac{2\pi}{5\alpha}\right), \; \Lambda = m \exp\left(-\frac{2\pi}{6\alpha}\right) \;\; . \tag{2.208}$$

This gives us

$$\Lambda'^5 m = \Lambda^6 \;\; . \tag{2.209}$$

From this rotation we can deduce the gaugino condensate to be

$$\langle \lambda^a \lambda^a \rangle = \pm \tilde{C} \Lambda^3 \ , \tag{2.210}$$

with some non-zero constant $\tilde{C}$.
For general $G = SU(N)$ we get [87]

$$\langle \lambda^a \lambda^a \rangle = C(N) \exp\left(2\pi i \frac{j}{N}\right) \Lambda^3 \ , \tag{2.211}$$

leading to N distinct vacua with $j = 1, \dots, N-1$. The exact N dependence is given by [83]

$$\langle \lambda^a \lambda^a \rangle = N \Lambda^3 \exp\left(2\pi i \frac{j}{N}\right) \ . \tag{2.212}$$

2.7 SYM Domain Walls

In this chapter we describe some exact results for SYM domain walls, which cannot be obtained in a non-supersymmetric framework. Especially the wall tension can be calculated exactly under the assumption that the walls are BPS states.
Then we will generalize the topological theory to the bosonic sector of the SYM domain walls and point out the differences.
In the last chapter we have seen that $\mathcal{N} = 1$ SYM with gauge group $SU(N)$ in four dimensional spacetime has N vacua, which arise due to spontaneous breaking of a $\mathbb{Z}_{2N}$ symmetry to $\mathbb{Z}_2$.
Domain walls between these vacua were first discussed by Dvali and Shifman in [42, 92] and later studied by a number of researchers. These domain walls were also studied in a M-theory setup by Witten [83], where he argued that those walls should be D-brane analogs. As in pure YM theories the chromo-electric flux tubes of the SYM theories can end on the walls. This closely mimics fundamental open strings, which can end on D-branes. Furthermore in 't Hoofts large N limit the tension of these domain walls is proportional to N rather to N^2 which is very odd for solitons arising in a theory of glueballs with coupling $1/N^2$. This scaling also closely resembles D-Branes, whose tension also behaves strangely when considered as solitons of the closed string sector. In the non-supersymmetric case the only argument for a linear N dependence of the wall tension come from an embedding of the theory into string theory and subsequent identification with bona fide D-Branes. This situation is not very satisfying, since it doesn't really answer what the mechanism is for D-Brane analogues to arise in field theories. In SYM we are in much better shape, since we can calculate their tension from quantum field theory.
In order to calculate the tension of the domain walls we start by looking at the central extension of the superalgebra. The central charges can then be related to topological quantum numbers and due to the assumed BPS property of the states, the tensions for BPS-walls is directly related to the central charge [93]. BPS-states are states which preserve part of the supersymmetry, in the case of domain walls in four dimensions they preserve two of the four real supercharges. So far there is no explicit construction of the walls and therefore no explicit proof, that they saturate the BPS bound, however there is some evidence that they do saturate the BPS bound (see e.g. [94–97]).

2.7.1 Exact Domain Wall Tension

In the following we will assume that the SYM domain walls are BPS saturated. If this is not be the case the presented calculations lead to a lower bound on the wall tension. Even if they don't saturate the bound the physical configuration will probably have an energy close to that [98]. The centrally extended superalgebra is given by [34]

$$\{Q_\alpha, Q_\beta\} = -4\Sigma_{\alpha\beta}\bar{Z} \ , \tag{2.213}$$

with the wall area tensor defined by

$$\Sigma_{\alpha\beta} = -\frac{1}{2}\int \mathrm{d}x_{[\mu}\mathrm{d}x_{\nu]}\, (\sigma^\mu)_{\alpha\dot{\alpha}}\, (\sigma^\nu)^{\dot{\alpha}}_{\beta} \ , \tag{2.214}$$

where the integral runs over an equal time slice of the worldvolume of the wall. By rearranging the superalgebra, we see, that there are two supercharges in the domain wall background $Q^{(w)}_\alpha$ with anticommutation relation

$$\left\{Q^{(w)}_\alpha, Q^{(w)}_\beta\right\} = 8\Sigma_{\alpha\beta}(T - |Z|) \ , \tag{2.215}$$

where T is the domain wall tension. If we want to preserve those, the anticommutator should vanish, i.e. the wall tension is equal to the absolute value of the central charge $|Z|$.
For SYM with matter the central charge (up to total superderivatives) is given by [99, 100]

$$Z = \frac{2}{3}\Delta\left\{3\mathcal{W} - \frac{3N - \frac{1}{2}N_i}{16\pi^2}W^{a\alpha}W^a_\alpha\right\}_{\theta=0} \ , \tag{2.216}$$

where $\mathcal{W}$ denotes the superpotential. Δ denotes the difference of the expression in brackets at spatial infinity perpendicular to the domain wall. For pure gluodynamics there is no superpotential and $N_i = 0$. So we see that the central charge is an effect that arises due to the anomaly and has no analogue in the classical theory.
Now we can use the BPS bound to deduce the tension of the wall as

$$T = |Z| = \frac{N}{8\pi^2}\,|\langle\lambda^a\lambda^a\rangle_{k_1} - \langle\lambda^a\lambda^a\rangle_{k_2}| \ , \tag{2.217}$$

for a domain wall interpolating between vacua where the gaugino phase is labeled by k_j. Let us now consider SYM theory in the large N limit with

$$\lambda \equiv g^2 N \ , \tag{2.218}$$

as before. The Lagrangian density now changes to

$$\mathcal{L} = \frac{N}{4\lambda} \int \mathrm{d}\theta^2 W^{a\alpha} W^a_\alpha + h.c. \ . \tag{2.219}$$

Using this we get for the tension of a domain wall interpolating between two supersymmetric vacua labeled by k_1 and k_2

$$T = \frac{N^2}{8\pi^2} \Lambda^3 \left| \exp\left(2\pi i \frac{k_1}{N}\right) - \exp\left(2\pi i \frac{k_2}{N}\right) \right| \ . \tag{2.220}$$

We can rewrite this to obtain the following result [94]

$$\begin{aligned} T =& \frac{N^2}{8\pi^2} \Lambda^3 \left| \exp\left(\pi i \frac{k_1 + k_2}{N}\right) \left[\exp\left(\pi i \frac{k_1 - k_2}{N}\right) - \exp\left(-\pi i \frac{k_1 - k_2}{N}\right) \right] \right| = \\ =& \frac{N^2}{4\pi^2} \Lambda^3 \left| \sin\left(\pi \frac{k_1 - k_2}{N}\right) \right| \ . \end{aligned} \tag{2.221}$$

For an elemetary domain wall with $k_1 - k_2 = 1$ we get

$$T = \frac{N}{4\pi} \Lambda^3 + \mathcal{O}\left(\frac{1}{N}\right), \tag{2.222}$$

one should note, that there is no $\mathcal{O}(1)$ contribution, since the Taylor expansion of a sine only contains odd powers. If we now try to interpret this in terms of a non-critical string theory of closed strings with string coupling $g_s \propto \frac{1}{N}$ we see that the tension of the elemntary domain walls scales as

$$T \propto \frac{1}{g_s} \ , \tag{2.223}$$

which is the same scaling behaviour as for D-branes [101].

2.7.2 TFT for SYM Domain Walls

In principle all the relevant topological field theories can be made supersymmetric [102, 103], here we will only look at the bosonic sector however. Formulating our construction in an explicitly supersymmetric language would be an interesting future project.

TFT for SYM

The general construction of the non-supersymmetric case still is valid and just gets modified in some minor details. Confinement should still be due to condensation of charge N monopoles. In SYM theories we have an axion, which effectively acts as a dynamical θ, this allows us make monopoles out of the condensed dyons in the YM quasivacua by using the Witten effect, at the expense of generating a vev for the axion. This suggests the identification of the quasivacua of YM with the vacua of SYM, with the difference, that the quasivacua now become degenerate, since we have an axion. Therefore the TFT describing out system is

$$S = \frac{i}{2\pi} \int \left[\tilde{F} \wedge (F - NB) - \frac{N\theta}{4\pi} B \wedge B \right] \ , \tag{2.224}$$

with 1-form gauge transformations

$$\begin{aligned} B &\to B + \mathrm{d}\lambda \ , \\ A &\to A + N\lambda \ , \\ \tilde{A} &\to \tilde{A} - \frac{\theta}{2\pi}\lambda \ . \end{aligned} \tag{2.225}$$

From our previous discussion we see, that the fundamental domain walls interpolating between vacua with $\Delta k = 1$ are equally well described by a jump of θ by 2π from the viewpoint of the TFT. The difference which will only become apparent in the dynamical theory is that in the YM case both sides of the wall differ by the charges of the condensate, while here they differ by the value of the effective θ angle.
Once we let θ vary in space the action is again not gauge invariant anymore

$$\Delta S = \int \left[-\frac{i}{8\pi^2} \mathrm{d}(2\theta\lambda \wedge F + N\theta\lambda \wedge \mathrm{d}\lambda) + \frac{iN}{8\pi^2} \mathrm{d}\theta \wedge (2\lambda \wedge B + \lambda \wedge \mathrm{d}\lambda) \right] \ . \tag{2.226}$$

If we take a sharp jump of θ by 2π on a codimension one surface $\mathcal{V}$

$$\mathrm{d}\theta = 2\pi\delta_{\mathcal{V}} \ , \tag{2.227}$$

and on a spacetime manifold without boundaries we get an additional contribution on the domain wall

$$\Delta S_{\text{wall}} = \frac{iN}{4\pi} \int_{\mathcal{V}} [2\lambda \wedge B + \lambda \wedge \mathrm{d}\lambda] \ . \tag{2.228}$$

This again implies that we have to include additional degrees of freedom on the wall given by

$$S_{\mathcal{V}} = \frac{iN}{4\pi} \int_{\mathcal{V}} [2\mathcal{A} \wedge B + \mathcal{A} \wedge \mathrm{d}\mathcal{A}] \ , \tag{2.229}$$

with gauge transformations

$$\mathcal{A} \to \mathcal{A} - \lambda \ . \tag{2.230}$$

We see again a coupling of the wall gauge field $\mathcal{A}$ to the Kalb-Ramond field B as well as a $U(1)$ CS-term of level N is present. These were predicted by both stringy constructions [78] as well as from embedding into $\mathcal{N} = 2$ theories [82], however both approaches suffer from an inability to take the limit of pure SYM. Therefore our construction can be regarded as an independent check of this result.

2.8 Summary and Outlook

2.8.1 Summary of results presented so far

Let us shortly summarize what we have seen so far in both YM as well as in SYM. We saw, that in YM we can expect $\mathcal{O}(N)$ quasivacua, which become stable at $N = \infty$. They are still nondegenerate however, furthermore there should be walls interpolating between different quasivacua, whose tension is $\mathcal{O}(N)$, analogously to D-Branes. We gave an explanation for these quasivacua in terms of dyon condensates. From this we constructed the low energy effective action for YM theory. Then we studied the domain walls in this low energy effective action, where we saw why electric flux tubes can end on the walls and we constructed the low energy effective action of the worldvolume theory of the domain wall, this turned out to be a level N abelian $U(1)$ CS theory.

In the case of SYM there were N exactly degenerate ground states. We explained why it is natural to identify these with the quasivacua seen earlier and explained a mechanism how it can be understood that they can become degenerate in the supersymmetric case. The mechanism makes use of gaugino condensation, the phase of the gaugino condensation effectively acting as an axion. The axion vev can turn dyons into monopoles via the Witten effect, so the different vacua effectively correspond to monopole condensates, however at different values of θ. We again studied the low energy effective theory and the effective theories for domain walls, which in turned out to be the same as in the YM case.

Now let us further comment on the dynamics of the walls. In the YM case the different quasivacua differ by the vev of the 4-form electric field, where the corresponding 3-form gauge potential is the composite CS 3-form. An explanation for the vev of the field strength is as follows: The 4-Form is effectively $\vec{E} \cdot \vec{B}$, which is non-zero in the presence of a dyon and it has the same sign if we take an antidyon. Since dyon condensation corresponds to a ground state populated by dyons and antidyons it is natural to expect an expectation value for the 4-form in the presence of a dyon condensate. Since we can expect strongly coupled dynamics to generate a kinetic term for the CS 3-form this explains the difference in energy between the different states, however estimating the magnitude of the splitting seems to be out of reach given our present understanding. We also see that the domain walls act as charges for the CS 3-form, the force between two walls being linear if there

is a kinetic term for the 3-form. The potential between two walls due to the 3-form grows linearly, since the field strength between them is constant. Furthermore the force between wall and anti-wall will be attractive. This shows, that in the YM theory the walls will be confined and we can expect to see neutral wall anti-wall bound states in the spectrum. These objects might also be extremely relevant to dynamics of real world QCD as Monte-Carlo studies on the lattice of pure Yang-Mills show that similar kind of objects appear in the YM vacuum [104–107], these were also seen in lattice studies of QCD [108–112]. Studying those objects in more detail in pure YM as well as QCD might be a very fruitful avenue for new research.
Let us now contrast this with the situation in SYM, where the domain walls are not confined and there is no long range force between them. This can be easily understood in two complementary ways. Both are due to the presence of an composite axionic degree of freedom. From the point of view of Dyon condensation, we expect, that the electric charge of the dyon condensate can effectively be screened via the Witten effect by giving a vev to the axion. Since the constant 4-form field strength could be understood as resulting from the dyons having a combination of electric and magnetic charge we immediately see, that the axion will effectively screen the 4-form field strength, which explains the absence of a long-range force. We can also understand this effect from a more field theoretic viewpoint as follows: The axion/CS Lagrangian should have the form (we assume here, that strong dynamics generates kinetic terms for both the axion and the CS 3-form)

$$\mathcal{L} = da \wedge \star da + da \wedge dC + dC \wedge \star dC, \tag{2.231}$$

after dualizing the axion into a 2-form B we see, that B becomes the longitudinal part of C in a form of Higgs mechanism and C acquires a mass, this explains from a field theoretic point of view why the 4-form field strength is screened. In the case of SYM it can also be seen, that the tension should scale like $\mathcal{O}(N)$ as there is a correspoding central charge allowing strong control over the tension, should the state be BPS.
It should also be pointed out, that our discussion sheds some new light on an intuitive picture of why flux tubes can end on walls suggested by Soo-Jong Rey and presented by Witten in [83]. This picture suggests, that one should think of the SYM vacuum on one side of the wall as a condensate of monopoles with charge $(1, 0)$ and on the original side of the wall as a condensate of charge $(1, 1/N)$, now it was suggested that on the wall bound states of anti-monopoles and dyons can form, which carry no magnetic charge and

electric charge of a quark and that these bound states serve as ends of flux tubes. The picture we advocate, which was already suggested in [98] is as follows: In the usual abelian Higgs mechanism the Higgs field can screen any charge, even fractional charges. Given this knowledge we can try to apply an electric field orthogonal to the wall (this would be the flux tube), from the side of the monopole condensate,now the dyon condensate on the other side can get polarized and screen the electric field at the expense of producing a magnetic field, however this magnetic field can again get screened on the other side by polarizing the monopoles. This implies that flux tubes can end on the walls and furthermore shows, that at the ends of flux tubes there will be a net magnetic flux across the wall precisely mimicking the behaviour of the CS term. However [98] didn't give any arguments in favor of the latter mechanism compared to the former, neither did they explain how this mechanism connects to the CS term on the wall. In our construction we see clearly why the Rey construction can not be correct. Once we assume that there are electric flux tubes with a flux corresponding to the flux produced by particles in the fundamental and we furthermore assume, that these are the flux tubes with the least amount of flux in the spectrum this immediately implies, that the condensing monopoles have to carry the analog of adjoint charges under the magnetic group, this in turn implies that the condensing dyons on the other side of the wall should carry gluon charges and not quark charges.

In the following chapter we will try to interpret these results in terms of a string theoretic picture, which was already hinted at during much of the discussion so far.

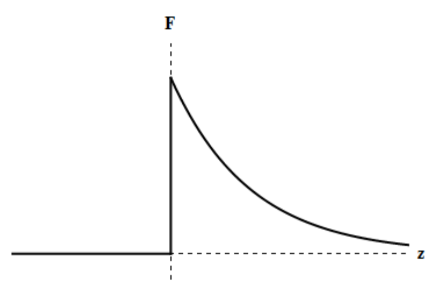


Figure 2.20: Profile of the electric field in the presence of a domain wall

2.8.2 Interpretation in terms of noncritical string theory

As we have seen the planar limit of YM theory can be understood in a certain sense as a noncritical string theory. We will first review some standard textbook results of critical string theory, the results reviewed can be found for example in [113]. In this section we will primarily focus on how to interpret certain D-Brane like objects and see how they arise, part of this will be a review of existing results, while other aspects will be new interpretations of the results presented in the previous parts. We will then conclude with some very interesting analogies to quantum hall systems, which might be fruitful for understanding certain properties of the SYM domain wall system.

Short Caricature of Critical String Theory

This section will review the relevant results from critical string theory, for the original references see [113]. Superstring theory can be understood as a theory of interacting strings, it is only consistent in $D = 10$ and contains various interesting objects. In fact there are five theories known as superstring theories, which are type I, type IIA, type IIB and two types of heterotic string theories, it is understood that those are in fact not distinct theories, but can be understood as particular limits of an 11 dimensional theory dubbed M-Theory. M-Theory has remained mysterious and understanding its dynamics is still an open problem. Here we will mostly focus on the different weakly coupled limits, i.e. on the five superstring theories.
Every string theory has a massless spin-2 particle in its spectrum, which corresponds to the lowest vibrational mode of a closed string. Since low energy dynamics for an interacting massless spin-2 particle is forced to be general relatitivity by consistency, it follows that string theory is a theory of quantum gravity. Also each string theory has a spin-0 particle called dilaton, whose vev determines the self coupling of the strings, furthermore there is always a Kalb-Ramond two form B for which the string acts as an electric charge, similar to our previous discussion of flux tubes. There is also a fermionic sector enforced by supersymmetry, which we will not discuss in detail.
Let us start by recapitulating some features of type II string theories. In these theories there are additional massless (p+1)-form gauge fields called Ramond-Ramond fields, which we denote by C_{p+1}. These forms couple electrically to objects with a $(p + 1)$ dimensional worldvolume, by dualizing the

RR fields we see that they couple magnetically to $(q+1)$ dimensional objects, with

$$p + q = 6 \ . \tag{2.232}$$

We should note that this is just the formula we discussed earlier for dualization with $D = 10$. These objects are called Dp-Branes [114] and have a $(p+1)$ dimensional worldvolume, in general their tension scales like $1/g_s$ where g_s is the closed string coupling, which is a crucial difference to solitons in weakly coupled theories whose tension should scale as $1/g_s^2$. Also since they couple to a massless gauge field, they carry an exactly conserved additive charge and again this charge enters into the BPS bound, which means, that they preserve part of the supersymmetry and in turn implies, that their worldvolume theory is supersymmetric. Furthermore there are additional curious properties; the worldvolume theory of $p - branes$ in general allow strings to end on them, furthermore the worldvolume theory contains massless gauge fields and the endpoints of strings look like fundamental charges to those gauge fields. A single brane carries a $U(1)$ gauge theory on its worldvolume, if we now stack several branes on top of each other, there are string excitations corresponing to strings stretching between the branes, which become light as the branes approach each other and they carry spin-1 and charge $(+1, -1)$ under the gauge group of the two branes, this leads to an enhancement of the symmetry on the worldvolume of a stack of branes. If we take k parallel branes separated from each other they are described by a $U(1)^k$ theory, however when they are put on top of each other we get a $U(k)$ symmetry. A second relevant process is the annihilation of a brane anti-brane pair, in this case the spectrum of strings stretching between the branes contains a tachyon of charge $(+1, -1)$, condensation of this tachyon leads to a higgsing down to the diagonal $U(1)$, it is then conjectured, that the remaining $U(1)$ becomes confining and that the flux tubes stretching between the confined charges (which are string endpoints) become fundamental strings [115]. This whole process goes under the name of tachyon condensation and can also explain how one can produce lower dimensional branes by annihilation of higher dimensional ones, this area of research was originated by Sen [116] and later significantly expanded in subsequent works.
In type IIA the RR fields exist for $p = 1, 3$, while for type IIB we have $p = -1, 2, 4$, i.e. in type IIA we have odd dimensional branes and in type IIB we have even dimensional branes. There is also an object dual to the fundamental string, which is an NS5 brane, this has the tension of a conventional

soliton, i.e. $1/g_s^2$. There is another relevant oddity of type IIA, namely there exists a D8 brane, which couples to a 9-form, a massless 9-form however is non propagating in 10 dimensions, its field strength is dual to a zero-form field strength F, which is called Romans mass. Note that the 0-form field strength can not be written as the outer derivative of a corresponding connection. It was shown by Polchinski, that upon crossing a D8 brane the properly normalized F jumps by an integer value. In the case of type IIB the 0-form C_0 and the dilaton can be combined into a complexified coupling τ analogously to the case of YM, where we combine θ and the coupling into a complexified coupling. Type IIB is mapped into itself under S-Duality[10], here τ is transformed to

$$\tau \to -\frac{1}{\tau} \ . \tag{2.233}$$

S-Duality is thus a string theoretic generalization of Montonen-Olive duality. To be explicit the dilaton Φ is identified with the real part of the complexified coupling via

$$\mathrm{Re}(\tau) = g_s = e^{\Phi} \ . \tag{2.234}$$

S-Duality also acts non-trivially on the two 2-form fields B and C_2, which get exchange when $\tau \to -1/\tau$. S-Duality also acts on the other string theories, however in a somewhat nontrivial manner, which we will not discuss here.
We can also discuss type-I theory, here one crucial difference is, the existence of open strings, furthermore the strings are unoriented. The open strings are charged under an $SO(32)$ gauge group, type I string theory also contains certain BPS branes, which exist for $p\,mod\,4 = 1$. These carry either symplectic or orthogonal gauge groups, there are also non BPS branes, these are charged under discrete Z_2 groups, i.e. two branes can annihilate. It was suggested by Witten [117], that for these branes instead of the usual connection between the dimensions of electric and magnetic objects one gets

$$p + q = 7 \ . \tag{2.235}$$

One should note, that this is precisely what we expect for Z_N charges, as these can be obtained by higgsing a theory with $\mathbb{Z}$ charges. One should note that this precisely reduces to our formula for the massive case with $D = 10$.

[10]S-Duality is actually a $SL(2,\mathbb{Z})$ group, this is one the generators, the other one is $\tau \to \tau + 1$

Glueballs, RR-forms and fluxtubes

Now we will try to interpret the results obtained for Yang Mills theories in terms of a string theory using analogies to critical superstring theory. The flux tubes are analogs of the fundamental strings in a critical string theory. However we should notice one crucial difference to critical string theory already at this point, N flux tubes can annihilate by forming Baryon vertices, this implies that the Kalb-Ramond field should be massive. This however is necessarily the case, since the YM theory has a mass gap, the massive Kalb-Ramond field should correspond to a 1^{+-} glueball, where we label glueballs by J^{PC} as usual. From lattice studies it can also be seen, that the 1^{+-} glueball is the lightest glueball of spin-1, see e.g. [118]. In the pure YM case we will also find a scalar 0^{++} glueball, which one would naturally identify with the dilaton of a critical string theory, this identification of the 0^{++} glueball with the dilaton was already made in [119], where the authors also identified the pseudoscalar gluino-gluino glueball an analog of the RR 0-form. There will also be a 2^{++} glueball, which will be the analog of the graviton in the present case, note again that this glueball is fairly light.
We can here suggest one more identification, we expect a 1^{--} glueball made from two gluinos and a gluon in the spectrum of SYM, this should be degenerate with the 1^{+-} glueball and live in the same multiplet, it seems natural to identify this glueball with RR 2-form.
The precise identification of (pseudo)-scalar glueballs with the dilaton and the RR 0-form can also be seen fairly explicitly from explicit stringy constructions, in these construction one gets a term [119]

$$e^{-\Phi}\,\mathrm{Tr}\,(F\wedge *F)+C_0\,\mathrm{Tr}\,(F\wedge F)\ \ , \tag{2.236}$$

naturally leading to a relation between the RR-field C_0 and the axion. Furthermore, this axionic field can be combined with the dilaton to a complexified coupling as in SYM. This can be used to relate them to the pseudoscalar and scalar glueballs of the field theory respectively, [120].

Duality

We can try to identify the magnetic object corresponding to the flux tube, following the reasoning of [117] for critical 10d type I string theory these should be objects, which acquire a phase after encircling the string. This immediately singles out a natural magnetic object. Monopoles of a fundamental charge under the dual magnetic group acquire an Aharanov Bohm phase upon encircling a string. Note that depending on whether we are studyin $SU(N)$ or $SU(N)/Z_N$ these might or might not be in the spectrum. Now we can look at the other non-perturbative objects we have in the theory. We have the domain walls, these couple electrically to the massive CS 3-form. Now let us note that this massive 3-form is the dual of the massive RR 0-form, which is the analog of the axion in our case. Therefore the object, that is paired up with the domain wall should be 0 dimensional, i.e. it should behave like an instanton. Futhermore again following the analogy with type I theory [117] we are looking for an object, which picks up a phase when it is moved across the wall. This again singles out a natural object, which is the string intersection. Intersections of strings on different sides of the wall induce phases differing precisely by $2\pi/N$. Now let us take a look at actual electric magnetic duality on the field theory side. In this case we should not just apply the duality transformations alluded to before, which are just a rewriting of the theory, but instead we should also exchange the magnetic charges making up the condensate with electric charges. Under this duality we can expect the massive Kalb-Ramond field B to be exchanged with a 2-form field C_2, where C_2 couples to magnetic flux and not to electric flux, this is very analogous to how S-Duality acts in type IIB, therefore magnetic flux should be analogous to D1-Branes.

Walls and Romans Mass

Let us again look at the domain walls in pure YM theory, here the walls couple electrically to the massless CS 3-form. When one crosses the domain wall it can be checked that the 4-form field strength of the CS 3-form jumps by a finite amount. [121] [11] Here we see a close analogy to the behaviour of a D8 brane in type IIA string theory, which has codimension one as well. When crossing a D8 brane the Romans mass jumps, this makes it natural to think of the 4-form field strength as an analog of Romans mass in type IIA. There is a further very suggestive relation between a CS term and the Romans mass in an AdS/CFT setup, the AdS/CFT dual of type IIA on $AdS_3 \times \mathbb{C}P^3$ is a theory with gauge group $U(k)^2$ and CS terms of level $(N, -N)$ [122], if we add a Romans in the bulk supergravity this changes the sum of the levels of

[11] This is closely analogous to the discontinuous electric field in the presence of electric charges in the 1+1d Schwinger model

the CS terms to be non-zero (amongst other things) [123], this is very similar to the behaviour we encounter here, where the 4-form field strength controls the CS level on the wall.

SYM, intersecting strings and worldsheet instantons

Let us turn our attention now more closely to SYM, here we will get a pseudoscalar glueball corresponding to variations in the phase of the gaugino condensate, as seen before this behaves like an axion and the CS 3-form becomes massive, furthermore this state will be exactly degenerate with the dilaton identified earlier, making it natural to combine them again into a single complex field, the natural multiplet for this set of glueballs is a three-form multiplet [124]. As already discussed earlier the objects coupling magnetically to the 3-form are naturally identified with string intersections, remember that the phase two fundamental flux tubes pick up when they cross each other is θ/N, which implies, that the crossings behave like fractional instantons. This also naturally explains why these couple linearly to the axion, the axion is a bound state of two gauginos, a single instanton has $2N$ gaugino zero modes, i.e. it produces $2N$ gauginos. This suggests, that a fractional instanton should have 2 gaugino zero modes, which is precisely what is needed to produce one axion.

These string intersection events that we obtain this way also have a close analog in critical string theory. Here we should realise, that we do not expected them to be surpressed in large N as opposed to instantons. One should remind oneself, that in string theory there are two types of instanton like objects. There are brane instantons, which are $D(-1)$ branes and these will be suppressed by the string coupling i.e. in our case $1/N$. There are also worldsheet instantons, these are not supressed by g_s. Worldsheet instantons can be constructed by wrapping a string worldsheet around some 2-cycles of the manifold the strings propagate in. From the viewpoint of the worldsheet theory these wrapped worldsheets look like vortices, when one views the worldsheet theory as a gauged linear sigma model [125]. these instantons are supressed by α' and the Volume of the corresponding 2-cycle the worldsheet is wrapping, however they are not g_s suppressed.

In our case we can do an analogous construction, we put our YM theory on a manifold with non contractible 2-cycles. Now we wrap a flux tube around one of the cycles and we let the worldsheet of a second flux tube intersect the worldsheet wrapping the 2-cycle. The intersection of two strings in four dimensions can be understood as a vortex in the normal bundle, closely resembling the way the worldsheet instantons appear in the worldsheet theory in critical string theory.[12], The action of this wrapped flux tube will be proportional to the volume of the 2-cycle measured in units of Λ and will not be supressed by N in close analogy to critical string theory. One should note however that the objects for which we see vortices on the string world-sheet are somewhat different, nevertheless the analogy seems quite striking.

Tachyon Condensation

Let us now discuss a seemingly close analog of the confinement mechanism in tachyon condensation in our setup: If we have a wall anti-wall pair, monpoles can tunnel across the pair. If the walls are close this process will not be strongly supressed, since the total effective CS term for both walls vanishes. Equivalently one can note, that in this case the condensates on both sides are identical. For a single wall the CS term prevents monopole tunnelling from happening as discussed earlier, equivalently again we can note, that the condensates on both sides carry different charges. Using the arguments given in [74], this implies that the wall anti-wall theory gets confined by the mechanism introduced by Polyakov in [73]. The picture therefore seems to be as follows: When a wall and an anti-wall annihilate strings stretching between them become tachyonic in a similar manner to string theory[13], this Higgses the theory to a single $U(1)$, this $U(1)$ then gets confined by monopole tunnelling, where the flux tubes of the theory on the wall should become the confining flux tubes in the bulk, understanding this mechanism in more detail is an interesting future project.

Hints from Holography

Now we will give another motivation from an embedding into critical superstring theory why the picture advocated here makes sense: Since for us all the fields are massive and we are in $D = 4$ we expect string-like objects with

[12] For a precise construction of the normal bundle and the corresponding gauge field see e.g. [126]

[13] It would be extremely interesting to have a field theoretic source for this instability

$p = 1$ to be dual to line operators with $q = 0$, the string theory construction of [120] relates these operators to worldlines of electric and magnetic charges. This is further supported from a different viewpoint, by the famous Polchinski-Strassler construction [127], which tried to construct an AdS/CFT dual for a confining theory.[14] In this case the p=0 vacuum, which in our picture should correspond to condensation of pure monopoles is realized by an $NS5$-brane. The states which in our picture correspond to dyon condensates are realized via an $NS5$-$D5$ boundstate. So in a sense we can understand the $NS5$ as comprising the monopoles and the $D5$ as creating the electric charges of the condensate which can be interpreted as baryons. Here we see that the condensate is dual to the corresponding strings in a similar way as in our case. The instantons on the other hand are dual to objects with worldvolumes of dimension 3, exactly fitting the domain walls, just as argued before on field theoretic grounds. So we see that most of our field theoretic arguments reduce to their standard string theory analogs when constructing the field theory via AdS/CFT duality. Note that this is somewhat surprising and relies strongly on the fact that the duality relations get modified in a subtle way when the corresponding form fields acquire a mass.
One of the most important differences to critical string theory is the presence of a mass gap and correspondingly massive RR forms. This leads to charges which are only conserved mod N. Appreciating this point might also lead new into insights into non-critical string theories describing confined strings, which is a fairly active area of research (for a review see [126])

Charge Fractionalization and Quantum Hall Effect

We saw in the previous sections, that one common theme seems to be the appearance of fractional charges. We saw ends of flux tubes appearing on the worldvolume theory of domain walls. This looks like it has fractional charge with respect to the objects appearing in our theory (remember that all objects in the theory are uncharged under the center of the gauge group). We also saw objects similar to fractional instantons appear when flux tubes cross. Fractional charges have a long history in physics (for a review see [128]) and especially in strongly correlated systems in condensed matter physics it is fairly common to see excitations that behave as if the objects building

[14] The $\mathcal{N} = 1^*$ theory discussed by them is somewhat different from our constructions however we expect certain qualitative features to survive

up the system split into several parts. The most famous example for this is the fractional quantum hall effect, where one puts electrons into a magnetic field and the quasiparticle excitations of the system carry fractional electron charges. Here we will restrict ourselves to a simple description of the fractional quantum hall effect(FQH) and describe some features which seem analogous to properties we conjecture to see in YM theory. The FQH effect describes a two dimensional electron gas in the presence of a magnetic field perpendicular to the plane as depicted in 2.10. A FQH system with filling factor ν is described by a CS term at level $1/\nu$ [129]

$$S_{\mathrm{FQH}} = \frac{1}{4\pi\nu} \int \mathcal{A} \wedge \mathrm{d}\mathcal{A} \ . \tag{2.237}$$

The filling factor counts the number of electrons per unit of fundamental magnetic flux, $\Phi_0 = \frac{2\pi}{e}$. In this system one electron binds $1/\nu$ fluxes of the magnetic field corresponding to $\mathcal{A}$. The groundstate of the system is well described by a composite state of fractionalized electrons. These quasi-particles are comprised of one fundamental flux bound to ν electrons (remember that for FQH $\nu < 1$).In the case of a filling factor of $\nu = 1/N$ we recover the level N CS term of the (S)YM domain wall action. In this case the electrons fractionalize into quasiparticles, where each has charge $1/N$ and each carries one unit of magnetic flux [130].

There is a close analog to this in our domain wall theory in (S)YM: take a baryonic object, i.e. a junction of N fundamental flux tubes. In the bulk the junction can not split as the configuration wouldn't be gauge invariant after the splitting. On the wall, however, they can split up into single flux tubes ending on the wall, by producing Wilson lines of the gauge field $\mathcal{A}$, this is depicted in figure 2.21. The ends of the flux tubes behave like electric charges of $1/N$ much like the fractionalized electrons and due to the CS term have similar statistical properties as well.

The exterior magnetic field in the FQH setup is equivalent to the jump in the θ-angle for the domain walls, this generates a magnetic flux for electric charges located on the wall, for an explicit construction in a toy model in terms of image charges see our publication [15].

The FQH groundstates can be described by so-called Laughlin-states, introduced in [131]. An interesting idea would be to check whether a similar construction can be used to gain further insights into the structure of the state corresponding to the domain wall.

If these walls truly behave like D-Branes of YM theory, we might expect

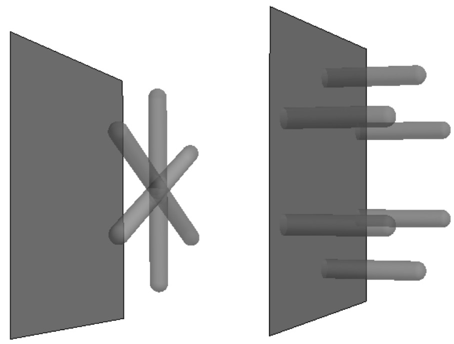


Figure 2.21: Baryonic vertex in the bulk and on the wall respectively

an enhancement of gauge symmetry, when we stack several walls. Especially when we have a wall with $\Delta k = 2$ we expect to see a $U(2)$ gauge symmetry of level N.From the topological field theory point of view this cannot be understood in our model, because a factor of two in front of the domain wall action would be sufficient to render the action gauge invariant.This phenomenon on the other hand is realized for FQH systems. In a bilayer configurations with two FQH states the quasiparticles develop non-Abelian statistics, which is described by a non-Abelian CS theory [132–134].
It is a priori not clear, that it is valid or useful to treat our system as a FQH state, however if this is the case for a single wall, it is very natural to expect, that a stack of two walls behaves like a bilayer system and therefore has an enhanced gauge symmetry. As we stressed earlier charge fractionlization might also be relevant for other objects in the theory like for example instantons. Thinking of some of the objects observed in critical string theory in terms of charge fractionalization might also yield interesting insights into its microscopic dynamics.

Chapter 3

Bose Einstein Condensates and Black Holes

3.1 Classical Black Holes

Blacks holes were originally introduced by John Mitchell in a letter to Henry Cavendish [135]:

> If there should really exist in nature any bodies whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us, or of there should exist any other bodies of a somewhat smaller size which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from sight; yet, if any luminous bodies infer their existence of the central ones with some degree of probability, as this might afford some of the apparent irregularities of the revolving bodies, which would not be easily explicable on any other hypothesis; but as the consequences of such a supposition are very obvious, I shall not prosecute them any further.

Mitchell's result, although it was conceived within newtonian ideas about gravity and light, were very deep and profound, ultimately leading to some of the deepest mysterious about gravity. He came up with the idea that objects whose escape velocity exceeds the velocity of light would be completely dark, which were later named black holes by J. A. Wheeler almost 200 years after. Furthermore he proposed a method to detect their presence closely mimicing the ones currently used by experiments. On top of being highly interesting astrophysical objects they have become of great interest for different reasons. Their highly unusual properties made the clash between classical general relativity and quantum mechanics most obvious. Therefore black holes can be described as 'the hydrogen atom of quantum gravity' [136]. Amongst the puzzling properties of quantum black holes are a tremendous number of microstates, very fast thermalization and their speculated impact on high energy scattering in gravitational theories.

3.1.1 Schwarzschild Black Holes

Let us first give a slightly more detailed version of Mitchell's argument, which is due to Laplace [137]. We simply take the escape velocity of a gravitating body of mass M and size R and equate it to the velocity of light, this gives a cricital size $R_{\rm S}(M)$. If the body is smaller than this a particle will need to travel fast than the speed of light in order to escape from the surface of the body. The critical size $R_{\mathrm{S}}(M)$ is given by

$$R_{\mathrm{S}}(M) = 2G_N M/c^2. \tag{3.1}$$

However we should note that in the context of Newtonian gravity this does not mean that nothing can escape. It merely means that in order to escape, we have to move faster than the speed of light c, which in a Galilean theory is possible without any restriction.
In order to get a better account of the actual dynamics we should take into account relativity, i.e. we should describe the black hole in terms of general relativity. In general relativity the gravitational field is not sourced by mass alone, but instead the full energy-momentum tensor $T_{\mu\nu}$ acts as a source for the gravitational field. The natural variable which couples to the energy momentum tensor is a symmetric 2-tensor. It turns out that this 2-tensor corresponds to the metric of spacetime $g_{\mu\nu}$, therefore in order to have dynamical gravity we have to give the metric dynamics. This is perhaps the most profound insight of general relativity, namely that in a gravitational theory spacetime itself is not a mute but instead the dramatis persona. The dynamics of general relativity can be most concisely summarized in the Einstein Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\mathcal{R}, \tag{3.2}$$

where $\mathcal{R}$ denotes the Riemannian curvature of spacetime, g denotes the determinant of the metric and G_N is Newton's constant. By varying this action we can derive Einstein's equations in vacuum

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 0. \tag{3.3}$$

For $d > 2$ taking the trace of this equation leads to

$$\mathcal{R}_{\mu\nu} = 0, \tag{3.4}$$

i.e. the corresponding spacetime should be Ricci flat. As with any other set of highly non linear partial differential equations it is very hard to find solutions. We can simplify that task by imposing symmetries, hopefully reducing the equations to a set of simpler equations. Continous symmetries of spacetime are called isometries and are generated by a set of Killing vectors ξ. In order for a killing vector $\xi = xi_\mu \partial^\mu$ to generate an isometry it has to satisfy the killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{3.5}$$

Birkhoff's theorem [138], which is due to Jebsen [139] states that any spherically symmetric spacetime is asymptotically flat and stationary, i.e. it has a timelike Killing vector field. This leads to the simplest black hole solution, which was originally found by Schwarzschild [140] and the corresponding line element is given by given by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.6}$$

here M denotes the mass of the black hole and $d\Omega$ is the line element on the unit two-sphere. This metric has a singularity at $R_\mathrm{S} = 2GM$ and at $r = 0$. However the singularity at R_S is only a coordinate singularity, while the solution is perfectly regular. As we cross the Schwarzschild radius R_S the radial coordinate becomes timelike, this is the same property of trapping of particles we encountered earlier: As an object crosses the horizon the radial coordinate becomes it's time and it has to move along decreasing r, eventually reaching the central singularity at $r = 0$. The central singularity at $r = 0$ however is a true singularity as can be seen by computing curvature invariants like $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = 48(GM)^2/r^6$, which diverges at $r = 0$. Birkhoff's theorem allows us to establish the uniqueness of the Schwarzschild solution amongst spherically symmetric solutions to the vacuum Einstein equations. By looking at the large r asymptotics of the metric and matching to newtonian gravity, we can see, that M can be interpreted as the mass of the black hole.

3.1.2 Thermodynamics of Classical Black holes

There are more general black holes, which are rotating. Furthermore black holes can also be charged under gauge fields. If we still require the spacetime to be stationary there is a set of theorems, which generally run under the name "No-hair theorem" establishing certain uniqueness properties. [1] Namely, if we have Einstein gravity coupled to some matter fields, among which might be an abelian gauge field A_μ, described by standard Maxwell

[1] For a review of no hair theorems and references to the original literature see [141]

dynamics then the black hole solution is uniquely determined by a set of parameters which describe mass, angular momentum, electric charge and magnetic charge.[2] This can already be interpreted as a hint towards a thermodynamic description. Thermodynamics systems in equilibrium can be described in terms of a few macroscopic variables although they have a very complex description in terms of microscopic degrees of freedom. As we will see, there are close analogues to the basic laws of thermodynamics for the mechanics of black holes. [142]

Let us begin by identifying an analogue of the zeroth law of thermodynamics, which states, that temperature in thermodynamics is a global property of a thermal system. In order to get an analogue of the zeroth law we will define a quantity called surface gravity κ as follows. In general relativity all horizons of stationary black holes turn out to be Killing horizons, i.e. the vector field ξ normal to the codimension-1 horizon is a Killing vector. Since the horizon is a null surface the norm of the Killing vector is constant on the horizon. Therefore its gradient has to be normal to the horizon, i.e. parallel to the Killing:

$$\nabla_\mu(\xi_\nu \xi^\nu) = -2\kappa\xi^\mu, \tag{3.7}$$

where κ is the surface gravity, which is constant on the horizon. The surface gravity can be physically understood as the acceleration of a static particle on the horizon of the black hole as measured at spatial infinity.

The first law of thermodynamics state that

$$dE = TdS + \text{work}. \tag{3.8}$$

There is an analogous law for Black hole mechanics, where κ plays the role of the temperature as already hinted at in the zeroth law, mass naturally plays the role of energy, while it turns out that area plays the role of entropy. To be more precise

$$dM = \frac{\kappa}{8\pi G_N} dA + \dots, \tag{3.9}$$

where the dots denote terms depending on charge and angular momentum, analogous to the work terms in the first law.

There is also a close analog of the second law of thermodynamics in the context of black holes. It states that no physical process can reduce the total

[2] This can be trivially generalized to the case of multiple abelian gauge fields, in the case of non-abelian gauge fields we can label them by charges under the Cartan subalgebra.

area of all black hole horizons, i.e.

$$\frac{dA}{dt} \geq 0. \tag{3.10}$$

This holds in classical general relativity under the assumption of the weak energy condition.[3]
There is also an analogue of the third law of thermodynamics in the black hole context. It is impossible to achieve a surface gravity $\kappa = 0$ in any physical process.[4]
So we see that there are close analogues to all laws of thermodynamics in the context of black hole dynamics. This begs the question whether these laws can actually be connected to the conventional laws of thermodynamics. Here it turns out that quantum mechanics seems to provide the missing link. It has been shown by Hawking that black holes should radiate and thereby lose mass causing their event horizons to shrink. This implies that quantum mechanically (3.10) can not be quite correct. However once one takes the combined entropy of the radiation and the entropy corresponding to the now smaller horizon it can be shown that this always increases, thereby connecting the horizon area and the entropy of a usual system of radiation. It turns out also that the temperature of the radiation is proportional to the surface gravity mentioned ealier, linking the surface gravity to a conventional temperature. A property which is very curious from a thermodynamic viewpoint is that the surface gravity of a black holes grows as it gets lighter, i.e. it gets hotter as it radiates. From a thermodynamic viewpoint this implies that black holes have a negative heat capacity and are thermodynamically unstable. We will review semiclassical properties of black holes in the next section.

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[3]The weak energy condition states, that for every timelike vector field X the following inequality holds

$$T^{\mu\nu} X_\mu X_\nu \geq 0. \tag{3.11}$$

Physically this means, that an observer corresponding to the vector field X will measure a positive energy density.

[4]Black holes with a vanishing surface gravity are usually called extremal black holes

3.2 Semiclassical Black Holes

Let us start now by looking at how to describe gravity quantum mechanically, it has been established already for a long time, that local off-shell observables can not be defined in a theory of quantum gravity. However it is crucial to realize that this is not related to reparametrization invariance, but instead is a consequence of dynamical gravity, especially all these problems disappear in the limit $G_N \to 0$, a beautiful explanation of this fact was given in [143] and our discussion will only differ in some details. Canonical quantization of general relativity leads to the so called Wheeler-de Witt [144, 145] equations

$$\begin{aligned} \mathcal{H}|\Psi > &= 0 \\ \mathcal{H}_i|\Psi > &= 0, \end{aligned} \tag{3.12}$$

where the first is a consequence of time reparametrization invariance and the second one encodes invariance under spatial diffeomorphisms Here it should be noted that neither of these involves time explicitly, which seems to make a notion of time evolution hard to define in quantum gravity. Naively this argument seems to strongly rely on time reparametrization invariance and not on the dynamics of gravity. However, this naive conclusion is incorrect. In order to see this explicitly let us take the so-called decoupling limit $M_{\mathrm{Pl}} \to \infty$, making gravity nondynamical. Let us first consider the case where we are in flat space. In this case, the metric $g_{\alpha\beta}$ must be diffeomorphic to the standard lorentz metric $\eta_{\alpha\beta}$:

$$g^{\alpha\beta} = \frac{\partial X^\alpha}{\partial x^\mu}\frac{\partial X^\beta}{x^\nu}\eta^{\mu\nu}, \tag{3.13}$$

where the X_μ can be thought of as components of diffeomorphisms $x^\mu \to X^\mu(x)$. Now the theory is still reparametrization invariant, where the matter fields transform in the usual way, while the X fields transform via

$$X^\mu \to (f \circ X)^\mu, \tag{3.14}$$

where $\circ$ denotes the standard concatenation of diffeomorphisms. In the limit we took the Hamiltonian constraint (3.12) becomes

$$\left(\frac{\partial X^\mu}{\partial x^0}P_\mu + \mathcal{H}_{\mathrm{matter}}\right)|\Psi(X^\mu, \phi) >= 0, \tag{3.15}$$

where $\mathcal{H}_{\mathrm{matter}}$ is the matter Hamiltonian. Once we realize, that $P_\mu = i\frac{\partial}{\partial Y^\mu}$ we immediately see that this is nothing else than the usual time dependent

Schrödinger equation. This shows very clearly that the problems with defining local observables in gravity are related to dynamics of gravity and not merely reparametrization invariance, which is just a redundancy in the way we describe physics.

3.2.1 Hawking radiation & information paradox

As has been first pointed out by Hawking [8], black holes although they are classically stable objects should radiate once quantum mechanics is taken into account. This effect can be thought of as virtual pair creation close to the horizon, where one particle of the virtual pair falls into the black hole, while the other particle escapes to infinity. In order to study Hawking radiation one looks at the dynamics of a quantum field on a classical Schwarzschild metric. The starting point of Hawking was to look at the dynamics of a field theory on a Schwarzschild metric. What we should note is that in this setup the metric is nondynamical, i.e. in order to arive at this from a dynamical theory of gravity we have to decouple gravity. To achieve this we have to take the limit $M_{\mathrm{Pl}} \to \infty$, while keeping the geometry, i.e. the Schwarzschild radius R_s fixed. This implies that we have to take the limit

$$M_{\mathrm{Pl}} \to \infty,\ M \to \infty,\ R_s = \frac{M}{M_{\mathrm{Pl}}^2}\text{fixed.} \tag{3.16}$$

When treating a quantum field on a Schwarzschild background in this way one sees that an observer at infinity sees a state corresponding to thermal radiation of temperature

$$T_H = \frac{\kappa}{2\pi}, \tag{3.17}$$

where κ is the aforementioned surface gravity. For a Schwarzschild black hole we have

$$T_H = \frac{M_{Pl}^2}{8\pi M}. \tag{3.18}$$

The information paradox now arises as follows: If we take Hawking's calculation at face value we conclude that Hawking radiation corresponds to black body radiation whose density matrix is described by a mixed state. However the black hole can be formed from a pure initial state of sufficiently high center of mass energy. Therefore if the black hole evaporates completely we conclude that a pure state evolves into a mixed state. However, quantum mechanical evolution is unitary, which implies that pure states always evolve into pure states. This led Hawking to suggest that quantum mechanics should break down in the context of gravity. [7] This apparent incompatibility of gravity and quantum mechanics is commonly dubbed the information paradox. In discussing this argument, however, we should keep in mind that we considered the dynamics of our system in a classical Schwarzschild background, i.e. we sidestepped all the notorious difficulties related to quantum gravity. Upon closer inspection it is clear that what we actually did in the derivation of Hawking radiation is a procedure similar to the one in the preceding section, i.e. we took the limit $M_{\text{Pl}} \to \infty$ limit while keeping the geometry fixed. This is the double scaling limit explained earlier. We also see, that in this limit the life time of the black hole and its entropy become infinitely large. This shows that just from Hawking's result we cannot conclude that there is really a paradox, in order to see whether there is any clash between quantum mechanics and gravity we should actually study the black hole as a quantum system.

3.2.2 Scrambling

The notion of fast scrambling for black holes was introduced by Sekino and Susskind [146] following work by Preskill and Hayden [147]. Preskill and Hayden asked the question how long it takes for a bit of information that is thrown into a black hole to leak out of the black hole via Hawking radiation. To be more precise they asked how long it takes for an observer at infinity to be able to almost certainly decode the bit originally thrown in by just observing the emitted Hawking radiation. The naive answer, which was originally suggested by Page [148, 149] is the time necessary to radiate half the entropy, which is the so called Page time and is of order M^3, i.e. of order of the evaporation time. Preskill and Hayden showed that the relevant time scale should not be related to the evaporation time, but instead to the time it takes to scramble the information, which was dubbed scrambling time. By scrambling time, we mean the time, which it takes for the bit of information to be fully diffused into the black hole,i.e. how long it takes until we have to observe a substantially large subsystem of the black hole Hilbert space to determine the bit thrown in. Preskill and Hayden suggested a lower bound for this scrambling time t_{scr} of

$$t_{\text{scr}} \gtrsim R_S \log\left(R_S\right), \tag{3.19}$$

where R_S is the Schwarzschild radius. Susskind and Sekino suggested then based on several qualitative arguments, that black holes should saturate this bound.

3.2.3 Black Holes & Scattering

A feature that is expected to appear in any consistent theory of quantum gravity is the dominance of black hole production in scattering processes with large momentum transfer. A central motivation for this comes from observing that in the collision of classical shock waves a black hole necessarily forms at sufficiently high energies. To be more precise we consider classical scattering of shockwaves in general relativity. Here one sees that once the impact parameter of the scattering process drops below the Schwarzschild radius corresponding to the process the formation of a black hole is guaranteed by singularity theorems[5] This suggest that ultraplanckian scattering should be dominated by black hole production. It has been recently suggested by Dvali and Gomez [151] that it would also be plausible that black holes give the only contribution to high energy scattering in gravity. This would imply, that transplanckian degrees of freedom are not needed to describe quantum gravity. Hence, this has been dubbed self-completion.
One should also note that black hole dominance implies that the cross section in ultra high energy processes should be equal to the horizon area of a black hole whose mass is the center of mass energy, i.e.

$$\sigma \sim E^2. \tag{3.20}$$

We should note that this behaviour seems to violate the so called Froissart bound [152], which arises in any consistent local Poincaré invariant QFT with a mass gap:

$$\sigma < c(\log E)^2. \tag{3.21}$$

This is commonly taken as a sign, that quantum gravity should violate locality, however we should note, that general relativity has massless gravitons and therefore no mass gap, which implies that Froissart bound would not apply in the first place. Recently it was suggested, that a similarly fast growth of scattering cross section at high energies might take place in other theories of massless particles as well, this was dubbed classicalization. [153]

[5]For a review on singularity theorems and the original references see [150].

3.3 Graviton condensates and Black holes

As we have seen most of the apparent paradoxes in black hole physics can be traced back to taking a semiclassical limit, any approach towards a resolution of these paradoxes should treat the space-time geometry in a quantum mechanical way as opposed to demoting it to a classical arena where quantum physics takes place. One such approach was introduced by Gia Dvali and Cesar Gomez in a sequence of publication [9–14]; we will now review the important points. Here the authors made the assumption that the black hole can be described as a bound state of N gravitons of wavelength R_S. Now let us note, that since the gravitational self-interaction contains two derivatives, the self coupling of gravitons will be momentum dependent, for gravitons of wavelength λ the effective coupling α_{gr} corresponding to this self-interaction will scale as

$$\alpha_{\mathrm{gr}} \sim \hbar G_N \lambda^2. \tag{3.22}$$

In order to determine N we should note, that each graviton has an energy of order $R_S^{-1} \sim (G_N M)^{-1}$, where M is the mass of the black hole. The total energy is thus given by

$$N (G_N M)^{-1}. \tag{3.23}$$

Equating this with the black hole mass allows us to determine N as

$$N \sim \frac{M^2}{M_{\mathrm{Pl}}^2}. \tag{3.24}$$

Here N will be a very large number for a macroscopic black hole, making it natural, to treat the gravitons composing the black hole as a Bose Einstein condensate. Now let us note, that $\alpha N \sim 1$, this indicates that the collective self-coupling of the gravitons is strong and might lead to relevant collective effects. By exploiting an analogy to non-relativistic Bose Einstein condensates with attractive self-interactions the authors concluded in [12], that the graviton condensate should be close to a so called quantum critical point. This immediately suggests, that the condensate should not be well described by semiclassical methods.

A system near a quantum critical point has at least one gapless excitation. In the context of Bose Einstein condensates excitations are called Bogoliubov modes and since we have a finite number of bosons, the excitation will actually not be exactly gapless, but can be expected to have a gap of $\mathcal{O}(1/N)$, this similar to conventional critical points, where in finite Volume, excitations are also not exactly gapless, but have a gap controlled by the inverse volume. Since the states generated by these excitations are almost degenerate it is natural to use them to give a microscopic explanation of the black hole entropy. However one almost gapless excitations is not enough to reproduce the large entropy of a black hole. As in this picture the entropy is extensive in N, this led Dvali and Gomez to conjecture, that in the case of a graviton BEC there should be N almost gapless Bogoliubov excitations, this is presumably related to the particular gravitational interactions.

Hawking radiation is in this picture explained as the quantum depletion, which is well known in the context of Bose Einstein condensates, which leads to the excitation of quanta compared to the naive ground state, these quanta are then interpreted as Hawking radiation.

The information paradox is here resolved by new $1/N$ corrections, which dissapear in the semi classical $N \to \infty$ limit, this can be easily seen by noting, that the $N \to \infty$ limit is just the semi-classical limit alluded to earlier.

In the next section we will review the dynamics of a non-relativistic toy model for black hole dynamics. This non-relativistic toy model formed the basis for the two publications of the author, where one was concerned with making the statement, that the system behaves very quantum mechanical at the critical point quantitative. The second publication was concerned with the question of whether there is a phenomenon analogous to scrambling in this toy model and what is the origin of this phenomenon.

3.4 Attractive Bosons as a Toy Model for Black Holes

Motivated by the intuitive picture described earlier we are looking for a simple toy model which captures some of the properties of the condensate picture of black holes. In order to simplify the picture our model is going to drop two properties of general relativity. In order for the statement that the black hole corresponds to a collection of N particles to be made more precise we want to have a system that preserves particle number. This also simplifies numerical analyses we will do later by significantly reducing the size of the relevant Hilbert space. Particle number conservation enforces us to give up on Lorentz invariance, since a Poincaré invariant, interacting theory in more than two spacetime dimensions necessarily allows for particle creation and annihilation [154]. Furthermore we will only consider two-body interactions. This is again a huge technical simplification over general relativity which allows for vertices with an arbitrary number of external lines. Looking at the general mechanism of condensate black holes it does not seem that either one of those points is particularly important for quasistatic properties of the black hole.

Therefore we are going to look at the dynamics of a large number N of non-relativistic bosons described by the Hamiltonian operator

$$H = \sum_{i=1}^{N} -\Delta_{x_i} + \frac{1}{N} \sum_{i<j}^{N} V(x_i - x_j), \tag{3.25}$$

where Δ_{x_i} denotes the Laplacian with respect to x_i and $V(x_i - x_j)$ is a two body interaction potential. For a generic state described by the density matrix $\rho(t, x_1, \ldots, x_N; x'_1, \ldots, x'_N)$ the time evolution is determined by the von Neumann equation

$$\partial_t \rho = \frac{i}{\hbar} [H, \rho] \,. \tag{3.26}$$

Note that we put a peculiar factor of $1/N$ in front of the two-body interaction. We do this because we want to mimic the balance between potential energy and kinetic energy in the large N limit seen in the graviton picture. In order to achieve that we have to realize that for momenta $\mathcal{O}(1)$ the total kinetic energy will be $\mathcal{O}(N)$, while if the two particle potential would be $\mathcal{O}(1)$, then the total sum would be $\mathcal{O}(N^2)$. Therefore in order to have a competition between potential and kinetic energy the two body potential should be $\mathcal{O}(1/N)$. In order to describe the system we are now going to focus on a subsystem of k particles. This seems to be a good idea, since for a semiclassical system we do not expect there to be too strong quantum mechanical correlations. In order to make the equations more readable we introduce the following notation:

$$\mathbf{x} = (x_1, \dots, x_N), \quad \mathbf{x}_k = (x_1, \dots, x_k), \quad \mathbf{x}_{N-k} = (x_{k+1}, \dots, x_N). \tag{3.27}$$

We define the k-particle reduced density matrix by

$$\rho_k(t, \mathbf{x}_k, \mathbf{x}'_k) = \int d\mathbf{x}_{N-k} \rho(t, \mathbf{x}_k, \mathbf{x}_{N-k}, \mathbf{x}'_k, \mathbf{x}'_{N-k}). \tag{3.28}$$

Under this normalization we have $Tr\rho_k = 1$ for all k.

From here we can easily derive the equations of motion for the k-particle density matrix by just tracing (3.26) and using indistinguishability of the bosons:

$$\begin{aligned} i\partial_t \rho_k(t, \mathbf{x}_k, \mathbf{x}'_k) = & \quad (\textstyle\sum_{i=1}^k (-\Delta_{x_i} + \Delta_{x'_i}) \\ & + \tfrac{1}{N} \textstyle\sum_{i<j}^k (V(x_i - x_j) - V(x'_i - x'_j))) \rho_k(t, \mathbf{x}_k, {}'_k) \\ & + \tfrac{N-k}{N} \textstyle\sum_{j=1}^k \int dx_{k+1} \big(V(x_j - x_{k+1}) - V(x'_j - x_{k+1})\big) \\ & \quad \rho_{k+1}(t, {}_k, x_{k+1}, \mathbf{x}'_k, x_{k+1}). \end{aligned} \tag{3.29}$$

This is a system of equations which is in statistical physics commonly known as the BBKGY hierarchy. For $N \to \infty$ it converges to

$$\begin{aligned} i\partial_t \rho_k(t, \mathbf{x}_k, \mathbf{x}'_k) &= \sum_{i=1}^k (-\Delta_{x_i} + \Delta_{x'_i}) \rho_k(t, \mathbf{x}_k, \mathbf{x}'_k) \\ &+ \sum_{j=1}^k \int dx_{k+1} (V(x_j - x_{k+1}) \\ &- V(x'_j - x_{k+1})) \rho_{k+1}(t, \mathbf{x}_k, x_{k+1}, \mathbf{x}'_k, x_{k+1}). \end{aligned} \tag{3.30}$$

Now we make the additional assumption that in the limit $N \to \infty$ any two particles particles are uncorrelated, i.e. that we can factorize the two particle density matrix into a product of two one particle density matrices (and analogously for higher particle density matrices):

$$\rho_2(t, x_1, x_2, x'_1, x'_2) \approx \rho_1(t, x_1, x'_1) \rho_1(t, x_2, x'_2). \tag{3.31}$$

In statistical mechanics this is Boltzmann's celebrated molecular chaos. In this context this should be better understood as a mean field approximation, where we model the interaction of an arbitrary particle with all the other particles by a "mean field" ρ_1.

This leads to the equation

$$\begin{aligned} i\partial_t \rho_1(t, x_1, x_1') &= (-\Delta_{x_1} + \Delta_{x_1'})\rho_1(t, x_1, x_1') \\ &+ \int dx \rho_1(t, x, x) \left(V(x_1 - x) - V(x_1' - x) \right) \rho_1(t, x_1, x_1'). \end{aligned} \tag{3.32}$$

In case the one particle density matrix corresponds to a pure state, i.e. that

$$\rho_1(t, x_1, x_1') = \psi(t, x_1)\bar{\psi}(t, x_1'), \tag{3.33}$$

then $\psi(t, x)$ obeys the Hartree equation

$$i\partial_t \psi(t, x) = -\Delta\psi(t, x) + \int dx' V(x - x')|\psi(t, x')|^2 \psi(t, x). \tag{3.34}$$

In order to simplify things further we are now going to look at a very short ranged potential, which an be approximated by $\alpha\delta(x - x')$. Then (3.34) becomes the Gross-Pitaevskii equation

$$i\partial_t \psi(t, x) = -\Delta\psi(t, x) + \alpha|\psi(t, x)|^2 \psi(t, x). \tag{3.35}$$

Note that the factorization condition (3.31) really encodes two properties of the system. One is the assumption that the initial state fullfills condition (3.31) at $t = 0$, the other one is that the factorization condition is stable under time evolution for $N \to \infty$. The second point is rather subtle and has recently been shown to be realized for a repulsive Bose gas in a series of papers by Erdös, Schlein and Yau [155] . Here however we are interested in attractive systems and we will not try to prove that this holds but rather assume it.

Before proceeding to solving the Gross-Pitaevskii equation for the case of interest to us we will give a different derivation which in spirit is closer to deriving the classical equations of motion from a quantum field theory. In order to do that we will go to a multiparticle formalism, i.e. we will use a field operator $\hat{\psi}(x)$, where applying the (conjugate) field operator once on the vacuum yields a single particle state. Expanding the field operator into a complete set gives:

$$\hat{\psi}(x) = \sum_i u_i(x)\hat{a}_i, \tag{3.36}$$

where a_i are annihilation operators and $u_i(x)$ are a set of basis functions.

Now we are going to consider the system with an attractive δ interaction. In order to get even closer to the standard field theory description we are going to describe our quantum theory using a path integral. This leads us to define the partition function in the standard way:

$$Z = \int \mathcal{D}\psi(x,t)\mathcal{D}\psi^\dagger(x,t)e^{i\frac{S}{\hbar}}, \tag{3.37}$$

with the action given by

$$S = \int dxdt \left(\psi^\dagger i\hbar\partial_t\psi - |\hbar\partial_x\psi|^2 + g|\psi|^4\right). \tag{3.38}$$

In this system there is a symmetry corresponding to rotations in the phase of the field ψ. Physically this corresponds to particle number conservation. The corresponding conserved charge is given by

$$\int dx|\psi|^2. \tag{3.39}$$

In the case at hand we want to fix the number of particles to be N. This can be achieved via a Lagrange multiplier, i.e. we add the term

$$\mu\left(\int dx|\psi|^2 - N\right) \tag{3.40}$$

to the action.[6] In order to understand the exactness of the Gross-Pitaevskii equation in the $N \to \infty$ limit we should first note that the constraint implies that the field will generically have an amplitude $\mathcal{O}(\sqrt{N}))$. Now in order to easier understand the large N limit it is more convenient to rescale the field such that the typical values are $\mathcal{O}(1)$, i.e. we define $\tilde{\psi} = \psi/\sqrt{N}$. This gives

$$S = N\left[\int dxdt\left(\tilde{\psi}^\dagger i\partial_t\tilde{\psi} - |\partial_x\tilde{\psi}|^2 + gN|\tilde{\psi}|^4\right) + \mu\left(\int dx|\tilde{\psi}|^2 - 1\right)\right]. \tag{3.41}$$

Here we see that once we fix $\alpha = gN$ as we take the large N limit, then N just appears as a multiplier of the total action, i.e. in the path integral it takes the role of $\hbar$. This also immediately implies that the $N \to \infty$ limit is truly

[6]This actually fixes the expectation value of the particle number to be N, however for our purposes this is sufficient.

a classical limit in the sense, that it completely localizes the contributions from the path integral to its saddle points. Doing a saddle point expansion of 3.41 corresponds to an expansion in $1/N$. Looking for saddle points of 3.41 on S^1 leads us to the Gross Pitaevskii equation

$$\left[\partial_\theta^2 + \pi\alpha|\Psi_0(\theta)|^2\right]\Psi_0(\theta) = \mu\Psi_0(\theta)\,, \tag{3.42}$$

with the constraint $\int|\psi|^2 = 1$, here μ is the chemical potential. Here we chose the size of S^1 to be 1. The minimal energy solutions are given by [156]

$$\Psi_0(\theta) = \begin{cases} \sqrt{\frac{1}{2\pi}} & \mu = \frac{\alpha N}{2} \\ \alpha < 1 & \\ \sqrt{\frac{K(m)}{2\pi E(m)}}\mathrm{dn}\left(\frac{E(m)}{\pi}(\theta-\theta_0)|m\right) & \mu = \frac{(2-m)K^2(m)}{\pi^2} \\ \alpha > 1 & \end{cases}. \tag{3.43}$$

Here the former describes a homogeneous system, while the latter describes a localized soliton. θ_0 is the center of the soliton, while m is determined by the nonlinear equation

$$K(m)E(m) = \left(\frac{\pi}{2}\right)^2\alpha N\,. \tag{3.44}$$

So we see, that there is a phase transition at $\alpha N = 1$, where for $\alpha N < 1$ the system is described by a homogeneous ground state, while for $\alpha N > 1$ the ground state is given by a localized soliton. In order to see the appearance of a gapless mode let us expand the system around the homogeneous state for $\alpha N < 1$ and approach $\alpha N = 1$.
We write $\hat{\Psi} = \Psi_0 + \delta\hat{\Psi}$, then we expand to quadratic order in $\delta\hat{\Psi}$ and look for the energies of the corresponding excitations. This procedure results in an energy E_k for an excitation with momentum k as follows

$$E_k = \sqrt{k^2(k^2-\alpha N)}. \tag{3.45}$$

From this it is obvious, that for $\alpha N = 1$ the mode with $k = 1$ becomes gapless. The presence of this gapless mode leads to a non classical behaviour of the corresponding ground state. As for any quantum mechanical system the ground state of the system will contain quantum fluctuations around the classical ground state. In our publication we expressed these flucuations in terms of the original non-diagonal excitations and computed their entanglement entropy. As is to be expected this entanglement entropy showed a divergence at the quantum critical point, the exact construction and a detailed description of a numerical simulation for finite N is subject of the publication [16], which is reproduced at the end of the thesis.

3.5 Chaos & Instabilities

The property of fast scrambling in the context of black hole physics seems highly mysterious. We should note that scrambling naturally seems to imply that quantum effects already become important after a time scale as short as $R_S \log(R_S)$. Let us introduce the *quantum break time*[7], which is the time up to which the difference of semiclassical time evolution and full quantum time evolution is small. However for typical textbook quantum systems the time scale for breakdown of a semiclassical description typically scales with some typical classical time scale λ for the given classical configuration times the action of this configuration, i.e. λS. In the case of black holes it is natural to identify the typical time scale with the Schwarzschild radius R_S, since it is the only quantity surviving in the classical limit. This means naively we would conclude that the semiclassical description is valid for $R_S * S_{\mathrm{BH}} \sim R_S^3 M_{\mathrm{Pl}}^2$, which is comparable to the evaporation time of the black hole and certainly much larger than the scrambling time. However as we will review now for unstable systems the quantum break time is significantly shorter and given by $\lambda \log(S)$, where λ is the Lyapunov exponent corresponding to the instability. This is especially relevant for chaotic systems, where the instability is not at isolated points of phase space but rather covers whole regions of phase space.

3.5.1 A simple quantum mechanical system near an instability

In this section we are going to look at quantum systems near an unstable point. As a simple toy model let us look at a single quantum mechanical particle with a potential

$$V(x) = -\frac{m\omega^2}{2}x^2 + \frac{\lambda}{4}x^4. \tag{3.46}$$

Our main focus will be to understand why an unstable system is very badly described by semiclassical methods. An intuitive explanation for this goes as follows: The way to describe a quantum mechanical system of a single particle semiclassically is through the WKB approximation, which is known not to be a good approximation close to turning points. For problems typically considered in a simple quantum mechanics course this is no problem, since

[7] This also runs under the name Ehrenfest time

loosely speaking in these problems the system only spends very little time at the turning points of motion. However when we have an instability and look for trajectories probing the instability; in the example this would be one starting on the top of the hill. Here the particle becomes very slow around the top, which at the same time is a turning point. Furthermore this slowing down implies that the classical period of motion will diverge as we tune the energy close to the top of the hill and we can expect the system to become more and more localized near the top. Now we will try to make this more quantitative. In order to do this we look at the spectrum of energy eigenstates in a semiclassical approximation. This is most easily done by doing a Bohr-Sommerfeld type calculation. In order to look for the eigenstates we look for solutions to

$$S(E) = (n + \frac{1}{2})2\pi \tag{3.47}$$

for integer n, here one should note that this WKB type approximation includes the contribution of the Maslov index due to the turning points. S(E) is the integral of the action over a full classical period of motion $S = \oint pdx$, this is

$$S(E) = 2\int_{x_0(E)}^{x_1(E)} 2m\sqrt{E - V(x)}dx, \tag{3.48}$$

where $x_0(E), x_1(E)$ are the turning points of the classical motion. The quantity we will be interested in will be the gap between consecutive eigenvalues Δ or equivalently the level density ρ. Assuming that the energy levels are close this is well approximated by

$$\Delta = \frac{dE_n}{dn} = \left(\frac{\partial S}{\partial E}\right)^{-1} 2\pi. \tag{3.49}$$

Explicitly this gives us

$$\Delta = \left(2\int_{x_0(E)}^{x_1(E)} m/\sqrt{E - V(x)}\right)^{-1} 2\pi = 2\pi\tau^{-1}. \tag{3.50}$$

So we see that close to the top of hill in the potential we expect the density of states to diverge. Evaluating this more precisely shows that the density of states diverges logarithmically close to $E = 0$:

$$\rho(E) \sim -\log(E). \tag{3.51}$$

Since the classical motion is most of the time near the top of the hill, we can expect that a wave packet centered around $x = 0$ will have a substantial overlap with most of these states. Since the energy difference between these different states vanishes logarithmically we can expect the time scale over which we see non classical behaviour to behave logarithmically in the characteristics action as alluded to earlier.

3.6 Quantum break time in unstable systems

We can also understand this behaviour naturally from a more dynamical point of view by looking at the time evolution of the quantum system in question. In the following we will look at a rather generic unstable system and not focus on the example of the inverted hilltop. However For simplicity we will again focus on the dynamics of a single degree of freedom. In order to understand the deviation from a (semi-)classical behaviour it is most useful to employ a description of the quantum dynamics which is as close as possible to the phase space description of classical dynamics. What we want to do is to describe the dynamics of the quantum system in terms of a quasi probability distribution on phase space instead of describing it via a wave function. One well-known quasiprobability distribution is the Wigner function P, which for state described by the density operator $\hat{\rho}$ is defined by

$$P(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy \langle x+y|\hat{\rho}|x-y\rangle e^{-2ipy/\hbar}. \tag{3.52}$$

This distribution is similar to a probability distribution in phase space. However it differs in crucial details. First of all, the Wigner function can and in fact does become negative for non classical states. Observing a negative Wigner function is in fact a way to check for non classical behaviour. Furthermore for each state there is a unique Wigner function. The Wigner function can be used to compute the expectation values of operators. To be concrete let us define an operator $\hat{G}$, which we take to be Weyl ordered. Then we can associate a function $g(x,p)$ to this operator via the Weyl transform

$$g(x,p) = \int_{-\infty}^{\infty} dy \langle x-y/2|\hat{G}|x+y/2\rangle e^{ipy/\hbar}. \tag{3.53}$$

This function will be real if $\hat{G}$ is hermitian. The expectation value of the corresponding operator is now determined via

$$\bar{\psi}\hat{G}|\psi\rangle = \mathrm{Tr}(\hat{\rho}\hat{G}) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp P(x,p) g(x,p). \tag{3.54}$$

In order to describe the time evolution of the system we should take the Wigner transform of the von Neumann equation, which leads to the *Moyal evolution equation*

$$\frac{\partial P(x,p,t)}{\partial t} = -\left\{P(x,p,t), H(x,p)\right\}_{\mathrm{MB}}. \tag{3.55}$$

Here $\{,\}_{\mathrm{MB}}$ denotes the so called Moyal bracket, which encodes the noncommutative nature of quantum mechanics. The Moyal bracket is defined as

$$\{f(x,p), g(x,p)\}_{\mathrm{MB}} = \frac{2}{\hbar} f(x,p) \sin\left(\frac{\hbar}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right) g(x,p). \tag{3.56}$$

It should be noted that if we formally expand the Moyal bracket in terms of $\hbar$ the leading term corresponds to the familiar Poisson bracket $\{,\}_{\mathrm{PB}}$. This allows us to clearly see how the classical description emerges for suitable states from the $\hbar \to 0$ limit. The simple correspondence to the classical description also allows us to easily identify, when the classical description breaks down. This will be precisely the case once the terms nonlinear in $\hbar$ in the Moyal bracket become important. This will be the case if the derivatives of one of the phase space variables becomes small compared to $\hbar$. Now we can argue for a very short quantum break time in an unstable system. For a state which can be well described classically the Wigner function can be thought of as the classical probability distribution in phase space.The instability implies that the Wigner function will stretch in one direction in phase space. However as long as the time evolution is well described by a classical evolution Liouville's theorem holds and we are guaranteed that the corresponding flow on phase space is volume preserving. This means that as the Wigner function is getting stretched in one direction in phase space it has to get squeezed in another direction. As it gets squeezed the gradient of the Wigner function grows exponentially fast with a rate given by the Lyapunov exponent λ corresponding to the instability. Since the evolution is exponentially fast the time scale corresponding to the quantum break time will be logarithmically short, i.e. we expect the quantum break time t_{br} to be given by

$$t_{\mathrm{br}} \sim \lambda^{-1} \log \frac{S}{\hbar}. \tag{3.57}$$

It has recently been shown that in simple chaotic models the time scale for full thermalization has the same parametric behaviour as the quantum break time [157]. The main difference to the model considered before is the presence of a persistent instability, i.e. the system is unstable in a region of phase space. In the model considered before, there is only an instability on top of the hill. Since in the black hole condensate picture the instability should be related to the evaporation process we expect there to be an instability during the full dynamical history of the black hole, which makes it natural to expect fast thermalization, i.e. scrambling. Therefore detecting a local instability can be seen as a precursor to scrambling. This relation between scrambling and instability was the main subject of our publication which is reproduced ad verbatim in the final part of the thesis. There we showed for the 1+1d toy model discussed earlier that there is a natural instability appearing after a so called quantum quench across the phase transition, i.e. when time evolving the homogeneous state in the solitonic phase. Then we showed explicitly using numerical simulation that this instability leads to a short quantum break time. However we did not show that this model is a fast scrambler. Here we should notice that the corresponding instability is not persistent. Therefore we don't expect this toy model to capture this behaviour. We can also understand why the system won't scramble from a different viewpoint. As we have seen earlier the density of states near the instability diverges logarithmically. In our case this will be cut off due to N being finite. Therefore it is natural to expect the density of states to behave as $\log(N)$ near the instability. The dynamics of the system will then be mostly happening in a subspace of size $\log(N)$. This immediately gives an upper bound on any entanglement entropy we can possibly observe of $\log \log N$, which is too small to lead to full scrambling. In the black hole case, however we expect the density of state to be exponential in N. Therefore this seems to be a problem of the simplified toy model we are considering. Construction of an improved toy model, which tackles this problem should be a very promising and fruitful area for future research. Exploring this new connection between quantum chaos, quantum break time and fast scrambling in other contexts might lead to new insights into a variety of areas of physics like quantum information theory and quantum gravity. A particular problem worth studying would be a connection to the recent discovery of fast thermalization in matrix models [158], which are supposed to describe black holes in M-Theory.

Chapter 4

Papers

4.1 CS Localization

4.1.1 Abstract

We propose an explicit model, where an axionic domain wall dynamically localizes a U(1)-component of a nonabelian gauge theory living in a 3+1 dimensional bulk. The effective theory on the wall is 2+1d Maxwell-Chern-Simons theory with a compact U(1) gauge group. This setup allows us to understand all key properties of MCS theory in terms of the dynamics of the underlying 3+1 dimensional gauge theory. Our findings can also shed some light on branes in supersymmetric gluodynamics.

4.1.2 Introduction

Maxwell-Chern-Simons (MCS) Theory in 2+1 dimensions is a theory leading to a plethora of interesting phenomena, it has found a wide array of applications ranging from fractional quantum hall effect [159] to knot theory [160].

Some of the more interesting implications of adding a Chern-Simons (CS) term include screening of the electric field, the association of electric charges with magnetic fluxes and the appearance fractional statistics for charged particles [161]. Compact U(1) gauge theories, which ordinarily are confining [73] in 2+1 dimensions are altered in an even more dramatic way. The confinement disappears [162] and magnetic events (the analog of sphaleron transitions) start producing electrically charged particles [75]. Also, the coefficient of the CS term can only take discrete values [163].

In this letter, we are going to propose an explicit setup in the Dvali-Shifman gauge field localization framework [42], in which a 2+1 dimensional MCS theory appears as the effective world volume theory of a field theoretic brane in a 3+1d spacetime. This is achieved by promoting the field theoretic brane into an axionic domain wall. We can then clarify many of the interesting features of 2+1d MCS theory in terms of 3+1d bulk physics.

4.1.3 2+1d Chern Simons Electrodynamics

We begin by summarizing various properties found in 2+1d Maxwell-Chern-Simons theory with a $U(1)_c$ gauge group. Its action is given by

$$S = \int_{2+1\text{d}} -\frac{1}{2g^2} F \wedge \star F + \frac{k}{2} A \wedge dA + A \wedge \star j \tag{4.1}$$

Here the Chern-Simons term acts as a mass term with mass proportional to kg^2 and screens the electric field. Furthermore by explicitly solving the equations of motion for a point charge we can see that a point particle with electric charge qe at $\vec{x}_0$ produces a magnetic field B of

$$B(\vec{x},t) = \frac{q}{k} e\delta(\vec{x} - \vec{x}_0). \tag{4.2}$$

A further peculiarity is the appearance of fractional statistics first noticed by Wen and Zee [161], i.e. exchange of two particles of electric charges q_1, q_2 leads to an additional phase of

$$\delta = \frac{1}{2k} q_1 q_2 \tag{4.3}$$

in the wavefunction.

It was shown by Polyakov [73] that QED with a compact U(1) gauge group in 2+1d is confining at exponentially large scales due to instantons, which effectively act as a plasma of magnetic charges and lead to Debye screening, which in turn implies the existence of a mass gap. However in [162] it was noted that confinement is absent if one introduces a CS term.

Gauge invariance under large gauge transformations demands that k must be quantized [58] as

$$k = \frac{n}{8\pi} \qquad n \in \mathbb{N} \tag{4.4}$$

4.1.4 Dvali-Shifman localization mechanism

In [42] Dvali and Shifman proposed a field theoretic mechanism for localizing 3+1d gauge fields to 2+1d solitons. Their setup includes a strongly interacting nonabelian gauge theory, e.g. SU(2), living in 3+1d space (bulk) and a Higgs field in the adjoint representation. In the bulk, their Higgs has a vanishing vacuum expectation value (vev) and the gauge theory is in the confining phase. Their explicit theory allows for a domain wall (brane) on which the Higgs acquires a nonzero vev — the SU(2) is broken down to $\mathrm{U}(1)_\mathrm{c}$ on the wall. Because electric flux cannot escape into the bulk, this setup actually produces an effective 2+1 dimensional $\mathrm{U}(1)_\mathrm{c}$ gauge theory with compact gauge group on the brane.

Their theory can also be understood as the electric-magnetic dual of a Josephson junction [74], where a layer of insulator is sandwiched between two superconductors. Magnetic flux is confined to the junction while charges inside the junction are still screened.

This duality lends itself to a straightforward interpretation of confinement in 2+1d QED with compact U(1). In the Dvali-Shifman picture, monopoles from the condensate in the bulk can tunnel across the brane (the dual effect of the Josephson current). The tunneling monopoles effectively act as a dilute plasma of magnetic charges on the brane and induce confinement in the 2+1d theory.

4.1.5 Induced Chern-Simons Term

It is well known that the effective world volume theory of an axionic domain wall contains a Chern-Simons term. Starting with a (3+1)d Yang-Mills theory, which contains an axion $\theta(x)$ coupling to the SU(2) gauge fields

$$S_\theta^{(3+1)\mathrm{d}} = -\frac{1}{8\pi^2}\int \theta \,\mathrm{Tr}\, F\wedge F \tag{4.5}$$

This can be written as a total derivative of the Chern-Simons form

$$S_\theta^{(3+1)\mathrm{d}} = -\frac{1}{8\pi^2}\int \theta \,\mathrm{d}\,\mathrm{Tr}[A\wedge \mathrm{d}\, A + \frac{2}{3}A\wedge A\wedge A] \tag{4.6}$$

and after integrating by parts gives a contribution on an axionic domain wall, across which the axion VEV changes by $\Delta\theta$,

$$S_\theta^{(3+1)\mathrm{d}} = \frac{\Delta\theta}{8\pi^2}\int_{(2+1)\mathrm{d}} \mathrm{Tr}[A\wedge \mathrm{d}\, A + \frac{2}{3}A\wedge A\wedge A] \tag{4.7}$$

Comparing to (4.1), we find the coefficient of the induced CS term to be

$$k = \frac{\Delta\theta}{8\pi^2} \tag{4.8}$$

4.1.6 Localization of Chern-Simons theory

We will now consider the situation, where a gauge field is localized to an axionic domain wall. For concreteness, consider again a confining SU(2) gauge theory in the bulk, with both an axion field and a Higgs in the adjoint representation. The explicit potential presented below allows for an axionic domain wall and gives a nonzero Higgs-vev on the domain wall only.

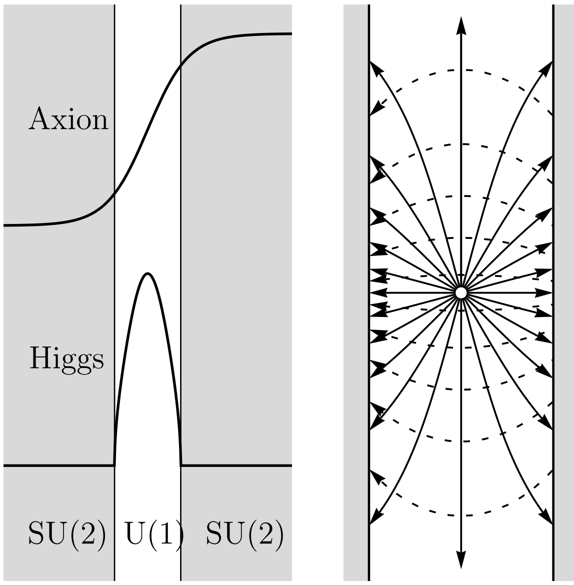

Figure 4.1: Axion and Higgs field configuration across the domain wall on the left. And U(1)-electric (solid) and magnetic field lines (dashed) of an electric point charge within the wall on the right.

Assuming the confinement scale in the bulk to be sufficiently high, there will again be a $\mathrm{U}(1)_{\mathrm{c}}$ gauge theory localized to the brane. The jump in the axion value across the brane however contributes a CS term to the 2+1d effective action, with a coefficient given by (4.8).

We now elaborate on the 2+1d effective field theory that we expect to see in our model. We will then take a complimentary perspective and compare the 2+1d arguments to the 3+1d setup in an accessible limit. Along these lines we will clarify the appearance of a photon mass.

In 2+1 dimensions, pure Maxwell-theory contains a massless photon, which corresponds to one propagating degree of freedom(dof). As the 2+1d theory that we are interested in comes from a 3+1d model, we also expect to see a tower of heavy states. Their characteristic mass scale is given by the inverse localization width and they appear as Proca-fields in the Lagrangian, containing 2 dof.

When a Chern-Simons term is added to the 2+1d Maxwell theory, the previously massless photon acquires a mass $m_{\mathrm{cs}} = kg^2$. A vector field with a Proca-mass of m_{p} on the other hand splits up. One of its degrees of freedom becomes lighter, the other one heavier, with masses

$$m_\pm = \sqrt{m_{\mathrm{p}}^2 + \frac{m_{\mathrm{cs}}^2}{4}} \pm \frac{m_{\mathrm{cs}}}{2} \tag{4.9}$$

The number of degrees of freedom per field however does not change in either case.

For the effective 2+1d theory, we are interested in the lowest energy degrees of freedom. The field at long distances will be completely dominated by the lightest mode in the spectrum. If the CS term is small, the previously-massless-photon will be this lightest dof with mass m_{cs}. If we imagine to increase the CS term, there will be a crossover at some point and the m_- mode of the first level Proca field becomes the lightest mode, as illustrated in fig. 4.2.

Let us now return to our 3+1d model and study the electromagnetic field sourced by a charge on the brane. Consider the brane on which SU(2) is broken to $\mathrm{U}(1)_{\mathrm{c}}$ to be of finite thickness. Further assume a simplified axion field configuration, where the change of the axion vev is localized at the left and right boundary of the brane (unlike the axion field depicted in fig. 4.1, left). Then the U(1) field on the brane worldvolume just obeys Maxwell equations and has some peculiar boundary conditions at the interfaces to the bulk. The static field of a point charge can be calculated along the lines of the image charge techniques developed for topological insulators in [164]. The results are in perfect agreement with the expectations from the lower dimensional effective theory. For a configuration corresponding to the lowest CS-level the electric field $\vec{E}$ decays for large distances like a Yukawa potential, i.e.

$$\vec{E} \sim \frac{1}{\sqrt{r}} e^{-kg^2 r}, \tag{4.10}$$

while the image charges produce a localized magnetic flux across the brane as expected. The flux lines corresponding to this configuration are depicted in fig. 4.1, right. Furthermore if we check for more general jumps of θ across the brane, we see that the effective mass varies as predicted, i.e. it grows linearly until $kg^2 \sim 1$ and then drops again as $\Delta\theta$ is raised further, reproducing the expected $1/\Delta\theta$ behavior. See the appendix for more details.

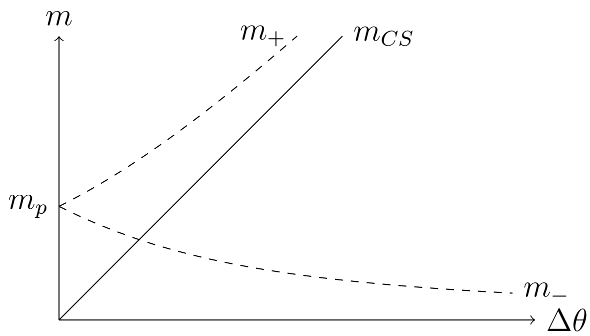

Figure 4.2: Mass of the zero mode and the first Kaluza-Klein modes as a function of $\Delta\theta$

4.1.7 Magnetic events and confinement

Let us shortly recall magnetic events and then elaborate on their physics in our gauge field localization setup. In 3+1 dimensions, magnetic monopoles in $U(1)_c$ gauge theory are well known. These static field configuration in 3 space dimensions however can also be interpreted as instantons of a euclidean 2+1d gauge theory. Such tunneling transitions between topologically inequivalent vacua of the classical gauge theory play a crucial role in Polyakov's derivation of confinement. The associated classical transitions in Minkowski-space were recently dubbed magnetic events. In pure $U(1)_c$ theory, their field evolution is singular at one spacetime point, yet this singularity is lifted, when the theory emerges from a broken SU(2). Moreover, the events carry magnetic charge. Finally, in the context of MCS theory, consistency demands that an electrically charged particle remains as a "remnant" of a magnetic event [75].

In our model, one can understand the magnetic events as actual monopoles flying across the brane. A monopole crossing the brane will acquire an electric charge of

$$q = \frac{\Delta\theta}{2\pi}, \tag{4.11}$$

as shown by Witten [28]. The electric flux in the bulk however is confined into flux tubes, therefore the monopole will stay connected to the brane by a flux tube. From the lower dimensional point of view the throat of that tube looks like an electric point charge, precisely as seen in [75]. Note that this is consistent with conservation of the electric current, as from the (2+1) d effective field theory point of view the magnetic event generates a Chern-

Simons current. In the lower dimensional theory only the sum of electric and Chern-Simons current is conserved.

This also explains the absence of confinement seen in the lower dimensional theory [162]: The monopoles on one side of the brane cannot tunnel to the other side, since on the other side, they are charged dyons, which can not be absorbed by the condensate. So there is no plasma of magnetic charges on the brane and no confinement.

4.1.8 Chern-Simons coefficient quantization

Our explicit construction – detailed in the appendix – allows for domain walls with $\Delta\Theta$ a multiple of 2π. This is compatible with the famous theorem by Vafa and Witten [30], which restricts the possible minima of the vacuum energy to occur for $\theta \in 2\pi\mathbb{Z}$. This reproduces the same quantization condition obtained from breaking 2+1d SU(2) Chern-Simons theory to $U(1)_c$ [163].

The model described so far gives a nice understanding, of why magnetic events in the 2+1d theory lead to remnant electric charges. These electric charges represent the throats of electric flux tubes. We can also give a construction where the remnant charges have a simple description in the 2+1d effective theory. Let us add a doublet of fermions to the theory. Because of their color charge, they are confined in the bulk and we expect a localized fermion mode on the brane. As is well known, the monopoles acquire a fermion zero mode and monopoles with half integer charge appear in the spectrum. Flux tubes on the other hand are unstable to Schwinger-type pair creation in the extended theory. When a monopole crosses the brane in the extended theory — instead of making a flux tube — it's baryon number changes by one unit and a localized fermion of opposite baryon number is left on the brane.

Notice that from the point of view of Yang-Mills theory we get a different periodicity in θ due to the axial anomaly. Therefore the Vafa-Witten theorem implies a different quantization condition for the Chern-Simons coefficient, namely $8\pi k \in \mathbb{Z}$. This is still consistent with the quantization condition obtained in [75] from pure MCS theory. It does however allow values for k that do not obey the quantization condition obtained by putting the 2+1d SU(2) on a 3-sphere and considering large gauge transformations. It is important to note, however that in our localization model, a large gauge transformation on the spatial sphere changes the number of instantons in the enclosed bulk. Due to the axial anomaly an instanton produces a fermionic zero mode and therefore does not leave the ground state invariant.

4.1.9 Etc.

We can also try to understand what happens in the context of multiple branes, as those, when brought on top of each other, merge into a single brane. In other words from the view point of the lower dimensional EFT the $U(1) \times U(1)$ symmetry gets broken to a single diagonal $U(1)$ as the branes come on top of each other, as was first proposed in [165]. In our context it is evident that one ends up with a jump in the axion field that is the sum of the jumps in the original branes, i.e. as two branes are brought on top of each other their CS levels add up.

Now we take a look at two monopoles of unit charge. Moving them across a brane of $\Delta\theta = \pi$, each produces a fermion of charge $1/2$ on the brane. When we now exchange the fermions on the brane, the wavefunction picks up an additional anyonic phase

$$\delta = \pi. \tag{4.12}$$

Therefore the magnetic event remnant fermions actually behave like bosons. This is to be expected, because the interchange of monopoles before moving them across the brane would not have caused a nontrivial phase either.

4.1.10 Conclusions and Outlook

We have found an explicit mechanism to localize a MCS to a field-theoretic brane in 3+1 dimensions. Building on the Dvali-Shifman gauge field localization mechanism, we constructed a model in which the localization happens on an axionic domain wall. This naturally leads to a CS term in the 2+1d low energy theory.

As expected our low energy theory exhibits the features deduced from classical MCS, namely a photon mass and the appearance of magnetic flux in the presence of electric charge. In terms of bulk physics those features can be understood as modified boundary conditions in Maxwell's equations due to the varying θ-term across the bulk-brane interface.

Quite generally for MCS theory obtained from some compactification, we have found that for large CS level the massive Kaluza-Klein-tower states can become lighter than the zero mode. I.e. for large CS level the effective theory behaves vastly different than one would expect from just a single gauge field with CS coupling.

Even more interestingly we have developed a nice understanding of the disappearance of confinement and the properties of magnetic events. Also the discreteness of the Chern–Simons coupling is related in a surprising way to bulk axion physics.

This work opens up several possible directions for future research, the most obvious being a generalization to more complicated gauge groups in the bulk and on the brane. Other interesting questions concern supersymmetric generalizations of our model with connections to string theory. In the original SQCD model given by Dvali-Shifman [42], the gluino-condensate order parameter $\langle\lambda\lambda\rangle$ plays the role of the axion. This implies that the conclusions drawn in our paper should also apply to the above mentioned field theoretic branes found in SQCD. This observation can shed some new light on the conjecture [83] that in the large N limit the SQCD domain walls should assume a role analogous to D-branes in QCD string theory. Using string dualities the effective world volume theory has already been conjectured to be a Chern–Simons gauge theory [78]. It might also be worth exploring a possible realization of our setup in Seiberg-Witten theory [166], where it could bear a relation to the phenomenon of wall-crossings. It would also be interesting to check whether a dual of our setup can lead to non-trivial applications in condensed matter systems.

4.1.11 appendix

Explicit potential

Here we propose an explicit potential that can be used to realize the localization of Chern-Simons theory as described above. The potential for the Axion $\Theta(x)$ and Higgs $\phi(x)$ fields is

$$V = A\cos(\Theta) + (B + C\cos(\Theta))\,(\phi^a)^2 + \lambda(\phi^a)^4 + \cdots \qquad (4.13)$$

As long as $|B| < C$ and A and C are of the same sign, the potential has global minima where θ is a multiple of 2π and $\phi = 0$ in this theory. This corresponds to the confining vacuum in the bulk. Clearly, axionic domain walls exist and near the center of such a wall, the Higgs field must take a nonzero vev to minimize the potential. An exemplary field configuration for such a domain wall is shown in fig. 4.1 (left). Terms of higher order in ϕ or $\cos(\Theta)$ will not change the story much as long as their coefficients are sufficiently small. Note however that $B-C$ must be sufficiently negative to allow for a Higgs-mode to condense.

Image charge details

Here we will give some details of our calculation of the electromagnetic field sourced by a point charge on the domain wall. Consider the simplified situation described in the text, where the axion vev does not change continuously by $\Delta\theta$, but in two discrete jumps at the left and right boundary of the U(1) domain. For the purpose of determining the correct boundary conditions, the dual superconductor in the bulk can be modeled by a medium of singular permeability/permittivity. For one charge on the brane, an infinite series of image charges is recursively built up. For a given electric and magnetic charge, the charge-vector of its mirror image is determined by the following matrix:

$$\begin{pmatrix} \frac{-\theta_{\rm dsc}^2+\theta_{\rm vac}^2+\pi^2}{(\theta_{\rm vac}-\theta_{\rm dsc})^2+\pi^2} & \frac{\theta_{\rm dsc}\left(\theta_{\rm vac}(\theta_{\rm vac}-\theta_{\rm dsc})+\pi^2\right)}{\pi((\theta_{\rm vac}-\theta_{\rm dsc})^2+\pi^2)} \\ \frac{4\pi(\theta_{\rm dsc}-\theta_{\rm vac})}{(\theta_{\rm vac}-\theta_{\rm dsc})^2+\pi^2} & -\frac{-\theta_{\rm dsc}^2+\theta_{\rm vac}^2+\pi^2}{(\theta_{\rm vac}-\theta_{\rm dsc})^2+\pi^2} \end{pmatrix} \tag{4.14}$$

where $\theta_{\rm vac}$ is the axion-vev in the U(1) vacuum on the brane and $\theta_{\rm dsc}$ is the axion-vev in the dual-superconductor on the other side of the respective boundary.

For the simplest case, when $\Delta\theta = 2\pi$ is symmetrically distributed and $\theta_{\rm vac} = 0$, the mirror charges contributing to the electric field at the center of the brane are just an evenly spaced sequence of alternating sign. After Fourier expanding the charge density, the field created by the Fourier mode $\rho_k \propto \cos((2k+1)\pi z/(2L))$ can be evaluated exactly. At asymptotically large distance from the charges, the $k = 0$ mode dominates and reproduces the expected Yukawa form

$$|E(r,\theta,z=0)| \propto \left(\frac{1}{\sqrt{r}} + \mathcal{O}(1/r^{3/2})\right)\exp\left(-\frac{\pi r}{2L}\right) \tag{4.15}$$

For generic $\Delta\theta$ and when $\Delta\theta$ is distributed unevenly to the left and right boundary, the effective mass of the lightest mode can be evaluated numerically. The results are displayed in 4.3. As expected, for a small values of $\Delta\theta$ we see a linear growth of the mass with $\Delta\theta$. This is precisely the Chern-Simons mass of the previously-massless 2+1d photon. In the case of larger $\Delta\theta$ however, the situation is more complicated. When $\Delta\theta$ is distributed unevenly, we see a falloff in the effective mass proportional to $\Delta\theta^{-1}$. This can

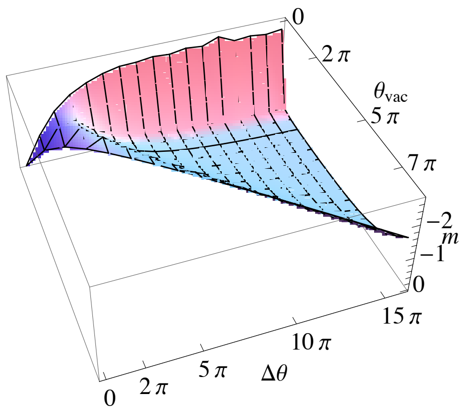

Figure 4.3: Effective mass of the lightest mode seen by a brane observer as a function of the discontinuities in θ

be interpreted as one of the two dof of a massive KK-mode becoming light, as remarked in the text. The evenly distributed case however is peculiar, as the effective mass saturates for large $\Delta\theta$ at the value of the mass of the lightest KK-mode in the absence of a Chern-Simons term. In this case there is an enhanced symmetry, implemented by first performing a reflection about the brane and then inverting the axion field. It implies that both dof in each KK level have to stay degenerate.

4.2 Black Holes and Quantumness on Macroscopic Scales

4.2.1 Abstract

It has recently been suggested that black holes may be described as condensates of weakly interacting gravitons at a critical point, exhibiting strong quantum effects. In this paper, we study a model system of attractive bosons in one spatial dimension which is known to undergo a quantum phase transition. We demonstrate explicitly that indeed quantum effects are important at the critical point, even if the number of particles is macroscopic. Most prominently, we evaluate the entropy of entanglement between different momentum modes and observe it to become maximal at the critical point. Furthermore, we explicitly see that the leading entanglement is between long wavelength modes and is hence a feature independent of ultraviolet physics. If applicable to black holes, our findings substantiate the conjectured breakdown of semiclassical physics even for large black holes. This can resolve long standing mysteries, such as the information paradox and the no-hair theorem.

4.2.2 Introduction

In a recent series of papers [167–170], Dvali and Gomez have proposed a new conceptual framework for the understanding of black hole physics. The first of two key claims is that the black hole is a bound state or condensate of many weakly interacting (i.e. long-wavelength) gravitons. Secondly, it was suggested that this condensate is at a quantum critical point and therefore exhibits properties that are not apparent in the traditional description in terms of (semi-)classical general relativity. Most importantly, the underlying quantum physics could be able to resolve the mysteries of the information paradox. Hawking evaporation is described as the depletion and evaporation of the condensate and its purification is thus a natural result. To the same extent, black holes could carry quantum hair [169]. These effects are not visible in the semiclassical approximation, since this limit corresponds to an infinite number of black hole constituents.

In this letter, we will review the Dvali-Gomez proposal and will elaborate in more detail why one can expect Bose condensates at a critical point to display qualitatively new phenomena. In particular, we will discuss how quantum physics can be relevant on macroscopic scales in such systems. To this end, we are going to investigate in detail the quantum phase transition of the attractive Bose gas in $1+1$ dimensions.

The idea that black holes may be described by the quantum physics of N weakly interacting gravitons was first put forward in [167]. There it was observed that in pure Einstein gravity a black hole of mass M and hence size $R_s = G_N M$ primarily consists of gravitons of wavelength $\lambda \sim R_s$. As each of these long wavelength particles contributes energy $E_1 = \hbar/\lambda$, one obtains

$$N \sim \frac{M}{E_1} = \frac{M^2}{M_p^2}. \tag{4.16}$$

The idea of a black hole as a Bose condensate of gravitons can also be motivated in a bottom-up approach. As gravitons are self coupled, they can potentially form a self sustained bound state. The properties of such a bound state can be estimated via the virial theorem,

$$\langle E_{\text{kin}} \rangle \sim \langle V \rangle \,. \tag{4.17}$$

The kinetic energy E_{kin} of N gravitons of wavelength λ is given by

$$\langle E_{\text{kin}} \rangle = N \frac{\hbar}{\lambda} \,, \tag{4.18}$$

while a naive estimate of the potential energy of the configuration of size R is

$$\langle V \rangle \sim N^2 \frac{G_N \hbar^2}{\lambda^2 R} \,. \tag{4.19}$$

Assuming the size to be of the order of the wavelength, $R \sim \lambda$, one obtains

$$\lambda \sim \sqrt{N} L_p \,. \tag{4.20}$$

It is easily verified that this relation is nothing but Eq. (4.16). An order of magnitude estimate of graviton-emission gives a result consistent with the rate of emission of Hawking radiation. Consequently, a self sustained bound state of gravitons, if it exists, will likely behave like a black hole.

One should observe that the typical interaction strength between two gravitons is $\alpha \sim 1/(\lambda M_{Pl})^2$. However, all mutual interactions add up and their total effect should be quantified by $N\alpha$. This implies that the self bound graviton condensate is at $\alpha N \sim 1$, where interactions start to dominate over the kinetic term. This condition characterizes the critical point of a zero temperature phase transition or quantum phase transition (QPT) in a simple bosonic model system [170] and is considered to ensure that quantum effects are important even for macroscopic black holes.

To substantiate this idea, it is our interest to gain more qualitative insight into bosonic systems at a critical point by a detailed study of this $1+1$-dimensional nonrelativistic attractive Bose gas on a ring. The transition in this system was discovered and first studied in [171–174]. We will substantiate the existence of this critical point by studying appropriate characteristics.

We will then focus on the quantum behavior. As a measure of quantumness, we calculate the entanglement of different momentum modes applying analytical as well as numerical techniques. We observe that it becomes maximal at the critical point and for low momentum modes. We interpret this as further evidence that the black hole condensate picture can be successful independent of the ultraviolet physics that completes Einstein theory.

The remainder of the paper will be organized as follows. In section 2 we will introduce in detail the $1+1$-dimensional attractive Bose gas, remind the reader of mean g and introduce the basis of our numerical studies. Further evidence for the existence of a quantum critical point is provided in section 3. We will then introduce the fluctuation entanglement as a relevant measure of quantumness and present our results in 4. Finally, in the conclusions, we discuss the qualitative consequences of our findings with regards to the physics of macroscopic black holes.

4.2.3 The $1+1$-dimensional Bose Gas

Throughout this paper, we consider a Bose gas on a $1D$-circle of radius R with attractive interactions. The Hamiltonian is given by

$$\begin{aligned} \hat{H} &= \frac{1}{R}\int_0^{2\pi} d\theta \left[-\frac{\hbar^2}{2m}\hat{\psi}^\dagger(\theta)\partial_\theta^2\hat{\psi}(\theta)\right. \\ &\left. -\frac{\hbar^2}{2m}\frac{\pi\alpha R}{2}\hat{\psi}^\dagger(\theta)\hat{\psi}^\dagger(\theta)\hat{\psi}(\theta)\hat{\psi}(\theta)\right], \end{aligned} \tag{4.21}$$

where α is a dimensionless, positive coupling constant. This Hamiltonian can be cast into a more convenient form by decomposing $\hat{\psi}(\theta)$ in terms of annihilation operators:

$$\hat{\psi}(\theta) = \frac{1}{\sqrt{2\pi R}} \sum_{k=-\infty}^{\infty} \hat{a}_k e^{ik\theta}\,, \tag{4.22}$$

which leads to

$$\hat{H} = \sum_{k=-\infty}^{\infty} k^2 \hat{a}_k^\dagger \hat{a}_k - \frac{\alpha}{4} \sum_{k,l,m=-\infty}^{\infty} \hat{a}_k^\dagger \hat{a}_l^\dagger \hat{a}_{m+k} \hat{a}_{l-m} \tag{4.23}$$

Note that in order to improve readability we have now switched to units $R = \hbar = 2m = 1$. The total number operator is

$$\hat{N} = \int_0^{2\pi} d\theta \hat{\psi}^\dagger(\theta)\hat{\psi}(\theta) = \sum_{k=-\infty}^{\infty} \hat{a}_k^\dagger \hat{a}_k\,. \tag{4.24}$$

It was first shown in [171–174] that an increase of the effective coupling αN on the ring leads to a transition from a homogenous ground state to a solitonic phase, where the critical point is reached for $\alpha N = 1$.

Mean Field Analysis

A mean field approach to the hamiltonian (4.21) leads to the Gross-Pitaevskii energy functional

$$E[\Psi_{GP}] = \int_0^{2\pi} d\theta \left[|\partial_\theta \Psi(\theta)|^2 - \frac{\alpha}{2} |\Psi(\theta)|^4 \right] \tag{4.25}$$

The ground state wavefunction Ψ_0 is obtained through minimization of the energy functional subject to the constraint $\int d\theta |\Psi(\theta)|^2 = N$. This leads to the time independent Gross-Pitaevskii equation

$$\left[\partial_\theta^2 + \pi\alpha |\Psi_0(\theta)|^2\right] \Psi_0(\theta) = \mu \Psi_0(\theta)\,, \tag{4.26}$$

where $\mu = dE/dN$ is the chemical potential. Solutions to this equation are given by (see e.g. [175])[1]

$$\Psi_0(\theta) = \begin{cases} \sqrt{\frac{N}{2\pi}} \\ \sqrt{\frac{NK(m)}{2\pi E(m)}} \mathrm{dn}\left(\frac{E(m)}{\pi}(\theta - \theta_0)|m\right) \end{cases} . \tag{4.27}$$

[1] Here, $\mathrm{dn}(u|m)$ is a Jacobi elliptic function and $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively.

Here, θ_0 denotes the center of the soliton and m is determined by the equation

$$K(m)E(m) = \left(\frac{\pi}{2}\right)^2 \alpha N\,. \tag{4.28}$$

For small $\alpha N < 1$, (4.25) is minimized by the homogenous wavefunction. On the other hand, for $\alpha N > 1$ the solitonic solution has a lower energy. At $\alpha N = 1$, both configurations are degenerate in energy - a clear indication for a quantum phase transition.

On a side note, one may wonder whether the one-soliton solution is stable for arbitrary $\alpha N > 1$ or if multi-soliton solutions may eventually be energetically favored. This can be checked in a simple argument. A soliton of size R_s has a total energy

$$E \sim \frac{N}{R_s^2} - \alpha \frac{N^2}{R_s}\,. \tag{4.29}$$

Minimization with respect to R yields $R_s = \frac{2}{\alpha N}$ and $E_1 = -\frac{1}{4}\alpha^2 N^3$. A split into two stable solitons of boson number rN and $(1-r)N$ yields a total energy $E_2 = -\frac{1}{4}\alpha^2 N^3[1 - 3r(1-r)]$. This is bigger than E_1 for any $r < 1$. This can be straightforwardly generalized two multi-soliton solutions; therefore, the single soliton is stable.

Finally, let us note that the apparent spontaneous breaking of translation symmetry in the solitonic phase is in no contradiction to known theorems about the absence of finite volume symmetry breaking. The Gross-Pitaevskii ground state only becomes exact in the $N \to \infty$ limit. In this limit, translated Gross-Pitaevskii states are orthogonal and do not mix under time evolution. Technically, symmetry breaking is made possible because expecation values of composite operators made out of the fields diverge in the large N limit. We comment on this in more detail in the Appendix.

This again emphasizes how the classical limit really emerges as a large N limit from quantum mechanics. Exactly how this argument breaks down at the critical point and what the implications of this breakdown are will be the focus of the remainder of this manuscript.

Bogoliubov Approximation

The Gross-Pitaevskii equation is the zeroth-order equation in an expansion of the field operator into its mean value and quantum (and, in more general setups, thermal) fluctuations around it:

$$\hat{\psi}(\theta) = \langle \hat{\psi}(\theta) \rangle + \delta\hat{\psi}(\theta)\,. \tag{4.30}$$

The spectrum of these small excitations around the mean field can then be found in the Bogoliubov approximation. Generally, this corresponds to approximating the fluctuation Hamiltonian by its quadratic term and subsequent diagonalization through canonical transformations of the field.

For $\alpha N < 1$, i.e. on the homogeneous background, it is convenient to stick to the momentum decomposition (4.78) and replace $\hat{a}_0 = \hat{a}_0^\dagger = \sqrt{N_0} \sim \sqrt{N}$. In words, one assumes that the zero mode is macroscopically occupied and all commutators $[\hat{a}_0, \hat{a}_0^\dagger]$ in the Hamiltonian are suppressed by relative powers of $1/N$; the quantum fluctuations of the zero mode may therefore be neglected. This, in combination with taking into account the constraint

$$\hat{N} = N_0 + \sum_{k\neq 0} \hat{a}_k^\dagger \hat{a}_k \tag{4.31}$$

leads to the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \sum_{k\neq 0} \left(k^2 - \alpha N/2\right) a_k^\dagger a_k \\ &\quad -\frac{1}{4}\alpha N \sum_{k\neq 0} \left(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}\right) + \mathcal{O}(1/N)\,. \end{aligned} \tag{4.32}$$

All interaction terms are suppressed by $1/N$ and go to zero in the double scaling limit $N \to \infty$, $\alpha \to 0$ with αN finite. The Hamiltonian can be diagonalized

$$\mathcal{H} = \sum_{k\neq 0} \epsilon_k b_k^\dagger b_k,\ \epsilon_k = \sqrt{k^2(k^2 - \alpha N)} \tag{4.33}$$

with a Bogoliubov transformation

$$a_k = u_k b_k + v_k^\star b_{-k}^\dagger, \tag{4.34}$$

where the Bogoliubov coefficients are

$$u_k^2 = \frac{1}{2}\left[1 + \frac{k^2 - \frac{\alpha N}{2}}{\epsilon_k}\right], \tag{4.35}$$

$$v_k^2 = \frac{1}{2}\left[-1 + \frac{k^2 - \frac{\alpha N}{2}}{\epsilon_k}\right]. \tag{4.36}$$

The Boguliubov approximation breaks down whenever an ϵ_k becomes too small. In that case the initial assumption that only the zero mode is macroscopically occupied is no longer justified. Obviously, it is ϵ_1 that first goes to zero, namely when $\alpha N \to 1$. Right at the phase transition, the Boguliubov approximation is never valid. It is worth noting however, that for any finite distance δ from the critical point, there exists a minimal N for which the approximation is valid. In other words, for any finite δ, the Boguliubov approximation becomes exact in the limit $N \to \infty$. This is due to the fact that both the interaction terms as well as v_k^2/N vanish in this limit for any finite δ. For $\delta = 0$, however, this is never true.

In the $\alpha N > 1$ case, the classical background is not homogenous any more, but is given by the bright soliton solution (4.27). In this case, the background induces an additional nontrivial mixing between momentum eigenmodes of different $|k|$. A decomposition into momentum eigenmodes requires an (unknown) analytic expression for the Fourier components of the soliton and is thus no longer convenient. On the other hand, an analytic Bogoliubov treatment is still possible by directly decomposing $\delta\hat{\psi}$ into normal modes:

$$\delta\hat{\psi}(\theta) = \sum_i \left(u_i(\theta)\hat{b}_i^\dagger + v_i^\star(\theta)\hat{b}_i \right) . \tag{4.37}$$

If the mode functions obey the Boguliubov-de Gennes equations

$$\partial_\theta^2 u_j + \alpha\Psi_0^2(2u_j + v_j) + \mu u_j = E_j u_j \tag{4.38}$$

$$\partial_\theta^2 v_j + \alpha\Psi_0^2(2v_j + u_j) + \mu u_j = -E_j u_j \tag{4.39}$$

and are normalized such that they form a complete set and the transformation (4.37) is canonical, the Hamiltonian is diagonalized. The first excited Bogoliubov modes have the form

$$u_1(\theta) = N_1 \mathrm{sn}^2\left(\frac{K(m)}{\pi}(\theta - \theta_0) \middle| m \right) \tag{4.40}$$

$$v_1(\theta) = -N_1 \mathrm{cn}^2\left(\frac{K(m)}{\pi}(\theta - \theta_0) \middle| m \right) . \tag{4.41}$$

The coefficient N_1 is defined by

$$N_1^2 = \frac{mK(m)}{2\pi\left[(2-m)K(m) - 2E(m)\right]} . \tag{4.42}$$

Numerical Diagonalization

While the Bogoliubov treatment provides an approximative description of the Bose gas deep in the respective phases, it fails, as we have reasoned above, around the critical point.

A complementary method to explore the quantum properties of the system is numerical diagonalization of the Hamiltonian. Of course, numerical techniques are only applicable for sufficiently small Hilbert spaces. The Hamiltonian (4.21) is number conserving. This allows for exact diagonalization of (4.23) by considering a subspace of fixed N. However, to make any numerical procedure feasible, we need to limit the allowed momenta. In the spirit of [171–174], we truncate the basis of free states in which we perform the diagonalization to $|l| = 0, 1$. This gives a very good approximation to the low energy spectrum of the theory well beyond the phase transition. Analytically, this can be seen by analyzing the spectrum of the soliton solution (4.27). Only for $\alpha N \gtrsim 1.5$, higher l modes start giving relevant contributions. We have further verified this numerically by allowing for $|l| = 2, 3$; the low energy modes are only marginally affected up until $\alpha N \sim 2$. Our code allowed us to consider particle numbers $N \lesssim 10000$. In order to illustrate scaling properties, all analyses are performed for various particle numbers.

Since the normalized coupling αN is the relevant quantity for a phase transition, one can analyze all interesting properties for a fixed N by varying α. The corresponding spectrum of excitations above the ground state as a function of αN is shown in Fig.4.4 for $N = 5000$ and $-1 \leq k \leq 1$. One observes a decrease in the energy gap between the low lying excitations due to the attractive interactions as αN is increased. At the quantum critical point, the spacing between levels reaches its minimum. Its magnitude depends on the particle number N; the energy of the lowest lying excitation decreases with N. By further increasing the coupling α one reaches the solitonic phase. The spectrum corresponds to that of translations and deformations of a soliton.

Obviously, (4.21) is invariant under translations; since we are considering a finite length ring, the ground state obtained by exact diagonalization can never correspond to a localized soliton. It will instead contain a superposition of solitons centered around arbitrary θ. This problem can be overcome by superposing a weak symmetry breaking potential to break the degeneracy between states with a different soliton position:

$$\hat{H}_{sb} = \hat{H} + \hat{V}_\epsilon \tag{4.43}$$

$$\hat{V}_\epsilon = \frac{\epsilon}{N^2} \int d\theta \hat{\psi}^\dagger(\theta) \cos\theta \hat{\psi}(\theta) \,. \tag{4.44}$$

The higher ϵ, the deeper the symmetry breaking potential, and the more localized the soliton will be.

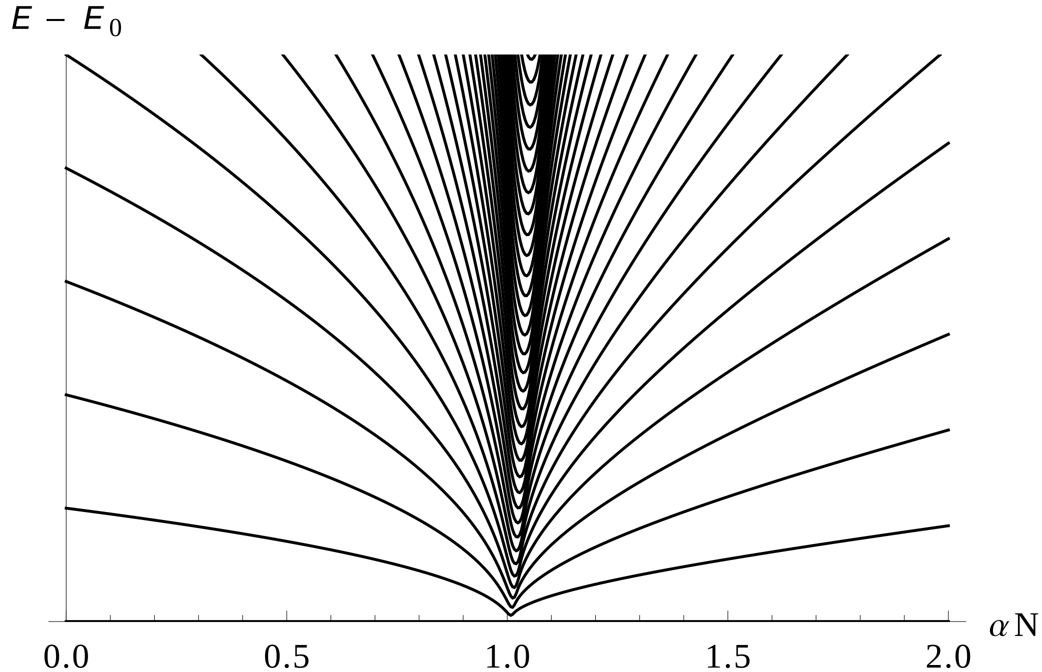

Figure 4.4: Energy spectrum for $N = 5000$ as a function of the effective coupling αN

4.2.4 Quantum Phase Transition in the 1D-Bose gas

The mean field treatment of the attractive 1D Bose gas above has signalled a quantum phase transition. The degeneration of the Bogoliubov modes at $\alpha N = 1$ supports the existence of a critical point. Although, by definition, a phase transition can only occur for infinite N, indications for it should already be visible for large but finite N. Here we will focus on two observations:

(i) The one-particle entanglement entropy displays a sharp increase close to the critical point.

(ii) The ground state fidelity peaks at the critical point; the height of the peak grows with N.

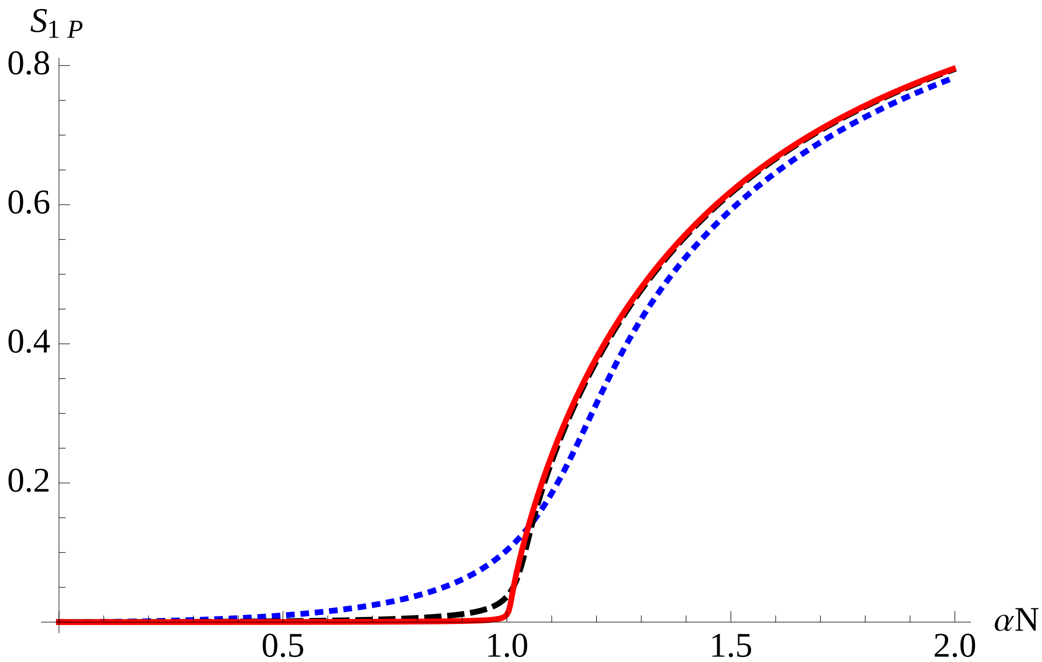

Figure 4.5: One-particle entanglement entropy for $N = 50$ (blue, dotted), 500 (black, dashed), 5000 (red, solid).

One-Particle Entanglement

The one particle entanglement entropy is defined as the von Neumann entropy $S_1 = \mathrm{Tr}[\hat{\rho}_{1\mathrm{p}} \log \hat{\rho}_{1\mathrm{p}}]$ of the one particle density matrix $\hat{\rho}_{1\mathrm{p}}$ of the ground state, obtained by singling out one particle and tracing over all $N-1$ other. As long as the ground state of the system is well described by a Hartree, i.e. product state, $\hat{\rho}_{1\mathrm{p}}$ describes a pure state; the entanglement entropy vanishes. When the critical point is approached, collective effects become important. No longer is the ground state described by a product state; consequently the entanglement entropy increases - a single particle becomes strongly entangled with the rest of the system.

The one particle density matrix is defined via

$$\hat{\rho}_{1\mathrm{p}} = \underset{(N-1)\mathrm{P}}{\mathrm{Tr}}\, \hat{\rho} = \underset{(N-1)\mathrm{p}}{\mathrm{Tr}}\, |0_{GP}\rangle\langle 0_{GP}|\,, \tag{4.45}$$

or, explicitly, in the one particle momentum eigenbasis

$$(\hat{\rho}_{1\mathrm{P}})_{ij} = \delta_{ij} \sum_{\{n_k\}} |\alpha_{\{n_k\}}|^2 \frac{n_i}{N} \,. \tag{4.46}$$

Here, n_k is the occupation number of the k-th momentum mode and we have used

$$|0_{GP}\rangle = \sum_{\{n_k\}} \alpha_{\{n_k\}} |\{n_k\}\rangle \,. \tag{4.47}$$

We have plotted the numerically evaluated one particle entanglement as a function of αN for different N in Fig.4.5. The increase close to the critical point gets profoundly sharper for larger N. Independent of N, the entropy is bounded by $S_{\max} = \log 3$, due to the truncation of the one-particle Hilbert space to a three level system.

The entanglement entropy becomes maximal for large αN. This, as argued before, is due to the fact that the numerical groundstate is given by a superposition of solitons localized at arbitrary positions [176].[2]

Ground State Fidelity

Ground State Fidelity (GSF) was introduced in [177] as a characteristic of a QPT. It is defined as the modulus of the overlap of the exact ground states for infinitesimally different effective couplings.

$$F(\alpha N, \alpha N + \delta) = |\langle 0_{\alpha N} | 0_{\alpha N + \delta} \rangle| \tag{4.48}$$

Far away from the critical point, this overlap will be very close to unity. For small αN, the ground state is dominated by the homogeneous state, and while coefficients may change slightly, no important effect will be seen. The analogous statement holds deep in the solitonic regime. While the shape of the soliton changes, it will so smoothly; in the infinitesimal limit, the overlap is one. Right at the critical point, however, the ground state changes in a *non-analytic* way. The homogeneous state ceases to be the ground state and becomes an excited state, while the soliton becomes the new ground state. As energy eigenstates with different eigenvalue are orthogonal, the ground state fidelity across the phase transition is exactly zero.

[2] This quantum behavior is not expected to survive in the large N limit if the symmetry breaking potential is turned on. In this case, a vacuum is selected which does not mix with translated states (see Appendix). The entanglement entropy is therefore much smaller.

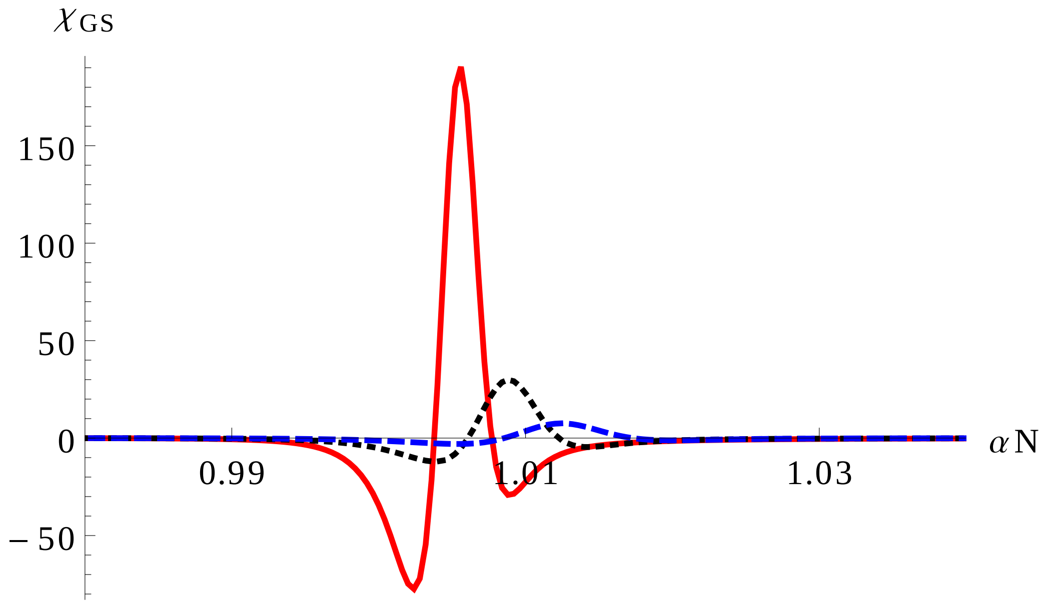

Figure 4.6: Numerical ground state fidelity susceptibility for $N = 3000$ (blue, dashed), $N = 5000$ (black, dotted) and $N = 10000$ (red, solid).

The GSF has the disadvantage of depending on the arbitrary choice of the small parameter δ. This can be cured by introducing the fidelity susceptibility $\chi_{\rm gs}(\alpha N)$ as the second derivative of the GSF.

$$\chi_{\rm gs}(\alpha N) = \lim_{\delta\to 0} \frac{F(\alpha N, \alpha N + \delta) - F(\alpha N, \alpha N - \delta)}{\delta^2} \, . \tag{4.49}$$

It has been shown [178] that singular behavior of the fidelity susceptibility directly signals a discontinuity of the first or second derivative of the ground state energy - a quantum phase transition.

The aforementioned behavior is of course idealized for an infinite system, where ground state degeneracy and thus level crossing become an exact property. In the finite N systems we examined numerically, the overlap cannot go to zero, because there is anticrossing which allows the energy levels to degenerate only for $N \to \infty$.

Still we can observe a drop in the fidelity which deepens with N but is of magnitude much smaller than 1 for all N we were able to simulate. The fidelidy susceptibility as obtained from the exact diagonalization is plotted in Fig.4.6 for different N. In the limit $N \to \infty$, we expect a behavior $\chi_{\rm gs} \to -\delta''(\alpha N - 1)$. This tendency can be clearly observed. Both the negative and the positive peak move towards $\alpha N = 1$, they become narrower, and their modulus diverges with growing N.

4.2.5 Fluctuation Entanglement

We will now consider the entanglement between the fluctuation $\delta\hat{a}_k = \hat{a}_k - a_k^{\rm c}$ of a given original momentum mode and the fluctuations of the rest of the system. The motivation for studying this quantity is twofold. We imagine, that an external observer would couple linearly to the bosonic field (so that the situation has some minimal resamblance with the gravity case). It has been pointed out [179] that for such a coupling, field values (or their Fourier components) will be the environment-selected pointer states[3] and not localized single particle states. This leads us to consider the entanglement of a momentum mode, rather than single-particle entanglement, as a measure of relevant quantum correlations of the given state. Furthermore, the observer couples to the original field $\hat{a}_k$ and hence its fluctuations as opposed to coupling to the Bogoliubov modes $\hat{b}_k$.

More technically speaking, the quantity we calculate is the von Neumann entropy of the reduced density matrix for a given $\delta\hat{a}_k$

$$(\delta\rho_k)_{nm} = \underset{\text{modes } k' \neq k}{\mathrm{Tr}} \left[\rho\, (\delta\hat{a}_k^\dagger)^m |0^{\rm c}\rangle\langle 0^{\rm c}| (\delta\hat{a}_k)^n\right] \tag{4.50}$$

where $|0^{\rm c}\rangle$ denotes the state that would be observed classically.

Fluctuation entanglement provides a measure for the quantum correlations between a single momentum mode with the rest of the system. It hence gives a direct handle of the quantumness of our ground state as measured by an outside observer if coupled linearly to the field. Note also that due to the fact that we are considering a closed system, the fluctuation entanglement is exactly equivalent to the Quantum Discord introduced in works [180, 181] as a measure of quantumness.

[3]Pointer states denote those states that are stable with respect to interactions with the environment and therefore correspond to classically observable states.

Calculation in the Bogoliubov Approximation

In order to calculate the fluctuation entanglement in the Bogoliubov case, note that the sought-for density matrix is Gaussian[4]. The ground state in terms of $\hat{b}_k$ is Gaussian and the Bogoliubov transformation amounts to squeezing - which leaves a Gaussian state Gaussian. Also integrating out modes in a Gaussian state does not change this property. Hence the reduced density matrix in terms of $\delta\hat{a}_k$ must have the form

$$\rho_k = C_k \exp\Big\{ -\lambda_k\Big(\delta\hat{a}_k^\dagger\delta\hat{a}_k - \frac{1}{2}\tau_k\left[\delta\hat{a}_k^\dagger\delta\hat{a}_k^\dagger + \delta\hat{a}_k\delta\hat{a}_k\right]\Big)\Big\}, \tag{4.51}$$

with real coefficients λ_k and τ_k and normalization C_k such that $\mathrm{Tr}\,\rho_k = 1$. This density matrix has a von Neumann entropy

$$S_k = \frac{\lambda_k\sqrt{1-\tau_k^2}}{2}\left(\coth\frac{\lambda_k\sqrt{1-\tau_k^2}}{2} - 1\right) - \ln\left(1 - e^{-\lambda_k\sqrt{1-\tau_k^2}}\right) \tag{4.52}$$

We can fix the unknown coefficients by imposing

$$\langle\psi|\delta\hat{a}_k^\dagger\delta\hat{a}_k|\psi\rangle = \mathrm{Tr}[\rho_k\delta\hat{a}_k^\dagger\delta\hat{a}_k]$$

and

$$\langle\psi|\delta\hat{a}_k\delta\hat{a}_k|\psi\rangle = \mathrm{Tr}[\rho_k\delta\hat{a}_k\delta\hat{a}_k]\,, \tag{4.53}$$

where $|\psi\rangle$ is the groundstate of the Bogoliubov modes.

Homogenous Phase

In the homogenous case, imposing (4.53) and evaluating the left hand side by inserting the Bogoliubov transformation (4.34) leads to

$$\lambda_k = \ln\left(\frac{u_k}{v_k}\right)^2, \qquad \tau_k = 0 \qquad \text{and} \qquad C_k = 1/u_k^2. \tag{4.54}$$

[4]A density matrix is called Gaussian, when its Wigner function $W(\alpha,\alpha^\star) = \frac{1}{\pi^2}\int d^2\beta\exp(-i\beta\alpha^\star - i\beta^\star\alpha)\,\mathrm{Tr}[\rho\exp(i\beta a^\dagger + i\beta^\star a)]$ is Gaussian

Thus, the fluctuation entanglement entropy is

$$S_k = u_k^2 \ln u_k^2 - v_k^2 \ln v_k^2. \tag{4.55}$$

The entanglement of the first momentum mode S_1 diverges near the critical point $\alpha N = 1 - \delta$ as

$$S_1 \approx 1 - \ln(4) - \frac{1}{2} \ln \delta. \tag{4.56}$$

A similar divergence of an entanglement entropy has been pointed out in spin chain (and analogous) systems undergoing a phase transition [182, 183]. In contrast to these cases however, where the entanglement is between nearest neighbour sites, the diverging entanglement in our case is between different low-momentum modes and not between localized sites. So one may say, that the entanglement in our case is long-range. Furthermore it should be noted, that the entanglement of the higher modes $|k| > 1$ stays finite near the critical point, showing that the diverging entanglement is an infrared effect, which can be expected to be independent of short distance physics.

Solitonic Phase

The relevant expectation values in the Bogoliubov ground state are given by

$$\langle\psi| \quad \delta\hat{a}_m^\dagger \delta\hat{a}_n|\psi\rangle = \sum_k \left(\int e^{im\theta} v_k(\theta) d\theta\right)\left(\int e^{-in\theta} v_k(\theta)^\star d\theta\right), \tag{4.57}$$

$$\langle\psi| \quad \delta\hat{a}_m \delta\hat{a}_n|\psi\rangle = \sum_k \left(\int e^{-im\theta} u_k(\theta) d\theta\right)\left(\int e^{-in\theta} v_k(\theta)^\star d\theta\right). \tag{4.58}$$

It can be checked that close to the phase transition the first excited mode gives the leading contribution to the aforementioned entanglement entropy. The quantities $\langle\psi|\delta\hat{a}_1^\dagger \delta\hat{a}_1|\psi\rangle$ and $\langle\psi|\delta\hat{a}_1 \delta\hat{a}_1|\psi\rangle$ can be obtained by numerical integration. The parameters λ, τ of the reduced gaussian density matrix can then be determined. The final von Neumann entropy again shows a divergence[5] close to the phase transition. Fig.4.7 shows the fluctuation entanglement obtained in the Bogoliubov approximation on both sides of the phase transition.

[5] The shape of the divergence obtained by numeric integration seems to be consistent with a logarithm with a coefficient close to 0.33 near the phase transition.

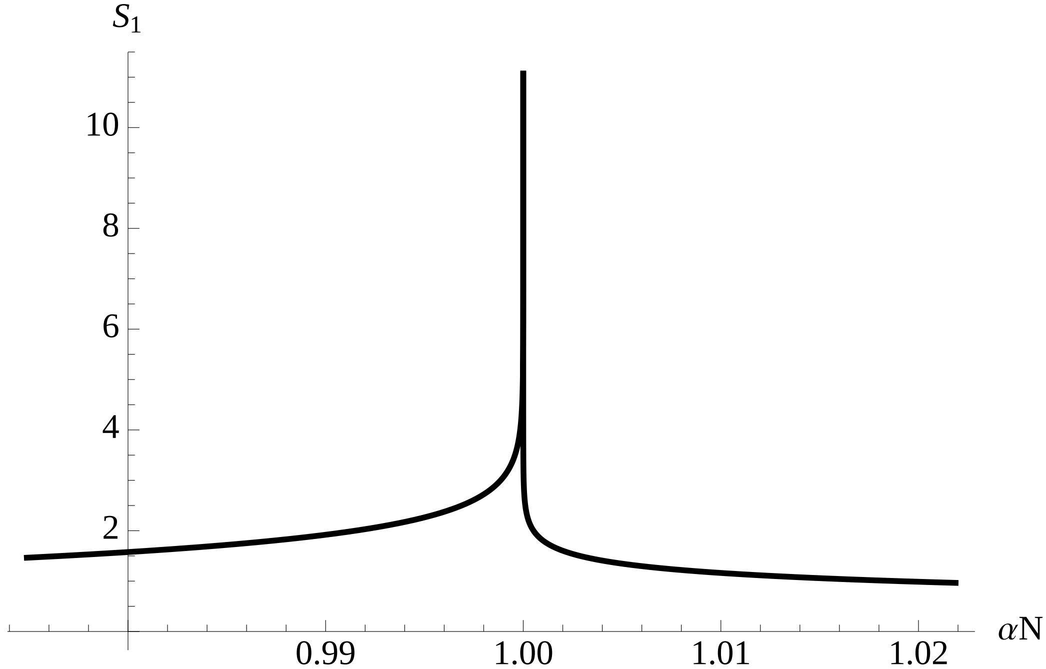

Figure 4.7: Analytical fluctuation entanglement

Numerical Treatment

As discussed before, for any given finite size system, the Bogoliubov approximation should not be trusted close to the critical point. Therefore it is important to study the exact behavior of finite size systems numerically, in order to substantiate the claim that the fluctuation entanglement entropy becomes large.

Within an exact treatment this quantity is considerably more difficult to extract, because in contrast to the Bogoliubiv analysis, one does not have direct access to a "classical background" which one could use to disentangle classical correlations. Instead, the seperation can be obtained through the following procedure.

Since all numerical solutions are obtained for a fixed particle number N, the field expectation value in the exact ground state $|0_N\rangle$ will necessarily vanish, $\langle 0_N|\hat{\psi}(\theta)|0_N\rangle = 0$. Obviously, the ground state $|0_N\rangle$ can hence never correspond to the classical (coherent) state with a wave function corresponding to the soliton solution of the Gross-Pitaevskii equation, $\langle\psi_{\rm cl}|\hat{\psi}(\theta)|\psi_{\rm cl}\rangle = \Psi_{\rm GP}(\theta)$. In order to define a mapping from $|0_N\rangle$ to $|\psi_{\rm cl}\rangle$,

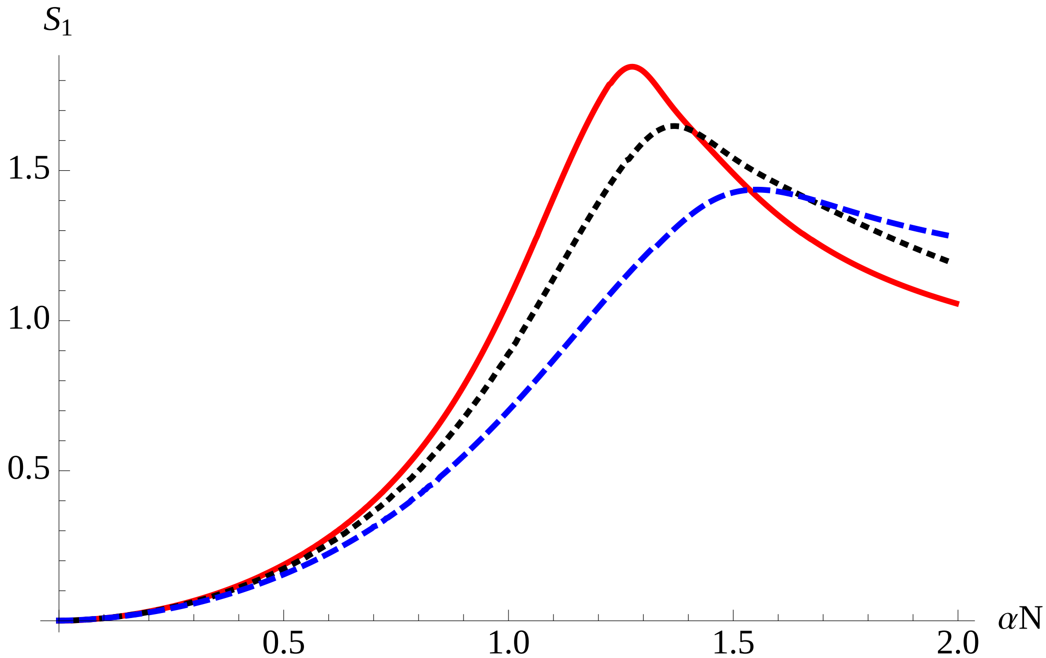

Figure 4.8: Numerical fluctuation entanglement for $N = 15$ (blue, dashed), $N = 20$ (black, dotted) and $N = 25$ (red, solid).

we numerically search for the coherent state $|\alpha\rangle$ with maximal overlap with $|0_N\rangle$. This state is expected to be annihilated by the perturbations of the Gross-Pitaevskii ground state,

$$\delta\hat{a}_k|\alpha_k\rangle = 0\,, \tag{4.59}$$

where

$$\delta\hat{a}_k = \hat{a}_k - c_k \tag{4.60}$$

and the c_k are the Fourier coefficient of $\Psi_{\rm GP}(\theta)$. From Eq.(4.59) it directly follows that $\alpha_k = c_k$. There is now an obvious measure of correlations which excludes those of the Gross-Pitaevskii background: The entanglement entropy of the $\delta\hat{a}_1$ modes, described by the density matrix

$$(\tilde{\rho}_1)_{kl} = \mathrm{Tr}\left[\hat{\rho}|\delta l_1\rangle\langle\delta k_1|\right]\,. \tag{4.61}$$

Now, $|\delta l_1\rangle$ denotes the eigenstate of $\delta\hat{N}_1 = \delta\hat{a}_1^\dagger\delta\hat{a}_1$ with eigenvalue δl_1. Eq. (4.61) directly corresponds to the density matrix (4.51) in the Bogoliubov approximation. Using the relations (4.59) and (4.60), it can be directly recast to take on the form

$$(\tilde{\rho}_1)_{kl} = \mathrm{Tr}\left[\hat{\rho}\frac{(\hat{a}_1^\dagger - \alpha_1^*)^l}{\sqrt{l!}}|\alpha_1\rangle\langle\alpha_1|\frac{(\hat{a}_1 - \alpha_1)^k}{\sqrt{k!}}\right], \tag{4.62}$$

which, by the definition of a coherent state, can be straightforwardly evaluated.

The resulting fluctuation entanglement is shown in Fig.4.8 for different particle number. It has a clear maximum at the would–be–phase–transition. The maximum value becomes larger and the peak narrower with increasing particle number, so the divergence in the Bogoliubov case seems a plausible limit. The fact that at $\alpha N = 2$, the fluctuation entanglement is still quite high is not surprising. Only in the limit $N \to \infty$ do we expect to see the behavior of the Boguliubov analysis. This is supported by the fact that a decrease is observed for increasing N, as well as for stronger localization potentials.

4.2.6 Conclusions and Outlook

In this paper, we have considered properties of the 1+1-dimensional attractive Bose gas around its critical point. By analyzing important indicators for QPTs, we provided further evidence that a tuning of the effective coupling gN leads to a phase transition in the system. More importantly, we have shown that quantum correlations become very important close to the critical point - contrary to the naive intuition that at sufficiently large particle number, systems should behave approximately classical. We have also pointed out that the quantum entanglement of the bosons close to the critical point is "long range" - in contrast to the observations in spin-chain systems that display nearest neighbour entanglement at criticality.

The motivation for our study of this model system, however, was the conjecture that black holes are bound states of a large number of weakly interacting gravitons. It has been claimed that the graviton condensates behave significantly different with respect to the semiclassical black hole analysis due to their being at a quantum critical point. It was argued that criticality allows quantum effects to only be suppressed by the perturbative coupling $\alpha_g \sim 1/N$ as opposed to the usual exponential suppression. If the qualitative insights from our simple toy model are valid for graviton condensates our results can back up several of the claims. We can argue that quantum effects become important for attractive Bose condensates at their critical point - even though the perturbative coupling is very small. Moreover, the entanglement of the true state is long range, consistent with the notion of a condensate of gravitons of wavelength comparable to the Schwarzschild radius. This would imply that for a black hole, the semiclassical treatment with a background geometry that obeys classical general relativity and quantization of fields on top of this rigid background becomes invalid much earlier than what the standard lore tells. Although curvature invariants in the horizon region of large Schwarzschild black hole are small, the semiclassical treatment is not applicable. Instead, quantum correlations in the graviton bound state become relevant. Importantly, our results point in the direction that the physics is dominated by large wavelengths. Therefore the description of black holes as graviton condensates has the attractive feature of being independent of the ultraviolet completion of gravity. The only requirement being that the low energy theory resembles perturbatively quantized Einstein theory with a massless spin two graviton.

The 1+1-dimensional Bose gas can indeed capture quite a few of the intriguing features of black holes and their possibly quantum nature. To understand in more detail time dependent features, such as Hawking evaporation, resolutions of the information paradox or scrambling, the implementation of dynamical methods will be amongst the aims of immediate future work. This will then necessarily also address possible couplings to external systems in order to be able to model the evaporation process. Working with more spatial dimensions may prove feasible to model the collapse induced by Hawking evaporation. This could alternatively be achieved by considering couplings that show further resemblance with gravitational self- interactions. Steps in these directions also include generalizations to non-number conserving, and ultimately relativistic theories.

Instabilities can in turn be countered by adding repulsive interactions that dominate at very short scales. Stable configurations of that sort would correspond to extremal black holes. Their properties also provide a vast playground for future investigation.

4.2.7 Spontaneous symmetry breaking in finite volume

Standard lore states that there can be no spontaneous symmetry breaking in finite volume with a finite number of fields. This may seem puzzling since we claim here explicitly that our ground state is a localized soliton for $gN > 1$, clearly signaling spontaneous breaking of translation invariance. This puzzle is resolved by noticing that the mean field approximation can only be exact in the limit $N \to \infty$.

To illustrate this, consider a number conserving Hamiltonian containing a finite number $2m$ of fields in the potential. Mean field approximation becomes exact if the corresponding state can be approximated arbitrarily well by a Hartree-type state, i.e.

$$|\psi_0\rangle = \otimes \sum_k c_k |k\rangle \,. \tag{4.63}$$

A translated state is given by

$$|\psi_{\delta\theta}\rangle = \otimes \sum_k c_k e^{ik\delta\theta} |k\rangle \,. \tag{4.64}$$

Now it can be seen that spontaneous symmetry breaking is indeed possible as we take the limit $N \to \infty$. It can be shown that

$$\lim_{N\to\infty} |\langle\psi_0|\psi_{\delta\theta}\rangle|^2 = \lim_{N\to\infty} \left|\sum_k c_k e^{ik\delta\theta}\right|^{2N} \,, \tag{4.65}$$

$$\lim_{N\to\infty} |\langle\psi_0|H|\psi_{\delta\theta}\rangle|^2 \leq \lim_{N\to\infty} CN^{2m} \left|\sum_k c_k e^{ik\delta\theta}\right|^{2(N-m)} \,. \tag{4.66}$$

In both expressions the right hand side vanishes as long as $c_k \neq 0$ for at least two different k. In other words, in the large N limit localized objects, localized at different points are orthogonal and, furthermore, do not mix under time evolution.

In the proof for absence of finite volume symmetry breaking enters the assumption that expectation values of composite operators made out of the fields are finite. This assumption breaks down in the large N limit. While this implies unboundedness of the energy in this limit, it is nothing to worry about. The energy diverges linearly in N and hence the energy per particle remains bounded, reminiscent of a thermodynamic limit.

4.3 Scrambling in the Black Hole Portrait

4.3.1 abstract

Recently a quantum portrait of black holes was suggested according to which a macroscopic black hole is a Bose-Einstein condensate of soft gravitons stuck at the critical point of a quantum phase transition. We explain why quantum criticality and instability are the key for efficient generation of entanglement and consequently of the scrambling of information. By studying a simple Bose-Einstein prototype, we show that the scrambling time, which is set by the quantum break time of the system, goes as $\log N$ for N the number of quantum constituents or equivalently the black hole entropy.

4.3.2 Introduction

The present state of affairs in black hole physics is somewhat paradoxical. On one side, it is widely believed that the final state of the black hole evaporation process is a pure state, while on the other side, the standard Hawking's model of evaporation does not account for the purification mechanism. Obviously the missing ingredient is a microscopic quantum model of the black hole beyond its pure geometrical definition.

In the present paper, we shall focus on a specific microscopic description, put forward in [9, 11–13, 16] (for different aspects of this portrait, see [184–188]). In this picture, black holes of arbitrarily large size R are treated as self-sustained bound states of a large number of long wavelength ($\sim R$) gravitons. From the quantum physics point of view, such a bound-state represents a Bose-Einstein condensate stuck at the critical point of a quantum phase transition. This quantum criticality is the key to the understanding of the mysterious properties of black holes that emerge in the naive semi-classical treatment.

In this respect, our approach sharply differs from previous attempts, such as D-brane models for extremal black holes [6], models based on Matrix theory [189–191] and fuzzballs [192]. These approaches heavily rely on a particular UV-completion of gravity at short distances, such as string or Planck (L_p) length-scales. Our key postulate is fundamentally different. We state that physics of macroscopic black holes of size $R \gg L_P$, must be largely insensitive to the properties of UV-completion at Planck-distances and must be governed solely by the quantum physics of long wavelength gravitons with their quantum interaction strength being fully determined by the graviton-graviton interaction vertices of Einstein theory. All the seemingly-mysterious properties of the black holes must originate from *collective quantum* phenomena of these constituent soft gravitons. To put it shortly, in our picture large black holes are not governed by UV-physics, but rather by the quantum collective effects of IR-physics.

These collective effects render the entire macroscopic system extremely sensitive to quantum effects. A fundamental aspect is the appearance of large number of almost gapless collective modes (Bogolyubov modes), which can be thought of as the quantum holographic degrees of freedom. They are responsible for the instability of the condensate, for its quantum depletion as well as for a large (near)degeneracy of the quantum states. These phenomena provide the underlying quantum-mechanical dynamics for black hole evaporation, entropy and holography.

An accompanying property of the quantum phase transition is a very efficient generation of entanglement. Sharp rise of one-particle ground-state entanglement was already confirmed by numerical studies of a prototype model [16].

In this paper, we shall discuss how the instability of the BEC is the key for understanding the efficient generation of entanglement and information-scrambling by a black hole in a logarithmic time,

$$t_{\text{scrambling}}/R \propto \log N\,. \tag{4.67}$$

Noticing that in our treatment N measures the number of constituents, this result is in full agreement with the semi-classical prediction originally made in [146, 147].

Let us briefly review some of the key ingredients of the black hole quantum portrait. In the picture of [12] we track the formation of a black hole as bringing the graviton condensate to *the critical point of a quantum phase transition*. At this point the BE condensate is nearly self-sustained with mass M and size R related to the total number N of constituents as $M = \sqrt{N}L_P^{-1}$, $R = \sqrt{N}L_P$. However, the condensate is unstable both with respect to collapse as well as to quantum depletion. The two effects balance each other in such a way that although the condensate slowly collapses and loses its gravitons, it stays at the quantum critical point. This process can be parametrized as a self-similar decrease of N,

$$\frac{dN}{dt} = -\frac{1}{\sqrt{N}L_P}\,. \tag{4.68}$$

Note that this instability survives in the semi-classical limit ($L_P \to 0, N \to \infty, \sqrt{N} L_P =$ fixed), which corresponds to the Gross-Pitaevskii limit of the graviton condensate.

One of the most important outputs of the black hole N-quantum portrait is to allow us to identify important quantum corrections that are not resolvable within the standard semi-classical approximation. In the semi-classical picture one works with the notion of classical metric. Irrespectively whether the metric is derived from the loop-corrected effective action, it is an intrinsically classical entity and its quantum constituents are not resolved. The only non-perturbative quantum corrections that one can visualise in this limit for a black hole of action S are of the form $e^{-\frac{S}{\hbar}}$. These sort of corrections take into the account only the total black hole action and are blind to any form of microscopic constituency. Such corrections, for instance, can measure the transition amplitudes between black hole and thermal topologies [193, 194].

On the other hand there exist more important quantum corrections that scale as $\hbar/S$, but they are unaccountable in the semi-classical treatment. The key problem lies in unveiling their *microscopic meaning* as well as in understanding under what conditions these quantum corrections can effectively lead to order-one effects for macroscopic black holes. In the quantum N-portrait these corrections naturally appear as $1/N$ corrections, since the occupation number of gravitons measures the black hole action (as well as the entropy),

$$N = \frac{S}{\hbar} \,. \tag{4.69}$$

Thus, the quantity $1/N$ is a measure of quantum effects that are much more important then the e^{-N}-type effects captured by the semi-classical analysis. In particular, it was shown that $1/N$-corrections account for the deviations from thermality of black hole radiation [9] as well as for the quantum hair of black holes [11]. Existence of these corrections was also confirmed for the string holes [195].[6] These $1/N$-corrections are the key for abolishing the black hole "information paradox", since over the black hole half-lifetime they give order-one effect for arbitrarily-large black holes $N \gg 1$ [13].

A Bose-Einstein condensate represents a very natural setup for identifying the physical meaning of $1/N$-corrections. In a nutshell, for BE condensates

[6] The similarly large corrections are also indicated in a different treatment in which one prescribes a wave-function to the horizon [196–198], This approach differs from ours since the metric is still treated semi-classically and its quantum constituents are not resolved. Nevertheless the largeness of the corrections is in a qualitative agreement.

the small quantum deviations from the mean field Gross-Pitaevskii (GP) description are $1/N$-corrections, with $1/N$ replacing the role of the Planck constant $\hbar$. Moreover, as we will discuss in this paper, instabilities of the GP equation can naturally lead to *fast* enhancement of these quantum corrections. More concretely, around instabilities of the GP equation the *quantum break time* (i.e. the time needed to depart significantly ($O(1)$) from the mean field approximation) scales with N as $\log N$. Nicely enough, the BE portrait of black holes implies instabilities of the GP equation. The root of these instabilities lies in the mean-field instability of the condensate at the quantum critical point due to the attractive nature of the interaction. As we will show in this note, the quantum break time for BE condensates fits naturally with the notion of scrambling time for black holes.

4.3.3 Scrambling and Quantum Break Time

The notion of black holes as scramblers was first introduced in [147], where it was realized that perturbed black holes should thermalize in a time $t \geq R \log S_{BH}$ for S_{BH} the black hole entropy and R the black hole radius. In [146] it was then suggested that black holes may saturate this bound, a property that has become known as fast scrambling. The associated timescale is now known as scrambling time.[7]

The concept of scrambling is intimately related to entanglement of subsystems. Consider a quantum mechanical system whose Hilbert space is a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in a state described by the density matrix ρ. The conventional measure of entanglement between the subsystems is the Von Neumann entropy of the reduced density matrix:

$$S_A = \operatorname*{Tr}_A (\rho_A \log \rho_A) \quad \rho_A = \operatorname*{Tr}_B \rho \tag{4.70}$$

A system is called a scrambler if it dynamically thermalizes in the sense that, if prepared in an atypical state, it evolves towards typicality. That is, even for an initial state that has little or no entanglement between subsystems, the time evolution is such that the reduced density matrices are finally close to thermal density matrices. The scrambling time is simply the characteristic time scale associated to this process. It can be described as the time it takes for a perturbed system, one that is described by a product state, to evolve back into a strongly entangled state. It can also be interpreted as the time

[7] For several attempts to understand the physics of scrambling, see [158, 199–201].

necessary to distribute any information entering the system amongst all its constituents.

The quantum meaning of the scrambling time becomes more transparent if we rewrite it as

$$t_{\text{scrambling}} \sim R \log\left(\frac{S}{\hbar}\right) \tag{4.71}$$

with S now denoting the action of the black hole. This is the typical expression for the *quantum break time* provided the system is near an instability, where quantum break time denotes the timescale for the breakdown of the classical (mean field) description. Hence we will identify as a necessary condition for a system to behave as a fast scrambler to have a quantum break time scaling logarithmically with the number of constituents.

Logarithmic Quantum Break Time

In the context of quantum chaos, it has long been known that under certain conditions, the classical description breaks down much quicker than the naively expected polynomial quantum break time. Specifically, in the vicinity of an instability for the classical description, i.e. positive local Lyapunov exponent λ, the quantum break time usually goes as

$$t_{\text{break}} \sim \lambda^{-1} \log \frac{S}{\hbar} \tag{4.72}$$

This exactly resembles the logarithmic scaling of the scrambling time. In fact, the black hole scrambling time coincides with the typical quantum break time if the microscopic description of the black hole contains an instability characterized by a Lyapunov exponent $\lambda \sim 1/R$. The black hole quantum portrait contains such an instability which survives in the semi-classical limit ($L_P = 0$, $N = \infty$, with $\sqrt{N} L_P$ fixed) and is described by equation (4.68). The characteristic timescale is given by $R = \sqrt{N} L_P$ which classically becomes the black hole radius. Hence we expect the Lyapunov exponent to be set by $1/R$. This is precisely the way we will identify scrambling in the BE portrait of black holes.

For the convenience of the interested reader, in appendix 4.3.7 we reproduce a general argument for logarithmic quantum break time at an instability. In the next section we show specifically for Bose-Einstein condensate systems that they exhibit quantum breaking in the scrambling time. We will also comment on the instability there. In section 4.3.5, we perform a numerical analysis that confirms this reasoning.

Chaos and Thermalization

The relation between scrambling and quantum break time is even stronger if the classical limit of the relevant system not only contains a local instability, but also exhibits classical chaos. For such systems it has been claimed - and checked to some extent - that the time scale of thermalization is of the same order as t_{break} [202]. By taking a pure quantum state it was shown that the time evolution not only stretches and folds the quasi-probability distribution, but also smoothens it out. Of course the quantum state stays pure, but it is thermalized in the sense of being smeared out over the accessible classical phase space volume. This would presumably imply scrambling as defined above. Although, at this point we cannot prove that this is indeed how scrambling actually takes place in the graviton condensates of the BH portrait, we do take it as further evidence that the quantum break time is intimately related with scrambling time.

4.3.4 Quantum Break Time in BE Condensates

Prototype Models

It has been pointed out [9, 12, 13] that many of the seemingly mysterious properties of black holes can be resolved when considering them as Bose-Einstein condensates of long wavelength gravitons that interact with a critical coupling strength. Indeed, it has been realized that a vast amount of those properties can already be explored in much simpler systems. These systems share the crucial property that they contain bifurcation or quantum critical points.

Within this work we will follow that route and further explore models of attractive cold bosons both in one and three spatial dimensions. We will show that they exhibit a logarithmic quantum break time, again intimately related to the existence of instabilities and quantum critical or bifurcation points.

The explicit models under consideration in d+1 dimensions are described by the Hamiltonian

$$H = \int_V d^d x \left(\frac{\hbar^2}{2m} (\nabla \phi^\dagger)(\nabla \phi) - \frac{g}{2} (\phi^\dagger \phi)^2 \right). \tag{4.73}$$

Here, ϕ carries the dimension length$^{-d/2}$, while the coupling constant g carries dimension energy × length^d. The integral is taken over the volume of a d-dimensional torus V.

Expanding ϕ into mean field and quantum fluctuations $\phi = \phi_{\rm mf} + \delta\phi$ and subsequent minimization of the energy functional leads, at zeroth order, to the Gross-Pitaevskii (GP) equation for stationary solutions:

$$i\hbar\partial_t\phi_{\rm mf} = \left(\frac{\hbar^2}{2m}\Delta + g|\phi_{\rm mf}|^2\right)\phi_{\rm mf} = \mu\phi_{\rm mf}. \quad (4.74)$$

The chemical potential μ appears as a Lagrange multiplier that imposes a constraint on the particle number N, $\int_V d^dx\phi^\dagger\phi = N$.

An intuitive understanding of the physics of these Bose-Einstein condensates may be gained by considering the behavior of the energy when rescaling the characteristic size of the condensate R:

$$E \sim \frac{N}{R^2} - gN\frac{N}{R^d}\,, \quad (4.75)$$

where the coefficients of both terms naturally depend on the shape of the condensate. As illustrated in Fig. 4.9, the behavior depends strongly on the dimension under consideration. For $d = 1$, the energy is always bounded from below. The (stable) ground state solution is given by a homogeneous condensate for $gN < 1$ and a localized soliton for $gN > 1$. A quantum phase transition is observed [203] at $gN = 1$. On the other hand, for $d \geq 3$, there is a classically stable homogeneous solution for $gN < 1$, while the condensate is unstable for $gN > 1$.

Quantum Breaking in Bose Condensates

We will now apply the notion of quantum breaking to a Bose-Einstein condensate system of N identical particles. In general, we want to study k-particle subsets (although k particles do not form a proper subspace, this technicality will not disturb us much) and use the conventional k-particle sub-density matrices

$$\rho^{(k)}_{mn} = \mathcal{N}\ \mathrm{Tr}\left[\rho\left(\prod_l (a_l^\dagger)^{m_l}\right)\left(\prod_l a_l^{n_l}\right)\right] \quad (4.76)$$

where m and n label k-particle states, a_l is the annihilation operator for one Boson in the l orbital and n_l is the occupation number in state n of orbital l, which satisfy $\sum n_l = k$. The normalization $\mathcal{N}$ is chosen so that

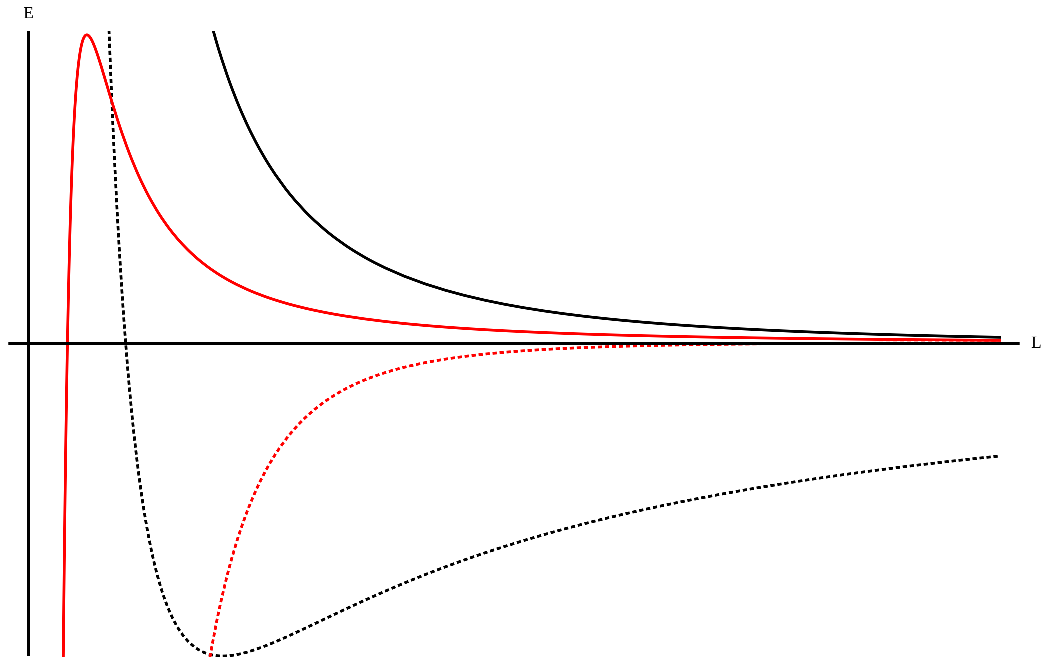

Figure 4.9: Energy as a function of the condensate width for $gN \ll 1$ (solid) and $gN \gg 1$ (dashed) for a condensate in 1 (black) and 3 (red) spatial dimensions.

$\mathrm{Tr}\,\rho^{(1)} = 1$. We would identify a Bose gas as a fast scrambler, *if its time evolution would create a large entropy in each* ρ_k *for* $k \ll N$ *on a timescale that scales logarithmically with* N.

More precisely, we do not expect generic atomic Bose-Einstein condensates - as available in the laboratory - to be scramblers in the sense of the previous paragraph. We do however identify the scrambling timescale to be the relevant thermalization scale for the quantum time evolution of the system in the following restricted way. If we do not insist on the thermalization of all sub-density matrices, but restrict our attention to $\rho^{(1)}$, then the time in which a state with pure $\rho^{(1)}$ develops a large von Neumann entropy in $\rho^{(1)}$ is exactly the quantum break time. This is because a pure $\rho^{(1)}$ represents a condensate-like state with all bosons in one orbital. This state can be completely described by a classical field representing the wave function of the relevant orbital. Therefore as soon as $\rho^{(1)}$ develops a large entropy, the gas can no longer be expected to have a classical description.

The one-particle density matrix may be diagonalized

$$\rho^{(1)} = \sum_i \lambda_i |\Phi_i\rangle\langle\Phi_i| \,. \tag{4.77}$$

with eigenvectors $|\Phi_i\rangle$, λ_i and eigenvalues $\rho^{(1)}(\Psi)$.

A true BE condensate state $|\Psi_{BE}\rangle$ is characterized by possessing one eigenvalue $\lambda_{max} = O(1)$ with the sum of all other eigenvalues suppressed as $1/N$. If a many-body ground state is of this type, we will say that the system is a BE condensate.

In the limit $N \to \infty$ the corresponding reduced one-particle density matrix $\rho^{(1)}$ defines a pure state $|\Phi_{GP}\rangle$ in the one-particle Hilbert space, which is the eigenvector corresponding to the unique maximal eigenvalue. The BE many-body state corresponds to having all the N constituents in the same state $|\Phi_{GP}\rangle$. The wave function $\Phi_{GP}(x,t)$ of this one-particle state is the Gross-Pitaevskii wave function and its evolution is described by the Gross-Pitaevskii equation (4.74).

For finite N and finite gN, the Gross-Pitaevskii equation is never exact. In fact, any exact BE condensate state will, by quantum mechanical time evolution, deplete. This is reflected by the fact that the other eigenvalues of $\rho^{(1)}$ grow. In what follows, we are interested in tracking precisely this growth for some concrete initial conditions, as this allows us to quantify how quickly the Gross-Pitaevskii description breaks down.

Under these conditions the quantum break time t_b appears as the time in which the difference between the exact many-body evolution and the mean field time evolution surpasses a threshold value. Note that the scaling of t_b with N is independent of the choice of threshold value, therefore rendering it effectively arbitrary for our purposes.

Before going into more concrete details let us briefly discuss the physical meaning of this timescale. Let us denote by $\rho^{(1)}(t)$ the exact many-body evolution of the reduced density matrix, whereas by $\rho^{(1)}_{GP}(t)$ we label the mean field GP time evolution for the same initial conditions at $t = 0$. Since $\rho^{(1)}_{GP}(t)$ is a pure state, we can use as a measure of the difference with respect to $\rho^{(1)}(t)$ the entanglement entropy $S(\rho^{(1)}(t))$. We will define t_b as the time needed to reach a certain threshold entropy. This time will generically depend both on the initial condition as well as on the number N of constituents.

The potential growth of the entanglement with time means that the one-particle density matrix is losing *quantum coherence*. On the other hand, and from the point of view of the many body wave function, this loss of quantum coherence is reflected in the form of *quantum depletion*, i.e. in the growth of the number of constituents that are not in the condensate state. Note, that since at the time t_b the number of constituents away from the condensate is significant, this time also sets the limit of applicability of the Bogolyubov approximation.

For regular quantum systems we can expect the time t_b to depend on N as some power [204]. However, as we will show, some attractive BE condensates exhibit a quantum breaking time scaling with N as $t_b \sim \log N$ i.e., they generate entanglement in a time depending on the effective Planck constant as $\log(1/\hbar)$.

In this sense BE condensates – under those conditions – effectively behave as *fast scramblers*. Hence our task will be, on one side to identify the above conditions and on the other side to relate those fast scrambler BE condensates with the sort of BE condensates we have put forward as microscopic portraits of black holes.

4.3.5 Scrambling and Quantumness in BE Condensates

A necessary condition for having a quantum break time t_b scaling like $\log N$ for some initial many body state Ψ_0 is the exponential growth with time of small fluctuations $\delta\Psi(t)$ where $\Psi = \Psi_0 + \delta\Psi$. In linear approximation the equation controlling $\delta\Psi$ is the Bogolyubov-De-Gennes equation. As discussed above, a significant departure from the mean field approximation as well as generation of entanglement for the reduced one particle density matrix requires a growth in time of the depleted i.e of the non-condensed particles. Nicely enough the equations controlling the growth of depleted particles are the same as the ones controlling the small fluctuations of the Gross-Pitaevskii equation and therefore we can translate the problem of finding a time t_b scaling like $\log N$ into the simpler problem of the *stability of the Gross-Pitaevskii equation*. For a detailed discussion and the related technicalities, see [205].

We can understand the short break time more concretely if we think about the difference between the exact evolution and the mean field evolution as the addition of a small perturbation to the exact Hamiltonian. Since an unstable system is exponentially sensitive to perturbations of the Hamiltonian then the time for the evolution of states to differ substantially is very short. The instability is controlled by the Lyapunov exponent λ, while the preexponential factor will depend on the size of the perturbation. The quantum break time is the time when this becomes important, so we can naturally expect it to scale like $t_b \sim \lambda^{-1} \log N$.

Numerical Analysis

Quantum Break Time of One Dimensional Condensates

In this section we will verify the logarithmic quantum break time numerically for the (1+1)-d Bose condensate.

The theory (4.73) in $1+1$ dimensions undergoes a quantum phase transition for $gN = 1$. When surpassing the critical coupling, the homogeneous state becomes dynamically unstable.

As we expect the black hole to lie at such a point of instability, due to its collapse going in hand with Hawking evaporation, we will model the behavior of the black hole by considering the homogeneous state past the point of quantum phase transition.

We consider $gN > 1$ and prepare as initial condition a perfect condensate in the homogeneous one-particle orbital . The linear stability analysis (simply expanding the classical Hamiltonian (4.73) around a the background) at once indicates an instability: the energy of the first Bogolyubov mode becomes imaginary; its magnitude corresponds to λ, the Lyapunov coefficient for the unstable direction.

Note that this setup may be interpreted as preparing the system in a supercooled phase. Or as the result of a quench across the phase transition, suddenly increasing the coupling from $gN = 0$ to $gN > 1$ [206, 207]. The system finds itself in a classically instable configuration and quantum fluctuations ensure that a rapid depletion of the condensate and simultaneous entanglement generation take place.

Would we evolve the same initial state for $gN < 1$, very little entanglement would be generated (because it overlaps with very few energy eigenstates there) and the relevant timescale of evolution would not scale logarithmically in N (as can be checked by studying the spectrum).

Decomposition of ϕ in terms of annihilation and creation operators

$$\hat{\phi} = \frac{1}{\sqrt{L_b}} \sum_{k=-\infty}^{\infty} \hat{a}_k e^{ikx} , \tag{4.78}$$

leads to the more convenient form for (4.73)[8]

$$\hat{H} = \sum_{k=-\infty}^{\infty} k^2 \hat{a}_k^\dagger \hat{a}_k - \frac{g}{4} \sum_{k,l,m=-\infty}^{\infty} \hat{a}_k^\dagger \hat{a}_l^\dagger \hat{a}_{m+k} \hat{a}_{l-m} \tag{4.79}$$

Bogolyubov diagonalization around the homogeneous background $\phi_{\text{hom}} = \sqrt{N}$ yields for the energy of the first Bogolyubov mode [13, 16, 203]

$$\epsilon_1 = \sqrt{1 - gN} \tag{4.80}$$

Parametrizing the effective coupling as $gN = 1 + \delta$, we obtain $\epsilon_1 = i\sqrt{\delta}$. Applying the above argument, we therefore expect the system to break from mean field on a timescale $t_{\text{break}} \sim \Im(\epsilon_1)^{-1} \log N \sim \sqrt{\delta}^{-1} \log N$. The argument of the logarithm is proportional to N because the action of the mean field solution scales as $S \sim N$ for fixed gN.

Within this setup, the departure from classical evolution is expected to go in hand with the generation of large entanglement. This allows us to identify the quantum break time directly with the scrambling time.

Since we are interested in finite N effects in a regime where we expect semi-classical methods to fail, we will use a method not relying on any kind of perturbation theory. We will diagonalize the Hamiltonian (4.79) explicitly. Then, in order to time evolve the homogeneous Hartree state

$$|\phi_{\text{hom}}\rangle = (\hat{a}_0^\dagger)^N |0\rangle, \tag{4.81}$$

we will project $|\phi_{\text{hom}}\rangle$ onto energy eigenstates and apply the time evolution operator $U(t) = \exp(iHt)$ on the state. Finally, we project the time evolved state onto a k-particle subspace and compute the von Neumann entropy

$$\begin{aligned} S_1 &= -\operatorname{Tr} \rho_1 \log \rho_1 \\ (\rho_1)_{ij} &= \langle \phi_{\text{hom}} | \hat{a}_i^\dagger \hat{a}_j | \phi_{\text{hom}} \rangle \end{aligned} \tag{4.82}$$

as a function of time.

In order to make this task computationally feasible we will make use of several properties of the system [203]. Since the Hamiltonian is translationally invariant and number conserving we can restrict ourselves to fixed total momentum and fixed total particle number. In our case, only the total momentum zero sector is relevant, since this contains the homogeneous state.

[8] For improved readability, we have now set $\hbar = 2m = V = 1$

Furthermore, from the Bogolyubov analysis we see that the modes with $k > 2$ have a fairly large gap for gN not much bigger than 1. Therefore, we can truncate the momentum modes l we take into account to $l = -1, 0, 1$.

In Fig. 4.10 we plot S_1 as functions of time for different values of N. In order to see the break time, we evaluate the time when S_1 is higher than some threshold value S_{th}. We plot this time as a function of particle number N in Fig. 4.11, where the solid line is the result of fitting a logarithm to the data points. This clearly shows a logarithmic break time.

A clearer understanding for the observed behavior emerges if we look at the density of states. In Fig. 4.12 we show a plot of the density of states in the zero-momentum sector for given energy and coupling. It can be clearly observed that there is a large density of states for low energies near the phase transition, which is due to the light Bogolyubov mode appearing at the quantum critical point. Furthermore, we clearly see a band of a high density of states for large couplings. The state we time-evolve in the numerics overlaps only with the modes in this band. We have checked that the density of states in this band varies logarithmically with N, i.e. the gap Δ between states in this band will typically go as

$$\Delta \sim 1/(\lambda \log N). \tag{4.83}$$

Given that the time scale for the time evolution will be set by this gap we naturally see the logarithmic break time emerging.

Three Dimensional Condensates and Connection with Black Hole

In the previous section, we have studied a Bose condensate in one spatial dimension as a prototype model. In that case it was viable to perform numerical simulations of the quantum time evolution. For an attractive Bose condensate, one dimension is special however insofar as the classical GP system has a well defined lowest energy configuration after the phase transition - the bright soliton. In higher dimensions, however, there is no classical solution in the would-be solitonic phase. Instead when increasing the effective coupling gN past 1, the stable lowest energy solution of the Gross-Pitaevskii equation and another (unstable) solution disappear together in a saddle node bifurcation [208] (see a sketch of the phase diagram in Fig. 4.13[9]). Thus,

[9] This can also be understood intuitively from Fig.4.9 and Eq.(4.75). The two solutions for small gN correspond to the maximum of the energy functional and the infinitely

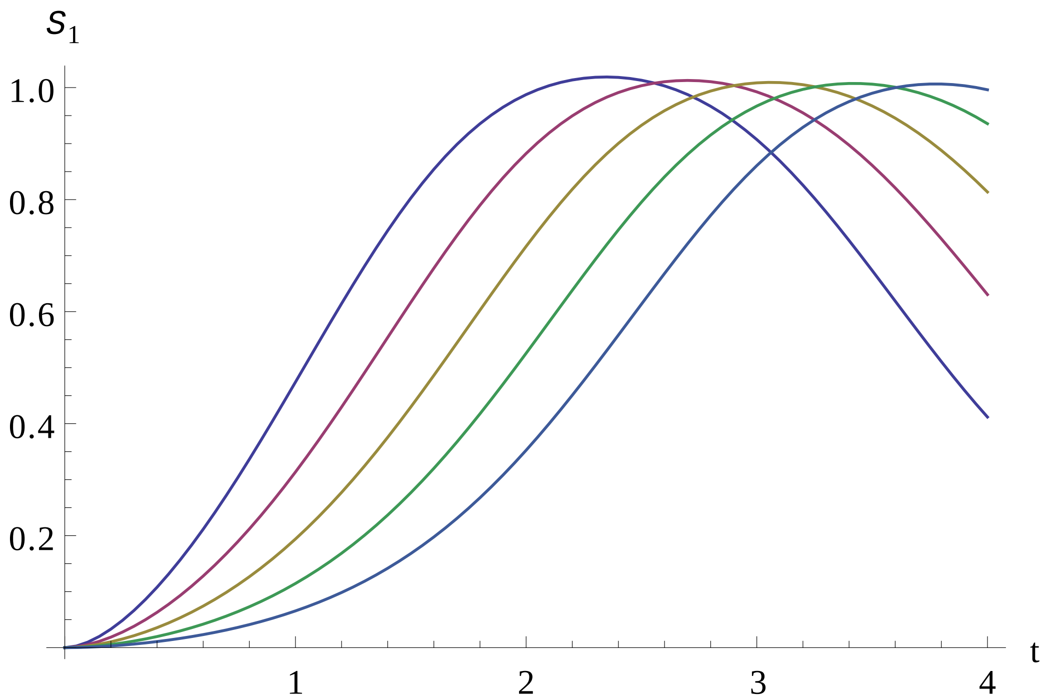

Figure 4.10: One particle entanglement entropy as a function of time for $N = 16, 32, 64, 128$ and 256.

while we willfully prepared an unstable initial state for the (1+1)-d bosons, when a perfectly stable ground state was available, in (3+1)-d it is inevitable to enter the instability when going past the bifurcation point.

It is precisely this instability that we believe to be responsible for the fast scrambling of information in black holes.

There, the relevant coupling controlling the mean field approximation is gN with $g = \frac{L_P^2}{l^2}$ for l the wave length of the constituent gravitons. In the weak coupling regime $gN < 1$ the condensate cannot be self-sustained and we should therefore imagine some external trapping potential that sets the wavelength of the constituent gravitons. The many body wave function is a stretched condensate in the corresponding trap. At the critical point $gN = 1$ the system of gravitons becomes self-sustained in the sense that

stretched condensate. For large gN, no stable points exist. This analysis assumes the presence of a trapping potential. As we will argue below, this is in close analogy to the black hole.

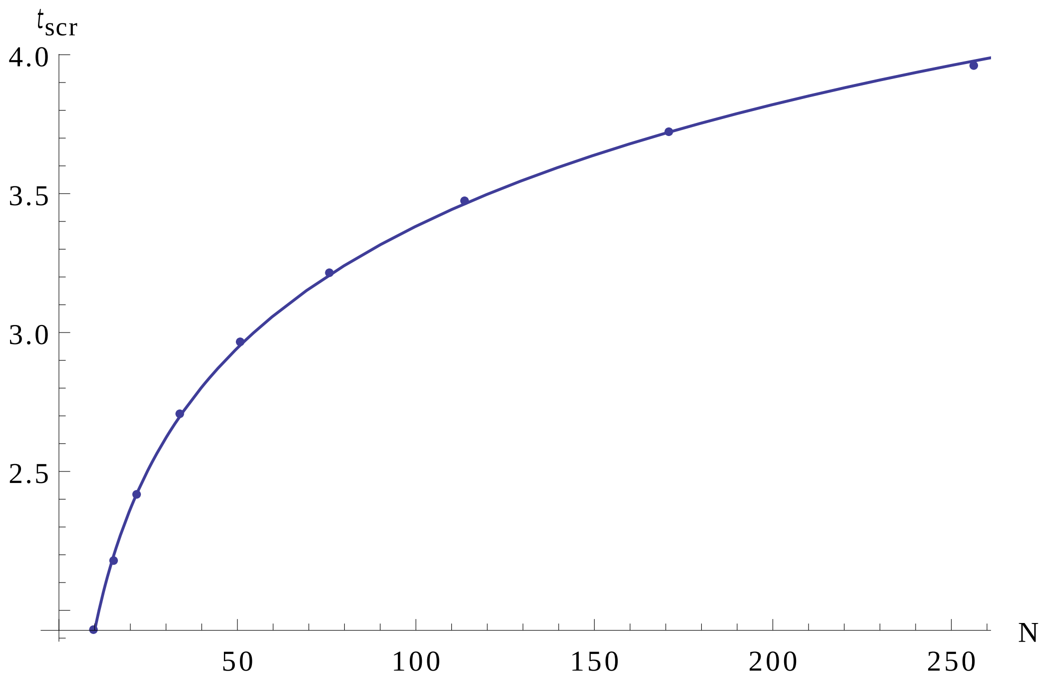

Figure 4.11: Quantum break time as a function of N.

the quantum pressure compensates the gravitational attraction. However, although at this point we can satisfy the virial condition of self-sustainability, the system is not stable in the mean field approximation and will tend to collapse - reducing its size and consequently decreasing the typical wavelength of the constituent gravitons. As we have elaborated, this mean field picture dramatically changes once we take appropriately into account $1/N$ quantum effects. Based on our prototypes, we expect the quantum evolution to break from mean field in a time $O(R \log N)$. This is reflected in the generation of large entanglement entropy for the corresponding one particle density matrix as a function of gN.

The evolution of black holes is different from that of laboratory condensates because of Hawking evaporation. While collapse usually puts a condensate off the critical point, this is prevented by the decrease of the number of gravitons N. As the condition of instability persists along the collapse, we also expect larger-k-density matrices to be efficiently scrambled.

4.3.6 Summary and Outlook

The purpose of this note was to stress that the properties of unstable Bose-Einstein condensates are crucial in understanding the efficient generation of quantum entanglement and scrambling.

The idea of black holes as maximal scramblers is a very interesting hypothesis. Its verification requires a microscopic quantum theory and the goal of this paper is to set some ideas in this direction.

The very conservative assumption of our work lies in modeling black holes as many-body quantum systems governed by weakly-coupled IR gravity. The semi-classical one-particle collective behavior appears as a consequence of the many-body system being in a BE condensate state. Quantum fluctuations relative to this state are measured by $1/N$ with N being the number of graviton constituents (and, equivalently, the BH action in Planck units). Some special features of BHs, as, for instance, fast scrambling, are understood in this frame as the reflection of a logarithmic quantum break time.

These observations provide the clue for solving some recalcitrant BH paradoxes. In particular, the assumption of purity of the final evaporation state seems to lead to strong departures from semi-classicality at least in Page's time [149], meaning that a breakdown of semi-classicality takes place after this time irrespective of the size of the black hole. This is very puzzling, since naively one expects the semi-classical approximation to be valid for large macroscopic black holes. The approach to these sort of puzzles that we can extract from the present work lies in identifying the root of this breakdown of semi-classicality in the existence of a logarithmic quantum break time. Because BHs are unstable BECs, the quantum evolution takes over much sooner then what would be naively expected.

Furthermore, we consider the properties of quantum criticality and quantum instability as crucial for fast scrambling. We take the ensuing logarithmic quantum break time as a very encouraging sign. However, we would refrain from making strong statements about implications of additional black hole properties, such as, for example, their age or the embedding spacetime.[10] To address such questions the prototype model must be refined, which is the subject of future work.

As a marginal comment let us just note that the quantum time coordinate λt, with λ the Lyapunov exponent, is the natural candidate for the Rindler time. This leads to a potential connection between the physics of the time-

[10]We thank the referee for reminding us of these issues.

coordinate inside the black hole and the entanglement flow for the reduced density matrix.

Finally, it would be very interesting to study some of the phenomena discussed in this work, in particular the appearance of logarithmic quantum break time, for realistic Bose-Einstein condensates in the laboratory. This would give an exciting prospect of simulating some aspects of quantum black hole physics in the labs.

4.3.7 appendices

Instability and Quantum Break Time

The question of how long a mean field (i.e. classical) trajectory faithfully reproduces the quantum evolution of a dynamical system has been studied a very long time ago [204]. Only much later however has it been noticed, that under certain circumstances, the quantum evolution can deviate from mean field in sub-polynomial time. Good arguments have been given [209] that where the classical phase space of a system exhibits a dynamical instability, i.e. a Lyapunov exponent $\lambda > 1$, the quantum dynamics will deviate from mean field after a time that goes like

$$t_{\text{break}} = \lambda^{-1} \log(S/\hbar) \tag{4.84}$$

where S is the typical action.

Local Instability Argument

The general argument (following [209]) that leads to the logarithmic break time can be summarized as follows: Assume that the classical phase space of the system contains a region with a local Lyapunov exponent $\lambda > 0$. For the sake of the argument, let us represent every pure quantum state $|\psi\rangle$ as a Glauber Q or Husimi quasi-probability distribution[11] on phase space

$$Q_\psi(\alpha) = \frac{1}{\pi} \left|\langle\alpha|\psi\rangle\right|^2 , \tag{4.85}$$

where the $|\alpha\rangle$ form an overcomplete basis of coherent states. As one would expect for a real probability distribution, the Q distribution moves along with

[11] It is similar to the better known Wigner quasi-probability function, but has some properties that make it favorable for the study of chaotic systems.

the classical trajectories. There are however intrinsically quantum terms that contain additional derivatives and act diffusively on phase space.

Imagine that we initially prepare a close analog to classical state - a coherent state - and localize it in the unstable region of phase space. The Q distribution, initially well localized, will be stretched in the unstable direction. However, because Hamiltonian flows are volume preserving, there must also be a “stable” direction with a local Lyapunov exponent $-\lambda$. The Q distribution is exponentially compressed in the stable direction. When its width gets smaller than a given phase space distance (that involves $\hbar$), the diffusive quantum terms become important. From that point on, the quantum time evolution departs even from the physics of a classical phase space ensemble. The time scale for the departure naturally goes like

$$t_{\text{break}} = \lambda^{-1} \log(1/\hbar). \tag{4.86}$$

It can be argued that the dimensionless ratio in the exponent should be $S/\hbar$ with S being the typical action [204]. The quantum break time t_{break} is also referred to as Ehrenfest time in the literature on quantum chaos.

The quick break time has been explicitly verified numerically, e.g. in tractable two level systems that are well motivated experimentally [210].

4.3.8 Quantum Break Time for a Wave Packet

In order to illustrate the arguments of the previous section, we show the phase space evolution of the simplest possible system with an instability, a nonrelativistic particle of mass m in the potential

$$V(x) = -\alpha x^2 + \beta x^4 \tag{4.87}$$

Around $x = 0$, there is an instability in phase space with positive local Lyapunov exponent $\lambda = \sqrt{2\alpha/m}$. We evolve a minimum uncertainty wave packet centered around $x = 0$, $p = 0$. Three snapshots of the Husimi function at different instances of time are shown in Fig. 4.14, top row. The bottom row shows the classical Liouville time evolution of the same initial functional shape. Evidently, the contraction of the Husimi function in the stable direction is limited compared to the classical evolution. As explained above, this is due to quantum diffusive terms and generically limits the applicability of the classical approximation to the quantum break time.

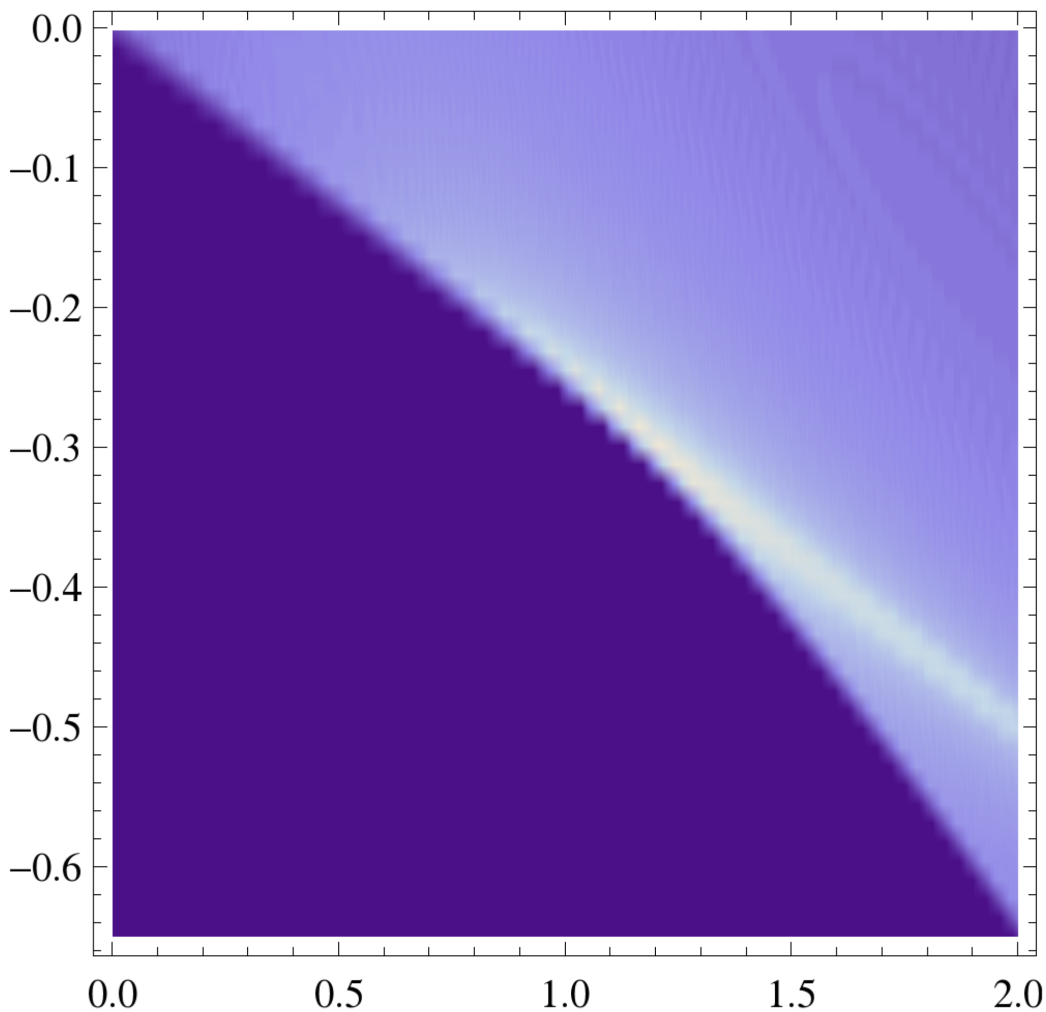

Figure 4.12: Density of states as a function of gN and E/N for N=1500.

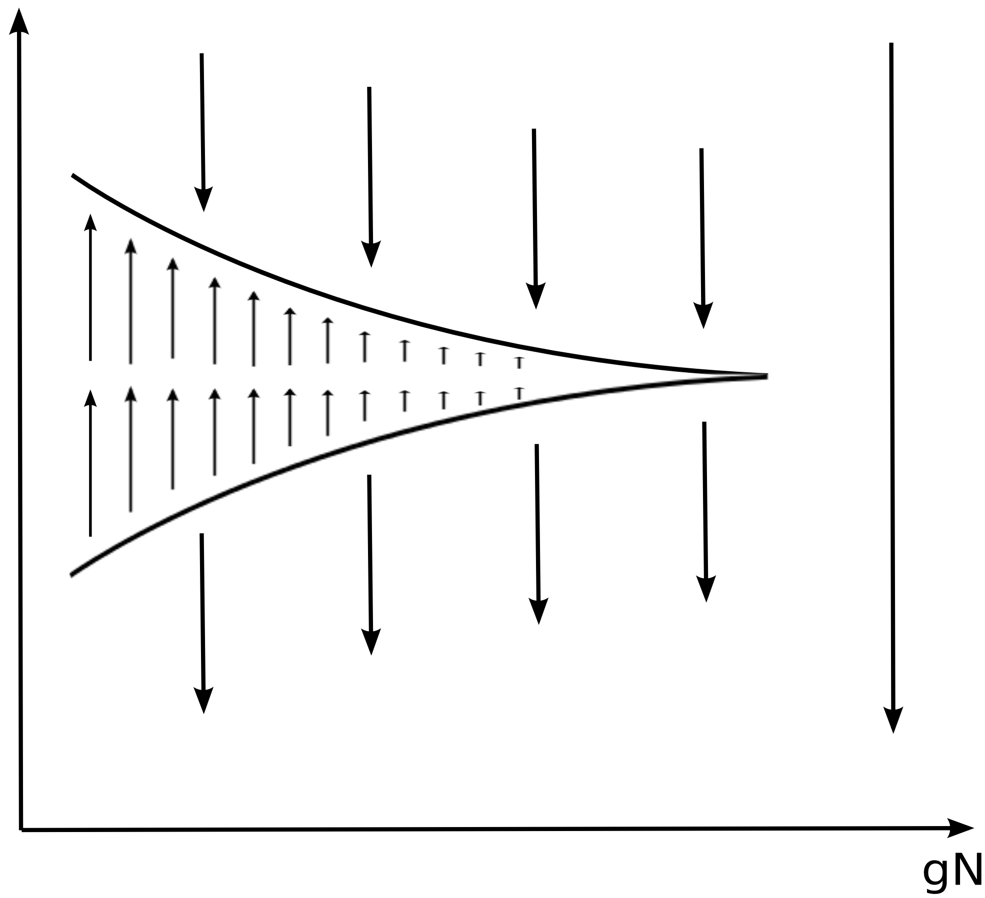

Figure 4.13: Phase diagram for the three-dimensional condensate. For small gN two solutions exist; one is stable while the other one is unstable. At the critical point, both solutions disappear.

Figure 4.14: Phase space (x,p) evolution of the quantum mechanical Husimi function starting from an instability (top row). Classical Liouville evolution of the same initial function. (bottom row)

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