On the stability of a de Sitter universe with self-interacting massive particles

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Zusammenfassung

In dieser Arbeit untersuche ich den Einfluss massiver Quantenfelder auf einen reinen de Sitter Hintergrund. Nach einer kurzen Zusammenfassung der neuesten Entwicklungen zu diesem Thema gebe ich eine Einführung in die klassische Geometrie von de Sitter Räumen. Darin behandle ich die physikalischen Eigenschaften und die verschiedenen Koordinatensysteme, die unterschiedliche Teile des de Sitter Raumes bedecken. Im anschließenden Kapitel wiederhole ich die Quantenfeldtheorie freier Skalarfelder auf gekrümmten Hintergründen im Allgemeinen und auf de Sitter im Speziellen. Hier gebe ich die Lösungen für die Modenfunktionen in geschlossenen und flachen Koordinaten an und diskutiere das Problem der richtigen Wahl des Vakuums auch im Hinblick auf die Eigenschaften der zugehörigen Green Funktionen. Da sich der Hintergrund für die Quantenfeldtheorie auf de Sitter mit der Zeitentwicklung ändert, verwende ich den in/in (Keldysh) Formalismus zur Berechnung von Observablen. Ich fasse den Formalismus zusammen und erläutere die für Rechnungen benötigten Methoden. Die Einführung des Wechselwirkungspotentials und der Feynmanregeln für Wechselwirkungsdiagramme bilden schliesslich den Abschluss des einleitenden Teils.

Mit Hilfe des effektiven Potentials für das reskalierte Skalarfeld zeige ich, dass jede Theorie mit ungeraden Wechselwirkungspotentialen Probleme mit der Stabilität des freien Vakuums aufweist, falls der Skalenfaktor in der Vergangenheit verschwindet. Dies ist auch ein Argument, auf de Sitter die globalen Koordinaten anstelle der flachen zu verwenden, da sie im Gegensatz zu diesen den ganzen Raum bedecken und der Skalenfaktor nur einen nicht verschwindenden Minimalwert annimmt. Ich beweise weiterhin, dass aus der Betrachtung der Vakuumpersistenz kein Einwand gegen Wechselwirkungen auf de Sitter folgt, da die resultierende Entwicklung immer unitär ist, falls die Kopplung klein genug gewählt wird. Für die Schleifenkorrekturen zum Keldyshpropagator in globalen Koordinaten ergeben meine Berechnungen keine problematischen Divergenzen. Insbesondere finde ich keine Divergenz, die es verbietet, den adiabatischen Limes in Berechnung zu nehmen, was den Ergebnissen von Polyakov und Krotov widerspricht. Zusammenfassend ist meine Schlussfolgerung, dass die wechselwirkenden Quantenfelder zu keinen offensichtlichen Instabilitäten des de Sitter Hintergrundes führen.

Abstract

In this work I discuss the influence of interacting massive quantum fields on a pure de Sitter background. After a short review of recent developments on the topic, I give an introduction to the classical geometry of de Sitter. I discuss the physical properties and the different coordinate charts covering parts of de Sitter. In the next chapter I recapitulate free quantum fields on curved backgrounds in general and on de Sitter in particular. Subsequently I give the solutions to the mode equation for closed and flat coordinates and cover the problem of the correct choice of the vacuum also with respect to the properties of the corresponding Green function. As the background for quantum field theory on de Sitter is changing with time I use the in/in (Keldysh) formalism to calculate observables. I review this formalism and give the mathematical tools to perform calculations. The introduction of the interaction potential and the Feynman rules for interaction diagrams concludes the introductory part.

Using the effective potential for the rescaled field, I show that any theory with odd interaction potential has problems with the stability of the free vacuum on a dynamic background if the scale factor vanishes in the early past. In particular this is one argument for using the global closed coordinate chart on de Sitter instead of the flat one covering only half of de Sitter. I also prove that from the vacuum persistence there is no objection to taking interactions on de Sitter, i.e. the resulting evolution is unitary for small enough coupling. For the loop corrections to the Keldysh propagator in global coordinates I calculate no problematic divergence, especially I find no divergence prohibiting the adiabatic limit in calculations in contrast to Polyakov and Krotov's result. Summarizing, this shows that there is no obvious instability of the de Sitter background inferred by interacting quantum fields.

Contents

1. Introduction, motivation and current state

De Sitter space is one of the earliest solutions to Einstein's equations of motion for gravity. Although its name originates from Willem de Sitter, the basic concept was introduced by Albert Einstein in 1917. The equations of motion for the scale factor of the universe derived from his theory of gravity always required the size of our universe to be dynamically changing. Einstein was very uncomfortable with the notion of a dynamic universe as at that time no evidence indicated an expanding or contracting universe. He therefore added a constant energy term to the equations of motion to allow for a static universe. This constant had the desired effect, but allowed a static universe only for a spatially closed universe and a value of the constant which is very fine-tuned to the matter content of the universe. Willem de Sitter discussed the general solution [1, 2] which leads to an exponentially contracting or expanding universe, depending on the dominating part of the energy contribution and the initial condition. Nowadays de Sitter space refers to a space-time in which the major energy contribution comes from the cosmological constant and other matter, if present, is treated as a perturbation. However, after the discovery of the expansion of the universe by Hubble in 1929 [3], the cosmological constant was less popular, as the expansion of the universe could be explained by the known matter sources and there was no need to introduce an artificial new form of energy. This changed with the measurement of the cosmic microwave radiation background by Penzias and Wilson in 1965 [4]. To explain this uniform background radiation and other cosmological problems an early phase of accelerated expansion was proposed, which is strongly supported by the recent observations of Planck [5, 6] and BICEP2 [7, 8]. Although the exact mechanism is still not fully determined today, this early quasi de Sitter stage renewed interest in de Sitter space-time. Futhermore the discovery that today's expansion of the universe is accelerating in 1998 by Riess et al. and Perlmutter et al. [9, 10] increased the interest in the study of de Sitter space. Today it is one of the most studied space-times aside from Minkowski space.

The first discussions of quantum fields on de Sitter space are by Chernikov and Tagirov in 1968 [11, 12]. Further discussion on the possible vacua, the Green functions and the renormalization of the energy momentum tensor and the effective action followed [13–18]. In 1984 Ford [19] discussed toy models of massless interacting scalar fields and showed them to lead to an energy momentum tensor that could decrease the cosmological constant. The evolution of a universe with a decreasing cosmological constant and other matter was considered by Freese et al. shortly thereafter [20]. Tsamis and Woodard showed by pseudo-quantizing gravity on a de Sitter background that the quantum gravitational backreaction can lead to a decrease of the cosmological constant as well [21, 22]. The equations of motion for the mode function allows for a plethora of solutions. From any set of mode functions corresponding to a definite vacuum, another set can be constructed by a Bogolyubov transformation. Unlike in Minkowski space, the condition of minimisation of the Hamiltonian does not work in de Sitter globally, so there is no unique guideline for the choice of the correct set of mode functions for the vacuum which led to an extended discussion on the properties of the different vacua which includes particle production [16, 17], thermal properties [18, 23], application to transplanckian physics [24–27] and analytic properties in perturbation theory [28–32]. Most analysis concentrated on massless or very light scalar fields as they mimic the behaviour of gravitons in quantized gravity [17–19, 33–51]. In 2010 Burgess et al. [52] showed that for very light or massless self-interacting fields on a de Sitter background, standard perturbation theory breaks down for small enough masses. They argue that the semiclassical treatment is therefore no longer applicable and one has to resort to non-perturbative methods.

As the de Sitter background does not have a global timelike Killing vector, energy conservation no longer holds. This leads to the possibility of particle decays that are forbidden in Minkowski space-time. E.g. the decay of one particle into multiple particles of the same type is allowed if self-interactions are switched on. The exact amplitude for these processes was calculated by Bros et al. [53–57]. In 2007 Polyakov [30] conjectured that de Sitter space is intrinsically unstable when the self interaction of massive scalar fields is switched on and this leads to a screening of the cosmological constant, similar to the screening of the electric charge. He also argued for a composition principle of the propagator which is satisfied in Minkowski space, but not for the propagator resulting from the standard euclidean vacuum in de Sitter space. This would support a different choice of vacuum. In the following publication [31] he calculated vacuum and particle stability in the in/out formalism and found an averaged decay rate after the use of Fermi's golden rule, which leads to a catastrophic increase of the particle density. However, on a changing background the application of Fermi's rule is not sensible [53] and one has to find the dynamic Boltzmann equations for the particle number densities. This has been attempted by several authors [39, 58–69]. Together with Krotov [70], Polyakov showed that a cubic interaction on a contracting de Sitter space leads to a divergency depending on the time the interaction is active, i. e. the adiabatic limit of sending the time of switching on to past infinity, which is a crucial element of perturbation theory, is not valid in these coordinates. For expanding de Sitter, Jatkar et al. [71] showed that the propagators get loop corrections depending on the logarithm of the geodesic distance of the two events but this contribution can be resumed to a shift of the mass and thus not leading to a problem of perturbation theory. Several other authors also investigated this possible decay of de Sitter space e.g. by studying the corrections to the propagators and the kinetic equations for the particle occupation number [32, 44, 58–66, 72–91].

These references should give a glimpse that the study of de Sitter space and especially quantum fields on it is a very actively discussed subject. The importance of de Sitter geometry for the early and late evolution of our universe as emphasized in recent observations Planck and BICEP [5–8] is my motivation to study it in more detail. To investigate the interactions of different particles, it is important to know whether perturbation theory as we are familiar with from Minkowski space can be applied in de Sitter space. I discuss the influence of a self interaction potential of a massive scalar field on a pure de Sitter background up to one loop level. The coupling to gravity is supposed to be minimal, but on a constant de Sitter background a nonminimal coupling will only lead to a shift of the mass. I consider only fields in the principle series, i.e. the effective mass is larger than a threshold depending on the Hubble scale. Particles from the principle series exhibit a behaviour distinctly different from light particles. In contrast to Minkowski space, loop corrections on de Sitter are much more cumbersome due to the more complicated form of the mode functions. Therefore I follow [70] and use a toy model of cubic interaction, which should nevertheless give us insights into the relevant physics. Although Hamiltonians with cubic interactions are unbounded, small perturbations around the free vacuum are stable. By adding a quartic interaction the potential can then be made bounded again but the resulting loop diagrams are much more complicated to evaluate than in the cubic case. As I expect particle excitations I use the closed coordinate chart of de Sitter which covers the complete de Sitter space in contrast to the flat coordinates which cover only half of it. This is to make sure that no particles can escape or enter through the border of the geodesically incomplete coordinate system. Moreover the effective potential indicates that for cubic interaction in the expanding patch of de Sitter the free vacuum is destabilized if the initial hypersurface is sent to past infinity. As for short distances the behaviour of the mode functions is similar to Minkowski space, I investigate corrections to propagators with small external momenta which is another argument for a coordinate system that covers the whole Cauchy surface. In the global coordinates, I show that contrary to [70] the adiabatic limit is possible and there is no objection to letting the interaction act over an infinitely long time. In conclusion I do not find any objection to using perturbation theory for massive particles on de Sitter from this point of view. Nevertheless, if divergencies should appear in future analysis, they first of all signal a breakdown of perturbation theory but not an instability of de Sitter itself.

2. Geometry of de Sitter space

2.1. De Sitter space

In this chapter I review the basic properties of D-dimensional de Sitter space-time (dS_D) based on [54,92–95]. De Sitter space is a maximally symmetric space-time of constant curvature discovered by Einstein and discussed by Willem de Sitter and independently by Levi-Civita in 1917 [1,2,96]. D dimensional de Sitter space can be expressed as a hyperboloid of constant curvature embedded in a $D+1$ dimensional Minkowski space

$$\eta_{AB} X^A X^B = -H^{-2}. \tag{2.1}$$

I use the metric signature $(+,-,-,\dots)$ throughout this thesis. It is useful to consider dS_D as the analytic continuation of the euclidean sphere S^D of radius H^{-1} embedded in $\mathbb{R}^{D+1}$ by

$$\delta_{ij} E^i E^j = H^{-2}. \tag{2.2}$$

The continuation is achieved by the Wick rotation $E^0 \to iX^0$ which transforms (2.2) into (2.1), (see figure 2.1). Wick rotation is a useful tool to easily derive certain properties of de Sitter space from those of the sphere. However some care has to be taken when using this in the context of quantum field theory on de Sitter, as the analytic continuation may be hindered by cuts in the complex time plane. From the

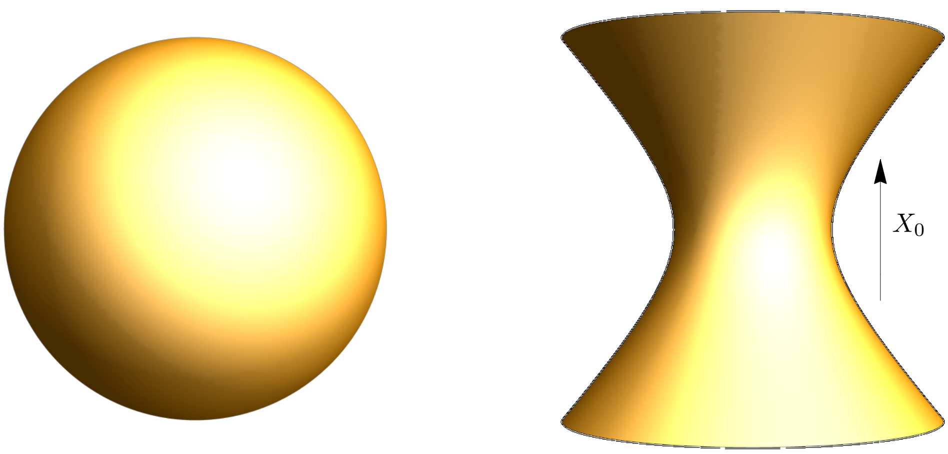


Figure 2.1.: Embedding of dS_2 in $\mathbb{R}^3$ in contrast to the two-dimensional sphere.

definition (2.1) it is obvious that dS is invariant under rotations in the embedding Minkowski spacetime with symmetry group $SO(1,D)$. Therefore it has the same number of symmetry generators generators as a Minkowski spacetime of the same dimension and is maximally symmetric [92]. For maximally symmetric spaces the Riemann tensor is given by

$$R_{\alpha\beta\mu\nu} = \frac{1}{D(D-1)} R \left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \right).$$

From this the Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is given by

$$G_{\alpha\beta} = \frac{2-D}{2D} R g_{\alpha\beta}.$$

The Bianchi identity (or conservation of the energy momentum tensor) therefore requires that R is constant. This space is a solution to the Einstein equations for a cosmological constant with value $\Lambda = \frac{D-2}{2D} R$ or a general matter distribution with $p = -\epsilon$. It is related to the radius of de Sitter space by $R = D(D-1)H^2$.

Another useful property is the antipodal transformation sending a point to its “inverse”. In the embedding space the transformation rule is

$$x \to \bar{x} : X^A(x) \to X^A(\bar{x}) = -X^A(x),$$

i.ei just the reflection through the origin. The antipodal point has some special properties which will be important later.

2.2. Geodesic distance

In order to determine the geodesic distance between two points we first have to solve the geodesic equation and then calculate the length of the curve between two points. However, for de Sitter we can make use of the analytic continuation to the sphere. For two points on the euclidean sphere the geodesic joining the two points is the segment of the great circle through these two points [93, 94]. The geodesic distance d between these two points is then proportional to the angle between the two points.

$$d(E,F) = H^{-1} \theta(E,F), \tag{2.3}$$

where the angle θ is defined by

$$\delta_{ij} E^i F^j = H^{-2} \cos\theta. \tag{2.4}$$

Instead of the geodesic distance, it is often more convenient to use directly the quantity

$$P(E,F) = H^2 \delta_{ij} E^i F^j. \tag{2.5}$$

On dS it is not obvious but a similar relation holds here [93, 94]. The geodesic between two points is the intersection of a plane through the two points and the origin with the de Sitter hyperboloid. Some geodesics are shown in figure 2.2. We can relate the geodesic distance to the quantity Z defined as

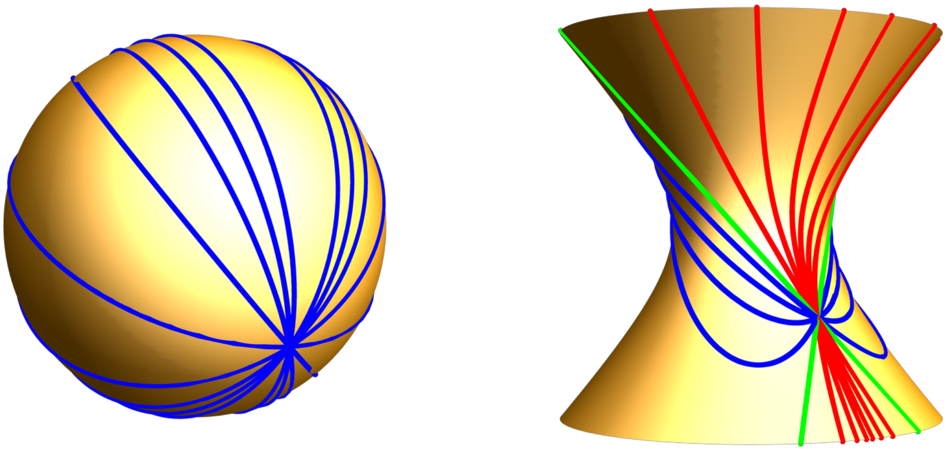


Figure 2.2.: Geodesics for the sphere S^2 and dS_2. Time-like geodesics are red, space-like blue and the Null geodesic is marked in green.

$$Z(X,Y) = -H^2 \eta_{AB} X^A(x) X^B(y). \tag{2.6}$$

We can formally define the geodesic distance as $d(X,Y) = H^{-1} \arccos Z(X,Y)$. The type of separation can be inferred from Z. If X and Y are time-like separated we have $Z(X,Y) > 1$ and the geodesic distance is imaginary (similar to Minkowski space). Null separation corresponds to $Z(X,Y) = 1$ and for space-like separation we have $Z(X,Y) < 1$ (cf. [17]). For $Z(X,Y) < -1$ no geodesic exists even though the space-time is geodesically complete [28]. The points with $Z(X,Y) = -1$ correspond to geodesics from a point to its antipodal point $Y = \bar{X}$ for which the lightcones cross only in the asymptotic future (or past). There is no geodesic from X to any point that lies in the future or past lightcone of $\bar{X}$.

As it is impossible in this form to determine whether one point lies to the future or the past of another, it is convenient to modify the geodesic distance via the following prescription [17]

$$\tilde{Z}(X,Y) = -H^2 \eta_{AB} X^A(x) X^B(y) \begin{cases} +i\epsilon & \text{for x in the future lightcone of y} \\ -i\epsilon & \text{for x in the past lightcone of y} \end{cases} \tag{2.7}$$

to recover the same ϵ prescription as in Minkowski space.

2.3. Coordinate systems and conformal diagrams

There are several useful coordinate systems covering different parts of de Sitter space. They arise from different spatial foliations of de Sitter space. Some examples are given in figure 2.3.

I now give a summary of the most commonly used ones, cf. [93, 97].

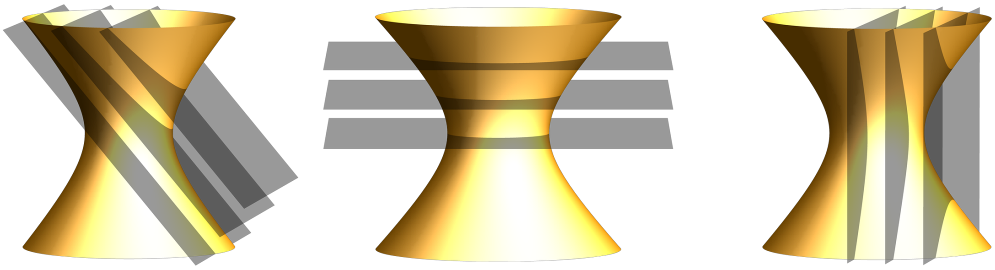


Figure 2.3.: Slicing of dS into flat, closed and open spatial sections.

2.3.1. Global coordinates

The global coordinates are defined by

$$X^0 = \frac{1}{H}\sinh(Ht),$$
$$X^i = \frac{1}{H}\cosh(Ht)n^i, \tag{2.8}$$

with $\vec{n} \in S^{D-1}$ and $t \in (-\infty, \infty)$. The spatial sections for constant t are spheres, so these coordinates are often called closed coordinates. The coordinates on the sphere S^{D-1} are given by

$$n^1 = \cos\theta_1,$$
$$n^2 = \sin\theta_1 \cos\theta_2,$$
$$\dots$$
$$n^D = \sin\theta_1 \dots \sin\theta_{D-1},$$

and the line element on S^{D-1} is

$$\mathrm{d}\Omega^2_{D-1} = \sum(\mathrm{d}n^i)^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2 + \dots + \sin^2\theta_1 \cdots \sin^2\theta_{D-2}\mathrm{d}\theta^2_{D-1}.$$

The induced line element for dS in these coordinates is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{1}{H^2}\cosh^2(Ht)\mathrm{d}\Omega^2_{D-1}.$$

This is the metric of a FRLW universe with closed spatial sections and $a(t) = \cosh(Ht)$. They cover the whole hyperboloid of de Sitter space, cf. figure 2.4. Introducing conformal time η in the global coordinates via $\mathrm{d}\eta = H\frac{\mathrm{d}t}{a(t)}$, i.e. $\cosh(Ht) = \frac{1}{\sin\eta}$, we get the line element

$$\mathrm{d}s^2 = \frac{1}{H^2\sin^2\eta}\left(\mathrm{d}\eta^2 - \mathrm{d}\Omega^2_{D-1}\right) \tag{2.9}$$

with $\eta \in [0, \pi]$. This metric is conformal to the metric with line element

$$\mathrm{d}s^2 = \mathrm{d}\eta^2 - \mathrm{d}\Omega^2_{D-1}. \tag{2.10}$$

The conformal factor $H^2\sin^2\eta$ does not change the causal structure, i.e. if a geodesic is Null or time-/space-like in the metric (2.9) it will be so in the conformally scaled metric (2.10), as well. To analyse the

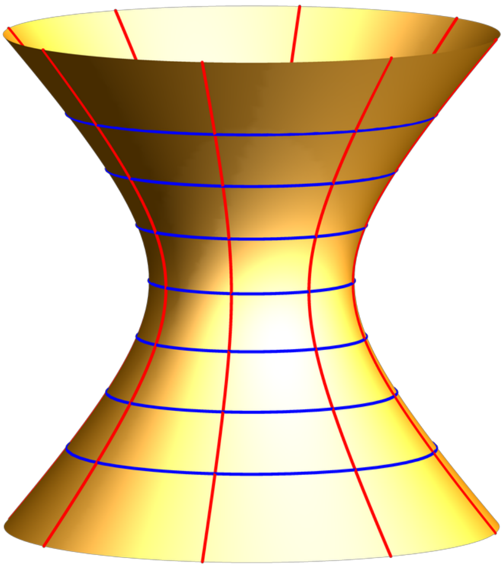


Figure 2.4.: Covering of dS_2 by global coordinates. Blue lines are surfaces of constant time t. Red lines show the evolution of fixed spatial points with time.

causal structure of de Sitter space, we can therefore draw the conformal diagram (Penrose diagram) based on (2.10) as it covers the whole space. It is shown in figure 2.5. I plot only the first spatial angle and every line of constant η corresponds to a sphere S^{D-1}. Each point in the diagram therefore corresponds to a sphere S^{D-2}, except left and right border which correspond to the north/ south pole respectively. Lightrays propagate at 45° as is required for conformal diagrams. Apparently, no observer can access the whole space-time. By using the symmetry rotations on the sphere we can always orient our coordinate system such that we rest at the south pole. A signal from an observer at the south pole will reach the north pole only at future infinity if the signal is emitted at past infinity. This visualizes that the lightcone

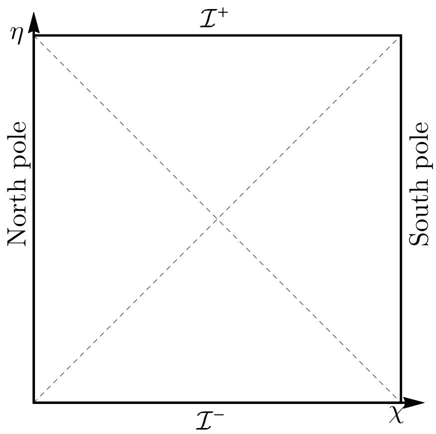


Figure 2.5.: Conformal diagram for global coordinates. $\mathcal{I}^+$ and $\mathcal{I}^-$ correspond to future and past infinity. The dashed lines correspond to the past and future horizon of observers on the north and south pole

of a point and a lightcone from its antipodal point only cross in the asymptotic future or past. The total space this observer is able to send signals to is marked by $\mathcal{O}^+$ in figure 2.6. Correspondingly, all signals he can receive emanate from the region marked by $\mathcal{O}^-$ in figure 2.7. This is a distinct difference to Minkowski space, where an observer can access the whole universe if she waits long enough.

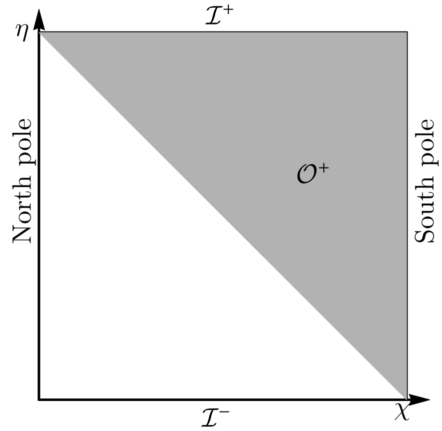


Figure 2.6.: Conformal diagram for global coordinates with region $\mathcal{O}^+$

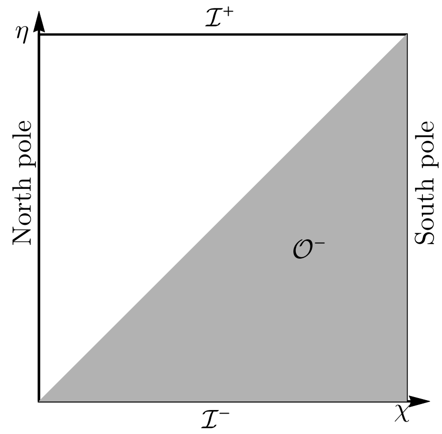


Figure 2.7.: Conformal diagram for global coordinates with region $\mathcal{O}^-$

In this coordinate system the geodesic distance (2.6) is given by

$$Z(x,y) = \frac{1}{\sin\eta_1 \sin\eta_2}\left(1 - \cos\eta_1 \cos\eta_2 - \frac{1}{2}|\vec{n}_1 - \vec{n}_2|^2\right).$$

2.3.2. Flat coordinates

The flat slicing of figure 2.3 is obtained by the following coordinate chart

$$\begin{aligned} X^0 &= \frac{1}{H}\sinh(Ht) + \frac{1}{2}He^{Ht}|\vec{x}|^2, \\ X^D &= \frac{1}{H}\cosh Ht - \frac{1}{2}He^{Ht}|\vec{x}|^2, \\ X^i &= e^{Ht}x^i, \qquad i = 1, \dots, D-1, \end{aligned} \tag{2.11}$$

with the coordinate ranges $(t, \vec{x}) \in \mathbb{R}^D$. These coordinates cover half of deSitter space, $(X^0 + X^D > 0)$ [97], cf. figure 2.8. They are also referred to as steady state coordinates or Poincare patch.

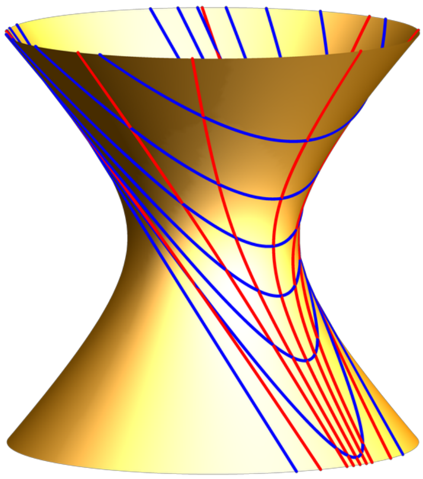


Figure 2.8.: Covering of dS_2 by flat coordinates. Blue lines are surfaces of constant time t. Red lines show the evolution of fixed spatial points with time.

The induced line element is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2Ht}\mathrm{d}\vec{x}^2.$$

This is the metric for a Friedmann universe with flat spatial sections and a scale factor $a(t) = \mathrm{e}^{Ht}$. There are two branches corresponding to contracting or expanding de Sitter, depending on the sign of H. Both together cover the whole de Sitter space. This is made more clear in the conformal diagram in these coordinates, see figure 2.9.

The change to conformal time is given by $\eta = -\frac{1}{H}\mathrm{e}^{-Ht}$. For an expanding universe the range of the conformal time is $\eta = -\infty \to 0$, for a contracting universe, $\eta = 0 \to \infty$. The line element in these coordinates

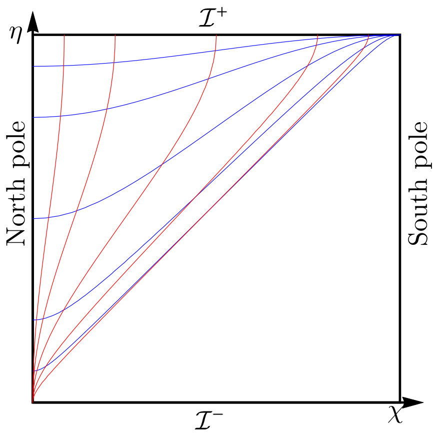


Figure 2.9.: Conformal diagram for flat coordinates. Blue lines are surfaces of constant time t. Red lines show the evolution of fixed spatial points with time.

is

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(\mathrm{d}\eta^2 - \mathrm{d}\vec{x}^2\right).$$

In this coordinate system the geodesic distance (2.6) is given by

$$Z(x,y) = 1 + \frac{1}{2\eta_1\eta_2}\left(\Delta\eta^2 - |\Delta\vec{x}|^2\right).$$

2.3.3. Static coordinates

They are defined by

$$\begin{aligned} X^0 &= \left(\frac{1}{H^2} - r^2\right)^{1/2} \sinh(Ht), \\ X^D &= \left(\frac{1}{H^2} - r^2\right)^{1/2} \cosh(Ht), \\ X^i &= r n^i, \end{aligned} \tag{2.12}$$

where $\vec{n} \in S^{D-2}$ and $r \in [0, \frac{1}{H}]$. They span only a quarter of de Sitter space, $(X^0 + X^D > 0, X^D > X^0)$, figure 2.10, but can be analytically continued to $r > \frac{1}{H}$ to cover half of de Sitter space [97]. The surface $r = \frac{1}{H}$ is a horizon limiting the radius of observation of any observer sitting in $r < \frac{1}{H}$. The line element is

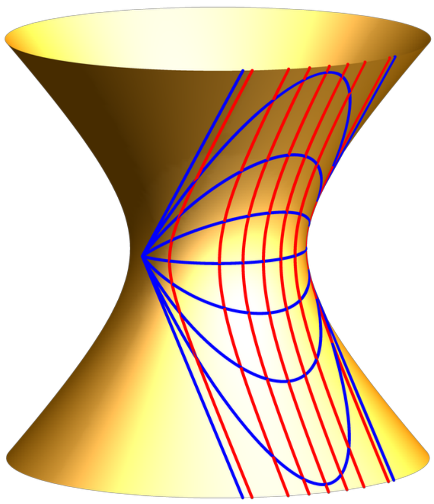


Figure 2.10.: Covering of dS_2 by static coordinates. Blue lines are surfaces of constant time t. Red lines show the evolution of fixed spatial points with time.

given by

$$\mathrm{d}s^2 = (1 - H^2 r^2)\mathrm{d}t^2 - (1 - H^2r^2)^{-1}\mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2_{D-2}.$$

This coordinate systems possesses a timelike Killing vector $\partial/\partial t$. Unfortunately the norm of this vector vanishes at the horizon $r = \frac{1}{H}$, so there is no global Killing vector.

2.3.4. Open coordinates

If one chooses the open slicing of figure 2.3 we get the coordinate chart

$$\begin{aligned} X^0 &= \frac{1}{H}\sinh(Ht)\cosh\xi, \\ X^D &= \frac{1}{H}\cosh(Ht), \\ X^i &= \frac{1}{H}\sinh(Ht)\sinh\xi\; n^i, \end{aligned} \tag{2.13}$$

with $\vec{n} \in S^{D-2}$ and $t, \xi \in \mathbb{R}$. These coordinates cover one quarter of de Sitter space, figure 2.11. The metric

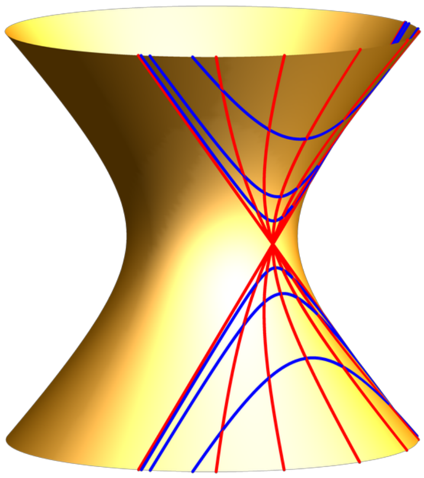


Figure 2.11.: Covering of dS_2 by open coordinates. Blue lines are surfaces of constant time t. Red lines show the evolution of fixed spatial points with time.

becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{1}{H^2}\sinh(Ht)\left(\mathrm{d}\xi^2 + \sinh^2\xi\mathrm{d}\Omega^2_{D-2}\right).$$

2.3.5. Kruskal coordinates

They are given by the coordinates

$$\begin{aligned} X^0 &= \frac{U+V}{1-UV}, \\ X^i &= \frac{1+UV}{1-UV}n^i, \\ X^D &= \frac{U-V}{1-UV}, \end{aligned} \tag{2.14}$$

with $n^i \in S^{D-2}$ and $U, V \in \mathbb{R}$. They cover the whole de Sitter space [93]. The line element is

$$\mathrm{d}s^2 = \frac{1}{(1-UV)^2}\left(4\mathrm{d}U\mathrm{d}V - (1+UV)^2\mathrm{d}\Omega^2_{D-2}\right).$$

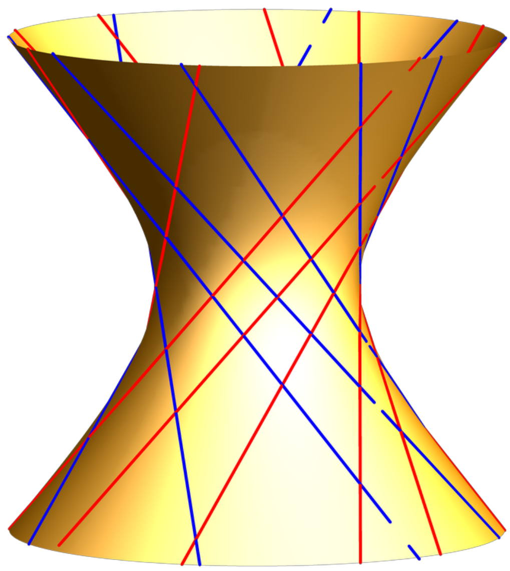


Figure 2.12.: Covering of dS_2 by Kruskal coordinates. Blue lines are surfaces of constant U. Red lines are surfaces of constant V.

3. Quantum field theory on curved space

Quantum field theory on Minkowski space-time is a well studied and understood theory. The transfer of the classical equations of motion to curved space-time is straightforward, but upon quantisation problems arise. The main point is that in Minkowski space-time the quantisation relies on the unique choice of the vacuum as the state of minimal energy. In most curved space-times however, there exists no global notion of energy as no global time-like Killing vector can be constructed. This leads to a plethora of vacua to choose from and it is not always obvious which one is the correct. In this chapter I give a short introduction to quantum field theory on curved space-times. For detailed discussions see e.g. [97–99].

3.1. Free scalar fields on Friedmann background

3.1.1. Quantisation

The action for a scalar field of mass m is

$$S = \int \mathrm{d}^D x \; \mathcal{L} = \int \mathrm{d}\mathcal{V}_x \frac{1}{2} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - (m^2 + \xi R) \phi^2 \right], \tag{3.1}$$

where $\mathrm{d}\mathcal{V}_x = \mathrm{d}^D x$
$sqrt|g(x)|$ and the background Friedmann metric is given by $\mathrm{d}s^2 = (\mathrm{d}t)^2 - a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$. Here $a(t)$ is the scale factor and γ_{ij} the spatial metric. ξ is a constant giving an additional coupling of the field to gravity beyond the one encoded in the covariant derivatives. In flat Minkowski background this term is not present, as there $R \equiv 0$. The case $\xi = 0$ is referred to as minimal coupling. If the field is massless, the case $\xi = \frac{1}{4}\frac{D-2}{D-1}$ is called conformal coupling, as in this case the field equations derived from action (3.1) are invariant under conformal transformations of the metric. Variation with respect to the field leads to the Klein Gordon equation

$$\sqrt{|g|} \left(\Box + m^2 + \xi R \right) \phi = 0. \tag{3.2}$$

The only difference to Minkowski space is the explicit form of the d'Alembert operator which encodes the information about the metric via the covariant derivatives. It is given by $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{|g|}g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right)$. I have kept the factor $\sqrt{|g|}$ for the moment as it will be important for the discussion of Green functions in section 3.1.3. In the FRW background the field equations can be expanded to

$$\ddot{\phi} + (D-1)\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\Delta\phi + (m^2 + \xi R)\phi = 0,$$

or in conformal time

$$\frac{1}{a^2}\phi'' + (D-2)\frac{a'}{a}\frac{1}{a^2}\phi' - \frac{1}{a^2}\Delta\phi + (m^2 + \xi R)\phi = 0, \tag{3.3}$$

with the Laplacian $\Delta = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\gamma^{ij}\frac{\partial}{\partial x^j}\right)$. To quantise the field, we would like to express it as a mode expansion

$$\hat{\phi} = \sum\!\!\!\!\!\!\int w_i \hat{a}_i^\dagger + w_i^* \hat{a}_i. \tag{3.4}$$

Here i is a generic index, containing continuous and/or discrete parameters and the mode functions are normalised with respect to the generalized Klein-Gordon inner product

$$(w_1, w_2) = \mathrm{i}\int_\Sigma \mathrm{d}\Sigma^\mu \sqrt{|g|} f_1^* \overleftrightarrow{\partial}_\mu f_2 = \mathrm{i}\int_\Sigma \mathrm{d}\Sigma^\mu \sqrt{|g|}\left(f_1^*\partial_\mu f_2 - \partial_\mu f_1^* f_2\right) = -\delta_{1,2} \tag{3.5}$$

to provide the standard commutation relations for the field and its conjugate momentum:

$$\begin{aligned} &[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0 \quad \text{if } x \text{ and } y \text{ are separated spacelike,} \\ &[\phi(x), \pi(y)] = \mathrm{i}\delta(x-y). \end{aligned} \tag{3.6}$$

Note that my definition of the mode functions differs from [97, 98] by complex conjugation. It is consistent with [99]. The mode functions w_i are more easily obtained if we first consider a rescaled field $\chi = \phi a^{\frac{D-2}{2}}$ for which the equations of motion are

$$\ddot{\chi} - \frac{1}{a^2}\Delta\chi + \dot{\chi}\frac{\dot{a}}{a} + \left(m^2 + \xi R + \frac{2-D}{2}\frac{\ddot{a}}{a} - \frac{(D-2)^2}{4}\frac{\dot{a}^2}{a^2}\right)\chi = 0$$

In conformal time this is

$$\chi'' - \Delta\chi + \left(a^2 m^2 + a^2\xi R + \frac{2-D}{2}\frac{a''}{a} + \frac{(4-D)(D-2)}{4}\frac{a'^2}{a^2}\right)\chi = 0.. \tag{3.7}$$

From (3.7) is becomes apparent why it is useful to consider the equation for the rescaled field. The friction term proportional to $\dot{\phi}$ disappears and we get the differential equation for a harmonic oscillator with time dependent frequency. As is standard in Minkowski space, to obtain the mode expansion, we first expand the field in term of eigenfunctions $\mathcal{Y}_k(\vec{x})$ of the Laplacian

$$\Delta\mathcal{Y}_k = -\kappa^2\mathcal{Y}_k,$$

with the proper normalisation

$$\int \mathrm{d}^{D-1}x\ \gamma^{\frac{1}{2}}\mathcal{Y}_{\vec{k}}(\vec{x})\mathcal{Y}^*_{\vec{k}'}(\vec{x}) = \delta(\vec{k}, \vec{k}'),$$

(cf. [97],p. 121). The specific form of the Laplacian and therefore the eigenfunctions depend on the foliation and coordinates chosen. A different foliation will in general lead also to a different scale factor. For the flat case the eigenfunctions are the usual exponential Fourier functions with continuous $D-1$ dimensional $\vec{k}$. For closed spatial sections they are higher dimensional analogues of the spherical harmonics with discrete $\vec{k}$. We can then express the field χ as a sum over modes

$$\chi(x) = \frac{1}{\sqrt{2}}\sum\!\!\!\!\!\!\int v_{\vec{k}}(t)\mathcal{Y}_{\vec{k}}(\vec{x}), \tag{3.8}$$

where $\displaystyle\sum\!\!\!\!\!\!\int$ has to be replaced by a sum or integral with appropriate measure for the specific cases. Inserting the expansion (3.8) into (3.7) we get differential equations for the mode functions. The normalisation condition (3.5) translates to

$$v^* v' - v^{*\prime} v = 2\mathrm{i}. \tag{3.9}$$

Assuming we have obtained solutions to this equations, we can then quantise the field by expanding it in terms of creation and annihilation operators

$$\begin{aligned}\hat{\phi}(x) &= a(t)^{\frac{2-D}{2}} \frac{1}{\sqrt{2}} \sum\!\!\!\!\!\!\int \left(v_k(t) \hat{a}_k^\dagger \mathcal{Y}_k + v_k^*(t) \hat{a}_k \mathcal{Y}_k^* \right) = \\ &\equiv \sum\!\!\!\!\!\!\int \left(f_k(t) \hat{a}_k^\dagger \mathcal{Y}_k + f_k^*(t) \hat{a}_k \mathcal{Y}_k^* \right),\end{aligned} \tag{3.10}$$

where I have made the operator dependence explicit. The creation and annihilation operators have to satisfy the standard commutation relations:

$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger \right] = \delta(k, k').$$

The perturbative vacuum is defined by

$$\hat{a} \left| 0_v \right\rangle = 0,$$

where I have added the subscript v to make the dependence of the vacuum on the mode function explicit. This dependence arises as (3.7) is a second order differential equation with a two parameter family of solutions. If v_k is a mode function, then

$$u_k^* = \alpha_k v_k^* + \beta_k v_k,$$

with $\alpha_k, \beta_k \in \mathbb{C}$ is a solution to the equations of motion as well. It is correctly normalised, if the parameters obey the Bogolyubov condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1.$$

Then we could equally well build our Fock space using the operators defined by the expansion of the field in terms of the u mode functions:

$$\hat{\phi}(x) = a(t)^{\frac{2-D}{2}} \frac{1}{\sqrt{2}} \sum\!\!\!\!\!\!\int \left(u_k(t) \hat{b}_k^\dagger \mathcal{Y}_k + u_k^*(t) \hat{b}_k \mathcal{Y}_k^* \right).$$

The relation between the different creation and annihilation operators is

$$\hat{a}^\dagger = \alpha^* \hat{b}^\dagger + \beta \hat{b}, \qquad \hat{a} = \alpha \hat{b} + \beta^* \hat{b}^\dagger. \tag{3.11}$$

As long as $\beta_k \neq 0$ these two Fock spaces will be different. This is most easily seen by calculating the expectation value of the particle numbers operator with respect to v in the u vacuum:

$$\left\langle 0_u \left| \hat{a}_k^\dagger \hat{a}_k \right| 0_u \right\rangle = |\beta_k|^2.$$

There is no general selection rule singling out a vacuum as the correct one. In Minkowski space we choose the vacuum to minimise the Hamiltonian, i.e. to contain only positive frequency modes, $v_k = \frac{1}{\sqrt{E_k}} \mathrm{e}^{\mathrm{i}Et}, E =$

$\sqrt{m^2 + |p|^2}$. If a general space-time has asymptotically flat regions, a possible choice of vacuum is to specify the vacuum by those mode functions which have the Minkowski vacuum as limit in these asymptotic regions. As any space-time is locally flat, a similar reasonable argument is to demand the Minkowski behaviour for short distances, i.e. for large momenta. This leads to the Bunch-Davies vacuum proposal in de Sitter space-time.

3.1.2. Scalar fields on de Sitter background

Let us now focus on the de Sitter background. I will give the mode functions for flat and closed slicing. The Ricci scalar appearing in the action is constant and in terms of the Hubble constant H given by $R = D(D-1)H^2$. This motivates the introduction of an effective mass parameter $\tilde{m}^2 = m^2 + \xi D(D-1)H^2$ and I will omit the tilde in the following for simplicity. As the curvature H of de Sitter space sets a natural energy scale, I will measure all other dimensionful quantities in units of H. The correct powers of H can be reconstructed by dimensional analysis. I will however write them explicitly in certain points to make the dependence on the Hubble scale clear.

Closed coordinates

For closed spatial sections of section 2.3.1, the eigenfunctions of the Laplacian are orthonormal surface harmonics $\mathcal{Y}_k(\vec{x}) = Y_{k,\vec{\sigma}}(\vec{x})$. Their definitions and useful properties can be found in appendix B.1. The scale factor is $a(\eta) = \frac{1}{\sin\eta}$ and (3.7) becomes

$$v_k'' + \left(k + \frac{D}{2} - 1\right)^2 v_k + \frac{1}{\sin^2\eta}\left(\frac{m^2}{H^2} + \frac{D}{2}\left(1 - \frac{D}{2}\right)\right) v_k = 0. \tag{3.12}$$

The mode functions only depend on the absolute value k of the momentum and not on the direction σ, which is consistent with the $O(D)$ symmetry in space. A solution for the mode function was first found by Chernikov and Tagirov [11] to be

$$v_k(\eta) = \frac{1}{\Gamma(p+1)}\sqrt{\Gamma(p+h_+)\Gamma(p+h_-)}\mathrm{e}^{\mathrm{i}p\eta}{}_2\mathrm{F}_1\left(h_+, h_-, p+1, \frac{1}{2} - \frac{\mathrm{i}}{2}\cot\eta\right), \tag{3.13}$$

where $p = k + \frac{D}{2} - 1, h_\pm = \frac{1}{2} \pm n, n = \sqrt{(D-1)^2/4 - (m/H)^2}$. The general solution can be built from the mode function and its complex conjugate as

$$u_k(\eta) = \cosh\alpha_k v_k(\eta) + \mathrm{e}^{\mathrm{i}\beta_k}\sinh\alpha_k v_k^*, \tag{3.14}$$

where the Bogolyubov conditions have been implemented and an overall phase omitted [17]. In these coordinates there is no asymptotically flat region but I can take the limit of high momenta, $p \to \infty$ in which case the mode function approaches the limit

$$v_k(\eta) \stackrel{p\to\infty}{\to} \frac{1}{\sqrt{p}}\mathrm{e}^{\mathrm{i}p\eta},$$

which is exactly the Minkowski mode function. This corresponds to the choice of $\alpha_k = 0$ for $k \to \infty$ and de Sitter symmetry enforces it for all k. An additional argument for this mode function is the connection of de Sitter space with the euclidean sphere. We could derive the mode function in de Sitter space by analytic continuation from the sphere and demanding that the mode function are regular there. This again leads to $\alpha = 0$. For this reason the mode functions (3.13) are often called Euclidean vacuum. Further arguments for this choice of vacuum will be given in section 3.1.3.

The expansion is then

$$\phi = a^{\frac{2-D}{2}} \frac{1}{\sqrt{2}} \sum_{k,\vec{\sigma}} v_k(\eta) a^\dagger_{k,\vec{\sigma}} Y_{k,\vec{\sigma}}(\vec{n}) + v^*_k(\eta) a_{k,\vec{\sigma}} Y^*_{k,\vec{\sigma}}(\vec{n}).$$

Flat coordinates

In this case the eigenfunctions of the Laplacian are $\mathcal{Y}_k(\vec{x}) = \mathrm{e}^{-\mathrm{i}\vec{k}\vec{x}}, \vec{k} \in \mathbb{R}^{D-1}$, the conformal scale factor is given by $a(\eta) = \frac{1}{H|\eta|}$ and (3.7) becomes

$$v''_k + k^2 v_k + \frac{1}{\eta^2}\left(\frac{m^2}{H^2} + \frac{D}{2}\left(1 - \frac{D}{2}\right)\right) v_k = 0, \tag{3.15}$$

where $\vec{k}$ is the d dimensional spatial momentum. We have to distinguish two cases: for the expanding patch, η runs from $-\infty \to 0$, for the contracting patch, from $0 \to \infty$. The solutions to (3.15) are Bessel or Hankel functions, i.e. linear combinations of

$$v_{1,k}(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left(\mathrm{H}^{(1)}_n(k|\eta|)\right) \mathrm{e}^{\mathrm{i}n\pi},$$
$$v_{2,k}(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left(\mathrm{H}^{(2)}_n(k|\eta|)\right) \mathrm{e}^{-\mathrm{i}n\pi}.$$

The exponential factor will be just an irrelevant phase for $n \in \mathbb{R}$ but will matter for large mass as n becomes complex. Because of the absolute value in the expanding case attention has to be paid when determining the correctly normalised mode functions. For the expanding case the function $v_k = v_{2,k}$ exhibit the correct high momentum behaviour, for the contracting case it is the modes $v_k = v_{1,k}$. The general solution is then built from Bogolyubov transformation which I parametrized identical to (3.14)

$$u_k(\eta) = \cosh \alpha v_k(\eta) + \mathrm{e}^{\mathrm{i}\beta} \sinh \alpha v^*_k.$$

Here it is not clear, that the $\alpha = 0$ vacua in flat and closed case are identical. We only know that both have the same high energy behaviour. That they are indeed equivalent will become clear in the next section. In the flat case the mode expansion is

$$\phi(x) = a^{\frac{2-D}{2}} \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^{D-1}k}{\sqrt{(2\pi)^{D-1}}} \left(v_k(t) a^\dagger_k \mathrm{e}^{-i\vec{k}\vec{x}} + v^*_k(t) a_k \mathrm{e}^{i\vec{k}\vec{x}}\right).$$

In these coordinates the quantum fluctuations on a given physical scale do not depend on time [15, 97, 99].

3.1.3. Green functions

The Green function or propagator is a solution to the inhomogeneous Klein Gordon equation (3.2) with a point source

$$\sqrt{|g|}\left(\Box_x + (m^2 + \xi R)\right) \; G(x,y) = \delta(x,y) \tag{3.16}$$

subject to boundary conditions of retarded, advanced and Feynman Green functions. In Minkowski space, the propagator can only depend on the geodesic distance between two points because of Poincare symmetry. It can be calculated using Fourier transform or summation over the modes. The result in position space is given by

$$D_F(d^2) = \langle 0|T\phi(x)\phi(y)|0\rangle = -\frac{m^{\frac{D}{2}-1}}{(2\pi)^{\frac{D}{2}}(-d^2+\mathrm{i}\epsilon)^{\frac{D-2}{4}}} K_{\frac{D}{2}-1}(m\sqrt{-d^2+\mathrm{i}\epsilon}),$$

[100–102] where $d = \sqrt{\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)}$ is the geodesic distance. It has a pole at coinciding points

$$D_F \overset{x\to 0}{\to} \begin{cases} -\mathrm{i}\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}}\left(\frac{1}{-x^2+\mathrm{i}\epsilon}\right)^{\frac{D}{2}-1} & D \text{ odd} \\ -\mathrm{i}\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}}\left(\frac{1}{-x^2+\mathrm{i}\epsilon}\right)^{\frac{D}{2}-1} - \frac{\mathrm{i}m^{D-2}}{\pi^{\frac{D}{2}}2^{D-1}\Gamma(\frac{D}{2})}(-1)^{\frac{D}{2}}\log m\sqrt{-x^2+\mathrm{i}\epsilon} & D \text{ even} \end{cases}.$$

Most commonly the Fourier transform of the Green function is used in Minkowski space. In general space-times we cannot make use of the Fourier transform, as the Klein Gordon equation in general contains time dependent coefficients. We can however make use of symmetries of the space-time. As de Sitter is maximally symmetric, a function of two points can only depend on the geodesic distance between them. The homogeneous Klein Gordon equation can be recast in terms of the geodesic distance $Z(x,y)$ to

$$H^2\sqrt{|g(x_1)|}\left((Z^2-1)\frac{\mathrm{d}^2}{\mathrm{d}Z^2} + DZ\frac{\mathrm{d}}{\mathrm{d}Z} + \frac{m^2}{H^2}\right) D_0(Z) = 0, \tag{3.17}$$

where $D_0(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle$ denotes the Wightman function [17,103]. The solutions to this differential equations are hypergeometric functions

$$D_0(Z(x,y)) = A\,{}_2\mathrm{F}_1\left(\frac{D}{2}-\frac{1}{2}+n, \frac{D}{2}-\frac{1}{2}-n, \frac{D}{2}, \frac{1+Z}{2}\right) + B\,{}_2\mathrm{F}_1\left(\frac{D}{2}-\frac{1}{2}+n, \frac{D}{2}-\frac{1}{2}-n, \frac{D}{2}, \frac{1-Z}{2}\right). \tag{3.18}$$

The hypergeometric function ${}_2\mathrm{F}_1(a,b,c,z)$ has a pole at $z = 1$ and a branch cut going from $z = 1$ along the positive real axis to $z \to \infty$. For the first term in (3.18) the pole correspond to points separated by null geodesics similiar to Minkowski. The second term has the pole if x and y are antipodal points. This behaviour is not present in Minkowski space-time. I will discuss the choice of the constants in shortly.

To specify the behaviour along the cut we consider the commutator function

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = D_0(x,y) - D_0(y,x).$$

We can only achieve a non vanishing result for time-like separation ($Z > 1$), if we go above and below the cut, depending on the order of time. We get this by replacing $Z \to Z + \mathrm{i}\epsilon\,\mathrm{sgn}(x^0 - y^0)$ (see (2.7)) [17,30]:

$$D_0(Z(x,y)) = D(Z + \mathrm{i}\epsilon\,\mathrm{sgn}(x^0 - y^0)).$$

The time ordered propagator can then be built by

$$D_F(x,y) = \theta(x^0 - y^0)D_0(x,y) + \theta(y^0 - x^0)D_0(y,x) = D(Z + \mathrm{i}\epsilon).$$

This propagator has a pole at null separated points ($Z = 1$) of strength

$$D_F(Z) \overset{Z\to 1}{\to} A\frac{\Gamma\left(\frac{D}{2}\right)2^{1+\frac{D}{2}}\pi^{\frac{D}{2}}\mathrm{i}H^{2-D}}{\Gamma\left(\frac{D}{2}-n\right)\Gamma\left(\frac{D}{2}+n\right)}\left(-\mathrm{i}\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}}\left(\frac{1}{-d^2+\mathrm{i}\epsilon}\right)^{\frac{D}{2}-1}\right) + [\log H^2(d^2 - \mathrm{i}\epsilon)].$$

It has an additional pole at $Z = -1$ of same strength with A replaced by B. If we demand that our propagator has the same pole and cut behaviour as in Minkowski space-time, we have to fix the constants to

$$A = \frac{-\mathrm{i}}{\left(\frac{4\pi}{H^2}\right)^{\frac{D}{2}}}\frac{1}{H^2}\frac{\Gamma(\frac{D}{2}-\frac{1}{2}+n)\Gamma(\frac{D}{2}-\frac{1}{2}-n)}{\Gamma(\frac{D}{2})}, \tag{3.19}$$

$$B = 0. \tag{3.20}$$

The values of the constant also determine the vacuum chosen, as the Green function can be calculated as sum over modes as well. This has been carried out in [12, 14, 15]. For closed and flat coordinates the choice $\alpha = 0$ leads to the above derived Green function with Minkowski behaviour ($B = 0$).

Allen [17] derived the change of the Green function under Bogolyubov transformation. Only for $\beta = 0$ the resulting Green function respects the symmetry group of de Sitter. The family of mode functions leading to invariant Green functions are called α-vacua [16, 17].

3.2. Path Integral

The path integral or functional integral is a well known method in quantum field theory to describe the physics of a system. The methods goes back to Dirac and Feyman [104, 105]. The core is the evolution of an initial state to a final state via all possible intermediate states and weighting each path using the classical action along that path [102, 106]. Often the initial and the final state are taken to be the vacuum of the quantized theory. However, unlike in Minkowski space in general the initial and final vacuum do not coincide. This is even less the case if interactions are introduced. For such theories one has to use the in/in or Schwinger-Keldysh formalism which does only depend on the initial vacuum and is independent of any point to the future of the observables [107–109]. In the following section I give a short introduction to this formalism based on [110–119].

3.2.1. Schwinger-Keldysh Formalism

In quantum field theory an observable O is associated to a hermitian operator $\hat{O}$. The value of the observable is given by the expectation value of the operator with respect to the state under consideration.

If our system is specified at the initial time t_i, to get the value at time t we have to evolve our system to time t using the evolution operator $\hat{U}(t,t')$ satisfying

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t,t') = \hat{H}(t)\hat{U}(t,t'), \qquad \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t'}\hat{U}(t,t') = -\hat{U}(t,t')\hat{H}(t'),$$

with the initial condition $\hat{U}(t,t) = 1$. It can be formally solved by

$$\hat{U}(t,t') = \begin{cases} \hat{T}\exp\left[-\mathrm{i}\int_{t'}^{t}\mathrm{d}\bar{t}\ \hat{H}(\bar{t})\right] & t > t' \\ \hat{\bar{T}}\exp\left[-\mathrm{i}\int_{t'}^{t}\mathrm{d}\bar{t}\ \hat{H}(\bar{t})\right] & t < t' \end{cases},$$

where $\hat{T}$ denotes time ordering with later times to the left and $\hat{\bar{T}}$ denotes anti-time ordering. For actions with quadratic kinetic term for the field, the evolution operator can be expressed in terms of the Lagrangian

$$\hat{U}(t,t') = \begin{cases} \hat{T}\exp\left[-\mathrm{i}S\right] & t > t' \\ \hat{\bar{T}}\exp\left[-\mathrm{i}S\right] & t < t' \end{cases}.$$

It satisfies the composition principle

$$\hat{U}(t_3,t_2)\hat{U}(t_2,t_1) = \hat{U}(t_3,t_1).$$

Then the value of the observable is

$$O(t) = \langle\psi|\,\hat{U}(t_i,t)\hat{O}\hat{U}(t,t_i)\,|\psi\rangle = \frac{\mathrm{Tr}\left\{\hat{U}(t_i,t)\hat{O}\hat{U}(t,t_i)\rho\right\}}{\mathrm{Tr}\{\rho\}}$$

for a pure or mixed state respectively. In the following my system will always start from a pure state, specified at time t_i, mainly my chosen vacuum $|0_i\rangle = |\mathrm{in}\rangle$. We can insert additional time evolution operators to get

$$\begin{aligned} O(t) &= \langle 0_i|\,\hat{U}(t_i,t)\hat{U}(t,t_f)\hat{U}(t_f,t)\hat{O}\hat{U}(t,t_i)\,|0_i\rangle \\ &= \langle 0_i|\,\hat{U}(t_i,t_f)\hat{U}(t_f,t)\hat{O}\hat{U}(t,t_i)\,|0_i\rangle\,. \end{aligned} \tag{3.21}$$

The evolution of the vacuum over all time changes the initial vacuum to

$$|0_f\rangle \equiv \hat{U}(t_f,t_i)\,|0_i\rangle\,,$$

where $|0_f\rangle$ is only the notation for the time evolved vacuum. In general it is no longer the physical vacuum valid at the final time. In Minkowski space-time there is only one vacuum, so the time evolution with the free action will only contribute a phase (Gell-Mann-Low theorem [120])

$$|0_f\rangle = \mathrm{e}^{\mathrm{i}\alpha}\,|0_i\rangle = \mathrm{e}^{\mathrm{i}\alpha}\,|\mathrm{in}\rangle\,. \tag{3.22}$$

Therefore we can calculate the observables using

$$O(t) = \frac{\langle\mathrm{in}|\,\hat{U}(t_f,t)\hat{O}\hat{U}(t,t_i)\,|\mathrm{in}\rangle}{\langle\mathrm{in}|\,\hat{U}(t_f,t_i)\,|\mathrm{in}\rangle}. \tag{3.23}$$

This is not valid in general, but of course this is the case for Minkowski space-time or space-times which are asymptotically flat with the same scale factor. For other spaces we can close our eyes and blindly calculate the matrix element between the *in*-vacuum and the *out*-vacuum. With

$$|\text{out}_f\rangle = \hat{U}(t_f, t_i)\,|\text{out}_i\rangle\,, \quad |\text{in}_f\rangle = \hat{U}(t_f, t_i)\,|\text{in}_i\rangle\,,$$

we get the matrix element

$$O(t) = \frac{\langle\text{out}_i|\,\hat{U}(t_i, t_f)\hat{U}(t_f, t)\hat{O}\hat{U}(t, t_i)\,|\text{in}_i\rangle}{\langle\text{out}_i\,|\,\text{in}_i\rangle}. \tag{3.24}$$

In the case $|\text{out}\rangle = \mathrm{e}^{\mathrm{i}\alpha}\,|\text{in}\rangle$ we recover the above prescription. Mind however, that in general this procedure only generates matrix elements and not physical expectation values. In terms of diagrammatic language (which I will cover in more detail later) the purpose of the denominator is to cancel the unconnected vacuum loop diagrams. This is the standard procedure in Minkowski space.

The evolution operator can be expressed as a functional integral in field space [97, 99, 106, 121]

$$\langle\text{in}|\,\hat{U}(t_f, t_i)\,|\text{in}\rangle = \int \mathcal{D}\Phi \mathrm{e}^{\mathrm{i}S[\Phi]}. \tag{3.25}$$

To calculate observables which can be expressed as powers of the field operator, we introduce a source term to the action and get the generating functional

$$Z[J] = \int \mathcal{D}\Phi \exp\left[\mathrm{i}(S[\Phi] + J\Phi)\right], \tag{3.26}$$

where $J\phi \equiv \int \mathrm{d}\mathcal{V}_x\; J(x)\phi(x)$. The observables can then be calculated by functional differentiation, taking into account the denominator in (3.23)

$$\langle 0_i\,|\phi(x_1)\dots\phi(x_n)|\,0_i\rangle = \frac{1}{Z[0]}\frac{1}{\mathrm{i}\sqrt{|g(x_1)|}}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{\mathrm{i}\sqrt{|g(x_n)|}}\frac{\delta}{\delta J(x_n)}\;Z[J]\big|_{J=0}\,. \tag{3.27}$$

Note that here again the denominator $\frac{1}{Z[0]}$ appear which cancels unconnected loop vacuum diagrams.

If adiabatic time evolution does change the initial state, i.e. either the underlying space-time changes our vacuum or the system does not start in equilibrium, relation (3.22) does not hold anymore. We have to calculate observables starting from (3.21) where we introduce the additional time evolution operators towards late times and back to the initial time. Schematically we have to evolve along the contour depicted in figure 3.1. This formalism is called *closed time path* or Schwinger-Keldysh formalism [107, 108]. The

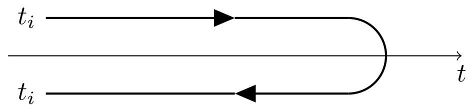


Figure 3.1.: Representation of the contour of integration for the closed time path formalism

maximal time can be chosen arbitrarily as long as it is larger than the largest time appearing in the observable. Using this contour we know all the relevant states, i.e. only the initial state matters and we do not have to know any state in the future. We can therefore use path integrals to generate observables with the generating functional

$$Z_C[J] = \int \mathcal{D}\Phi \exp\left[\mathrm{i}(S_C[\Phi] + J\Phi)\right], \tag{3.28}$$

where the integration for the action has to be performed along the contour. To simplify the calculation it is useful to introduce two fields Φ^+, Φ^- which live on the forward (upper) and backward (lower) branch respectively. They are not completely independent as they have to coincide at future infinity: for $t \to \infty$, $\Phi^+(x) = \Phi^-(x), \partial_t\Phi^+(x) = \partial_t\Phi^-(x)$. The functional integral representation is

$$Z[J^+, J^-] = \int \mathcal{D}\Phi^+(x)\mathcal{D}\Phi^-(x) \exp\left\{\mathrm{i}(S[\Phi^+] + J^+\Phi^+ - S^*[\Phi^-] - J^-\Phi^-)\right\}, \tag{3.29}$$

where S^* indicates that mass terms carry $+\mathrm{i}\epsilon$ and $J\Phi = \int \mathrm{d}\mathcal{V}_x J(x)\Phi(x)$. Note that for $J^+ = J^-$ the evolution on the forward and backward branch is identical. Therefore from the construction we have

$$Z[J, J] = 1.$$

This is an important point as vacuum loop diagrams are cancelled automatically in this prescription without having to remove them by hand. Differentiation with respect to the sources $J^\pm$ generates contour ordered expectation values of fields. We have only one physical field ϕ, so the $\pm$-fields are only auxiliary to keep track of the forward and backward branches and have be set equal at the end. From the field ϕ^+ we get the usual time ordered expectation value.

$$\langle 0|\, T[\phi(x_1)\dots\phi(x_n)]\,|0\rangle = \frac{(-\mathrm{i})^n}{\sqrt{|g(x_1)|\dots|g(x_n)|}} \left.\frac{\partial^n Z[J^+, J^-]}{\partial J^+(x_1)\dots\partial J^+(x_n)}\right|_{J^+=J^-=0}. \tag{3.30}$$

As the fields ϕ^- live on the backward branch we get anti-time ordered expectation values from this field

$$\langle 0|\, \bar{T}[\phi(x_1)\dots\phi(x_m)]\,|0\rangle = \frac{(\mathrm{i})^m}{\sqrt{|g(x_1)|\dots|g(x_m)|}} \left.\frac{\partial^m Z[J^+, J^-]}{\partial J^-(x_1)\dots\partial J^-(x_m)}\right|_{J^+=J^-=0}. \tag{3.31}$$

Finally the combination of J^+ and J^- generates observables with ϕ^- fields always appearing in front of ϕ^+ fields.

$$\begin{aligned}
&\langle 0|\, \bar{T}[\phi(x_1)\dots\phi(x_n)]\cdot T[\phi(y_1)\dots\phi(y_n)]\,|0\rangle = \\
&= \frac{(-1)^n(\mathrm{i})^{n+m}}{\sqrt{|g(x_1)|\dots|g(x_m)||g(y_1)|\dots|g(y_n)|}} \left.\frac{\partial^{m+n} Z[J^+, J^-]}{\partial J^-(x_1)\dots\partial J^-(x_m)\partial J^+(y_1)\dots\partial J^+(y_n)}\right|_{J^+=J^-=0}.
\end{aligned} \tag{3.32}$$

For a definite theory we can perform the functional integration over the field and simplify the generating functional. The procedure is standard [106], but I give it in detail, as it is complicated by the appearance of the two fields and the boundary conditions. For Lagrangians quadratic in the fields we can rewrite the action as

$$S_{free}[\phi] = \frac{1}{2}\int \mathrm{d}\mathcal{V}_x\ \phi(x)\left[-\Box - M^2\right]\phi(x) = \frac{1}{2}\int \mathrm{d}\mathcal{V}_x \mathrm{d}\mathcal{V}_{x'}\ \phi(x)D(x,x')\phi(x'),$$

(cf. [98] p. 186, [97] p. 156) where the kernel is

$$D(x,x') = \left[-\Box_x - M^2\right]\delta(x-x')\frac{1}{\sqrt{|g(x')|}}.$$

The classical fields $\phi^{\pm}$ have to satisfy the Klein-Gordon equation with source terms

$$\left[-\Box_x - M^2\right]\phi^{\pm}(x) = -J^{\pm}(x).$$

We introduce Green functions satisfying

$$\int \mathrm{d}\mathcal{V}_{x'}\ D(x,x')G^{\pm\pm}(x',x'') = \pm\frac{\delta(x-x'')}{\sqrt{|g(x'')|}},$$

or equivalently in differential form

$$\left[-\Box_x - M^2\right]G^{\pm\pm}(x,x') = \pm\frac{\delta(x-x')}{\sqrt{|g(x')|}}.$$

Using these Green functions we can solve the inhomogeneous equations with source terms. But we have to keep in mind that the fields are not independent but are related by the boundary condition at future infinity. Therefore we need the homogeneous Green functions

$$\left[-\Box_x - M^2\right]G^{\pm\mp}(x,x') = 0.$$

Together with the boundary condition that the fields coincide in the far future, $t \to \infty$, $\phi^+(x) \to \phi^-(x)$ and $\nabla_\mu\phi^+(x) = \nabla_\mu\phi^-(x)$ [111], the solution for the classical fields is [118]:

$$\begin{aligned}\phi^+(x) &= \int \mathrm{d}\mathcal{V}_{x'}\ [-G^{++}(x,x')J^+(x') + G^{+-}J^-(x')],\\ \phi^-(x) &= \int \mathrm{d}\mathcal{V}_{x'}\ [G^{--}(x,x')J^-(x') - G^{-+}J^+(x')].\end{aligned}$$

The first term in each line is to produce the source term and the second one to satisfy the boundary at future infinity. We shift the field in the integration in (3.29) by this classical solution (i.e. a constant from the point of view of the functional integral)

$$\begin{aligned}\phi^+(x) &= \phi'^+(x) + \int \mathrm{d}\mathcal{V}_{x'}\ [-G^{++}(x,x')J^+(x') + G^{+-}J^-(x')],\\ \phi^-(x) &= \phi'^-(x) + \int \mathrm{d}\mathcal{V}_{x'}\ [G^{--}(x,x')J^-(x') - G^{-+}J^+(x')].\end{aligned}$$

This is equivalent to completing the square in the gaussian integral and we get

$$\begin{aligned}Z_{free}[J^+,J^-] = \exp\Bigg[\frac{\mathrm{i}}{2}\int \mathrm{d}\mathcal{V}_x\mathrm{d}\mathcal{V}_{x'}\ \big(&-J^+(x)G^{++}(x,x')J^+(x') + J^+(x)G^{+-}(x,x')J^-(x') +\\ &+ J^-(x)G^{-+}(x,x')J^+(x') - J^-(x)G^{--}(x,x')J^-(x')\big)\Bigg].\end{aligned} \tag{3.33}$$

If both sources are identical, i.e. the change during the forward evolution is compensated by the backward evolution we should get the identity. This is satisfied if

$$G^{++} + G^{--} = G^{+-} + G^{-+}. \tag{3.34}$$

Calculating two point correlation functions, we can relate the $\pm$ Green functions to the well known propagators. We find four different Green functions: the chronological, the anti-chronological Green function and the positive and negative "frequency" Wightman functions:

$$
\begin{aligned}
G^{++}(x_1,x_2) &= -\mathrm{i}\,\langle T_C\phi^+(x_1)\phi^+(x_2)\rangle = -\mathrm{i}\,\langle T\phi(x_1)\phi(x_2)\rangle = G^T(x_1,x_2),\\
G^{--}(x_1,x_2) &= -\mathrm{i}\,\langle T_C\phi^-(x_1)\phi^-(x_2)\rangle = -\mathrm{i}\,\langle \bar{T}\phi(x_1)\phi(x_2)\rangle = G^{\bar{T}}(x_1,x_2),\\
G^{-+}(x_1,x_2) &= -\mathrm{i}\,\langle T_C\phi^-(x_1)\phi^+(x_2)\rangle = -\mathrm{i}\,\langle \phi(x_1)\phi(x_2)\rangle = G^+(x_1,x_2) = G^>(x_1,x_2),\\
G^{+-}(x_1,x_2) &= -\mathrm{i}\,\langle T_C\phi^+(x_1)\phi^-(x_2)\rangle = -\mathrm{i}\,\langle \phi(x_2)\phi(x_1)\rangle = G^-(x_1,x_2) = G^<(x_1,x_2) = G^{-+}(x_2,x_1).
\end{aligned}
$$

Adding the Green functions in this form, we see that (3.34) holds.

3.2.2. Keldysh base

It is convenient to perform the Keldysh rotation to a different basis in field space e.g. [36, 115, 116]

$$\phi^{\mathrm{cl}} = \frac{1}{\sqrt{2}}(\phi^+ + \phi^-), \qquad \phi^{\mathrm{q}} = \frac{1}{\sqrt{2}}(\phi^+ - \phi^-),$$

the superscripts "cl" and "q" stand for *classical* and *quantum* components of the fields, respectively. The reason for this notation will become clear shortly.

$$\begin{pmatrix}\phi^{\mathrm{cl}}\\ \phi^{\mathrm{q}}\end{pmatrix} = R\begin{pmatrix}\phi^+\\ \phi^-\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\phi^+\\ \phi^-\end{pmatrix}$$

This transformation takes the action to

$$S[\phi^{\mathrm{cl}},\phi^{\mathrm{q}}] = S[\phi^+] - S[\phi^-] = \frac{1}{2}\int \mathrm{d}\mathcal{V}_x \mathrm{d}\mathcal{V}_{x'}\; D(x,x')\left(\phi^{\mathrm{cl}}(x)\phi^{\mathrm{q}}(x') + \phi^{\mathrm{q}}(x)\phi^{\mathrm{cl}}(x')\right). \tag{3.35}$$

We see that for purely classical configurations $\phi^{\mathrm{q}} = 0$ the action vanishes. Actually here it is the case for ϕ^{cl} as well, but for interacting theories it is only the case for ϕ^{q} thus the names. The propagators are [36, 61, 115, 116]

$$G^{\mathrm{cl,q}} = -\mathrm{i}\left\langle \phi^{cl}\phi^{q}\right\rangle = G^R = G^{-+} - G^{--} = \theta(\eta_1-\eta_2)(G^+ - G^-), \tag{3.36}$$

$$G^{\mathrm{q,cl}} = -\mathrm{i}\left\langle \phi^{q}\phi^{cl}\right\rangle = G^A = G^{+-} - G^{--} = \theta(\eta_2-\eta_1)(G^- - G^+), \tag{3.37}$$

$$G^{\mathrm{cl,cl}} = -\mathrm{i}\left\langle \phi^{cl}\phi^{cl}\right\rangle = G^K = G^{++} + G^{--} = G^+ + G^-, \tag{3.38}$$

$$G^{\mathrm{q,q}} = \langle \phi^q\phi^q\rangle = 0. \tag{3.39}$$

Or, equivalently

$$G_K = \begin{pmatrix}G^K & G^R\\ G^A & 0\end{pmatrix} = RGR^T.$$

The components are called retarded, advanced and Keldysh Green function. The retarded and advanced components depend only on the spectrum, whereas the Keldysh function depends on the occupation number [115–117]. We can introduce a graphical representation for the Green functions. We depict the classical field by a full line and the quantum field by a dashed line. The three propagators are shown in figure 3.2. In this base the generating functional is

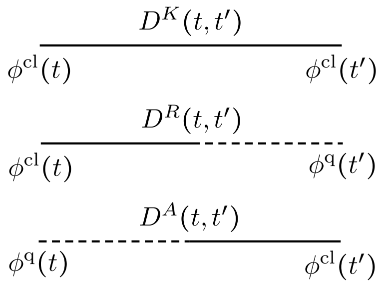


Figure 3.2.: Graphic representation of the D^K, D^R and D^A Green function.

$$Z_{free}[J^{\mathrm{cl}}, J^{\mathrm{q}}] = \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}\mathcal{V}_x \mathrm{d}\mathcal{V}_{x'} \ \Big(J^{\mathrm{q}}(x)G^{\mathrm{clcl}}(x,x')J^{\mathrm{q}}(x') + J^{\mathrm{q}}(x)G^{\mathrm{clq}}(x,x')J^{\mathrm{cl}}(x') + \right.$$
$$\left. + J^{\mathrm{cl}}(x)G^{\mathrm{qcl}}(x,x')J^{\mathrm{q}}(x')\Big)\right], \quad (3.40)$$

where $J^{\mathrm{cl}} = 1/\sqrt{2}(J^+ + J^-), J^{\mathrm{q}} = 1/\sqrt{2}(J^+ - J^-)$. For physical sources $J^{\mathrm{q}} = 0$ and we have $Z[J^{\mathrm{cl}}, 0] \equiv 1$. The quantum source is purely fictitious and only needed to generate observables. Expectation values of the fields are obtained by differentiating with respect to the sources.

$$\left\langle\phi^{\mathrm{cl}}(x)\right\rangle = \frac{1}{\mathrm{i}}\frac{\partial}{\partial J^{\mathrm{q}}(x)} Z_{free}[J^{\mathrm{cl}}, J^{\mathrm{q}}]\bigg|_{J^{\mathrm{cl}}=J^{\mathrm{q}}=0},$$
$$\left\langle\phi^{\mathrm{q}}(x)\right\rangle = \frac{1}{\mathrm{i}}\frac{\partial}{\partial J^{\mathrm{cl}}(x)} Z_{free}[J^{\mathrm{cl}}, J^{\mathrm{q}}]\bigg|_{J^{\mathrm{cl}}=J^{\mathrm{q}}=0}.$$

From (3.38) we can express the Keldysh function using the mode expansion (3.10) as

$$G^K(x_1,x_2) = -\mathrm{i}(\langle\phi(x_1\phi(x_2)\rangle + \langle\phi(x_2)\phi(x_1)\rangle) =$$
$$= -\mathrm{i}\sum\!\!\!\!\!\!\int \mathcal{Y}_k\mathcal{Y}_k^* \left[(f_k^*(\eta_1)f_k(\eta_2) + f_k^*(\eta_2)f_k(\eta_1))\left(1 + 2\left\langle\hat{a}_{\vec{k}}\hat{a}_{\vec{k}}^\dagger\right\rangle\right) + \right.$$
$$\left. + f_k(\eta_1)f_k(\eta_2)\left\langle\hat{a}_{\vec{k}}^\dagger\hat{a}_{-\vec{k}}^\dagger\right\rangle + f_k^*(\eta_1)f_k^*(\eta_2)\left\langle\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}}\right\rangle\right]. \quad (3.41)$$

The expectation value $\left\langle\hat{a}_{\vec{k}}\hat{a}_{\vec{k}}^\dagger\right\rangle$ is the number of particles with momentum $\vec{k}$. The averages $\left\langle\hat{a}_{\vec{k}}^\dagger\hat{a}_{-\vec{k}}^\dagger\right\rangle$ and $\left\langle\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}}\right\rangle$ are called anomalous quantum averages (cf. [60]). In contrast, the retarded and advanced Green functions can be expressed as

$$G^R(x_1,x_2) = -\mathrm{i}\theta(\eta_1 - \eta_2)\sum\!\!\!\!\!\!\int \mathcal{Y}_k\mathcal{Y}_k^*\left[f_l^*(\eta_1)f_k(\eta_2) - f_k^*(\eta_2)f_k(\eta_1)\right], \quad (3.42)$$
$$G^A(x_1,x_2) = -\mathrm{i}\theta(\eta_2 - \eta_1)\sum\!\!\!\!\!\!\int \mathcal{Y}_k\mathcal{Y}_k^*\left[f_k^*(\eta_2)f_k(\eta_1) - f_k^*(\eta_1)f_k(\eta_2)\right]. \quad (3.43)$$

In the Keldysh base. we see that the retarded and advanced functions only depend on the spectrum of the theory, whereas the Keldysh function contains the information about the occupation number of the system, cf. [116]. We also see, that the Keldysh Green function is pure imaginary, but the advanced and retarded functions are pure real. In the ±-base these relations are obstructed. In this thesis I will always initially start with the vacuum, so occupation number and anomalous expectation value are zero. However, interaction can lead to the appearance of non-vanishing values.

Using the Wightman function we can express all other Green functions

$$D_0^{++}(x,y) = D_0(Z + \mathrm{i}\epsilon), \qquad D_0^{+-}(x,y) = D_0(Z - \mathrm{i}\epsilon\,\mathrm{sgn}(X_x^0 - X_y^0)),$$
$$D_0^{--}(x,y) = D_0(Z - \mathrm{i}\epsilon), \qquad D_0^{-+}(x,y) = D_0(Z + \mathrm{i}\epsilon\,\mathrm{sgn}(X_x^0 - X_y^0)).$$

In the Keldysh base the Green functions can be expressed by

$$D_0^R(x,y) = \theta(X_x^0 - X_y^0) D_0(Z + \mathrm{i}\epsilon), \qquad D_0^A(x,y) = \theta(X_y^0 - X_x^0) D_0(Z - \mathrm{i}\epsilon),$$
$$D_0^K(x,y) = D_0(Z + \mathrm{i}\epsilon) + D_0(Z - \mathrm{i}\epsilon).$$

3.3. Interacting scalar fields

As a toy model for interactions between different particles I will consider a power law self-interaction of the scalar field. To be able to use the results from our free field theory, I assume that the interaction is switched on adiabatically at some point in the far past. This allows me to define the vacuum at past infinity with respect to the free theory. The explicit time dependence introduced this way spoils covariance so I have to check a posteriori, that this adiabatic limit is justified. I use the standard interaction picture in which the evolution of the states is governed by the free action and the interaction lagrangian determines the evolution of the operators. My interaction has the form

$$S_{\text{int}}[\phi] = -\sum_n g_n \frac{\phi^n}{n!},$$

The interaction is supposed to be small, so I can treat it perturbatively. The linear and quadratic term are possible counter terms canceling infinities in loop corrections. Perturbatively the generating functional is given by

$$Z[J^+, J^-]_{\text{full}} = \mathrm{e}^{\mathrm{i}\int \mathrm{d}\mathcal{V}\; S_{\text{int}}\left(\frac{-\mathrm{i}}{\sqrt{|g(x)|}}\frac{\partial}{\partial J^+}\right) - \mathrm{i}\int \mathrm{d}\mathcal{V}\; S_{\text{int}}\left(\frac{\mathrm{i}}{\sqrt{|g(x)|}}\frac{\partial}{\partial J^-}\right)} Z_{\text{free}}[J^+, J^-]$$

(cf. [111, 112]). This allows us to read of the Feynman rules for a diagrammatic calculation of expectation values. It proceeds in analogy to the flat space case, [106], with the following rules:

The propagators are defined as

$$D_F^{ab}(x,y) = \langle 0| T_C \phi^a(x)\phi^b(y) |0\rangle$$

1) Each propagator between two fields $\phi^a(x)$ and $\phi^b(y)$ gives a factor $D_F^{ab}(x,y)$.

2) Each vertex with n lines gives a factor $(-\mathrm{i}g_n)\int \mathrm{d}^D x\ \sqrt{|g(x)|}$.

3) Each $-$ vertex gives an additional factor -1.

4) Divide by the symmetry factor.

As the Keldysh base makes the relation between the Green functions explicit and only has the three independent ones, it is more convenient to calculate diagrams in this base. In the Keldysh base, the action is

$$\begin{aligned} S_{\text{int}}[\phi^{\text{cl}}, \phi^{\text{q}}] &= S_{\text{int}}[\phi^+] - S_{\text{int}}[\phi^-] = \\ &= -\sum_{n,l} \frac{g_n}{n!}\frac{1}{\sqrt{2}^n}\binom{n}{l}\left(\phi^{\text{cl}}\right)^{n-l}(\phi^{\text{q}})^l\left(1-(-1)^l\right) \\ &= -\sum_{k,l}\frac{g_{k,l}}{k!l!}\left(\phi^{\text{cl}}\right)^k(\phi^{\text{q}})^l \end{aligned}$$

with $g_{k,l} = \frac{g_{k+l}}{\sqrt{2}^{k+l}}\left(1-(-1)^l\right)$. We see, that we have only vertices with odd numbers of quantum fields i.e. for purely classical configuration the action vanishes. For a cubic interaction this gives $S_{\text{int}}[\phi^{\text{cl,q}}] = -\frac{g}{3!\sqrt{2}}\left(3(\phi^{\text{cl}})^2\phi^{\text{q}} + (\phi^q)^3\right)$. The Feynman rules in the Keldysh base are [36]

1) Each propagator between the fields $\phi^{cl}(x)$ and $\phi^{cl}(y)$ gives a factor $D^K(x,y)$.

2) Each propagator between a field $\phi^{cl}(x)$ and $\phi^{q}(y)$ gives a factor $D^R(x,y) = D^A(y,x)$.

3) Each vertex with k,l fields $\phi^{\mathrm{cl}}, \phi^{\mathrm{q}}$ gives a factor $(-\mathrm{i})g_{k,l}\int \mathrm{d}^D x\ \sqrt{|g(x)|}$.

4) Divide by the symmetry factor.

Counter terms

To renormalize the theory I have to introduce the standard counter terms. Additional counter terms come from constraints satisfied by the classical field. The counter terms are [36]:

$$\delta\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(\delta_Z g^{\mu\nu}\phi_\mu\phi_\nu - \delta_m\phi^2 - \frac{\delta_g}{3}\phi^3 - \delta_1\phi\right) \tag{3.44}$$

The linear terms origins in the implicit condition for the quantisation of the scalar field that it rests in the minimum of the potential, i.e. the vacuum expectation value of the field should vanish $\langle\hat{\phi}(x)\rangle = 0$. Interactions with odd powers of the interaction allow for tadpole diagrams leading to a non-vanishing vacuum expectation value. For cubic interaction diagram in figure 3.3 contributes at first order. If the

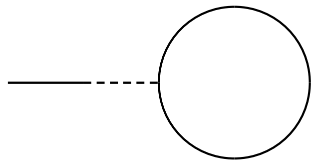
$\langle\phi^{\mathrm{cl}}\rangle = -\frac{g}{2\sqrt{2}}$ 

Figure 3.3.: Diagram contributing to the vacuum expectation value in cubic interaction

value for the classical field is to be kept at zero I have to compensate for this diagram by introducing a linear counter term in the interaction potential. This has to been done for every order, but I am not concerned with the exact form but just ignore tadpole diagrams in the following calculations. There is no diagram contributing to the expectation value of the quantum field as all of those contain advanced Green functions at identical points which therefore vanish. This is necessary for consistency as the linear term in the action is by construction only ϕ^{q} which cannot couple to the quantum field.

4. Vacuum energy decay

4.1. Effective potential

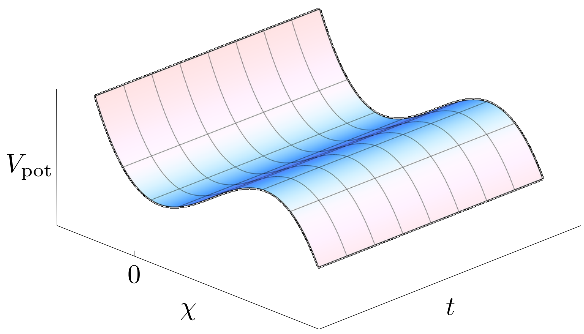
The addition of an interaction to the Lagrangian can lead to a shift of the minimum. As perturbative quantum field theory relies on the expansion about a classical minimum, this can generate problems with the stability of the theory. The addition of a cubic interaction removes the global minimum and the classical Hamiltonian becomes unbounded. In Minkowski space-time this is of no great concern as there is still a local minimum which is a meta stable vacuum. Its instability in the global context will only appear in non-perturbative effects. In de Sitter space-time the situation is different, as here the changing scale factor increases the influence of the interaction potential. An interaction of the form $\frac{g}{3!}\phi^3$ amounts to the following effective potential in the equations of motion for the rescaled field $\chi = \phi a^{\frac{D-2}{2}}$ which is used for quantization:

$$V_{\text{eff}} = \frac{\frac{1}{4} - n^2}{2}\chi^2 a(\eta)^2 + \frac{g}{3!}\chi^3 a(\eta)^{2+\frac{2-D}{2}} = \alpha\frac{\chi^2}{2} + \beta\frac{\chi^3}{6}. \tag{4.1}$$

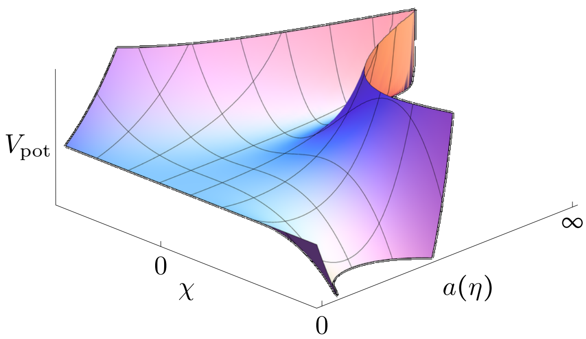
This potential has a local minimum at $\chi = 0$ but in addition a local maximum at $\chi_2 = -\frac{\alpha}{\beta} \propto a^{\frac{D-2}{2}}$. The value of the potential at this maximum is $V(\chi_2) = \frac{\alpha^3}{6\beta^2} \propto a^D$. For Minkowski space-time the scale factor is constant and the position and value of the maximum are constant. The local minimum at zero field values is always stable against small perturbations, see figure 4.1a. In Friedman-Robertson-Walker spaces the behaviour is different. For $D > 2$ the minimum and the maximum coincide for a vanishing scale factor and the value of the potential in these cases is zero, figure 4.1b. This is an indication that the BD vacuum is not stable against cubic interactions in flat de Sitter coordinates when the scale factor can vanish. In global coordinates the maximum is always separated from the minimum so the vacuum should be stable against small perturbations, see figure 4.1c. Keeping this in mind, I can discuss the stability of de Sitter space-time against interactions of the scalar field using cubic instead of quartic interactions as here the calculations are easier because fewer loop momenta appear.

4.2. Vacuum persistence

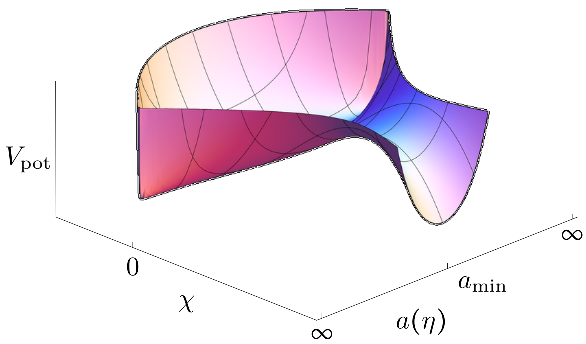
The first quantity to consider when discussing the stability of a quantum field theory is the vacuum persistence or vacuum-vacuum transition amplitude. It is a measure for the probability of the vacuum



(a) Effective potential for cubic interaction in Minkowski space-time



(b) Effective potential for cubic interaction in flat expanding de Sitter coordinates, $D = 4$



(c) Effective potential for cubic interaction in global de Sitter coordinates, $D = 4$

Figure 4.1.: Plot of the effective potential for ϕ^3 interaction, for the rescaled field χ ($\chi = a^{\frac{D-2}{2}}\phi$) in conformal time η for Minkowski and de Sitter space-time. For flat coordinates the local minimum at $\phi = 0$ is unstable for vanishing scale factor.

evolving to a different state under the influence of external sources or interactions. In the Schwinger Keldysh formalism it is given by the generating functional and is interpreted as the overlap of the initial vacuum time evolved under the influence of different external sources, $J^{\mathrm{q}} \neq 0$.

$$\langle 0_J \,|\, 0 \rangle = Z[J^{\mathrm{cl}}, J^{\mathrm{q}}]. \tag{4.2}$$

In the free theory the generating functional is given by (3.40). Using the mode expansion of the free Green function (3.41) to (3.43) we see that the only contribution to the absolute value of the partition function comes from the Keldysh Green function

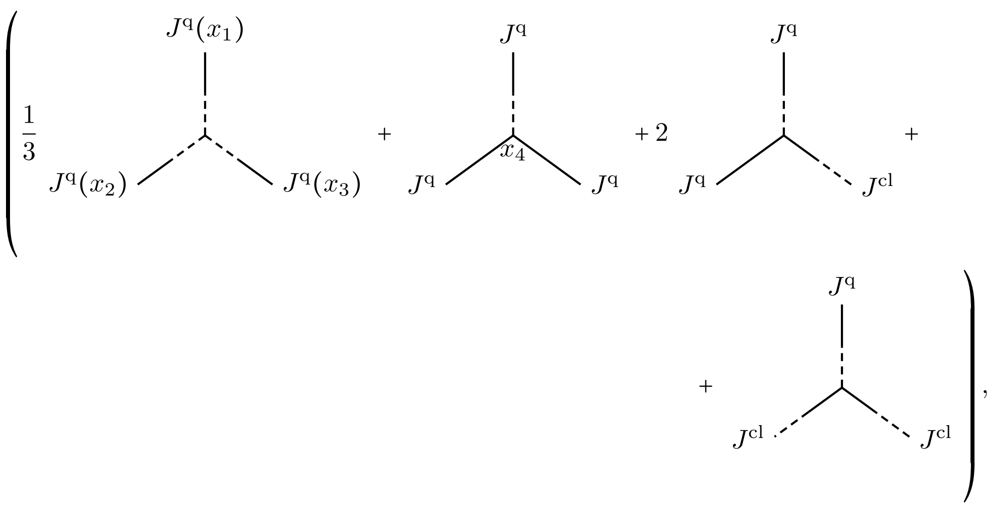
$$\begin{aligned} \left|Z[J^{\mathrm{cl}}, J^{\mathrm{q}}]\right| &= \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}\mathcal{V}_x \mathrm{d}\mathcal{V}_{x'} \ \left(J^{\mathrm{q}}(x) G^{\mathrm{clcl}}(x,x') J^{\mathrm{q}}(x')\right)\right] = \\ &= \exp\left[-\sum\!\!\!\!\!\!\int_k \left|\int \mathrm{d}\mathcal{V}_x \ J^{\mathrm{q}}(x) \mathcal{Y}_k f_k(\eta)\right|^2\right] \leq 1. \end{aligned}$$

So the vacuum persistence indicates no problems on the free level. Moreover, for identical external sources on the upper and lower branch ($J^+ = J^-, J^{\mathrm{q}} = 0$) the vacuum persistence is exactly zero by construction as an expectation value. This corresponds to a real physical source acting on the vacuum. For interacting theories, the vacuum persistence can not be calculated exactly, but I am only interested if perturbation theory is consistent up to a certain order. Perturbatively the generating functional is given by

$$\begin{aligned} Z[J^{\mathrm{cl}}, J^{\mathrm{q}}] &= \exp\left[\mathrm{i} S_{\mathrm{int}}\left[-\mathrm{i}\frac{\partial}{\partial J^{\mathrm{cl}}}, -\mathrm{i}\frac{\partial}{\partial J^{\mathrm{q}}}\right]\right] Z_{\mathrm{free}}[J^{\mathrm{cl}}, J^{\mathrm{q}}] = \\ &= Z_{\mathrm{free}}[J^{\mathrm{cl}}, J^{\mathrm{q}}] + Z_1[J^{\mathrm{cl}}, J^{\mathrm{q}}] + Z_2[J^{\mathrm{cl}}, J^{\mathrm{q}}] + \ldots . \end{aligned}$$

Using this equation we can calculate the generating functional to any desired order in perturbation theory. The first order correction is given by

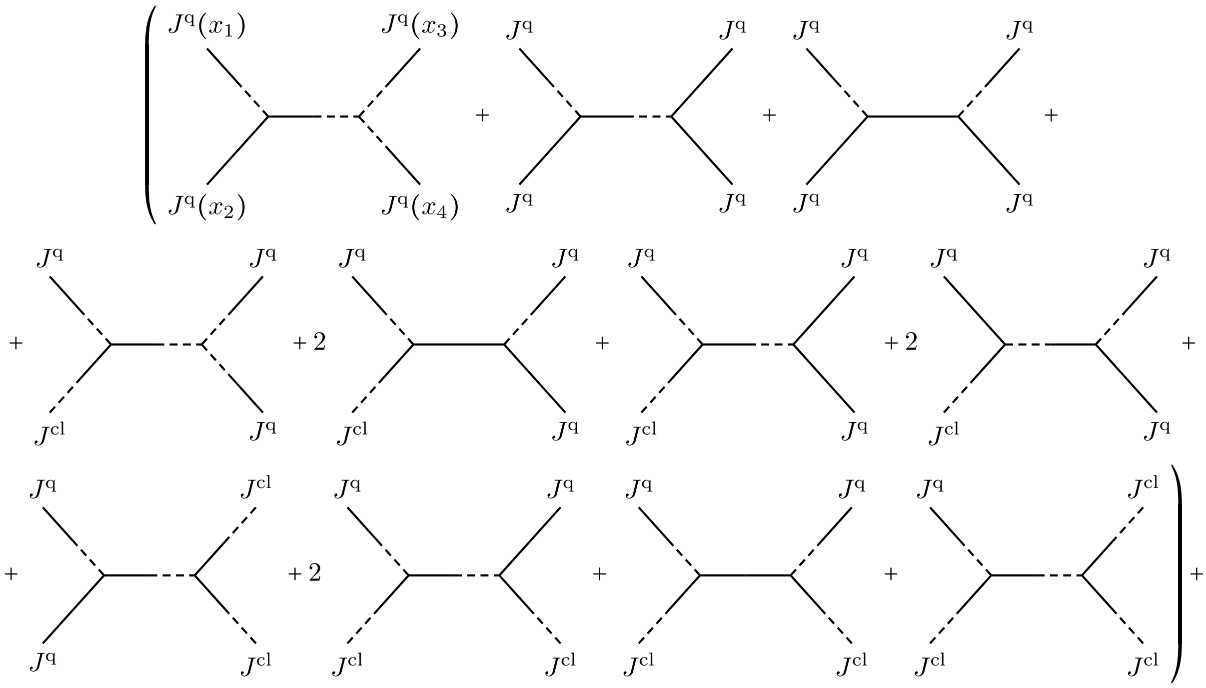
$$Z_1[J^{\mathrm{cl}}, J^{\mathrm{q}}] = -Z_{\mathrm{free}}[J^{\mathrm{cl}}, J^{\mathrm{q}}] \frac{\mathrm{i}g}{2\sqrt{2}} \int \mathrm{d}\mathcal{V}_{x_1 \ldots x_4}$$



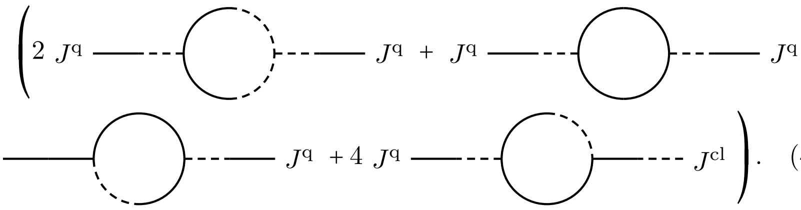
, (4.3)

where the diagrams only depict the connection between the Green functions and the numerical factors and coupling constants have been extracted. The second order is

$$Z_2[J^{\mathrm{cl}}, J^{\mathrm{q}}] = \frac{1}{2} Z_1[J^{\mathrm{cl}}, J^{\mathrm{q}}]^2 - Z_{\mathrm{free}}[J^{\mathrm{cl}}, J^{\mathrm{q}}] \frac{\mathrm{i}g^2}{4} \int \mathrm{d}\mathcal{V}_{x_1 \ldots x_6}$$



$$+ Z_{\mathrm{free}}[J^{\mathrm{cl}}, J^{\mathrm{q}}] \frac{g^2}{8} \int \mathrm{d}\mathcal{V}_{x_1 \ldots x_4}$$



(4.4)

From the above diagrams we see that each correction is multiplied by a factor of Z_{free}. If $|Z_{\mathrm{free}}| < 1$ we can always choose the coupling constant small enough such that the full vacuum persistence amplitude does not exceed unity. If $J^{\mathrm{q}} = 0$, Z_{free} vanishes, but then also each correction vanishes as each diagram contains at least one quantum source, so $Z_{\mathrm{int}}[J^{\mathrm{cl}}, 0] = 1$. This is a robust property in the Schwinger-Keldysh formalism (cf. [116]) where the generating functional is always unity for vanishing quantum source, i. e. a physical source.

4.3. Correction to the Correlation functions

Switching on the interaction leads to corrections of the tree level Green functions (3.41) to (3.43). As the Keldysh Green function contains information about the occupation number of particles, any correction to it can be interpreted as a change in the occupation number and the anomalous quantum average. This is only an interpretation, if one regards the instantaneous Green function as coming from a state with a fixed occupation number. We cannot extract the Boltzmann equations for the particle number densities in this way.

4.3.1. Corrections to the Green functions

Using the Dyson-Schwinger equation we can write the loop correction to the propagators in the following way

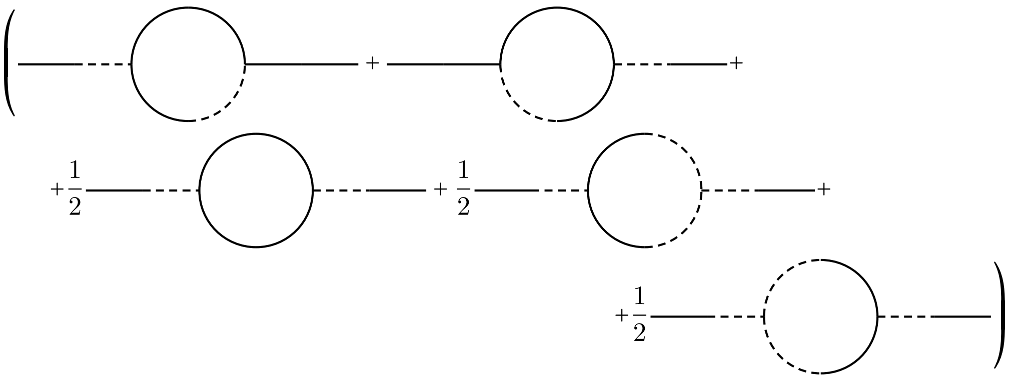
$$D = D_0 + D_0 \Sigma D,$$

where matrix multiplication and integration over the intermediate points with the space time measure is implicit. The self energy can be calculated from all diagrams contributing to a certain order using the above stated Feynman rules. e.g. in second order in the coupling we get for ϕ^3

$$\Sigma_1(x_1, x_2) = -\frac{g^2}{2} \begin{pmatrix} D^R(x_1,x_2) D^A(x_1,x_2) & D^K D^A \\ D^K D^R & \frac{1}{2}((D^K)^2 + (D^R)^2 + (D^A)^2) \end{pmatrix}.$$

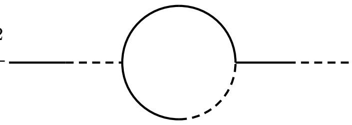
The product $D^A D^R$ is zero, as it has no support in the time domain [116]. This ensures that Σ^K is alway zero. From this (or from (4.4)) the corrections to the tree level Green functions are at second order (with coupling constants and numerical factors extracted)

$D^K_{(2)} = -\frac{g^2}{2}$



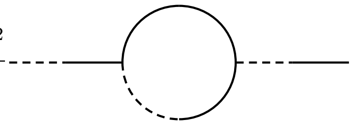
$$= -\frac{g^2}{2} \int \mathrm{d}\mathcal{V}_{u,v} \Big(D_0^R(x,u) D_0^K(u,v) D_0^R(u,v) D_0^K(v,y) + D_0^K(x,u) D_0^K(u,v) D_0^A(u,v) D_0^A(v,y) + \frac{1}{2} D_0^R(x,u) D_0^K(u,v) D_0^K(u,v) D_0^A(v,y) + \frac{1}{2} D_0^R(x,u) D_0^R(u,v) D_0^R(u,v) D_0^A(v,y) + \frac{1}{2} D_0^R(x,u) D_0^A(u,v) D_0^A(u,v) D_0^A(v,y) \Big), \tag{4.5}$$

$$D^R_{(2)} = -\frac{g^2}{2}$$



$$= -\frac{g^2}{2}\int \mathrm{d}\mathcal{V}_{u,v}\; D^R_0(x,u)D^K_0(u,v)D^R_0(u,v)D^R_0(v,y), \tag{4.6}$$

$$D^A_{(2)} = -\frac{g^2}{2}$$



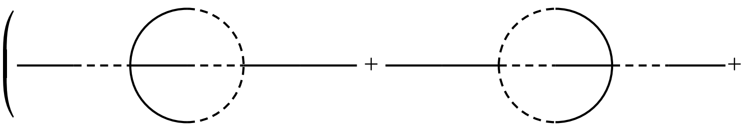
$$= -\frac{g^2}{2}\int \mathrm{d}\mathcal{V}_{u,v}\; D^A_0(x,u)D^K_0(u,v)D^A_0(u,v)D^A_0(v,y). \tag{4.7}$$

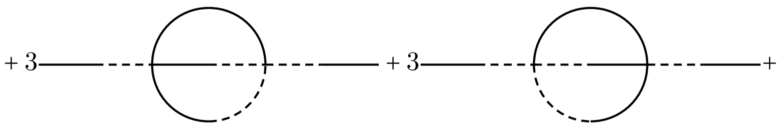
For ϕ^4 interaction the self energy is given by

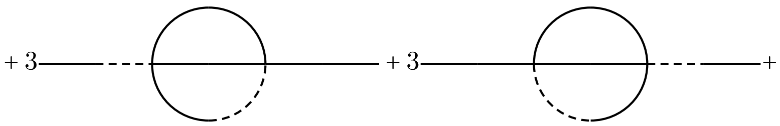
$$\Sigma_1(x_1,x_2) = -\frac{\lambda^2}{12}\begin{pmatrix} 6D^RD^AD^K & (D^A)^3 + 3(D^K)^2D^A + 3(D^R)^2D^A \\ (D^R)^3 + 3(D^K)^2D^R + 3(D^A)^2D^R & (D^K)^3 + 3D^K((D^R)^2 + (D^A)^2) \end{pmatrix}.$$

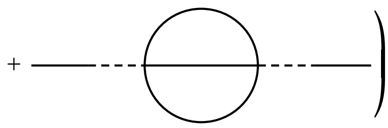
This corresponds the following diagramatic correction to the Green functions

$$D^K_{(2)} = -\frac{\lambda^2}{12}\Bigg($$





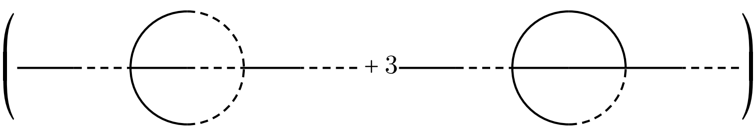




$$\Bigg) =$$

$$= -\frac{\lambda^2}{12}\int \mathrm{d}\mathcal{V}_{u,v}\big(D^R_0(x,u)G^R_{(0)}(u,v)^3G^K_{(0)}(v,y) + D^K_0(x,u)D^A_0(u,v)^3D^A_0(v,y) +$$
$$+ 3D^R_0(x,u)D^K_0(u,v)D^R_0(u,v)^2D^A_0(v,y) + 3D^R_0(x,u)D^K_0(u,v)D^A_0(u,v)^2D^A_0(v,y) +$$
$$+ 3D^R_0(x,u)D^K_0(u,v)^2D^R_0(u,v)D^K_0(v,y) + 3D^K_0(x,u)D^K_0(u,v)^2D^A_0(u,v)D^A_0(v,y) +$$
$$+ D^R_0(x,u)D^K_0(u,v)^3D^A_0(v,y)\big),$$

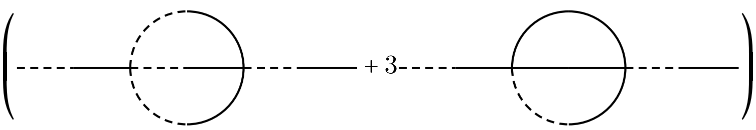
$$D^R_{(2)} = -\frac{\lambda^2}{12}\Bigg($$



$$\Bigg) =$$

$$= -\frac{\lambda^2}{12}\int \mathrm{d}\mathcal{V}_{u,v}\big(D^R_0(x,u)G^R_{(0)}(u,v)^3G^R_{(0)}(v,y) + D^R_0(x,u)D^K_0(u,v)^2D^R_0(u,v)D^R_0(v,y)\big),$$

$$D^A_{(2)} = -\frac{\lambda^2}{12}\Bigg($$



$$\Bigg) =$$

$$= -\frac{\lambda^2}{12}\int \mathrm{d}\mathcal{V}_{u,v}\big(D^A_0(x,u)G^A_{(0)}(u,v)^3G^A_{(0)}(v,y) + D^A_0(x,u)D^K_0(u,v)^2D^A_0(u,v)D^A_0(v,y)\big).$$

These corrections are general and do not depend on the space-time parametrisation or the mode functions. I will calculate the integrals in the following sections. I will focus on the closed coordinates and cubic interaction.

4.3.2. Calculation in closed coordinates

To calculate the corrections to the Green functions explicitly, I insert the expansions (3.41) to (3.43) into (4.5) to (4.7). For the correction to the Keldysh function this yields

$$
\begin{aligned}
D^K_{(2)} = &-\frac{1}{2}\sum_{k_1,k_2,k_3,k_4}\frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_2,d-1)}{C_{k_2}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_3,d-1)}{C_{k_3}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_4,d-1)}{C_{k_4}{}^{1/2(d-1)}(1)\Omega_d}\\
&\int_{S^d}\mathrm{d}^d u\ \mathrm{d}^d w C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{u})C^{1/2(d-1)}_{k_2}(\vec{u}.\vec{v})C^{1/2(d-1)}_{k_3}(\vec{u}.\vec{v})C^{1/2(d-1)}_{k_4}(\vec{v}.\vec{y})\\
&\int \mathrm{d}\xi\ \mathrm{d}\chi\ g(\xi)g(\chi)a(\xi)^D a(\chi)^D\\
&\big[\theta(\eta_1-\xi)\theta(\xi-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\xi)-f_{k_1}(\eta_1)f^*_{k_1}(\xi)\right)\left(f^*_{k_2}(\xi)f_{k_2}(\chi)+f_{k_2}(\xi)f^*_{k_2}(\chi)\right)\\
&\quad\left(f^*_{k_3}(\xi)f_{k_3}(\chi)-f_{k_3}(\xi)f^*_{k_3}(\chi)\right)\left(f^*_{k_4}(\chi)f_{k_4}(\eta_2)+f_{k_4}(\chi)f^*_{k_4}(\eta_2)\right)\\
&+\theta(\chi-\xi)\theta(\eta_2-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\xi)+f_{k_1}(\eta_1)f^*_{k_1}(\xi)\right)\left(f^*_{k_2}(\xi)f_{k_2}(\chi)+f_{k_2}(\xi)f^*_{k_2}(\chi)\right)\\
&\quad\left(f^*_{k_3}(\xi)f_{k_3}(\chi)-f_{k_3}(\xi)f^*_{k_3}(\chi)\right)\left(f^*_{k_4}(\chi)f_{k_4}(\eta_2)-f_{k_4}(\chi)f^*_{k_4}(\eta_2)\right)\\
&-\frac{1}{2}\theta(\eta_1-\xi)\theta(\eta_2-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\xi)-f_{k_1}(\eta_1)f^*_{k_1}(\xi)\right)\left(f^*_{k_2}(\xi)f_{k_2}(\chi)+f_{k_2}(\xi)f^*_{k_2}(\chi)\right)\\
&\quad\left(f^*_{k_3}(\xi)f_{k_3}(\chi)+f_{k_3}(\xi)f^*_{k_3}(\chi)\right)\left(f^*_{k_4}(\chi)f_{k_4}(\eta_2)-f_{k_4}(\chi)f^*_{k_4}(\eta_2)\right)\\
&-\frac{1}{2}\theta(\eta_1-\xi)\theta(\xi-\chi)\theta(\eta_2-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\xi)-f_{k_1}(\eta_1)f^*_{k_1}(\xi)\right)\left(f^*_{k_2}(\xi)f_{k_2}(\chi)-f_{k_2}(\xi)f^*_{k_2}(\chi)\right)\\
&\quad\left(f^*_{k_3}(\xi)f_{k_3}(\chi)-f_{k_3}(\xi)f^*_{k_3}(\chi)\right)\left(f^*_{k_4}(\chi)f_{k_4}(\eta_2)-f_{k_4}(\chi)f^*_{k_4}(\eta_2)\right)\\
&-\frac{1}{2}\theta(\eta_1-\xi)\theta(\chi-\xi)\theta(\eta_2-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\xi)-f_{k_1}(\eta_1)f^*_{k_1}(\xi)\right)\left(f^*_{k_2}(\xi)f_{k_2}(\chi)-f_{k_2}(\xi)f^*_{k_2}(\chi)\right)\\
&\quad\left(f^*_{k_3}(\xi)f_{k_3}(\chi)-f_{k_3}(\xi)f^*_{k_3}(\chi)\right)\left(f^*_{k_4}(\chi)f_{k_4}(\eta_2)-f_{k_4}(\chi)f^*_{k_4}(\eta_2)\right)\big].
\end{aligned}
$$

The Heaviside functions set the upper limit of the time integrations to the largest external time appearing, thus the Keldysh formalism enforces causality. Using the results from the momentum conserving spatial integrals (B.4) and rearranging the terms I get

$$
\begin{aligned}
D^K_{(2)} = &-\frac{1}{2}\sum_{k_1,k_2,k_3=0}^{|k_1-k_2|\le k_3\le k_1+k_2}\frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_2,d-1)}{C_{k_2}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_3,d-1)}{C_{k_3}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d}\\
&B(k_1,k_2,k_3,d)C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y})\int \mathrm{d}\xi\ \mathrm{d}\chi\ g(\xi)g(\chi)a(\xi)^D a(\chi)^D\\
&\big[\theta(\eta_1-\xi)\theta(\xi-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\xi)-f_{k_1}(\eta_1)f^*_{k_1}(\xi)\right)\left(f^*_{k_1}(\chi)f_{k_1}(\eta_2)+f_{k_1}(\chi)f^*_{k_1}(\eta_2)\right)\\
&\quad\left(f^*_{k_2}(\xi)f_{k_2}(\chi)+f_{k_2}(\xi)f^*_{k_2}(\chi)\right)\left(f^*_{k_3}(\xi)f_{k_3}(\chi)-f_{k_3}(\xi)f^*_{k_3}(\chi)\right)\\
&+\theta(\chi-\xi)\theta(\eta_2-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\xi)+f_{k_1}(\eta_1)f^*_{k_1}(\xi)\right)\left(f^*_{k_1}(\chi)f_{k_1}(\eta_2)-f_{k_1}(\chi)f^*_{k_1}(\eta_2)\right)\\
&\quad\left(f^*_{k_2}(\xi)f_{k_2}(\chi)+f_{k_2}(\xi)f^*_{k_2}(\chi)\right)\left(f^*_{k_3}(\xi)f_{k_3}(\chi)-f_{k_3}(\xi)f^*_{k_3}(\chi)\right)\\
&-\theta(\eta_1-\xi)\theta(\eta_2-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\xi)-f_{k_1}(\eta_1)f^*_{k_1}(\xi)\right)\left(f^*_{k_1}(\chi)f_{k_1}(\eta_2)-f_{k_1}(\chi)f^*_{k_1}(\eta_2)\right)\\
&\quad\left(f^*_{k_2}(\xi)f_{k_2}(\chi)f^*_{k_3}(\xi)f_{k_3}(\chi)+f_{k_2}(\xi)f^*_{k_2}(\chi)f_{k_3}(\xi)f^*_{k_3}(\chi)\right)\big].
\end{aligned}
$$

Expanding the factors in each line I have

$$
\begin{aligned}
D^K_{(2)} = -\frac{1}{2} \sum_{k_1,k_2,k_3=0}^{|k_1-k_2|\leq k_3 \leq k_1+k_2} & \frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_2,d-1)}{{C_{k_2}}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_3,d-1)}{{C_{k_3}}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d} \\
& B(k_1,k_2,k_3,d) C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y}) \int \mathrm{d}\xi \ \mathrm{d}\chi \ g(\xi) g(\chi) a(\xi)^D a(\chi)^D \\
& \big[\theta(\eta_1-\xi)\theta(\xi-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\eta_2)f_{k_1}(\xi)f^*_{k_1}(\chi) + f^*_{k_1}(\eta_1)f^*_{k_1}(\eta_2)f_{k_1}(\xi)f_{k_1}(\chi) - c.c.\right) \\
& \qquad \left(f^*_{k_2}(\xi)f^*_{k_3}(\xi)f_{k_2}(\chi)f_{k_3}(\chi) + f_{k_2}(\xi)f^*_{k_3}(\xi)f^*_{k_2}(\chi)f_{k_3}(\chi) - c.c.\right) \\
& + \theta(\chi-\xi)\theta(\eta_2-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\eta_2)f_{k_1}(\xi)f^*_{k_1}(\chi) - f^*_{k_1}(\eta_1)f^*_{k_1}(\eta_2)f_{k_1}(\xi)f_{k_1}(\chi) - c.c.\right) \\
& \qquad \left(f^*_{k_2}(\xi)f^*_{k_3}(\xi)f_{k_2}(\chi)f_{k_3}(\chi) + f_{k_2}(\xi)f^*_{k_3}(\xi)f^*_{k_2}(\chi)f_{k_3}(\chi) - c.c.\right) \\
& - \theta(\eta_1-\xi)\theta(\eta_2-\chi)\left(f^*_{k_1}(\eta_1)f_{k_1}(\eta_2)f_{k_1}(\xi)f^*_{k_1}(\chi) - f^*_{k_1}(\eta_1)f^*_{k_1}(\eta_2)f_{k_1}(\xi)f_{k_1}(\chi) + c.c.\right) \\
& \qquad \left(f^*_{k_2}(\xi)f^*_{k_3}(\xi)f_{k_2}(\chi)f_{k_3}(\chi) + f_{k_2}(\xi)f_{k_3}(\xi)f^*_{k_2}(\chi)f^*_{k_3}(\chi)\right)\big] .
\end{aligned}
$$

I introduce the shorthand notation $f^*_{k_1}(\xi)f_{k_2}(\xi)f^*_{k_3}(\xi)f_{k_1}(\chi)f_{k_2}(\chi)f_{k_3}(\chi) = \ast\circ\ast\circ\circ\circ$ with star denoting the complex conjugate function and the circle denoting the pure function. The order of arguments and of momenta is always the same. Sorting the terms to resemble the structure of (3.41) the result is

$$
\begin{aligned}
D^K_{(2)} = -\frac{1}{2} \sum_{k_1,k_2,k_3=0}^{|k_1-k_2|\leq k_3 \leq k_1+k_2} & \frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_2,d-1)}{{C_{k_2}}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_3,d-1)}{{C_{k_3}}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d} \\
& B(k_1,k_2,k_3,d) C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y}) \int \mathrm{d}\xi \ \mathrm{d}\chi \ g(\xi) g(\chi) a(\xi)^D a(\chi)^D \\
& \big[f^*_{k_1}(\eta_1)f_{k_1}(\eta_2) \\
& \qquad ((\theta(\eta_1-\xi)\theta(\xi-\chi) + \theta(\chi-\xi)\theta(\eta_2-\chi))\,(\circ\ast\ast\ast\circ\circ + \circ\circ\ast\ast\ast\circ - \circ\circ\circ\ast\ast\ast - \circ\ast\circ\ast\circ\ast) + \\
& \qquad + \theta(\eta_1-\xi)\theta(\eta_2-\chi)(-\circ\ast\ast\ast\circ\circ - \circ\circ\circ\ast\ast\ast)) - \\
& - f_{k_1}(\eta_1)f^*_{k_1}(\eta_2) \\
& \qquad ((\theta(\eta_1-\xi)\theta(\xi-\chi) + \theta(\chi-\xi)\theta(\eta_2-\chi))\,(\ast\ast\ast\circ\circ\circ + \ast\circ\ast\circ\ast\circ - \ast\circ\circ\circ\ast\ast - \ast\ast\circ\circ\circ\ast) - \\
& \qquad - \theta(\eta_1-\xi)\theta(\eta_2-\chi)(-\ast\ast\ast\circ\circ\circ - \ast\circ\circ\circ\ast\ast)) + \\
& + f^*_{k_1}(\eta_1)f^*_{k_1}(\eta_2) \\
& \qquad ((\theta(\eta_1-\xi)\theta(\xi-\chi) - \theta(\chi-\xi)\theta(\eta_2-\chi))\,(\circ\ast\ast\circ\circ\circ + \circ\circ\ast\circ\ast\circ - \circ\circ\circ\circ\ast\ast - \circ\ast\circ\circ\circ\ast) + \\
& \qquad + \theta(\eta_1-\xi)\theta(\eta_2-\chi)(\circ\ast\ast\circ\circ\circ + \circ\circ\circ\circ\ast\ast)) - \\
& - f_{k_1}(\eta_1)f_{k_1}(\eta_2) \\
& \qquad ((\theta(\eta_1-\xi)\theta(\xi-\chi) - \theta(\chi-\xi)\theta(\eta_2-\chi))\,(\ast\ast\ast\ast\circ\circ + \ast\circ\ast\ast\ast\circ - \ast\circ\circ\ast\ast\ast - \ast\ast\circ\ast\circ\ast) + \\
& \qquad + \ \theta(\eta_1-\xi)\theta(\eta_2-\chi)(\ast\ast\ast\ast\circ\circ + \ast\circ\circ\ast\ast\ast))\big] .
\end{aligned}
\tag{4.8}
$$

Partially exchanging k_2 and k_3 I can further simplify this to

$$
\begin{aligned}
D^K_{(2)} = &-\frac{1}{2}\sum_{k_1,k_2,k_3=0}^{|k_1-k_2|\le k_3\le k_1+k_2} \frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_2,d-1)}{C_{k_2}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_3,d-1)}{C_{k_3}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d} \\
&B(k_1,k_2,k_3,d)C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y})\int \mathrm{d}\xi\ \mathrm{d}\chi\ g(\xi)g(\chi)a(\xi)^D a(\chi)^D \\
&\Big\{-2\theta(\eta_1-\xi)\theta(\eta_2-\chi)\left[\left(f^*_{k_1}(\eta_1)f_{k_1}(\eta_2)(\circ\circ\circ***)+f_{k_1}(\eta_1)f^*_{k_1}(\eta_2)(***\circ\circ\circ)\right)+\right. \\
&+ f^*_{k_1}(\eta_1)f^*_{k_1}(\eta_2)\left(2\theta(\xi-\chi)\circ**\circ\circ\circ+2\theta(\chi-\xi)\circ\circ\circ\circ**\right)+ \\
&\left.+ f_{k_1}(\eta_1)f_{k_1}(\eta_2)\left(2\theta(\xi-\chi)*\circ\circ***+2\theta(\chi-\xi)****\circ\circ\right)\right]+ \\
&+\left[\theta(\eta_1-\xi)\theta(\xi-\chi)\theta(\chi-\eta_2)+\theta(\eta_2-\chi)\theta(\chi-\xi)\theta(\xi-\eta_1)\right]\cdot \\
&\left[f^*_{k_1}(\eta_1)f_{k_1}(\eta_2)\left(\circ***\circ\circ-\circ\circ\circ***\right)+f_{k_1}(\eta_1)f^*_{k_1}(\eta_2)\left(*\circ\circ\circ**-***\circ\circ\circ\right)\right]+ \\
&+\left[\theta(\eta_1-\xi)\theta(\xi-\chi)\theta(\chi-\eta_2)-\theta(\eta_2-\chi)\theta(\chi-\xi)\theta(\xi-\eta_1)\right]\cdot \\
&\left.\left[f^*_{k_1}(\eta_1)f^*_{k_1}(\eta_2)\left(\circ**\circ\circ\circ-\circ\circ\circ\circ**\right)+f_{k_1}(\eta_1)f_{k_1}(\eta_2)(*\circ\circ***-****\circ\circ)\right]\right\}.
\end{aligned}
\tag{4.9}
$$

For the retarded and advanced Green functions, I get similar results:

$$
\begin{aligned}
D^R_{(2)} = &-\frac{1}{2}\sum_{k_1,k_2,k_3=0}^{|k_1-k_2|\le k_3\le k_1+k_2} \frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_2,d-1)}{C_{k_2}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_3,d-1)}{C_{k_3}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d} \\
&B(k_1,k_2,k_3,d)C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y})\int \mathrm{d}\xi\ \mathrm{d}\chi\ g(\xi)g(\chi)a(\xi)^D a(\chi)^D\theta(\eta_1-\xi)\theta(\xi-\chi)\theta(\chi-\eta_2) \\
&\left[f^*_{k_1}(\eta_1)f_{k_1}(\eta_2)(\circ***\circ\circ-\circ\circ\circ***)+\right. \\
&+ f_{k_1}(\eta_1)f^*_{k_1}(\eta_2)(***\circ\circ\circ-*\circ\circ\circ**)+ \\
&+ f^*_{k_1}(\eta_1)f^*_{k_1}(\eta_2)(-\circ**\circ\circ\circ+\circ\circ\circ\circ**)+ \\
&\left.+f_{k_1}(\eta_1)f_{k_1}(\eta_2)(-****\circ\circ+*\circ\circ***)\right].
\end{aligned}
$$

$$
\begin{aligned}
D^A_{(2)} = &\frac{1}{2}\sum_{k_1,k_2,k_3=0}^{|k_1-k_2|\le k_3\le k_1+k_2} \frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_2,d-1)}{C_{k_2}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_3,d-1)}{C_{k_3}{}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d} \\
&B(k_1,k_2,k_3,d)C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y})\int \mathrm{d}\xi\ \mathrm{d}\chi\ g(\xi)g(\chi)a(\xi)^D a(\chi)^D\theta(\xi-\eta_1)\theta(\chi-\xi)\theta(\eta_2-\chi) \\
&\left[f^*_{k_1}(\eta_1)f_{k_1}(\eta_2)(\circ***\circ\circ-\circ\circ\circ***)+\right. \\
&+ f_{k_1}(\eta_1)f^*_{k_1}(\eta_2)(***\circ\circ\circ-*\circ\circ\circ**)+ \\
&+ f^*_{k_1}(\eta_1)f^*_{k_1}(\eta_2)(-\circ**\circ\circ\circ+\circ\circ\circ\circ**)+ \\
&\left.+f_{k_1}(\eta_1)f_{k_1}(\eta_2)(-****\circ\circ+*\circ\circ***)\right].
\end{aligned}
$$

The integrals cannot be calculated in closed form analytically, so as a first step I evaluate the integrals for coinciding times $\eta_1 = \eta_2 \equiv \eta$. In this case, using the symmetry in ξ and χ and k_2 and k_3, (4.9) simplifies to

$$\begin{aligned} D^K_{(2)}(t,\vec{x},t,\vec{y}) = & -\frac{1}{2}\sum_{k_1,k_2,k_3=0}^{|k_1-k_2|\le k_3\le k_1+k_2} \frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_2,d-1)}{{C_{k_2}}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_3,d-1)}{{C_{k_3}}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d} \\ & B(k_1,k_2,k_3,d)C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y}) \int \mathrm{d}\xi\ \mathrm{d}\chi\ g(\xi)g(\chi)a(\xi)^D a(\chi)^D \theta(\eta-\xi)\theta(\eta-\chi) \\ & \left[\left(f^*_{k_1}(\eta)f_{k_1}(\eta)\right)(-4 \circ\circ\circ * * *) + \right. \\ & + f^*_{k_1}(\eta)f^*_{k_1}(\eta)\theta(\chi-\xi)\,(4 \circ\circ\circ\circ * *) + \\ & \left. + f_{k_1}(\eta)f_{k_1}(\eta)\theta(\chi-\xi)\,(4 * * * * \circ\circ)\right] = \\ = & -2\sum_{k_1,k_2,k_3=0}^{|k_1-k_2|\le k_3\le k_1+k_2} \frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_2,d-1)}{{C_{k_2}}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_3,d-1)}{{C_{k_3}}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d} \\ & B(k_1,k_2,k_3,d)C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y}) \\ & \left[\left(f^*_{k_1}(\eta)f_{k_1}(\eta)\right)A + f^*_{k_1}(\eta)f^*_{k_1}(\eta)B + f_{k_1}(\eta)f_{k_1}(\eta)B^*\right] = \\ = & \sum_{k_1} D^K_{(2)}(t,t,k_1)\frac{C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y})}{C^{1/2(d-1)}_{k_1}(1)}\frac{h(k_1,d-1)}{\Omega_d}, \end{aligned} \tag{4.10}$$

with

$$D^K_{(2)}(t,t,k_1) = \sum_{k_2,k_3=0}^{|k_1-k_2|\le k_3\le k_1+k_2} 2Q_{k_2,k_3}\left[\left(f^*_{k_1}(\eta)f_{k_1}(\eta)\right)A + f^*_{k_1}(\eta)f^*_{k_1}(\eta)B + f_{k_1}(\eta)f_{k_1}(\eta)B^*\right] \tag{4.11}$$

and $Q_{k_2,k_3} = (-1)\frac{h(k_1,d-1)}{{C_{k_1}}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_2,d-1)}{{C_{k_2}}^{1/2(d-1)}(1)\Omega_d}\frac{h(k_3,d-1)}{{C_{k_3}}^{1/2(d-1)}(1)\Omega_d}B(k_1,k_2,k_3,d)$.

Comparing (4.10) to (3.41) the corrections can be interpreted as occupation number and anomalous expectation value (cf. [62]). Note however that I use this interpretation in the case of equal external times. For distinct external times the correction (4.8) is not of the form allowing a comparison to (3.41).

Calculation for Minkowski space-time

To get a first estimate and base to compare to, I calculate the above corrections for a Minkowski background. In this case the mode functions are

$$f_k(\eta) = \frac{1}{2\sqrt{E_k}}\mathrm{e}^{\mathrm{i}E_k t}, E = \sqrt{m^2 + H^2 k(k+D-2)},$$

and the scale factor is just unity. When the switching on of the interaction is adiabatically moved to past infinity, the integrals from (4.10) are

$$\int_{-\infty}^{t} \mathrm{d}\eta\ f_{k_1}(\eta)f_{k_2}(\eta)f_{k_3}(\eta) = \frac{1}{2^{3/2}\sqrt{E_1E_2E_3}}\begin{cases}\frac{-\mathrm{i}}{\sum E_j - \mathrm{i}\epsilon}\mathrm{e}^{\mathrm{i}t\sum E_j} & t < \infty \\ \delta(\sum E_j) & t \to \infty\end{cases},$$

$$\int_{-\infty}^{t} \mathrm{d}\xi \int_{-\infty}^{\xi} \mathrm{d}\chi \; f_{k_1}(\xi) f_{k_2}(\xi) f_{k_3}(\xi) f_{k_1}(\chi) f^*_{k_2}(\chi) f^*_{k_3}(\chi) =$$

$$= -\frac{1}{2^3 E_1 E_2 E_3} \frac{1}{E_1 - E_2 - E_3 - \mathrm{i}\epsilon} \begin{cases} \frac{-\mathrm{i}}{2E_1 - \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}t2E_1} & t < \infty \\ \delta(2E_1) & t \to \infty \end{cases} .$$

With these results, the correction (4.10) becomes for finite t

$$\begin{aligned} D^K_{(2)} = &-2 \sum_{k_1,k_2,k_3=0}^{|k_1-k_2| \le k_3 \le k_1+k_2} \frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_2,d-1)}{C_{k_2}{}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_3,d-1)}{C_{k_3}{}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d} \\ &B(k_1,k_2,k_3,d) C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y}) \frac{1}{8E_1E_2E_3} \\ &\Big[\big(f^*_{k_1}(\eta) f_{k_1}(\eta) \big) \left(-\frac{1}{(\sum E_j - \mathrm{i}\epsilon)^2} \right) + \\ &+ f^*_{k_1}(\eta) f^*_{k_1}(\eta) \left(-\frac{1}{E_1 - E_2 - E_3 - \mathrm{i}\epsilon} \frac{1}{2E_1 - \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}2\eta E_1} \right) + \\ &+ f_{k_1}(\eta) f_{k_1}(\eta) \left(-\frac{1}{E_1 - E_2 - E_3 + \mathrm{i}\epsilon} \frac{1}{2E_1 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}2\eta E_1} \right) \Big]. \end{aligned} \tag{4.12}$$

For $t \to \infty$ all corrections vanish because of the delta functions and the non-vanishing mass. This result can now be compared to the calculation for the de Sitter background. I already give the result for $k_1 \ll k_2 \approx k_3$ as this will be most important later on. In Minkowski this limit corresponds the following result for the coefficient A:

$$A \propto p_2^{-4}. \tag{4.13}$$

Corrections over the complete time domain

The integrals still cannot be calculated for general times, but if the interaction is switched on in the asymptotic past and the correction is evaluated in the asymptotic future I can make use of the branch cuts of the mode functions to evaluate the change to the occupation number. I will calculate the following integral in the limit $\epsilon \to 0, \eta \to \pi$, i.e. the interaction starts in the infinite past and I calculate the two point function of points in the far future:

$$\begin{aligned} I_1 = &\int_{\epsilon}^{\eta} \mathrm{d}\xi \; a(\xi)^D f_{k_1}(\xi) f_{k_2}(\xi) f_{k_3}(\xi) = \\ = &N \int_{\epsilon}^{\eta} \mathrm{d}\xi \; (\sin\xi)^{\frac{D}{2}-3} \mathrm{e}^{\mathrm{i}\xi \sum_j p_j} \prod_j {}_2\mathrm{F}_1 \left(h_+, h_-, p_j + 1, \frac{1}{2} - \frac{\mathrm{i}}{2} \cot\xi \right) \end{aligned}$$

where $N = \frac{1}{2^{3/2}} \prod \frac{\sqrt{\Gamma(p_j+h_+)\Gamma(p_j+h_-)}}{\Gamma(p_j+1)}$. To use the branch cuts, I change variables to $z = \cot\xi$ with the relations

$$\mathrm{d}z = -\frac{1}{\sin^2\eta} \mathrm{d}\eta,$$

$$\sin\eta = (z+\mathrm{i})^{-\frac{1}{2}} (z-\mathrm{i})^{-\frac{1}{2}},$$

$$\mathrm{e}^{\mathrm{i}\eta} = \frac{(z+\mathrm{i})^{\frac{1}{2}}}{(z-\mathrm{i})^{\frac{1}{2}}},$$

which results in:

$$\begin{aligned} I_1 =&N \int \mathrm{d}z \ (z-\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}-\Sigma\frac{p_j}{2}} (z+\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}+\Sigma\frac{p_j}{2}} \prod {}_2F_1\left(h_+, h_-, p_i+1, \frac{1}{2}-\mathrm{i}\frac{z}{2}\right) = \\ =&N \int \mathrm{d}z \ (z-\mathrm{i})^{-D+2-\Sigma\frac{k_j}{2}} (z+\mathrm{i})^{\frac{D}{2}-1+\Sigma\frac{k_j}{2}} \prod {}_2F_1\left(h_+, h_-, p_i+1, \frac{1}{2}-\mathrm{i}\frac{z}{2}\right) = \int \mathrm{d}z \ J_1(z). \end{aligned} \tag{4.14}$$

Each hypergeometric function has a branch cut from $z = \mathrm{i}$ to infinity. From the momentum conservation I have the condition $\sum k_j = 2s, s \in \mathbb{N}$ so the first factor has a pole at $z = \mathrm{i}$ while the second factor has a branch cut for odd dimensions from $z = -\mathrm{i}$ to infinity. For real z the integrand has the following reflection symmetry:

$$J_1(-x) = J(x)^* \mathrm{e}^{3\mathrm{i}\pi\left(\frac{D}{2}-1\right)}, \tag{4.15}$$

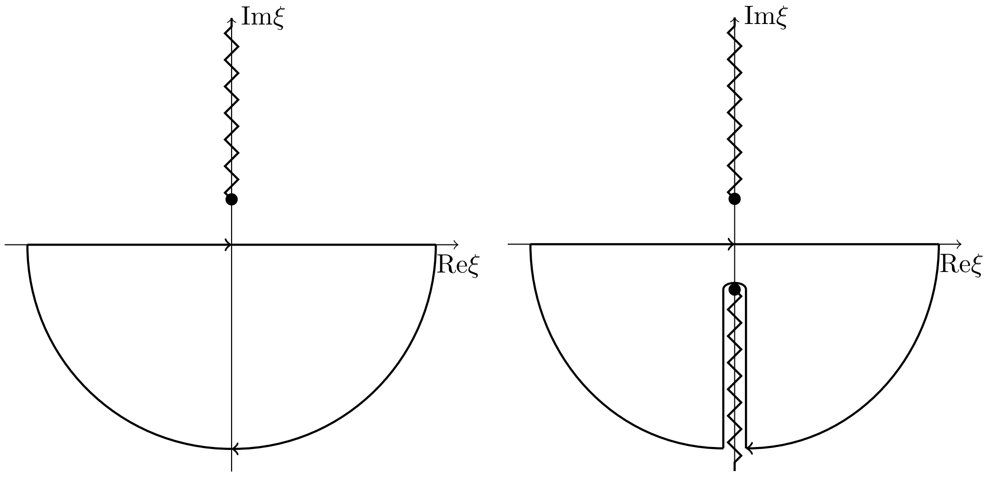
The value along the lower cut $(z = \mathrm{i}x, x < -1)$ is

$$J_1(\mathrm{i}x+\epsilon) \propto \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\left(\frac{D}{2}-1\right)}, \qquad J_1(\mathrm{i}x-\epsilon) = (-1)^D J_1(-\mathrm{i}x+\epsilon), \tag{4.16}$$

where I have used that the hypergeometric functions are real for negative real arguments. At infinity the integrand behaves like

$$J_1 \propto z^{-\frac{D}{2}+1-3h_-} = z^{-\frac{D+1}{2}+3n} \tag{4.17}$$

(cf. [122]2.3.2.(9)), so for the very massive case I consider ($n \in \mathrm{i}\mathbb{R}$) it drops off faster than $\frac{1}{z}$. This allows us to use contour integration to calculate the integral. The complete time limit corresponds to $z(\epsilon) \to \infty$ and $z(\eta) \to -\infty$ so for even D I can close the contour on the lower half-plane, cf. figure 4.2a. For odd D I cannot close the contour at infinity, but have to integrate around the branch cut, see figure 4.2b. From



(a) Contour for even D (b) Contour for odd D

Figure 4.2.: Contours for integration over the complete time

the two contour integrations I get for even and odd D

$$I_{1,D\text{ even}} = \int_{-\infty}^{0} \mathrm{d}x\ \left(J_1(x) + \mathrm{e}^{3\mathrm{i}\pi(\frac{D}{2}-1)} J_1^*(x)\right) = 0 \tag{4.18}$$

$$I_{1,D\text{ odd}} = \int_{-\infty}^{0} \mathrm{d}x\ \left(J_1(x) + \mathrm{e}^{3\mathrm{i}\pi(\frac{D}{2}-1)} J_1^*(x)\right) = -2\mathrm{i} \int_{-\infty}^{-1} \mathrm{d}x\ J_1(\mathrm{i}x + \epsilon)) = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\frac{D}{2}} \rho \tag{4.19}$$

where $\rho \in \mathbb{R}$. From this we see, that for even dimensions the net number density of particles produced vanishes, while for odd dimension no result can be drawn. (4.19) only gives a consistent relation but the value of ρ cannot be determined. Numerical calculations indicate that the value of ρ is non-vanishing for all momenta. For even dimensions the question arises, whether particles are produced in the contracting half of the evolution and annihilated in the expanding half or if in both phases no particles are produced. Unfortunately, contour integrals lead to an equation similar to (4.19) which yields no analytic result for the occupation number, but again numerical analysis shows a non-vanishing result in either half.

The second integrals for the anomalous expectation value are more tricky. They are of the form

$$I_2 = \int_0^{\pi} \mathrm{d}\xi\ a(\xi)^D f_{k_1}(\xi) f_{k_2}(\xi) f_{k_3}(\xi) \int_0^{\xi} \mathrm{d}\chi\ a(\chi)^D f_{k_1}(\chi) f_{k_2}^*(\chi) f_{k_3}^*(\chi).$$

Using the symmetry properties of the mode functions I find that the integral has to be purely imaginary:

$$\begin{aligned}
&\int_0^{\pi} \mathrm{d}\xi\ a(\xi)^D f_{k_1}(\xi) f_{k_2}(\xi) f_{k_3}(\xi) \int_0^{\pi} \mathrm{d}\chi\ a(\chi)^D f_{k_1}(\chi) f_{k_2}^*(\chi) f_{k_3}^*(\chi) = \\
&= \int_0^{\pi} \mathrm{d}\xi\ a(\xi)^D f_{k_1}(\xi) f_{k_2}(\xi) f_{k_3}(\xi) \int_0^{\xi} \mathrm{d}\chi\ a(\chi)^D f_{k_1}(\chi) f_{k_2}^*(\chi) f_{k_3}^*(\chi) + \\
&\quad + \int_0^{\pi} \mathrm{d}\xi\ a(\xi)^D f_{k_1}(\xi) f_{k_2}(\xi) f_{k_3}(\xi) \int_{\xi}^{\pi} \mathrm{d}\chi\ a(\chi)^D f_{k_1}(\chi) f_{k_2}^*(\chi) f_{k_3}^*(\chi) = \\
&\overset{(4.15)}{=} \int_0^{\pi} \mathrm{d}\xi\ a(\xi)^D f_{k_1}(\xi) f_{k_2}(\xi) f_{k_3}(\xi) \int_0^{\xi} \mathrm{d}\chi\ a(\chi)^D f_{k_1}(\chi) f_{k_2}^*(\chi) f_{k_3}^*(\chi) + \\
&\quad + \mathrm{e}^{\mathrm{i}\pi D} \int_0^{\pi} \mathrm{d}\xi\ a(\xi)^D f_{k_1}^*(\xi) f_{k_2}^*(\xi) f_{k_3}^*(\xi) \int_0^{\xi} \mathrm{d}\chi\ a(\chi)^D f_{k_1}^*(\chi) f_{k_2}(\chi) f_{k_3}(\chi).
\end{aligned}$$

For even D the first line vanishes by the contour integration discussed above, so the final line forces the real part of I_2 to vanish. Analytically there is no reason for this to vanish and numerical calculations indicate that in general it does not vanish. So generically also for even D I get a non-vanishing anomalous expectation value.

Calculation by approximation of the mode functions in the adiabatic limit

I am interested in the influence of the adiabatic switching of the interaction. In [70] it was claimed, that the limit of switching on the interaction in the infinite past leads to a fatal IR/UV mixing. In this section I will calculate this effect in a different way, with a contradicting result. I find that there is no divergence upon taking the adiabatic limit.

The integrals in the coefficients A and B in (4.10) are not easily integrated. I concentrate only on the calculation of the coefficient A, the calculation for B will proceed similarly but is technically more complicated. For finite time I approximate the mode function by integrable functions, by simplifying the hypergeometric functions. The approximations used can be found in B.5. They can be thought of as modes being inside or outside the horizon.

I am interested in the case of infrared external momenta, i.e. all internal momenta are larger than the external one $k_1 \ll k_2 \approx k_3$. The interaction is switched on adiabatically in the past, so I will let $\epsilon \to 0$. During the evolution of time the modes start with super-horizon behaviour and cross the horizon at a time η_p. To calculate the integrals I split the sum over momenta and the time intervals such that always a fixed number of modes is outside or inside the horizon and the corresponding approximations hold. The time of horizon crossing is determined from the approximations of the mode functions to be

$$p_i < p_\xi = \sqrt{\frac{|n^2 - 1|}{\sin^2 \xi}}. \tag{4.20}$$

For A after the split we can have three different cases:

1) All modes are super-horizon.

2) One mode is super-horizon (p_1), and two are sub-horizon.

3) All modes are sub-horizon

To perform the integration I introduce intermediate auxiliary integrals. I also do the limit of large internal momenta p_2, p_3 and ignore all lower order corrections. The detailed calculations are in appendix B.6. Depending on the final time and the physical wavelenghts of the modes at that time I find different momentum dependence of the coefficient. Together with the approximation for the momentum dependent coefficient Q_{p_2,p_3} an approximation for the first order correction to the Green function (4.11) can be given. For large internal momenta the sums can be replaced by integrals and as a consequence of the triangle inequality from momentum conservation it is required that $k_3 \approx k_2$. Corresponding to the different horizon behaviour of the mode functions at different times, the integrals are split according to the different approximation for the coefficients made in (B.17) to (B.21). At large momenta the integral has to be limited using an ultraviolet cutoff. I impose a cutoff in the physical momentum space as

$$q_{\text{phys}}(\eta) = \frac{Hp}{a(\eta)} \leq \Lambda_{UV}$$

and as the global de Sitter has a minimal scale factor at the bounce we have an absolute limit of

$$p \leq \frac{\Lambda_{UV}}{H} a_{\min} \leq \frac{\Lambda_{UV}}{H} a(\eta).$$

For small final times $\eta \approx 0$, when the modes with momenta p_1 are still outside the horizon, the correction which can be thought of as the number density is

$$\sum_{k_2,k_3} Q_{k_2,k_3} A(\eta_{\text{early}}) = \int_k^{k_\eta} \mathrm{d}\tilde{k}\ Q_{\tilde{p},\tilde{p}} A_1 + \int_{k_\eta}^{k_\epsilon} \mathrm{d}\tilde{k}\ Q_{\tilde{p},\tilde{p}} A_2 + \int_{k_\epsilon}^{\frac{\Lambda_{UV}}{H} a(\eta)} \mathrm{d}\tilde{k}\ Q_{\tilde{p},\tilde{p}} A_3.$$

For large final time, when the external momenta modes cross the horizon during the evolution of the universe ($p_1 > p_\eta$), the expression is

$$\sum_{k_2,k_3} Q_{k_2,k_3} A(\eta_{\text{late}}) = \int_k^{k_\epsilon} \mathrm{d}\tilde{k}\ Q_{\tilde{p},\tilde{p}} A_4 + \int_{k_\epsilon}^{\frac{\Lambda_{UV}}{H} a(\eta)} \mathrm{d}\tilde{k}\ Q_{\tilde{p},\tilde{p}} A_5.$$

Before inserting the approximations from (B.17) to (B.22) let us discuss the terms containing A_3 and A_5. Comparing with (B.19) and (B.21) we notice that both terms are of the form

$$\int_{k_\epsilon}^{\Lambda_{UV} a(\eta)} \mathrm{d}\tilde{k}\ Q_{\tilde{p},\tilde{p}} A_{3,5} \propto \left(\frac{\Lambda_{UV}}{H} \frac{a(\eta)}{a(\epsilon)} \right)^{D-5}.$$

This seems to indicate the mixing of IR and UV divergencies and lead to a divergent correction to the two point function in the limit when the switching on of the interaction is sent to past infinity. However, we have to note, that this contribution is only valid as long as $k_\epsilon < \Lambda_{UV} a(\eta)$. That means that in the limit when I send $\epsilon \to 0$ the physical momentum of those modes that are sub-horizon at time ϵ crosses the physical cutoff and the term is no longer valid. In this case k_ϵ has to be replaced by $\Lambda_{UV} a(\eta)$ in the above integrals. With the approximations (B.17) to (B.22) the leading order correction always comes from the last integral up to the UV cutoff, but the precise form depends on the number of dimensions. As the appearance of the initial scale factor $a(\epsilon)$ in the remaining coefficients is always in negative powers and terms containing them will therefore vanish in the limit of adiabatic switching, I omit them immediately. Therefore the A-part of the correction to the two point function is given by:

$$D_{2,A}^{K}(\eta,\eta,k) \propto \sum_{k_2,k_3} Q_{k_2,k_3} A(\eta_{\text{early}}) \propto \left(\frac{k_\eta}{a(\eta)} \right)^{D-1} + \left(\left(\frac{\Lambda_{UV}}{H} \right)^{D-5} - \left(\frac{k_\eta}{a(\eta)} \right)^{D-5} \right) + \log \frac{\Lambda_{UV} a(\eta)}{H p_\eta}$$

where only the time dependent contributions to the coefficients of each term are kept. For late times when the external momentum has entered the horizon the result is similar:

$$D_{2,A}^{K}(\eta,\eta,k) \propto \sum_{k_2,k_3} Q_{k_2,k_3} A(\eta_{\text{late}}) \propto (a(\eta)^{6-D} + C) \left((\frac{\Lambda_{UV}}{H} a(\eta))^{D-5} - k_\eta^{D-5} \right) + \log \frac{\Lambda_{UV} a(\eta)}{H p_\eta}.$$

Using the time evolution of the scale factor, both results have the same ultraviolet behaviour and time dependence:

$$\sum_{k_2,k_3} Q_{k_2,k_3} A(\eta) \propto \left(\frac{\Lambda_{UV}}{H} \right)^{D-5} + \log \frac{\Lambda_{UV}}{H}. \tag{4.21}$$

4.3.3. Discussion

My first remark is that my result for large internal momenta is consistent with the result for Minkwoski space-time (4.13). This is not surprising, as in this limit the internal momenta modes behave to leading order just like plane waves. The important result is that I do not have a divergent dependence on the scale factor at the time of switch-on of the interaction. This means that I can safely take the adiabatic limit and start in the interaction in the infinite past. This is in contrast to the results of [70, 72] where they find a logarithmic dependence on the time since the interaction was switched on. In [70] their logarithmic correction to the two point function is $D_{(2)}(t,t,q) \propto \log a(\epsilon)$ whereas in [72] their correction is $D_{(2)}(t,t,q) \propto \log \frac{a(\epsilon)}{a(t)}$. I believe their result is wrong for several reasons. To obtain the result in [70] they employ the scaling behaviour after the change of integration variable in an integral over the mode functions,

$$\sigma_q(\tau_1,\tau_2) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} h^*(k\tau_1)h(k\tau_2)h^*(|\vec{k}-\vec{q}|\tau_1)h(|\vec{k}-\vec{q}|\tau_2)$$

where $h(x)$ is proportional to the mode functions in flat coordinates. For large $k \gg q$ they claim that this integral has the scaling behaviour

$$\sigma_q(\tau_1,\tau_2) = (\tau_1\tau_2)^{-d/2}\Phi\left(\frac{\tau_1}{\tau_2}\right),$$

which seems to be the case on the first glance. Unfortunately this change of integration variable with unlimited boundaries is only valid if the integrand decays sufficiently fast. A similar problem is addressed in the calculation of the chiral anomaly in [121] IV.7.(2). Here this condition is not satisfied and the integral diverges which is clear from the ultraviolet behaviour of the mode functions. The integral has therefore to be regularized and the desired scaling behaviour cannot be extracted. In [72] the result is obtained by taking cutoffs for the momentum integral $a(t) < k < a(\epsilon)$. These bounds translate to the condition that the internal modes should cross the horizon during the evolution from ϵ to t. The upper bound however operates as ultraviolet cutoff and the limit $a(\epsilon) \to \infty$ leads naturally to a divergent UV contribution, i.e. the divergence in initial time ϵ is in faction the UV divergence in disguise. Therefore this issue cannot be treated without proper UV regularization.

In my calculation I approximated the mode functions for super- and sub-horizon behaviour. I integrated these approximations over time and expanded the result for large internal momenta. In this approximation, all dependence on the initial time vanishes and I conclude that there is no obstruction to taking the interaction over all time. Similiar expressions should be obtained for the remaining component of the Keldysh function and the other Green functions.

5. Alternative Attempts

5.1. Integration by power series

As we saw in the last chapter, the approximative calculation breaks down for the first order correction. To see whether this breakdown comes from the approximation method used, the breakdown of perturbation theory or from the background itself it would be useful if the correction to the propagator could be solved analytically in another way. Therefore I expressed the hypergeometric functions appearing in the definition of the mode functions by their power series definition. This has the advantage, that the resulting integrals can be calculated fully analytically. Nevertheless the final result is combined of eight infinite summations, so I did not pursue this attempt any further for the moment. The full calculation is in appendix C.

5.2. Solution using the Dyson-Schwinger equation and symmetries

Because of the complexity of the mode functions appearing in the quantisation of quantum field theory on de Sitter space, it is difficult to calculate loop integrals analytically. Using the Dyson-Schwinger equation for the full propagator and the symmetries of the space time we can however attempt to find a more general solution than the expansions above. For the retarded Green function, the correction due to one loop in the cubic interaction is (4.6)

$$D^R_{(2)}(x,y) = -\frac{g^2}{2}\int \mathrm{d}\mathcal{V}_{u,v}\; D^R_0(x,u)D^K_0(u,v)D^R_0(u,v)D^R_0(v,y). \tag{5.1}$$

From the symmetries of the involved propagators we can conclude that the final result can only depend on the geodesic distance (plus theta functions ensuring the retardedness). Thus we can act with the Klein-Gordon operator on the equation. On the right hand side I use the form of (3.16) while on the left hand side I use (3.17). This results in the following differential equation

$$\left[(Z^2-1)\frac{\mathrm{d}^2}{\mathrm{d}x^2}Z + DZ\frac{\mathrm{d}}{\mathrm{d}x}Z + \frac{(D-1)^2}{4} - n^2\right]^2 D^R_{(2)}(Z(x,y)) = -\frac{g^2}{2}D^K_0(Z(x,y))D^R_0(Z(x,y)). \tag{5.2}$$

For large geodesic distance of x and y the righthand side decays as Z^{-d} while the homogeneous equations allows for solutions of the form

$$D^R_{(2)}(Z) \propto \log(Z)D^R_0(Z).$$

This is precisely the behaviour found in [71] for the loop corrections to the propagator for large distances. As it is only the homogeneous solution in our case we still have to prove that it is indeed part of the solution. This can be obtained by acting on (5.1) only once with the Klein-Gordon operator. As the left hand side allows only the free large distance behaviour, the logarthmic part has to be contained in the special solution satisfying the right hand side. By determining the large distance behaviour of the righthand side it should be possible to confirm the validity of the solution.

This shows the power of the use of symmetries when determining the solution to loop corrections. It seems promising to pursue this attempt further e.g. by expanding the solution as a power series in the geodesic distance, determining the coefficients and finding a corresponding analytic function to the series.

6. Conclusion and Outlook

In this work I discussed the implications of a cubic self-interaction for a massive scalar field from the principal series in global de Sitter space. I showed that from the vacuum persistence amplitude there is no obvious breakdown of perturbation theory on de Sitter space, as in the in/in (Keldysh) formalism the vacuum persistence is always unity for physical sources. For arbitrary sources, the coupling can always be chosen small enough to ensure unitarity. This is in contrast to Polyakov [31] where he finds an imaginary part in the effective action spoiling the vacuum persistence. The effective potential for the conformally rescaled scalar field indicates that for the toy model of a cubic interaction the vacuum is destabilized in the expanding flat coordinate chart if the interaction is switched on in the infinite past, as there is no local maximum preventing the field from escaping from the free vacuum. It is not the case for the closed coordinate chart or even interaction potentials. This is another argument for studying interactions on de Sitter space only in the global coordinate chart instead of the flat expanding one. In the loop corrections to the propagators due to a cubic interaction I approximated the integrals by expanding the modes around their leading sub- or super-horizon behaviour and recast the result into a form analytically integrable. I found that for small internal momenta the leading order correction does not suffer from divergencies depending on the time the interaction has been in effect, in contradiction to Krotov and Polyakov [70]. This is mainly due to a different treatment of the ultraviolet divergent loop momenta integrals. The time divergence Krotov and Polyakov discover is in fact an unregularized ultra-violet momentum divergence in disguise. The conclusion is that perturbation theory on closed or contracting flat coordinate charts in de Sitter space can proceed just as is known from Minkowski with UV renormalisation. It should be possible to extend my analysis to higher interactions and determine the leading corrections due to the interaction. For an exact calculation of the loop corrections one has to refine the mathematical treatment of the involved mode functions and make use of the underlying symmetry.

Future research should investigate the contribution to large momenta propagators as well and show that all appearing divergencies can be resummed into a shift of the parameters, i.e. renormalisation. If this is not possible, the divergencies do not indicate an instability of de Sitter space per se, but first of all only the breakdown of the perturbation theory used and one has to resort to non-perturbative methods including the explicit backreaction. Otherwise we cannot distinguish divergencies of the background from those of the mathematical treatment. This also applies to any attempt interpreting the divergencies as the explosive creation of particles which is only justified with a rigorous definition of the occupation number and the kinetic equation. The study of quantum field theory on de Sitter space remains an important and exciting topic especially after the strong support of an early de Sitter phase by recent observations like Plank and BICEP2 [5–8].

A. Definitions and Conventions

A.1. Conventions

In cosmology, most fundamental constants enter the equations for the evolution of the universe. In the following, I will use Planckian (natural) units to simplify calculations, i.e. I will set $G = \hbar = c = 1$. The numerical value in common units for a quantity can be obtained via the Planckian elementary units.

$$l_{Pl} = \sqrt{\frac{G\hbar}{c^3}} = 1.616 \times 10^{-33} cm,$$
$$t_{Pl} = \frac{l_{Pl}}{c} = 5.391 \times 10^{-44} s,$$
$$m_{Pl} = \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-5} g.$$

The units for other quantities can be derived by combining the above elementary units.

Throughout this thesis the signature for my metric is $(+,-,-,\ldots)$. As I will study only homogeneous, isotropic universes the metric is the Friedmann-Robertson-Walker metric. The line element in this metric is given by

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right).$$

A.2. Fourier transform and Delta distribution

I use the following definition for the Fourier transform in d–dimensional flat space-time (Minkowksi or Euclidean)

$$F[f](k) = \int \frac{\mathrm{d}^d x}{(2\pi)^{\frac{d}{2}}} f(x) \mathrm{e}^{-\mathrm{i}\vec{k}\vec{x}}.$$

The inverse Fourier transform is given by

$$f[F](x) = \int \frac{\mathrm{d}^d k}{(2\pi)^{\frac{d}{2}}} F(k) \mathrm{e}^{\mathrm{i}\vec{k}\vec{x}}.$$

One representation of the delta distribution is

$$\delta^d(\vec{x}) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \mathrm{e}^{\mathrm{i}\vec{p}\vec{x}}.$$

B. Formula and Calculations

B.1. Surface harmonics

The surface harmonics $Y_{k,\vec{\sigma}}(\vec{n})$ are eigenfunctions of the Laplacian on S^d embedded in $D = d+1$ dimensional space, $|\vec{n}| = 1, \vec{n} \in \mathbb{R}^D$. See Bateman [123] for an extended definition, but note that the surface harmonics are S_n^l in their notation. The complex harmonics Y can be related to the real harmonics S of Bateman via $S^R_{k,\sigma} = \frac{1}{\sqrt{2}}(Y_{k,\sigma} + Y^*_{k\sigma}), S^I_{k,\sigma} = \frac{1}{\mathrm{i}\sqrt{2}}(Y_{k,\sigma} - Y^*_{k\sigma})$. The surface harmonics obey

$$\Delta Y_{k,\vec{\sigma}} = -k(k + D - 2)Y_{k,\vec{\sigma}},$$

where $k \in \mathbb{N}$. $k, \vec{\sigma}$ is the angular momentum of a given mode. The components of $\vec{\sigma}$ are limited similar to quantum mechanics by $k \geq \sigma_1 \geq \ldots \geq \sigma_{d-1}$. Total multiplicity of surface harmonics for a given k is (Bateman, Vol 2, 11.2(2))

$$h(k, d-1) = \frac{(2k + d - 1)(k + d - 2)!}{(d-1)!k!}.$$

The surface harmonics are orthonormal:

$$\int_{S^d} Y_{k,\vec{\sigma}}(\vec{n})Y^*_{k',\vec{\sigma}'}(\vec{n}) = \delta_{k,k'}\delta_{\vec{\sigma},\vec{\sigma}'}$$

They satisfy the useful addition relation

$$\sum_\sigma Y_{n,\sigma}(\vec{n})Y^*_{n,\sigma}(\vec{n}') = \frac{C_n^{1/2(d-1)}(\vec{n}.\vec{n}')}{C_n^{1/2(d-1)}(1)} \frac{h(n, d-1)}{\Omega_d}.$$

B.2. Solution to the Klein Gordon equation in closed coordinates

The differential equation for the mode functions in closed de Sitter coordinates is

$$v_k'' + \left(k + \frac{D}{2} - 1\right)^2 v_k + \frac{1}{\sin^2\eta}\left(\frac{m^2}{H^2} + \xi D(D-1) + \frac{D}{2}\left(1 - \frac{D}{2}\right)\right) v_k = 0.$$

This differential equation can be transformed using $v_k(\eta) = g_k(z)\sqrt{\sin\eta}, z = \cos\eta$ into the differential equation for Legendre functions for f

$$(1 - z^2)g_k'' - 2zg_k' + \left[p^2 - \frac{1}{4} + \frac{1}{1 - z^2}\left(M^2 + \frac{D}{2}\left(1 - \frac{D}{2}\right) - \frac{1}{4}\right)\right] g_k = 0.$$

Via the transformation $v_k(\eta) = \mathrm{e}^{\pm \mathrm{i} p\eta} h_k(z), z = \frac{1}{2} \mp \mathrm{i}\frac{\cot\eta}{2}$, we get the differential equation for hypergeometric functions

$$z(1-z)h_k'' + [p+1-2z]\,h_k' - \left[M^2 + \frac{D}{2}\left(1-\frac{D}{2}\right)\right] h_k = 0.$$

Solutions are therefore Legendre functions on the cut or equivalently hypergeometric functions:

$$\begin{aligned} v_k(\eta) &= \frac{1}{\Gamma(p+1)}\sqrt{\Gamma(p+h_+)\Gamma(p+h_-)}\mathrm{e}^{\mathrm{i}p\eta}{}_2F_1(h_+,h_-,p+1,-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\eta}}{2\sin\eta}) = \\ &= \sqrt{\sin\eta}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(p+h_-)}{\Gamma(p+h_+)}}e^{\mathrm{i}\frac{\pi}{4}}\mathrm{e}^{-\frac{\mathrm{i}}{2}n\pi}\left(\mathrm{P}^n_{k+\frac{D}{2}-\frac{3}{2}}(\cos\eta) - \frac{2\mathrm{i}}{\pi}\mathrm{Q}^n_{k+\frac{D}{2}-\frac{3}{2}}(\cos\eta)\right) \end{aligned}$$

with $n = \frac{1}{2}\sqrt{(D-1)^2-(2m/H)^2}$, $p = k+\frac{D}{2}-1$ and $h_\pm = \frac{1}{2}\pm n$. The equivalence of both functions follows from [122], 3.2.(12), 3.2.(44), and 3.4.(9). An alternative form can be achieved by using [122] 3.3.(13) and 3.3.1.(12) giving

$$\frac{1}{\Gamma(p+1)}\mathrm{e}^{\mathrm{i}p\eta}{}_2F_1(h_+,h_-,p+1,-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\eta}}{2\sin\eta}) = \left[\theta(\pi/2-\eta) + \mathrm{e}^{\mathrm{i}\pi p}\theta(\eta-\pi/2)\right] P^{-p}_{-h_+}(\mathrm{i}\cot\eta+\epsilon)\,,$$

where the Legendre function has a cut from 1 to $-\infty$ along the real axis, but the additional theta functions correct for this cut.

B.3. Different forms of the mode functions and Green function

The Wightman function can also expressed in terms of Gegenbauer functions

$$D_0(Z) = \left[AC^{\frac{D-1}{2}}_{n-\frac{D-1}{2}}(-Z) + BC^{\frac{D-1}{2}}_{n-\frac{D-1}{2}}(Z)\right]\frac{\Gamma(-\frac{D-3}{2}+n)\Gamma(D-1)}{\Gamma(\frac{D-1}{2}+n)}.$$

In flat coordinates, in spatial momentum space, the Wightman function is in terms of the mode functions

$$D_0(\eta_1,\vec{x}_1,\eta_2,\vec{x}_2) = \frac{1}{(2\pi)^{d/2}}\int \mathrm{d}^d q f_q^*(\eta_1) f_q(\eta_2)\mathrm{e}^{\mathrm{i}\vec{q}(\vec{x}_1-\vec{x}_2)},$$

with the mode function $f_q(\eta) = \frac{1}{\sqrt{2}}(H\eta)^{\frac{D-2}{2}}\sqrt{\frac{\pi|\eta|}{2}}\left(\mathrm{H}^{(2/1)}_n(k|\eta|)\right)$ for the Bunch Davies vacuum. In global coordinates, the Laplace-transform is

$$D_0(\eta_1,\vec{n}_1,\eta_2,\vec{n}_2) = \sum_{k,\vec{\sigma}} f_k^*(\eta_1)f_k(\eta_2)Y^*_{k,\vec{\sigma}}(\vec{n}_1)Y_{k,\vec{\sigma}}(\vec{n}_2).$$

The sum over the angular directions can be performed using [123]11.4(2)

$$\sum_{\vec{\sigma}} S_{n,\vec{\sigma}}(\vec{n})S_{n,\vec{\sigma}}(\vec{n}') = \frac{C_n^{1/2(d-1)}(\vec{n}.\vec{n}')}{C_n^{1/2(d-1)}(1)}\frac{h(n,d-1)}{\Omega_d} = \sum_\sigma Y_{n,\sigma}(\vec{n})Y^*_{n,\sigma}(\vec{n}'),$$

where Ω_d is the area of the d-dimensional sphere: $\Omega_d = \frac{2\pi^{d/2+1/2}}{\Gamma(d/2+1/2)}, h(k,d-1) = \frac{(2k+d-1)(k+d-2)!}{(d-1)!k!}$ and $C_k^{1/2(d-1)}(1) = \frac{(k+d-2)!}{k!(d-2)!}$ ([123]11.1.(28)). Using this relation we can simplify the Green function to

$$D_0(\eta_1,\vec{x}_1,\eta_2,\vec{x}_2) = \sum_k f_k^*(\eta_1)f_k(\eta_2)\frac{C_k^{1/2(d-1)}(\vec{n}_1.\vec{n}_2)}{C_k^{1/2(d-1)}(1)}\frac{h(k,d-1)}{\Omega_d}.$$

This is equivalent to integrating out the angular part in flat spatial section.

B.4. Spatial integrals - Momentum conservation

Performing the spatial integrals in (4.3.2) over $\vec{u}$ and $\vec{v}$ utilising (B.1) we get integrals of the form

$$I = \int_{S^d} \mathrm{d}^d w \mathrm{d}^d u \; C_{k_1}^{1/2(d-1)}(\vec{x}.\vec{u}) C_{k_2}^{1/2(d-1)}(\vec{u}.\vec{w}) C_{k_3}^{1/2(d-1)}(\vec{u}.\vec{w}) C_{k_4}^{1/2(d-1)}(\vec{w}.\vec{y}).$$

Following the reasoning in [123] the result has to be a harmonic polynomial in $\vec{x}$ with degree k_1, and in $\vec{y}$ with degree k_4 (missing the radial part). It can also only depend on $\vec{x}.\vec{y}$ as rotations in $\vec{x}$ and $\vec{y}$ can be compensated by corresponding rotations in the integrals. Therefore we have

$$I = \delta_{k_1,k_4} B(k_1,k_2,k_3,d) C_{k_1}^{1/2(d-1)}(\vec{x}.\vec{y}).$$

To determine the coefficient, I consider

$$I_2 = \int_{S^d} \mathrm{d}^d u \; C_{k_1}^{1/2(d-1)}(\vec{x}.\vec{u}) C_{k_2}^{1/2(d-1)}(\vec{u}.\vec{w}) C_{k_3}^{1/2(d-1)}(\vec{u}.\vec{w}).$$

The result will be a harmonic in $\vec{x}$ with degree k_1. As the result is invariant under simultaneous rotations in $\vec{x}$ and $\vec{w}$, the final result can also only depend on $\vec{x}.\vec{w}$. Therefore

$$I_2 = \int_{S^d} \mathrm{d}^d u \; C_{k_1}^{1/2(d-1)}(\vec{x}.\vec{u}) C_{k_2}^{1/2(d-1)}(\vec{u}.\vec{w}) C_{k_3}^{1/2(d-1)}(\vec{u}.\vec{w}) = \frac{A_3(k_1,k_2,k_3,1/2(d-1))}{C_{k_1}^{1/2(d-1)}(1)} C_{k_1}^{1/2(d-1)}(\vec{x}.\vec{w}).$$

iff $|k_1 - k_2| \leq k_3 \leq k_1 + k_2$. To determine the coefficient choose $\vec{x} = \vec{w}$ and perform the integral. This has been done in [124].

$$\int_{-1}^{1} \mathrm{d}x \; (1-x^2)^{\lambda-1/2} C_l^\lambda(x) C_m^\lambda(x) C_n^\lambda(x) =$$
$$= \frac{2^{1-2\lambda}}{(\Gamma(\lambda))^2} \frac{\pi}{s+\lambda} \frac{\Gamma(s+2\lambda)}{\Gamma(s+1)} \frac{\begin{pmatrix} s-l+\lambda-1 \\ s-l \end{pmatrix}\begin{pmatrix} s-m+\lambda-1 \\ s-m \end{pmatrix}\begin{pmatrix} s-n+\lambda-1 \\ s-n \end{pmatrix}}{\begin{pmatrix} s+\lambda-1 \\ s \end{pmatrix}},$$

for $k+l+m = 2s$ with $s \in \mathbb{N}$ and if a triangle with sides k,l,m exists, i.e. $|k-l| \leq m \leq k+l$. Therefore

$$A_3(k,l,m,\lambda) = \frac{2^{1-2\lambda}}{(\Gamma(\lambda))^2} \frac{\pi}{s+\lambda} \frac{\Gamma(s+2\lambda)}{\Gamma(s+1)} \frac{\begin{pmatrix} s-l+\lambda-1 \\ s-l \end{pmatrix}\begin{pmatrix} s-m+\lambda-1 \\ s-m \end{pmatrix}\begin{pmatrix} s-k+\lambda-1 \\ s-k \end{pmatrix}}{\begin{pmatrix} s+\lambda-1 \\ s \end{pmatrix}},$$

with the above conditions. The second integration over $\vec{w}$ can be performed according to Bateman 11.4.(15). Therefore

$$B(k_1,k_2,k_3,d) = \frac{A_3(k_1,k_2,k_3,1/2(d-1)) A_2(k_1,d-1)}{C_{k_1}^{1/2(d-1)}(1)},$$

with $A(k_1,d-1)$ given in Bateman 11.4.(16):

$$A_2(k_1,d-1) = C_{k_1}^{1/2(d-1)}(1) \frac{\Omega_d}{h(k_1,d-1)} = \frac{2\pi^{1+1/2(d-1)}}{(k_1+(d-1)/2)\Gamma((d-1)/2)}.$$

So

$$B(k_1,k_2,k_3,d) = \frac{A_3(k_1,k_2,k_3,1/2(d-1))\Omega_d}{h(k_1,d-1)} =$$
$$= \frac{2\pi^{(d+1)/2}}{(k_1+d/2-1/2)\Gamma((d-1)/2)} \frac{1}{C_{k_1}^{1/2(d-1)}(1)} \frac{2^{2-d}}{(\Gamma(1/2(d-1)))^2} \frac{\pi}{s+1/2(d-1)} \frac{\Gamma(s+d-1)}{\Gamma(s+1)} \times$$
$$\times \frac{\binom{s-k_1+1/2(d-1)-1}{s-k_1}\binom{s-k_2+1/2(d-1)-1}{s-k_2}\binom{s-k_3+1/2(d-1)-1}{s-k_3}}{\binom{s+1/2(d-1)-1}{s}},$$

with $k_1+k_2+k_3=2s$ and $|k_1-k_2| \le k_3 \le k_1+k_2$.

B.5. Asymptotic form of the Bunch Davies mode functions

In flat and closed coordinates the mode functions are given by

$$w_{\vec{k}} = a^{\frac{2-D}{2}} \frac{1}{\sqrt{2}} \sqrt{\frac{\pi|\eta|}{2}} \left(\mathrm{H}_n^{(2)}(k|\eta|)\mathrm{e}^{-\mathrm{i}n\pi}\right),$$
$$w_k = a^{\frac{2-D}{2}} \frac{1}{\sqrt{2}} \frac{1}{\Gamma(p+1)} \sqrt{\Gamma(p+h_+)\Gamma(p+h_-)} \mathrm{e}^{\mathrm{i}p\eta} {}_2\mathrm{F}_1\left(h_+,h_-,p+1,\frac{1}{2}-\frac{\mathrm{i}}{2}\cot\eta\right).$$

The mode functions exhibit nice asymptotic features. Let us first consider the flat case. Using the asypmtotic behaviour of the Bessel functions [123], 7.13.1., we get for large physical momenta p_{phys}

$$w_{\vec{k}} \approx a^{\frac{2-D}{2}} \frac{1}{\sqrt{2k}} \mathrm{e}^{-\mathrm{i}k|\eta|+\mathrm{i}\frac{\pi}{4}}.$$

The condition is $p_{\text{phys}} = k|\eta| \gg H\sqrt{|n^2-\frac{1}{4}|}$, which can be seen directly from the differential equation (3.15) or by estimating the remainder of the expansion using [123]7.4.1. For small masses $m \ll H$ this is can be interpreted as the mode having a wave length smaller than the Hubble horzion. For small physical momenta $k|\eta|$, the asyptotic behaviour for very massive modes from the principal series $n \in \mathrm{i}\mathbb{R}$ we get

$$w_{\vec{k}} \approx a^{\frac{2-D}{2}} \frac{\mathrm{i}}{\sqrt{2}} \sqrt{\frac{|\eta|}{2\pi}} \left[\Gamma(-n)\left(\frac{k|\eta|}{2}\right)^n + \Gamma(n)\mathrm{e}^{-\mathrm{i}n\pi}\left(\frac{k|\eta|}{2}\right)^{-n}\right].$$

In the complementary series ($n \in (0,\frac{3}{2}]$) we have

$$w_{\vec{k}} \approx a^{\frac{2-D}{2}} \frac{\mathrm{i}}{\sqrt{2}} \sqrt{\frac{|\eta|}{2\pi}} \left[\Gamma(n)\mathrm{e}^{-\mathrm{i}n\pi}\left(\frac{k|\eta|}{2}\right)^{-n}\right].$$

In a similar manner, I can determine the asymptotic behaviour of the mode functions in the closed coordinates. From the differential equation (3.12) I extract the condition for sub or super-horizon modes to be

$$\frac{n^2-1}{p^2\sin^2\eta} \gtrless 1.$$

To determine the first order correction I employ the relations in [122]2.8.ff. Around the minimal scale factor (bounce at $\eta = \frac{\pi}{2}$) I do a taylor expansion of the hypergeometric functions.

$$\begin{aligned} g(\eta) &= {}_2F_1(h_+, h_-, p+1, \frac{1}{2} - \mathrm{i}\frac{\cot\eta}{2}) = \\ &= {}_2F_1(h_+, h_-, p+1, \frac{1}{2}) - \frac{\mathrm{i}}{2}\cot\eta \frac{h_+h_-}{p+1} {}_2F_1\left(h_+ + 1, h_- + 1, p+1+1, \tfrac{1}{2}\right). \end{aligned}$$

Using the relations for contiguous functions (cf. [122]2.8.(33),2.8.(42),2.8.(51)) I can transform the second part

$$\begin{aligned} {}_2F_1\left(h_+ + 1, h_- + 1, p+1+1, \tfrac{1}{2}\right) =& 2\frac{p+1-h_-}{h_+}\, {}_2F_1\left(h_+, h_-, p+1+1, \tfrac{1}{2}\right) - 2\frac{p}{h_+}\, {}_2F_1\left(h_+, h_- + 1, p+1+1, \tfrac{1}{2}\right) = \\ =& 2\frac{p+1-h_-}{h_+}\, {}_2F_1\left(h_+, h_-, p+1+1, \tfrac{1}{2}\right) + \\ & - 2\frac{p}{h_+}\left[\frac{p+1}{h_-}\, {}_2F_1\left(h_+, h_-, p+1, \tfrac{1}{2}\right) - \frac{p+1-h_-}{h_-}\, {}_2F_1\left(h_+, h_-, p+1+1, \tfrac{1}{2}\right)\right] = \\ =& 2\, {}_2F_1\left(h_+, h_-, p+1, \tfrac{1}{2}\right)\left[-\frac{p(p+1)}{h_+h_-} + \right. \\ & \left. + \left(\frac{p+1-h_-}{h_+} + \frac{(p)(p+1-h_-)}{h_+h_-}\right)\frac{{}_2F_1\left(h_+, h_-, p+1+1, \tfrac{1}{2}\right)}{{}_2F_1\left(h_+, h_-, p+1, \tfrac{1}{2}\right)}\right] = \\ =& \frac{2(p+1)}{h_+h_-}\, {}_2F_1\left(h_+, h_-, p+1, \tfrac{1}{2}\right)\left[-p + \right. \\ & \left. + \frac{(p+h_-)(p+1-h_-)}{p+1}\frac{{}_2F_1\left(h_+, h_-, p+1+1, \tfrac{1}{2}\right)}{{}_2F_1\left(h_+, h_-, p+1, \tfrac{1}{2}\right)}\right] = \\ =& \frac{2(p+1)}{h_+h_-}\, {}_2F_1\left(h_+, h_-, p+1, \tfrac{1}{2}\right)\left[-p + \right. \\ & \left. + \frac{(p+h_+)(p+h_-)}{p+1}\frac{1}{2}(p+1)\frac{\Gamma(\frac{p+1+h_+}{2})\Gamma(\frac{p+1+h_-}{2})}{\Gamma(\frac{p+h_+}{2}+1)\Gamma(\frac{p+h_-}{2}+1)}\right] = \\ =& \frac{2(p+1)}{h_+h_-}\, {}_2F_1\left(h_+, h_-, p+1, \tfrac{1}{2}\right)\left[-p + 2\frac{\Gamma(\frac{p+1+h_+}{2})\Gamma(\frac{p+1+h_-}{2})}{\Gamma(\frac{p+h_+}{2})\Gamma(\frac{p+h_-}{2})}\right]. \end{aligned}$$

We therefore have

$$g(\eta) \approx {}_2F_1(h_+, h_-, p+1, \tfrac{1}{2})\left[1 - \mathrm{i}\cot\eta\left(-p + 2\frac{\Gamma(\frac{p+1+h_+}{2})\Gamma(\frac{p+1+h_-}{2})}{\Gamma(\frac{p+h_+}{2})\Gamma(\frac{p+h_-}{2})}\right)\right].$$

For large p we can approximate the last gamma fractions using [125]6.1.47

$$\frac{\Gamma(\frac{p+1+h_+}{2})\Gamma(\frac{p+1+h_-}{2})}{\Gamma(\frac{p+1+h_+-1}{2})\Gamma(\frac{p+1+h_--1}{2})} = \frac{p}{2}\left(1 - \frac{1}{2p^2}(n^2 - \frac{1}{4}) + \mathcal{O}(p^{-3})\right).$$

This yields

$${}_2F_1\left(h_+ + 1, h_- + 1, p+1+1, \tfrac{1}{2}\right) = \frac{2(p+1)}{h_+h_-}{}_2F_1\left(h_+, h_-, p+1, \tfrac{1}{2}\right)\frac{1}{2p}(n^2 - \frac{1}{4}).$$

Summarizing we can approximate the hypergeometric function for sub-horizon modes by

$$\begin{aligned}
g(\eta) &= {}_2F_1(h_+, h_-, p+1, \tfrac{1}{2} - \mathrm{i}\tfrac{\cot\eta}{2}) = \\
&= {}_2F_1(h_+, h_-, p+1, \tfrac{1}{2}) - \tfrac{\mathrm{i}}{2}\cot\eta \frac{h_+h_-}{p+1} {}_2F^1(h_+ + 1, h_- + 1, p+1+1, \frac{1}{2}) = \\
&= {}_2F_1(h_+, h_-, p+1, \tfrac{1}{2}) \left[1 + \mathrm{i}\left(n^2 - \frac{1}{4}\right)\frac{1}{2p}\cot\eta\right] = \\
&= 2^{-p} \frac{\Gamma(p+1)\sqrt{\pi}}{\Gamma(\frac{p+1+h_+}{2})\Gamma(\frac{p+1+h_-}{2})} \left[1 + \mathrm{i}\left(n^2 - \frac{1}{4}\right)\frac{1}{2p}\cot\eta\right],
\end{aligned}$$

and the approximated mode functions are

$$w_k = a^{\frac{2-D}{2}} \mathrm{e}^{\mathrm{i}p\eta} \frac{1}{2\sqrt{2\pi}} \sqrt{\frac{\Gamma(\frac{p+h_+}{2})\Gamma(\frac{p+h_-}{2})}{\Gamma(\frac{p+1+h_+}{2})\Gamma(\frac{p+1+h_-}{2})}} \left[1 + \mathrm{i}\left(n^2 - \frac{1}{4}\right)\frac{1}{2p}\cot\eta\right]. \tag{B.1}$$

The approximation is only valid as long as $\frac{|n^2|}{p}\cot\eta \ll 1$ and $p^2 \gg |n|^2$.

For early and late time I use the Kummer relations between different solutions of the hypergeometric differential equation [122], 2.9. to continue the hypergeometric function outside their primary domain of definition. I use [122]2.11.(22) and 2.9.(34),(9,13) to get

$$\begin{aligned}
g(\eta) =& {}_2F_1(h_+, h_-, p+1, \frac{1}{2} - \mathrm{i}\frac{\cot\eta}{2}) = \\
=& \left(\frac{1}{2} + \frac{\mathrm{i}}{2}\cot\eta\right)^p {}_2F_1\left(\frac{p+1-h_+}{2}, \frac{p+h_+}{2}, p+1, \frac{1}{\sin^2\eta}\right) = \\
=& 2^{-p}\Gamma(p+1)\mathrm{e}^{-\mathrm{i}p\eta}(\mathrm{i}\sin\eta)^{\frac{1}{2}} \\
& \left[\frac{\Gamma(n)}{\Gamma(\frac{p+1+h_+}{2})\Gamma(\frac{p+h_+}{2})}(\mathrm{i}\sin\eta)^{-n} {}_2F_1\left(\frac{p+h_-}{2}, \frac{-p+h_-}{2}, 1-n, \sin^2\eta\right) + \right. \\
& \left. \frac{\Gamma(-n)}{\Gamma(\frac{p+1+h_-}{2})\Gamma(\frac{p+h_-}{2})}(\mathrm{i}\sin\eta)^{n} {}_2F_1\left(\frac{p+h_+}{2}, \frac{-p+h_+}{2}, 1+n, \sin^2\eta\right)\right] = \\
\doteq& \sqrt{\frac{1}{2\pi}}\Gamma(p+1)\mathrm{e}^{-\mathrm{i}p\eta}(\mathrm{i}\sin\eta)^{\frac{1}{2}} \\
& \left[\frac{\Gamma(n)}{\Gamma(p+h_+)}\left(\frac{\mathrm{i}}{2}\sin\eta\right)^{-n}\left(1 - \frac{p^2-(1/2-n)^2}{4(1-n)}\sin^2\eta\right) + \right. \\
& \left. \frac{\Gamma(-n)}{\Gamma(p+1-h_+)}\left(\frac{\mathrm{i}}{2}\sin\eta\right)^{n}\left(1 - \frac{p^2-(1/2+n)^2}{4(1+n)}\sin^2\eta\right)\right],
\end{aligned} \tag{B.2}$$

and the approximated mode functions are

$$\begin{aligned}
w_k =& a^{\frac{1-D}{2}} \frac{\sqrt{\mathrm{i}}}{2\sqrt{\pi}} \sqrt{\Gamma(p+h_+)\Gamma(p+h_-)} \\
& \left[\frac{\Gamma(n)}{\Gamma(p+h_+)}\left(\frac{\mathrm{i}}{2}\sin\eta\right)^{-n}\left(1 - \frac{p^2-(1/2-n)^2}{4(1-n)}\sin^2\eta\right) + \right. \\
& \left. \frac{\Gamma(-n)}{\Gamma(p+h_-)}\left(\frac{\mathrm{i}}{2}\sin\eta\right)^{n}\left(1 - \frac{p^2-(1/2+n)^2}{4(1+n)}\sin^2\eta\right)\right].
\end{aligned} \tag{B.3}$$

This is valid as long as $\sin^2\eta \frac{|p^2-(1/2-n)^2|}{4|1+n|} \ll 1$.

B.6. Approximate time integrals

To calculate the time integrals of section 4.3.2 I introduce the following auxiliary integrals:

$$K_1(\tau,\alpha) = \int^{\tau} \sin^{\alpha}\eta \mathrm{d}\eta,$$
$$K_2(\tau,\alpha,\beta) = \int^{\tau} \sin^{\alpha}\eta \mathrm{e}^{\mathrm{i}\beta\eta}\mathrm{d}\eta.$$

These integrals can be evaluated to give

$$K_1(\eta,\alpha) = -\cos\eta\sin\eta^{1+\alpha}{}_2F_1\left(1, 1+\frac{\alpha}{2}, \frac{3}{2}, \cos^2\eta\right),$$
$$K_2(\eta,\alpha,\beta) = 2^{-\alpha}\mathrm{e}^{\mathrm{i}(\beta+1)\eta}(2\sin\eta)^{1+\alpha}\frac{{}_2F_1\left(1, \frac{\alpha+\beta}{2}+1, \frac{\beta-\alpha}{2}+1, \mathrm{e}^{2\mathrm{i}\eta}\right)}{\alpha-\beta}.$$

I use the approximations from B.5 to calculate the integrals in the two cases. For my current purpose, the leading order suffices.

1) All modes are super-horizon at all intermediate times, $p_i < p_\xi = \sqrt{\frac{|n^2-1|}{\sin^2\xi}}$. We get:

$$M_1(\eta) = \int^{\eta} \mathrm{d}\xi\ a(\xi)^D f_{k_1}(\xi) f_{k_2}(\xi) f_{k_3}(\xi) =$$
$$= \frac{1}{8\pi\sqrt{\pi}}(\mathrm{i})^{3/2}\int^{\eta}\mathrm{d}\xi\ a^{3-\frac{D}{2}}\sin^{3/2}\xi$$
$$\prod_{j=1,2,3}\sqrt{\Gamma(p_j+h_+)\Gamma(p_j+h_-)}\left[\frac{\Gamma(n)}{\Gamma(p_j+h_+)}\left(\frac{\mathrm{i}}{2}\sin\xi\right)^{-n}\left(1-\frac{p_j^2-(1/2-n)^2}{4(1-n)}\sin^2\xi\right)+\right.$$
$$\left.\frac{\Gamma(-n)}{\Gamma(p_j+1-h_+)}\left(\frac{\mathrm{i}}{2}\sin\xi\right)^{n}\left(1-\frac{p_j^2-(1/2+n)^2}{4(1+n)}\sin^2\xi\right)\right].$$

Using the auxiliary integrals, we can express M_1 as

$$M_1 = \frac{1}{8}(-\mathrm{i}\pi)^{3/2}\prod_j\sqrt{\Gamma(p_j+h_+)\Gamma(p_j+h_-)}$$
$$\left[\prod_j\frac{\Gamma(n)}{\Gamma(p_j+h_+)}\left(\frac{\mathrm{i}}{2}\right)^{-3n}K_1\left(\eta,\frac{D-3}{2}-3n\right)+\right.$$
$$\left.+\sum_{j=1,2,3}\frac{\mathrm{i}}{2}^{-n}\frac{\Gamma(-n)}{\Gamma(p_j+h_-)}\prod_{l\neq j}\frac{\Gamma(n)}{\Gamma(p_l+h_+)}K_1\left(\eta,\frac{D-3}{2}-n\right)+(n\leftrightarrow -n)\right].\tag{B.4}$$

2) One mode is super-horizon ($p_1 < p_\xi$), and two are sub-horizon, ($p_2, p_3 > p_\xi$) in the integration range.

In this case we have.

$$M_2(\eta) = \frac{\sqrt{\mathrm{i}}}{8(\pi)^{3/2}}\sqrt{\Gamma(p_1+h_+)\Gamma(p_1+h_-)}\prod_{l=2,3}\sqrt{\frac{\Gamma(\frac{p_l+h_+}{2})\Gamma(\frac{p_l+h_-}{2})}{\Gamma(\frac{p_l+1+h_+}{2})\Gamma(\frac{p_l+1+h_-}{2})}}$$

$$\int^{\eta} \mathrm{d}\xi\ a^{3-\frac{D}{2}}(\sin\xi)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}(p_2+p_3)\xi}\left[1+\mathrm{i}\left(n^2-\frac{1}{4}\right)\frac{1}{2p_2}\cot\xi\right]\left[1+\mathrm{i}\left(n^2-\frac{1}{4}\right)\frac{1}{2p_3}\cot\xi\right]$$

$$\left[\frac{\Gamma(n)}{\Gamma(p_1+h_+)}\left(\frac{\mathrm{i}}{2}\sin\xi\right)^{-n}\left(1-\frac{p_1^2-(1/2-n)^2}{4(1-n)}\sin^2\xi\right)+\right.$$

$$\left.+\frac{\Gamma(-n)}{\Gamma(p_1+1-h_+)}\left(\frac{\mathrm{i}}{2}\sin\xi\right)^{n}\left(1-\frac{p_1^2-(1/2+n)^2}{4(1+n)}\sin^2\xi\right)\right] =$$

$$= \frac{\sqrt{\mathrm{i}}}{8(\pi)^{3/2}}2\pi\prod_i\sqrt{\Gamma(p_i+h_+)\Gamma(p_i+h_-)}\prod_{l=2,3}\frac{2^{-p_l}}{\Gamma(\frac{p_l+1+h_+}{2})\Gamma(\frac{p_l+1+h_-}{2})}$$

$$\left[\frac{\Gamma(n)}{\Gamma(p_1+h_+)}\left(\frac{\mathrm{i}}{2}\right)^{-n}K_2\left(\eta,\frac{D-5}{2}-n,p_2+p_3\right)+(n\leftrightarrow -n)\right].$$

3) All modes are sub-horizon. This integral is given by

$$M_3(\eta) = \frac{1}{8(\pi)^{3/2)}}\prod_j\sqrt{\frac{\Gamma(\frac{p_j+h_+}{2})\Gamma(\frac{p_j+h_-}{2})}{\Gamma(\frac{p_j+1+h_+}{2})\Gamma(\frac{p_j+1+h_-}{2})}}\int^{\eta}\mathrm{d}\xi\ a^{3-\frac{D}{2}}\mathrm{e}^{\mathrm{i}\xi\sum_j p_j}\prod_j\left[1+\mathrm{i}\left(n^2-\frac{1}{4}\right)\frac{1}{2p_j}\cot\xi\right] =$$

$$= \frac{1}{8(\pi)^{3/2)}}\prod_j\sqrt{\frac{\Gamma(\frac{p_j+h_+}{2})\Gamma(\frac{p_j+h_-}{2})}{\Gamma(\frac{p_j+1+h_+}{2})\Gamma(\frac{p_j+1+h_-}{2})}}K_2\left(\eta,\frac{D}{2}-3,\sum_l p_l\right).$$

Summarizing we can express the coefficient A in the following way for the different range of momenta

1) $p_1 < p_2 < p_\eta$:

$$A_1 = |M_1(\eta) - M_1(\epsilon)|^2 . \tag{B.5}$$

2) $p_1 < p_\eta < p_2 < p_\epsilon$:

$$A_2 = |M_2(\eta) - M_2(\eta_{p_2}) + M_1(\eta_{p_2}) - M_1(\epsilon)|^2 . \tag{B.6}$$

3) $p_1 < p_\eta < p_\epsilon < p_2$:

$$A_3 = |M_2(\eta) - M_2(\epsilon)|^2 . \tag{B.7}$$

4) $p_\eta < p_1 < p_2 < p_\epsilon$:

$$A_4 = |M_3(\eta) - M_3(\eta_{p_1}) + M_2(\eta_{p_1}) - M_2(\eta_{p_2}) + M_1(\eta_{p_2}) - M_1(\epsilon)|^2 . \tag{B.8}$$

5) $p_\eta < p_1 < p_\epsilon < p_2$:

$$A_5 = |M_3(\eta) - M_3(\eta_{p_1}) + M_2(\eta_{p_1}) - M_2(\epsilon)|^2 . \tag{B.9}$$

I now give the approximations for the functions M_1, M_2, M_3 for the times to be evaluated. My interest is in very early final times ($\eta \approx 0$), final times around the bounce ($\eta \approx \frac{\pi}{2}$) and very late final times ($\eta \approx \pi$). Making excessive use of the Kummer relation in [122] we find the following approximations.

- For K_1 for very early or very late times I use [122]2.9.(33,5,22) to get

$$K_1(\eta,\alpha) = -\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}+\frac{\alpha}{2})}{\Gamma(1)\Gamma(1+\frac{\alpha}{2})} - \cos\eta\sin\eta^{1+\alpha}\frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}-\frac{\alpha}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\frac{\alpha}{2})}{}_2F_1\left(1, 1+\frac{\alpha}{2}, \frac{3}{2}+\frac{\alpha}{2}, \sin^2\eta\right)$$
$$= -\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}+\frac{\alpha}{2})}{\Gamma(1)\Gamma(1+\frac{\alpha}{2})} + \frac{1}{1+\alpha}\cos\eta\sin\eta^{1+\alpha}\left(1+\frac{2+\alpha}{3+\alpha}\sin^2\eta + \mathcal{O}(\sin^4\eta)\right).$$

- For K_1 around the bounce we get

$$K_1(\eta,\alpha) = -\cos\eta\sin\eta^{1+\alpha}{}_2F_1\left(1, 1+\frac{\alpha}{2}, \frac{3}{2}, \cos^2\eta\right) =$$
$$= -\cos\eta\sin\eta^{1+\alpha}\left(1+\frac{2+\alpha}{3}\cos^2\eta + \mathcal{O}(\cos^4\eta)\right).$$

- For K_2 for very early or very late times I use [122]2.9.(34,11,14) to get

$$K_2(\eta,\alpha,\beta) = 2\mathrm{e}^{\mathrm{i}(\beta+1)\eta}\left(\sin\eta\right)^{1+\alpha}\frac{{}_2F_1\left(1, \frac{\alpha+\beta}{2}+1, \frac{\beta-\alpha}{2}+1, \mathrm{e}^{2\mathrm{i}\eta}\right)}{\alpha-\beta} =$$
$$= (-2\mathrm{i})^{-\alpha-1}\frac{\Gamma\left(\frac{\beta-\alpha}{2}\right)\Gamma(1+\alpha)}{\Gamma\left(\frac{\alpha+\beta}{2}+1\right)} + \frac{\mathrm{e}^{\mathrm{i}(\beta+1)\eta}\left(\sin\eta\right)^{1+\alpha}}{\alpha+1}\left(1-\mathrm{i}\frac{\alpha+\beta+2}{2+\alpha}\mathrm{e}^{\mathrm{i}\eta}\sin\eta + \mathcal{O}(\sin^2\eta)\right).$$

- For K_2 around the bounce I use [122]2.9.(34,11,14), which yields

$$K_2(\eta,\alpha,\beta) = \frac{2^{-\alpha}}{\alpha-\beta}\mathrm{e}^{\mathrm{i}(\beta+1)\eta}\left(2\sin\eta\right)^{1+\alpha}\left(\frac{\Gamma\left(\frac{\beta-\alpha}{2}+1\right)\Gamma(-\frac{\alpha+\beta}{2})}{\Gamma(-\alpha)}(-\mathrm{e}^{2\mathrm{i}\eta})^{\frac{\alpha-\beta}{2}}(1-\mathrm{e}^{2\mathrm{i}\eta})^{-1-\alpha} + \right.$$
$$\left. + \frac{\Gamma\left(\frac{\beta-\alpha}{2}+1\right)\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(\frac{\beta-\alpha}{2})\Gamma(\frac{\alpha+\beta}{2}+1)}(1-\mathrm{e}^{2\mathrm{i}\eta})^{-1}{}_2F_1\left(1, -\alpha, 1-\frac{\alpha+\beta}{2}, \frac{1}{2}+\frac{\mathrm{i}}{2}\cot\eta\right)\right) =$$
$$= \frac{2^{-\alpha}}{\alpha-\beta}(\mathrm{i})^{-\beta}(-1)^{-1-\alpha}\frac{\Gamma\left(\frac{\beta-\alpha}{2}+1\right)\Gamma(-\frac{\alpha+\beta}{2})}{\Gamma(-\alpha)} -$$
$$- \frac{\mathrm{i}}{\alpha+\beta}\mathrm{e}^{\mathrm{i}\beta\eta}\left(\sin\eta\right)^{\alpha}{}_2F_1\left(1, -\alpha, 1-\frac{\alpha+\beta}{2}, \frac{1}{2}+\frac{\mathrm{i}}{2}\cot\eta\right).$$

  As K_2 is always evaluated for times $\eta > \eta_{p_2}$ and the coefficient $\beta \propto p_2$ I can use the series approximation for the hypergeometric functions. This gives the result:

$$K_2(\eta,\alpha,\beta) = \frac{2^{-\alpha}}{\alpha-\beta}(\mathrm{i})^{-\beta}(-1)^{-1-\alpha}\frac{\Gamma\left(\frac{\beta-\alpha}{2}+1\right)\Gamma(-\frac{\alpha+\beta}{2})}{\Gamma(-\alpha)} -$$
$$- \frac{\mathrm{i}}{\alpha+\beta}\mathrm{e}^{\mathrm{i}\beta\eta}\left(\sin\eta\right)^{\alpha}\left(1-\mathrm{i}\frac{\alpha}{2-\alpha+\beta}\frac{e^{-\mathrm{i}\eta}}{\sin\eta} + \mathcal{O}(\beta^{-2}\sin\eta^{-2})\right).$$

The above approximations can be used to get the leading order results for the coefficients in the cases (B.5), (B.6), (B.8). The final time can be so early, that some modes with momenta p_2 and p_3 enter the horizon at some time, but the modes with momenta p_1 are still outside the horizon at the final time η. In this case (B.5) and (B.6) are relevant.

1) If the final time is so early, that no modes enter the horizon and all stay super-horizon we get with

(B.5)

$$\begin{aligned}
A_1 &= |M_1(\eta) - M_1(\epsilon)|^2 = \\
&= \frac{\pi^3}{64}\prod_j \Gamma(p_j+h_+)\Gamma(p_j+h_-)\Bigg|\cos^2\eta\sin\eta^{D-1}\Bigg[\prod_j \frac{\Gamma(n)}{\Gamma(p_j+h_+)}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{\sin\eta^{-3n}}{\frac{D-1}{2}-3n}+ \\
&\quad + \sum_{j=1,2,3}\left(\frac{\mathrm{i}}{2}\right)^{-n}\frac{\Gamma(-n)}{\Gamma(p_j+h_-)}\prod_{l\neq j}\frac{\Gamma(n)}{\Gamma(p_l+h_+)}\frac{\sin\eta^{-n}}{\frac{D-1}{2}-n}\Bigg]- \\
&\quad - \cos^2\epsilon\sin\epsilon^{D-1}\Bigg[\prod_j \frac{\Gamma(n)}{\Gamma(p_j+h_+)}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{\sin\epsilon^{-3n}}{\frac{D-1}{2}-3n}+ \\
&\quad + \sum_{j=1,2,3}\left(\frac{\mathrm{i}}{2}\right)^{-n}\frac{\Gamma(-n)}{\Gamma(p_j+h_-)}\prod_{l\neq j}\frac{\Gamma(n)}{\Gamma(p_l+h_+)}\frac{\sin\epsilon^{-n}}{\frac{D-1}{2}-n}\Bigg]+ \\
&\quad + (n\leftrightarrow -n)\Bigg|^2 .
\end{aligned} \tag{B.10}$$

The dependence on the initial time ϵ vanishes in the limit $\epsilon \to 0$.

2) When the high momenta modes are inside the horizon at the final time, (B.6) is valid with

$$\begin{aligned}
A_2 &= |M_2(\eta) - M_2(\eta_{p_2}) + M_1(\eta_{p_2}) - M_1(\epsilon)|^2 = \\
&= \frac{\pi^3}{64}\prod_i \Gamma(p_i+h_+)\Gamma(p_i+h_-)\Bigg|\frac{2}{\pi^2}\prod_{l=2,3}\frac{2^{-p_l}}{\Gamma(\frac{p_l+1+h_+}{2})\Gamma(\frac{p_l+1+h_-}{2})}\frac{\Gamma(n)}{\Gamma(p_1+h_+)} \\
&\left(\frac{\mathrm{i}}{2}\right)^{-n}\frac{\mathrm{i}}{p_2+p_3+\frac{D-5}{2}-n}\left(\mathrm{e}^{\mathrm{i}(p_2+p_3)\eta}\left(\sin\eta\right)^{\frac{D-5}{2}-n}-\mathrm{e}^{\mathrm{i}(p_2+p_3)\eta_{p_2}}\left(\sin\eta_{p_2}\right)^{\frac{D-5}{2}-n}\right)+ \\
&\quad + \cos\eta_{p_2}\sin\eta_{p_2}^{\frac{D-1}{2}}\Bigg[\prod_j \frac{\Gamma(n)}{\Gamma(p_j+h_+)}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{\sin\eta_{p_2}^{-3n}}{\frac{D-1}{2}-3n}+ \\
&\quad + \sum_{j=1,2,3}\left(\frac{\mathrm{i}}{2}\right)^{-n}\frac{\Gamma(-n)}{\Gamma(p_j+h_-)}\prod_{l\neq j}\frac{\Gamma(n)}{\Gamma(p_l+h_+)}\frac{\sin\eta_{p_2}^{-n}}{\frac{D-1}{2}-n}\Bigg]- \\
&\quad - \cos^2\epsilon\sin\epsilon^{D-1}\Bigg[\prod_j \frac{\Gamma(n)}{\Gamma(p_j+h_+)}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{\sin\epsilon^{-3n}}{\frac{D-1}{2}-3n}+ \\
&\quad + \sum_{j=1,2,3}\left(\frac{\mathrm{i}}{2}\right)^{-n}\frac{\Gamma(-n)}{\Gamma(p_j+h_-)}\prod_{l\neq j}\frac{\Gamma(n)}{\Gamma(p_l+h_+)}\frac{\sin\epsilon^{-n}}{\frac{D-1}{2}-n}\Bigg]+(n\leftrightarrow -n)\Bigg|^2 .
\end{aligned} \tag{B.11}$$

3) If the high momenta modes are always inside the horizon, even at the initial time, we get with (B.7)

$$\begin{aligned}
A_3 &= |M_2(\eta) - M_2(\epsilon)|^2 = \\
&= \frac{\pi^3}{64}\prod_i \Gamma(p_i+h_+)\Gamma(p_i+h_-)\Bigg|\frac{2}{\pi^2}\prod_{l=2,3}\frac{2^{-p_l}}{\Gamma(\frac{p_l+1+h_+}{2})\Gamma(\frac{p_l+1+h_-}{2})}\frac{\Gamma(n)}{\Gamma(p_1+h_+)}\left(\frac{\mathrm{i}}{2}\right)^{-n} \\
&\frac{\mathrm{i}}{p_2+p_3+\frac{D-5}{2}-n}\left(\mathrm{e}^{\mathrm{i}(p_2+p_3)\eta}\left(\sin\eta\right)^{\frac{D-5}{2}-n}-\mathrm{e}^{\mathrm{i}(p_2+p_3)\epsilon}\left(\sin\epsilon\right)^{\frac{D-5}{2}-n}\right)+(n\leftrightarrow -n)\Bigg|^2 .
\end{aligned} \tag{B.12}$$

4) In this case the universe has contracted by such an amount that all modes are inside the horizon at

the final time but are outside at the initial time, so (B.8) is valid. We get

$$\begin{aligned}
A_4 &= \left|M_3(\eta) - M_3(\eta_{p_1}) + M_2(\eta_{p_1}) - M_2(\eta_{p_2}) + M_1(\eta_{p_2}) - M_1(\epsilon)\right|^2 = \\
&= \frac{\pi^3}{64} \prod_i \Gamma(p_i + h_+)\Gamma(p_i + h_-) \\
&\left| \frac{e^{i\pi/4}\sqrt{2}}{\pi^{3/2}} \frac{1}{\sum p_i + \frac{D-6}{2}} \prod_j \left(\frac{2^{-p_j}}{\Gamma(\frac{p_j+1+h_+}{2})\Gamma(\frac{p_j+1+h_-}{2})} \right) \left(e^{i\eta \sum p_i} (\sin\eta)^{\frac{D-6}{2}} - e^{i\eta_{p_1} \sum p_i} (\sin\eta_{p_1})^{\frac{D-6}{2}} \right) + \right. \\
&\quad + \frac{2}{\pi^2} \prod_{l=2,3} \left(\frac{2^{-p_l}}{\Gamma(\frac{p_l+1+h_+}{2})\Gamma(\frac{p_l+1+h_-}{2})} \right) \frac{\Gamma(n)}{\Gamma(p_1+h_+)} \left(\frac{i}{2}\right)^{-n} \frac{i}{p_2+p_3+\frac{D-5}{2}-n} \times \\
&\quad \times \left(e^{i(p_2+p_3)\eta_{p_1}} (\sin\eta_{p_1})^{\frac{D-5}{2}-n} - e^{i(p_2+p_3)\eta_{p_2}} (\sin\eta_{p_2})^{\frac{D-5}{2}-n} \right) + \\
&\quad + \cos\eta_{p_2} \sin\eta_{p_2}^{\frac{D-1}{2}} \left(\prod_j \frac{\Gamma(n)}{\Gamma(p_j+h_+)} \left(\frac{i}{2}\right)^{-3n} \frac{\sin\eta_{p_2}^{-3n}}{\frac{D-1}{2}-3n} + \right. \\
&\quad \left. + \sum_{j=1,2,3} \left(\frac{i}{2}\right)^{-n} \frac{\Gamma(-n)}{\Gamma(p_j+h_-)} \prod_{l\neq j} \frac{\Gamma(n)}{\Gamma(p_l+h_+)} \frac{\sin\eta_{p_2}^{-n}}{\frac{D-1}{2}-n} \right) - \\
&\quad - \cos^2\epsilon \sin\epsilon^{D-1} \left[\prod_j \frac{\Gamma(n)}{\Gamma(p_j+h_+)} \left(\frac{i}{2}\right)^{-3n} \frac{\sin\epsilon^{-3n}}{\frac{D-1}{2}-3n} + \right. \\
&\quad \left.\left. + \sum_{j=1,2,3} \left(\frac{i}{2}\right)^{-n} \frac{\Gamma(-n)}{\Gamma(p_j+h_-)} \prod_{l\neq j} \frac{\Gamma(n)}{\Gamma(p_l+h_+)} \frac{\sin\epsilon^{-n}}{\frac{D-1}{2}-n} \right] + (n \leftrightarrow -n) \right|^2 .
\end{aligned} \tag{B.13}$$

5) In the last case only the p_1 mode crosses the horizon during the evolution of the universe, whereas the other two modes are always inside. With (B.9) we have

$$\begin{aligned}
A_5 &= \left|M_3(\eta) - M_3(\eta_{p_1}) + M_2(\eta_{p_1}) - M_2(\epsilon)\right|^2 = \\
&= \frac{\pi^3}{64} \prod_i \Gamma(p_i + h_+)\Gamma(p_i + h_-) \\
&\left| \frac{e^{i\pi/4}\sqrt{2}}{\pi^{3/2}} \frac{1}{\sum p_i + \frac{D-6}{2}} \prod_j \left(\frac{2^{-p_j}}{\Gamma(\frac{p_j+1+h_+}{2})\Gamma(\frac{p_j+1+h_-}{2})} \right) \left(e^{i\eta \sum p_i} (\sin\eta)^{\frac{D-6}{2}} - e^{i\eta_{p_1} \sum p_i} (\sin\eta_{p_1})^{\frac{D-6}{2}} \right) + \right. \\
&\quad + \frac{2}{\pi^2} \prod_{l=2,3} \left(\frac{2^{-p_l}}{\Gamma(\frac{p_l+1+h_+}{2})\Gamma(\frac{p_l+1+h_-}{2})} \right) \frac{\Gamma(n)}{\Gamma(p_1+h_+)} \left(\frac{i}{2}\right)^{-n} \frac{i}{p_2+p_3+\frac{D-5}{2}-n} \times \\
&\quad \left. \times \left(e^{i(p_2+p_3)\eta_{p_1}} (\sin\eta_{p_1})^{\frac{D-5}{2}-n} - e^{i(p_2+p_3)\epsilon} (\sin\epsilon)^{\frac{D-5}{2}-n} \right) + (n \leftrightarrow -n) \right|^2 .
\end{aligned} \tag{B.14}$$

I can now insert the approximated coefficients into (4.10). To be able to discuss the momentum behaviour of the first order correction to the Green function I still have to perform the sum over the internal momenta p_2, p_3. As we can see from the above structure of the coefficients, these sums can in general not be performed analytically. But as I am interested in large internal and small external momenta I can take the limit $p_1 \ll p_2 \approx p_3 \to \infty$, keep only the leading order in momenta and replace the sums by integrals after that.

The Gamma functions can be approximated for large arguments using [122]1.18(4)

$$\frac{\Gamma(p+h_+)}{\Gamma(p+h_-)} = p^{2n}\left(1+\mathcal{O}(p^{-1})\right). \tag{B.15}$$

Moreover I use [122]1.3(15) to get

$$\begin{aligned}
\frac{1}{\Gamma(\frac{p+\frac{3}{2}+n}{2})\Gamma(\frac{p+\frac{3}{2}-n}{2})} &= \frac{1}{\Gamma(\frac{p+\frac{3}{2}+n}{2})\Gamma(\frac{p+\frac{3}{2}+n}{2}+\frac{1}{2})}\frac{\Gamma(\frac{p+\frac{3}{2}+n}{2}+\frac{1}{2})}{\Gamma(\frac{p+\frac{3}{2}-n}{2})} = \\
&= \frac{2^{p+\frac{1}{2}+n}}{\sqrt{\pi}}\frac{1}{\Gamma(p+\frac{3}{2}+n)}\frac{\Gamma(\frac{p+\frac{3}{2}+n}{2}+\frac{1}{2})}{\Gamma(\frac{p+\frac{3}{2}-n}{2})} = \\
&\approx \frac{2^{p+\frac{1}{2}+n}}{\sqrt{\pi}}\frac{1}{\Gamma(p+\frac{3}{2}+n)}\left(\frac{p}{2}\right)^{\frac{1}{2}+n}\left(1+\mathcal{O}(p^{-1})\right) = \\
&= \frac{2^{p+\frac{1}{2}+n}}{\sqrt{\pi}}\frac{1}{(p+h_+)\Gamma(p+h_+)}\left(\frac{p}{2}\right)^{\frac{1}{2}+n} = \\
&= \sqrt{\frac{2^{p+\frac{1}{2}+n}}{\sqrt{\pi}}\frac{1}{(p+h_+)\Gamma(p+h_+)}\left(\frac{p}{2}\right)^{\frac{1}{2}+n}}\sqrt{\frac{2^{p+\frac{1}{2}-n}}{\sqrt{\pi}}\frac{1}{(p+h_-)\Gamma(p+h_-)}\left(\frac{p}{2}\right)^{\frac{1}{2}-n}} = \\
&= 2^p\sqrt{\frac{p}{\pi}\frac{1}{(p+h_+)(p+h_-)}\frac{1}{\Gamma(p+h_+)\Gamma(p+h_-)}},
\end{aligned} \tag{B.16}$$

where I have used the symmetry in n in the last steps. The sine functions at horizon crossing are given by (4.20), $\sin\eta_{p_2} = \sqrt{\frac{|n^2-1|}{p_2^2}}$.

Using these approximations for the coefficients in (B.10), (B.11) and (B.13) we get the following:

$$\begin{aligned}
A_1 = \frac{\pi^3}{64}\Bigg|&\cos\eta\sin\eta^{\frac{D-1}{2}}\left[\Gamma(n)^3(p_2)^{-2n}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{\sin\eta^{-3n}}{\frac{D-1}{2}-3n}+\right.\\
&\left.+\left(\frac{\mathrm{i}}{2}\right)^{-n}\Gamma(-n)\Gamma(n)^2\frac{\sin\eta^{-n}}{\frac{D-1}{2}-n}\left(\sqrt{\frac{\Gamma(p_1+h_+)}{\Gamma(p_1+h_-)}}(p_2)^{-2n}+2\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\right)\right]-\\
&-\cos\epsilon\sin\epsilon^{\frac{D-1}{2}}\left[\Gamma(n)^3(p_2)^{-2n}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{\sin\epsilon^{-3n}}{\frac{D-1}{2}-3n}+\right.\\
&\left.+\left(\frac{\mathrm{i}}{2}\right)^{-n}\Gamma(-n)\Gamma(n)^2\frac{\sin\epsilon^{-n}}{\frac{D-1}{2}-n}\left(\sqrt{\frac{\Gamma(p_1+h_+)}{\Gamma(p_1+h_-)}}(p_2)^{-2n}+2\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\right)\right]+(n\leftrightarrow -n)\Bigg|^2,
\end{aligned} \tag{B.17}$$

$$A_2 = \frac{\pi^3}{64}\Bigg|\frac{2p_2}{\pi^3}\frac{\Gamma(n)}{(p_2+h_+)(p_2+h_-)}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}$$
$$\left(\frac{\mathrm{i}}{2}\right)^{-n}\frac{\mathrm{i}}{p_2+p_3+\frac{D-5}{2}-n}\left(\mathrm{e}^{\mathrm{i}(p_2+p_3)\eta}\left(\sin\eta\right)^{\frac{D-5}{2}-n}-\mathrm{e}^{\mathrm{i}(p_2+p_3)\eta_{p_2}}|n^2-1|^{\frac{D-5}{4}-\frac{n}{2}}p_2^{n-\frac{D-5}{2}}\right)+$$
$$+|n^2-1|^{\frac{D-1}{4}}p_2^{-\frac{D-1}{2}}\Bigg[\Gamma(n)^3(p_2)^{-2n}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{|n^2-1|^{-3n}p_2^{3n}}{\frac{D-1}{2}-3n}+$$
$$+\left(\frac{\mathrm{i}}{2}\right)^{-n}\Gamma(-n)\Gamma(n)^2\frac{|n^2-1|^{-n}p_2^{n}}{\frac{D-1}{2}-n}\left(\sqrt{\frac{\Gamma(p_1+h_+)}{\Gamma(p_1+h_-)}}(p_2)^{-2n}+2\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\right)\Bigg]- \tag{B.18}$$
$$-\cos\epsilon\sin\epsilon^{\frac{D-1}{2}}\Bigg[\Gamma(n)^3(p_2)^{-2n}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{\sin\epsilon^{-3n}}{\frac{D-1}{2}-3n}+$$
$$+\left(\frac{\mathrm{i}}{2}\right)^{-n}\Gamma(-n)\Gamma(n)^2\frac{\sin\epsilon^{-n}}{\frac{D-1}{2}-n}\left(\sqrt{\frac{\Gamma(p_1+h_+)}{\Gamma(p_1+h_-)}}(p_2)^{-2n}+2\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\right)\Bigg]+(n\leftrightarrow -n)\Bigg|^2,$$

$$A_3 = \frac{\pi^3}{64}\Bigg|\frac{2p_2}{\pi^3}\frac{\Gamma(n)}{(p_2+h_+)(p_2+h_-)}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\left(\frac{\mathrm{i}}{2}\right)^{-n}\frac{\mathrm{i}}{p_2+p_3+\frac{D-5}{2}-n}\times$$
$$\times\left(\mathrm{e}^{\mathrm{i}(p_2+p_3)\eta}\left(\sin\eta\right)^{\frac{D-5}{2}-n}-\mathrm{e}^{\mathrm{i}(p_2+p_3)\epsilon}\left(\sin\epsilon\right)^{\frac{D-5}{2}-n}\right)+(n\leftrightarrow -n)\Bigg|^2, \tag{B.19}$$

$$A_4 = \frac{\pi^3}{64}\Bigg|\frac{\mathrm{e}^{-\mathrm{i}\pi/2}\sqrt{2}}{\pi^{3/2}}\frac{1}{\sum p_i+\frac{D-6}{2}}\frac{p_2}{\pi}\frac{1}{(p_2+h_+)(p_2+h_-)}\frac{2^{p_1}}{\sqrt{2\pi}}\sqrt{\Gamma\left(\frac{p_1+h_+}{2}\right)\Gamma\left(\frac{p_1+h_-}{2}\right)}\times$$
$$\times\left(\mathrm{e}^{\mathrm{i}\eta\sum p_i}\left(\sin\eta\right)^{\frac{D-6}{2}}-\mathrm{e}^{\mathrm{i}\eta_{p_1}\sum p_i}\left(\sin\eta_{p_1}\right)^{\frac{D-6}{2}}\right)+$$
$$+\frac{2p_2}{\pi^3}\frac{\Gamma(n)}{(p_2+h_+)(p_2+h_-)}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\left(\frac{\mathrm{i}}{2}\right)^{-n}\frac{\mathrm{i}}{p_2+p_3+\frac{D-5}{2}-n}\times$$
$$\times\left(\mathrm{e}^{\mathrm{i}(p_2+p_3)\eta_{p_1}}\left(\sin\eta_{p_1}\right)^{\frac{D-5}{2}-n}-\mathrm{e}^{\mathrm{i}(p_2+p_3)\eta_{p_2}}|n^2-1|^{\frac{D-3}{4}}p_2^{-\frac{D-5}{2}-n}\right)+$$
$$+|n^2-1|^{\frac{D-1}{4}}p_2^{-\frac{D-1}{2}}\Bigg[\Gamma(n)^3(p_2)^{-2n}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{|n^2-1|^{-3n}p_2^{3n}}{\frac{D-1}{2}-3n}+ \tag{B.20}$$
$$+\left(\frac{\mathrm{i}}{2}\right)^{-n}\Gamma(-n)\Gamma(n)^2\frac{|n^2-1|^{-n}p_2^{n}}{\frac{D-1}{2}-n}\left(\sqrt{\frac{\Gamma(p_1+h_+)}{\Gamma(p_1+h_-)}}(p_2)^{-2n}+2\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\right)\Bigg]-$$
$$-\cos\epsilon\sin\epsilon^{\frac{D-1}{2}}\Bigg[\Gamma(n)^3(p_2)^{-2n}\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\left(\frac{\mathrm{i}}{2}\right)^{-3n}\frac{\sin\epsilon^{-3n}}{\frac{D-1}{2}-3n}+$$
$$+\left(\frac{\mathrm{i}}{2}\right)^{-n}\Gamma(-n)\Gamma(n)^2\frac{\sin\epsilon^{-n}}{\frac{D-1}{2}-n}\left(\sqrt{\frac{\Gamma(p_1+h_+)}{\Gamma(p_1+h_-)}}(p_2)^{-2n}+2\sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}}\right)\Bigg]+(n\leftrightarrow -n)\Bigg|^2,$$

$$
\begin{aligned}
A_5 = \frac{\pi^3}{64}\Bigg| & \frac{e^{-i\pi/2}\sqrt{2}}{\pi^{3/2}} \frac{1}{\sum p_i + \frac{D-6}{2}} \frac{p_2}{\pi} \frac{1}{(p_2+h_+)(p_2+h_-)} \frac{2^{p_1}}{\sqrt{2\pi}} \sqrt{\Gamma\left(\frac{p_1+h_+}{2}\right)\Gamma\left(\frac{p_1+h_-}{2}\right)} \times \\
& \times \left(e^{i\eta\sum p_i}(\sin\eta)^{\frac{D-6}{2}} - e^{i\eta_{p_1}\sum p_i}(\sin\eta_{p_1})^{\frac{D-6}{2}}\right) + \\
& + \frac{2p_2}{\pi^3} \frac{\Gamma(n)}{(p_2+h_+)(p_2+h_-)} \sqrt{\frac{\Gamma(p_1+h_-)}{\Gamma(p_1+h_+)}} \left(\frac{i}{2}\right)^{-n} \frac{i}{p_2+p_3+\frac{D-5}{2}-n} \times \\
& \times \left(e^{i(p_2+p_3)\eta_{p_1}}(\sin\eta_{p_1})^{\frac{D-5}{2}-n} - e^{i(p_2+p_3)\epsilon}(\sin\epsilon)^{\frac{D-5}{2}-n}\right) + (n \leftrightarrow -n)\Bigg|^2 .
\end{aligned}
\tag{B.21}
$$

To make a complete approximation of the first order correction the momentum dependent factor Q_{k_2,k_3} in (4.11) has to be approximated for large momenta. It is given by

$$
\begin{aligned}
Q_{p_2,p_3} &= (-1) \frac{h(k_1,d-1)}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_2,d-1)}{C_{k_2}{}^{1/2(d-1)}(1)\Omega_d} \frac{h(k_3,d-1)}{C_{k_3}{}^{1/2(d-1)}(1)\Omega_d} B(k_1,k_2,k_3,d) = \\
&= (-1) \frac{1}{C_{k_1}{}^{1/2(d-1)}(1)\Omega_d^2} \times \frac{2k_2+d-1}{d-1} \frac{2k_3+d-1}{d-1} \frac{2^{2-d}}{(\Gamma(1/2(d-1)))^2} \frac{\pi}{s+1/2(d-1)} \frac{\Gamma(s+d-1)}{\Gamma(s+1)} \times \\
&\quad \times \frac{\binom{s-k_1+1/2(d-1)-1}{s-k_1}\binom{s-k_2+1/2(d-1)-1}{s-k_2}\binom{s-k_3+1/2(d-1)-1}{s-k_3}}{\binom{s+1/2(d-1)-1}{s}}
\end{aligned}
$$

where $s = \frac{1}{2}\sum k_i$. For large $k_2 \approx k_3 \gg k_1$ this can be approximated by

$$
\begin{aligned}
Q_{p_2,p_3} &= (-1) \frac{1}{C_{k_1}^{1/2(d-1)}(1)\Omega_d^2} \frac{2^{2-d}\pi}{\left(\Gamma\left(\frac{d+1}{2}\right)\right)^2} \binom{\frac{k_1}{2}+1/2(d-1)-1}{\frac{k_1}{2}}^2 k_2^{d-1} = \\
&= k_2^{d-1}(-1) \frac{1}{(2\pi)^d C_{k_1}^{1/2(d-1)}(1)\Omega_d} \binom{\frac{k_1}{2}+1/2(d-1)-1}{\frac{k_1}{2}}^2 .
\end{aligned}
\tag{B.22}
$$

C. Alternative attempts

I try solving the integrals for the first order correction to $D^{++}(x,y)$ analytically by expressing the hypergeometric functions appearing in the definition of the mode function by their power series. The correction to first order is

$$G^{++}_{(1)}(Z_{XY}) = \sum_{k_1,k_2,k_3=0}^{|k_3-k_2|\leq k_1 \leq k_2+k_3} P \cdot C^{1/2(d-1)}_{k_1}(\vec{x}.\vec{y}) B(k_1,k_2,k_3,d) \times \\ \times \left(\frac{h(k_1,d-1)}{C^{1/2(d-1)}_{k_1}(1)\Omega_d} \right)^2 \frac{h(k_2,d-1)}{C^{1/2(d-1)}_{k_2}(1)\Omega_d} \frac{h(k_3,d-1)}{C^{1/2(d-1)}_{k_3}(1)\Omega_d}, \tag{C.1}$$

where the part depending on the integrals of mode functions has been denoted by

$$\begin{aligned} P = \int_0^\pi & \mathrm{d}\xi_1 \mathrm{d}\xi_2 \; C(\xi_1)C(\xi_2)a(\xi_1)^D a(\xi_2)^D \\ & \big[f^*_{k_1}(\eta_{>(\eta_1,\xi_1)}) f_{k_1}(\eta_{<(\eta_1,\xi_1)}) f^*_{k_2}(\eta_{>(\xi_1,\xi_2)}) f_{k_2}(\eta_{<(\xi_1,\xi_2)}) \times \\ & \quad \times f^*_{k_3}(\eta_{>(\xi_1,\xi_2)}) f_{k_3}(\eta_{<(\xi_1,\xi_2)}) f^*_{k_1}(\eta_{>(\eta_2,\xi_2)}) f_{k_1}(\eta_{<(\eta_2,\xi_2)}) + \\ & + f^*_{k_1}(\xi_1) f_{k_1}(\eta_1) f^*_{k_2}(\eta_{<(\xi_1,\xi_2)}) f_{k_2}(\eta_{>(\xi_1,\xi_2)}) f^*_{k_3}(\eta_{<(\xi_1,\xi_2)}) f_{k_3}(\eta_{>(\xi_1,\xi_2)}) f^*_{k_1}(\xi_2) f_{k_1}(\eta_2) + \\ & - f^*_{k_1}(\eta_{>(\eta_1,\xi_1)}) f_{k_1}(\eta_{<(\eta_1,\xi_1)}) f^*_{k_2}(\xi_2) f_{k_2}(\xi_1) f^*_{k_3}(\xi_2) f_{k_3}(\xi_1) f^*_{k_1}(\xi_2) f_{k_1}(\eta_2) + \\ & - f^*_{k_1}(\xi_1) f_{k_1}(\eta_1) f^*_{k_2}(\xi_1) f_{k_2}(\xi_2) f^*_{k_3}(\xi_1) f_{k_3}(\xi_2) \; f^*_{k_1}(\eta_{>(\eta_2,\xi_2)}) f_{k_1}(\eta_{<(\eta_2,\xi_2)}) \big]. \end{aligned} \tag{C.2}$$

Without loss of generality I can take $\eta_1 \geq \eta_2$. Evaluating the lesser and greater times, collecting all terms and using the substitution of (4.14), this can be simplified to

$$\begin{aligned} P = & f_{k_1}(\eta_1) f^*_{k_1}(\eta_2) \left[- \int_{\cot\eta_1}^{\cot\epsilon} \mathrm{d}u \int_{\cot\eta_2}^{\cot\epsilon} \mathrm{d}v \right] \frac{(a(u)a(v))^D}{(1+u^2)(1+v^2)} f^*_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) f_{k_3}(v) f_{k_2}(v) f_{k_1}(v) + \\ & + f^*_{k_1}(\eta_1) f_{k_1}(\eta_2) \left[- \int_{\cot\eta_1}^{\cot\eta_2} \mathrm{d}u \int_u^{\cot\eta_2} \mathrm{d}v - \int_{\cot\eta_1}^{\cot\epsilon} \mathrm{d}u \int_{\cot\eta_2}^{\cot\epsilon} \right] \\ & \quad \frac{(a(u)a(v))^D}{(1+u^2)(1+v^2)} f_{k_1}(u) f_{k_2}(u) f_{k_3}(u) f^*_{k_3}(v) f^*_{k_2}(v) f^*_{k_1}(v) + \\ & + f^*_{k_1}(\eta_1) f_{k_1}(\eta_2) \left[\int_{\cot\eta_1}^{\cot\eta_2} \mathrm{d}u \int_u^{\cot\eta_2} \mathrm{d}v \right] \frac{(a(u)a(v))^D}{(1+u^2)(1+v^2)} f_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) f_{k_3}(v) f_{k_2}(v) f^*_{k_1}(v) + \\ & + f^*_{k_1}(\eta_1) f^*_{k_1}(\eta_2) \left[2 \int_{\cot\eta_2}^{\cot\epsilon} \mathrm{d}u \int_u^{\cot\epsilon} \mathrm{d}v + \int_{\cot\eta_1}^{\cot\eta_2} \mathrm{d}u \int_{\cot\eta_2}^{\cot\epsilon} \mathrm{d}v \right] \\ & \quad \frac{(a(u)a(v))^D}{(1+u^2)(1+v^2)} f_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) f_{k_3}(v) f_{k_2}(v) f_{k_1}(v) + \\ & + f_{k_1}(\eta_1) f_{k_1}(\eta_2) \left[2 \int_{\cot\eta_1}^{\cot\epsilon} \mathrm{d}u \int_{\cot\eta_1}^{u} \mathrm{d}v - \int_{\cot\eta_1}^{\cot\epsilon} \mathrm{d}u \int_{\cot\eta_1}^{\cot\eta_2} \mathrm{d}v \right] \\ & \quad \frac{(a(u)a(v))^D}{(1+u^2)(1+v^2)} f^*_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) f_{k_3}(v) f_{k_2}(v) f^*_{k_1}(v). \end{aligned} \tag{C.3}$$

Now the mode functions can be expressed using the definition for the hypergeometric functions ([122] 2.1.1.(2))

$$ {}_2F_1\left(a,b,c,\zeta\right) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!}\zeta^n, $$

where $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$. This is absolutely convergent for $|\zeta| < 1$ and for $|\zeta| = 1, \zeta \neq 1$ if $\mathrm{Re}(c-a-b) > 0$, and convergent if $0 \geq \mathrm{Re}(c-a-b) > -1$ ([126]). For the coefficient used for the mode functions, the series for ${}_2F^1\left(h_+, h_-, p+1, \frac{1}{2} \pm \mathrm{i}\frac{z}{2}\right)$ is absolutely convergent for $|z| \leq \sqrt{3}$. For $|z| > \sqrt{3}$ I have to use Kummer's relations to perform the analytic continuation outside the original domain. The relation [122] 2.9.(4) transform the hypergeometric function to a form which is always convergent except for $|z| \to \infty$. The result is

$$ {}_2F_1\left(a,b,c,\zeta\right) = (1-\zeta)^{-b}{}_2F_1\left(c-a,b,c,\frac{\zeta}{\zeta-1}\right) $$

$$ \begin{aligned} f_k(\eta) &= (H\sin\eta)^{\frac{D-2}{2}} \frac{1}{\sqrt{2}} \frac{\sqrt{\Gamma(p+h_+)\Gamma(p+h_-)}}{\Gamma(p+1)} \mathrm{e}^{\mathrm{i}p\eta} \left(\frac{\mathrm{i}}{2}(z-\mathrm{i})\right)^{-h_-} {}_2F_1\left(p+h_-, h_-, p+1, \frac{z+\mathrm{i}}{z-\mathrm{i}}\right) = \\ &= (H\sin\eta)^{\frac{D-2}{2}} \frac{1}{\sqrt{2}} \frac{\sqrt{\Gamma(p+h_+)\Gamma(p+h_-)}}{\Gamma(p+1)} \mathrm{e}^{\mathrm{i}p\eta} \left(\frac{\mathrm{i}}{2}\right)^{-h_-} \frac{\mathrm{e}^{-(-h_-)\mathrm{i}\eta}}{\sin\eta^{-h_-}} {}_2F_1\left(p+h_-, h_-, p+1, \mathrm{e}^{2\mathrm{i}\eta}\right). \end{aligned} $$

In this case $\tilde{c} - \tilde{a} - \tilde{b} = 2n$, so unless $\zeta = 1$ the series is convergent, even for imaginary n. For real n it is absolutely convergent for all z. Transforming all mode function in the integral in (4.10) using this relation I get two different types of integrands:

$$ \begin{aligned} L_1 &= (z-\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}-\Sigma\frac{p_i}{2}} (z+\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}+\Sigma\frac{p_i}{2}} \prod_{i=1}^{3} {}_2F_1\left(h_+, h_-, p_i+1, -\frac{\mathrm{i}}{2}(z+\mathrm{i})\right) = \\ &= (z-\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}-\Sigma\frac{p_i}{2}} (z+\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}+\Sigma\frac{p_i}{2}} \prod_{i=1}^{3} (z-\mathrm{i})^{-h_-} \left(\frac{\mathrm{i}}{2}\right)^{-h_-} {}_2F_1\left(p_i+h_-, h_-, p_i+1, \frac{z+\mathrm{i}}{z-\mathrm{i}}\right) = \\ &= \left(\frac{\mathrm{i}}{2}\right)^{-3h_-} (z-\mathrm{i})^{-3h_- -\frac{D}{4}+\frac{1}{2}-\Sigma\frac{p_i}{2}} (z+\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}+\Sigma\frac{p_i}{2}} \times \\ &\times \sum_{j_1,j_2,j_3=0}^{\infty} (z-\mathrm{i})^{-\Sigma_i j_i} (z+\mathrm{i})^{\Sigma_i j_i} \prod_{i=1}^{3} \frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!} \end{aligned} $$

and

$$ \begin{aligned} L_2 &= (z-\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}+\frac{p_1}{2}-\Sigma\frac{p}{2}} (z+\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}-\frac{p_1}{2}+\Sigma\frac{p_i}{2}} {}_2F_1\left(h_+, h_-, p_1+1, \frac{\mathrm{i}}{2}(z-\mathrm{i})\right) \\ &\times \prod_{i=2,3} {}_2F_1\left(h_+, h_-, p_i+1, -\frac{\mathrm{i}}{2}(z+\mathrm{i})\right) = \\ &= (z-\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}+\frac{p_1}{2}-\Sigma\frac{p_i}{2}} (z+\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}-\frac{p_1}{2}+\Sigma\frac{p_i}{2}} \times \\ &\times \left(-\frac{\mathrm{i}}{2}\right)^{-3h_-} (z+\mathrm{i})^{-h_-} {}_2F_1\left(p_1+h_-, h_-, p_1+1, \frac{z-\mathrm{i}}{z+\mathrm{i}}\right) \prod_{i=2,3} (z-\mathrm{i})^{-h_-} {}_2F_1\left(p_i+h_-, h_-, p_i+1, \frac{z+\mathrm{i}}{z-\mathrm{i}}\right) \\ &= (-1)^{-h_-} \left(\frac{\mathrm{i}}{2}\right)^{-3h_-} (z-\mathrm{i})^{-2h_- -\frac{D}{4}+\frac{1}{2}+\frac{p_1}{2}-\Sigma\frac{p_i}{2}} (z+\mathrm{i})^{-h_- -\frac{D}{4}+\frac{1}{2}-\frac{p_1}{2}+\Sigma\frac{p_i}{2}} \times \\ &\times \sum_{j_1,j_2,j_3=0}^{\infty} (z-\mathrm{i})^{j_1-j_2-j_3} (z+\mathrm{i})^{-j_1+j_2+j_3} \prod_{i=1}^{3} \frac{\Gamma(h_-+j_i)\Gamma(h_-+p_i+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(h_-+p_i)\Gamma(p_i+1+j_i)j_i!}. \end{aligned} $$

These integrals of powers of $z+\mathrm{i}$ and $z-\mathrm{i}$ can be integrated using [122] 2.12.(1) and with 2.9.(4) can be brought into the convergence radius

$$\int^x (z-\mathrm{i})^\alpha (z+\mathrm{i})^\beta \mathrm{d}z = (-2\mathrm{i})^\alpha \frac{(x+\mathrm{i})^{\beta+1}}{\beta+1} {}_2F_1\left(-\alpha, \beta+1, \beta+2, -\frac{\mathrm{i}}{2}(x+\mathrm{i})\right) =$$
$$= \frac{(-2\mathrm{i})^{\alpha+\beta+1}}{\beta+1}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{\beta+1} {}_2F_1\left(\alpha+\beta+2, \beta+1, \beta+2, \frac{x+\mathrm{i}}{x-\mathrm{i}}\right) = \tag{C.4}$$
$$= (-2\mathrm{i})^{\alpha+\beta+1} B_z(\beta+1, \gamma). \tag{C.5}$$

For real x the condition for convergence is $c-a-b = -\alpha-\beta-1 = \frac{D}{2} - \frac{1}{2} - 3n > 0$, so for massive particles with $n \in \mathrm{i}\mathbb{R}$ this is satisfied. The parameters α and β for the two different integrals are

$$\alpha_1 = -3h_- - \frac{D}{4} + \frac{1}{2} - \sum \frac{p_i}{2} - \sum_i j_i, \qquad \beta_1 = -\frac{D}{4} + \frac{1}{2} + \sum \frac{p_i}{2} + \sum_i j_i,$$
$$\alpha_2 = -2h_- - \frac{D}{4} + \frac{1}{2} + \frac{p_1}{2} - \sum_{i=2,3} \frac{p_i}{2} + j_1 - \sum_{i=2,3} j_i, \qquad 2 = -h_- - \frac{D}{4} + \frac{1}{2} - \frac{p_1}{2} + \sum_{i=2,3} \frac{p_i}{2} - j_1 + \sum_{i=2,3} j_i.$$

Therefore the antiderivatives of the integrals are

$$I_1(x) = I_1(\cot\chi) = N(-1)^{\frac{1}{2}-\frac{D}{2}+3n}(2)^{2-\frac{D}{2}}(\mathrm{i})^{6n-1-\frac{D}{2}} \sum_{j_1,j_2,j_3=0}^{\infty} \frac{1}{\beta_1+1}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{\beta_1+1} \times$$
$$\times {}_2F_1\left(\gamma+1, \beta_1+1, \beta_1+2, \frac{x+\mathrm{i}}{x-\mathrm{i}}\right) \prod_{i=1}^{3} \frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!} =$$
$$= N(-1)^{\frac{1}{2}-\frac{D}{2}+3n}(2)^{2-\frac{D}{2}}(\mathrm{i})^{6n-1-\frac{D}{2}} \sum_{j_1,j_2,j_3,j_4=0}^{\infty} \frac{1}{\beta_1+1}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{j_4+\beta_1+1} \times$$
$$\times \frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_1+1+j_4)\Gamma(\beta_1+2)}{\Gamma(\gamma+1)\Gamma(\beta_1+1)\Gamma(\beta_1+2+j_4)j_4!} \prod_{i=1}^{3} \frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}$$

and

$$I_2(x) = I_2(\cot\chi) = N \sum_{j_1,j_2,j_3=0}^{\infty} (-1)^{-h_-}(-1)^{\frac{1}{2}-\frac{D}{2}+3n}(2)^{2-\frac{D}{2}}(\mathrm{i})^{6n-1-\frac{D}{2}} \frac{1}{\beta_2+1}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{\beta_2+1} \times$$
$$\times {}_2F_1\left(\gamma+1, \beta_2+1, \beta_2+2, \frac{x+\mathrm{i}}{x-\mathrm{i}}\right) \prod_{i=1}^{3} \frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!} =$$
$$= N \sum_{j_1,j_2,j_3=0}^{\infty} (-1)^{-h_-}(-1)^{\frac{1}{2}-\frac{D}{2}+3n}(2)^{2-\frac{D}{2}}(\mathrm{i})^{6n-1-\frac{D}{2}} \frac{1}{\beta_2+1}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{j_4+\beta_2+1} \times$$
$$\times \frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!} \prod_{i=1}^{3} \frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!},$$

where $\gamma = \alpha_i + \beta_i + 1 = -\frac{D}{2} + \frac{1}{2} + 3n$. Substituting all these results into (C.3) we get

$$
\begin{aligned}
P =& - f_{k_1}(\eta_1) f^*_{k_1}(\eta_2) \left[I^*_1(\cot\epsilon) - I^*_1(\cot\eta_1)\right]\left[I_1(\cot\epsilon) - I_1(\cot\eta_2)\right] - \\
&- f^*_{k_1}(\eta_1) f_{k_1}(\eta_2) \Bigg(\left[I_1(\cot\epsilon) - I_1(\cot\eta_1)\right]\left[I^*_1(\cot\epsilon) - I^*_1(\cot\eta_2)\right] + \left[I_1(\cot\eta_2) - I_1(\cot\eta_1)\right] I^*_1(\cot\eta_2) - \\
&\quad - \int_{\cot\eta_1}^{\cot\eta_2} \mathrm{d}u \ \frac{a(u)^D}{1+u^2} f_{k_1}(u) f_{k_2}(u) f_{k_3}(u) I^*_1(u) \Bigg) + \\
&+ f^*_{k_1}(\eta_1) f_{k_1}(\eta_2) \left(\left[I^*_2(\cot\eta_2) - I^*_2(\cot\eta_1)\right] I_2(\cot\eta_2) - \int_{\cot\eta_1}^{\cot\eta_2} \mathrm{d}u \ \frac{a(u)^D}{1+u^2} f_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) I_2(u) \right) + \\
&+ f^*_{k_1}(\eta_1) f^*_{k_1}(\eta_2) \Bigg(2\left[I^*_2(\cot\epsilon) - I^*_2(\cot\eta_2)\right] I_1(\cot\epsilon) - 2\int_{\cot\eta_2}^{\cot\epsilon} \mathrm{d}u \ \frac{a(u)^D}{1+u^2} f_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) I_1(u) + \\
&\quad + \left[I^*_2(\cot\eta_2) - I^*_2(\cot\eta_1)\right]\left[I_1(\cot\epsilon) - I_1(\cot\eta_2)\right] \Bigg) \\
&+ f_{k_1}(\eta_1) f_{k_1}(\eta_2) \Big(-\left[I^*_1(\cot\epsilon) - I^*_1(\cot\eta_1)\right]\left[I_2(\cot\eta_2) - I_2(\cot\eta_1)\right] - 2\left[I^*_1(\cot\epsilon) - I^*_1(\cot\eta_1)\right] I_2(\cot\eta_1) \\
&\quad + 2\int_{\cot\eta_1}^{\cot\epsilon} \mathrm{d}u \ \frac{a(u)^D}{1+u^2} f^*_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) I_2(u) \Big).
\end{aligned}
$$

The remaining integrals are of the form

$$
\begin{aligned}
I_4(x) &= \int^x \mathrm{d}u \ \frac{a(u)^D}{1+u^2} f_{k_1}(u) f_{k_2}(u) f_{k_3}(u) I^*_1(u), \\
I_{5,m}(x) &= \int^x \mathrm{d}u \ \frac{a(u)^D}{1+u^2} f_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) I_m(u), \\
I_6(x) &= \int^x \mathrm{d}u \ \frac{a(u)^D}{1+u^2} f^*_{k_1}(u) f^*_{k_2}(u) f^*_{k_3}(u) I_2(u),
\end{aligned}
$$

and can be expressed in the same manner using power series. The result is

$$
\begin{aligned}
I_4(x) =& \int^x \mathrm{d}u \ N \left(\frac{\mathrm{i}}{2}\right)^{-3h_-} (u-\mathrm{i})^{-3h_- - \frac{D}{4} + \frac{1}{2} - \Sigma \frac{p_i}{2}} (u+\mathrm{i})^{-\frac{D}{4} + \frac{1}{2} + \Sigma \frac{p_i}{2}} \times \\
&\times \sum_{j'_1, j'_2, j'_3 = 0}^{\infty} (u-\mathrm{i})^{-\Sigma_i j'_i} (u+\mathrm{i})^{\Sigma_i j'_i} \prod_{i=1}^{3} \frac{\Gamma(h_- + j'_i)\Gamma(p_i + h_- + j'_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i + h_-)\Gamma(p_i + 1 + j'_i) j'_i!} \times \\
&\times N(-1)^{\frac{1}{2} - \frac{D}{2} + 3n^*} (2)^{2-\frac{D}{2}} (-\mathrm{i})^{6n^* - 1 - \frac{D}{2}} \sum_{j_1, j_2, j_3, j_4 = 0}^{\infty} \frac{1}{\beta^*_1 + 1} \left(\frac{u-\mathrm{i}}{u+\mathrm{i}}\right)^{j_4 + \beta^*_1 + 1} \times \\
&\times \frac{\Gamma(\gamma^* + 1 + j_4)\Gamma(\beta^*_1 + 1 + j_4)\Gamma(\beta^*_1 + 2)}{\Gamma(\gamma^* + 1)\Gamma(\beta^*_1 + 1)\Gamma(\beta^*_1 + 2 + j_4) j_4!} \prod_{i=1}^{3} \frac{\Gamma(h^*_- + j_i)\Gamma(p_i + h^*_- + j_i)\Gamma(p_i + 1)}{\Gamma(h^*_-)\Gamma(p_i + h^*_-)\Gamma(p_i + 1 + j_i) j_i!} = \\
&= N^2 (-1)^{\frac{1}{2} - \frac{D}{2} + 3n^*} \left(\frac{\mathrm{i}}{2}\right)^{-3h_-} (2)^{2-\frac{D}{2}} (-\mathrm{i})^{6n^* - 1 - \frac{D}{2}} \sum_{j_1, j_2, j_3, j_4, j'_1, j'_2, j'_3 = 0}^{\infty} \frac{1}{\beta^*_1 + 1} \\
&\times \frac{\Gamma(\gamma^* + 1 + j_4)\Gamma(\beta^*_1 + 1 + j_4)\Gamma(\beta^*_1 + 2)}{\Gamma(\gamma^* + 1)\Gamma(\beta^*_1 + 1)\Gamma(\beta^*_1 + 2 + j_4) j_4!} \\
&\times \prod_{i=1}^{3} \left[\frac{\Gamma(h_- + j'_i)\Gamma(p_i + h_- + j'_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i + h_-)\Gamma(p_i + 1 + j'_i) j'_i!} \frac{\Gamma(h^*_- + j_i)\Gamma(p_i + h^*_- + j_i)\Gamma(p_i + 1)}{\Gamma(h^*_-)\Gamma(p_i + h^*_-)\Gamma(p_i + 1 + j_i) j_i!} \right] \times \\
&\times \int^x \mathrm{d}u \ (u-\mathrm{i})^{-3h_- - \frac{D}{4} + \frac{1}{2} - \Sigma \frac{p_i}{2} - \Sigma_i j'_i + j_4 + \beta^*_1 + 1} \times \\
&\times (u+\mathrm{i})^{-\frac{D}{4} + \frac{1}{2} + \Sigma \frac{p_i}{2} + \Sigma_i j'_i - j_4 - \beta^*_1 - 1},
\end{aligned}
$$

$$
\begin{aligned}
I_{5,m}(x) = &\int^x \mathrm{d}u\ N(-1)^{-2h_-}\left(\frac{\mathrm{i}}{2}\right)^{-3h_-}(u-\mathrm{i})^{-h_- - \frac{D}{4}+\frac{1}{2}-\frac{p_1}{2}+\sum_{i=2,3}\frac{p_i}{2}}(u+\mathrm{i})^{-2h_- - \frac{D}{4}+\frac{1}{2}+\frac{p_1}{2}-\sum_{i=2,3}\frac{p_i}{2}}\times\\
&\times\sum_{j_1',j_2',j_3'=0}^{\infty}(u-\mathrm{i})^{-j_1+j_2+j_3}(u+\mathrm{i})^{j_1-j_2-j_3}\prod_{i=1}^{3}\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\times\\
&\times(-1)^{-h_-(m-1)}N(-1)^{\frac{1}{2}-\frac{D}{2}+3n}(2)^{2-\frac{D}{2}}(\mathrm{i})^{6n-1-\frac{D}{2}}\sum_{j_1,j_2,j_3,j_4=0}^{\infty}\frac{1}{\beta_m+1}\left(\frac{u+\mathrm{i}}{u-\mathrm{i}}\right)^{(j_4+\beta_m+1)}\times\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_m+1+j_4)\Gamma(\beta_m+2)}{\Gamma(\gamma+1)\Gamma(\beta_m+1)\Gamma(\beta_m+2+j_4)j_4!}\prod_{i=1}^{3}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}=\\
&=(-1)^{-h_-(m-1)}N^2(-1)^{-2h_-}(-1)^{\frac{1}{2}-\frac{D}{2}+3n}\left(\frac{\mathrm{i}}{2}\right)^{-3h_-}(2)^{2-\frac{D}{2}}(\mathrm{i})^{6n-1-\frac{D}{2}}\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3'=0}^{\infty}\frac{1}{\beta_m+1}\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_m+1+j_4)\Gamma(\beta_m+2)}{\Gamma(\gamma+1)\Gamma(\beta_m+1)\Gamma(\beta_m+2+j_4)j_4!}\\
&\times\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\right]\times\\
&\times\int^x \mathrm{d}u\ \ (u-\mathrm{i})^{-h_- - \frac{D}{4}+\frac{1}{2}-\frac{p_1}{2}-j_1'+\sum_{i=2,3}\left(\frac{p_i}{2}+j_i'\right)-j_4-\beta_m-1}\times\\
&\times(u+\mathrm{i})^{-2h_- - \frac{D}{4}+\frac{1}{2}+\frac{p_1}{2}+j_1'-\sum_{i=2,3}\left(\frac{p_i}{2}+j_i'\right)+j_4+\beta_m+1},
\end{aligned}
$$

and

$$
\begin{aligned}
I_6(x) = &\int^x \mathrm{d}u\ N(-1)^{-3h_-}\left(\frac{\mathrm{i}}{2}\right)^{-3h_-}(u+\mathrm{i})^{-3h_- - \frac{D}{4}+\frac{1}{2}-\sum\frac{p_i}{2}}(u-\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}+\sum\frac{p_i}{2}}\times\\
&\times\sum_{j_1',j_2',j_3'=0}^{\infty}(u+\mathrm{i})^{-\sum_i j_i'}(u-\mathrm{i})^{\sum_i j_i'}\prod_{i=1}^{3}\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\times\\
&\times(-1)^{-h_-}N(-1)^{\frac{1}{2}-\frac{D}{2}+3n}(2)^{2-\frac{D}{2}}(\mathrm{i})^{6n-1-\frac{D}{2}}\sum_{j_1,j_2,j_3,j_4=0}^{\infty}\frac{1}{\beta_2+1}\left(\frac{u+\mathrm{i}}{u-\mathrm{i}}\right)^{j_4+\beta_2+1}\times\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\prod_{i=1}^{3}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}=\\
&=(-1)^{-4h_-}N^2(-1)^{\frac{1}{2}-\frac{D}{2}+3n}\left(\frac{\mathrm{i}}{2}\right)^{-3h_-}(2)^{2-\frac{D}{2}}(\mathrm{i})^{6n-1-\frac{D}{2}}\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3'=0}^{\infty}\frac{1}{\beta_2+1}\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\\
&\times\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\right]\times\\
&\times\int^x \mathrm{d}u\ \ (u+\mathrm{i})^{-3h_- - \frac{D}{4}+\frac{1}{2}-\sum\frac{p_i}{2}-\sum_i j_i'+j_4+\beta_2+1}(u-\mathrm{i})^{-\frac{D}{4}+\frac{1}{2}+\sum\frac{p_i}{2}+\sum_i j_i'-j_4-\beta_2-1},
\end{aligned}
$$

with the parameters α_i

$$
\begin{aligned}
\alpha_4 &= -3h_- - \frac{D}{4}+\frac{1}{2}-\sum\frac{p_i}{2}-\sum_i j_i'+j_4+\beta_1^*+1 = -3h_- - \frac{D}{2}+2+\sum_{i=1,2,3}(j_i-j_i')+j_4,\\
\alpha_{5,m} &= -1+(m-2)h_-+(m-2)p_1+(-1)^m j_1-\sum_{i=2,3,4}j_i-j_1'+\sum_{i=2,3}j_i',\\
\alpha_6 &= -\frac{D}{4}+\frac{1}{2}+\sum\frac{p_i}{2}+\sum_i j_i'-j_4-\beta_2-1 = h_- -1+p_1+j_1-\sum_{i=2,3,4}j_i+\sum_{i=1,2,3}j_i',
\end{aligned}
$$

and β_i

$$\beta_4 = -\frac{D}{4} + \frac{1}{2} + \sum \frac{p_i}{2} + \sum_i j_i' - j_4 - \beta_1^* - 1 = -1 - j_4 + \sum_{i=1,2,3} (j_i' - j_i),$$

$$\beta_{5,m} = -2h_- - \frac{D}{4} + \frac{1}{2} + \frac{p_1}{2} + j_1' - \sum_{i=2,3} \left(\frac{p_i}{2} + j_i'\right) + j_4 + \beta_{1,m} + 1 = .$$

$$= -(1+m)h_- - \frac{D}{2} + 2 + (2-m)p_1 - (-1)^m j_1 + \sum_{i=2,3,4} j_i + j_1' - \sum_{i=2,3} j_i',$$

$$\beta_6 = -3h_- - \frac{D}{4} + \frac{1}{2} - \sum \frac{p_i}{2} - \sum_i j_i' + j_4 + \beta_2 + 1 = -4h_- - \frac{D}{2} + 2 - p_1 - j_1 + \sum_{i=2,3,4} j_i - \sum_{i=1,2,3} j_i'.$$

and $\gamma = \alpha_i + \beta_i + 1 = -\frac{D}{2} + \frac{1}{2} + 3n$. Again these integrals can be integrated using (C.4) to yield

$$\begin{aligned}
I_4(x) =& N^2(-1)^{-\frac{1}{2}-D+9n^*}(2)^{\frac{7}{2}-\frac{D}{2}-3n}(\mathrm{i})^{6n^*+3n-\frac{5}{2}-\frac{D}{2}} \sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3'=0}^{\infty} \frac{1}{\beta_1^*+1} \\
&\times \frac{\Gamma(\gamma^*+1+j_4)\Gamma(\beta_1^*+1+j_4)\Gamma(\beta_1^*+2)}{\Gamma(\gamma^*+1)\Gamma(\beta_1^*+1)\Gamma(\beta_1^*+2+j_4)j_4!} \\
&\times \prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-^*+j_i)\Gamma(p_i+h_-^*+j_i)\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i)j_i!}\right]\times \\
&\times \frac{(-2\mathrm{i})^{\gamma}}{\beta_4+1}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{\beta_4+1} {}_2F_1\left(\gamma+1,\beta_4+1,\beta_4+2,\frac{x+\mathrm{i}}{x-\mathrm{i}}\right) = \\
&= N^2(-1)^{-D+9n^*+3n}(2)^{4-D}(\mathrm{i})^{6n^*+6n-2-D} \sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty} \frac{1}{\beta_1^*+1}\frac{1}{\beta_4+1} \\
&\times \frac{\Gamma(\gamma^*+1+j_4)\Gamma(\beta_1^*+1+j_4)\Gamma(\beta_1^*+2)}{\Gamma(\gamma^*+1)\Gamma(\beta_1^*+1)\Gamma(\beta_1^*+2+j_4)j_4!}\frac{\Gamma(\gamma+1+j_4')\Gamma(\beta_4+1+j_4')\Gamma(\beta_4+2)}{\Gamma(\gamma+1)\Gamma(\beta_4+1)\Gamma(\beta_4+2+j_4')j_4'!} \\
&\times \prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-^*+j_i)\Gamma(p_i+h_-^*+j_i)\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i)j_i!}\right]\times \\
&\times \left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{j_4'+\beta_4+1} = \sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty} V_4\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{j_4'+\beta_4+1},
\end{aligned}$$

$$\begin{aligned}
I_{5,m}(x) =& (-1)^{-h_-(m-1)}N^2(-1)^{-\frac{1}{2}-\frac{D}{2}+5n}(2)^{\frac{7}{2}-\frac{D}{2}-3n}(\mathrm{i})^{9n-\frac{5}{2}-\frac{D}{2}} \sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3'=0}^{\infty} \frac{1}{\beta_m+1} \\
&\times \frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_m+1+j_4)\Gamma(\beta_m+2)}{\Gamma(\gamma+1)\Gamma(\beta_m+1)\Gamma(\beta_m+2+j_4)j_4!} \\
&\times \prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\right]\times \\
&\times \frac{(-2\mathrm{i})^{\gamma}}{\beta_{5,m}+1}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{\beta_{5,m}+1} {}_2F_1\left(\gamma+1,\beta_4+1,\beta_{5,m}+2,\frac{x+\mathrm{i}}{x-\mathrm{i}}\right) = \\
=& (-1)^{-h_-(m-1)}N^2(-1)^{-D+8n}(2)^{4-D}(\mathrm{i})^{12n-2-D} \sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty} \frac{1}{\beta_m+1}\frac{1}{\beta_{5,m}+1} \\
&\times \frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_m+1+j_4)\Gamma(\beta_m+2)}{\Gamma(\gamma+1)\Gamma(\beta_m+1)\Gamma(\beta_m+2+j_4)j_4!}\frac{\Gamma(\gamma+1+j_4')\Gamma(\beta_{5,m}+1+j_4')\Gamma(\beta_{5,m}+2)}{\Gamma(\gamma+1)\Gamma(\beta_{5,m}+1)\Gamma(\beta_{5,m}+2+j_4')j_4'!} \\
&\times \prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\right]\times \\
&\times \left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{j_4'+\beta_{5,m}+1} = \sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty} V_{5,m}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{j_4'+\beta_{5,m}+1}
\end{aligned}$$

and

$$
\begin{aligned}
I_6(x) =&N^2(-1)^{-\frac{3}{2}-\frac{D}{2}+7n}(2)^{\frac{7}{2}-\frac{D}{2}-3n}(\mathrm{i})^{9n-\frac{5}{2}-\frac{D}{2}}\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty}\frac{1}{\beta_2+1}\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\\
&\times\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\right]\times\\
&\times\frac{(-2\mathrm{i})^{\gamma}}{\beta_6+1}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{\beta_6+1}{}_2F_1\left(\gamma+1,\beta_6+1,\beta_6+2,\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)=\\
&=N^2(-1)^{-1-D+10n}\left(\frac{\mathrm{i}}{2}\right)^{-3h_-}(2)^{4-D}(\mathrm{i})^{6n-1-\frac{D}{2}}\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3'=0}^{\infty}\frac{1}{\beta_2+1}\frac{1}{\beta_6+1}\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\frac{\Gamma(\gamma+1+j_4')\Gamma(\beta_6+1+j_4')\Gamma(\beta_6+2)}{\Gamma(\gamma+1)\Gamma(\beta_6+1)\Gamma(\beta_6+2+j_4')j_4'!}\\
&\times\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\right]\times\\
&\times\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{j_4'+\beta_6+1}=\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty}V_6\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{j_4'+\beta_6+1}.
\end{aligned}
$$

Plugging all of the above integrals into (C.3) and collecting terms, we get:

$$
\begin{aligned}
P =&f_{k_1}(\eta_1)f_{k_1}^*(\eta_2)\left[I_1^*(\cot\eta_1)-I_1^*(\cot\epsilon)\right]\left[I_1(\cot\epsilon)-I_1(\cot\eta_2)\right]\\
&+f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)\left(\left[I_1(\cot\eta_1)-I_1(\cot\epsilon)\right]I_1^*(\cot\epsilon)+\left[I_1(\cot\epsilon)-I_1(\cot\eta_2)\right]I_1^*(\cot\eta_2)+\right.\\
&+\left[I_2^*(\cot\eta_1)-I_2^*(\cot\eta_2)\right]I_2(\cot\eta_2)+I_4(\cot\eta_2)-I_4(\cot\eta_1)-I_{5,2}(\cot\eta_2)+I_{5,2}(\cot\eta_1)\left.\right)+\\
&+f_{k_1}^*(\eta_1)f_{k_1}^*(\eta_2)\left(\left[2I_2^*(\cot\epsilon)-I_2^*(\cot\eta_1)-I_2^*(\cot\eta_2)\right]I_1(\cot\epsilon)-2I_{5,1}(\cot\epsilon)+2I_{5,1}(\cot\eta_2)+\right.\\
&+\left.\left[I_2^*(\cot\eta_1)-I_2^*(\cot\eta_2)\right]I_1(\cot\eta_2)\right)\\
&+f_{k_1}(\eta_1)f_{k_1}(\eta_2)\left(\left[I_1^*(\cot\eta_1)-I_1^*(\cot\epsilon)\right]I_2(\cot\eta_2)+\left[I_1^*(\cot\eta_1)-I_1^*(\cot\epsilon)\right]I_2(\cot\eta_1)\right.\\
&\left.+2I_6(\cot\epsilon)-2I_6(\cot\eta_1)\right).
\end{aligned}
$$

Each of these terms will have 8 sums over $j_1,\dots,j_4,j_1',\dots,j_4'$. I have three different type of products, $I_1I_1^*$, $I_2I_2^*$ und $I_1^*I_2$. I collect the x independent factors in constants defined in the following way:

$$
\begin{aligned}
I_1(x_1)I_1^*(x_2) =&N^2(-1)^{1-D+3(n+n^*)+6n^*}(2)^{4-D}(\mathrm{i})^{6(n+n^*)}\times\\
&\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty}\frac{1}{\beta_1+1}\frac{1}{\beta'^*_1+1}\left(\frac{x_1+\mathrm{i}}{x_1-\mathrm{i}}\right)^{j_4+\beta_1+1}\left(\frac{x_2-\mathrm{i}}{x_2+\mathrm{i}}\right)^{j_4'+\beta'^*_1+1}\times\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_1+1+j_4)\Gamma(\beta_1+2)}{\Gamma(\gamma+1)\Gamma(\beta_1+1)\Gamma(\beta_1+2+j_4)j_4!}\times\frac{\Gamma(\gamma^*+1+j_4')\Gamma(\beta'^*_1+1+j_4')\Gamma(\beta'^*_1+2)}{\Gamma(\gamma^*+1)\Gamma(\beta'^*_1+1)\Gamma(\beta'^*_1+2+j_4')j_4'!}\\
&\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\times\frac{\Gamma(h_-^*+j_i')\Gamma(p_i+h_-^*+j_i')\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i')j_i'!}\right]=\\
&=:\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty}V_1\left(\frac{x_1+\mathrm{i}}{x_1-\mathrm{i}}\right)^{j_4+\beta_1+1}\left(\frac{x_2-\mathrm{i}}{x_2+\mathrm{i}}\right)^{j_4'+\beta'^*_1+1},
\end{aligned}
$$

$$
\begin{aligned}
I_2(x_1)I_2^*(x_2) =& N^2(-1)^{2-D+4(n+n^*)+6n^*}(2)^{4-D}(\mathrm{i})^{6(n+n^*)}\times\\
&\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty}\frac{1}{\beta_2+1}\frac{1}{{\beta'}_2^*+1}\left(\frac{x_1+\mathrm{i}}{x_1-\mathrm{i}}\right)^{j_4+\beta_2+1}\left(\frac{x_2-\mathrm{i}}{x_2+\mathrm{i}}\right)^{j_4'+{\beta'}_2^*+1}\times\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\times\frac{\Gamma(\gamma^*+1+j_4')\Gamma({\beta'}_2^*+1+j_4')\Gamma({\beta'}_2^*+2)}{\Gamma(\gamma^*+1)\Gamma({\beta'}_2^*+1)\Gamma({\beta'}_2^*+2+j_4')j_4'!}\\
&\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\times\frac{\Gamma(h_-^*+j_i')\Gamma(p_i+h_-^*+j_i')\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i')j_i'!}\right]=\\
=:&\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty}V_2\left(\frac{x_1+\mathrm{i}}{x_1-\mathrm{i}}\right)^{j_4+\beta_2+1}\left(\frac{x_2-\mathrm{i}}{x_2+\mathrm{i}}\right)^{j_4'+{\beta'}_2^*+1},
\end{aligned}
$$

$$
\begin{aligned}
I_2(x_1)I_1^*(x_2) =& N^2(-1)^{\frac{1}{2}-D+4(n+n^*)+5n^*}(2)^{4-D}(\mathrm{i})^{6(n+n^*)}\times\\
&\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty}\frac{1}{\beta_2+1}\frac{1}{{\beta'}_1^*+1}\left(\frac{x_1+\mathrm{i}}{x_1-\mathrm{i}}\right)^{j_4+\beta_2+1}\left(\frac{x_2-\mathrm{i}}{x_2+\mathrm{i}}\right)^{j_4'+{\beta'}_1^*+1}\times\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\times\frac{\Gamma(\gamma^*+1+j_4')\Gamma({\beta'}_1^*+1+j_4')\Gamma({\beta'}_1^*+2)}{\Gamma(\gamma^*+1)\Gamma({\beta'}_1^*+1)\Gamma({\beta'}_1^*+2+j_4')j_4'!}\\
&\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\times\frac{\Gamma(h_-^*+j_i')\Gamma(p_i+h_-^*+j_i')\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i')j_i'!}\right]=\\
=:&\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty}V_3\left(\frac{x_1+\mathrm{i}}{x_1-\mathrm{i}}\right)^{j_4+\beta_2+1}\left(\frac{x_2-\mathrm{i}}{x_2+\mathrm{i}}\right)^{j_4'+{\beta'}_1^*+1}.
\end{aligned}
$$

Summarizing with the other prefactors from I_4, I_5, I_6 we have:

$$
\begin{aligned}
V_1 =& N^2(-1)^{1-D+3(n+n^*)+6n^*}(2)^{4-D}(\mathrm{i})^{6(n+n^*)}\frac{1}{\beta_1+1}\frac{1}{{\beta'}_1^*+1}\times\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_1+1+j_4)\Gamma(\beta_1+2)}{\Gamma(\gamma+1)\Gamma(\beta_1+1)\Gamma(\beta_1+2+j_4)j_4!}\times\frac{\Gamma(\gamma^*+1+j_4')\Gamma({\beta'}_1^*+1+j_4')\Gamma({\beta'}_1^*+2)}{\Gamma(\gamma^*+1)\Gamma({\beta'}_1^*+1)\Gamma({\beta'}_1^*+2+j_4')j_4'!}\\
&\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\times\frac{\Gamma(h_-^*+j_i')\Gamma(p_i+h_-^*+j_i')\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i')j_i'!}\right],
\end{aligned}
$$

$$
\begin{aligned}
V_2 =& N^2(-1)^{2-D+4(n+n^*)+6n^*}(2)^{4-D}(\mathrm{i})^{6(n+n^*)}\frac{1}{\beta_2+1}\frac{1}{{\beta'}_2^*+1}\times\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\times\frac{\Gamma(\gamma^*+1+j_4')\Gamma({\beta'}_2^*+1+j_4')\Gamma({\beta'}_2^*+2)}{\Gamma(\gamma^*+1)\Gamma({\beta'}_2^*+1)\Gamma({\beta'}_2^*+2+j_4')j_4'!}\\
&\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\times\frac{\Gamma(h_-^*+j_i')\Gamma(p_i+h_-^*+j_i')\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i')j_i'!}\right],
\end{aligned}
$$

$$
\begin{aligned}
V_3 =& N^2(-1)^{\frac{1}{2}-D+4(n+n^*)+5n^*}(2)^{4-D}(\mathrm{i})^{6(n+n^*)}\frac{1}{\beta_2+1}\frac{1}{{\beta'}_1^*+1}\times\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\times\frac{\Gamma(\gamma^*+1+j_4')\Gamma({\beta'}_1^*+1+j_4')\Gamma({\beta'}_1^*+2)}{\Gamma(\gamma^*+1)\Gamma({\beta'}_1^*+1)\Gamma({\beta'}_1^*+2+j_4')j_4'!}\\
&\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\times\frac{\Gamma(h_-^*+j_i')\Gamma(p_i+h_-^*+j_i')\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i')j_i'!}\right],
\end{aligned}
$$

$$
\begin{aligned}
V_4 =& N^2(-1)^{-D+9n^*+3n}(2)^{4-D}(\mathrm{i})^{6n^*+6n-2-D}\frac{1}{\beta_1^*+1}\frac{1}{\beta_4+1}\\
&\times\frac{\Gamma(\gamma^*+1+j_4)\Gamma(\beta_1^*+1+j_4)\Gamma(\beta_1^*+2)}{\Gamma(\gamma^*+1)\Gamma(\beta_1^*+1)\Gamma(\beta_1^*+2+j_4)j_4!}\frac{\Gamma(\gamma+1+j_4')\Gamma(\beta_4+1+j_4')\Gamma(\beta_4+2)}{\Gamma(\gamma+1)\Gamma(\beta_4+1)\Gamma(\beta_4+2+j_4')j_4'!}\\
&\times\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-^*+j_i)\Gamma(p_i+h_-^*+j_i)\Gamma(p_i+1)}{\Gamma(h_-^*)\Gamma(p_i+h_-^*)\Gamma(p_i+1+j_i)j_i!}\right],
\end{aligned}
$$

$$
\begin{aligned}
V_{5,m} =& (-1)^{-h_-(m-1)}N^2(-1)^{-D+8n}(2)^{4-D}(\mathrm{i})^{12n-2-D}\frac{1}{\beta_m+1}\frac{1}{\beta_{5,m}+1}\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_m+1+j_4)\Gamma(\beta_m+2)}{\Gamma(\gamma+1)\Gamma(\beta_m+1)\Gamma(\beta_m+2+j_4)j_4!}\frac{\Gamma(\gamma+1+j_4')\Gamma(\beta_{5,m}+1+j_4')\Gamma(\beta_{5,m}+2)}{\Gamma(\gamma+1)\Gamma(\beta_{5,m}+1)\Gamma(\beta_{5,m}+2+j_4')j_4'!}\\
&\times\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\right],
\end{aligned}
$$

$$
\begin{aligned}
V_6 =& N^2(-1)^{-1-D+10n}\left(\frac{\mathrm{i}}{2}\right)^{-3h_-}(2)^{4-D}(\mathrm{i})^{6n-1-\frac{D}{2}}\frac{1}{\beta_2+1}\frac{1}{\beta_6+1}\\
&\times\frac{\Gamma(\gamma+1+j_4)\Gamma(\beta_2+1+j_4)\Gamma(\beta_2+2)}{\Gamma(\gamma+1)\Gamma(\beta_2+1)\Gamma(\beta_2+2+j_4)j_4!}\frac{\Gamma(\gamma+1+j_4')\Gamma(\beta_6+1+j_4')\Gamma(\beta_6+2)}{\Gamma(\gamma+1)\Gamma(\beta_6+1)\Gamma(\beta_6+2+j_4')j_4'!}\\
&\times\prod_{i=1}^{3}\left[\frac{\Gamma(h_-+j_i')\Gamma(p_i+h_-+j_i')\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i')j_i'!}\frac{\Gamma(h_-+j_i)\Gamma(p_i+h_-+j_i)\Gamma(p_i+1)}{\Gamma(h_-)\Gamma(p_i+h_-)\Gamma(p_i+1+j_i)j_i!}\right].
\end{aligned}
$$

With these prefactors and $\frac{\cot\chi+\mathrm{i}}{\cot\chi-\mathrm{i}}=\mathrm{e}^{2i\chi}$, P can be expressed as:

$$
\begin{aligned}
P = &\sum_{j_1,j_2,j_3,j_4,j_1',j_2',j_3',j_4'=0}^{\infty} \\
&f_{k_1}(\eta_1)f_{k_1}^*(\eta_2)V_1\mathrm{e}^{\mathrm{i}2\epsilon(j_4+\beta_1+1)}\mathrm{e}^{-\mathrm{i}2\eta_1(j_4'+\beta_1'+1)}- \\
&-f_{k_1}(\eta_1)f_{k_1}^*(\eta_2)V_1\mathrm{e}^{\mathrm{i}2\epsilon(j_4+\beta_1+1)}\mathrm{e}^{-\mathrm{i}2\epsilon(j_4'+\beta_1'+1)}- \\
&-f_{k_1}(\eta_1)f_{k_1}^*(\eta_2)V_1\mathrm{e}^{\mathrm{i}2\eta_2(j_4+\beta_1+1)}\mathrm{e}^{-\mathrm{i}2\eta_1(j_4'+\beta_1'+1)}+ \\
&+f_{k_1}(\eta_1)f_{k_1}^*(\eta_2)V_1\mathrm{e}^{\mathrm{i}2\eta_2(j_4+\beta_1+1)}\mathrm{e}^{-\mathrm{i}2\epsilon(j_4'+\beta_1'+1)} \\
&+f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_1\mathrm{e}^{\mathrm{i}2\eta_1(j_4+\beta_1+1)}\mathrm{e}^{-\mathrm{i}2\epsilon(j_4'+\beta_1'+1)}- \\
&-f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_1\mathrm{e}^{\mathrm{i}2\epsilon(j_4+\beta_1+1)}\mathrm{e}^{-\mathrm{i}2\epsilon(j_4'+\beta_1'+1)}+ \\
&+f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_1\mathrm{e}^{\mathrm{i}2\epsilon(j_4+\beta_1+1)}\mathrm{e}^{-\mathrm{i}2\eta_2(j_4'+\beta_1'+1)}- \\
&-f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_1\mathrm{e}^{\mathrm{i}2\eta_2(j_4+\beta_1+1)}\mathrm{e}^{-\mathrm{i}2\eta_2(j_4'+\beta_1'+1)}+ \\
&+f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_2\mathrm{e}^{\mathrm{i}2\eta_2(j_4+\beta_2+1)}\mathrm{e}^{-\mathrm{i}2\eta_1(j_4'+\beta_2'^*+1)}- \\
&-f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_2\mathrm{e}^{\mathrm{i}2\eta_2(j_4+\beta_2+1)}\mathrm{e}^{-\mathrm{i}2\eta_2(j_4'+\beta_2'^*+1)}+ \\
&+f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_4\mathrm{e}^{\mathrm{i}2\eta_2(j_4'+\beta_4+1)}- \\
&-f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_4\mathrm{e}^{\mathrm{i}2\eta_1(j_4'+\beta_4+1)}- \\
&-f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_{5,2}\mathrm{e}^{\mathrm{i}2\eta_2(j_4'+\beta_{5,2}+1)}+ \\
&+f_{k_1}^*(\eta_1)f_{k_1}(\eta_2)V_{5,2}\mathrm{e}^{\mathrm{i}2\eta_1(j_4'+\beta_{5,2}+1)}+ \\
&+2f_{k_1}^*(\eta_1)f_{k_1}^*(\eta_2)V_3^*\mathrm{e}^{-\mathrm{i}2\epsilon(j_4+\beta_2^*+1)}\mathrm{e}^{\mathrm{i}2\epsilon(j_4'+\beta_1'+1)}- \\
&-f_{k_1}^*(\eta_1)f_{k_1}^*(\eta_2)V_3^*\mathrm{e}^{-\mathrm{i}2\eta_1(j_4+\beta_2^*+1)}\mathrm{e}^{\mathrm{i}2\epsilon(j_4'+\beta_1'+1)}- \\
&-f_{k_1}^*(\eta_1)f_{k_1}^*(\eta_2)V_3^*\mathrm{e}^{-\mathrm{i}2\eta_2(j_4+\beta_2^*+1)}\mathrm{e}^{\mathrm{i}2\epsilon(j_4'+\beta_1'+1)}- \\
&-2f_{k_1}^*(\eta_1)f_{k_1}^*(\eta_2)V_{5,1}\mathrm{e}^{\mathrm{i}2\epsilon(j_4'+\beta_{5,1}+1)}+ \\
&+2f_{k_1}^*(\eta_1)f_{k_1}^*(\eta_2)V_{5,1}\mathrm{e}^{\mathrm{i}2\eta_2(j_4'+\beta_{5,1}+1)}+ \\
&+f_{k_1}^*(\eta_1)f_{k_1}^*(\eta_2)V_3^*\mathrm{e}^{-\mathrm{i}2\eta_1(j_4+\beta_2^*+1)}\mathrm{e}^{\mathrm{i}2\eta_2(j_4'+\beta_1'+1)}- \\
&-f_{k_1}^*(\eta_1)f_{k_1}^*(\eta_2)V_3^*\mathrm{e}^{-\mathrm{i}2\eta_2(j_4+\beta_2^*+1)}\mathrm{e}^{\mathrm{i}2\eta_2(j_4'+\beta_1'+1)}+ \\
&+f_{k_1}(\eta_1)f_{k_1}(\eta_2)V_3\mathrm{e}^{\mathrm{i}2\eta_2(j_4+\beta_2+1)}\mathrm{e}^{-\mathrm{i}2\eta_1(j_4'+\beta_1'+1)}- \\
&-f_{k_1}(\eta_1)f_{k_1}(\eta_2)V_3\mathrm{e}^{\mathrm{i}2\eta_2(j_4+\beta_2+1)}\mathrm{e}^{-\mathrm{i}2\epsilon(j_4'+\beta_1'+1)}+ \\
&+f_{k_1}(\eta_1)f_{k_1}(\eta_2)V_3\mathrm{e}^{\mathrm{i}2\eta_1(j_4+\beta_2+1)}\mathrm{e}^{-\mathrm{i}2\eta_1(j_4'+\beta_1'+1)}- \\
&-f_{k_1}(\eta_1)f_{k_1}(\eta_2)V_3\mathrm{e}^{\mathrm{i}2\eta_1(j_4+\beta_2+1)}\mathrm{e}^{-\mathrm{i}2\epsilon(j_4'+\beta_1'+1)}- \\
&+2f_{k_1}(\eta_1)f_{k_1}(\eta_2)V_6\mathrm{e}^{\mathrm{i}2\epsilon(j_4'+\beta_6+1)}- \\
&-2f_{k_1}(\eta_1)f_{k_1}(\eta_2)V_6\mathrm{e}^{\mathrm{i}2\eta_1(j_4'+\beta_6+1)}.
\end{aligned}
$$

Unfortunately, the resulting power series is not of a form easily summed to a closed form.

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