GEOMETRIC ALGEBRA AND AN ACOUSTIC SPACE TIME FOR PROPAGATION IN NON-UNIFORM FLOW

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This study aims to make use of two concepts in the field of aeroacoustics; an analogy with relativity, and Geometric Algebra. The analogy with relativity has been investigated in physics and cosmology, but less has been done to use this work in the field of aeroacoustics. Despite being successfully applied to a variety of fields, Geometric Algebra has yet to be applied to acoustics. Our aim is to apply these concepts first to a simple problem in aeroacoustics, sound propagation in uniform flow, and the more general problem of acoustic propagation in non-uniform flows. By using Geometric Algebra we are able to provide a simple geometric interpretation to a transformation commonly used to solve for sound fields in uniform flow. We are then able to extend this concept to an acoustic space-time applicable to irrotational, barotropic background flows. This geometrical framework is used to naturally derive the requirements that must be satisfied by the background flow in order for us to be able to solve for sound propagation in the non-uniform flow using the simple wave equation. We show that this is not possible in the most general situation, and provide an explicit expression that must be satisfied for the transformation to exist. We show that this requirement is automatically satisfied if the background flow is incompressible or uniform, and for both these cases derive an explicit transformation. In addition to a new physical interpretation for the transformation, we show that unlike previous investigations, our work is applicable to any frequency.

1 The Acoustic Space-Time

It was noted by Unruh [1] that there is an analogy between the equations of general relativity and acoustics in the presence of background flow. This analogy has since been expanded on by Visser [2] and Barceló et al [3] providing some illuminating space-time diagrams of acoustic propagation in idealised flows. At the heart of this analogy is the geometry within which the problem can be considered, rather than a direct correspondence between governing equations. Einstein's equations of gravitation (or the Einstein field equations) [4] are not the same as those that govern acoustic perturbations (which we shall come to in a moment), however, the geometry of the problem is usually considered to be a 4-dimensional space of mixed signature which can in general be curved. It is possible to consider acoustic perturbations in terms of a similar space, and it is in this sense that there is an analogy.

If we consider an acoustic perturbation to a background flow, described in terms of a perturbation velocity potential ϕ, and the background flow is irrotational and barotropic, then ϕ is governed by [1, 2, 5, 6],

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\phi-\frac{1}{\rho}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\phi)=0, \tag{1}$$

where $\boldsymbol{u}$, ρ and c are the background flow velocity, density and speed of sound, and t and $\boldsymbol{\nabla}$ are the standard time and gradient operator of Newtonian mechanics. Conservation of mass for the background flow tells us that,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0. \tag{2}$$

Using this along with Eq. (1) we obtain a slightly altered expression for ϕ,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{u}\right) \frac{\rho}{c^2} \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \phi - \boldsymbol{\nabla} \cdot (\rho \boldsymbol{\nabla} \phi) = 0, \tag{3}$$

where $\boldsymbol{\nabla}$ acts on *everything* to its right. This may seem like a needless complication , but as pointed out by, for example, Visser [2], we can write this as the d'Alembertian of a curved, 4-dimensional space-time of mixed signature,

$$\Delta \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right) = 0. \tag{4}$$

$g^{\mu\nu}$ is the inverse of the metric $g_{\mu\nu}$ of the space, and g is the determinant of $g_{\mu\nu}$. Hence it becomes clear that we can regard ϕ as a scalar field on a 4-dimensional space with the metric $g_{\mu\nu}$ defined by the background flow. This space is the *acoustic space-time*. The authors who have pointed this out have largely used it as a tool to illuminate what parts of general relativity are due to Einstein's field equations specifically, and what phenomena remain when the equations are changed but the form of the geometry remains (this is where black holes are of interest). We, however, would like to turn this on its head, making use of the acoustic space-time to illuminate how variable transformations can be used to understand sound propagation in the presence of background flow.

The idea of these transformations is to use a change of variables to convert a challenging equation, such as Eq. (1), into, say, the classic wave equation. To the authors knowledge the most general transformation to date of this kind was first presented by Taylor [5], and is valid for irrotational, barotropic, low Mach number, steady background flows for acoustic fields where the wavelength is of the same order of magnitude or smaller than the length scale of variations of the background flow [7]. The presentation of this transformation is fairly ad hoc, and we would like to use the acoustic space-time to derive and generalise such transformations in a more systematic way.

2 Calculus on Curved Manifolds

In order to proceed further we shall introduce some new tools to deal with calculus on general manifolds. More precisely we shall introduce Geometric Algebra (GA).

GA deals with general manifolds through the concept of embedding. A flat vector space of large dimension is first defined, within which the, possibly curved, manifold of interest is placed. The embedded manifold then inherits a metric from the extrinsic space, and this allows a study of Riemannian geometry. Some readers may wonder why we use GA instead of the more widely used differential forms and differential geometry (see for example Nakahara [8]); ultimately this a preference of the authors. We find that the approach gives more streamlined and intuitive proofs of some key results, and also allows more to be expressed independently of coordinates. A debate of the relative benefits is beyond the scope of the current paper. As was proven by John Nash in 1956, any finite dimensional Riemannian manifold can be embedded in a larger dimensional flat space in such a way that the metric is generated by the embedding [9]. Therefore our approach is easily adequate for our purposes, and all the results used should still be familiar to anyone with knowledge of differential geometry. For a full introduction to geometric calculus we direct the reader to Hestenes [10] and Doran [11]. Key to GA is the geometric product between vectors $ab = a \cdot b + a \wedge b$. If two vectors appear adjacent then the geometric product is implied.

We consider an n dimensional curved manifold embedded within a flat space of higher dimension. Let x^i be a set of n scalar coordinate functions defined over the part of the manifold of interest and x be some point on the manifold. A basis of the tangent space (frame) of the manifold is given by $e_i = \partial x / \partial x^i$. The reciprocal frame $\{e^i\}$ of the tangent space is defined to satisfy,

$$e_j \cdot e^i = \delta^i_j \tag{5}$$

where the inner product is inherited from the flat embedding space. The vector derivative intrinsic to the manifold ∂ is defined as the projection of the vector derivative of the embedding space onto the manifold, and can be written as,

$$\partial = e^i \frac{\partial}{\partial x^i} = e^i \partial_{x^i}. \tag{6}$$

In addition we can show that the reciprocal frame $\{e^i\}$ is given by,

$$e^i = \partial x^i. \tag{7}$$

Let the local pseudoscalar of the manifold be $I(x)$. We define the scalar field $V(x)$, which is specific to the frame $\{e_i\}$, such that it satisfies,

$$e_1 \wedge e_2 \wedge \cdots \wedge e_n = VI. \tag{8}$$

The covariant derivative DA of a multivector field A (defined on the manifold) is defined as the projection onto the manifold of ∂A. A useful result for a scalar field α is that,

$$D \cdot (D\alpha) = D^2 \alpha = \frac{1}{V} \frac{\partial}{\partial x^i} \left(V g^{ij} \frac{\partial \alpha}{\partial x^j} \right), \tag{9}$$

where $g^{ij} = e^i \cdot e^j$. Similarly $g_{ij} = e_i \cdot e_j$, and this satisfies $g^{ik} g_{kj} = \delta^i_j$. We can also show that $V = \sqrt{|\det g_{ij}|}$. g_{ij} is the metric of the frame.

3 A Transformation Derived from the Acoustic Space-Time

3.1 A Preliminary Investigation into Uniform Flow

To begin we consider a 4-dimensional flat acoustic space-time. We define three frames within this which satisfy,

$$\begin{array}{ccc} \{e_i\} & \{e'_i\} & \{e''_i\} \\ V = 1 & V' = 1 & V'' = 1 \\ g^{ij} = \begin{bmatrix} 1 & 0 & 0 & M \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ M & 0 & 0 & M^2 - 1 \end{bmatrix} & g^{ij\prime} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & g^{ij\prime\prime} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \tag{10}$$

and are related to each other by,

$$e_0 = e''_0 - M e''_z, \quad e_1 = e''_1, \quad e_2 = e''_2, \quad e_3 = e''_3, \tag{11a}$$

$$e'_0 = \frac{1}{\beta} \left(e''_0 - M e''_3 \right), \quad e'_1 = e''_1, \quad e'_2 = e''_2, \quad e'_3 = \frac{1}{\beta} \left(e''_3 - M e''_0 \right), \tag{11b}$$

where M is a constant and $\beta = \sqrt{1-M^2}$ (it is simple to check that these transformations are consistent with the metrics defined above). From this we can derive the following coordinate transformations,

$$x^0 = x^{0''}, \quad x^1 = x^{1''}, \quad x^2 = x^{2''}, \quad x^3 = x^{3''} + Mx^{0''}, \tag{12a}$$

$$x^{0'} = \frac{1}{\beta}\left(x^{0''} + Mx^{3''}\right), \quad x^{1'} = x^{1''}, \quad x^{2'} = x^{2''}, \quad x^{3'} = \frac{1}{\beta}\left(x^{3''} + Mx^{0''}\right), \tag{12b}$$

If c_0 is a constant, and we define the following coordinates of the three frames,

$$x^0 = c_0 t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \tag{13}$$

with similar relations for $\{x^{i'}\}$ and $\{x^{i''}\}$. From Eq. (9) $D^2\phi = 0$ may be written in the 3 frames as,

$$0 = D^2\phi = \left[\tfrac{1}{c_0^2}\left(\partial_t + U\partial_z\right)^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right]\phi \tag{14a}$$

$$= \left[\tfrac{1}{c_0^2}\partial_{t'}^2 - \partial_{x'}^2 - \partial_{y'}^2 - \partial_{z'}^2\right]\phi \tag{14b}$$

$$= \left[\tfrac{1}{c_0^2}\partial_{t''}^2 - \partial_{x''}^2 - \partial_{y''}^2 - \partial_{z''}^2\right]\phi, \tag{14c}$$

where $U = Mc_0$. From this we see that Eq. (14a) is identical to Eq. (1) when the background flow is uniform, while Eq. (14c) is identical to Eq. (1) when there is no flow. This result is expected since the transformation from $\{e_i\}$ to $\{e_i''\}$ is a Galilean transformation. $\{e_i\}$ represents a frame moving with Mach number M relative to the fluid in the $-z$ direction, while $\{e_i''\}$ represents the fluid frame.

Inspecting Eq. (12b) we see that the transformation from $\{e_i''\}$ to $\{e_i'\}$ is a Lorentz transform, and so it is not surprising that the metric is unchanged, and from Eq. (9) we see that $D^2\phi$ will take the simplest form. However, by deriving the relations between the x^i and $x^{i'}$ coordinates,

$$x^{0'} = \beta x^0 + \frac{M}{\beta}x^3, \quad x^{1'} = x^1, \quad x^{2'} = x^2, \quad x^{3'} = \frac{x^3}{\beta}, \tag{15}$$

we can see that if an observer is stationary in the $\{e_i\}$ frame, they will also be stationary in the $\{e_i'\}$ frame. The value of the $\{e_i'\}$ frame lies in the fact that it has a simple metric, and so a simple form of the wave equation, but moves with the observer frame. The transformation in Eq. (15) has been presented before by, for example, Chapman [12], but by interpreting it though an acoustic space time we are able to see why the transformation works, and precisely what frame we are transforming to when we use it.

3.2 Motivation: Generalised Galilean and Lorentz Transforms

Now we would like to try to generalise this process to non-uniform flows. We start from the frame defined in Section 3.1, $\{e_i''\}$. Note that for now we are still dealing with a flat acoustic space-time. We would like to now find a frame in which Eq. (9) gives Eq. (1) in the general case.

3.2.1 Generalised Galilean Transform

We redefine the $\{e_i\}$ frame and its reciprocal as,

$$e_0 = h_\tau\left(e_0'' - M_1 e_1'' - M_2 e_2'' - M_3 e_3''\right), \quad e_1 = h e_1'', \quad e_2 = h e_2'', \quad e_3 = h e_3'', \tag{16}$$

where $h_\tau = \sqrt{\rho c/\rho_0 c_0}$ and $h = \sqrt{\rho c_0/\rho_0 c}$. We interpret the M_i as Mach number components, ρ and c as the (non-uniform) background density and speed of sound, and ρ_0 and c_0 as the (constant) free-field density and speed of sound. Comparing this to Eq. (11a) we see that this is a generalised form of a Galilean transform. For this frame we can show that $V = h^3 h_\tau$ and,

$$Vg^{ij} = \frac{\rho}{\rho_0 c^2}\begin{bmatrix} c_0^2 & c_0 u_1 & c_0 u_2 & c_0 u_3 \\ c_0 u_1 & u_1^2 - c^2 & u_1 u_2 & u_1 u_3 \\ c_0 u_2 & u_1 u_2 & u_2^2 - c^2 & u_2 u_3 \\ c_0 u_3 & u_1 u_3 & u_2 u_3 & u_3^2 - c^2 \end{bmatrix}, \tag{17}$$

where $u_i = M_i c$ are interpreted as the velocity components of $\boldsymbol{u}$ in Eq. (1). If we denote the coordinates associated with the $\{e_i\}$ frame as $x^0 = c_0 t, x^1 = x, x^2 = y, x^3 = z$, then by Eq. (9) $D^2\phi = 0$ may be written as,

$$\begin{aligned} D^2\phi = \frac{1}{V\rho_0}\Big[&\partial_t(\rho/c^2)\left(\partial_t + u_1\partial_x + u_2\partial_y + u_3\partial_z\right) \\ &+ \partial_x(\rho/c^2)\left(u_1\partial_t + (u_1^2 - c^2)\partial_x + u_1u_2\partial_y + u_1u_3\partial_z\right) \\ &+ \partial_y(\rho/c^2)\left(u_2\partial_t + u_1u_2\partial_x + (u_2^2 - c^2)\partial_y + u_2u_3\partial_z\right) \\ &+ \partial_z(\rho/c^2)\left(u_3\partial_t + u_1u_3\partial_x + u_2u_3\partial_y + (u_3^2 - c^2)\partial_z\right)\Big]\phi = 0, \end{aligned} \tag{18}$$

which is a multiple of Eq. (1), as required.

3.2.2 Generalised Lorentz Transform

Following the methodology of Section 3.1, we will now try to produce a frame that moves with $\{e_i\}$, but that has a simple metric, and so simplifies the governing equation. Here the notation of geometric algebra will be very useful. To produce a frame with a simple metric we will apply a Lorentz transform to the $\{e_i''\}$ frame, since a Lorentz transform preserves the metric. Furthermore we shall apply a Lorentz transform that produces a frame whose time vector (e_0') is parallel to e_0 from the previous section. It is in this sense that the $\{e_i'\}$ frame will move with the $\{e_i\}$ frame.

In geometric algebra, Lorentz transforms have a very neat representation in the form of rotors (see [11, §5.4]). Hence we can produce the frame just described using the simple relation,

$$e_i' = R e_i'' \tilde{R}, \tag{19}$$

where R is the rotor and $\tilde{R}$ denotes the reverse of R (see [11, §4.1.3]). R is given explicitly by the expression,

$$\begin{gathered} R = \exp\left(-\frac{\alpha}{2} e_m e_0''\right), \\ \tanh\alpha = M, \quad e_m = (M_1 e_1'' + M_2 e_2'' + M_3 e_3'')/M, \quad M^2 = M_1^2 + M_2^2 + M_3^2. \end{gathered} \tag{20}$$

Applying this transformation we obtain the $\{e_i'\}$ frame,

$$\begin{gathered} e_0' = \frac{1}{\beta}\left(e_0'' - M e_m\right), \quad e_1' = -\frac{M_1}{\beta} e_0'' + e_1 + \frac{M_1}{M}\frac{1-\beta}{\beta} e_m, \\ e_2' = -\frac{M_2}{\beta} e_0'' + e_2 + \frac{M_2}{M}\frac{1-\beta}{\beta} e_m, \quad e_3' = -\frac{M_3}{\beta} e_0'' + e_3 + \frac{M_3}{M}\frac{1-\beta}{\beta} e_m. \end{gathered} \tag{21}$$

Comparing these definitions with Eq. (16) we see that e_0' is indeed parallel to e_0. Furthermore it is relatively simple to show that the reciprocal frame $\{e^{i'}\}$, volume form V' and metric $g^{ij'}$

are the same as in Eq. (10). From this it follows that $D^2\phi = 0$ can be written as Eq. (14b) in this frame.

So far we have been working in a flat acoustic space-time. Despite this, we appear to have found a frame where we can write Eq. (1) independently of the frame. However, we have so far skipped over deriving the explicit coordinate transformations from (say) $\{x^{i''}\}$ to $\{x^i\}$. To do this, we must use Eq. (7). In fact, by assuming a flat space-time, we have made it possible for the coordinate transformation to not exist. To get around this problem, we will now generalise to a curved space time. The reader may wonder why we went to the trouble of deriving the results for a flat space time, but we will see in the next section that the frame transformations in Eq. (16) and Eq. (21) will in fact be very useful, and without taking our detour through the flat space-time, there is little chance that we would have arrived at these results.

3.3 Transforming in a Curved Acoustic Space-Time

We consider a 4-dimensional (possibly curved) manifold with an associated coordinate system $\{x^i\}$ and frame $\{e_i\}$ defined such that the metric g_{ij} is given by,

$$e_i \cdot e_j = g_{ij} = \begin{bmatrix} h_\tau^2\beta^2 & M_1 h_\tau h & M_2 h_\tau h & M_3 h_\tau h \\ M_1 h_\tau h & -h^2 & 0 & 0 \\ M_2 h_\tau h & 0 & -h^2 & 0 \\ M_3 h_\tau h & 0 & 0 & -h^2 \end{bmatrix}, \tag{22}$$

where symbols have been defined in Section 3.2.1. We can show that the inverse metric g^{ij} and the volume form V are given in this case by,

$$V = h^3 h_\tau = \frac{\rho^2 c_0}{\rho_0^2 c}, \quad g^{ij} = \frac{\rho_0}{\rho c c_0} \begin{bmatrix} c_0^2 & c_0 u_1 & c_0 u_2 & c_0 u_3 \\ c_0 u_1 & u_1^2 - c^2 & u_1 u_2 & u_1 u_3 \\ c_0 u_2 & u_1 u_2 & u_2^2 - c^2 & u_2 u_3 \\ c_0 u_3 & u_1 u_3 & u_2 u_3 & u_3^2 - c^2 \end{bmatrix} \tag{23}$$

From this it is simple to show that $D^2\phi = 0$ written out in this frame is Eq. (18) which we have already noted is a multiple of Eq. (1) if we define the coordinates $x^0 = c_0 t, x^1 = x, x^2 = y, x^3 = z$.

We now need to define the frame $\{e_i'\}$ that spans the tangent space spanned by $\{e_i\}$ and has a simple metric. Now the reader should see why we went to all the trouble of generalising the Galilean and Lorentz transforms. Using Eq. (16) and Eq. (21) we can find precisely this frame transformation, which we can still use locally in the tangent space of our newly defined curved manifold. Hence we define the frame $\{e_i'\}$ as,

$$\begin{gathered} e_0' = \frac{1}{h_\tau \beta} e_0, \quad e_1' = \frac{1}{h} e_1 + \frac{M_1}{\beta h}\left(\frac{1-\beta}{M^2} - 1\right) m - \frac{M_1}{\beta h_\tau} e_0, \\ e_2' = \frac{1}{h} e_2 + \frac{M_2}{\beta h}\left(\frac{1-\beta}{M^2} - 1\right) m - \frac{M_2}{\beta h_\tau} e_0, \quad e_3' = \frac{1}{h} e_3 + \frac{M_3}{\beta h}\left(\frac{1-\beta}{M^2} - 1\right) m - \frac{M_3}{\beta h_\tau} e_0, \end{gathered} \tag{24}$$

where m is defined as $m = M_1 e_1 + M_2 e_2 + M_3 e_3$. We can show that the volume measure V' and metric $g^{ij'}$ for this frame are the same as those given in Eq. (10), as expected. $D^2\phi = 0$ written in this frame gives Eq. (14b).

Now we relate the coordinates of the $\{e_i\}$ frame to those of the $\{e_i'\}$ frame. From Eqs. (5), (6) and (7),

$$e^{i'} = \partial(x^{i'}) = e^k \partial_{x^k}(x^{i'}) \Rightarrow \frac{\partial x^{i'}}{\partial x^j} = e^{i'} \cdot e_j. \tag{25}$$

Therefore, using the frame definitions given in this section, we can derive the set of relationships given in Table 1. The coordinate transformation from $\{x^i\}$ to $\{x^{i'}\}$ must satisfy these relationships.

Table 1. Relationships between the $\{x^i\}$ and $\{x^{i'}\}$ coordinates, found using Eq. (25).

$\partial x^{0'}/\partial x^0$	$h_\tau \beta$	$\partial x^{0'}/\partial x^1$	hM_1/β
$\partial x^{1'}/\partial x^0$	0	$\partial x^{1'}/\partial x^1$	$h(M_1^2+\beta M_2^2+\beta M_3^2)/(M^2\beta)$
$\partial x^{2'}/\partial x^0$	0	$\partial x^{2'}/\partial x^1$	$h[M_1M_2(1-\beta)]/(M^2\beta)$
$\partial x^{3'}/\partial x^0$	0	$\partial x^{3'}/\partial x^1$	$h[M_1M_3(1-\beta)]/(M^2\beta)$
$\partial x^{0'}/\partial x^2$	hM_2/β	$\partial x^{0'}/\partial x^3$	hM_3/β
$\partial x^{1'}/\partial x^2$	$h[M_1M_2(1-\beta)]/(M^2\beta)$	$\partial x^{1'}/\partial x^3$	$h[M_1M_3(1-\beta)]/(M^2\beta)$
$\partial x^{2'}/\partial x^2$	$h(\beta M_1^2+M_2^2+\beta M_3^2)/(M^2\beta)$	$\partial x^{2'}/\partial x^3$	$h[M_2M_3(1-\beta)]/(M^2\beta)$
$\partial x^{3'}/\partial x^2$	$h[M_2M_3(1-\beta)]/(M^2\beta)$	$\partial x^{3'}/\partial x^3$	$h(\beta M_1^2+\beta M_2^2+M_3^2)/(M^2\beta)$

4 Further Manipulations of the Transform

We have derived the 16 partial differential equations that a transformation must satisfy (see Table 1) in terms of the observable coordinates $\{x^i\}$ and the transformed coordinates $\{x^{i'}\}$. These are all scalar fields, and we now map these to a 3-dimensional Euclidean space plus time (the standard manifold of Newtonian mechanics). This allows us to write the relations given in Table 1 in a more illuminating way.

We treat x^1, x^2, x^3 as Cartesian coordinates in the three dimensional space, $t = x^0/c_0$ as time, and the transformed coordinates $\{x^{i'}\}$ as scalar fields over this space (note that this map between the acoustic space-time and a Euclidean space is precisely the map needed to take us from the equation $D^2\phi = 0$ to Eq. (1)). We denote the Cartesian basis vectors of this space $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. If we denote the vector derivative of the 3-dimensional space as $\boldsymbol{\nabla}$, and define the Mach number vector $\boldsymbol{M}$ as $\boldsymbol{M} = M_i\boldsymbol{e}_i$, then we can show that the relations in Table 1 can be written,

$$\begin{gathered}\frac{\partial x^{0'}}{\partial x^0} = h_\tau\beta, \quad \frac{\partial x^{1'}}{\partial x^0} = 0, \quad \boldsymbol{\nabla} x^{0'} = \frac{h}{\beta}\boldsymbol{M}, \quad \boldsymbol{\nabla} x^{1'} = h\frac{1-\beta}{M^2\beta}M_1\boldsymbol{M} + h\boldsymbol{e}_1, \\ \frac{\partial x^{2'}}{\partial x^0} = 0, \quad \frac{\partial x^{3'}}{\partial x^0} = 0, \quad \boldsymbol{\nabla} x^{2'} = h\frac{1-\beta}{M^2\beta}M_2\boldsymbol{M} + h\boldsymbol{e}_2, \quad \boldsymbol{\nabla} x^{3'} = h\frac{1-\beta}{M^2\beta}M_3\boldsymbol{M} + h\boldsymbol{e}_3,\end{gathered} \tag{26}$$

Combining the expressions for $\boldsymbol{\nabla} x^{0'}$, $\boldsymbol{\nabla} x^{1'}$, $\partial x^{0'}/\partial x^0$ and $\partial x^{1'}/\partial x^0$ we can derive,

$$\frac{\partial}{\partial x^0}\left(\frac{1-\beta}{M^2}M_1\right)\frac{h}{\beta}\boldsymbol{M} + \frac{1-\beta}{M^2}M_1\boldsymbol{\nabla}(h_\tau\beta) + \frac{\partial h}{\partial x^0}\boldsymbol{e}_1 = 0. \tag{27}$$

This is a constraint that the background flow must satisfy if the transformation is to exist. We have not been able to show that this is satisfied in the general case, for instance we cannot find any equivalence between this and the compressible potential flow equations. This would seem to suggest that a transformation is not possible in the general case, however, there are some special cases still of interest. If we assume that the background flow is steady then the condition that must be satisfied in order for the transformation to exist becomes,

$$\boldsymbol{\nabla}(h_\tau\beta) = 0 \quad \Rightarrow \quad \boldsymbol{\nabla}\left(\sqrt{\rho c(1-M^2)}\right) = 0. \tag{28}$$

Again, we have been unable to show any equivalence between this and the compressible potential flow equations, however it is clear that, if the flow is uniform, or incompressible (low Mach number), then this will be satisfied. In both of these cases $h = h_\tau = 1$. For uniform flow the relations in Table 1 can be integrated directly. If we also assume that flow is only in the z

direction then we obtain the transformation given in Eq. (15), as expected. For incompressible steady flow, the relations in Eq. (26) simplify to become,

$$\frac{\partial x^{0'}}{\partial x^0} = 1, \quad \frac{\partial x^{i'}}{\partial x^0} = 0, \quad \boldsymbol{\nabla} x^{0'} = \boldsymbol{M}, \quad \boldsymbol{\nabla} x^{i'} = \boldsymbol{e}_i, \tag{29}$$

where $i = 1, 2, 3$. The background flow is potential, so we may write $\boldsymbol{M} = \boldsymbol{\nabla}\Phi$. From this we see that the transformation for an incompressible, irrotational, steady, barotropic background flow is,

$$x^{0'} = x^0 + \Phi, \quad x^{1'} = x^1, \quad x^{2'} = x^2, \quad x^{3'} = x^3. \tag{30}$$

This agrees with a result presented by Taylor [5], however, as pointed out by Astley [7], Taylor had to make an assumption about the frequency of the acoustic field in order to derive the transform. We have removed this requirement, generalising Taylor's transform to all frequencies.

5 Conclusions

An acoustic space-time was used to derive the requirements that must be satisfied by a variable transformation to solve for sound propagation in the presence of irrotational, barotropic background flow. By comparison to previous work deriving such transformations our approach is more systematic, generalising the combined Galilean and Lorentz transformations used when the flow is uniform.

We have shown that in order for the transformation to exist the background flow must satisfy additional constraints to the compressible potential flow equations, and that these additional constraints are satisfied automatically if the background flow is steady, and either uniform, or incompressible. In the former case we have shown that the derived transformation agrees with the standard transformation for uniform flow. In the latter case we show that our transformation agrees with one presented before by Taylor; but, unlike Taylor, we do not need to make any assumption about the frequency of the acoustic field, and so we have extended the validity to low frequency acoustic problems.

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