Becker, Christian and Benini, Marco and Schenkel, Alexander and Szabo, Richard J. (2017) Abelian duality on globally hyperbolic spacetimes. Communications in Mathematical Physics, 349 (1). pp. 361-392. ISSN 1432-0916

Abelian duality on globally hyperbolic spacetimes

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May 2016

Abstract

We study generalized electric/magnetic duality in Abelian gauge theory by combining techniques from locally covariant quantum field theory and Cheeger-Simons differential cohomology on the category of globally hyperbolic Lorentzian manifolds. Our approach generalizes previous treatments using the Hamiltonian formalism in a manifestly covariant way and without the assumption of compact Cauchy surfaces. We construct semi-classical configuration spaces and corresponding presymplectic Abelian groups of observables, which are quantized by the CCR-functor to the category of C^*-algebras. We demonstrate explicitly how duality is implemented as a natural isomorphism between quantum field theories. We apply this formalism to develop a fully covariant quantum theory of self-dual fields.

Report no.: EMPG–15–13

Keywords: Abelian gauge theory, differential cohomology, Dirac charge quantization, Abelian duality, self-dual Abelian gauge fields, algebraic quantum field theory

MSC 2010: 81T13, 81T05, 53C08, 55N20

Contents

1 Introduction and summary

Dualities in string theory have served as a rich source of conjectural relations between seemingly disparate situations in mathematics and physics, particularly in some approaches to quantum field theory. Heuristically, a 'duality' is an equivalence between two descriptions of the same quantum theory in different classical terms, and it typically involves an interchange of classical and quantum data. The prototypical example is electric/magnetic duality of Maxwell theory on a four-manifold M: Magnetic flux is discretized at the classical level by virtue of the fact that it originates as the curvature of a line bundle on M, whereas electric flux discretization is a quantum effect arising via Dirac charge quantization. The example of electric/magnetic duality in Maxwell theory has a generalization to any spacetime dimensionality, of relevance to the study of fluxes in string theory, which we may collectively refer to as 'Abelian duality'. The configuration spaces of these (generalized) Abelian gauge theories are mathematically modeled by suitable (generalized) differential cohomology groups, see e.g. [Fre00, Sza12] for reviews.

In this paper we will describe a new perspective on Abelian duality by combining methods from Cheeger-Simons differential cohomology and locally covariant quantum field theory; this connection between Abelian gauge theory and differential cohomology was originally pursued by [BSS14]. The quantization of Abelian gauge theories was described from a Hamiltonian perspective by [FMS07a, FMS07b], where the representation theory of Heisenberg groups was used to define the quantum Hilbert space of an Abelian gauge theory in a manifestly duality invariant way. In the present work we shall instead build the semi-classical configuration space for dual gauge field configurations in a fully covariant fashion, which agrees with that proposed by [FMS07a, FMS07b] upon fixing a Cauchy surface Σ in a globally hyperbolic spacetime M, but which is manifestly independent of the choice of Σ. Following the usual ideas of algebraic quantum field theory, we construct not a quantum Hilbert space of states but rather a C^*-algebra of quantum observables; the requisite natural presymplectic structure also agrees with that of [FMS07a, FMS07b] upon fixing a Cauchy surface Σ, but is again independent of the choice of Σ. Our approach thereby lends a new perspective on the phenomenon of Abelian duality, and it enables a rigorous (functorial) definition of quantum duality as a natural isomorphism between quantum field theory functors. An alternative rigorous perspective on Abelian duality has been recently proposed by [Ell14] using the factorization algebra approach to (Euclidean) quantum field theory. We do not yet understand how to describe the full duality groups, i.e. the analogues of the $SL(2,\mathbb{Z})$ S-duality group of Maxwell theory, as this in principle requires a detailed understanding of the automorphism groups of our quantum field theory functors [Few13], which is beyond the scope of the present paper.

Our approach also gives a novel and elegant formulation of the quantum theory of self-dual fields, which is an important ingredient in the formulation of string theory and supergravity: In two dimensions the self-dual gauge field is a worldsheet periodic chiral scalar field in heterotic string theory whose quantum Hilbert space carries representations of the usual (affine) Heisenberg algebra; in six dimensions the self-dual field is an Abelian gerbe connection which lives on the worldvolume of M5-branes and NS5-branes, and in the evasive superconformal $(2,0)$ theory whose quantum Hilbert space should similarly carry irreducible representations of the corresponding Heisenberg group; in ten dimensions the self-dual field is the Ramond-Ramond four-form potential of Type IIB supergravity. The two generic issues associated with the formulation of the self-dual field theory are: (a) The lack of covariant local Lagrangian formulation of the theory (without certain choices, cf. [BM06]); and: (b) The reconciliation of the self-duality equation with Dirac quantization requires the simultaneous discretization of both electric and magnetic fluxes in the same semi-classical theory. Our quantization of Abelian gauge theories at the level of algebras of quantum observables eludes both of these problems. In particular, the noncommutativity of torsion fluxes observed by [FMS07a, FMS07b] is also straightforwardly evident in our approach. As in [FMS07a, FMS07b], our quantization of the self-dual field does not follow from the approach developed in the rest of this paper. Other Abelian self-dual gauge theories can be analyzed starting from generalized differential cohomology theories fulfilling a suitable self-duality property, e.g. differential K-theory, see [FMS07a, FMS07b] for the Hamiltonian point of view. An approach closer to the one pursued in the present paper is possible also in these cases provided one has suitable control on the properties of the relevant generalized differential cohomology theory.

In addition to being cast in a manifestly covariant framework, another improvement on the development of [FMS07a, FMS07b] is that our approach does not require the spacetime to admit compact Cauchy surfaces. Our main technical achievement is the development of a suitable theory of Cheeger-Simons differential characters with compact support and Pontryagin duality, in a manner which does not destroy the Abelian duality. As the mathematical details of this theory are somewhat involved and of independent interest, they have been delegated to a companion paper [BBSS15] to which we frequently refer. The present paper focuses instead on the aspects of interest in physics.

The outline of the remainder of this paper is as follows. In Section 2 we introduce and analyze the semi-classical configuration spaces of dual gauge fields in the language of differential cohomology; our main result is the identification of this space with the space of solutions of a well-posed Cauchy problem which agrees with the description of [FMS07a, FMS07b], but in a manifestly covariant fashion and without the assumption of compactness of Cauchy surfaces. In Section 3 we analogously study a suitable space of dual gauge field configurations of spacelike compact support, and show in Section 4 that it is isomorphic to a suitable Abelian group of observables defined in the spirit of smooth Pontryagin duality as in [BSS14]. In Section 5 we consider the quantization of the semi-classical gauge theories and the extent to which they satisfy the axioms of locally covariant quantum field theory [BFV03]; we show that, just as in [BSS14], our quantum field theory functors satisfy the causality and time-slice axioms but violate the locality axiom.[1] In Section 6 we show that dualities extend to the quantum field theories thus defined. In Section 7 we apply our formalism to give a proper covariant formulation of the quantum field theory of a self-dual field. An appendix at the end of the paper provides some technical details of constructions which are used in the main text.

2 Dual gauge fields

In this section we describe and analyze the configuration spaces of the (higher) gauge theories that will be of interest in this paper. Their main physical feature is a discretization of both electric and magnetic fluxes, which is motivated by Dirac charge quantization. To simplify notation, we normalize both electric and magnetic fluxes so that they are quantized in the same integer lattice $\mathbb{Z} \subset \mathbb{R}$. Because Dirac charge quantization arises as a quantum effect (i.e. it depends on Planck's constant $\hbar$, which in our conventions is equal to 1), we shall use the attribution "semi-classical" for the gauge field configurations introduced below. In this paper all manifolds are implicitly assumed to be smooth, connected, oriented and of finite type, i.e. they admit a finite good cover.

2.1 Semi-classical configuration space

Let M be a manifold. The integer cohomology group $\mathrm{H}^k(M;\mathbb{Z})$ of degree k is an Abelian group which has a (non-canonical) splitting $\mathrm{H}^k(M;\mathbb{Z}) \simeq \mathrm{H}^k_{\mathrm{free}}(M;\mathbb{Z}) \oplus \mathrm{H}^k_{\mathrm{tor}}(M;\mathbb{Z})$ into free and torsion subgroups, respectively. Let $\Omega^k_{\mathbb{Z}}(M) \subset \Omega^k(M)$ denote the closed differential k-forms on M with integer periods. Below we recall the definition of Cheeger-Simons differential characters [CS85].

Definition 2.1. A degree k Cheeger-Simons differential character on a manifold M is a group homomorphism $h : Z_{k-1}(M) \to \mathbb{T}$ from the group $Z_{k-1}(M)$ of $k-1$-cycles on M to the circle group $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for which there exists a differential form $\omega_h \in \Omega^k(M)$ such that

$$h(\partial\gamma) = \int_\gamma \omega_h \mod \mathbb{Z} \ , \qquad \forall \gamma \in C_k(M) \ , \tag{2.1}$$

where $\partial\gamma$ denotes the boundary of the k-chain γ. The Abelian group of Cheeger-Simons differential characters is denoted by $\hat{\mathrm{H}}^k(M;\mathbb{Z})$.

For a modern perspective on differential cohomology which includes the Cheeger-Simons model see [SS08, BB14]. We use the degree conventions of [BB14] in which the curvature of a differential character in $\hat{\mathrm{H}}^k(M;\mathbb{Z})$ is a k-form. The assignment of $\hat{\mathrm{H}}^k(M;\mathbb{Z})$ to each manifold M is a contravariant functor

$$\hat{\mathrm{H}}^k(-;\mathbb{Z}) : \mathsf{Man}^{\mathrm{op}} \longrightarrow \mathsf{Ab} \tag{2.2}$$

from the category Man of manifolds to the category Ab of Abelian groups. For notational convenience, we simply denote by f^* the group homomorphism $\hat{\mathrm{H}}^k(f;\mathbb{Z}) : \hat{\mathrm{H}}^k(M';\mathbb{Z}) \to \hat{\mathrm{H}}^k(M;\mathbb{Z})$ for any smooth map $f : M \to M'$. The functor (2.2) comes together with four natural transformations which are given by the curvature map $\mathrm{curv} : \hat{\mathrm{H}}^k(-;\mathbb{Z}) \Rightarrow \Omega^k_{\mathbb{Z}}(-)$, the characteristic class map $\mathrm{char} : \hat{\mathrm{H}}^k(-;\mathbb{Z}) \Rightarrow \mathrm{H}^k(-;\mathbb{Z})$, the inclusion of topologically trivial fields

[1] The violation of locality is due to topological properties of the spacetime M and owes to the fact that differential cohomology constructs the pertinent configuration spaces as gauge orbit spaces. As a matter of fact, all approaches to gauge theory in the context of general local covariance [BFV03] exhibit at least some remnant of the failure of locality, see [BSS14, BDHS14, BDS14a, DL12, FL14, DS13, SDH14]. There are indications that the tension between locality and gauge theory can be solved by means of homotopical techniques (in the context of model categories), see [BSS15] for the first steps towards this goal.

$\iota : \Omega^{k-1}(-)/\Omega^{k-1}_{\mathbb{Z}}(-) \Rightarrow \hat{\mathrm{H}}^k(-;\mathbb{Z})$ and the inclusion of flat fields $\kappa : \mathrm{H}^{k-1}(-;\mathbb{T}) \Rightarrow \hat{\mathrm{H}}^k(-;\mathbb{Z})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group. The (functorial) diagram of Abelian groups

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathrm{H}^{k-1}(M;\mathbb{R})}{\mathrm{H}^{k-1}_{\mathrm{free}}(M;\mathbb{Z})} & \xrightarrow{\tilde{\kappa}} & \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathbb{Z}}(M)} & \xrightarrow{\mathrm{d}} & \mathrm{d}\Omega^{k-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \subseteq} & & \\
0 & \longrightarrow & \mathrm{H}^{k-1}(M;\mathbb{T}) & \xrightarrow{\kappa} & \hat{\mathrm{H}}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}} & \Omega^k_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{char}} & & \downarrow{\scriptstyle [\cdot]} & & \\
0 & \longrightarrow & \mathrm{H}^k_{\mathrm{tor}}(M;\mathbb{Z}) & \longrightarrow & \mathrm{H}^k(M;\mathbb{Z}) & \longrightarrow & \mathrm{H}^k_{\mathrm{free}}(M;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.3}$$

is a commutative diagram whose rows and columns are short exact sequences.

In the remainder of this paper we shall take M to be a time-oriented m-dimensional globally hyperbolic Lorentzian manifold, which we regard as 'spacetime'; for a thorough discussion of Lorentzian geometry including global hyperbolicity see e.g. [BEE96, O'N83], while a brief overview can be found in e.g. [BGP07, Section 1.3]. The semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$ of interest to us is obtained as the pullback

$$\begin{array}{ccc}
\mathfrak{C}^k(M;\mathbb{Z}) & \dashrightarrow & \hat{\mathrm{H}}^{m-k}(M;\mathbb{Z}) \\
\vdots & & \downarrow{\scriptstyle *\,\mathrm{curv}} \\
\hat{\mathrm{H}}^k(M;\mathbb{Z}) & \xrightarrow[\mathrm{curv}]{} & \Omega^k(M)
\end{array} \tag{2.4}$$

By definition, any element $(h,\tilde{h}) \in \mathfrak{C}^k(M;\mathbb{Z}) \subseteq \hat{\mathrm{H}}^k(M;\mathbb{Z}) \times \hat{\mathrm{H}}^{m-k}(M;\mathbb{Z})$ has the property that the curvature of h is the Hodge dual of the curvature of $\tilde{h}$, i.e. $\mathrm{curv}\, h = *\,\mathrm{curv}\,\tilde{h}$. We may interpret this condition as being responsible for the quantization of electric fluxes: the de Rham cohomology class of the Hodge dual curvature $*\,\mathrm{curv}\, h$ is also an element in $\mathrm{H}^{m-k}_{\mathrm{free}}(M;\mathbb{Z})$ and hence electric fluxes are quantized in the same lattice $\mathbb{Z} \subset \mathbb{R}$ as magnetic fluxes. In a similar fashion, we introduce the semi-classical topologically trivial fields $\mathfrak{T}^k(M;\mathbb{Z})$ as the pullback

$$\begin{array}{ccc}
\mathfrak{T}^k(M;\mathbb{Z}) & \dashrightarrow & \frac{\Omega^{m-k-1}(M)}{\Omega^{m-k-1}_{\mathbb{Z}}(M)} \\
\vdots & & \downarrow{\scriptstyle *\,\mathrm{d}} \\
\frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathbb{Z}}(M)} & \xrightarrow[\mathrm{d}]{} & \Omega^k(M)
\end{array} \tag{2.5}$$

To simplify notation we will adopt the following useful convention: For any graded Abelian group $A^\sharp = \bigoplus_{k\in\mathbb{Z}} A^k$, we introduce

$$A^{p,q} := A^p \times A^q \ . \tag{2.6}$$

Using (2.3) we introduce a new commutative diagram of Abelian groups with exact rows and columns, whose central object is the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$.

Theorem 2.2. *Consider the two group homomorphisms*

$$\mathrm{curv}_1 : \mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M) \ , \qquad (h,\tilde{h}) \longmapsto \mathrm{curv}\, h = *\,\mathrm{curv}\, \tilde{h} \tag{2.7a}$$

and

$$\mathrm{d}_1 : \mathfrak{T}^k(M;\mathbb{Z}) \longrightarrow \mathrm{d}\Omega^{k-1} \cap *\mathrm{d}\Omega^{m-k-1}(M) \ , \qquad ([A],[\tilde{A}]) \longmapsto \mathrm{d}A = *\mathrm{d}\tilde{A} \ . \tag{2.7b}$$

Then the diagram of Abelian groups

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{\mathrm{H}^{k-1,m-k-1}(M;\mathbb{R})}{\mathrm{H}^{k-1,m-k-1}_{\mathrm{free}}(M;\mathbb{Z})} & \xrightarrow{\tilde{\kappa}\times\tilde{\kappa}} & \mathfrak{T}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{d}_1} & \mathrm{d}\Omega^{k-1} \cap *\,\mathrm{d}\Omega^{m-k-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \iota\times\iota}\downarrow & & \downarrow{\scriptstyle \subseteq} & & \\
0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(M;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \mathfrak{C}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}_1} & \Omega^k_{\mathbb{Z}} \cap *\,\Omega^{m-k}_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \mathrm{char}\times\mathrm{char}}\downarrow & & \downarrow{\scriptstyle ([\,\cdot\,],[*^{-1}\,\cdot\,])} & & \\
0 & \longrightarrow & \mathrm{H}^{k,m-k}_{\mathrm{tor}}(M;\mathbb{Z}) & \longrightarrow & \mathrm{H}^{k,m-k}(M;\mathbb{Z}) & \longrightarrow & \mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.8}$$

is a commutative diagram whose rows and columns are short exact sequences.

Proof. Commutativity of this diagram follows by construction. Hence we focus on proving that the rows and columns are exact. The bottom row and the left column are exact because they are Cartesian products of exact sequences. Injectivity of $\iota \times \iota$, $\kappa \times \kappa$ and $\tilde{\kappa} \times \tilde{\kappa}$ is immediate by (2.3).

Let us now show that d_1 and curv_1 are surjective. Given $\mathrm{d}A = *\,\mathrm{d}\tilde{A}$ for $A \in \Omega^{k-1}(M)$ and $\tilde{A} \in \Omega^{m-k-1}(M)$, we note that $([A],[\tilde{A}])$ is an element of $\mathfrak{T}^k(M;\mathbb{Z})$ and $\mathrm{d}_1([A],[\tilde{A}]) = \mathrm{d}A = *\,\mathrm{d}\tilde{A}$, thus showing that d_1 is surjective. A similar argument applies to curv_1 using surjectivity of $\mathrm{curv} : \hat{\mathrm{H}}^p(M;\mathbb{Z}) \to \Omega^p_{\mathbb{Z}}(M)$ for $p = k$ and for $p = m-k$.

To show that $([\,\cdot\,],[*^{-1}\,\cdot\,])$ is also surjective, let us take any $(z,\tilde{z}) \in \mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z}) \subseteq \mathrm{H}^{k,m-k}(M;\mathbb{R})$ and recall that by de Rham's theorem it can be presented as $(z,\tilde{z}) = ([\omega],[\tilde{\omega}])$, for some $\omega \in \Omega^k_{\mathbb{Z}}(M)$ and $\tilde{\omega} \in \Omega^{m-k}_{\mathbb{Z}}(M)$. Let $\delta = (-1)^{m\,(k-1)} *\mathrm{d}\, * : \Omega^k(M) \to \Omega^{k-1}(M)$ denote the codifferential. We solve the equations $[\delta\theta] = [\omega] \in \mathrm{H}^k_{\mathrm{dR}}(M)$ and $[\delta\tilde{\theta}\,] = [\tilde{\omega}] \in \mathrm{H}^{m-k}_{\mathrm{dR}}(M)$ for $\theta \in \Omega^{k+1}(M)$ and $\tilde{\theta} \in \Omega^{m-k+1}(M)$.[2] Introducing $F = \delta\theta + *\,\delta\tilde{\theta}$, we find $[F] = [\delta\theta] = [\omega] \in \mathrm{H}^k_{\mathrm{dR}}(M)$ and $[*^{-1}F] = [\delta\tilde{\theta}\,] = [\tilde{\omega}] \in \mathrm{H}^{m-k}_{\mathrm{dR}}(M)$; in particular, both F and $*^{-1}F$ have integral periods since so do ω and $\tilde{\omega}$. We conclude that $F \in \Omega^k_{\mathbb{Z}} \cap *\,\Omega^{m-k}_{\mathbb{Z}}(M)$.

Surjectivity of $\mathrm{char} \times \mathrm{char}$ follows from what we have already shown above and by using a diagram chasing argument. Take any $(x,\tilde{x}) \in \mathrm{H}^{k,m-k}(M;\mathbb{Z})$. Mapping to the corresponding free group and recalling that both $([\,\cdot\,],[*^{-1}\,\cdot\,])$ and curv_1 are epimorphisms, we find $(h,\tilde{h}) \in \mathfrak{C}^k(M;\mathbb{Z})$ whose image along $([\,\cdot\,],[*^{-1}\,\cdot\,]) \circ \mathrm{curv}_1$ matches the image of $(x,\tilde{x})$ in $\mathrm{H}^{k,m-k}_{\mathrm{free}}(M;\mathbb{Z})$. By exactness of the bottom row, $(\mathrm{char}\, h, \mathrm{char}\, \tilde{h})$ differs from $(x,\tilde{x})$ by an element $(t,\tilde{t}\,)$ of the

[2]To show that a solution exists, let us introduce the d'Alembert operator $\Box = \delta\,\mathrm{d} + \mathrm{d}\,\delta$ and consider its retarded/advanced Green's operators $G^{\pm}$, cf. [Bär15, BGP07]. Let us also consider a partition of unity $\{\chi_+,\chi_-\}$ on M such that $\chi_{\pm}$ has past/future compact support, see [Bär15] for a definition of these support systems. Then $\theta = G(\mathrm{d}\chi_+ \wedge \omega)$ is a solution, where $G = G^+ - G^-$ is the causal propagator. In fact $\delta\theta = \delta\,\mathrm{d}(G^+(\chi_+\,\omega) + G^-(\chi_-\,\omega)) = \omega - \mathrm{d}\,\delta(G^+(\chi_+\,\omega) + G^-(\chi_-\,\omega))$. A similar argument applies to $\tilde{\theta}$.

torsion subgroup $\mathrm{H}^{k,m-k}_{\mathrm{tor}}(M;\mathbb{Z})$, i.e. $(x,\tilde{x}) = (\operatorname{char} h + t, \operatorname{char} \tilde{h} + \tilde{t})$. Exactness of the left column allows us to find a preimage $(u,\tilde{u}) \in \mathrm{H}^{k-1,m-k-1}(M;\mathbb{T})$ for $(t,\tilde{t})$. Commutativity of the diagram then implies that $(h + \kappa\, u, \tilde{h} + \kappa\, \tilde{u})$ is a preimage of $(x,\tilde{x})$ via $\operatorname{char} \times \operatorname{char}$.

We still have to check that the first two rows and the last two columns are exact at their middle objects. This is a straightforward consequence of the exactness of the corresponding rows and columns in (2.3). □

Remark 2.3. To better motivate the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$ we establish below its relation with Maxwell theory. For this purpose we consider the case $m = 4$ and $k = 2$. The usual Maxwell equations (without external sources) for the Faraday tensor $F \in \Omega^2(M)$ are $\mathrm{d}F = 0$ and $\mathrm{d} * F = 0$. These equations are invariant under electric-magnetic duality, i.e. under the exchange of F and $*F$. The standard approach to gauge theory consists in the replacement of F with the curvature of (the isomorphism class of) a circle bundle with connection (equivalently, a differential cohomology class in degree 2). In this framework, however, $*F$ does not have any geometric interpretation, hence the original electric-magnetic duality of Maxwell theory is lost passing to gauge theory. Nevertheless, one can present Maxwell equations in an equivalent way, which is however better suited for a gauge theoretic extension preserving electric-magnetic duality:

$$F = *\tilde{F} \ , \qquad \mathrm{d}F = 0 \ , \qquad \mathrm{d}\tilde{F} = 0 \ . \tag{2.9}$$

Interpreting both F and $\tilde{F}$ as the curvatures of circle bundles with connections, the semi-classical configuration space $\mathfrak{C}^2(M;\mathbb{Z})$ is obtained and the original electric-magnetic duality of Maxwell theory is lifted to $\mathfrak{C}^2(M;\mathbb{Z})$, see Section 6 for the situation in arbitrary spacetime dimension and degree. Notice that the semi-classical configuration space has the same local "degrees of freedom" as Maxwell theory. In fact, on a contractible spacetime $\mathfrak{C}^2(M;\mathbb{Z})$ reduces to the top-right corner in diagram (2.8). Since exact and closed forms are the same on a contractible manifold, Maxwell theory is recovered. In conclusion, the semi-classical configuration space $\mathfrak{C}^2(M;\mathbb{Z})$ is a gauge theoretic extension of Maxwell theory that carries the same local information, however preserving electric-magnetic duality by matching the relevant topological (as opposed to local) data in a suitable way. As a by-product, any configuration $(h,\tilde{h}) \in \mathfrak{C}^2(M;\mathbb{Z})$ realizes the discretization of magnetic and electric fluxes, which arise as the characteristic classes $\operatorname{char} h, \operatorname{char} \tilde{h} \in \mathrm{H}^2(M;\mathbb{Z})$. This argument can be made general for higher gauge theories in arbitrary spacetime dimension.

Remark 2.4. The semi-classical configuration space is a contravariant functor

$$\mathfrak{C}^k(-;\mathbb{Z}) : \mathsf{Loc}_m^{\mathrm{op}} \longrightarrow \mathsf{Ab} \tag{2.10}$$

from the category Loc_m of time-oriented m-dimensional globally hyperbolic Lorentzian manifolds with causal embeddings[3] as morphisms to the category Ab of Abelian groups. For notational convenience, we simply denote by f^* the group homomorphism $\mathfrak{C}^k(f;\mathbb{Z}) : \mathfrak{C}^k(M';\mathbb{Z}) \to \mathfrak{C}^k(M;\mathbb{Z})$ associated with a morphism $f : M \to M'$ in Loc_m.

2.2 Cauchy problem

We will now show that the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$ is the space of solutions of a well-posed Cauchy problem. Let us start by recalling a well-known result for the Cauchy

[3]A causal embedding $f : M \to M'$ between time-oriented m-dimensional globally hyperbolic Lorentzian manifolds is an orientation and time-orientation preserving isometric embedding, whose image is open and causally compatible, i.e. $J^{\pm}_{M'}(f(p)) \cap f(M) = f(J^{\pm}_M(p))$ for all $p \in M$; here $J^{\pm}_M(p)$ denotes the causal future/past of $p \in M$ consisting of all points of M which can be reached by a future/past-directed smooth causal curve stemming from p, see [BGP07].

problem of the Faraday tensor, see e.g. [DL12, FL14] and also [BF09, Chapter 3, Corollary 5] for details on how to treat initial data of not necessarily compact support. For the related Cauchy problem of the gauge potential see [SDH14]. Throughout this paper Σ will denote a smooth spacelike Cauchy surface of M with embedding $\iota_\Sigma : \Sigma \to M$ into M.

Theorem 2.5. *For each $(B, \tilde{B}) \in \Omega^{k,m-k}_{\mathrm{d}}(\Sigma)$ (where the subscript $_{\mathrm{d}}$ denotes closed forms), there exists a unique solution $F \in \Omega^k(M)$ to the initial value problem*

$$\mathrm{d}F = 0 \ , \qquad \iota_\Sigma^* F = B \ , \tag{2.11a}$$

$$\mathrm{d} *^{-1} F = 0 \ , \qquad \iota_\Sigma^* *^{-1} F = \tilde{B} \ , \tag{2.11b}$$

whose support is contained in the causal future and past of the support of the initial data, i.e. $\operatorname{supp} F \subseteq J(\operatorname{supp} B \cup \operatorname{supp} \tilde{B})$.

We consider also the similar well-posed initial value problem for $\tilde{F} \in \Omega^{m-k}(M)$ given by

$$\mathrm{d}\tilde{F} = 0 \ , \qquad \iota_\Sigma^* \tilde{F} = \tilde{B} \ , \tag{2.12a}$$

$$\mathrm{d} * \tilde{F} = 0 \ , \qquad \iota_\Sigma^* * \tilde{F} = B \ , \tag{2.12b}$$

where the initial data are also specified by $(B, \tilde{B}) \in \Omega^{k,m-k}_{\mathrm{d}}(\Sigma)$.

Given now any initial data $(B, \tilde{B}) \in \Omega^{k,m-k}_{\mathrm{d}}(\Sigma)$, let us consider the corresponding unique solutions F and $\tilde{F}$ of the Cauchy problems (2.11) and (2.12). This implies that $F - *\tilde{F}$ solves the Cauchy problem (2.11) with vanishing initial data, and therefore $F = *\tilde{F}$. We further show that, given initial data $(B, \tilde{B}) \in \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma)$ with integral periods, the corresponding solution F of the Cauchy problem (2.11) is such that both F and $*^{-1}F$ have integral periods. For this, using the results of Lemma A.1 (i) we can express each k-cycle $\gamma \in Z_k(M)$ as $\gamma = \iota_{\Sigma\,*}\, \pi_{\Sigma\,*}\gamma + \partial h_\Sigma \gamma$, and hence

$$\int_\gamma F = \int_{\pi_{\Sigma\,*}\gamma} \iota_\Sigma^* F + \int_{h_\Sigma \gamma} \mathrm{d}F = \int_{\pi_{\Sigma\,*}\gamma} B \ \in \ \mathbb{Z} \ . \tag{2.13}$$

Similarly, for each $m{-}k$-cycle $\tilde{\gamma} \in Z_{m-k}(M)$ we have

$$\int_{\tilde{\gamma}} *^{-1} F = \int_{\pi_{\Sigma\,*}\tilde{\gamma}} \iota_\Sigma^* *^{-1} F + \int_{h_\Sigma \tilde{\gamma}} \mathrm{d} *^{-1} F = \int_{\pi_{\Sigma\,*}\tilde{\gamma}} \tilde{B} \ \in \ \mathbb{Z} \ . \tag{2.14}$$

Conversely, given $F \in \Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M)$ we have $\iota_\Sigma^* F \in \Omega^k_{\mathbb{Z}}(\Sigma)$ and $\iota_\Sigma^* *^{-1} F \in \Omega^{m-k}_{\mathbb{Z}}(\Sigma)$. Summing up, we obtain

Corollary 2.6. *The embedding $\iota_\Sigma : \Sigma \to M$ of Σ into M induces an isomorphism of Abelian groups*

$$\Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M) \underset{\mathrm{solve}_\Sigma}{\overset{(\iota_\Sigma^*, \iota_\Sigma^* *^{-1})}{\rightleftarrows}} \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma) \quad , \tag{2.15}$$

whose inverse solve_Σ *is the map assigning to initial data $(B, \tilde{B}) \in \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma)$ the corresponding unique solution $F \in \Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M)$ of the Cauchy problem* (2.11).

Let us consider the central row of the diagram (2.8). Taking into account also naturality of κ and curv, one finds that the diagram of Abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(M;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \mathfrak{C}^k(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}_1} & \Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \iota_\Sigma^*\times\iota_\Sigma^*} & & \Big\downarrow{\scriptstyle \iota_\Sigma^*\times\iota_\Sigma^*} & & \Big\downarrow{\scriptstyle (\iota_\Sigma^*,\iota_\Sigma^* *^{-1})} & & \\ 0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(\Sigma;\mathbb{T}) & \underset{\kappa\times\kappa}{\longrightarrow} & \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) & \underset{\mathrm{curv}\times\mathrm{curv}}{\longrightarrow} & \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma) & \longrightarrow & 0 \end{array} \tag{2.16}$$

commutes and its rows are short exact sequences. Using also Lemma A.1 (ii), Corollary 2.6 and the five lemma, we obtain

Theorem 2.7. *The embedding $\iota_\Sigma : \Sigma \to M$ induces an isomorphism of Abelian groups*

$$\mathfrak{C}^k(M;\mathbb{Z}) \xrightarrow{\iota_\Sigma^\ast \times \iota_\Sigma^\ast} \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) \ . \tag{2.17}$$

We can interpret the result of Theorem 2.7 as establishing the well-posedness of the initial value problem for $(h,\tilde{h}) \in \hat{\mathrm{H}}^{k,m-k}(M;\mathbb{Z})$ given by

$$\mathrm{curv}\, h = \ast\, \mathrm{curv}\, \tilde{h} \ , \qquad \iota_\Sigma^\ast h = h_\Sigma \ , \quad \iota_\Sigma^\ast \tilde{h} = \tilde{h}_\Sigma \ , \tag{2.18}$$

for initial data $(h_\Sigma, \tilde{h}_\Sigma) \in \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$. It follows that the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$ arises as the space of solutions of this Cauchy problem.

Remark 2.8. If M has compact Cauchy surfaces Σ, we can easily endow $\mathfrak{C}^k(M;\mathbb{Z})$ with the structure of a presymplectic Abelian group induced by the ring structure $\cdot$ on differential characters, see [CS85, SS08, BB14]. For this, we define the circle-valued presymplectic structure

$$\sigma : \mathfrak{C}^k(M;\mathbb{Z}) \times \mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \mathbb{T} \ , \qquad \big((h,\tilde{h})\, ,\, (h',\tilde{h}'\,)\big) \longmapsto \big(\iota_\Sigma^\ast(\tilde{h}\cdot h' - \tilde{h}'\cdot h)\big)[\Sigma] \ , \tag{2.19}$$

where $[\Sigma] \in \mathrm{H}_{m-1}(\Sigma)$ denotes the fundamental class of Σ. Using compatibility between the ring structure on differential characters and the natural transformations ι, κ, curv and char, one can show that σ is in fact independent of the choice of Σ. Fixing any Cauchy surface Σ and using the isomorphism given in Theorem 2.7, the presymplectic structure (2.19) can be induced to initial data and thereby agrees with the one constructed by [FMS07b, FMS07a] from a Hamiltonian perspective. However, in contrast to [FMS07b, FMS07a] our construction does *not* depend on the choice of a Cauchy surface, i.e. it is generally covariant. As we show in Section 4, the assumption of compactness of the Cauchy surfaces can be dropped, provided that one introduces a suitable support restriction on the semi-classical gauge fields.

3 Dual gauge fields with spacelike compact support

In this section we introduce and analyze a suitable Abelian group $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ of semi-classical gauge fields of spacelike compact support. Similarly to the case of the usual quantum field theories on curved spacetimes, such as Klein-Gordon theory, the role played by $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ will be dual to that of the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$; in fact, we shall show in Section 4 that elements in $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ define functionals (i.e. classical observables) on $\mathfrak{C}^k(M;\mathbb{Z})$ which are group characters $\mathfrak{C}^k(M;\mathbb{Z}) \to \mathbb{T}$. This dual role of the semi-classical gauge fields of spacelike compact support will be reflected mathematically in the fact that $\mathfrak{C}^k_{\mathrm{sc}}(-;\mathbb{Z}) : \mathsf{Loc}_m \to \mathsf{Ab}$ is a covariant functor, while $\mathfrak{C}^k(-;\mathbb{Z}) : \mathsf{Loc}_m^{\mathrm{op}} \to \mathsf{Ab}$ is contravariant. The correct definition of $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ is a very subtle point because, in contrast to the standard examples like Klein-Gordon theory, the Abelian group $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ *cannot* be presented as a subgroup of $\mathfrak{C}^k(M;\mathbb{Z})$, see Remark 3.1 below. We give a definition of $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ in terms of relative differential cohomology and frequently refer to the companion paper [BBSS15] for further technical details.

3.1 Semi-classical configuration space

Let $K \subseteq M$ be a compact subset. In analogy to (2.4), we define the Abelian group $\mathfrak{C}^k(M, M\setminus J(K);\mathbb{Z})$ of semi-classical gauge fields on M relative to $M \setminus J(K)$ as the pullback

$$\begin{array}{ccc} \mathfrak{C}^k(M,M\setminus J(K);\mathbb{Z}) & \dashrightarrow & \hat{\mathrm{H}}^{m-k}(M,M\setminus J(K);\mathbb{Z}) \\ \downarrow & & \downarrow{\scriptstyle \ast\,\mathrm{curv}} \\ \hat{\mathrm{H}}^{k}(M,M\setminus J(K);\mathbb{Z}) & \xrightarrow[\mathrm{curv}]{} & \Omega^k(M,M\setminus J(K)) \end{array} \tag{3.1}$$

where $\hat{\mathrm{H}}^p(M, M \setminus J(K);\mathbb{Z})$ denote the relative differential cohomology groups and $\Omega^k(M, M \setminus J(K))$ denotes the group of relative differential forms, see [BB14, BBSS15] for the definitions and our conventions. We shall make frequent use of the short exact sequence

$$0 \longrightarrow \mathrm{H}^{k-1,m-k-1}(M, M \setminus J(K);\mathbb{T}) \xrightarrow{\kappa\times\kappa} \mathfrak{C}^k(M, M \setminus J(K);\mathbb{Z}) \xrightarrow{\mathrm{curv}_1} \Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M, M \setminus J(K)) \longrightarrow 0 \tag{3.2}$$

for relative semi-classical gauge fields, which immediately follows from [BB14, Part II, Section 3.3] and [BBSS15, Theorem 3.2] by imitating the proof of Theorem 2.2.

Remark 3.1. One may heuristically think of semi-classical gauge fields on M relative to $M \setminus J(K)$ as fields on M which "vanish" outside of the closed light-cone $J(K)$ of K. However, strictly speaking this interpretation is not correct: There is a group homomorphism $I : \mathfrak{C}^k(M, M \setminus J(K);\mathbb{Z}) \to \mathfrak{C}^k(M;\mathbb{Z})$ which is induced by the group homomorphisms (denoted with abuse of notation by the same symbols) $I : \hat{\mathrm{H}}^p(M, M \setminus J(K);\mathbb{Z}) \to \hat{\mathrm{H}}^p(M,\mathbb{Z})$ that restrict relative differential characters from relative cycles to cycles by precomposing them with the homomorphism $Z_{p-1}(M) \to Z_{p-1}(M, M\setminus J(K))$, cf. [BBSS15, Section 3.1]. By [BBSS15, Remark 3.3] and Theorem 3.4 below, we observe that the homomorphism $I : \mathfrak{C}^k(M, M \setminus J(K);\mathbb{Z}) \to \mathfrak{C}^k(M;\mathbb{Z})$ is not necessarily injective, which implies that $\mathfrak{C}^k(M, M \setminus J(K);\mathbb{Z})$ is in general not a subgroup of $\mathfrak{C}^k(M;\mathbb{Z})$.

We define the Abelian group $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ of semi-classical gauge fields of spacelike compact support by formalizing the intuition that for any element $(h, \tilde{h}) \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ there should exist a sufficiently large compact subset $K \subseteq M$ such that $(h, \tilde{h})$ can be represented as an element in $\mathfrak{C}^k(M, M \setminus J(K);\mathbb{Z})$. Let us denote by $\mathcal{K}_M$ the directed set of compact subsets of M and notice that the assignment $\mathfrak{C}^k(M, M \setminus J(-);\mathbb{Z}) : \mathcal{K}_M \to \mathsf{Ab}$ is a diagram of shape $\mathcal{K}_M$.[4] Then the intuition is formalized by taking the colimit of this diagram, i.e. we define the semi-classical gauge fields of spacelike compact support by

$$\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) := \mathrm{colim}\big(\mathfrak{C}^k(M, M \setminus J(-);\mathbb{Z}) : \mathcal{K}_M \to \mathsf{Ab}\big) \ . \tag{3.3}$$

Remark 3.2. The colimit in (3.3) can be equally well computed by restricting to the directed set $\mathcal{K}_\Sigma$ of compact subsets of any smooth spacelike Cauchy surface Σ of M. In fact, denoting by $\mathcal{C}_M$ the directed set of closed subsets of M, one notices that the map $\mathcal{K}_M \to \mathcal{C}_M$, $K \mapsto J(K)$, preserves the preorder relation induced by inclusion. In particular, we may interpret the functor $\mathfrak{C}^k(M, M \setminus J(-);\mathbb{Z}) : \mathcal{K}_M \to \mathsf{Ab}$ as the composition of the functors $\mathfrak{C}^k(M, M \setminus -;\mathbb{Z}) : \mathcal{C}_M \to \mathsf{Ab}$ and $J : \mathcal{K}_M \to \mathcal{C}_M$; then $\mathcal{K}_\Sigma \subseteq \mathcal{K}_M$ is cofinal with respect to $J : \mathcal{K}_M \to \mathcal{C}_M$. In fact, for each $K \subseteq M$, we have $J(K) \subseteq J(K_\Sigma)$ for $K_\Sigma = J(K) \cap \Sigma$, which is by construction a compact subset of Σ. This observation provides the isomorphism

$$\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \simeq \mathrm{colim}\big(\mathfrak{C}^k(M, M \setminus J(-);\mathbb{Z}) : \mathcal{K}_\Sigma \to \mathsf{Ab}\big) \ . \tag{3.4}$$

Similarly to Remark 3.1, there is a group homomorphism (denoted with abuse of notation by the same symbol)

$$I : \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \longrightarrow \mathfrak{C}^k(M;\mathbb{Z}) \ , \tag{3.5}$$

which is however in general not injective, see [BBSS15, Remark 4.4] and Corollary 3.5 below. Hence semi-classical gauge fields of spacelike compact support cannot in general be faithfully represented as elements in the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$.

[4] For this, we use the group homomorphisms $Z_{p-1}(M, M \setminus J(K')) \to Z_{p-1}(M, M \setminus J(K))$ of relative cycles which exist for any $K \subseteq K'$.

3.2 Cauchy problem

Consider any compact subset $K \subseteq \Sigma$. Taking into account the support property of the Cauchy problem considered in Theorem 2.5 and applying arguments similar to those in Section 2.2 to the relative case, in particular (2.13) and (2.14) (see also Lemma A.2 (i)), one concludes that, given initial data $(B,\tilde{B}) \in \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma,\Sigma\setminus K)$, the Cauchy problem (2.11) has a unique solution $F \in \Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M, M\setminus J(K))$. This observation leads us to the relative version of Corollary 2.6.

Corollary 3.3. *The embedding $\iota_\Sigma : \Sigma \to M$ induces an isomorphism of Abelian groups*

$$\Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M, M\setminus J(K)) \underset{\mathrm{solve}_\Sigma}{\overset{(\iota_\Sigma^*,\iota_\Sigma^* *^{-1})}{\rightleftarrows}} \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma,\Sigma\setminus K)\ , \tag{3.6}$$

whose inverse solve_Σ *is the map assigning to initial data $(B,\tilde{B}) \in \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma,\Sigma\setminus K)$ the corresponding unique solution $F \in \Omega^k_{\mathbb{Z}} \cap *\Omega^{m-k}_{\mathbb{Z}}(M, M\setminus J(K))$ of the Cauchy problem* (2.11).

Using (3.2) and [BBSS15, Theorem 3.2], and the fact that relative differential cohomology is a functor (in a suitable sense, see [BBSS15, Section 3.1]), we conclude that the diagram of Abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(M,M\setminus J(K);\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \mathfrak{C}^k(M,M\setminus J(K);\mathbb{Z}) & \xrightarrow{\mathrm{curv}_1} & \Omega^k_{\mathbb{Z}}\cap *\Omega^{m-k}_{\mathbb{Z}}(M,M\setminus J(K)) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \iota_\Sigma^*\times\iota_\Sigma^*} & & \Big\downarrow{\scriptstyle \iota_\Sigma^*\times\iota_\Sigma^*} & & \Big\downarrow{\scriptstyle (\iota_\Sigma^*,\iota_\Sigma^* *^{-1})} & & \\ 0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}(\Sigma,\Sigma\setminus K;\mathbb{T}) & \xrightarrow[\kappa\times\kappa]{} & \hat{\mathrm{H}}^{k,m-k}(\Sigma,\Sigma\setminus K;\mathbb{Z}) & \xrightarrow[\mathrm{curv}\times\mathrm{curv}]{} & \Omega^{k,m-k}_{\mathbb{Z}}(\Sigma,\Sigma\setminus K) & \longrightarrow & 0 \end{array} \tag{3.7}$$

commutes and its rows are short exact sequences. Using also Lemma A.2 (ii), Corollary 3.3 and the five lemma, we obtain the relative version of Theorem 2.7.

Theorem 3.4. *The embedding $\iota_\Sigma : \Sigma \to M$ induces an isomorphism of Abelian groups*

$$\mathfrak{C}^k(M,M\setminus J(K);\mathbb{Z}) \xrightarrow{\iota_\Sigma^*\times\iota_\Sigma^*} \hat{\mathrm{H}}^{k,m-k}(\Sigma,\Sigma\setminus K;\mathbb{Z})\ . \tag{3.8}$$

Taking the colimit of (3.7) over the directed set $\mathcal{K}_\Sigma$ of compact subsets of Σ and recalling Remark 3.2 we find that the diagram of Abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(M;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}_1} & \Omega^k_{\mathrm{sc},\mathbb{Z}}\cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \iota_\Sigma^*\times\iota_\Sigma^*} & & \Big\downarrow{\scriptstyle \iota_\Sigma^*\times\iota_\Sigma^*} & & \Big\downarrow{\scriptstyle (\iota_\Sigma^*,\iota_\Sigma^* *^{-1})} & & \\ 0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}_{\mathrm{c}}(\Sigma;\mathbb{T}) & \xrightarrow[\kappa\times\kappa]{} & \hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z}) & \xrightarrow[\mathrm{curv}\times\mathrm{curv}]{} & \Omega^{k,m-k}_{\mathrm{c},\mathbb{Z}}(\Sigma) & \longrightarrow & 0 \end{array} \tag{3.9}$$

commutes, its rows are short exact sequences and its vertical arrows are isomorphisms. The subscript $_\mathrm{c}$ denotes compact support and the various groups of this diagram are *defined* by these colimits.[5] This shows that $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ is the space of solutions of the Cauchy problem (2.18) for $(h,\tilde{h}) \in \hat{\mathrm{H}}^{k,m-k}_{\mathrm{sc}}(M;\mathbb{Z})$ with initial data in $\hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z})$.

Corollary 3.5. *The embedding $\iota_\Sigma : \Sigma \to M$ induces an isomorphism of Abelian groups*

$$\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \xrightarrow{\iota_\Sigma^*\times\iota_\Sigma^*} \hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z})\ . \tag{3.10}$$

[5] For a detailed presentation of differential characters with compact support, see [BBSS15, Section 4].

The assignment of the Abelian groups $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ to objects M in Loc_m is a covariant functor

$$\mathfrak{C}^k_{\mathrm{sc}}(-;\mathbb{Z}) : \mathsf{Loc}_m \longrightarrow \mathsf{Ab} \ . \tag{3.11}$$

The group homomorphism $f_* := \mathfrak{C}^k_{\mathrm{sc}}(f;\mathbb{Z}) : \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \to \mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z})$ associated with a morphism $f : M \to M'$ in Loc_m is constructed in Lemma A.3.

Remark 3.6. With a similar construction as in Lemma A.3, we obtain two more functors

$$\mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(-;\mathbb{T}) : \mathsf{Loc}_m \longrightarrow \mathsf{Ab} \ , \qquad \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(-) : \mathsf{Loc}_m \longrightarrow \mathsf{Ab} \ . \tag{3.12}$$

Using these constructions one can further show that the short exact sequence in the first row of the diagram (3.9) is natural, i.e. for any morphism $f : M \to M'$ in Loc_m, the diagram of Abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(M;\mathbb{T}) & \xrightarrow{\kappa\times\kappa} & \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}_1} & \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ 0 & \longrightarrow & \mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(M';\mathbb{T}) & \xrightarrow[\kappa\times\kappa]{} & \mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z}) & \xrightarrow[\mathrm{curv}_1]{} & \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M') & \longrightarrow & 0 \end{array} \tag{3.13}$$

commutes. Here we also use the notation f_* for the group homomorphisms $\mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(f;\mathbb{T})$ and $\Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(f)$.

4 Observables for dual gauge fields

In this section we introduce and analyze a suitable Abelian group $\mathfrak{O}^k(M;\mathbb{Z})$ of basic semi-classical observables. In general, observables are given by functionals on the configuration space of the field theory. Recalling that the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$ is an Abelian group, there is a distinguished class of observables given by the group characters $\mathfrak{C}^k(M;\mathbb{Z})^\star := \mathrm{Hom}_{\mathsf{Ab}}(\mathfrak{C}^k(M;\mathbb{Z}),\mathbb{T})$. However, generic group characters define observables that are too singular for quantization, hence it is reasonable to impose a suitable regularity condition in the spirit of smooth Pontryagin duality [HLZ03, BSS14]. After defining and analyzing the smooth Pontryagin dual $\mathfrak{O}^k(M;\mathbb{Z})$ of $\mathfrak{C}^k(M;\mathbb{Z})$, we shall show that it is isomorphic to the Abelian group $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ of semi-classical gauge fields of spacelike compact support. Generalizing the constructions of Remark 2.8 to the case of not necessarily compact Cauchy surfaces, we obtain a natural presymplectic structure on the Abelian group of semi-classical observables $\mathfrak{O}^k(M;\mathbb{Z})$. We shall analyze properties of these presymplectic Abelian groups in view of the axioms of locally covariant quantum field theory [BFV03].

4.1 Semi-classical observables

We shall begin by imposing a suitable regularity condition on the Abelian group of group characters $\mathfrak{C}^k(M;\mathbb{Z})^\star$ of the semi-classical configuration space.

Definition 4.1. The Abelian group $\mathfrak{O}^k(M;\mathbb{Z})$ of semi-classical observables is the following subgroup of $\mathfrak{C}^k(M;\mathbb{Z})^\star$: A group character $\varphi \in \mathfrak{C}^k(M;\mathbb{Z})^\star$ is an element in $\mathfrak{O}^k(M;\mathbb{Z})$ if and only if there exists $\omega = *\tilde{\omega} \in \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M)$ and a smooth spacelike Cauchy surface Σ of M such that

$$\varphi\big((\iota\times\iota)([A],[\tilde{A}])\big) = \int_\Sigma \big(\tilde{A}\wedge\omega - (-1)^{k\,(m-k)}\,A\wedge\tilde{\omega}\big) \quad \mathrm{mod}\ \mathbb{Z} \ , \tag{4.1}$$

for all semi-classical topologically trivial fields $([A],[\tilde{A}]) \in \mathfrak{T}^k(M;\mathbb{Z})$.

We now prove that Definition 4.1 does not depend on the choice of Cauchy surface Σ used to evaluate the integral (4.1). For this, notice that $\omega = *\tilde{\omega} \in \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M)$ implies $\Box\,\omega = 0$ and $\Box\,\tilde{\omega} = 0$, where $\Box = \delta\,\mathrm{d} + \mathrm{d}\,\delta$ is the d'Alembert operator. By [BGP07, Bär15] there exists $\tilde{\beta} \in \Omega^{m-k}_{\mathrm{c}}(M)$ such that $\tilde{\omega} = G\tilde{\beta}$, where $G = G^+ - G^-$ is the causal propagator and $G^\pm$ are the retarded/advanced Green's operators of $\Box$. We further have $\omega = *\tilde{\omega} = G*\tilde{\beta}$. Because of $\mathrm{d}\omega = 0$ and $\mathrm{d}\tilde{\omega} = 0$, there exist $\alpha \in \Omega^{k+1}_{\mathrm{c}}(M)$ and $\tilde{\alpha} \in \Omega^{m-k+1}_{\mathrm{c}}(M)$ such that $\mathrm{d}*\tilde{\beta} = \Box\alpha$ and $\mathrm{d}\tilde{\beta} = \Box\tilde{\alpha}$. Using these observations, and realizing Σ as the boundary of $J^-(\Sigma) \subseteq M$ and also as the boundary of $J^+(\Sigma) \subseteq M$ (with opposite orientation), we can rewrite (4.1) as

$$
\begin{aligned}
\varphi\big((\iota\times\iota)([A],[\tilde{A}])\big) &= \int_\Sigma \big(\tilde{A}\wedge G*\tilde{\beta} - (-1)^{k\,(m-k)}\,A\wedge G\tilde{\beta}\big) \mod \mathbb{Z}\\
&= \int_{J^-(\Sigma)} \mathrm{d}\big(\tilde{A}\wedge G^+*\tilde{\beta} - (-1)^{k\,(m-k)}\,A\wedge G^+\tilde{\beta}\big) \mod \mathbb{Z}\\
&\qquad + \int_{J^+(\Sigma)} \mathrm{d}\big(\tilde{A}\wedge G^-*\tilde{\beta} - (-1)^{k\,(m-k)}\,A\wedge G^-\tilde{\beta}\big) \mod \mathbb{Z}\\
&= \int_M \big((-1)^{m-k}\,\tilde{A}\wedge\alpha - (-1)^{k\,(m-k)}\,(-1)^k\,A\wedge\tilde{\alpha}\big) \mod \mathbb{Z}\ , \qquad (4.2)
\end{aligned}
$$

where we have also used $\mathrm{d}A = *\mathrm{d}\tilde{A}$. It then follows that (4.1) is independent of the choice of Cauchy surface because (4.2) shows that it can be written as an integral over spacetime M.

Remark 4.2. There is an alternative but equivalent definition of the Abelian group $\mathfrak{O}^k(M;\mathbb{Z})$ of semi-classical observables which employs the notion of smooth Pontryagin duality developed in [HLZ03, BSS14]. Taking the smooth Pontryagin dual of the pullback diagram (2.4) which defines the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$, we define an Abelian group $\mathfrak{C}^k(M;\mathbb{Z})^\star_\infty$ (called the smooth Pontryagin dual of $\mathfrak{C}^k(M;\mathbb{Z})$) via the pushout

$$
\begin{array}{ccc}
\Omega^k_{\mathrm{c}}(M) & \xrightarrow{(*\,\mathrm{curv})^\star} & \hat{\mathrm{H}}^{m-k}(M;\mathbb{Z})^\star_\infty\\
{\scriptstyle \mathrm{curv}^\star}\downarrow & & \downarrow\\
\hat{\mathrm{H}}^k(M;\mathbb{Z})^\star_\infty & \dashrightarrow & \mathfrak{C}^k(M;\mathbb{Z})^\star_\infty
\end{array} \qquad (4.3)
$$

where $\hat{\mathrm{H}}^p(M;\mathbb{Z})^\star_\infty$ denotes the smooth Pontryagin dual of $\hat{\mathrm{H}}^p(M;\mathbb{Z})$. This pushout may be realized explicitly as the quotient

$$
\mathfrak{C}^k(M;\mathbb{Z})^\star_\infty = \frac{\hat{\mathrm{H}}^k(M;\mathbb{Z})^\star_\infty \oplus \hat{\mathrm{H}}^{m-k}(M;\mathbb{Z})^\star_\infty}{\big\{\mathrm{curv}^\star\omega \oplus -(*\,\mathrm{curv})^\star\omega \ :\ \omega\in\Omega^k_{\mathrm{c}}(M)\big\}}\ . \qquad (4.4)
$$

One can show that the Abelian group $\mathfrak{C}^k(M;\mathbb{Z})^\star_\infty$ is isomorphic to the Abelian group $\mathfrak{O}^k(M;\mathbb{Z})$ of semi-classical observables given in Definition 4.1. As we do not need this isomorphism in this paper, we refrain from writing it out explicitly. Let us just point out that the elements of the smooth Pontryagin dual are in particular continuous group characters. In fact, on account of [BSS14, Appendix A], all differential cohomology groups on a manifold of finite type are Fréchet-Lie groups that are (non-canonically) isomorphic to the Cartesian product of a torus, a torsion group, a discrete lattice in a Euclidean space (all finite dimensional) and a Fréchet vector space of differential forms. This observation allows one to conclude that the elements of the smooth Pontryagin dual are continuous group characters with respect to the Fréchet topology mentioned above.

We now show that the assignment of the Abelian groups $\mathfrak{O}^k(M;\mathbb{Z})$ of semi-classical observables to objects M in Loc_m is a covariant functor

$$
\mathfrak{O}^k(-;\mathbb{Z}) : \mathsf{Loc}_m \longrightarrow \mathsf{Ab}\ . \qquad (4.5)
$$

For this, note that the assignment of character groups $\mathfrak{C}^k(M;\mathbb{Z})^\star = \mathrm{Hom}_{\mathsf{Ab}}(\mathfrak{C}^k(M;\mathbb{Z}),\mathbb{T})$ (without the regularity condition of Definition 4.1) to objects M in Loc_m is a covariant functor $\mathfrak{C}^k(-;\mathbb{Z})^\star : \mathsf{Loc}_m \to \mathsf{Ab}$: Given any morphism $f : M \to M'$ in Loc_m, functoriality of the semi-classical configuration spaces provides us with a group homomorphism $f^* = \mathfrak{C}^k(f;\mathbb{Z}) : \mathfrak{C}^k(M';\mathbb{Z}) \to \mathfrak{C}^k(M;\mathbb{Z})$, which we can dualize to a group homomorphism (called pushforward) $f_* := \mathfrak{C}^k(f;\mathbb{Z})^\star = (f^*)^\star : \mathfrak{C}^k(M;\mathbb{Z})^\star \to \mathfrak{C}^k(M';\mathbb{Z})^\star$ between the character groups. It remains to show that these group homomorphisms induce group homomorphisms $f_* : \mathfrak{O}^k(M;\mathbb{Z}) \to \mathfrak{O}^k(M';\mathbb{Z})$, i.e. that pushforwards preserve the regularity condition of Definition 4.1. Let $\varphi \in \mathfrak{O}^k(M;\mathbb{Z})$ and $\omega = *\tilde{\omega} \in \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M)$ be as in Definition 4.1. Exploiting the Cauchy problem described by Corollary 3.3, we can easily push forward ω and $\tilde{\omega}$ to $f_*\omega = *f_*\tilde{\omega} \in \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M')$ by pushing forward the initial data from a Cauchy surface $\Sigma \subseteq M$ to a suitable Cauchy surface $\Sigma' \subseteq M'$.[6] By construction, we have

$$
\begin{aligned}
f_*\varphi\big((\iota\times\iota)([A],[\tilde{A}])\big) &= \varphi\big((\iota\times\iota)([f^*A],[f^*\tilde{A}])\big)\\
&= \int_\Sigma \big(f^*\tilde{A}\wedge\omega - (-1)^{k\,(m-k)}\, f^*A\wedge\tilde{\omega}\big) \mod \mathbb{Z}\\
&= \int_{\Sigma'} \big(\tilde{A}\wedge f_*\omega - (-1)^{k\,(m-k)}\, A\wedge f_*\tilde{\omega}\big) \mod \mathbb{Z}\ ,
\end{aligned}
\tag{4.6}
$$

which shows that $f_*\varphi \in \mathfrak{O}^k(M';\mathbb{Z})$ as required.

4.2 Observables from spacelike compact gauge fields

We shall now show that the Abelian group $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ of semi-classical gauge fields of spacelike compact support is isomorphic to the Abelian group $\mathfrak{O}^k(M;\mathbb{Z})$ of semi-classical observables introduced in Definition 4.1. Using the techniques which allow us to establish this isomorphism, we shall also prove that $\mathfrak{O}^k(M;\mathbb{Z})$ is large enough to separate points of the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$, i.e. for $(h,\tilde{h}),(h',\tilde{h}') \in \mathfrak{C}^k(M;\mathbb{Z})$ we have $\varphi\big((h,\tilde{h})\big) = \varphi\big((h',\tilde{h}')\big)$ for all $\varphi \in \mathfrak{O}^k(M;\mathbb{Z})$ if and only if $(h,\tilde{h}) = (h',\tilde{h}')$.

By [BBSS15, Section 5.2], for any smooth spacelike Cauchy surface Σ of M there is a $\mathbb{T}$-valued pairing $\langle\cdot,\cdot\rangle_{\mathrm{c}} : \hat{\mathrm{H}}^{m-p}(\Sigma;\mathbb{Z}) \times \hat{\mathrm{H}}^p_{\mathrm{c}}(\Sigma;\mathbb{Z}) \to \mathbb{T}$ between differential cohomology and compactly supported differential cohomology. Using the isomorphisms given in Theorem 2.7 and Corollary 3.5, we define a $\mathbb{T}$-valued pairing between $\mathfrak{C}^k(M;\mathbb{Z})$ and $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ by

$$
\begin{aligned}
\langle\cdot,\cdot\rangle : \mathfrak{C}^k(M;\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) &\longrightarrow \mathbb{T}\ ,\\
\big((h,\tilde{h})\,,\,(h',\tilde{h}')\big) &\longmapsto \langle\iota^*_\Sigma\tilde{h},\iota^*_\Sigma h'\rangle_{\mathrm{c}} - (-1)^{k\,(m-k)}\,\langle\iota^*_\Sigma h,\iota^*_\Sigma\tilde{h}'\rangle_{\mathrm{c}}\ .
\end{aligned}
\tag{4.7}
$$

In Lemma A.4 we show that this pairing does not depend on the choice of Cauchy surface Σ and we prove its naturality in the sense that for any morphism $f : M \to M'$ in Loc_m the diagram of Abelian groups

$$
\begin{array}{ccc}
\mathfrak{C}^k(M';\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) & \xrightarrow{\ f^*\times\mathrm{id}\ } & \mathfrak{C}^k(M;\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\\
{\scriptstyle \mathrm{id}\times f_*}\big\downarrow & & \big\downarrow{\scriptstyle \langle\cdot,\cdot\rangle}\\
\mathfrak{C}^k(M';\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z}) & \xrightarrow[\ \langle\cdot,\cdot\rangle\ ]{} & \mathbb{T}
\end{array}
\tag{4.8}
$$

[6] A suitable Cauchy surface $\Sigma' \subseteq M'$ can be constructed follows: Let $U \subseteq \Sigma$ be an open neighborhood of $\mathrm{supp}\,\omega\cap\Sigma$ with compact closure. Then the image $f(\overline{U}) \subseteq M'$ of the closure $\overline{U}$ of U is a spacelike and acausal compact submanifold with boundary, and we can take Σ' to be any Cauchy surface extending $f(\overline{U})$; see [BS06, Theorem 1.1] for the existence of such a Cauchy surface.

commutes.

By partial evaluation, the pairing (4.7) allows us to define group characters on $\mathfrak{C}^k(M;\mathbb{Z})$: For any $(h',\tilde{h}') \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ there is a group character

$$\langle \,\cdot\, , (h',\tilde{h}')\rangle : \mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \mathbb{T} \ , \quad (h,\tilde{h}) \longmapsto \langle (h,\tilde{h}) \, , \, (h',\tilde{h}')\rangle \ . \tag{4.9}$$

The next result in particular allows us to separate points of the semi-classical configuration space $\mathfrak{C}^k(M;\mathbb{Z})$ by using only such group characters.

Proposition 4.3. *The pairing* $\langle \cdot, \cdot \rangle$ *introduced in* (4.7) *is weakly non-degenerate.*

Proof. Recalling Theorem 2.7 and Corollary 3.5, the pullback along the embedding $\iota_\Sigma : \Sigma \to M$ provides isomorphisms $\mathfrak{C}^k(M;\mathbb{Z}) \simeq \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})$ and $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \simeq \hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z})$. Using these isomorphisms, the pairing (4.7) corresponds precisely to the pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle_\Sigma : \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z}) \times \hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z}) &\longrightarrow \mathbb{T} \ , \\ \big((h,\tilde{h}) \, , \, (h',\tilde{h}')\big) &\longmapsto \langle \tilde{h}, h' \rangle_{\mathrm{c}} - (-1)^{k\,(m-k)} \, \langle h, \tilde{h}' \rangle_{\mathrm{c}} \end{aligned} \tag{4.10}$$

between initial data on Σ. The proof then follows from weak non-degeneracy of the pairing $\langle \cdot, \cdot \rangle_{\mathrm{c}} : \hat{\mathrm{H}}^{m-p}(\Sigma;\mathbb{Z}) \times \hat{\mathrm{H}}^p_{\mathrm{c}}(\Sigma;\mathbb{Z}) \to \mathbb{T}$, cf. [BBSS15, Corollary 5.6]. □

Finally, we show that the partial evaluation (4.9) establishes an isomorphism between $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ and the Abelian group $\mathfrak{O}^k(M;\mathbb{Z})$ of semi-classical observables introduced in Definition 4.1.

Proposition 4.4. *The group homomorphism*

$$\mathcal{O} : \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \longrightarrow \mathfrak{O}^k(M;\mathbb{Z}) \ , \qquad (h',\tilde{h}') \longmapsto \langle \,\cdot\, , (h',\tilde{h}')\rangle \ , \tag{4.11}$$

is an isomorphism which provides a natural isomorphism between the functors $\mathfrak{C}^k_{\mathrm{sc}}(-;\mathbb{Z}) : \mathsf{Loc}_m \to \mathsf{Ab}$ *and* $\mathfrak{O}^k(-;\mathbb{Z}) : \mathsf{Loc}_m \to \mathsf{Ab}$.

Proof. We first have to show that the group character $\langle \,\cdot\, , (h',\tilde{h}')\rangle$ satisfies the regularity condition of Definition 4.1, for any $(h',\tilde{h}') \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$. This follows from [BBSS15, eq. (4.12)]: For any $([A],[\tilde{A}]) \in \mathfrak{T}^k(M)$, we find

$$\big\langle (\iota \times \iota)([A],[\tilde{A}]) \, , \, (h',\tilde{h}')\big\rangle = \int_\Sigma \big(\tilde{A} \wedge \mathrm{curv}\, h' - (-1)^{k\,(m-k)} \, A \wedge \mathrm{curv}\, \tilde{h}'\big) \mod \mathbb{Z} \ . \tag{4.12}$$

Moreover, (4.11) is injective due to Proposition 4.3.

To show that (4.11) is surjective, let us take any $\varphi \in \mathfrak{O}^k(M;\mathbb{Z})$ and choose a smooth spacelike Cauchy surface Σ of M. Using the isomorphism established in Theorem 2.7 and recalling Definition 4.1, there exists a unique smooth character $\varphi_\Sigma \in \hat{\mathrm{H}}^{k,m-k}(\Sigma;\mathbb{Z})^\star_\infty$ such that $\varphi_\Sigma \circ (\iota^*_\Sigma \times \iota^*_\Sigma) = \varphi$. Using further the character duality proven in [BBSS15, Theorem 5.4], there exists a unique $(h'_\Sigma, \tilde{h}'_\Sigma) \in \hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z})$ such that $\langle \,\cdot\, , (h'_\Sigma,\tilde{h}'_\Sigma)\rangle_\Sigma = \varphi_\Sigma$ by (4.10). Using Corollary 3.5, we introduce $(h',\tilde{h}') \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ as the unique element with $(\iota^*_\Sigma \times \iota^*_\Sigma)(h',\tilde{h}') = (h'_\Sigma,\tilde{h}'_\Sigma)$. Then the definition of $\langle \cdot, \cdot \rangle$ given in (4.7) implies that $\varphi = \langle \,\cdot\, , (h',\tilde{h}')\rangle$.

It remains to prove that the established isomorphism is natural. Let $f : M \to M'$ be a morphism in Loc_m. Recall that the pushforward $f_* : \mathfrak{O}^k(M;\mathbb{Z}) \to \mathfrak{O}^k(M';\mathbb{Z})$ is given by the Pontryagin dual of the pullback $f^* : \mathfrak{C}^k(M';\mathbb{Z}) \to \mathfrak{C}^k(M;\mathbb{Z})$ and that Lemma A.4

establishes naturality of the pairing $\langle\cdot,\cdot\rangle$. Therefore, for all $(h',\tilde{h}')\in\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$, one has $f_*\langle\cdot\,,(h',\tilde{h}')\rangle=\langle f^*\cdot\,,(h',\tilde{h}')\rangle=\langle\cdot\,,f_*(h',\tilde{h}')\rangle$, i.e. the diagram

$$\begin{array}{ccc}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) & \xrightarrow{\ \mathcal{O}\ } & \mathfrak{O}^k(M;\mathbb{Z}) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z}) & \xrightarrow[\ \mathcal{O}\ ]{} & \mathfrak{O}^k(M';\mathbb{Z})\end{array}\tag{4.13}$$

commutes and hence (4.11) is a natural isomorphism. □

4.3 Presymplectic structure

We will introduce a natural $\mathbb{T}$-valued presymplectic structure τ on the Abelian group $\mathfrak{O}^k(M;\mathbb{Z})$ of semi-classical observables. In this way we obtain a functor $(\mathfrak{O}^k(-;\mathbb{Z}),\tau):\mathsf{Loc}_m\to\mathsf{PSAb}$ valued in the category PSAb of presymplectic Abelian groups (with group homomorphisms preserving the presymplectic structures as morphisms). This will be the main input for Section 5, where the quantization of the semi-classical model described by $(\mathfrak{O}^k(-;\mathbb{Z}),\tau)$ will be addressed.

Proposition 4.5. *Let $\mathcal{O}:\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\to\mathfrak{O}^k(M;\mathbb{Z})$ be the isomorphism introduced in Proposition 4.4 and $I:\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\to\mathfrak{C}^k(M;\mathbb{Z})$ the group homomorphism given in* (3.5)*. Then*

$$\tau:\mathfrak{O}^k(M;\mathbb{Z})\times\mathfrak{O}^k(M;\mathbb{Z})\longrightarrow\mathbb{T}\ ,\qquad(\varphi,\varphi')\longmapsto\langle I(\mathcal{O}^{-1}\varphi),\mathcal{O}^{-1}\varphi'\rangle\tag{4.14}$$

defines a presymplectic structure on $\mathfrak{O}^k(M;\mathbb{Z})$ whose radical is $\mathcal{O}(\ker I)$.

Proof. Up to the isomorphism $\mathcal{O}^{-1}:\mathfrak{O}^k(M;\mathbb{Z})\to\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$, the mapping τ is given by

$$\sigma:\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\times\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\longrightarrow\mathbb{T}\ ,\qquad\big((h,\tilde{h})\,,(h',\tilde{h}')\big)\longmapsto\langle I(h,\tilde{h})\,,(h',\tilde{h}')\rangle\ .\tag{4.15}$$

Recalling (4.7), we observe that σ is $\mathbb{Z}$-bilinear and hence so is τ. Antisymmetry follows from [BBSS15, Proposition 5.9]. Since the pairing $\langle\cdot,\cdot\rangle:\mathfrak{C}^k(M;\mathbb{Z})\times\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\to\mathbb{T}$ is weakly non-degenerate by Proposition 4.3, the radical of σ coincides with the kernel of I which implies that the radical of τ is $\ker(I\circ\mathcal{O}^{-1})=\mathcal{O}(\ker I)$. □

Remark 4.6. If M has compact Cauchy surfaces $\Sigma\subseteq M$, the group homomorphism $I:\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\to\mathfrak{C}^k(M;\mathbb{Z})$ is the identity. In fact, with Σ compact, $J(\Sigma)=M$ entails that the diagram whose colimit defines $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ has $\mathfrak{C}^k(M,M\setminus J(\Sigma);\mathbb{Z})=\mathfrak{C}^k(M;\mathbb{Z})$ as its terminal object. In particular, τ is actually weakly symplectic for globally hyperbolic Lorentzian manifolds with compact Cauchy surfaces. In this case (4.15) coincides with the (pre)symplectic structure described in Remark 2.8.

Our next task is to prove that the presymplectic structure τ introduced in Proposition 4.5 is natural, so that we can interpret $(\mathfrak{O}^k(-;\mathbb{Z}),\tau)$ as a functor from Loc_m to PSAb.

Proposition 4.7. *Let $f:M\to M'$ be a morphism in Loc_m. Then the diagram of Abelian groups*

$$\begin{array}{ccc}\mathfrak{O}^k(M;\mathbb{Z})\times\mathfrak{O}^k(M;\mathbb{Z}) & & \\ & \searrow^{\tau} & \\ {\scriptstyle f_*\times f_*}\big\downarrow & & \mathbb{T} \\ & \nearrow_{\tau} & \\ \mathfrak{O}^k(M';\mathbb{Z})\times\mathfrak{O}^k(M';\mathbb{Z}) & & \end{array}\tag{4.16}$$

commutes.

Proof. Recalling from Proposition 4.4 that $\mathcal{O} : \mathfrak{C}^k_{\mathrm{sc}}(-;\mathbb{Z}) \Rightarrow \mathfrak{O}^k(-;\mathbb{Z})$ is a natural isomorphism, it is enough to prove commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) & & \\ & \overset{\sigma}{\searrow} & \\ {\scriptstyle f_* \times f_*}\big\downarrow & & \mathbb{T} \\ & \underset{\sigma}{\nearrow} & \\ \mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z}) & & \end{array} \tag{4.17}$$

for σ given in (4.15). Making use of naturality of the pairing $\langle \cdot , \cdot \rangle$ (cf. Lemma A.4), we obtain

$$\begin{aligned} \sigma\big(f_*(h,\tilde{h})\,,\, f_*(h',\tilde{h}'\,)\big) &= \langle I\, f_*(h,\tilde{h})\,,\, f_*(h',\tilde{h}'\,)\rangle \\ &= \langle f^*\, I\, f_*(h,\tilde{h})\,,\, (h',\tilde{h}'\,)\rangle \\ &= \langle I(h,\tilde{h})\,,\, (h',\tilde{h}'\,)\rangle = \sigma\big((h,\tilde{h})\,,\,(h',\tilde{h}'\,)\big)\ , \end{aligned} \tag{4.18}$$

for all $(h,\tilde{h}),(h,\tilde{h}'\,) \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$. In the third equality we used $f^* \circ I \circ f_* = I$ which is proven in Lemma A.5. □

4.4 Locally covariant field theory

We analyze properties of the functor $(\mathfrak{O}^k(-;\mathbb{Z}),\tau) : \mathsf{Loc}_m \to \mathsf{PSAb}$ from the point of view of the axioms of locally covariant field theory [BFV03].

Proposition 4.8 (Causality axiom). *Let $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$ be a diagram in Loc_m such that the images of f_1 and f_2 are causally disjoint, i.e. $J(f_1(M_1)) \cap f_2(M_2) = \emptyset$. Then the presymplectic structure $\tau : \mathfrak{O}^k(M;\mathbb{Z}) \times \mathfrak{O}^k(M;\mathbb{Z}) \to \mathbb{T}$ vanishes on $f_{1*}(\mathfrak{O}^k(M_1;\mathbb{Z})) \times f_{2*}(\mathfrak{O}^k(M_2;\mathbb{Z}))$.*

Proof. Using again the natural isomorphism $\mathcal{O} : \mathfrak{C}^k_{\mathrm{sc}}(-;\mathbb{Z}) \Rightarrow \mathfrak{O}^k(-;\mathbb{Z})$, it is equivalent to prove the analogous statement for σ given in (4.15). Given $(h,\tilde{h}) \in \mathfrak{C}^k_{\mathrm{sc}}(M_1;\mathbb{Z})$ and $(h',\tilde{h}'\,) \in \mathfrak{C}^k_{\mathrm{sc}}(M_2;\mathbb{Z})$, naturality of the pairing $\langle \cdot , \cdot \rangle$ implies

$$\sigma\big(f_{1*}(h,\tilde{h})\,,\, f_{2*}(h',\tilde{h}'\,)\big) = \langle f_2^*\, I\, f_{1*}(h,\tilde{h})\,,\,(h',\tilde{h}'\,)\rangle \tag{4.19}$$

and the proof follows from $f_2^* \circ I \circ f_{1*} = 0$, see Lemma A.5. □

Proposition 4.9 (Time-slice axiom). *Let $f : M \to M'$ be a Cauchy morphism, i.e. a Loc_m-morphism whose image $f(M)$ contains a smooth spacelike Cauchy surface of M'. Then $f_* : \mathfrak{O}^k(M;\mathbb{Z}) \to \mathfrak{O}^k(M';\mathbb{Z})$ is an isomorphism.*

Proof. Take any smooth spacelike Cauchy surface $\Sigma' \subseteq f(M)$ of M' and note that its preimage $\Sigma = f^{-1}(\Sigma'\,)$ is a smooth spacelike Cauchy surface of M. The diagram of Abelian groups

$$\begin{array}{ccc} \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) & \xrightarrow{\quad f_* \quad} & \mathfrak{C}_{\mathrm{sc}}(M';\mathbb{Z}) \\ {\scriptstyle \iota^*_\Sigma \times \iota^*_\Sigma}\big\downarrow & & \big\downarrow{\scriptstyle \iota^*_{\Sigma'} \times \iota^*_{\Sigma'}} \\ \hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z}) & \xrightarrow[f_{\Sigma *} \times f_{\Sigma *}]{} & \hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma';\mathbb{Z}) \end{array} \tag{4.20}$$

commutes, its vertical arrows are isomorphisms (cf. Corollary 3.5) and its bottom horizontal arrow is an isomorphism since, by restriction, f induces an orientation preserving isometry $f_\Sigma : \Sigma \to \Sigma'$. Hence f_* is an isomorphism and, by using again the natural isomorphism $\mathcal{O} : \mathfrak{C}^k_{\mathrm{sc}}(-;\mathbb{Z}) \Rightarrow \mathfrak{O}^k(-;\mathbb{Z})$, we find that $f_* : \mathfrak{O}^k(M;\mathbb{Z}) \to \mathfrak{O}^k(M';\mathbb{Z})$ is an isomorphism. □

Proposition 4.10 (Violation of the locality axiom)**.** *Let $f : M \to M'$ be a morphism in Loc_m. Then $f_* : \mathfrak{O}^k(M;\mathbb{Z}) \to \mathfrak{O}^k(M';\mathbb{Z})$ is injective if and only if $f_* : \mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(M;\mathbb{T}) \to \mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(M';\mathbb{T})$ is injective. For $m = 2$ and $k = 1$ the latter is always the case, while for $m \geq 3$ and $k \in \{1,\dots,m-1\}$ there is at least one morphism in Loc_m violating injectivity.*

Proof. Using again the natural isomorphism $\mathcal{O} : \mathfrak{C}^k_{\mathrm{sc}}(-;\mathbb{Z}) \Rightarrow \mathfrak{O}^k(-;\mathbb{Z})$, we can replace $f_* : \mathfrak{O}^k(M;\mathbb{Z}) \to \mathfrak{O}^k(M';\mathbb{Z})$ in the statement by $f_* : \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \to \mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z})$. The proof of the first part follows easily from the commutative diagram of short exact sequences given in Remark 3.6 and the fact that $f_* : \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M) \to \Omega^k_{\mathrm{sc},\mathbb{Z}} \cap *\Omega^{m-k}_{\mathrm{sc},\mathbb{Z}}(M')$ is always injective.

To prove the second part, we notice that there is a chain of isomorphisms

$$\mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(M;\mathbb{T}) \simeq \mathrm{H}^{k-1,m-k-1}_{\mathrm{c}}(\Sigma;\mathbb{T}) \simeq \mathrm{H}^{m-k,k}(\Sigma;\mathbb{Z})^\star \simeq \mathrm{H}^{m-k,k}(M;\mathbb{Z})^\star \ , \tag{4.21}$$

where * denotes Pontryagin duality. The first isomorphism is from (3.9), the second is presented in [BBSS15, Remark 5.7] and the third simply follows from homotopy invariance of cohomology and $M \simeq \mathbb{R} \times \Sigma$. Hence the counterexamples to injectivity provided in [BSS14, Example 6.9] can be used to prove the present claims. For the case $m = 2$ and $k = 1$, see the argument preceding [BSS14, Proposition 6.11]. □

The next theorem summarizes the results obtained in this section in view of the standard axioms of locally covariant field theory [BFV03]. In particular, we stress that the locality axiom, which requires $f_* : \mathfrak{O}^k(M;\mathbb{Z}) \to \mathfrak{O}^k(M';\mathbb{Z})$ to be injective for all Loc_m-morphisms $f : M \to M'$, does not hold in general.

Theorem 4.11. *The functor $(\mathfrak{O}^k(-;\mathbb{Z}),\tau) : \mathsf{Loc}_m \to \mathsf{PSAb}$ satisfies the causality and time-slice axioms of locally covariant field theory, however the locality axiom is satisfied only in the case $m = 2$ and $k = 1$, while it is violated for $m \geq 3$ and $k \in \{1,\dots,m-1\}$.*

5 Quantization

The quantization of the semi-classical gauge theory $(\mathfrak{O}^k(-;\mathbb{Z}),\tau) : \mathsf{Loc}_m \to \mathsf{PSAb}$ can be easily performed by using the well-established techniques of CCR-algebras, see [M$^+$73] and also [BDHS14, Appendix A] for details. Loosely speaking, given any m-dimensional spacetime M, we assign to the presymplectic Abelian group $(\mathfrak{O}^k(M;\mathbb{Z}),\tau)$ the C^*-algebra $\mathfrak{CCR}(\mathfrak{O}^k(M;\mathbb{Z}),\tau)$ that is generated by Weyl symbols $\mathrm{W}(\varphi)$, for all $\varphi \in \mathfrak{O}^k(M;\mathbb{Z})$, which satisfy the Weyl relations

$$\mathrm{W}(\varphi)\,\mathrm{W}(\varphi') = \exp\big(2\pi\,\mathrm{i}\,\tau(\varphi,\varphi')\big)\ \mathrm{W}(\varphi+\varphi') \ , \qquad \mathrm{W}(\varphi)^* = \mathrm{W}(-\varphi) \ , \tag{5.1}$$

for all $\varphi,\varphi' \in \mathfrak{O}^k(M;\mathbb{Z})$; by $\exp(2\pi\,\mathrm{i}\,(\,\cdot\,)) : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ we denote the embedding of the circle group into the complex numbers.

More precisely, the CCR-functor $\mathfrak{CCR} : \mathsf{PSAb} \to C^*\mathsf{Alg}$ from the category of presymplectic Abelian groups to the category of C^*-algebras is constructed in detail in [BDHS14, Appendix A]. Composing the functor $(\mathfrak{O}^k(-;\mathbb{Z}),\tau) : \mathsf{Loc}_m \to \mathsf{PSAb}$ with the CCR-functor, we obtain a functor from Loc_m to $C^*\mathsf{Alg}$ which, according to [BFV03], should be interpreted as a quantum field theory. We can thereby define a family of quantum field theories by setting

$$\mathfrak{A}^k := \mathfrak{CCR} \circ (\mathfrak{O}^k(-;\mathbb{Z}),\tau) : \mathsf{Loc}_m \longrightarrow C^*\mathsf{Alg} \ , \tag{5.2}$$

which depend on the degree $k \in \{1,\dots,m-1\}$ of the gauge theory. The properties of the semi-classical gauge theory from Theorem 4.11 are preserved by quantization (see e.g. the arguments in [BDS14b, BSS14]), which leads us to

Theorem 5.1. *The functor $\mathfrak{A}^k : \mathsf{Loc}_m \to C^*\mathsf{Alg}$ enjoys the following properties:*

- *Quantum causality axiom: Let $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$ be a diagram in Loc_m such that the images of f_1 and f_2 are causally disjoint. Then the subalgebras $f_{1*}(\mathfrak{A}^k(M_1))$ and $f_{2*}(\mathfrak{A}^k(M_2))$ of $\mathfrak{A}^k(M)$ commute.*

- *Quantum time-slice axiom: Let $f : M \to M'$ be a Cauchy morphism. Then $f_* : \mathfrak{A}^k(M) \to \mathfrak{A}^k(M')$ is an isomorphism.*

- *Violation of the quantum locality axiom: Let $f : M \to M'$ be a morphism in Loc_m. Then $f_* : \mathfrak{A}^k(M) \to \mathfrak{A}^k(M')$ is injective if and only if $f_* : \mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(M;\mathbb{T}) \to \mathrm{H}^{k-1,m-k-1}_{\mathrm{sc}}(M';\mathbb{T})$ is injective. For $m = 2$ and $k = 1$ the latter is always the case, while for $m \geq 3$ and $k \in \{1,\dots,m-1\}$ there is at least one morphism in Loc_m violating injectivity.*

6 Quantum duality

In this section we show that there exist dualities between the quantum field theories defined in (5.2). These dualities will be described at the functorial level and therefore hold true for all spacetimes M in a coherent (natural) way. In order to motivate our definition of duality given below, let us recall that a quantum field theory is a functor $\mathfrak{A} : \mathsf{Loc}_m \to C^*\mathsf{Alg}$ from the category of m-dimensional spacetimes to the category of C^*-algebras. The collection of *all* m-dimensional quantum field theories is therefore described by the functor category $[\mathsf{Loc}_m, C^*\mathsf{Alg}]$; objects in this category are functors $\mathfrak{A} : \mathsf{Loc}_m \to C^*\mathsf{Alg}$ and morphisms are natural transformations $\eta : \mathfrak{A} \Rightarrow \mathfrak{A}'$. In physics one calls the functor category $[\mathsf{Loc}_m, C^*\mathsf{Alg}]$ the "theory space" of m-dimensional quantum field theories which, being a category, comes with a natural notion of equivalence of theories.

Definition 6.1. A duality between two quantum field theories $\mathfrak{A}, \mathfrak{A}' : \mathsf{Loc}_m \to C^*\mathsf{Alg}$ is a natural isomorphism $\eta : \mathfrak{A} \Rightarrow \mathfrak{A}'$.

We shall now construct explicit dualities between the quantum field theories $\mathfrak{A}^k$ and $\mathfrak{A}^{m-k}$ given in (5.2), for all $m \geq 2$ and $k \in \{1,\dots,m-1\}$. Our strategy is to define first the dualities at the level of the semi-classical configuration spaces (2.4), and then lift them to the presymplectic Abelian groups and ultimately to the corresponding quantum field theories. For any object M in Loc_m we define a group homomorphism

$$\zeta : \mathfrak{C}^{m-k}(M;\mathbb{Z}) \longrightarrow \mathfrak{C}^k(M;\mathbb{Z}) ~, \qquad (h,\tilde{h}) \longmapsto \big(\tilde{h}, -(-1)^{k\,(m-k)}\,h\big) ~, \tag{6.1}$$

which interchanges (up to a sign) the roles of $h \in \hat{\mathrm{H}}^{m-k}(M;\mathbb{Z})$ and $\tilde{h} \in \hat{\mathrm{H}}^k(M;\mathbb{Z})$. We interpret the mapping (6.1) physically as exchanging the 'electric' and 'magnetic' sectors of the Abelian gauge theory. The map (6.1) defines a natural isomorphism $\zeta : \mathfrak{C}^{m-k}(-;\mathbb{Z}) \Rightarrow \mathfrak{C}^k(-;\mathbb{Z})$ because its components are clearly isomorphisms and for any morphism $f : M \to M'$ in Loc_m the diagram of Abelian groups

$$\begin{array}{ccc} \mathfrak{C}^{m-k}(M';\mathbb{Z}) & \xrightarrow{\ \zeta\ } & \mathfrak{C}^k(M';\mathbb{Z}) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \mathfrak{C}^{m-k}(M;\mathbb{Z}) & \xrightarrow[\ \zeta\ ]{} & \mathfrak{C}^k(M;\mathbb{Z}) \end{array} \tag{6.2}$$

commutes.

We now dualize (6.1) with respect to the weakly non-degenerate pairing (4.7): Define a group homomorphism $\zeta^\star : \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \to \mathfrak{C}^{m-k}_{\mathrm{sc}}(M;\mathbb{Z})$ by the condition

$$\big\langle (h,\tilde{h})\,,\, \zeta^\star(h',\tilde{h}'\,) \big\rangle := \big\langle \zeta(h,\tilde{h})\,,\, (h',\tilde{h}'\,) \big\rangle ~, \tag{6.3}$$

for all $(h', \tilde{h}') \in \mathfrak{C}^k_{\rm sc}(M;\mathbb{Z})$ and $(h,\tilde{h}) \in \mathfrak{C}^{m-k}(M;\mathbb{Z})$. A quick calculation shows that

$$\zeta^\star : \mathfrak{C}^k_{\rm sc}(M;\mathbb{Z}) \longrightarrow \mathfrak{C}^{m-k}_{\rm sc}(M;\mathbb{Z}) ~, \qquad (h',\tilde{h}') \longmapsto \big(-(-1)^{k\,(m-k)}\,\tilde{h}', h'\big) ~. \tag{6.4}$$

The mapping (6.4) defines a natural isomorphism $\zeta^\star : \mathfrak{C}^k_{\rm sc}(-;\mathbb{Z}) \Rightarrow \mathfrak{C}^{m-k}_{\rm sc}(-;\mathbb{Z})$, i.e. for any morphism $f : M \to M'$ in Loc_m the diagram of Abelian groups

$$\begin{array}{ccc} \mathfrak{C}^k_{\rm sc}(M;\mathbb{Z}) & \xrightarrow{\ \zeta^\star\ } & \mathfrak{C}^{m-k}_{\rm sc}(M;\mathbb{Z}) \\ {\scriptstyle f_\ast}\big\downarrow & & \big\downarrow{\scriptstyle f_\ast} \\ \mathfrak{C}^k_{\rm sc}(M';\mathbb{Z}) & \xrightarrow[\ \zeta^\star\ ]{} & \mathfrak{C}^{m-k}_{\rm sc}(M';\mathbb{Z}) \end{array} \tag{6.5}$$

commutes. We next observe that (6.4) preserves the presymplectic structure (4.15): A quick calculation shows that

$$\sigma\big(\zeta^\star(h,\tilde{h})\,,\,\zeta^\star(h',\tilde{h}')\big) = \sigma\big((h,\tilde{h})\,,\,(h',\tilde{h}')\big) ~, \tag{6.6}$$

for all $(h,\tilde{h}),(h',\tilde{h}') \in \mathfrak{C}^k_{\rm sc}(M;\mathbb{Z})$. Using also the natural isomorphisms $\mathcal{O} : \mathfrak{C}^p_{\rm sc}(-;\mathbb{Z}) \Rightarrow \mathfrak{O}^p(-;\mathbb{Z})$ given in Proposition 4.4, for $p=k$ and $p=m-k$, we find that $\zeta^\star$ defines a natural isomorphism (denoted by the same symbol)

$$\zeta^\star : \big(\mathfrak{O}^k(-;\mathbb{Z}),\tau\big) \Longrightarrow \big(\mathfrak{O}^{m-k}(-;\mathbb{Z}),\tau\big) \tag{6.7}$$

between functors from Loc_m to PSAb.

We can now state the main result of this section.

Theorem 6.2. *The $C^\ast$-algebra homomorphism*

$$\eta := \mathfrak{CCR}(\zeta^\star) : \mathfrak{A}^k(M) \longrightarrow \mathfrak{A}^{m-k}(M) \tag{6.8}$$

defines a duality between the two quantum field theories $\mathfrak{A}^k, \mathfrak{A}^{m-k} : \mathsf{Loc}_m \to C^\ast\mathsf{Alg}$.

Proof. We need to show that η defines a natural isomorphism $\eta : \mathfrak{A}^k \Rightarrow \mathfrak{A}^{m-k}$. Naturality of η is a direct consequence of naturality of $\zeta^\star$ and the fact that $\mathfrak{CCR}$ is a functor, in particular it preserves compositions. As functors preserve isomorphisms it then follows that η is a natural isomorphism. □

Corollary 6.3. *For $m = 2k$ the duality of Theorem 6.2 becomes a self-duality, i.e. a natural automorphism $\eta : \mathfrak{A}^k \Rightarrow \mathfrak{A}^k$.*

7 Self-dual Abelian gauge theory

In dimension $m = 2k$ it makes sense to demand the self-duality condition

$$\mathrm{curv}\, h = \ast\, \mathrm{curv}\, h \tag{7.1}$$

for a differential character $h \in \hat{\mathrm{H}}^k(M;\mathbb{Z})$. Applying the Hodge operator $\ast$ to both sides of (7.1) we obtain

$$\ast\,\mathrm{curv}\, h = -(-1)^{k^2}\,\mathrm{curv}\, h = -(-1)^{k^2}\ \ast\ \mathrm{curv}\, h ~, \tag{7.2}$$

which implies that for k even the only solutions to (7.1) are flat fields $h = \kappa(t)$, for $t \in \mathrm{H}^{k-1}(M;\mathbb{T})$. In the following we shall focus on the physically much richer and interesting case where $k \in 2\mathbb{Z}_{\geq 0}+1$ is odd.

The Abelian group of solutions to the self-duality equation (7.1) is denoted

$$\mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z}) := \big\{ h \in \hat{\mathrm{H}}^k(M;\mathbb{Z}) \,:\, \mathrm{curv}\, h = *\,\mathrm{curv}\, h \big\} \ . \tag{7.3}$$

There is a monomorphism

$$\mathrm{diag} : \mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \mathfrak{C}^k(M;\mathbb{Z}) \ , \qquad h \longmapsto (h,h) \tag{7.4}$$

to the semi-classical configuration space introduced in (2.4). Given any smooth spacelike Cauchy surface Σ of M with embedding $\iota_\Sigma : \Sigma \to M$, we compose (7.4) with the isomorphism of Theorem 2.7 and obtain a monomorphism

$$(\iota_\Sigma^* \times \iota_\Sigma^*) \circ \mathrm{diag} : \mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^{k,k}(\Sigma;\mathbb{Z}) \ , \qquad h \longmapsto (\iota_\Sigma^* h, \iota_\Sigma^* h) \ , \tag{7.5}$$

whose image is given by the diagonal in $\hat{\mathrm{H}}^{k,k}(\Sigma;\mathbb{Z})$: Given any $(h_\Sigma, h_\Sigma) \in \hat{\mathrm{H}}^{k,k}(\Sigma;\mathbb{Z})$ in the diagonal, consider the unique solution $(h,\tilde{h}) \in \mathfrak{C}^k(M;\mathbb{Z})$ of $\mathrm{curv}\, h = *\,\mathrm{curv}\, \tilde{h}$ with initial data $\iota_\Sigma^* h = h_\Sigma$ and $\iota_\Sigma^* \tilde{h} = h_\Sigma$, cf. Theorem 2.7. Then $(h-\tilde{h}, \tilde{h}-h) \in \mathfrak{C}^k(M;\mathbb{Z})$ satisfy $\iota_\Sigma^*(h-\tilde{h}) = 0$ and $\iota_\Sigma^*(\tilde{h}-h) = 0$, hence $\tilde{h} = h$ by using again Theorem 2.7. We have thereby obtained an isomorphism of Abelian groups

$$\iota_\Sigma^* : \mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^k(\Sigma;\mathbb{Z}) \ , \tag{7.6}$$

which we may interpret as in (2.18) as establishing the well-posedness of the initial value problem for $h \in \hat{\mathrm{H}}^k(M;\mathbb{Z})$ given by

$$\mathrm{curv}\, h = *\,\mathrm{curv}\, h \ , \qquad \iota_\Sigma^* h = h_\Sigma \ , \tag{7.7}$$

with initial datum $h_\Sigma \in \hat{\mathrm{H}}^k(\Sigma;\mathbb{Z})$.

Similar statements hold true for the Abelian group of solutions of spacelike compact support to the self-duality equation (7.1), denoted by

$$\mathfrak{sd}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) := \big\{ h \in \hat{\mathrm{H}}^k_{\mathrm{sc}}(M;\mathbb{Z}) \,:\, \mathrm{curv}\, h = *\,\mathrm{curv}\, h \big\} \ . \tag{7.8}$$

In particular, there is a monomorphism

$$\mathrm{diag} : \mathfrak{sd}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \longrightarrow \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \ , \qquad h \longmapsto (h,h) \tag{7.9}$$

to the Abelian group of semi-classical gauge fields of spacelike compact support introduced in (3.3). Using Corollary 3.5, one easily shows that

$$\iota_\Sigma^* : \mathfrak{sd}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^k_{\mathrm{c}}(\Sigma;\mathbb{Z}) \tag{7.10}$$

is an isomorphism, which we may interpret as establishing the well-posedness of the initial value problem (7.7) for $h \in \hat{\mathrm{H}}^k_{\mathrm{sc}}(M;\mathbb{Z})$ of spacelike compact support and initial datum $h_\Sigma \in \hat{\mathrm{H}}^k_{\mathrm{c}}(\Sigma;\mathbb{Z})$ of compact support.

Similarly to (4.7), there is a weakly non-degenerate $\mathbb{T}$-valued pairing

$$\langle\cdot,\cdot\rangle_{\mathfrak{sd}} : \mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z}) \times \mathfrak{sd}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \longrightarrow \mathbb{T} \ , \qquad (h,h'\,) \longmapsto \langle \iota_\Sigma^* h, \iota_\Sigma^* h'\,\rangle_{\mathrm{c}} \ , \tag{7.11}$$

which is independent of the choice of Cauchy surface Σ of M.[7] Thus there is a monomorphism

$$\mathcal{O}_{\mathfrak{sd}} : \mathfrak{sd}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \longrightarrow \mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z})^\star \ , \qquad h' \longmapsto \langle\cdot, h'\,\rangle_{\mathfrak{sd}} \tag{7.12}$$

[7] This is demonstrated by a proof similar to that of Lemma A.4.

to the character group of $\mathfrak{sdC}^k(M;\mathbb{Z})$, whose image is denoted $\mathfrak{sdO}^k(M;\mathbb{Z})$ and called the Abelian group of semi-classical observables on $\mathfrak{sdC}^k(M;\mathbb{Z})$.

Analogously to Proposition 4.5 we define a $\mathbb{T}$-valued presymplectic structure

$$\tau_{\mathfrak{sd}} : \mathfrak{sdO}^k(M;\mathbb{Z}) \times \mathfrak{sdO}^k(M;\mathbb{Z}) \longrightarrow \mathbb{T}\ , \qquad (\varphi,\varphi') \longmapsto \langle I(\mathcal{O}_{\mathfrak{sd}}^{-1}\varphi), \mathcal{O}_{\mathfrak{sd}}^{-1}\varphi' \rangle_{\mathfrak{sd}}\ . \tag{7.13}$$

Up to the isomorphism $\mathcal{O}_{\mathfrak{sd}}^{-1} : \mathfrak{sdO}^k(M;\mathbb{Z}) \to \mathfrak{sdC}^k_{\mathrm{sc}}(M;\mathbb{Z})$ induced by (7.12), the presymplectic structure reads as

$$\sigma_{\mathfrak{sd}} : \mathfrak{sdC}^k_{\mathrm{sc}}(M;\mathbb{Z}) \times \mathfrak{sdC}^k_{\mathrm{sc}}(M;\mathbb{Z}) \longrightarrow \mathbb{T}\ , \qquad (h,h') \longmapsto \langle I(h), h' \rangle_{\mathfrak{sd}}\ . \tag{7.14}$$

The radical of $\sigma_{\mathfrak{sd}}$ coincides with the kernel of $I : \mathfrak{sdC}^k_{\mathrm{sc}}(M;\mathbb{Z}) \to \mathfrak{sdC}^k(M;\mathbb{Z})$, hence the radical of $\tau_{\mathfrak{sd}}$ is $\mathcal{O}_{\mathfrak{sd}}(\ker I)$.

Using arguments similar to those of Section 4, one can show that the presymplectic Abelian groups $(\mathfrak{sdO}^k(M;\mathbb{Z}), \tau_{\mathfrak{sd}})$ for the self-dual field theory are functorial, i.e. we have constructed a functor

$$(\mathfrak{sdO}^k(-;\mathbb{Z}), \tau_{\mathfrak{sd}}) : \mathsf{Loc}_{2k} \longrightarrow \mathsf{PSAb}\ . \tag{7.15}$$

Composing with the CCR-functor from Section 5 we obtain quantum field theories

$$\mathfrak{sdA}^k := \mathfrak{CCR} \circ (\mathfrak{sdO}^k(-;\mathbb{Z}), \tau_{\mathfrak{sd}}) : \mathsf{Loc}_{2k} \longrightarrow C^\ast\mathsf{Alg}\ , \tag{7.16}$$

for all $k \in 2\mathbb{Z}_{\geq 0} + 1$, which quantize the self-duality equation (7.1). Using similar arguments as those of Section 4.4, one can show that these quantum field theories satisfy the same properties as those listed in Theorem 5.1.

Theorem 7.1. *The functor $\mathfrak{sdA}^k : \mathsf{Loc}_{2k} \to C^\ast\mathsf{Alg}$ enjoys the following properties:*

- *Quantum causality axiom: Let $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$ be a diagram in Loc_{2k} such that the images of f_1 and f_2 are causally disjoint. Then the subalgebras $f_{1\ast}(\mathfrak{sdA}^k(M_1))$ and $f_{2\ast}(\mathfrak{sdA}^k(M_2))$ of $\mathfrak{sdA}^k(M)$ commute.*
- *Quantum time-slice axiom: Let $f : M \to M'$ be a Cauchy morphism. Then $f_\ast : \mathfrak{sdA}^k(M) \to \mathfrak{sdA}^k(M')$ is an isomorphism.*
- *Violation of the quantum locality axiom: Let $f : M \to M'$ be a morphism in Loc_{2k}. Then $f_\ast : \mathfrak{sdA}^k(M) \to \mathfrak{sdA}^k(M')$ is injective if and only if $f_\ast : \mathrm{H}^{k-1}_{\mathrm{sc}}(M;\mathbb{T}) \to \mathrm{H}^{k-1}_{\mathrm{sc}}(M';\mathbb{T})$ is injective. For $k=1$ the latter is always the case, while for $k \in 2\mathbb{Z}_{\geq 0}+3$ there is at least one morphism in Loc_{2k} violating injectivity.*

Remark 7.2. We address the question how the self-dual quantum field theories $\mathfrak{sdA}^k$, which quantize the self-duality equation (7.1), are related to the self-dualities of the quantum field theories $\mathfrak{A}^k$ established in Corollary 6.3. Let $k \in 2\mathbb{Z}_{\geq 0}+1$ and consider any object M in Loc_{2k}. The self-duality (6.4) on $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ then reduces to $\zeta^\star(h',\tilde{h}') = (\tilde{h}', h')$, i.e. it simply interchanges h' and $\tilde{h}'$. The Abelian group of invariants under this self-duality is given by the diagonal

$$\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})^{\mathrm{inv}} := \big\{ (h',\tilde{h}') \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \,:\, \zeta^\star(h',\tilde{h}') = (h',\tilde{h}') \big\} = \big\{ (h',h') \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \big\}\ , \tag{7.17}$$

which by (7.9) is isomorphic to $\mathfrak{sdC}^k_{\mathrm{sc}}(M;\mathbb{Z})$. Restricting the presymplectic structure (4.15) to the invariants $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})^{\mathrm{inv}}$ then yields

$$\sigma : \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})^{\mathrm{inv}} \times \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})^{\mathrm{inv}} \longrightarrow \mathbb{T}\ , \qquad \big((h,h)\,,\,(h',h')\big) \longmapsto 2\,\sigma_{\mathfrak{sd}}(h,h')\ , \tag{7.18}$$

where $\sigma_{\mathfrak{sd}}$ is the presymplectic structure on $\mathfrak{sd}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ given in (7.14). Due to the prefactor 2, it follows that $(\mathfrak{sd}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}),\sigma_{\mathfrak{sd}})$ and $(\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})^{\mathrm{inv}},\sigma)$ are *not* isomorphic as presymplectic Abelian groups, but only as Abelian groups. Moreover, the C^*-algebras $\mathfrak{sd}\mathfrak{A}^k(M)$ and $\mathfrak{A}^k(M)^{\mathrm{inv}}$ (i.e. the C^*-subalgebra of $\mathfrak{A}^k(M)$ which is generated by the invariant Weyl symbols $\mathrm{W}(\mathcal{O}(h',h'))$, for all $(h',h')\in\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})^{\mathrm{inv}}$) are in general not isomorphic. Thus even though the quantum field theories $\mathfrak{sd}\mathfrak{A}^k:\mathsf{Loc}_{2k}\to C^*\mathsf{Alg}$ and $\mathfrak{A}^k(-)^{\mathrm{inv}}:\mathsf{Loc}_{2k}\to C^*\mathsf{Alg}$ are similar, they are strictly speaking not isomorphic. In particular, due to effects which are caused by $\mathbb{Z}_2$-torsion elements in the cohomology groups $\mathrm{H}^k(M;\mathbb{Z})$, the latter theory typically has a bigger center than the former theory. An explicit example of this fact is illustrated below.

Example 7.3. Fix any $k\in 2\mathbb{Z}_{\geq 0}+3$ and consider the lens space $L=\mathbb{S}^{2k-3}/\mathbb{Z}_2$ obtained as the quotient of the $2k{-}3$-sphere $\mathbb{S}^{2k-3}$ by the antipodal $\mathbb{Z}_2$-action. Take any object M in Loc_{2k} which admits a smooth spacelike Cauchy surface Σ diffeomorphic to $\mathbb{T}^2\times L$, where $\mathbb{T}^2$ is the 2-torus. Since the Cauchy surface Σ is compact, the notion of spacelike compact support becomes irrelevant for this spacetime M and in particular the homomorphism $I:\mathfrak{sd}\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\to\mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z})$ reduces to the identity. Using standard results on the homology groups of lens spaces, see e.g. [Hat02, Chapter 2, Example 2.43], and the universal coefficient theorem for cohomology, one shows that $\mathrm{H}^{k-1}(L;\mathbb{Z})\simeq\mathbb{Z}_2$. Using the Künneth theorem we find that $\mathrm{H}^k(\Sigma;\mathbb{Z})$ has a direct summand $(\mathbb{Z}_2)^2$. In particular, there exists $t\in\mathrm{H}^k(\Sigma;\mathbb{Z})$ such that $t\neq 0$, but $2t=0$. Recalling that char is surjective (cf. (2.3)), we find $f\in\hat{\mathrm{H}}^k(\Sigma;\mathbb{Z})$ such that $\mathrm{char}\, f=t$. It follows that there exists $A\in\Omega^{k-1}(\Sigma)$ such that $\iota[A]=2f$. Introducing $h_\Sigma=f-\iota[A/2]\in\hat{\mathrm{H}}^k(\Sigma;\mathbb{Z})$, by construction we obtain $h_\Sigma\neq 0$ (otherwise t would be trivial) and $2h_\Sigma=0$. Solving the initial value problem (7.7) provides $h\in\mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z})$ with $h\neq 0$, but $2h=0$. In fact, $2h\in\mathfrak{sd}\mathfrak{C}^k(M;\mathbb{Z})$ solves (7.7) with initial datum $2h_\Sigma=0$. Since Σ is compact, the presymplectic structure (7.14) is weakly non-degenerate. In particular, being non-zero, $h\in\mathfrak{sd}\mathfrak{C}^k(M)$ is not in the radical. Conversely, taking into account also $(h,h)\in\mathfrak{C}^k(M;\mathbb{Z})^{\mathrm{inv}}$, we find $\sigma((h,h),(h',h'))=2\sigma_{\mathfrak{sd}}(h,h')=\sigma_{\mathfrak{sd}}(2h,h')=0$ for all $(h',h')\in\mathfrak{C}^k(M)^{\mathrm{inv}}$. This shows that the center of $\mathfrak{A}^k(M)^{\mathrm{inv}}$ is bigger than that of $\mathfrak{sd}\mathfrak{A}^k(M)$ for this particular spacetime M.

Acknowledgments

We thank Ulrich Bunke for helpful discussions and the anonymous referees for their comments and remarks. This work was supported in part by the Action MP1405 QSPACE from the European Cooperation in Science and Technology (COST). The work of C.B. is partially supported by the Collaborative Research Center (SFB) "Raum Zeit Materie", funded by the Deutsche Forschungsgemeinschaft (DFG, Germany). The work of M.B. is supported partly by a Research Fellowship of the Della Riccia Foundation (Italy) and partly by a Postdoctoral Fellowship of the Alexander von Humboldt Foundation (Germany). The work of A.S. is supported by a Research Fellowship of the Deutsche Forschungsgemeinschaft (DFG, Germany). The work of R.J.S. is partially supported by the Consolidated Grant ST/L000334/1 from the UK Science and Technology Facilities Council.

A Technical lemmas

In this appendix we prove five lemmas which are used in the main text. Some of these proofs are rather technical and also make use of results in the companion paper [BBSS15], in which case we give precise references.

Lemma A.1. *Let M be a time-oriented m-dimensional globally hyperbolic Lorentzian manifold and Σ a smooth spacelike Cauchy surface of M. Consider the embedding $\iota_\Sigma:\Sigma\to M$ of Σ into M, and the projection $\pi_\Sigma:M\to\Sigma$ of M onto Σ which is induced by a choice of diffeomorphism $M\simeq\mathbb{R}\times\Sigma$ such that $\iota_\Sigma(\Sigma)\simeq\{0\}\times\Sigma$, cf. [BS05, BS06]. Then:*

(i) Inducing ι_Σ and π_Σ to smooth singular chains, i.e. $\iota_{\Sigma\,*} : C_\sharp(\Sigma) \to C_\sharp(M)$ and $\pi_{\Sigma\,*} : C_\sharp(M) \to C_\sharp(\Sigma)$, we have $\pi_{\Sigma\,*} \circ \iota_{\Sigma\,*} = \mathrm{id}$ and $\iota_{\Sigma\,*} \circ \pi_{\Sigma\,*} - \mathrm{id} = \partial \circ h_\Sigma + h_\Sigma \circ \partial$, for a chain homotopy $h_\Sigma : C_\sharp(M) \to C_{\sharp+1}(M)$. In particular, ι_Σ and π_Σ induce isomorphisms on smooth singular homology:

$$\mathrm{H}_\sharp(M) \underset{\iota_{\Sigma\,*}}{\overset{\pi_{\Sigma\,*}}{\rightleftarrows}} \mathrm{H}_\sharp(\Sigma) \ . \tag{A.1}$$

(ii) Let G be an Abelian group. Inducing ι_Σ and π_Σ to smooth singular G-valued cochains, i.e. $\iota_\Sigma^* : C^\sharp(M;G) \to C^\sharp(\Sigma;G)$ and $\pi_\Sigma^* : C^\sharp(\Sigma;G) \to C^\sharp(M;G)$, we have $\iota_\Sigma^* \circ \pi_\Sigma^* = \mathrm{id}$ and $\pi_\Sigma^* \circ \iota_\Sigma^* - \mathrm{id} = \delta \circ h_\Sigma^\star + h_\Sigma^\star \circ \delta$, for a cochain homotopy $h_\Sigma^\star : C^\sharp(M;G) \to C^{\sharp-1}(M;G)$. In particular, ι_Σ and π_Σ induce isomorphisms on smooth singular cohomology with coefficients in G:

$$\mathrm{H}^\sharp(M;G) \underset{\iota_\Sigma^*}{\overset{\pi_\Sigma^*}{\rightleftarrows}} \mathrm{H}^\sharp(\Sigma;G) \ . \tag{A.2}$$

Proof. We shall denote points by $x \in \Sigma$ and $(t,x) \in M \simeq \mathbb{R} \times \Sigma$. By construction we have $\pi_\Sigma \circ \iota_\Sigma = \mathrm{id}_\Sigma$. Notice further that $\iota_\Sigma \circ \pi_\Sigma$ and the identity id_M are homotopic via

$$H_\Sigma : [0,1] \times M \longrightarrow M \ , \qquad \big(s\,,\,(t,x)\big) \longmapsto (s\,t, x) \ . \tag{A.3}$$

As usual, see for example the proof of [Hat02, Theorem 2.10], this homotopy induces the desired chain homotopy $h_\Sigma : C_\sharp(M) \to C_{\sharp+1}(M)$, which proves item (i). Item (ii) then follows by defining $h_\Sigma^\star = \mathrm{Hom}(h_\Sigma, G) : C^\sharp(M;G) \to C^{\sharp-1}(M;G)$. □

Lemma A.2. *Under the same hypotheses as in Lemma A.1, let $K \subseteq \Sigma$ be a compact subset. Then:*

(i) Inducing ι_Σ and π_Σ to relative smooth singular chains, i.e. $\iota_{\Sigma\,*} : C_\sharp(\Sigma, \Sigma \setminus K) \to C_\sharp(M, M \setminus J(K))$ and $\pi_{\Sigma\,*} : C_\sharp(M, M \setminus J(K)) \to C_\sharp(\Sigma, \Sigma \setminus K)$, we have $\pi_{\Sigma\,*} \circ \iota_{\Sigma\,*} = \mathrm{id}$ and $\iota_{\Sigma\,*} \circ \pi_{\Sigma\,*} - \mathrm{id} = \partial \circ h_\Sigma + h_\Sigma \circ \partial$, for a chain homotopy $h_\Sigma : C_\sharp(M, M \setminus J(K)) \to C_{\sharp+1}(M, M \setminus J(K))$. In particular, ι_Σ and π_Σ induce an isomorphism on relative smooth singular homology:

$$\mathrm{H}_\sharp(M, M \setminus J(K)) \underset{\iota_{\Sigma\,*}}{\overset{\pi_{\Sigma\,*}}{\rightleftarrows}} \mathrm{H}_\sharp(\Sigma, \Sigma \setminus K) \ . \tag{A.4}$$

(ii) Let G be an Abelian group. Inducing ι_Σ and π_Σ to relative smooth singular G-valued cochains, i.e. $\iota_\Sigma^* : C^\sharp(M, M \setminus J(K);G) \to C^\sharp(\Sigma, \Sigma \setminus K;G)$ and $\pi_\Sigma^* : C^\sharp(\Sigma, \Sigma \setminus K;G) \to C^\sharp(M, M \setminus J(K);G)$, we have $\iota_\Sigma^* \circ \pi_\Sigma^* = \mathrm{id}$ and $\pi_\Sigma^* \circ \iota_\Sigma^* - \mathrm{id} = \delta \circ h_\Sigma^\star + h_\Sigma^\star \circ \delta$, for a cochain homotopy $h_\Sigma^\star : C^\sharp(M, M \setminus J(K);G) \to C^{\sharp-1}(M, M \setminus J(K);G)$. In particular, ι_Σ and π_Σ induce an isomorphism on relative smooth singular cohomology with coefficients in G:

$$\mathrm{H}^\sharp(M, M \setminus J(K);G) \underset{\iota_\Sigma^*}{\overset{\pi_\Sigma^*}{\rightleftarrows}} \mathrm{H}^\sharp(\Sigma, \Sigma \setminus K;G) \ . \tag{A.5}$$

Proof. Notice that $\iota_\Sigma : \Sigma \to M$ maps $\Sigma \setminus K$ to $M \setminus J(K)$ because Σ is by assumption spacelike. Moreover, $\pi_\Sigma : M \to \Sigma$ maps $M \setminus J(K)$ to $\Sigma \setminus K$: Assume there exists $(t,x) \in M \setminus J(K)$ such that $\pi_\Sigma(t,x) = x \in K$; then the curve $\gamma : [0,1] \to M\,,\ s \mapsto (s\,t, x)$ connecting $(0,x)$ with (t,x) is timelike, hence $(t,x) \in J(K)$ which is a contradiction. Similarly, the homotopy (A.3) restricts to $H_\Sigma : [0,1] \times \big(M \setminus J(K)\big) \to M \setminus J(K)$. The rest of the proof then follows that of Lemma A.1. □

Lemma A.3. *Let $f : M \to M'$ be a morphism in Loc_m and denote by $\mathcal{K}_M$ the directed set of compact subsets of M. Consider the natural transformation*

$$f^* : \mathfrak{C}^k(M', M' \setminus J(f(-)); \mathbb{Z}) \Longrightarrow \mathfrak{C}^k(M, M \setminus J(-); \mathbb{Z}) \tag{A.6}$$

between functors from $\mathcal{K}_M$ to Ab induced by f. Then, for each smooth spacelike Cauchy surface Σ of M, the restriction of f^* to the directed set $\mathcal{K}_\Sigma \subseteq \mathcal{K}_M$ of compact subsets of Σ is a natural isomorphism. In particular, we can consider the natural transformation

$$(f^*)^{-1} : \mathfrak{C}^k(M, M \setminus J(-); \mathbb{Z}) \Longrightarrow \mathfrak{C}^k(M', M' \setminus J(f(-)); \mathbb{Z}) \tag{A.7}$$

between functors from $\mathcal{K}_\Sigma$ to Ab. Then the pushforward for semi-classical gauge fields of spacelike compact support

$$f_* : \mathfrak{C}^k_{\mathrm{sc}}(M; \mathbb{Z}) \longrightarrow \mathfrak{C}^k_{\mathrm{sc}}(M'; \mathbb{Z}) \tag{A.8}$$

is canonically induced by the colimit prescription in (3.3) *restricted to $\mathcal{K}_\Sigma$, see also Remark 3.2, and by the universal property of colimits.*

Proof. For each $K \subseteq M$ compact, we note that $f : M \to M'$ in Loc_m induces an open embedding $f : (M, M \setminus J(K)) \to (M', M' \setminus J(f(K)))$ of pairs which is compatible with the inclusions of the given submanifolds. Looking at an inclusion $K \subseteq K'$ of compact subsets of M, one realizes that both $(M, M \setminus J(-))$ and $(M', M' \setminus J(f(-)))$ are functors from $\mathcal{K}_M$ to $\mathsf{Pair}^{\mathrm{op}}$, the opposite category of the category Pair of pairs of manifolds with submanifold preserving smooth maps as morphisms; moreover, $f : (M, M \setminus J(-)) \Rightarrow (M', M' \setminus J(f(-)))$ is a natural transformation between these functors. Therefore, applying the functor $\hat{\mathrm{H}}^{k,m-k}(-; \mathbb{Z}) : \mathsf{Pair}^{\mathrm{op}} \to \mathsf{Ab}$, cf. [BBSS15, Section 3.1], we obtain the pullback along f as a natural transformation

$$f^* : \hat{\mathrm{H}}^{k,m-k}(M', M' \setminus J(f(-)); \mathbb{Z}) \Longrightarrow \hat{\mathrm{H}}^{k,m-k}(M, M \setminus J(-); \mathbb{Z}) \tag{A.9}$$

between functors from $\mathcal{K}_M$ to Ab. Since f is a morphism in Loc_m, hence in particular an isometry, and curv is a natural transformation for relative differential characters, see [BBSS15, Section 3.1], f^* maps relative semi-classical gauge fields on M' to relative semi-classical gauge fields on M, so that we obtain the natural transformation displayed in (A.6).

We will now show that the restriction to $\mathcal{K}_\Sigma$ of the natural transformation (A.6) is a natural isomorphism. For each $K \subseteq \Sigma$ compact, we choose an open neighborhood $U \subseteq \Sigma$ of K with compact closure $\overline{U}$. We denote by $j : U \to \Sigma$ the open embedding induced by the inclusion. Observing that $f(\,\overline{U}\,) \subseteq M'$ is a spacelike and acausal compact submanifold with boundary of M', by [BS06, Theorem 1.1] there is a smooth spacelike Cauchy surface Σ' of M' extending $f(\,\overline{U}\,)$ whose embedding in M' is denoted by $\iota_{\Sigma'} : \Sigma' \to M'$. We also denote by $f_U : U \to \Sigma'$ the open embedding induced by the restriction of f. By construction, the diagram

$$\begin{array}{ccc}
(U, U \setminus K) & \xrightarrow{\quad f_U \quad} & (\Sigma', \Sigma' \setminus f(K)) \\
\Big\downarrow{\scriptstyle j} & & \\
(\Sigma, \Sigma \setminus K) & & \Big\downarrow{\scriptstyle \iota_{\Sigma'}} \\
\Big\downarrow{\scriptstyle \iota_\Sigma} & & \\
(M, M \setminus J(K)) & \xrightarrow[\quad f \quad]{} & (M', M' \setminus J(f(K)))
\end{array} \tag{A.10}$$

in the category Pair commutes. Therefore, applying the functor $\hat{\mathrm{H}}^{k,m-k}(-; \mathbb{Z}) : \mathsf{Pair}^{\mathrm{op}} \to \mathsf{Ab}$ and recalling that the pullback along f maps relative semi-classical gauge fields to relative semi-classical gauge fields, we obtain a new commutative diagram

$$\begin{array}{ccc} \mathfrak{C}^k(M',M'\setminus J(f(K));\mathbb{Z}) & \xrightarrow{\;f^*\;} & \mathfrak{C}^k(M,M\setminus J(K);\mathbb{Z}) \\ & & \Big\downarrow \iota_\Sigma^* \\ \Big\downarrow \iota_{\Sigma'}^* & & \hat{\mathrm{H}}^{k,m-k}(\Sigma,\Sigma\setminus K;\mathbb{Z}) \\ & & \Big\downarrow j^* \\ \hat{\mathrm{H}}^{k,m-k}(\Sigma',\Sigma'\setminus f(K);\mathbb{Z}) & \xrightarrow[\;f_U^*\;]{} & \hat{\mathrm{H}}^{k,m-k}(U,U\setminus K;\mathbb{Z}) \end{array} \tag{A.11}$$

in the category of Abelian groups Ab. Using Theorem 3.4 and the excision theorem [BBSS15, Theorem 3.8] we find that the vertical and bottom horizontal arrows are isomorphisms. In particular, note that f_U can be factored as the composition of a diffeomorphism onto its image followed by the inclusion of its image into the target, hence by excision $f_U^* : \hat{\mathrm{H}}^k(\Sigma',\Sigma'\setminus f(K);\mathbb{Z}) \to \hat{\mathrm{H}}^k(U,U\setminus K;\mathbb{Z})$ is an isomorphism. It follows that the top horizontal arrow is an isomorphism too. □

The proofs of our final two lemmas will rely extensively on Lemma A.3. In particular, we will adopt the following approach. Starting from a semi-classical gauge field of spacelike compact support and unraveling the directed colimit in (3.3), we will represent it by a gauge field relative to the complement of $J(K)$ for a suitable compact subset K of a smooth spacelike Cauchy surface. Then we use Lemma A.3 to represent the pushforward of the given spacelike compact gauge field by the image under the inverse of the pullback of the corresponding relative gauge field.

Lemma A.4. *The pairing* (4.7) *does not depend on the choice of Cauchy surface* Σ*. Moreover, for any morphism* $f : M \to M'$ *in* Loc_m *the diagram of Abelian groups*

$$\begin{array}{ccc} \mathfrak{C}^k(M';\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) & \xrightarrow{\;f^*\times\mathrm{id}\;} & \mathfrak{C}^k(M;\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z}) \\ \Big\downarrow \mathrm{id}\times f_* & & \Big\downarrow \langle\cdot,\cdot\rangle \\ \mathfrak{C}^k(M';\mathbb{Z}) \times \mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z}) & \xrightarrow[\;\langle\cdot,\cdot\rangle\;]{} & \mathbb{T} \end{array} \tag{A.12}$$

commutes.

Proof. We first prove independence of the pairing (4.7) on the choice of Cauchy surface Σ used to evaluate it. For this, we choose any two smooth spacelike Cauchy surfaces Σ and Σ' of M. Let $(h,\tilde{h}) \in \mathfrak{C}^k(M;\mathbb{Z})$, $K \subseteq \Sigma$ compact and $(h',\tilde{h}') \in \mathfrak{C}^k(M,M\setminus J(K);\mathbb{Z})$. Note that $K' = \Sigma' \cap J(K)$ is compact and that $J(K) \subseteq J(K')$. Let μ denote the unique element of $\mathrm{H}_{m-1}(\Sigma,\Sigma\setminus K)$ which agrees with the orientation of Σ for each point of K. Similarly, let μ' denote the unique element of $\mathrm{H}_{m-1}(\Sigma',\Sigma'\setminus K')$ which agrees with the orientation of Σ' at each point of K'. By means of the isomorphisms in Lemma A.2, we can compare μ with $\tilde{\mu} = \pi_{\Sigma\,*}\,\iota_{\Sigma'\,*}\mu' \in \mathrm{H}_{m-1}(\Sigma,\Sigma\setminus K)$. Since the orientations of both Σ' and Σ are chosen consistently with the orientation and time-orientation of M, for each point of K, $\tilde{\mu}$ agrees with the orientation of Σ. In particular [Hat02, Lemma 3.27] entails that $\tilde{\mu} = \mu \in \mathrm{H}_{m-1}(\Sigma,\Sigma\setminus K)$, therefore $\iota_{\Sigma'\,*}\mu' = \iota_{\Sigma\,*}\mu \in \mathrm{H}_{m-1}(M,M\setminus J(K))$ by Lemma A.2. Let $\nu \in Z_{m-1}(\Sigma,\Sigma\setminus K)$ and $\nu' \in Z_{m-1}(\Sigma',\Sigma'\setminus K')$ be cycles representing μ and μ' respectively. Hence we obtain $\gamma \in C_m(M,M\setminus J(K))$ such that $\iota_{\Sigma *}\nu - \iota_{\Sigma' *}\nu' = \partial\gamma$. Taking into account also [BBSS15, Definition 3.1], we get

$$\begin{aligned} \big\langle \iota_\Sigma^*(h,\tilde{h})\,,\,\iota_\Sigma^*(h',\tilde{h}')\big\rangle_\Sigma - \big\langle \iota_{\Sigma'}^*(h,\tilde{h})\,,\,\iota_{\Sigma'}^*(h',\tilde{h}')\big\rangle_{\Sigma'} &= \big(\tilde{h}\cdot h' - (-1)^{k\,(m-k)}\,h\cdot\tilde{h}'\big)(\partial\gamma) \\ &= \int_\gamma \mathrm{curv}\big(\tilde{h}\cdot h' - (-1)^{k\,(m-k)}\,h\cdot\tilde{h}'\big) \mod \mathbb{Z}\,, \end{aligned} \tag{A.13}$$

where the subscripts $_\Sigma$ and $_{\Sigma'}$ denote the Cauchy surfaces which have been used to evaluate the pairing (4.7). Furthermore, using [BBSS15, eq. (3.31)] together with the identities $\mathrm{curv}\, h = \ast\, \mathrm{curv}\, \tilde{h}$ and $\mathrm{curv}\, h' = \ast\, \mathrm{curv}\, \tilde{h}'$, one has

$$\mathrm{curv}\,\big(\tilde{h}\cdot h' - (-1)^{k\,(m-k)}\, h\cdot \tilde{h}'\,\big) = \mathrm{curv}\,\tilde{h}\wedge \ast\,\mathrm{curv}\,\tilde{h}' - \mathrm{curv}\,\tilde{h}'\wedge\ast\,\mathrm{curv}\,\tilde{h} = 0\ . \tag{A.14}$$

To complete the proof, we show that the diagram (A.12) commutes. Let $(h,\tilde{h})\in\mathfrak{C}^k(M';\mathbb{Z})$ and $(h',\tilde{h}')\in\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$. Consider a smooth spacelike Cauchy surface Σ of M and $K\subseteq\Sigma$ compact such that $(h',\tilde{h}')\in\mathfrak{C}^k(M,M\setminus J(K);\mathbb{Z})$. Since f is a morphism in the category Loc_m, the induced map $f:(M,M\backslash J(K))\to(M',M'\backslash J(f(K)))$ is a morphism in Pair, in particular an open embedding between the m-manifolds M and M' which maps the open subset $M\setminus J(K)$ to $M'\backslash f(J(K))$. We represent $f_\ast(h',\tilde{h}')\in\mathfrak{C}^k_{\mathrm{sc}}(M';\mathbb{Z})$ by $(f^\ast)^{-1}(h',\tilde{h}')\in\mathfrak{C}^k(M',M'\backslash J(f(K));\mathbb{Z})$, and interpret $(h',\tilde{h}')$ as its representative in $\mathfrak{C}^k(M,M\setminus J(K);\mathbb{Z})$. Recalling the proof of Lemma A.3, we consider an open neighborhood $U\subseteq\Sigma$ of K with compact closure $\overline{U}$ and we extend $f(\overline{U}\,)$ to a smooth spacelike Cauchy surface Σ' of M'. With the same notation, we then find

$$\big\langle(h,\tilde{h})\,,\,f_\ast(h',\tilde{h}')\big\rangle = \big\langle\iota_{\Sigma'}^\ast(h,\tilde{h})\,,\,\iota_{\Sigma'}^\ast\,(f^\ast)^{-1}(h',\tilde{h}')\big\rangle_{\Sigma'} = \big\langle\iota_{\Sigma'}^\ast(h,\tilde{h})\,,\,(f_U^\ast)^{-1}\,j^\ast\,\iota_\Sigma^\ast(h',\tilde{h}')\big\rangle_{\Sigma'}\ . \tag{A.15}$$

This follows from the definition of the pairing (4.7) and the diagram in (A.11). Note that $f_{U\ast}:\hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(U;\mathbb{Z})\to\hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma';\mathbb{Z})$ is defined as the colimit over the directed set $\mathcal{K}_U$ of compact subsets of U of the inverse of the natural isomorphism $f_U^\ast:\hat{\mathrm{H}}^{k,m-k}(\Sigma',\Sigma'\setminus f_U(-))\Rightarrow\hat{\mathrm{H}}^{k,m-k}(U,U\setminus -)$, see [BBSS15, Section 4]. We further use commutativity of the diagram

$$\begin{array}{ccc}
\hat{\mathrm{H}}^k(\Sigma';\mathbb{Z})\times\hat{\mathrm{H}}^{m-k}_{\mathrm{c}}(U;\mathbb{Z}) & \xrightarrow{\ f_U^\ast\times\mathrm{id}\ } & \hat{\mathrm{H}}^k(U;\mathbb{Z})\times\hat{\mathrm{H}}^{m-k}_{\mathrm{c}}(U;\mathbb{Z}) \\
{\scriptstyle \mathrm{id}\times f_{U\ast}}\big\downarrow & & \big\downarrow{\scriptstyle\langle\cdot,\cdot\rangle_{\mathrm{c}}} \\
\hat{\mathrm{H}}^k(\Sigma';\mathbb{Z})\times\hat{\mathrm{H}}^{m-k}_{\mathrm{c}}(\Sigma';\mathbb{Z}) & \xrightarrow[\ \langle\cdot,\cdot\rangle_{\mathrm{c}}\ ]{} & \mathbb{T}
\end{array} \tag{A.16}$$

which is shown in [BBSS15, Section 5.2]. It then follows that

$$\big\langle(h,\tilde{h})\,,\,f_\ast(h',\tilde{h}')\big\rangle = \big\langle\iota_{\Sigma'}^\ast(h,\tilde{h})\,,\,f_{U\ast}\,j^\ast\,\iota_\Sigma^\ast(h',\tilde{h}')\big\rangle_{\Sigma'} = \big\langle f_U^\ast\,\iota_{\Sigma'}^\ast(h,\tilde{h})\,,\,j^\ast\,\iota_\Sigma^\ast(h',\tilde{h}')\big\rangle_U\ . \tag{A.17}$$

From diagram (A.10) we have $\iota_{\Sigma'}\circ f_U = f\circ\iota_\Sigma\circ j$, and by using the analogue of the diagram (A.16) for the open embedding $j:U\to\Sigma$ we find

$$\big\langle(h,\tilde{h})\,,\,f_\ast(h',\tilde{h}')\big\rangle = \big\langle j^\ast\,\iota_\Sigma^\ast\,f^\ast(h,\tilde{h})\,,\,j^\ast\,\iota_\Sigma^\ast(h',\tilde{h}')\big\rangle_U = \big\langle\iota_\Sigma^\ast\,f^\ast(h,\tilde{h})\,,\,j_\ast\,j^\ast\,\iota_\Sigma^\ast(h',\tilde{h}')\big\rangle_\Sigma\ . \tag{A.18}$$

Similarly to $f_{U\ast}$, the group homomorphism $j_\ast:\hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(U;\mathbb{Z})\to\hat{\mathrm{H}}^{k,m-k}_{\mathrm{c}}(\Sigma;\mathbb{Z})$ is obtained as the colimit over $\mathcal{K}_U$ of the inverse of the natural isomorphism $j^\ast:\hat{\mathrm{H}}^{k,m-k}(\Sigma,\Sigma\setminus -;\mathbb{Z})\Rightarrow\hat{\mathrm{H}}^{k,m-k}(U,U\setminus -;\mathbb{Z})$. It follows that

$$\big\langle(h,\tilde{h})\,,\,f_\ast(h',\tilde{h}')\big\rangle = \big\langle\iota_\Sigma^\ast\,f^\ast(h,\tilde{h})\,,\,(j^\ast)^{-1}\,j^\ast\,\iota_\Sigma^\ast(h',\tilde{h}')\big\rangle_\Sigma = \big\langle f^\ast(h,\tilde{h})\,,\,(h',\tilde{h}')\big\rangle\ , \tag{A.19}$$

where for the last equality we used the definition of the pairing (4.7). □

Lemma A.5. *Recalling* (3.5)*, the group homomorphisms* $I:\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})\to\mathfrak{C}^k(M;\mathbb{Z})$*, for objects* M *in* Loc_m*, enjoy the following properties:*

(i) $f^\ast\circ I\circ f_\ast = I$*, for all morphisms* $f:M\to M'$ *in* Loc_m*.*

(ii) $f_2^* \circ I \circ f_{1*} = 0$, *for all diagrams* $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$ *in* Loc_m *such that the images of* f_1 *and* f_2 *are causally disjoint.*

Proof. Let us start with statement (i). For $(h', \tilde{h}') \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$, we show that $f^* \, I \, f_*(h', \tilde{h}') = I(h', \tilde{h}')$. For a fixed smooth spacelike Cauchy surface Σ, let $K \subseteq \Sigma$ be compact with $(h', \tilde{h}') \in \mathfrak{C}^k(M, M \backslash J(K);\mathbb{Z})$. By [BBSS15, eq. (3.13)] one has $f^* \circ I = I \circ f^* : \mathfrak{C}^k(M', M' \backslash J(f(K));\mathbb{Z}) \to \mathfrak{C}^k(M;\mathbb{Z})$, and so we find

$$f^* \, I \, (f^*)^{-1}(h', \tilde{h}') = I \, f^* \, (f^*)^{-1}(h', \tilde{h}') = I(h', \tilde{h}') \ . \tag{A.20}$$

This equation corresponds to $f^* \, I \, f_*(h', \tilde{h}') = I(h', \tilde{h}')$ when $(h', \tilde{h}')$ is regarded as an element of $\mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$.

For statement (ii), let $(h', \tilde{h}') \in \mathfrak{C}^k_{\mathrm{sc}}(M_1;\mathbb{Z})$. Choosing a smooth spacelike Cauchy surface Σ of M_1, we find a compact subset K of Σ such that $(h', \tilde{h}') \in \mathfrak{C}^k(M_1, M_1 \setminus J(K);\mathbb{Z})$. As above, we represent $f_{1*}(h', \tilde{h}') \in \mathfrak{C}^k_{\mathrm{sc}}(M;\mathbb{Z})$ by $(f_1^*)^{-1}(h', \tilde{h}') \in \mathfrak{C}^k(M, M \setminus J(f_1(K));\mathbb{Z})$. For each pair of cycles $(z, \tilde{z}) \in Z_{k-1,m-k-1}(M_2)$, the pushforward $f_{2*}(z, \tilde{z}) \in Z_{k-1,m-k-1}(M)$ is supported inside $f_2(M_2)$. By assumption $f_2(M_2) \subseteq M \setminus J(f_1(M_1)) \subseteq M \setminus J(f_1(K))$, hence $f_{2*}(z, \tilde{z}) = 0$ in $Z_{k-1,m-k-1}(M, M \setminus J(f_1(K)))$. In particular, $I \, (f_1^*)^{-1}(h', \tilde{h}')$ vanishes when evaluated on $f_{2*}(z, \tilde{z})$. Since this is the case for any $(z, \tilde{z}) \in Z_{k-1,m-k-1}(M_2)$, we conclude that $f_2^* \, I \, (f_1^*)^{-1} \, (h', \tilde{h}') = 0$ and hence also $f_2^* \, I \, f_{1*}(h', \tilde{h}') = 0$. □

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