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The Scarcity of Products in $\beta S \setminus S$

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Abstract

Let S be a discrete semigroup and let the Stone-Čech compactification βS of S have the operation extending that of S which makes βS a right topological semigroup with S contained in its topological center. Let $S^* = \beta S \setminus S$. Algebraically, the set of products S^*S^* tends to be rather large, since it often contains the smallest ideal of βS. We establish here sufficient conditions involving mild cancellation assumptions and assumptions about the cardinality of S for S^*S^* to be topologically small, that is for S^*S^* to be nowhere dense in S^*, or at least for $S^* \setminus S^*S^*$ to be dense in S^*. And we provide examples showing that these conditions cannot be significantly weakend. These extend results previously known for countable semigroups. Other results deal with large sets missing S^*S^* whose elements have algebraic properties, such as being right cancelable and generating free semigroups in βS.

Key words: Stone-Čech Compactification, nowhere dense, weak cancellation,

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1. Introduction

Let $(S, \cdot)$ be a discrete semigroup. We take the Stone-Čech compactification βS of S to be the set of ultrafilters on S with the points of S identified with the principal ultrafilters. Given $A \subseteq S$, we let $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets of βS as well as a basis for the closed sets. The operation on S extends uniquely to βS so that $(\beta S, \cdot)$ becomes a right topoological semigroup (meaning that the function ρ_p defined by $\rho_p(x) = x \cdot p$ is continuous for each $p \in \beta S$) with S contained in its topological center (meaning that the function λ_y defined by $\lambda_y(x) = y \cdot x$ is continuous for each $y \in S$). So, if $p, q \in \beta S$, $p \cdot q = \lim_{s \to p} \lim_{t \to q} st$, where s and t denote elements of S. Given p and q in βS and $A \subseteq S$, one has $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

If T is a subset of a set S, we shall regard βT as a subset of βS by identifying each $p \in \beta T$ with the unique ultrafilter in βS which contains p.

As does any compact Hausdorff right topological semigroup, βS has a smallest two sided ideal, $K(\beta S)$, which is the union of all of the minimal right ideals and is also the union of all of the minimal left ideals. For an elementary introduction to the algebraic structure of βS, see [5, Part I].

We let $S^* = \beta S \setminus S$. Let $\mathbb{N}$ be the set of positive integers. In an early paper [7] on the algebraic structure of βS, the second author of this paper showed that the elements of $\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)$ generate a semigroup which is almost free (subject only to the restriction that for $p \in \mathbb{N}^*$ and $n \in \mathbb{Z}$, p and $p+n$ commute), and that a corresponding assertion holds for $(\beta\mathbb{N}, \cdot)$. From a topological point of view, $\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)$ is most of $\mathbb{N}^*$, that is $\mathbb{N}^* + \mathbb{N}^*$ is nowhere dense in $\mathbb{N}^*$. (As we remarked in the abstract, $\mathbb{N}^* + \mathbb{N}^*$ is algebraically quite large since it contains $K(\beta\mathbb{N})$ and therefore contains $2^{\mathfrak{c}}$ minimal right ideals, $2^{\mathfrak{c}}$ minimal left ideals, and $2^{\mathfrak{c}}$ copies of the free group on $2^{\mathfrak{c}}$ generators.)

That is the point of departure of this paper. We investigate here what properties of a semigroup S guarantee that S^*S^* is nowhere dense in S^* or, failing that, that $S^* \setminus S^*S^*$ is dense in S^*. For example, we show that if S is embeddable in a group, then regardless of its size, S^*S^* is nowhere dense in S^*. But there is a cancellative semigroup with $|S| = \mathfrak{c}$ for which S^*S^* is not nowhere dense in S^*.

The points that we produce in $S^* \setminus S^*S^*$ tend to have the property that they are right cancelable in βS. Left and right cancellation properties have been extensively studied. See, for example, [1], [2], [4], and [6].

Definition 1.1. Let S be a semigroup and let $A \subseteq S$.

(a) A is a *left solution set* if and only if there exist a and b in S such that $A = \{x \in S : ax = b\}$.

(b) A is a *right solution set* if and only if there exist a and b in S such that $A = \{x \in S : xa = b\}$.

Notice that S is left cancellative if and only if each left solution set has at most one member.

We let $\omega = \mathbb{N} \cup \{0\}$. Then ω is the first infinite cardinal (and also the first infinite ordinal).

Definition 1.2. Let S be a semigroup with $|S| = \kappa \geq \omega$.

(a) S is *weakly left cancellative* if and only if every left solution set in S is finite.

(b) S is *weakly right cancellative* if and only if every right solution set in S is finite.

(c) S is *very weakly left cancellative* if and only if whenever $\mathcal{B}$ is a set of left solution sets in S with $|\mathcal{B}| < \kappa$, one has $|\bigcup \mathcal{B}| < \kappa$.

(d) S is *very weakly right cancellative* if and only if whenever $\mathcal{B}$ is a set of right solution sets in S with $|\mathcal{B}| < \kappa$, one has $|\bigcup \mathcal{B}| < \kappa$.

(e) Given $p \in \beta S$, the *norm* of p, $||p|| = \min\{|A| : A \in p\}$.

(f) $U_\kappa = U_\kappa(S) = \{p \in \beta S : ||p|| = \kappa\}$.

Notice that if κ is regular, then S is very weakly left cancellative if and only if every left solution set has cardinality less than κ. In particular, if $|S| = \omega$, then weakly left cancellative and very weakly left cancellative are equivalent notions.

In Section 2 of this paper we investigate conditions on S guaranteeing that S^*S^* is nowhere dense in S^* and conditions guaranteeing that S^*U_κ is nowhere dense in U_κ.

In Section 3 we deal with conditions guaranteeing that $S^* \setminus S^*S^*$ is dense in S^*, a conclusion weaker than the assertion that S^*S^* is nowhere dense in S^*.

Some of the results about S^*S^* include conclusions about right cancelable elements of U_κ. In Section 4 we investigate the problem of producing right cancelable elements p of S^* with $||p|| < \kappa = |S|$.

2. Nowhere dense products

We will be concerned first with determining whether S^*S^* is nowhere dense in S^*. For that, we would, of course, like to have $S^*S^* \subseteq S^*$, that is that S^* is a subsemigroup of βS. All of our results about S^*S^* involve semigroups that satisfy cancellation conditions, which are usually weaker than cancellativity. By [5, Theorem 4.31 and Corollary 4.33], if S is weakly left cancellative, then S^* is a left ideal of βS, while if S is right cancellative, then S^* is a right ideal of βS.

We remark that it was previously known [5, Theorem 6.35] that if $|S| = \omega$ and S is right cancellative and weakly left cancellative, then S^*S^* is nowhere dense in S^*. And it has been noted before that the assumption cannot be weakened to weakly right cancellative and weakly left cancellative. Indeed, given $n, m \in \mathbb{N}$, let $n \vee m = \max\{n, m\}$. Then $(\mathbb{N}, \vee)$ is both weakly right cancellative and weakly left cancellative, while every element of $(\beta\mathbb{N}, \vee)$ is an idempotent, so $\mathbb{N}^* \vee \mathbb{N}^* = \mathbb{N}^*$.

We aim to investigate this question for semigroups of arbitrary cardinality. We observe that the algebraic properties of uncountable semigroups are often far more challenging to handle.

For $A \subseteq S$, we write $A^* = \overline{A} \cap S^*$. The following is our only positive result which does not have a restriction on $|S|$.

Theorem 2.1. *Let S be an infinite semigroup which is embeddable in a group G. Then $S^* \cap (G^*G^*)$ is nowhere dense in S^*. In particular S^*S^* is nowhere dense in S^*.*

Proof. Suppose A is an infinite subset of S and $A^* \subseteq c\ell(G^*G^*)$. Pick $t_1 \in A$. Inductively, for $s > 1$, having chosen $\langle t_n \rangle_{n=1}^{s-1}$, choose

$$t_s \in A \setminus \{t_r t_n^{-1} t_m : m, n, r < s\}\,.$$

Note that $t_s \notin \big\{t_n : n \in \{1, 2, \ldots, s-1\}\big\}$. Let $V = \{t_s : s \in \mathbb{N}\}$. Then V is infinite so $V^* \cap G^*G^* \neq \emptyset$. So pick $p, q \in G^*$ such that $V \in pq$. Then $\{x \in G : x^{-1}V \in q\} \in p$ so pick distinct x_1 and x_2 in G such that $x_1^{-1}V \in q$ and $x_2^{-1}V \in q$. Pick $y \in x_1^{-1}V \cap x_2^{-1}V$ and let $n, r \in \mathbb{N}$ such that $x_1 y = t_n$ and $x_2 y = t_r$.

Now $\big\{w \in S : x_1 w \in \{t_1, t_2, \ldots, t_{\max\{n,r\}}\big\}$ is finite and $x_1^{-1}V \cap x_2^{-1}V$ is infinite, so pick $w \in x_1^{-1}V \cap x_2^{-1}V$ and $m, s > \max\{n, r\}$ in $\mathbb{N}$ such that $x_1 w = t_m$ and $x_2 w = t_s$. Assume without loss of generality that $s > m$. Then $t_s t_m^{-1} = x_2 x_1^{-1} = t_r t_n^{-1}$ so $t_s = t_r t_n^{-1} t_m$, a contradiction. □

We show now that cancellation is not necessary for a semigroup S with $|S| = \mathfrak{c}$ to have S^*S^* nowhere dense in S^*.

Given a set A and a cardinal κ, we let $[A]^\kappa = \{B \subseteq A : |B| = \kappa\}$.

Theorem 2.2. *Let $S = \{f : (\exists D \subseteq \mathbb{N})(|\mathbb{N} \setminus D| < \omega$ and $f : D \xrightarrow[\text{onto}]{\text{1-1}} \mathbb{N})\}$ with the operation of composition. Then S is a semigroup, $|S| = \mathfrak{c}$, S is not cancellative (though S is right cancellative and weakly left cancellative), and S^*S^* is nowhere dense in S^*.*

Proof. It is routine to verify that S is closed under composition, $|S| = \mathfrak{c}$, S is right cancellative, S is not left cancellative, and S is weakly left cancellative.

We establish first the following two assertions.

(*) *Let $t_n, t_r, t_m \in S$. Then*

$$|\mathbb{N} \setminus \{a \in \mathrm{dom}(t_m) : t_n^{-1}\big(t_m(a)\big) \in \mathrm{dom}(t_r)\}| \leq |\mathbb{N} \setminus \mathrm{dom}(t_m)| + |\mathbb{N} \setminus \mathrm{dom}(t_r)|\,.$$

In particular, $\{a \in \mathrm{dom}(t_m) : t_n^{-1}\big(t_m(a)\big) \in \mathrm{dom}(t_r)\}$ is cofinite.

(**) *Let A be a cofinite subset of $\mathbb{N}$ and let $v : A \xrightarrow{\text{1-1}} \mathbb{N}$. Then $C = \{t \in S : A \subseteq \mathrm{dom}(t)$ and $t_{|A} = v\}$ is finite.*

To verify (*), note that $\mathbb{N} \setminus \{a \in \mathrm{dom}(t_m) : t_n^{-1}\big(t_m(a)\big) \in \mathrm{dom}(t_r)\} = \big(\mathbb{N} \setminus \mathrm{dom}(t_m)\big) \cup \{a \in \mathrm{dom}(t_m) : t_n^{-1}\big(t_m(a)\big) \notin \mathrm{dom}(t_r)\}$ and $t_n^{-1} \circ t_m : \{a \in \mathrm{dom}(t_m) : t_n^{-1}\big(t_m(a)\big) \notin \mathrm{dom}(t_r)\} \xrightarrow{\text{1-1}} \mathbb{N} \setminus \mathrm{dom}(t_r)$.

To verify (**), note that if $|\mathbb{N} \setminus v[A]| > |\mathbb{N} \setminus A|$, then $C = \emptyset$, so we assume that $r = |\mathbb{N} \setminus v[A]| \leq |\mathbb{N} \setminus A| = m$. Then

$$C = \bigcup_{D \in [\mathbb{N} \setminus A]^r} \{t \in S : \mathrm{dom}(t) = A \cup D \text{ and } t_{|D} : D \xrightarrow[\text{onto}]{\text{1-1}} \mathbb{N} \setminus v[A]\}\,,$$

and for each $D \in [\mathbb{N} \setminus A]^r$, $\{t \in S : \mathrm{dom}(t) = A \cup D$ and $t_{|D} : D \xrightarrow[\text{onto}]{\text{1-1}} \mathbb{N} \setminus v[A]\}$ is finite.

Suppose we have infinite $V \subseteq S$ such that $V^* \subseteq c\ell(S^*S^*)$. We claim that we can choose a sequence $\langle t_n \rangle_{n=1}^{\infty}$ in V with the property that if $s > 1$ and $n, r, m \in \{1, 2, \ldots, s-1\}$, then there do not exist $x_1, x_2, y, w \in S$ such that $x_1 y = t_n$, $x_2 y = t_r$, $x_1 w = t_m$ and $x_2 w = t_s$.

So let $t_1 \in V$ and assume that $s > 1$ and $\langle t_n \rangle_{n=1}^{s-1}$ have been chosen. For $n, r, m \in \{1, 2, \ldots, s-1\}$, let

$$\begin{aligned} D_{n,r,m} = \{t \in S : \ & (\exists x_1, x_2, y, w \in S)(x_1 y = t_n\,,\ x_2 y = t_r\,, \\ & x_1 w = t_m, \text{ and } x_2 w = t)\}\,. \end{aligned}$$

We shall show that each $D_{n,r,m}$ is finite. So let $n, r, m \in \{1, 2, \ldots, s-1\}$. We will show that if $t \in D_{n,r,m}$, $a \in \mathrm{dom}(t_m)$, and $t_n^{-1}\big(t_m(a)\big) \in \mathrm{dom}(t_r)$, then $t(a) = t_r\big(t_n^{-1}\big(t_m(a)\big)\big)$ so that, by (*) and (**) (where $A = \{a \in \mathrm{dom}(t_m) : t_n^{-1}\big(t_m(a)\big) \in \mathrm{dom}(t_r)\}$ and $v = t_r \circ t_n^{-1} \circ t_m$), $D_{n,r,m}$ is finite. So let $t \in D_{n,r,m}$ and pick $x_1, x_2, y, w \in S$ such that $x_1 y = t_n$, $x_2 y = t_r$, $x_1 w = t_m$ and $x_2 w = t$ and let $a \in \mathrm{dom}(t_m)$ such that $t_n^{-1}\big(t_m(a)\big) \in \mathrm{dom}(t_r)$.

Since $a \in \mathrm{dom}(t_m)$, $x_1\big(w(a)\big) = t_m(a)$ and thus $w(a) = x_1^{-1}\big(t_m(a)\big)$. Since $t_n^{-1}\big(t_m(a)\big) \in \mathrm{dom}(t_n)$, $x_1\Big(y\big(t_n^{-1}\big(t_m(a)\big)\big)\Big) = t_n\big(t_n^{-1}\big(t_m(a)\big)\big) = t_m(a)$ so

$$y\big(t_n^{-1}\big(t_m(a)\big)\big) = x_1^{-1}\big(t_m(a)\big) = w(a)\,.$$

Since $t_n^{-1}\big(t_m(a)\big) \in \mathrm{dom}(t_r)$, we have

$$x_2\big(w(a)\big) = x_2\Big(y\big(t_n^{-1}\big(t_m(a)\big)\big)\Big) = t_r\big(t_n^{-1}\big(t_m(a)\big)\big)$$

so $t(a) = x_2\big(w(a)\big) = t_r\big(t_n^{-1}\big(t_m(a)\big)\big)$ as claimed.

Thus we may choose

$$t_s \in V \setminus \Big(\big\{t_n : n \in \{1, 2, \ldots, s\}\big\} \cup \bigcup \{D_{n,r,m} : n, r, m \in \{1, 2, \ldots, s-1\}\}\Big)\,.$$

Let $A = \{t_n : n \in \mathbb{N}\}$. Then $A^* \cap S^*S^* \neq \emptyset$ so pick $p, q \in S^*$ such that $A \in pq$. Pick distinct x_1 and x_2 in S such that $x_1^{-1}A \cap x_2^{-1}A \in q$. Pick $y \in x_1^{-1}A \cap x_2^{-1}A$ and pick $n, r \in \mathbb{N}$ such that $x_1 y = t_n$ and $x_2 y = t_r$. By right cancellation, assume without loss of generality that $n < r$.

Since S is weakly left cancellative, $\{w \in S : x_1 w = t_n \text{ or } x_2 w = t_r\}$ is finite, so pick $w \in x_1^{-1}A \cap x_2^{-1}A$ such that $x_1 w \neq t_n$ and $x_2 w \neq t_r$. Pick $m, s \in \mathbb{N}$ such that $x_1 w = t_m$ and $x_2 w = t_s$. We have that $\{z \in S : x_1 z = t_m \text{ or } x_2 z = t_s\}$ is finite so pick $z \in x_1^{-1}A \cap x_2^{-1}A$ such that $x_1 z \neq t_n$, $x_2 z \neq t_r$, $x_1 z \neq t_m$, and $x_2 z \neq t_s$. Pick $k, l \in \mathbb{N}$ such that $x_1 z = t_k$ and $x_2 z = t_l$. By right cancellation we have that $m \neq s$ and $k \neq l$.

Case 1. Either $m < s$ or $k < l$. Assume without loss of generality that $m < s$ and that $r < s$. But then $t_s \in D_{n,r,m}$, a contradiction.

Case 2. $m > s$ and $k > l$. Assume without loss of generality that $m > k$. Since $x_2 z = t_l$, $x_1 z = t_k$, $x_2 w = t_s$, and $x_1 w = t_m$ we have that $t_m \in D_{l,k,s}$, a contradiction. □

Lemma 2.3. *Let S be an infinite semigroup and assume that S^* is a subsemigroup of βS. The following statements are equivalent.*

(a) *$S^* S^*$ is not nowhere dense in S^*.*

(b) $(\exists V \in [S]^\omega)(\forall A \in [V]^\omega)(\exists \langle x_n \rangle_{n=1}^\infty)(\exists \langle y_n \rangle_{n=1}^\infty)\big(\langle x_n \rangle_{n=1}^\infty$ and $\langle y_n \rangle_{n=1}^\infty$ are injective sequences in S and $(\forall k \in \mathbb{N})(\forall n \in \mathbb{N})(n > k \Rightarrow x_k y_n \in A)\big)$.

Proof. To see that (a) implies (b), assume that we have $V \in [S]^\omega$ such that $V^* \subseteq c\ell(S^* S^*)$. Let $A \in [V]^\omega$ be given. Then $\overline{A} \cap (S^* S^*) \neq \emptyset$ so pick p and q in S^* such that $A \in pq$. Then $\{x \in S : x^{-1}A \in q\} \in p$ and is therefore infinite. Let $\langle x_n \rangle_{n=1}^\infty$ be an injective sequence in $\{x \in S : x^{-1}A \in q\}$. Pick $y_1 \in S$. Inductively for $n \in \mathbb{N}$, having chosen $\langle y_t \rangle_{t=1}^n$, pick $y_{n+1} \in (\bigcap_{k=1}^n x_k^{-1}A) \setminus \{y_1, y_2, \ldots y_n\}$.

For the other implication assume we have V as guaranteed by (b). We claim that $V^* \subseteq c\ell(S^* S^*)$. So let $r \in V^*$ be given. To see that $r \in c\ell(S^* S^*)$, let $A \in r$ be given with $A \subseteq V$. Pick injective sequences $\langle x_n \rangle_{n=1}^\infty$ and $\langle y_n \rangle_{n=1}^\infty$ as guaranteed for A. Pick $p \in S^*$ such that $\{x_n : n \in \mathbb{N}\} \in p$ and pick $q \in S^*$ such that $\big\{\{y_n : n > k\} : k \in \mathbb{N}\big\} \subseteq q$. Then $\{x_k : k \in \mathbb{N}\} \subseteq \{s \in S : s^{-1}A \in q\}$ so $A \in pq$. □

We see now that, unlike the countable situation, cancellation is not sufficient to guarantee that $S^* S^*$ is nowhere dense in S^*.

Theorem 2.4. *There is a cancellative semigroup S with $|S| = \mathfrak{c}$ such that $S^* S^*$ is not nowhere dense in S^*.*

Proof. Let $L = \{z_n : n \in \mathbb{N}\} \cup \{x_{\sigma,k} : \sigma < \mathfrak{c} \text{ and } k \in \mathbb{N}\} \cup \{y_{\sigma,k} : \sigma < \mathfrak{c} \text{ and } k \in \mathbb{N}\}$ where the z_n's, $x_{\sigma,k}$'s and $y_{\sigma,k}$'s are all distinct.

Let $V = \{z_n : n \in \mathbb{N}\}$. Enumerate $[V]^\omega$ as $\langle A_\sigma \rangle_{\sigma < \mathfrak{c}}$, and for $\sigma < \mathfrak{c}$, enumerate A_σ as $\big\langle \langle w_{\sigma,k,n} \rangle_{k=1}^\infty \big\rangle_{n=k+1}^\infty$.

Let S be the set of all words $a_1 \cdots a_t$ (with each a_i in L) that do not have any $i \in \{1, 2, \ldots, t-1\}$, any $\sigma < \mathfrak{c}$, and any $k < n$ in $\mathbb{N}$ such that $a_i = x_{\sigma,k}$ and $a_{i+1} = y_{\sigma,n}$. For words $u = a_1 \cdots a_t$ and $v = b_1 \cdots b_s$ in S, define $u \cdot v$ as ordinary concatenation unless there exist $\sigma < \mathfrak{c}$, and $k < n$ in $\mathbb{N}$ such that $a_t = x_{\sigma,k}$ and $b_1 = y_{\sigma,n}$, in which case $u \cdot v = a_1 \cdots a_{t-1} w_{\sigma,k,n} b_2 \cdots b_s$, where, for example, $a_1 \cdots a_{t-1}$ is the empty word if $t = 1$.

By Lemma 2.3 it suffices to verify that with this operation, S is a cancellative semigroup since $\langle x_{\sigma,n}\rangle_{n=1}^{\infty}$ and $\langle y_{\sigma,n}\rangle_{n=1}^{\infty}$ are as required by this lemma for $A_\sigma \in [V]^\omega$.

To verify associativity, let $u = a_1 \cdots a_t$, $v = b_1 \cdots b_s$, and $w = c_1 \cdots c_r$ be words in S. If it is not the case that $b_1 = y_{\sigma,n}$ or $b_s = x_{\tau,m}$ for some $\sigma < \mathfrak{c}$ and some $n, m \in \mathbb{N}$ then both $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$ are ordinary concatenation, hence equal.

<u>Case 1</u>. $b_1 = y_{\sigma,n}$ and $b_s = x_{\tau,m}$ for some $\sigma < \mathfrak{c}$ and some $n, m \in \mathbb{N}$. (In this case necessarily $s \geq 2$.)

<u>Case 1a</u>. $a_t \neq x_{\sigma,k}$ for any $k \in \{1, 2, \ldots, n-1\}$ and $c_1 \neq y_{\tau,l}$ for any $l > m$. In this case again both products are ordinary concatenation.

<u>Case 1b</u>. $a_t = x_{\sigma,k}$ for some $k \in \{1, 2, \ldots, n-1\}$ and $c_1 \neq y_{\tau,l}$ for any $l > m$. Then

$$(u \cdot v) \cdot w = a_1 \cdots a_{t-1} w_{\sigma,k,n} b_2 \cdots b_s c_1 \cdots c_r = u \cdot (v \cdot w)\,.$$

<u>Case 1c</u>. $a_t \neq x_{\sigma,k}$ for any $k \in \{1, 2, \ldots, n-1\}$ and $c_1 = y_{\tau,l}$ for some $l > m$. Then

$$(u \cdot v) \cdot w = a_1 \cdots a_t b_1 \cdots b_{s-1} w_{\tau,m,l} c_2 \cdots c_r = u \cdot (v \cdot w)\,.$$

<u>Case 1d</u>. $a_t = x_{\sigma,k}$ for some $k \in \{1, 2, \ldots, n-1\}$ and $c_1 = y_{\tau,l}$ for some $l > m$. Then

$$(u \cdot v) \cdot w = a_1 \cdots a_{t-1} w_{\sigma,k,n} b_2 \cdots b_{s-1} w_{\tau,m,l} c_2 \cdots c_r = u \cdot (v \cdot w)$$

unless $s = 2$ in which case

$$(u \cdot v) \cdot w = a_1 \cdots a_{t-1} w_{\sigma,k,n} w_{\tau,m,l} c_2 \cdots c_r = u \cdot (v \cdot w)\,.$$

Case 2 (namely $b_1 = y_{\sigma,n}$ for some $\sigma < \mathfrak{c}$ and some $n \in \mathbb{N}$ and $b_s \neq x_{\tau,m}$ for any $\tau < \mathfrak{c}$ and any $m \in \mathbb{N}$) and case 3 (namely $b_1 \neq y_{\sigma,n}$ for any $\sigma < \mathfrak{c}$ and any $n \in \mathbb{N}$ and $b_s = x_{\tau,m}$ for some $\tau < \mathfrak{c}$ and some $m \in \mathbb{N}$) are handled in a very similar fashion.

We will verify that S is left cancellative. The verification of right cancellativity is very similar. Let $u = a_1 \cdots a_t$, $v = b_1 \cdots b_s$, and $w = c_1 \cdots c_r$ be words in S and assume that $u \cdot v = u \cdot w$. If it is not the case that $a_t = x_{\sigma,k}$ for some $\sigma < \mathfrak{c}$ and some $k \in \mathbb{N}$, then $u \cdot v$ and $u \cdot w$ are ordinary concatenation so $v = w$.

So assume that $a_t = x_{\sigma,k}$ for some $\sigma < \mathfrak{c}$ and some $k \in \mathbb{N}$. If it is not the case that $b_1 = y_{\sigma,n}$ for some $n > k$ or $c_1 = y_{\sigma,m}$ for some $m > k$, then again $u \cdot v$ and $u \cdot w$ are ordinary concatenation. So assume that either $b_1 = y_{\sigma,n}$ for some $n > k$ or $c_1 = y_{\sigma,m}$ for some $m > k$.

If $b_1 = y_{\sigma,n}$ for some $n > k$, then $u \cdot v = a_1 \cdots a_{t-1} w_{\sigma,k,n} b_2 \cdots b_s$.

If $c_1 = y_{\sigma,m}$ for some $m > k$, then $u \cdot w = a_1 \cdots a_{t-1} w_{\sigma,k,m} c_2 \cdots c_r$.

Since neither $w_{\sigma,k,n} = a_t$ nor $w_{\sigma,k,m} = a_t$ for any n or m in $\mathbb{N}$, we must have that $b_1 = y_{\sigma,n}$ for some $n > k$ and $c_1 = y_{\sigma,m}$ for some $m > k$. Since $\big\langle \langle w_{\sigma,k,n}\rangle_{k=1}^{\infty} \big\rangle_{n=k+1}^{\infty}$ enumerates A_σ, we must have that $m = n$ and so $v = w$. $\square$

It is a consequence of Theorem 3.1 below that if S is cancellative and $|S| = \omega_1$, then $S^* \setminus (S^*S^*)$ is dense in S^*. So the continuum hypothesis implies that the semigroup of Theorem 2.4 has $S^* \setminus (S^*S^*)$ dense in S^*.

Question 2.5. *Is it consistent that for the semigroup S of Theorem 2.4, $S^* \setminus (S^*S^*)$ is not dense in S^*.*

We saw in Theorem 2.1 that if S is embeddable in a group, then G^*G^* is nowhere dense in S^*. We shall now see that considerably more can be said if $|S|^\omega < 2^{\mathfrak{c}}$.

Theorem 2.6. *Let S be a semigroup which is embeddable in a group G and assume that $|S| = \kappa \geq \omega$ and $\kappa^\omega < 2^{\mathfrak{c}}$. Let $\mathcal{V} = \{A \in [S]^\omega : \overline{A} \cap (G^*G^*) = \emptyset\}$ and let $T = \bigcup\{A^* : A \in \mathcal{V}\}$. Then*

(1) T is open and dense in S^*, $T \cap (G^*G^*) = \emptyset$, and every element of T is right cancelable in $\{p \in S^* : ||p|| \leq \omega\}$.

(2) Let H be the subgroup of G generated by S. Define an equivalence relation $\approx$ on S^* by $p \approx q$ if and only if there exist $a, b \in H$ such that $apb = q$. If $k, m \in \mathbb{N}$, $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_k \in T$, and $p_1 \cdots p_m \approx q_1 \cdots q_k$, then $k = m$ and for each $t \in \{1, 2, \ldots, m\}$, $p_t \approx q_t$.

(3) There exists $X \subseteq T$ such that X is dense in S^*, $|X| = 2^{\mathfrak{c}}$, and X generates a free subsemigroup of S^*.

Proof. Trivially, T is open in S^* and $T \cap (G^*G^*) = \emptyset$. Using Theorem 2.1, one easily sees that T is dense in S^*.

To see that T is right cancelable in $\{p \in S^* : ||p|| \leq \omega\}$, let $p \in T$ and suppose we have distinct q and r in βS such that $||q|| \leq \omega$, $||r|| \leq \omega$, and $qp = rp$. Pick $A \in q$ and $B \in r$ such that $|A| = |B| = \omega$ and $A \cap B = \emptyset$. Then $qp \in \overline{A}p \cap \overline{B}p = c\ell(Ap) \cap c\ell(Bp)$. Therefore by [5, Theorem 3.40] either $Ap \cap \overline{B}p \neq \emptyset$ or $\overline{A}p \cap Bp \neq \emptyset$. We assume without loss of generality that we have $a \in A$ and $b \in \overline{B}$ such that $ap = bp$. By [5, Corollary 8.2], $b \notin B$, so $b \in B^*$. Then $p = a^{-1}bp$ and $a^{-1}b \in G^*$ by [5, Corollary 4.33] so $p \in G^*G^*$, a contradiction.

To verify (2), assume that $k, m \in \mathbb{N}$, $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_k \in T$, and $p_1 \cdots p_m \approx q_1 \cdots q_k$. Suppose that the conclusion fails and assume that $k + m$ is minimal among all counterexamples. Note that $m > 1$ and $k > 1$. (If $m = k = 1$, we don't have a counterexample. If, say $m = 1$ and $k > 1$, then for some $a, b \in H$, $p_1 = (a^{-1}q_1 \cdots q_{k-1})q_k b^{-1} \in G^*G^*$.)

Pick $a, b \in H$ such that $ap_1 \cdots p_m b = q_1 \cdots q_k$. Note that $\{p \in S^* : ||p|| = \omega\}$ is a subsemigroup of S^*. (If $A \in p$ and $B \in q$, then $AB \in pq$.) So pick $A, B \in [S]^\omega$ such that $A \in p_1 \cdots p_{m-1}$ and $B \in q_1 \cdots q_{k-1}$. Then $a\overline{A}p_m b = c\ell_{\beta G}(aAp_m b)$ and $\overline{B}q_k = c\ell_{\beta G}(Bq_k)$ so by [5, Theorem 3.40], we may either

(a) pick $c \in A$ and $d \in \overline{B}$ such that $acp_m b = dq_k$ or

(b) pick $c \in A^*$ and $d \in B$ such that $acp_m b = dq_k$.

If (b) held, we would have $q_k = d^{-1}acp_m b \in G^*G^*$, a contradiction. So (a) holds. If $d \in B^*$, then $p_m = c^{-1}a^{-1}dq_k b^{-1} \in G^*G^*$, a contradiction. So $d \in B$. Since $d^{-1}acp_m b = q_k$ we have that $p_m \approx q_k$. Also $p_m b = c^{-1}a^{-1}dq_k$ so $ap_1 \cdots p_{m-1}c^{-1}a^{-1}dq_k = q_1 \cdots q_k$. By conclusion (1), we may cancel q_k so we conclude that $p_1 \cdots p_{m-1} \approx q_1 \cdots q_{k-1}$. By the minimality of $m + k$, we conclude that $m - 1 = k - 1$ and for each $t \in \{1, 2, \ldots, m-1\}$, $p_t \approx q_t$.

To verify conclusion (3) let $\lambda = |\mathcal{V}|$. Then $\mathcal{V} \subseteq [S]^\omega$ so $\lambda \leq \kappa^\omega < 2^\mathfrak{c}$. Enumerate $\mathcal{V}$ as $\langle A_\sigma \rangle_{\sigma<\lambda}$.

Since $|H| = |S|$, each $\approx$-equivalence class has at most $|S|$ members (in fact exactly $|S|$ members). And given $\sigma < \lambda$, $|A_\sigma^*| = 2^\mathfrak{c}$. So each A_σ^* hits $2^\mathfrak{c}$ equivalence classes. Inductively, choose $x_\sigma \in A_\sigma^*$ such that if $\sigma \neq \tau$, then $x_\sigma \not\approx x_\tau$.

For $x \in S^*$, let $[x]$ denote the $\approx$-equivalence class of x. Let $\mathcal{R} = \{[x] : x \in A_0^*\}$ and let $\mathcal{S} = \mathcal{R} \setminus \{[x_\sigma] : \sigma < \lambda\}$. Pick $Y \subseteq A_0^*$ such that $\mathcal{S} = \{[y] : y \in Y\}$ and if y and z are distinct members of Y, then $y \not\approx z$. Let $X = Y \cup \{x_\sigma : \sigma < \lambda\}$. Then $X \subseteq T$ so by conclusion (2), if $k, m \in \mathbb{N}$, $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_k \in X$, and $p_1 \cdots p_m = q_1 \cdots q_k$, then $k = m$ and for each $t \in \{1, 2, \ldots, m\}$, $p_t \approx q_t$. But since X includes at most one member from each equivalence class, one has that for each $t \in \{1, 2, \ldots, m\}$, $p_t = q_t$. That is the semigroup generated by X is free. Since X includes a representative of $[x]$ for each $x \in A_0^*$, $|X| = 2^\mathfrak{c}$.

Finally, to see that X is dense in S^*, let V be an infinite subset of S and pick $D \in [V]^\omega$. Pick $q \in D^* \setminus (G^*G^*)$ and pick $B \in q$ such that $\overline{B} \cap G^*G^* = \emptyset$. Let $A = B \cap D$. Then $A \in \mathcal{V}$ so for some $\sigma < \lambda$, $A = A_\sigma$ and thus $x_\sigma \in \overline{V} \cap X$. $\square$

We now turn our attention to determining whether S^*U_κ is nowhere dense in U^κ, and for this we would like to know that $S^*U_\kappa \subseteq U_\kappa$. All of our results about S^*U_κ involve semigroups that are very weakly left cancellative, and by [5, Lemma 6.34.3], if S is very weakly left cancellative, then U_κ is a left ideal of βS.

The hypotheses of the following theorem could be superficially weakened by replacing the assumption that S is right cancellative by the assumption that whenever a and b are distinct members of S, one has $|\{x \in S : ax = bx\}| < \kappa$. But that is not in fact a weakening since if S is very weakly left cancellative and $|\{x \in S : ax = bx\}| < \kappa$ whenever a and b are distinct members of S, then S is right cancellative. To see this, suppose we have a, b, and c in S with $a \neq b$ and $ac = bc$. Then $cS \subseteq \{x \in S : ax = bx\}$ so $|cS| < \kappa$. But then, $S = \bigcup_{d \in cS} \{x \in S : cx = d\}$, so S is the union of fewer than κ left solution sets, a contradiction.

Theorem 2.7. *Let S be a right cancellative and very weakly left cancellative semigroup with $|S| = \kappa \geq \omega$. Then S^*U_κ is nowhere dense in U_κ.*

Proof. Enumerate S as $\langle s_\alpha \rangle_{\alpha<\kappa}$. Suppose we have $V \in [S]^\kappa$ such that $\overline{V} \cap U_\kappa \subseteq c\ell(S^*U_\kappa)$. Pick $v_0 \in V$. Let $0 < \delta < \kappa$ and assume we have chosen $\langle v_\sigma \rangle_{\sigma<\delta}$ so that

(1) if $\alpha < \sigma < \delta$, then $v_\alpha \neq v_\sigma$ and

(2) if $\alpha < \sigma < \tau < \delta$, $\mu < \tau$, and $x \in S$, then either $s_\alpha x \neq v_\sigma$ or $s_\mu x \neq v_\tau$.

For $\alpha < \sigma < \delta$, let $C_{\alpha,\sigma} = \{x \in S : s_\alpha x = v_\sigma\}$ and let $B = \bigcup_{\alpha<\sigma<\delta} C_{\alpha,\sigma}$. Since each $C_{\alpha,\sigma}$ is a left solution set we have $|B| < \kappa$ so $|\{s_\mu x : \mu < \delta \text{ and } x \in B\}| < \kappa$. Pick $v_\delta \in V \setminus (\{v_\sigma : \sigma < \delta\} \cup \{s_\mu x : \mu < \delta \text{ and } x \in B\})$. Hypothesis (1) is trivially satisfied. Suppose we have $\alpha < \sigma < \delta$, $\mu < \delta$, and $x \in S$ such that $s_\alpha x = v_\sigma$ and $s_\mu x = v_\delta$. Then $x \in C_{\alpha,\sigma} \subseteq B$ so $v_\delta \in \{s_\mu x : \mu < \delta \text{ and } x \in B\}$, a contradiction.

Let $A = \{v_\sigma : \sigma < \kappa\}$ and for $\alpha < \kappa$, let $A_\alpha = \{v_\sigma : \alpha < \sigma < \kappa\}$. We claim that if $s \in S$, $q \in U_\kappa$, and $s^{-1}A \in q$, then for each $\alpha < \kappa$, $s^{-1}A_\alpha \in q$. To see this, let $\alpha < \kappa$ and suppose $s^{-1}A_\alpha \notin q$. Then $s^{-1}A \setminus s^{-1}A_\alpha \in q$ and $s^{-1}A \setminus s^{-1}A_\alpha = \bigcup_{\sigma \leq \alpha}\{x \in S : sx = v_\sigma\}$. Thus $s^{-1}A \setminus s^{-1}A_\alpha$ is a union of fewer than κ left solution sets, so $|s^{-1}A \setminus s^{-1}A_\alpha| < \kappa$, a contradiction.

Now $\overline{A} \cap U_\kappa \cap S^*U_\kappa \neq \emptyset$ so pick $p \in S^*$ and $q \in U_\kappa$ such that $A \in pq$. Then $\{x \in S : x^{-1}A \in q\} \in p$ so $\{x \in S : x^{-1}A \in q\}$ is infinite. Pick distinct $\alpha < \kappa$ and $\mu < \kappa$ such that $s_\alpha^{-1}A \in q$ and $s_\mu^{-1}A \in q$. Pick $x \in s_\alpha^{-1}A_\alpha \cap s_\mu^{-1}A_\mu$. Pick $\sigma < \kappa$ and $\delta < \kappa$ such that $\alpha < \sigma$, $\mu < \delta$, $s_\alpha x = v_\sigma$, and $s_\mu x = v_\delta$. Since $s_\alpha \neq s_\mu$, we have $\sigma \neq \delta$, so we assume without loss of generality that $\sigma < \delta$. This contradicts the choice of v_δ. □

In case S is countable, Theorem 2.7 is just [5, Theorem 6.35], since $U_\omega = S^*$ and for countable S, weakly cancellative and very weakly cancellative are the same.

We shall see in Theorem 3.2 that left cancellative and weakly right cancellative are not sufficient to force a countable semigroup to have S^*S^* nowhere dense in S^*.

Lemma 2.8. *Let S be a weakly left cancellative and very weakly right cancellative semigroup with $|S| = \kappa \geq \omega$. Enumerate S as $\langle s_\alpha\rangle_{\alpha<\kappa}$ and let $V = \{q \in S^* : (\exists \delta < \kappa)(\{s_\alpha : \alpha < \delta\} \in q)\}$. Then $(\beta S)V \cap U_\kappa$ is nowhere dense in U_κ.*

Proof. Suppose we have $C \in [S]^\kappa$ such that $\overline{C} \cap U_\kappa \subseteq c\ell\big((\beta S)V\big)$. Pick $t_0 \in C$. Let $0 < \alpha < \kappa$ and assume that we have chosen $\langle t_\delta\rangle_{\delta<\alpha}$ in C such that

(a) $\langle t_\delta\rangle_{\delta<\alpha}$ is injective and

(b) if $x \in S$, $\gamma < \sigma < \delta < \alpha$, and $\mu < \delta$, then either $xs_\gamma \neq t_\sigma$ or $xs_\mu \neq t_\delta$.

Let $D = \{x \in S : (\exists \gamma < \sigma < \alpha)(xs_\gamma = t_\sigma\}$. Then D is the union of fewer than κ right solution sets, so $|D| < \kappa$ and so $|\{xs_\mu : x \in D \text{ and } \mu < \alpha\}| < \kappa$. Pick $t_\alpha \in C \setminus (\{t_\delta : \delta < \alpha\} \cup \{xs_\mu : x \in D \text{ and } \mu < \alpha\})$. To verify hypothesis (b), suppose we have $x \in S$, $\gamma < \sigma < \alpha$, and $\mu < \alpha$ such that $xs_\gamma = t_\sigma$ and $xs_\mu = t_\alpha$. Then $x \in D$ so $t_\alpha \neq xs_\mu$, a contradiction.

Let $B = \{t_\alpha : \alpha < \kappa\}$. Then $\overline{B} \cap U_\kappa \cap (\beta S)V \neq \emptyset$, so pick $r \in \beta S$ and $q \in V$ such that $rq \in \overline{B} \cap U_\kappa$. Pick $\delta < \kappa$ such that $\{s_\alpha : \alpha < \delta\} \in q$ and let $H = \{t_\alpha : \alpha > \delta\}$. Then $H \in rq$ so $\{x \in S : x^{-1}H \in q\} \in r$. Pick $x \in S$ such that $x^{-1}H \in q$ and let $W = x^{-1}H \cap \{s_\alpha : \alpha < \delta\}$. Then $W \in q$ so W is infinite. We claim that $|xW| = 1$. Suppose instead we have $\delta < \sigma < \alpha$ and $\gamma < \delta$ and $\mu < \delta$ such that $xs_\gamma = t_\sigma$ and $xs_\mu = t_\alpha$. This contradicts hypothesis (b). Thus we have some $\alpha > \delta$ such that $xW = \{t_\alpha\}$. But now, $W \subseteq \{s \in S : xs = t_\alpha\}$ which is a left solution set, and is therefore finite. This is a contradiction. $\square$

Theorem 2.9. *Let S be a right cancellative and weakly left cancellative semigroup with $|S| = \kappa \geq \omega$. Assume that κ is regular. Then $S^*S^* \cap U_\kappa$ is nowhere dense in U_κ.*

Proof. Enumerate S as $\langle s_\alpha \rangle_{\alpha<\kappa}$ and let $V = \{q \in S^* : (\exists \delta < \kappa)(\{s_\alpha : \alpha < \delta\} \in q)\}$. Since κ is regular, $V = S^* \setminus U_\kappa$ so by Lemma 2.8, $(\beta S)(S^* \setminus U_\kappa) \cap U_\kappa$ is nowhere dense in U_κ. By Theorem 2.7, S^*U_κ is nowhere dense in U_κ. $\square$

We introduce some notation which is used in the following theorem. Given a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. We say that a set $\mathcal{C}$ of sets has the *κ-uniform finite intersection property* provided $|\bigcap \mathcal{F}| \geq \kappa$ whenever $\mathcal{F} \in \mathcal{P}_f(\mathcal{C})$. Given a sequence $\langle t_\alpha \rangle_{\alpha<\kappa}$ in a semigroup S, we let $FP(\langle t_\alpha \rangle_{\alpha<\kappa}) = \{\prod_{\alpha\in F} t_\alpha : F \in \mathcal{P}_f(\kappa)\}$, where $\prod_{\alpha\in F} t_\alpha$ is computed in increasing order of indices. (Recall that each ordinal is the set of its predecessors, so $\mathcal{P}_f(\kappa) = \mathcal{P}_f(\{\alpha : \alpha < \kappa\})$.)

Theorem 2.10. *Let S be a right cancellative and very weakly left cancellative semigroup with $|S| = \kappa \geq \omega$. Let $\mathcal{C}$ be a nonempty set of at most κ subsets of S with the κ-uniform finite intersection property. There exists an injective sequence $\langle t_\alpha \rangle_{\alpha<\kappa}$ in S such that, letting $B = \{t_\alpha : \alpha < \kappa\}$, the following statements hold.*

(1) $\overline{B} \cap U_\kappa \cap S^*U_\kappa = \emptyset$.

(2) If p and q are distinct members of $\overline{B} \cap U_\kappa$, then $\beta Sp \cap \beta Sq = \emptyset$.

(3) If $p \in \overline{B} \cap U_\kappa$, then p is right cancelable in βS.

(4) $|B \cap U_\kappa| = 2^{2^\kappa}$ and $\overline{B} \cap U_\kappa$ generates a free semigroup in U_κ.

(5) Let $T = \bigcap_{\alpha<\kappa} \overline{FP(\langle t_\sigma \rangle_{\alpha<\sigma<\kappa})}$. Then T is a compact subsemigroup of βS with the property that every maximal group in $K(T)$ contains a copy of the free group on 2^{2^κ} generators. In particular, $\overline{FP(\langle t_\alpha \rangle_{\alpha<\kappa})}$ contains a copy of the free group on 2^{2^κ} generators.

(6) If κ is regular and S is weakly left cancellative, then $\overline{B} \cap U_\kappa \cap S^*S^* = \emptyset$.

Further, there is a set $P \subseteq \overline{B} \cap U_\kappa$ such that $|P| = 2^{2^\kappa}$ and for every $p \in P$, $\mathcal{C} \subseteq p$.

Proof. Enumerate S as $\langle s_\alpha \rangle_{\alpha<\kappa}$. We may assume that $\mathcal{C}$ is closed under finite intersections. Let $\lambda = |\mathcal{C}|$, enumerate $\mathcal{C}$ as $\langle C_\alpha \rangle_{\alpha<\lambda}$, and let $f : \kappa \xrightarrow[\text{onto}]{\text{1-1}} \kappa \times \lambda$. Fix $a \in S$ and let $A = \{s \in S : as = a\}$. Then A is a left solution set so $|A| < \kappa$. Pick $t_0 \in C_{\pi_2(f(0))} \setminus A$, where π_2 is the projection from $\kappa \times \lambda$ onto λ. Let $0 < \alpha < \kappa$ and assume we have chosen $\langle t_\delta \rangle_{\delta<\alpha}$ such that

(a) if $\delta < \alpha$, then $t_\delta \notin FP(\langle t_\gamma \rangle_{\gamma<\delta})$;

(b) if $\delta < \alpha$ and $\nu = \pi_2\big(f(\delta)\big)$, then $t_\delta \in C_\nu$;

(c) if $\gamma < \delta < \alpha$ and $\mu < \sigma < \delta$, then $s_\gamma t_\delta \neq s_\mu t_\sigma$;

(d) if $\gamma < \sigma < \delta < \alpha$, $\mu < \delta$, and $x \in S$, then either $s_\gamma x \neq t_\sigma$ or $s_\mu x \neq t_\delta$;

(e) if $\delta < \alpha$ and $u, v \in FP(\langle t_\gamma \rangle_{\gamma<\delta})$, then $u \neq vt_\delta$;

(f) if $\delta < \alpha$, $u, v \in FP(\langle t_\gamma \rangle_{\gamma<\delta})$, and $u \neq v$ then $ut_\delta \neq vt_\delta$;

(g) if $\delta < \alpha$ and $u \in FP(\langle t_\gamma \rangle_{\gamma<\delta})$, then $ut_\delta \neq t_\delta$;

(h) $FP(\langle t_\delta \rangle_{\delta<\alpha}) \cap A = \emptyset$; and

(i) if κ is regular, S is weakly left cancellative, $x \in S$, $\gamma < \sigma < \delta < \alpha$, and $\mu < \delta$, then either $xs_\gamma \neq t_\sigma$ or $xs_\mu \neq t_\delta$.

All hypotheses are satisfied at $\alpha = 1$, all but (b) and (h) vacuously.

For $\mu < \sigma < \alpha$ and $\gamma < \alpha$, let $A_{\gamma,\mu,\sigma} = \{x \in S : s_\gamma x = s_\mu t_\sigma\}$. Then each $A_{\gamma,\mu,\sigma}$ is a left solution set, so $|\bigcup_{\gamma<\alpha}\bigcup_{\sigma<\alpha}\bigcup_{\mu<\sigma} A_{\gamma,\mu,\sigma}| < \kappa$.

For $\gamma < \sigma < \alpha$, let $F_{\gamma,\sigma} = \{x \in S : s_\gamma x = t_\sigma\}$. Then $|\bigcup_{\sigma<\alpha}\bigcup_{\gamma<\sigma} F_{\gamma,\sigma}| < \kappa$ so $|\{s_\mu x : \mu < \alpha \text{ and } x \in \bigcup_{\sigma<\alpha}\bigcup_{\gamma<\sigma} F_{\gamma,\sigma}\}| < \kappa$.

Let $V = FP(\langle t_\delta \rangle_{\delta<\alpha})$. Then $|V| < \kappa$. By hypothesis (h), if $u \in V$, then $au \neq a$ so $|\{x \in S : (\exists u \in V)(aux = ax)\}| < \kappa$.

Also, $|\bigcup_{u\in V}\{x \in S : aux = a\}| < \kappa$, $|\bigcup_{u\in V}\bigcup_{v\in V}\{x \in S : vx = u\}| < \kappa$, and $|\bigcup_{u\in V}\bigcup_{v\in V\setminus\{u\}}\{x \in S : vx = ux\}| < \kappa$.

If κ is regular and S is weakly left cancellative, let $D = \{x \in S : (\exists \gamma < \sigma < \alpha)(xs_\gamma = t_\sigma)\}$. Then $|D| < \kappa$ and consequently, $|\{xs_\mu : x \in D \text{ and } \mu < \alpha\}| < \kappa$.

Let $\nu = \pi_2\big(f(\alpha)\big)$ and pick $t_\alpha \in C_\nu$ such that t_α is not a member of

$$\begin{aligned}
&\textstyle\bigcup_{\gamma<\alpha}\bigcup_{\sigma<\alpha}\bigcup_{\mu<\sigma} A_{\gamma,\mu,\sigma} \cup V \cup A \,\cup \\
&\{s_\mu x : \mu < \alpha \text{ and } x \in \textstyle\bigcup_{\sigma<\alpha}\bigcup_{\gamma<\sigma} F_{\gamma,\sigma}\} \,\cup \\
&\{x \in S : (\exists u \in V)(aux = ax)\} \cup \textstyle\bigcup_{u\in V}\{x \in S : aux = a\} \,\cup \\
&\textstyle\bigcup_{u\in V}\bigcup_{v\in V}\{x \in S : vx = u\} \cup \bigcup_{u\in V}\bigcup_{v\in V\setminus\{u\}}\{x \in S : vx = ux\}\,.
\end{aligned}$$

If κ is regular and S is weakly left cancellative, require also that $t_\alpha \notin \{xs_\mu : x \in D \text{ and } \mu < \alpha\}$.

Hypotheses (a) and (b) hold directly. To verify hypothesis (c), assume that $\gamma < \alpha$ and $\mu < \sigma < \alpha$. Then $t_\alpha \notin A_{\gamma,\mu,\sigma}$ so $s_\gamma t_\alpha \neq s_\mu t_\sigma$.

To verify hypothesis (d), assume that $\gamma < \sigma < \alpha$, $\mu < \alpha$, and $x \in S$. If $s_\gamma x = t_\sigma$, then $x \in F_{\gamma,\sigma}$ so $t_\alpha \neq s_\mu x$.

Hypotheses (e) and (f) hold directly. To verify hypothesis (g), let $u \in FP(\langle t_\delta \rangle_{\delta<\alpha}$. Then $aut_\alpha \neq at_\alpha$ so $ut_\alpha \neq t_\alpha$.

To verify hypothesis (h), note that $t_\alpha \notin A$ and if $u \in V$, then $aut_\alpha \neq a$ so $ut_\alpha \notin A$.

To verify hypothesis (i), assume that κ is regular, S is weakly left cancellative, $x \in S$, $\gamma < \sigma < \alpha$, $\mu < \alpha$, and $xs_\gamma = t_\sigma$. Then $x \in D$, so $t_\alpha \neq xs_\mu$.

The inductive construction being complete, let $B = \{t_\alpha : \alpha < \kappa\}$ and for $\alpha < \kappa$, let $B_\alpha = \{t_\gamma : \alpha < \gamma < \kappa\}$.

To verify conclusion (1), let $p \in \overline{B} \cap U_\kappa$ and suppose we have $q \in S^*$ and $r \in U_\kappa$ such that $p = qr$. Pick $\gamma < \mu < \kappa$ such that $s_\gamma^{-1}B \in r$ and $s_\mu^{-1}B \in r$. We claim that $s_\gamma^{-1}B_\gamma \in r$. For otherwise, $\bigcup_{\delta \leq \gamma}\{x \in S : s_\gamma x = t_\delta\} = s_\gamma^{-1}B \backslash s_\gamma^{-1}B_\gamma \in r$ so the union of fewer than κ left solution sets is a member of r, a contradiction. Similarly, $s_\mu^{-1}B_\mu \in r$. Let $H = \{x \in S : s_\mu x = s_\gamma x\}$. Then $|H| < \kappa$ so $S \backslash H \in r$. Pick $x \in s_\gamma^{-1}B_\gamma \cap s_\mu^{-1}B_\mu \cap (S \backslash H)$. Pick $\sigma > \gamma$ and $\delta > \mu$ such that $s_\gamma x = t_\sigma$ and $s_\mu x = t_\delta$. Since $x \notin H$, $\sigma \neq \delta$. But then we get a contradiction to hypothesis (d) regardless of whether $\sigma < \delta$ or $\delta < \sigma$.

To verify conclusion (2), let p and q be distinct members of $\overline{B} \cap U_\kappa$. Pick F and G in $[B]^\kappa$ such that $F \in p$, $G \in q$, and $F \cap G = \emptyset$. Suppose we have u and v in βS such that $up = vq$. Let $D = \{s_\gamma t_\alpha : \gamma < \alpha < \kappa \text{ and } t_\alpha \in F\}$. We claim that $D \in up$. Indeed, given $\gamma < \kappa$, one has that $\{t_\alpha : \gamma < \alpha < \kappa \text{ and } t_\alpha \in F\} \subseteq s_\gamma^{-1}D$. Similarly $\{s_\mu t_\sigma : \mu < \sigma < \kappa \text{ and } t_\sigma \in G\} \in vq$. Pick $\gamma < \alpha < \kappa$ and $\mu < \sigma < \kappa$ such that $s_\gamma t_\alpha = s_\mu t_\sigma$, $t_\alpha \in F$, and $t_\sigma \in G$. Since $F \cap G = \emptyset$, $\alpha \neq \sigma$. But then we have a contradiction to hypothesis (c).

To verify conclusion (3), let $p \in \overline{B} \cap U_\kappa$ and suppose that we have $u \neq v$ in βS such that $up = vp$. Pick $F \in u$ and $G \in v$ such that $F \cap G = \emptyset$. Then $\{s_\gamma t_\alpha : \gamma < \alpha < \kappa \text{ and } s_\gamma \in F\} \in up$ and $\{s_\mu t_\sigma : \mu < \sigma < \kappa \text{ and } s_\mu \in G\} \in vp$ so pick $\gamma < \alpha < \kappa$ and $\mu < \sigma < \kappa$ such that $s_\gamma \in F$, $s_\mu \in G$, and $s_\gamma t_\alpha = s_\mu t_\sigma$. Then $\alpha \neq \sigma$ so this contradicts hypothesis (c).

To verify conclusion (4) note first that $|B \cap U_\kappa| = 2^{2^\kappa}$ by [5, Theorem 3.58]. We will show that if $k, m \in \mathbb{N}$, $p_1, \ldots, p_m, q_1, \ldots, q_k \in \overline{B} \cap U_\kappa$, and $p_1 \cdots p_m = q_1 \cdots q_k$, then $m = k$ and for each $i \in \{1, 2, \ldots, m\}$, $p_i = q_i$. Suppose that this fails and pick a counterexample with $m+k$ a minimum among all counterexamples. If $m = k = 1$, one does not have a counterexample, and $m + k = 3$ is out by conclusion (1). By conclusion (2) we have that $p_m = q_k$ and so by conclusion (3), $p_1 \cdots p_{m-1} = q_1 \cdots q_{k-1}$ and thus $m - 1 = k - 1$ and for all $i \in \{1, 2, \ldots, m-1\}$, $p_i = q_i$.

To verify conclusion (5), it suffices by [5, Theorem 7.35] to show that $\langle t_\alpha \rangle_{\alpha < \kappa}$ has distinct finite products. So suppose instead there exist $F \neq G$ in $\mathcal{P}_f(\kappa)$ such that $\prod_{\alpha \in F} t_\alpha = \prod_{\alpha \in G} t_\alpha$ and pick F and G with $|F \cup G|$ a minimum among all such examples. Assume without loss of generality that $\max F \leq \max G = \alpha$. Suppose first that $\max F < \alpha$. If $G = \{\alpha\}$ we have a contradiction to hypothesis (a) and if $|G| \geq 2$ we have a contradiction to hypothesis (e). So we must have $\alpha \in F$. If either F or G is a singleton, we get a contradiction to hypothesis (g) and otherwise we get a contradiction to hypothesis (f).

To verify conclusion (6), assume that κ is regular and S is weakly left cancellative. Suppose we have q and r in S^* such that $rq \in \overline{B} \cap U_\kappa$. By conclusion (1), $q \in S^* \setminus U_\kappa$. Since κ is regular, there is some $\delta < \kappa$ such that $\{s_\alpha : \alpha < \delta\} \in q$. One now derives a contradiction to hypothesis (j) exactly as in the last paragraph of the proof of Lemma 2.8.

Finally, let $\mathcal{B} = \{B \cap C_\alpha : \alpha < \lambda\}$. We claim that $\mathcal{B}$ has the κ-uniform finite intersection property. So let $F \in \mathcal{P}_f(\lambda)$. Since $\mathcal{C}$ is closed under finite intersections, pick $\gamma < \lambda$ such that $C_\gamma = \bigcap_{\alpha \in F} C_\alpha$. Then $\{t_\alpha : \pi_2(f(\alpha)) = \gamma\} \subseteq B \cap C_\gamma$. Let $P = \{p \in U_\kappa : \mathcal{B} \subseteq p\}$. By [5, Theorem 3.62], $|P| = 2^{2^\kappa}$. $\square$

Note that, by the proof of [5, Theorem 6.42], conclusion (2) of Theorem 2.10 does not require the assumption that S is right cancellative.

3. Dense nonproducts

We have one theorem guaranteeing that $S^* \setminus S^*S^*$ is dense in S^*. Recall that by Theorem 2.4 there is a cancellative semigroup with $|S| = \mathfrak{c}$ such that S^*S^* is not nowhere dense in S^*. So, if the continuum hypothesis holds, this is an example of a semigroup such that $S^* \setminus S^*S^*$ is dense in S^* while S^*S^* is not nowhere dense in S^*. On the other hand, as we will see, it is a consequence of Martin's Axiom and the negation of the continuum hypothesis, that any cancellative semigroup S with $|S| = \omega_1$ does have S^*S^* nowhere dense in S^*. For an elementary introduction to Martin's Axiom, see [5, Section 12.1].

Theorem 3.1. *Let S be a semigroup with $|S| = \omega_1$ and assume that S is right cancellative and weakly left cancellative. Then $S^* \setminus (S^*S^*)$ is dense in S^*. If $MA(\omega_1)$ holds, then S^*S^* is nowhere dense in S^*.*

Proof. Enumerate S as $\langle s_\sigma \rangle_{\sigma<\omega_1}$. For $\omega < \sigma < \omega_1$, let S_σ be the semigroup generated by $\{s_\tau : \tau < \sigma\}$. By [5, Theorem 6.35], for each σ with $\omega < \sigma < \omega_1$, $S_\sigma^*S_\sigma^*$ is nowhere dense in S_σ^*.

Let A be a countably infinite subset of S. We will show that $\overline{A} \cap (S^* \setminus S^*S^*) \neq \emptyset$ and that, if $MA(\omega_1)$ holds, then $A^* \setminus c\ell(S^*S^*) \neq \emptyset$. Pick $\delta < \omega_1$ such that $A \subseteq S_\delta$. Since $\overline{A} \setminus c\ell(S_\delta^*S_\delta^*) \neq \emptyset$, pick $V_\delta \in [A]^\omega$ such that $V_\delta^* \cap (S_\delta^*S_\delta^*) = \emptyset$. Now let $\delta < \sigma < \omega_1$ and assume that we have chosen $\langle V_\tau \rangle_{\delta \le \tau < \sigma}$ such that for each τ with $\delta \le \tau < \sigma$,

(a) $V_\tau \in [A]^\omega$,

(b) if $\mu < \tau$, then $V_\tau^* \subseteq V_\mu^*$; and

(c) $\overline{V_\tau} \cap (S_\tau^*S_\tau^*) = \emptyset$.

If $\sigma = \gamma + 1$ for some γ, we have that $V_\gamma^* \setminus c\ell(S_\sigma^*S_\sigma^*) \neq \emptyset$ so pick $V_\sigma \in [V_\gamma]^\omega$ such that $\overline{V_\sigma} \cap (S_\sigma^*S_\sigma^*) = \emptyset$.

Now assume that σ is a limit ordinal. Note that $\{V_\tau : \delta \le \tau < \sigma\}$ has the finite intersection property. Enumerate $\{\tau : \delta \le \tau < \sigma\}$ as $\langle \tau_n \rangle_{n<\omega}$. Choose $a_0 \in V_{\tau_0}$ and inductively for $n > 0$ choose $a_n \in \bigcap_{k=0}^n V_{\tau_k} \setminus \{a_0, a_1, \ldots, a_{n-1}\}$. Pick infinite $V_\sigma \subseteq \{a_n : n < \omega\}$ such that $\overline{V_\sigma} \cap (S_\sigma^*S_\sigma^*) = \emptyset$.

The inductive construction being complete, we have that $\{V_\sigma^* : \delta \le \sigma < \omega_1\}$ is a collection of closed subsets of S^* with the finite intersection property so pick $q \in \bigcap_{\delta \le \sigma < \omega_1} V_\sigma^*$. Then $q \in \overline{A}$. We claim that $q \notin S^*S^*$. So suppose instead we have $q = pr$ for some $p, r \in S^*$. Since U_{ω_1} is an ideal of βS by [5, Lemma 6.34.3], we have $||p|| = ||r|| = \omega$. So pick $B \in p$ and $C \in r$ such that $|B| = |C| = \omega$. Pick $\sigma < \omega_1$ such that $B \cup C \subseteq S_\sigma$. Then $q \in \overline{V_\sigma} \cap (S_\sigma^*S_\sigma^*)$, a contradiction.

Now assume that $MA(\omega_1)$ holds and suppose $A^* \subseteq c\ell(S^*S^*)$. By [5, Corollary 12.12], $\mathrm{int}_{A^*} \bigcap_{\delta \le \sigma < \omega_1} V_\sigma^* \neq \emptyset$ so pick $B \in [A]^\omega$ such that $B^* \subseteq \bigcap_{\delta \le \sigma < \omega_1} V_\sigma^*$. Picking $q \in B^* \cap (S^*S^*)$ we derive a contradiction as in the paragraph above. $\square$

Recall from [5, Theorem 6.35] that if S is a countably infinite right cancellative and weakly left cancellative semigroup, then S^*S^* is nowhere dense in S^*.

The semigroup produced in the following theorem is very similar to that of [5, Exercise 4.3.7].

Theorem 3.2. *There is a countably infinite left cancellative and weakly right cancellative semigroup S such that S^*S^* has nonempty interior. In particular, $S^* \setminus (S^*S^*)$ is not dense in S^*.*

Proof. Let $L = \{x_n : n \in \mathbb{N}\} \cup \{z_n : n \in \mathbb{N}\} \cup \{y\}$ be an alphabet of distinct letters. Let

$$\begin{aligned} S = \ & \{a_1 a_2 \cdots a_t : \text{ each } a_i \in L \text{ and if } i \in \{1, 2, \ldots, t-1\}, \\ & a_i = x_n, \text{ and } a_{i+1} = z_m, \text{ then } n \geq m\}. \end{aligned}$$

For $w_1 = a_1 \cdots a_t$ and $w_2 = b_1 \cdots b_s$ in S (with each a_i and b_i in L), let $w_1 w_2$ be ordinary concatenation unless $a_t = x_n$ and $b_1 = z_m$ with $n < m$ in which case $w_1 w_2 = a_1 \cdots a_{t-1} y b_1 \cdots b_s$.

It is routine (though at least mildly tedious) to verify that the operation on S is associative, that S is left cancellative, and that S is weakly right cancellative.

Let $A = \{y z_n : n \in \mathbb{N}\}$. We claim that $A^* \subseteq S^*S^*$, so let $p \in A^*$. For $B \in p$, let $C_B = \{z_n : y z_n \in B\}$. Then $\{C_B : B \in p\}$ has the ω-uniform finite intersection property, so pick $q \in S^*$ such that $\{C_B : B \in p\} \in q$. Let $r \in \{x_n : n \in \mathbb{N}\}^*$. We claim that $p = rq$. So let $B \in p$. We claim that $\{x_n : n \in \mathbb{N}\} \subseteq \{s \in S : s^{-1}B \in q\}$. So let $n \in \mathbb{N}$. Let $D = C_B \cap \{z_m : m > n\}$. Then $D \in q$ and $D \subseteq x_n^{-1}B$. □

4. Right cancelable elements

We saw in Theorem 2.10 that with mild cancellation assumptions on S we can get lots of uniform ultrafilters on S that are right cancelable in βS. And by [5, Lemma 8.1], if s is right cancelable in S, it is also right cancelable in βS. In [2, Theorem 3.2], M. Filali showed that if S is an infinite semigroup which can be embedded in a group and $\omega \leq \kappa \leq |S|$, then there exist right cancelable elements p of βS with $||p|| = \kappa$. We shall see as a consequence of Theorem 4.2, if p is right cancelable in in $\beta\mathbb{Z}$ or in $\beta\mathbb{Q}_d$, then p is right cancelable in $\beta\mathbb{R}_d$ and in $\beta\mathbb{C}_d$. Here, if X is a topological space, X_d denotes the set X with the discrete topology. As we have already remarked, if $T \subseteq S$, we ignore the subtle distinction between an ultrafilter on T and an ultrafilter on S with T as a member and pretend that $\beta T \subseteq \beta S$.

Lemma 4.1. *Let S be an infinite semigroup. An element $p \in \beta S$ is right cancelable in βS if and only if, for each $A \subseteq S$, there exists $B \subseteq S$ such that $A = \{x \in S : x^{-1}B \in p\}$.*

Proof. This is [5, Theorem 8.7]. □

In the following proof we use the notation

$$\begin{array}{ccc} P & \Leftrightarrow & Q \\ & \Leftrightarrow & R \end{array}$$

to abbreviate the statement "$P \Leftrightarrow Q$ and $Q \Leftrightarrow R$".

Theorem 4.2. *Let S be an infinite semigroup and let T be a semigroup with identity e. Let $p \in (S \times \{e\})^*$ be right cancelable in $\beta(S \times \{e\})$. Then p is right cancelable in $\beta(S \times T)$.*

Proof. We use Lemma 4.1. Let $A \subseteq S \times T$. We shall produce $B \subseteq S \times T$ such that $A = \{v \in S \times T : v^{-1}B \in p\}$.

We observe that the coordinate function $\pi_2 : S \times T \to T$ extends to a continuous homomorphism $\widetilde{\pi}_2 : \beta(S \times T) \to \beta T$.

For each $t \in T$, let $C_t = \{x \in S : (x,t) \in A\}$. By Lemma 4.1, for each $t \in T$, pick $B_t \subseteq S$ such that, for every $s \in S$,

$$\begin{array}{rcl} (s,e)p \in \overline{B_t \times \{e\}} & \Leftrightarrow & s \in C_t \\ & \Leftrightarrow & (s,t) \in A. \end{array}$$

Let $B = \bigcup_{t \in T} B_t \times \{t\}$. Since each $t \in T$ is an isolated point of βT, $\overline{B_t \times \{t\}} = \{x \in \overline{B} : \widetilde{\pi}_2(x) = t\}$.

Let $\langle v_i \rangle_{i \in D}$ be a net in S for which $\langle (v_i, e) \rangle_{i \in D}$ converges to p in $\beta(S \times T)$. Let $(s,t) \in S \times T$. We show that $(s,t)^{-1}B \in p$ if and only if $(s,t) \in A$. Given $i \in D$, the statements $(s,t)(v_i,e) \in B_t \times \{t\}$ and $(s,e)(v_i,e) \in B_t \times \{e\}$ are equivalent, because each is equivalent to the statement that $sv_i \in B_t$. Since $\lambda_{(s,t)}$ and $\lambda_{(s,e)}$ are continuous in $\beta(S \times T)$, we have $(s,t)p = \lim_{i \in D} (s,t)(v_i,e)$ and $(s,e)p = \lim_{i \in D} (s,e)(v_i,e)$ so $(s,t)p \in \overline{B_t \times \{t\}}$ if and only if $(s,e)p \in \overline{B_t \times \{e\}}$.

Since $\widetilde{\pi}_2\big((s,t)p\big) = t$, it follows that:

$$\begin{array}{rcl} (s,t)p \in \overline{B} & \Leftrightarrow & (s,t)p \in \overline{B_t \times \{t\}} \\ & \Leftrightarrow & (s,e)p \in \overline{B_t \times \{e\}} \\ & \Leftrightarrow & (s,t) \in A. \end{array}$$

□

A consequence of Theorem 4.2 is that if $p \in \mathbb{N}^*$ is right cancelable in $(\beta\mathbb{N}, +)$, T is any left cancelative semigroup with identity e, and $q \in \beta(\mathbb{N} \times T)$ has the property that $p = \{A \subseteq \mathbb{N} : A \times \{e\} \in q\}$, then q is right cancelable in $\beta(\mathbb{N} \times T)$. By way of contrast, by [5, Example 8.29], there exists $p \in \mathbb{N}^*$ which is right cancelable in $(\beta\mathbb{N}, +)$ but not in $(\beta\mathbb{Z}, +)$.

Corollary 4.3. *Suppose that a divisible abelian group H is a subgroup of an abelian group G. Then every element of βH which is right cancelable in βH is also right cancelable in βG.*

Proof. By [3, Theorem 18.1], there is a subgroup L of G such that $G = H \oplus L$. $\square$

Corollary 4.4. *Every right cancelable element of $\beta\mathbb{Q}_d$ is right cancelable in $\beta\mathbb{R}_d$, and every right cancelable element of $\beta\mathbb{R}_d$ is right cancelable in $\beta\mathbb{C}_d$.*

Proof. Corollary 4.3. $\square$

The next lemma will be used in Corollaries 4.6 and 4.7.

Lemma 4.5. *Let A be a countably infinite subset of a divisible group $(H, +)$. There is a countably infinite divisible subgroup D of H such that $A \subseteq D$.*

Proof. By [3, Theorem 20.1], we may presume that $H = \bigoplus_{\alpha\in I} K_\alpha$, where each K_α is either a copy of $\mathbb{Q}$ or a quasi-cyclic group. Let

$$J = \{\alpha \in I : (\exists x \in A)(x_\alpha \neq 0)\}$$

and let $D = \{x \in H : (\forall\alpha \in I\setminus J)(x_\alpha = 0)\}$. Then D is isomorphic to $\bigoplus_{\alpha\in J} K_\alpha$ so D is a countably infinite divisible subgroup of H containing A. $\square$

Corollary 4.6. *Let $(S, +)$ be an infinite cancellative commutative semigroup and let G denote the abelian group of differences of S. Let H denote any abelian group which contains S, and which therefore contains G. There is a subset V of S^* satisfying the following statements.*

(1) V is open and dense in S^* and every element of V is right cancelable in βH.

(2) For any two elements v_1 and v_2 of V, $\beta H + v_1$ and $\beta H + v_2$ intersect if and only if $S + v_1$ and $S + v_2$ intersect.

(3) Define an equivalence relation $\approx$ on S^* by $p \approx q$ if and only if $(S + p) \cap (S + q) \neq \emptyset$. If $k, m \in \mathbb{N}$, $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_k \in V$, and $p_1 + \ldots p_m \approx q_1 + \ldots q_k$, then $k = m$ and for each $t \in \{1, 2, \ldots, m\}$, $p_t \approx q_t$.

Proof. By [3, Theorem 20.1], H can be embedded in a divisible abelian group and so we may assume that H is divisible. Let $\mathcal{V} = \{A \subseteq [S]^\omega : \overline{A}\cap(H^*+H^*) = \emptyset\}$ and let $U = \bigcup\{A^* : A \in \mathcal{V}\}$. We claim that U is dense in S^*, so let $B \in [S]^\omega$. By Theorem 2.1, $B^* \setminus c\ell(H^* + H^*) \neq \emptyset$ so pick $p \in B^* \setminus c\ell(H^* + H^*)$ and pick $A \in p$ such that $\overline{A} \cap (H^* + H^*) = \emptyset$. Then $p \in B^* \cap U$.

Let $V = \bigcup\{U \cap D^* : D$ is a countable divisible subgroup of $H\}$. Then V is open in S^*. To see that V is dense in S^*, let $B \in [S]^\omega$. By Lemma 4.5, pick a countable divisible subgroup D of H containing B. Pick $p \in B^* \cap U$. Then $p \in U \cap D^*$. Note that each $p \in V$ has $||p|| = \omega$ and that $V \cap (H^* + H^*) = \emptyset$.

To see that every element of V is right cancelable in βH, let $p \in V$ and pick a countable divisible subgroup D of H such that $p \in U \cap D^*$. Since $p \in U$, $p \notin (D^* + D^*)$ and so by [5, Theorem 8.18], p is right cancelable in βD. By Corollary 4.3, p is right cancelable in βH.

To verify (2), let v_1 and v_2 be elements of V. By Lemma 4.5, we can choose a countable divisible subgroup D of H such that D is a member of v_1 and of v_2. By [3, Theorem 18.1], we can write H as a direct sum $H = D + E$ for some some subgroup E of H. Let π_D denote the projection of H onto D, and let $\widetilde{\pi}_D : \beta H \to \beta D$ denote its continuous extension. Suppose that $x + v_1 = y + v_2$ for some $x, y \in \beta H$. Then $\widetilde{\pi}_D(x) + v_1 = \widetilde{\pi}_D + v_2$. So $c\ell(D + v_1) \cap c\ell(D + v_2) \neq \emptyset$. By [5, Theorem 3.40], we may suppose that $s + v_1 = w + v_2$ for some $s \in D$ and some $w \in \beta D$. So $v_1 = z + v_2$, where $z = -s + w$. This equation implies that $z \in \beta G$ and hence that $z \in G$, becaause $v_1 \notin H^* + H^*$. So $z = s_2 - s_1$ for some $s_1, s_2 \in S$ and therefore $s_1 + v_1 = s_2 + v_2$.

Conclusion (3) can now be proved by an inductive argument similar to the proof of Theorem 2.6(2). □

Corollary 4.7. *Let G be a countably infinite subgroup of an abelian group $(H, +)$. Then every element of βG which is right cancelable in βG is also right cancelable in βH.*

Proof. Let p be a right cancelable element of βG. If $p \in G$, then $p \in H$ so by [5, Lemma 8.1], p is right cancelable in βH. So we assume that $p \in H^*$. By [3, Theorem 19.1], H can be embedded in a divisible abelian group K and if p is right cancelable in βK, it is also right cancelable in βH, so we may assume that H is divisible. Pick by Lemma 4.5 a countably infinite divisible subgroup D of H which contains G. We claim that p is right cancelable in βD. Suppose that p is not right cancelable in βD and pick by [5, Theorem 8.18] $q \in D^*$ such that $p = q + p$. Since $G \in p$, $\{x \in D : -x + G \in p\} \in q$ so $G \in q$. But then $p \in G^* + p$ so p is not right cancelable in βG. Finally, by Corollary 4.3, p is right cancelable in βH. □

Finally, we observe that, for example, there are many $p \in \beta\mathbb{R}_d$ that are right cancelable in $\beta\mathbb{R}_d$ and converge to a given point of $\mathbb{R}$ with respect to the usual topology.

Theorem 4.8. *Let S be a right cancellative and very weakly left cancellative semigroup with $|S| = \kappa \geq \omega$. Assume that $\mathcal{T}$ is a topology on S and $x \in S$ such that there is a neighborhood base $\mathcal{C}$ for x with respect to $\mathcal{T}$ such that $|\mathcal{C}| \leq \kappa$ and for each $C \in \mathcal{C}$, $|C| = \kappa$. Then there is a set P of 2^{2^κ} right cancelable elements of βS_d, each of which converges to x with respect to $\mathcal{T}$.*

Proof. This is an immediate consequence of Theorem 2.10. □

References

[1] M. Filali, *Weak p-points and cancellation in βS*, in Papers on General Topology and Applications, S. Andima et. al. eds., Annals of the New York Academy of Sciences **806** (1996), 130-139.

[2] M. Filali, *t-sets and some algebraic properties in βS and in $l_\infty(S)^*$*, Semigroup Forum **65** (2002), 285-300.

[3] L. Fuchs, *Abelian groups*, Pergamon Press, New York-Oxford-London-Paris, 1960.

[4] N. Hindman, *Minimal ideals and cancellation in* $\beta\mathbb{N}$, Semigroup Forum **25** (1982), 291-310.

[5] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, 2nd edition*, Walter de Gruyter & Co., Berlin, 2012.

[6] N. Hindman and D. Strauss, *Cancellation in the Stone-Čech compactification of a discrete semigroup*, Proc. Edinburgh Math. Soc. **37** (1994), 379-397.

[7] D. Strauss, *Semigroup structures on* $\beta\mathbb{N}$, Semigroup Forum **41** (1992), 238-244.