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# BADLY APPROXIMABLE POINTS IN TWISTED DIOPHANTINE APPROXIMATION AND HAUSDORFF DIMENSION

PALOMA BENGOECHEA[†] AND NIKOLAY MOSHCHEVITIN[*]

ABSTRACT. For any $j_1, \ldots, j_n > 0$ with $\sum_{i=1}^{n} j_i = 1$ and any $\theta \in \mathbb{R}^n$, let $\mathrm{Bad}_\theta(j_1, \ldots, j_n)$ denote the set of points $\eta \in \mathbb{R}^n$ for which $\max_{1 \le i \le n}(\|q\theta_i - \eta_i\|^{1/j_i}) > c/q$ for some positive constant $c = c(\eta)$ and all $q \in \mathbb{N}$. These sets are the 'twisted' inhomogeneous analogue of $\mathrm{Bad}(j_1, \ldots, j_n)$ in the theory of simultaneous Diophantine approximation. It has been shown that they have full Hausdorff dimension in the non-weighted setting, i.e provided that $j_i = 1/n$, and in the weighted setting when $\theta$ is chosen from $\mathrm{Bad}(j_1, \ldots, j_n)$. We generalise these results proving the full Hausdorff dimension in the weighted setting without any condition on $\theta$. Moreover, we prove $\dim(\mathrm{Bad}_\theta(j_1, \ldots, j_n) \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \mathrm{Bad}(0, \ldots, 0, 1)) = n$.

## 1. INTRODUCTION

The classical result due to Dirichlet: for any real number $\theta$ there exist infinitely many natural numbers $q$ such that

$$\|q\theta\| \le q^{-1}, \tag{1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer, has higher dimension generalisations. Consider any $n$-tuple of real numbers $(j_1, \ldots, j_n)$ such that

$$j_1, \ldots, j_n > 0 \quad \text{and} \quad \sum_{i=1}^{n} j_i = 1. \tag{2}$$

Then, for any vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, there exist infinitely many natural numbers $q$ such that

$$\max_{1 \le i \le n}(\|q\theta_i\|^{1/j_i}) \le q^{-1}. \tag{3}$$

The two results above motivate the study of real numbers and real vectors $\theta \in \mathbb{R}^n$ for which the right hand side of (1) and (3) respectively cannot be improved by an arbitrary constant. They respectively constitute the sets Bad

of badly approximable numbers and $\mathrm{Bad}(j_1, \ldots, j_n)$ of $(j_1, \ldots, j_n)$-badly approximable numbers. Hence

$$\mathrm{Bad}(j_1, \ldots, j_n) := \left\{ (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \ : \ \inf_{q \in \mathbb{N}} \max_{1 \leq i \leq n} \left( q^{j_i} \| q \theta_i \| \right) > 0 \right\}.$$

In the 1-dimensional case, it is well known that the set of badly approximable numbers has Lebesgue measure zero but maximal Hausdorff dimension. In the $n$-dimensional case, it is also a classical result that $\mathrm{Bad}(j_1, \ldots, j_n)$ has Lebesgue measure zero, and Schmidt proved in 1966 that the particular set $\mathrm{Bad}(1/2, 1/2)$ has full Hausdorff dimension. But the result of maximal dimension in the weigthed setting hasn't been proved until almost 40 years later, by Pollington and Velani [21]. In the 2-dimensional case, An showed in [1] that $\mathrm{Bad}(j_1, j_2)$ is in fact winning for the now famous Schmidt games -see [22]. Thus he provided a direct proof of a conjecture of Schmidt stating that any countable intersection of sets $\mathrm{Bad}(j_1, j_2)$ is non empty -see also [2].

Recently, interest in the size of related sets, usually referred to as the 'twists' of the sets $\mathrm{Bad}(j_1, \ldots, j_n)$, has developed. The study of these new sets started in the 1-dimensional setting: we fix $\theta \in \mathbb{R}$ and consider the twist of Bad:

$$\mathrm{Bad}_\theta := \left\{ \eta \in \mathbb{R} : \inf_{q \in \mathbb{N}} q \| q\theta - \eta \| > 0 \right\}.$$

The set $\mathrm{Bad}_\theta$ has a palpable interpretation in terms of rotations of the unit circle. Identifying the circle with the unit interval $[0, 1)$, the value $q\theta$ (modulo 1) may be thought of as the position of the origin after $q$ rotations by the angle $\theta$. If $\theta$ is rational, the rotation is periodic. If $\theta$ is irrational, a classical result of Weyl [25] implies that $q\theta$ (modulo 1) is equidistributed, so $q\theta$ visits any fixed subinterval of $[0, 1)$ infinitely often. The natural question of what happens if the subinterval is allowed to shrink with time arises. Shrinking a subinterval corresponds to making its length decay according to some specified function. The set $\mathrm{Bad}_\theta$ corresponds to considering, for any $\epsilon > 0$, the shrinking interval $(\eta - \epsilon/q, \eta + \epsilon/q)$ centred at the point $\eta$ and where the specified function is $\epsilon/q$. Khintchine showed in [14] that

$$(4) \qquad\qquad \| q\theta - \eta \| < \frac{1 + \delta}{\sqrt{5}q} \qquad (\delta > 0)$$

is satisfied for infinitely many integers $q$, and Theorem III in Chapter III of Cassels' book [5] shows that the right hand side of (4) cannot be improved by an arbitrary constant for every irrational $\theta$ and every real $\eta$. This motivates the study of the set $\mathrm{Bad}_\theta$. Kim [16] proved in 2007 that it has Lebesgue measure zero, and later it was shown by Tseng [23] that it has full Hausdorff

dimension (actually Tseng proved that $\mathrm{Bad}_\theta$ has the stronger property of being winning for any $\theta \in \mathbb{R}$).

By generalising circle rotations to rotations on torus of higher dimensions, i.e. by considering the sequence $q\theta$ (modulo 1) in $[0,1)^n$ where $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, we obtain the 'twists' of the sets $\mathrm{Bad}(j_1, \ldots, j_n)$:

(5)
$$\mathrm{Bad}_\theta(j_1, \ldots, j_n) = \left\{ (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n : \inf_{q \in \mathbb{N}} \max_{1 \leq i \leq n} (q^{j_i} \| q\theta_i - \eta_i \|) > 0 \right\}.$$

In [3] Bugeaud et al proved that the non-weighted set $\mathrm{Bad}_\theta(1/n, \ldots, 1/n)$ has full Hausdorff dimension. Recently, Einsiedler and Tseng [8] extended the results [3] and [23] by showing, among other results, that $\mathrm{Bad}_\theta(1/n, \ldots, 1/n)$ is also winning. It was shown in [18] that such results may be obtained by classical methods developed by Khintchine [15] and Jarník [12, 13] and discussed in Chapter V of Cassels' book [5]. Unfortunately, these methods cannot be directly extended to the weighted setting. For the weighted setting, less has heretofore been known. Harrap did the first contribution [10] in the 2-dimensional case, by proving that $\mathrm{Bad}_\theta(j_1, j_2)$ has full Hausdorff dimension provided that the fixed point $\theta \in \mathbb{R}^2$ belongs to $\mathrm{Bad}(j_1, j_2)$, which is a significantly restrictive condition. Recently, under the hypothesis $\theta \in \mathrm{Bad}(j_1, \ldots, j_n)$, Harrap and Moshchevitin have extended to weighted linear forms in higher dimension and improved to winning the result in [10] (see [11]).

In this paper, we prove that the weighted set $\mathrm{Bad}_\theta(j_1, \ldots, j_n)$ has full Hausdorff dimension for any $\theta \in \mathbb{R}^n$. Moreover, the following theorem holds.

**Theorem 1.1.** *For any $\theta \in \mathbb{R}^n$ and all $j_1, \ldots, j_n > 0$ with $\sum_{i=1}^n j_i = 1$,*

$$\dim(\mathrm{Bad}_\theta(j_1, \ldots, j_n) \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \mathrm{Bad}(0, \ldots, 0, 1)) = n.$$

The same type of theorem holds in the classical not twisted setting; it constitutes the work done in [21] (see Theorem 2).

Note that if $1, \theta_1, \ldots, \theta_n$ are linearly dependent over $\mathbb{Z}$, then Theorem 1.1 is obvious. Indeed, in this case $\{q\theta : q \in \mathbb{Z}\}$ is restricted to a hyperplane $H$ of $\mathbb{R}^n$, so $\mathrm{Bad}_\theta(j_1, \ldots, j_n) \supset \mathbb{R}^n \backslash H$ is winning. Hence $\mathrm{Bad}_\theta(j_1, \ldots, j_n) \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \mathrm{Bad}(0, \ldots, 0, 1)$ is winning and in particular has full dimension [1]. Therefore we suppose throughout the paper that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Z}$.

---

[1] We recall that winning sets in $\mathbb{R}^n$ have maximal Hausdorff dimension, and that countable intersections of winning sets are again winning. We refer the reader to [22] for all necessary definitions and results on winning sets.

The strategy for the proof of Theorem 1.1 is as follows. We start by defining a set $\mathcal{V} \subset \mathrm{Bad}_\theta(j_1, \ldots, j_n)$ related to the best approximations to the fixed point $\theta \in \mathbb{R}^n$. Then we construct a Cantor-type set $K(R)$ inside $\mathcal{V} \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \mathrm{Bad}(0, \ldots, 0, 1)$. Finally we describe a probability measure supported on $K(R)$ to which we can apply the mass distribution principle and thus find a lower bound for the dimension of $K(R)$.

Best approximations are defined in Section 2. In Section 3 we define $\mathcal{V}$ and give the proof of the inclusion $\mathcal{V} \subset \mathrm{Bad}_\theta(j_1, \ldots, j_n)$. We construct $K(R)$ in Section 4 and describe the probability measure in Section 5. Finally we compute the lower bound for the dimension of $K(R)$ in Section 6.

In the following, we let $n \in \mathbb{N}$, fix an $n$-tuple $(j_1, \ldots, j_n) \in \mathbb{R}^n$ satisfying (2) and a vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ such that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Z}$. We denote by $x \cdot y$ the scalar product of two vectors $x$ and $y$ in $\mathbb{R}^n$, and by $\| \cdot \|$ the distance to the nearest integer.

## 2. Best approximations

**Definition 2.1.** An n-dimensional vector $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \setminus \{0\}$ is called a best approximation to $\theta$ if for all $v \in \mathbb{Z}^n \setminus \{0, -m, m\}$ the following implication holds:

$$\max_{1 \leq i \leq n} (|v_i|^{1/j_i}) \leq \max_{1 \leq i \leq n} (|m_i|^{1/j_i}) \implies \|v \cdot \theta\| > \|m \cdot \theta\|.$$

Note that the condition $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Z}$-linearly independent allows us to demand a strict inequality in the right hand side of the implication above.

Note also that when $n = 1$ the best approximations to a real number $x$ are, up to the sign, the denominators of the convergents to $x$.

Since $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Z}$-linearly independent, we have an infinite number of best approximations to $\theta$. They can be arranged up to the sign -so that two vectors of opposite sign do not both appear- in an infinite sequence

$$(6) \qquad m_\nu = (m_{\nu,1}, \ldots, m_{\nu,n}) \qquad \nu \geq 1,$$

such that the values

$$(7) \qquad M_\nu = \max_{1 \leq i \leq n} (|m_{\nu,i}|^{1/j_i})$$

form a strictly increasing sequence, and the values

$$(8) \qquad \zeta_\nu = \|m_\nu \cdot \theta\|$$

form a strictly decreasing sequence. Hence each value $M_\nu$ corresponds to a single best approximation $m_\nu$. The quantity $M_\nu$ can be referred to as the 'height' of $m_\nu$.

Best approximations vectors have often been used in proofs, but not always explicitly. In particular, Voronoi [24] selected some points in a lattice that correspond exactly to the best approximation vectors (see also [7]). Similar constructions were introduced in [17] or Section 2 of [4]. Some important properties of the best approximation vectors are discussed in [19,20] and a recent survey on the topic is due to Chevallier [6].

For each $\nu \geq 1$, it is easy to see that the region

$$\left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} \left( |x_i|^{1/j_i} \right) < M_{\nu+1}, \left| x_0 + \sum_{i=1}^{n} x_i \theta_i \right| < \zeta_\nu \right\}$$

does not contain any integer point different from 0. Since this region has volume $2^{n+1} M_{\nu+1} \zeta_\nu$ (see Lemma 4 in Appendix B of [5]), it follows from Minkowski's convex body theorem that

$$(9) \qquad\qquad \zeta_\nu M_{\nu+1} \leq 1.$$

The inequality above will be used later as well as the following lemma, stating that the sequence of heights $M_\nu$ is lacunary.

**Lemma 2.2.** *For every $\nu \geq 1$, we have*

$$M_{\nu+2 \cdot 3^n} \geq 2M_\nu.$$

*Proof.* Given $\nu \geq 1$, we show that we have at most $2 \cdot 3^n$ vectors $m_{\nu+r}$ with $r \geq 0$ and $M_{\nu+r} < 2M_\nu$. The goal is to see that the 0-symmetric region

$$(10) \quad \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} \left( |x_i|^{1/j_i} \right) < 2M_\nu, \left| x_0 + \sum_{i=1}^{n} x_i \theta_i \right| \leq \zeta_\nu \right\}$$

contains at most $4 \cdot 3^n$ integer points other than 0. The region (10) is covered by sets of the form

$$T(\xi) = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\; : \begin{array}{l} \max_{1 \leq i \leq n} (|x_i - \xi_i|^{1/j_i}) \leq M_\nu, \\ \text{and } \left| x_0 - \xi_0 + \sum_{i=1}^{n} (x_i - \xi_i)\theta_i \right| \leq \zeta_\nu \end{array} \right\},$$

with

$$(11) \qquad \xi_i \in \left\{ -2M_\nu^{j_i}, 0, 2M_\nu^{j_i} \right\}, \qquad \xi_0 = -\sum_{i=1}^{n} \xi_i \theta_i.$$

Each region $T(\xi)$ is the translate by $(\xi_0, \ldots, \xi_n)$ of the set

$$\left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} \left( |x_i|^{1/j_i} \right) \leq M_\nu, \left| x_0 + \sum_{i=1}^{n} x_i \theta_i \right| \leq \zeta_\nu \right\},$$

which contains exactly three integer points: 0 and two best approximations with opposite sign. Hence each $T(\xi)$ contains at most four integer points. Since there are $3^n$ possible choices for $(\xi_0, \ldots, \xi_n)$ satisfying (11), the set (10) contains at most $4 \cdot 3^n$ integer points. $\qquad\square$

## 3. THE SET $\mathcal{V}$ INCLUDED IN $\mathrm{Bad}_\theta(j_1, \ldots, j_n)$

The following proposition allows us to work with a set defined by the best approximations to $\theta$ instead of working directly with $\mathrm{Bad}_\theta(j_1, \ldots, j_n)$.

**Proposition 3.1.** *If $\eta \in \mathbb{R}^n$ satisfies*

$$
(12) \qquad\qquad \inf_\nu \|m_\nu \cdot \eta\| > 0,
$$

*then $\eta \in \mathrm{Bad}_\theta(j_1, \ldots, j_n)$.*

*Proof.* Let $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$ satisfy

$$
\|m_\nu \cdot \eta\| > \gamma \qquad \forall \nu \geq 1
$$

for some $\gamma > 0$. For all $q \in \mathbb{N}$ and $\nu \geq 1$, we have the identity

$$
m_\nu \cdot \eta \;=\; m_\nu \cdot (\eta - q\theta) + q\,m_\nu \cdot \theta,
$$

from which we obtain the inequalities

$$
(13) \qquad \gamma < \|m_\nu \cdot \eta\| \leq n \max_{1 \leq i \leq n} (|m_{\nu,i}| \cdot \|\eta_i - q\theta_i\|) + q\zeta_\nu.
$$

Since $\zeta_\nu$ is strictly decreasing and $\zeta_\nu \to 0$ as $\nu \to \infty$, there exists $\nu \geq 1$ such that

$$
(14) \qquad\qquad \frac{\gamma}{2\zeta_\nu} \leq q \leq \frac{\gamma}{2\zeta_{\nu+1}}.
$$

On the one hand, from the inequalities (13) and the upper bound in (14), we deduce that

$$
(15) \qquad \max_{1 \leq i \leq n} (\|\eta_i - q\theta_i\| \cdot |m_{\nu+1,i}|) > \frac{\gamma}{2n}.
$$

On the other hand, from the lower bound in (14) and the inequality (9), it follows that

$$
q \geq \frac{\gamma}{2} M_{\nu+1}.
$$

We deduce that

$$
(16) \qquad q^{j_i} \geq c|m_{\nu+1,i}| \qquad \forall i = 1, \ldots, n,
$$

where

$$
c = \min_{1 \leq i \leq n} \left( \left(\frac{\gamma}{2}\right)^{j_i} \right).
$$

Finally, by combining (15) and (16), we have that

$$
\max_{1 \leq i \leq n} (\|\eta_i - q\theta_i\| q^{j_i}) > \frac{\gamma c}{2n}.
$$

This concludes the proof of the proposition. □

We define the set

$$\mathcal{V} := \left\{ \eta \in \mathbb{R}^n \ : \ \inf_{\nu \geq 1} \|m_\nu \cdot \eta\| > 0 \right\}.$$

Clearly

(17) $$\mathcal{V} \subset \mathrm{Bad}_\theta(j_1, \ldots, j_n).$$

## 4. The Cantor-type set $K(R)$

In this section we construct the Cantor-type set $K(R)$ inside $\mathrm{Bad}_\theta(j_1, \ldots, j_n) \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \mathrm{Bad}(0, \ldots, 0, 1)$. In order to lighten the notation, throughout this section we denote by $\mathcal{M}$ the set of best approximations in the sequence (6), and for each $m \in \mathcal{M}$, by $M_m$ the quantity defined by (7), i.e.

$$M_m = \max_{1 \leq i \leq n} (|m_i|^{1/j_i}).$$

Hence

$$\mathcal{V} = \left\{ \eta \in \mathbb{R}^n \ : \ \inf_{m \in \mathcal{M}} \|m \cdot \eta\| > 0 \right\}.$$

We define the following partition of $\mathcal{M}$:

(18) $$\mathcal{M}_k := \left\{ m \in \mathcal{M} : R^{k-1} \leq M_m < R^k \right\} \qquad (k \geq 0).$$

Note that $\mathcal{M}_0 = \emptyset$. We have that $\mathcal{M} = \bigcup_{k=0}^\infty \mathcal{M}_k$.

We also need, for each $1 \leq i \leq n$, the following partitions of $\mathbb{N}$:

(19) $$\mathcal{Q}_k^{(i)} := \left\{ q \in \mathbb{N} : R^{(k-1)j_i/2} \leq q < R^{kj_i/2} \right\} \qquad (k \geq 0).$$

Note that $\mathcal{Q}_0^{(i)} = \emptyset$ and for each $1 \leq i \leq n$, we have that $\mathbb{N} = \bigcup_{k=0}^\infty \mathcal{Q}_k^{(i)}$.

At the heart of the construction of $K(R)$ is constructing a collection $\mathcal{F}_k$ of hyperrectangles $H_k$ inside the hypercube $[0, 1]^n$ that satisfy the following $n$ conditions:

(0) $|m \cdot \eta + p| \geq \epsilon \qquad \forall \eta \in H_k, \ \forall m \in \mathcal{M}_{k-1}, \ \forall p \in \mathbb{Z};$

(1) $q|q\eta_1 - p| \geq \epsilon \qquad \forall \eta \in H_k, \ \forall q \in \mathcal{Q}_{k-1}^{(1)}, \ \forall p \in \mathbb{Z};$

$$\vdots$$

(n) $q|q\eta_n - p| \geq \epsilon \qquad \forall \eta \in H_k, \ \forall q \in \mathcal{Q}_{k-1}^{(n)}, \ \forall p \in \mathbb{Z}$

for some $\epsilon > 0$.

We start by constructing a collection $(\mathcal{G}_k^{(0)})_{k \geq 0}$ of hyperrectangles satisfying condition (0). This construction is done by induction. Then we define a subcollection $\mathcal{G}_k^{(1)} \subset \mathcal{G}_k^{(0)}$ of hyperrectangles that also satisfy condition (1), a subcollection $\mathcal{G}_k^{(2)} \subset \mathcal{G}_k^{(1)}$ that also satisfies condition (2), etc. This

process ends with a subcollection $\mathcal{G}_k^{(n)}$ that satisfies the $n$ conditions above. We would like to quantify $\#\mathcal{G}_k^{(n)}$. We can give a lower bound, but we cannot quantify the exact cardinal. So we refine the collection $\mathcal{G}_k^{(n)}$ by choosing a right and final subcollection $\mathcal{F}_k$ that we can quantify.

Let

$$j_{\min} = \min_{1 \leq i \leq n}(j_i), \qquad j_{\max} = \max_{1 \leq i \leq n}(j_i).$$

Let $R > 4^{1/j_{\min}}$ and $\epsilon > 0$ be such that

(20) $$\epsilon < \frac{1}{2R^{2j_{\max}}}.$$

The parameter $R$ will be chosen later to be sufficiently large in order to satisfy various conditions.

4.1. **The collection $\mathcal{G}_k^{(0)}$.** For each $m \in \mathcal{M}$ and $p \in \mathbb{Z}$, let

$$\Delta(m, p) := \{x \in \mathbb{R}^n : |m \cdot x + p| < \epsilon\}.$$

Geometrically, $\Delta(m, p)$ is the thickening of a hyperplane of the form

(21) $$\mathcal{L}(m, p) := \{x \in \mathbb{R}^n : m \cdot x + p = 0\}$$

with width $2\epsilon/m_i$ in all the $x_i$-coordinate directions.

Next we describe the induction procedure in order to define the collection $(\mathcal{G}_k^{(0)})_{k \geq 0}$. We work within the closed hypercube $H_0 = [0, 1]^n$ and set $\mathcal{G}_0^{(0)} = \{H_0\}$. For $k \geq 0$, we divide each $H_k \in \mathcal{G}_k^{(0)}$ into new hyperrectangles $H_{k+1}$ of size

$$R^{-(k+1)j_1} \times \ldots \times R^{-(k+1)j_n}.$$

Note that if $R^{j_i} \notin \mathbb{Z}$ for some $1 \leq i \leq n$, the division will not be exact, in the sense that the new hyperrectangles will not cover $H_k$. This division gives at least $\prod_{i=1}^n [R^{j_i}] > R - \sum_{i=1}^n R^{j_i}$ new hyperrectangles. Among these new hyperrectangles, we denote by $\mathcal{G}^{(0)}(H_k)$ the collection of hyperrectangles $H_{k+1} \subset H_k$ satisfying

$$H_{k+1} \cap \Delta(m, p) = \emptyset \qquad \forall m \in \mathcal{M}_k, \forall p \in \mathbb{Z}.$$

We define

$$\mathcal{G}_{k+1}^{(0)} := \bigcup_{H_k \in \mathcal{G}_k^{(0)}} \mathcal{G}^{(0)}(H_k).$$

Hence $\mathcal{G}_{k+1}^{(0)}$ is nested in $\mathcal{G}_k^{(0)}$ and it is a collection of 'good' hyperrectangles with respect to all the best approximations $m$ satisfying $M_m < R^k$ and all the integers $p$. The collection $\mathcal{G}^{(0)}(H_k)$ is the collection of 'good' hyperrectangles that we obtain from the division of $H_k$.

Next we give a lower bound for $\#\mathcal{G}_k^{(0)}$. Actually, for a fixed hyperrect-angle $H_k \in \mathcal{G}_k^{(0)}$, we give a lower bound for the number of hyperrectangles $H_{k+1} \in \mathcal{G}^{(0)}(H_k)$. Alternatively, we give an upper bound for the number of 'bad' hyperrectangles in $H_k$; these are the hyperrectangles $H_{k+1} \subset H_k$ that intersect the thickening $\Delta(m, p)$ of some hyperplane $\mathcal{L}(m, p)$ with $m \in \mathcal{M}_k$. Fact 1 and Fact 2 bound the number of thickenings $\Delta(m, p)$ with $m \in \mathcal{M}_k$ and $p \in \mathbb{Z}$ that intersect $H_k$. Fact 3 bounds the number of hyperrectangles $H_{k+1} \subset H_k$ that are intersected by a thickening $\Delta(m, p)$ with $m \in \mathcal{M}_k$ and $p \in \mathbb{Z}$.

**Fact 1.** We show that for each $k \geq 1$, the set $\mathcal{M}_k$ contains at most $2 \cdot 3^n (1 + \log_2(R))$ best approximations. Indeed, lemma 2.2 implies that

$$M_{\nu + 2 \cdot 3^n (1 + \log_2(R))} \geq 2^{1 + \log_2(R)} M_\nu$$
$$\overset{(18)}{\geq} 2^{1 + \log_2(R)} R^{k-1}$$
$$> R^k.$$

Therefore, there are at most $2 \cdot 3^n (1 + \log_2(R))$ best approximations in $\mathcal{M}_k$.

**Fact 2.** Fix $m \in \mathcal{M}_k$. We show that there are at most $2^n n$ thickenings $\Delta(m, p)$ that intersect $H_k$. Indeed, suppose that two different thickenings $\Delta(m, p)$ and $\Delta(m, p')$ intersect the same edge of $H_k$. This edge of $H_k$ is a segment of a line which is parallel to an $x_l$-axis. Let $P = (y_1, \ldots, y_n)$ and $P' = (y'_1, \ldots, y'_n)$ denote the points of intersection of this line parallel to the $x_l$-axis with $\mathcal{L}(m, p)$ and $\mathcal{L}(m, p')$ respectively. The fact that $P$ and $P'$ respectively belong to $\mathcal{L}(m, p)$ and $\mathcal{L}(m, p')$ is described by the equations

$$(22) \qquad m \cdot y + p = 0, \qquad m \cdot y' + p' = 0.$$

The fact that $P$ and $P'$ both belong to a line parallel to the $x_l$-axis implies that $y_i = y'_i \; \forall i \neq l$. Hence, by substracting the second equation in (22) to the first one, we have that

$$(23) \qquad |y_l - y'_l| - \frac{2\epsilon}{|m_l|} \geq \frac{|p - p'|}{|m_l|} - \frac{2\epsilon}{|m_l|} > \frac{1}{R^{kj_l}} - \frac{1}{2R^{kj_l}} = \frac{1}{2} R^{-kj_l}.$$

Since the length size of $H_k$ in the $x_l$-direction is $R^{-kj_l}$, the inequality (23) implies that there are not more than two thickenings intersecting the same edge of $H_k$. Thus the number of thickenings $\Delta(m, p)$ that intersect $H_k$ is at most twice the number of edges of $H_k$, and this is $2^n n$.

**Fact 3.** Given a thickening $\Delta(m,p)$, we give an upper bound for the number of hyperrectangles $H_{k+1} \subset H_k$ that intersect $\Delta(m,p)$. Fix $m \in \mathcal{M}_k$ and $p \in \mathbb{Z}$. Denote by $l$ the index such that $M_m = |m_l|^{1/j_l}$. Consider the projection of $\Delta(m,p) \cap H_k$ onto one of the faces of $H_k$ parallel to the plane given by the $x_l$-axis and an $x_i$-axis. We split this projected of $\Delta(m,p) \cap H_k$ into right triangles with perpendicular sides of length $2\epsilon/|m_l|$ and $2\epsilon/|m_i|$ respectively. From this splitting and the inequality

$$\frac{2\epsilon}{|m_l|} < \frac{1}{2R^{j_l(k+1)}},$$

we deduce that $\Delta(m,p)$ intersects at most $2[R^{1-j_{\min}}]$ hyperrectangles $H_{k+1} \subset H_k$.

**Conclusion.** There are at most $[2^{n+2}3^n n(1 + \log_2(R))R^{1-j_{\min}}]$ hyperrectangles $H_{k+1} \subset H_k$ that intersect some $\Delta(m,p)$ with $m \in \mathcal{M}_k$, $p \in \mathbb{Z}$. Hence

$$\#\mathcal{G}^{(0)}(H_k) \geq R - \sum_{i=1}^{n} R^{j_i} - [2^{n+2}3^n n(1 + \log_2(R))R^{1-j_{\min}}].$$

4.2. **The subcollections $\mathcal{G}_k^{(i)}$.** For each $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, consider the sets

(24)     $\Gamma_i(q,p) := \{x \in \mathbb{R}^n : q|qx_i - p| < \epsilon\}$     $(1 \leq i \leq n)$.

Geometrically, each $\Gamma_i(q,p)$ is a thickening of a hyperplane described by the equation $x_i = p/q$ with width $2\epsilon/q^2$ in the $x_i$-coordinate direction.

We construct a tower of subcollections

$$\mathcal{G}_k^{(n)} \subset \mathcal{G}_k^{(n-1)} \subset \ldots \subset \mathcal{G}_k^{(1)} \subset \mathcal{G}_k^{(0)},$$

where each $\mathcal{G}_k^{(i)}$ consists of hyperrectangles in $\mathcal{G}_k^{(i-1)}$ which points avoid each thickening $\Gamma_i(q,p)$ for $q \in \mathcal{Q}_{k-1}^{(i)}$. More precisely, for $1 \leq i \leq n$, we form $\mathcal{G}_k^{(i)}$ by letting

$$\mathcal{G}^{(i)}(H_k) := \left\{ H_{k+1} \in \mathcal{G}^{(i-1)}(H_k) : H_{k+1} \cap \Gamma_i(q,p) = \emptyset \ \forall q \in \mathcal{Q}_k^{(i)} \right\}$$

and

$$\mathcal{G}_{k+1}^{(i)} := \bigcup_{H_k \in \mathcal{G}_k^{(i-1)}} \mathcal{G}^{(i)}(H_k).$$

Clearly the hyperrectangles in $\mathcal{G}_{k+1}^{(i)}$ satisfy the conditions (0),(1),...,(i), so the collection $\mathcal{G}_k^{(n)}$ satisfies the $n$ conditions (0),...,(n).

Next, for each $1 \leq i \leq n$ and $H_k \in \mathcal{G}_k^{(i-1)}$, we give a lower bound of $\#\mathcal{G}^{(i)}(H_k)$. Suppose that there are two pairs $(q,p)$ and $(q',p')$ in $\mathcal{Q}_k^{(i)} \times \mathbb{Z}$ such that

$$H_k \cap \Gamma_i(q,p) \neq \emptyset, \qquad H_k \cap \Gamma_i(q',p') \neq \emptyset.$$

In other words, suppose there exist $\eta, \eta'$ in $H_k$ such that

(25) $$q|q\eta_i - p| < \epsilon, \qquad q'|q'\eta_i' - p'| < \epsilon.$$

Then, by (19) and (20), we have

(26) $$\left| \frac{p}{q} - \frac{p'}{q'} \right| - \frac{\epsilon}{q^2} - \frac{\epsilon}{q'^2} \geq \frac{1}{qq'} - \frac{\epsilon}{q^2} - \frac{\epsilon}{q'^2} > \frac{1}{R^{kj_i}} - \frac{1}{2R^{kj_i}} = \frac{1}{2}R^{-kj_i}.$$

Since the length sides of $H_k$ in the $x_i$-direction is $R^{-kj_i}$, the inequality (26) implies that at most two thickenings of the form (24) can intersect $H_k$.

Now, from (19) and (20), it follows that if $\eta \in \Gamma_i(q,p)$, then

$$\left| \eta_i - \frac{p}{q} \right| < \frac{\epsilon}{q^2} < \frac{1}{2}R^{-kj_i},$$

which implies that each thickening $\Gamma_i(q,p)$ intersects at most

$$2[R^{j_1}] \times \ldots \times [\widehat{R^{j_i}}] \times \ldots \times [R^{j_n}] \leq 2[R^{1-j_i}]$$

hyperrectangles $H_{k+1} \subset H_k$.

Therefore, there are at most $4[R^{1-j_{\min}}]$ hyperrectangles $H_{k+1} \subset H_k$ that do not satisfy condition (i). Hence

(27) $$\#\mathcal{G}^{(i)}(H_k) \geq R - \sum_{i=1}^{n} R^{j_i} - [2^{n+2}3^n n(1+\log_2(R))R^{1-j_{\min}}] - 4i[R^{1-j_{\min}}].$$

## 4.3. The right subcollection $\mathcal{F}_k$.

We choose a subcollection of $\mathcal{G}_k^{(n)}$ that we can exactly quantify in the following way. Let $\mathcal{F}_0 := \mathcal{G}_0^{(0)}$. Choose $R$ sufficiently large so that $[R - \sum_{i=1}^{n} R^{j_i} - 2^{n+2}3^n n(1 + \log_2(R)) \cdot R^{1-j_{\min}} - 4nR^{1-j_{\min}}] > 1$. For $k \geq 0$, for each $H_k \in \mathcal{F}_k$, we choose exactly $[R - \sum_{i=1}^{n} R^{j_i} - 2^{n+2}3^n n(1 + \log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}]$ hyperrectangles from the collection $\mathcal{G}^{(n)}(H_k)$ and denote this collection by $\mathcal{F}(H_k)$. Trivially,

(28)
$$\#\mathcal{F}(H_k) = [R - \sum_{i=1}^{n} R^{j_i} - 2^{n+2}3^n n(1 + \log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}] > 1,$$

so each hyperrectangle $H_k \in \mathcal{F}_k$ gives rise to exactly the same number of hyperrectangles $H_{k+1}$ in $\mathcal{F}(H_k)$. Finally, define

$$\mathcal{F}_{k+1} := \bigcup_{H_k \in \mathcal{F}_k} \mathcal{F}(H_k).$$

This completes the construction of the Cantor-type set

$$K(R) := \bigcap_{k=0}^{\infty} \mathcal{F}_k.$$

By construction, we have $K(R) \subset \mathcal{V} \cap \mathrm{Bad}(1, 0 \ldots, 0) \cap \ldots \cap \mathrm{Bad}(0, \ldots, 0, 1)$. Moreover, in view of (28), we have

(29)

$$\# \mathcal{F}_{k+1} = \# \mathcal{F}_k \, \# \mathcal{F}(H_k)$$

$$(30) \qquad = [R - \sum_{i=1}^{n} R^{j_i} - 2^{n+2} 3^n n (1 + \log_2(R)) R^{1-j_{\min}} - 4nR^{1-j_{\min}}]^{k+1}.$$

## 5. THE MEASURE $\mu$ ON $K(R)$

We now describe a probablity measure $\mu$ supported on the Cantor-type set $K(R)$ constructed in the previous section. The measure we define is analogous to the probability measure used in [21] and [2] on a Cantor-type set of $\mathbb{R}^2$. For any hyperrectangle $H_k \in \mathcal{F}_k$ we attach a weight $\mu(H_k)$ which is defined recursively as follows: for $k = 0$,

$$\mu(H_0) = \frac{1}{\# \mathcal{F}_0} = 1$$

and for $k \geq 1$,

$$\mu(H_k) = \frac{1}{\# \mathcal{F}(H_{k-1})} \mu(H_{k-1}) \qquad (H_k \in \mathcal{F}(H_{k-1})).$$

This procedure defines inductively a mass on any hyperrectangle used in the construction of $K(R)$. Moreover, $\mu$ can be further extended to all Borel subsets $X$ of $\mathbb{R}^n$, so that $\mu$ actually defines a measure supported on $K(R)$, by letting

$$\mu(X) = \inf \sum_{H \in \mathcal{C}} \mu(H)$$

where the infimum is taken over all coverings $\mathcal{C}$ of $X$ by rectangles $H \in \{\mathcal{F}_k : k \geq 0\}$. For further details, see [9], Proposition 1.7.

Notice that, in view of (29), we have

$$\mu(H_k) = \frac{1}{\# \mathcal{F}_k} \qquad (k \geq 0).$$

A classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set is the following mass distribution principle (see [9] p. 55).

**Lemma 5.1** (mass distribution principle)**.** *Let $\delta$ be a probability measure supported on a subset $X$ of $\mathbb{R}^n$. Suppose there are positive constants $c, s$ and $l_0$ such that*

$$\delta(S) \leq cl^s \tag{31}$$

*for any hypercube $S \subset \mathbb{R}^n$ with side length $l \leq l_0$. Then $\dim(X) \geq s$.*

The goal in the next section is to prove that there exist constants $c$ and $l_0$ satisfying (31) with $\delta = \mu$, $X = K(R)$ and $s = n - \lambda(R)$, where $\lambda(R) \to 0$ as $R \to \infty$. Then from the mass distribution principle it will follow that $\dim(K(R)) = n$.

## 6. A LOWER BOUND FOR $\dim(K(R))$

Recall that

$$j_{\min} = \min_{1 \leq i \leq n} (j_i).$$

Let $k_0$ be a positive integer such that

$$R^{-kj_i} < R^{-(k+1)j_{\min}} \qquad \forall j_i \neq j_{\min} \text{ and } k \geq k_0. \tag{32}$$

Consider an arbitrary hypercube $S$ of side length $l \leq l_0$ where $l_0$ satisfies

$$l_0 < R^{-(k_0+1)j_{\min}} \tag{33}$$

together with a second inequality to be determined later. We can choose $k > k_0$ so that

$$R^{-(k+1)j_{\min}} < l < R^{-kj_{\min}}. \tag{34}$$

From the inequality (32) it follows that

$$l > R^{-kj_i} \qquad \forall j_i \neq j_{\min}. \tag{35}$$

Then it is easy to see that $S$ intersects at most $2^n l^{n-1} \prod_{j_i \neq j_{\min}} R^{kj_i}$ hyperrectangles $H_k \in \mathcal{F}_k$, so

$$\mu(S) \leq 2^n l^{n-1} \prod_{j_i \neq j_{\min}} R^{kj_i} \mu(H_k) = 2^n l^{n-1} R^{k-kj_{\min}} \frac{1}{\#\mathcal{F}_k}.$$

Since $R^{(k+1)j_{\min}} > l^{-1}$ (see (34)), we have that

$$\mu(S) \leq 2^n l^n R^{j_{\min}} R^k \frac{1}{\#\mathcal{F}_k}.$$

Remember that we mentioned in Section 3 that later we would choose the parameter $R$ big enough so that it satisfies various conditions. We choose $R$ so that

$$R^{-1} \sum_{i=1}^{n} R^{j_i} - 2^{n+2} 3^n n (1 + \log_2(R)) R^{-j_{\min}} - 4n R^{-j_{\min}} - R^{-1} \leq 2^{-1}.$$

Then, by (29) we have that

$$\mu(S) \leq 2^n l^n R^{j_{\min}} 2^k.$$

We choose

$$k \geq \log(R) \quad \text{and} \quad \lambda(R) = \frac{1 + \log(2)}{j_{\min} \log(R)},$$

so

$$\mu(S) \leq 2^n l^n R^{k j_{\min} \lambda(R)}.$$

Since $R^{k j_{\min}} < l^{-1}$ (see (34)), it follows that

$$\mu(S) \leq 2^n l^{n - \lambda(R)}.$$

Finally, by applying the mass distribution principle we obtain

$$\dim K(R) \geq n - \lambda(R) \to n \qquad \text{as } R \to \infty.$$

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Department of Mathematics, University of York, York, YO10 5DD, United Kingdom
*E-mail address*: paloma.bengoechea@york.ac.uk

Department of Mathematics and Mechanics, Moscow State University, Leninskie Gory 1, GZ MGU, 119991 Moscow, Russia
*E-mail address*: moshchevitin@rambler.ru