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Hitting probabilities of random covering sets in tori and metric spaces*

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Abstract

We provide sharp lower and upper bounds for the Hausdorff dimension of the intersection of a typical random covering set with a fixed analytic set both in Ahlfors regular metric spaces and in the d-dimensional torus. In metric spaces, we consider covering sets generated by balls and, in tori, we deal with general analytic generating sets.

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1 Introduction

1.1 Background

Given a set X and a sequence of subsets $Z_n \subset X$, $n \in \mathbb{N}$, a general covering problem asks when X is covered or when X is covered infinitely often by the sets Z_n. If X is not covered, it is natural to study the size and structure of the uncovered set $X \setminus \bigcup_{n\in\mathbb{N}} Z_n$ as well as the limsup set $\limsup_{n\to\infty} Z_n = \bigcap_{k\in\mathbb{N}} \bigcup_{n=k}^{\infty} Z_n$. This kind of problems arise in many different fields of mathematics. For example, the generating sets Z_n can come from a predescribed, arbitrary sequence as in [3, 56]. Given more structure to the

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sequence, finer information on the limsup set can be obtained, as demonstrated for instance in the situations where the sets are defined dynamically [22, 23, 43], through shrinking targets [28] or in relation to continued fractions [8, 42].

A classical problem of this sort, and perhaps a great motivator in the study of limsup sets, is the problem of Diophantine approximation. In Diophantine approximation, the object under study is the set of points that can be approximated with a given approximation speed $\psi\colon \mathbb{N} \to \mathbb{R}_+$ by rationals, that is, the set

$$\mathcal{W}(\psi) = \{x \in \mathbb{R} \mid |x - \tfrac{p}{q}| \leq \psi(q) \text{ for infinitely many coprime pairs } (p,q) \in \mathbb{Z} \times \mathbb{N}\}. \quad (1.1)$$

This is a limsup set of shrinking balls centred at rational points, see [25, 53]. The question of which irrational points are 'close' to rational points is equally interesting in higher ambient dimensions, and in that case there is a certain amount of freedom in choosing the shapes of the generating sets, leading to various kinds of interesting problems. For example, approximation by cubes corresponds to simultaneous approximation at the same speed in all directions, aligned rectangles correspond to different approximation speeds and multiplicative approximation leads to approximating sets given by hyperbolic regions [1, 12, 50]. Lately, there has been an increasing amount of interest in the Diophantine approximation properties of points in fixed subsets of $\mathbb{R}^d$, for example, manifolds [2] or fractal sets [18, 57].

A further approach, and the one we will concentrate on, is to let the sets Z_n be random. In this context, the limsup set $E = \bigcap_{k\in\mathbb{N}} \bigcup_{n=k}^{\infty} Z_n$ is usually referred to as a *random covering set*. The study of random covering sets has a long and convoluted history; we refer the interested reader to [15, 20, 21, 33, 35, 36, 29, 51, 52] for the wide variety of interesting problems.

In the present paper, we study the *hitting probabilities* of a random covering set E, namely, the probability that E intersects a given subset $F \subset X$. For some classical random sets, such as Brownian paths and fractal percolation limit sets, the study of the hitting probabilities and the size of the intersections goes back to Dvoretzky, Erdös, Kakutani and Taylor [16, 17], Hawkes [26, 27] and Lyons [44], see also [33, 48]. When the randomness appears as a random transformation of a given subset of the Euclidean space, such results originate from the pioneering works of Kahane [34] and Mattila [45], see also [46]. A recent line of research concerns replacing the fixed set F by a suitable parameterised family Γ of sets and showing that $\mathbb{P}(E \cap F \neq \emptyset \text{ for all } F \in \Gamma) > 0$. See [54] and references therein.

The structures of the random covering sets and more general limsup random fractals are different from those considered in the above references. For instance, for random covering sets, the packing and Hausdorff dimensions are typically different. The hitting probabilities of a family of discrete random limsup sets were studied by Khoshnevisan, Peres and Xiao in [38], with applications to fast points of Brownian motion and other stochastic processes. For further results concerning dimensional properties of limsup random fractals, see [9, 11, 47, 58].

Applying the method of [38], Li, Shieh and Xiao [40] investigated the intersection properties of random covering sets in the circle $\mathbb{T}^1$. It is proved in [40] that if $(r_n)_{n\in\mathbb{N}}$ is a given sequence of positive numbers and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is a sequence of independent uniformly distributed random variables on the circle $\mathbb{T}^1$, then, denoting the closed ball with radius r and centre x by $B(x,r)$, the random covering set

$$E(\mathbf{x}) = \limsup_{n\to\infty} B(x_n, r_n)$$

avoids a given analytic set $F \subset \mathbb{T}^1$ for almost all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ provided that $\dim_{\mathrm{P}} F < 1 - \dim_{\mathrm{H}} E(\mathbf{x})$, while $E(\mathbf{x}) \cap F \neq \emptyset$ for almost all $\mathbf{x}$ if $\dim_{\mathrm{P}} F > 1 - \dim_{\mathrm{H}} E(\mathbf{x})$ and if $(r_n)_{n\in\mathbb{N}}$ satisfies a mild technical assumption. Here, $\dim_{\mathrm{H}} E(\mathbf{x}) = \inf\{s \leq 1 \mid$

$\sum_{n=1}^{\infty}(r_n)^s < \infty\}$ for almost all $\mathbf{x}$. In the latter case, [40] contains estimates for the Hausdorff dimension of $E(\mathbf{x}) \cap F$.

Several authors have considered random covering problems in the context of metric spaces (see [36] and the references therein), but their emphasis has been on the case of full covering and on the size of the uncovered set $X \setminus \bigcup_{n\in\mathbb{N}} Z_n$. To the best of our knowledge, the intersection properties of the random covering sets have been studied only for randomly placed balls on tori [13, 14, 40, 41]. This has motivated us to investigate hitting probabilities of random covering sets on more general metric spaces.

1.2 Overview of results and methods

This work contains two types of results on hitting probabilities of random covering sets. On one hand, we will consider the case when X is a compact Ahlfors regular metric space and $Z_n = B(x_n, r_n)$, where $(r_n)_{n\in\mathbb{N}}$ is a deterministic sequence of positive real numbers and $(x_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables distributed according to an Ahlfors regular probability measure μ on X. On the other hand, we will consider the random covering sets

$$\limsup_{n\to\infty}(x_n + A_n) \tag{1.2}$$

in the d-dimensional torus $\mathbb{T}^d$, where $(A_n)_{n\in\mathbb{N}}$ is a fixed sequence of analytic sets and $(x_n)_{n\in\mathbb{N}}$ are independent and uniformly distributed random variables in $\mathbb{T}^d$. Finally, in $\mathbb{T}^d$ we will also consider a model where the deterministic sets A_n are, in addition to translating, also randomly rotated.

Denoting the underlying space ($\mathbb{T}^d$ or a more general metric space) by X and, for all $\mathbf{x} \in X^{\mathbb{N}} =: \Omega$, the random covering set by $E(\mathbf{x})$, we will be interested in the probability that the random covering set intersects a fixed analytic set $F \subset X$. That is, we want to study the hitting probability

$$\mathbb{P}(\{\mathbf{x} \in \Omega \mid E(\mathbf{x}) \cap F \neq \emptyset\}),$$

and, if this is positive, we shall determine almost sure upper and lower bounds for $\dim_{\mathrm{H}}(E(\mathbf{x}) \cap F)$.

In Section 2, we will discuss the results of Li, Shieh and Xiao [40] in detail, and extend them from the Euclidean setting to Ahlfors regular metric spaces (see Theorems 2.2 and 2.4). These theorems give sharp lower and upper bounds for the Hausdorff dimension of the intersection of a typical random covering set with a fixed analytic set F in terms of the Hausdorff and packing dimensions of F and the covering sets. We remark that, in [40], Li, Shieh and Xiao derived their results from hitting probability estimates for the discrete limsup random fractals obtained in [38] and their proof could also be extended to metric spaces. However, in Section 2, we will give new proofs which avoid the use of the discrete limsup fractals. In Section 3.3, we consider a class of carpet type affine covering sets to demonstrate how our results apply in the Euclidean setting also when the results of [40] do not.

For the methods of the proof in Section 2, as well as in [40], it is essential that the generating sets are balls or ball-like. In Section 3, we consider the case of analytic generating sets $(A_n)_{n\in\mathbb{N}}$ in the d-dimensional torus and the random covering sets defined in (1.2). With methods completely different from those in Section 2, applying classical intersection results of Mattila [45], we prove in Section 3 that the upper bounds given in Theorems 2.2 and 2.4 are valid also in this general setting (see Theorem 3.2). However, the counterparts of the lower bounds are not true in this generality as shown by Example 3.3. To overcome the problem presented in Example 3.3, at the end of Section 3, we give a modification of the model where the generating sets are randomly rotated. In this modified model, under a classical extra assumption on the dimensions, a sharp almost sure lower bound for the Hausdorff dimension of the intersection of a random covering set with a fixed analytic set is discovered. For full details, see Theorem 3.8.

Intersection properties of random sets very similar to ours have been under consideration in the context of Diophantine approximation, see Bugeaud and Durand [6]. They use them to support a conjecture [6, Conjecture 1] related to the irrationality exponent of points in the middle third Cantor set. We discuss this connection in detail in Remarks 2.6 and 3.10.

2 Random covering sets in metric spaces

2.1 Notations and definitions

In this section, we consider random covering sets in the context of t-regular metric spaces. Assume that (X,ρ) is a compact metric space endowed with a Borel probability measure μ which is Ahlfors t-regular for some constant $0 < t < \infty$. Recall that μ is Ahlfors t-regular if there exists a constant $0 < C < \infty$ such that

$$C^{-1}r^t \le \mu(B(x,r)) \le Cr^t \tag{2.1}$$

for all $x \in X$ and $0 < r < \operatorname{diam} X$, where $B(x,r)$ is the closed ball in metric ρ centred at x with radius r and $\operatorname{diam} X$ is the diameter of X. Given $A \subset X$ and $\varepsilon > 0$, an *ε-net of A* is a subset $Y \subset A$ such that $A \subset \bigcup_{y\in Y} B(y,\varepsilon)$ and $\rho(x,y) \ge \varepsilon$ for all $x,y \in Y$ with $x \ne y$. We denote Hausdorff, packing and upper box counting dimensions in the metric space (X,ρ) by $\dim_{\mathrm{H}}$, $\dim_{\mathrm{P}}$ and $\overline{\dim}_{\mathrm{B}}$, respectively. In a couple of places we have two different metrics in the same space. Then we put the metric as a superscript to emphasise which metric is used to calculate the dimension.

In X, we will need an analogue of dyadic cubes, which we define next. For all integers $n \ge 1$, assume that $\mathcal{Q}_n = \{Q_{n,i}\}_{i\in I_n}$ is a finite family of pairwise disjoint Borel subsets of X such that $\bigcup_{i\in I_n} Q_{n,i} = X$ and, moreover, for each $Q_{n,i} \in \mathcal{Q}_n$, there is $x_{n,i} \in X$ such that

$$B(x_{n,i}, 2^{-n}) \subset Q_{n,i} \subset B(x_{n,i}, C2^{-n}), \tag{2.2}$$

where $C > 0$ is a universal constant. Further, we assume that the families $\mathcal{Q}_n$ are nested: for $i \ne j$ and $m \ge n$, either $Q_{m,j} \subset Q_{n,i}$ or $Q_{m,j} \cap Q_{n,i} = \emptyset$. For convenience, we also define $\mathcal{Q}_0 = \{X\}$. Set $\mathcal{Q} = \bigcup_{n=0}^{\infty} \mathcal{Q}_n$. We recall that, starting from a nested family of (2^{-n})-nets, such finite families $\mathcal{Q}_n$ may be constructed in any metric space which satisfies a mild doubling condition, in particular, in any t-regular metric space, see [32].

Next we define random covering sets. Let $(\Omega,\mathcal{A},\mathbb{P})$ be the completion of the infinite product of $(X,\mathcal{B}(X),\mu)$, where $\mathcal{B}(X)$ is the Borel σ-algebra on X. Let $(r_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers tending to zero. For all $\mathbf{x} \in \Omega$, the covering set is defined as

$$E(\mathbf{x}) = \limsup_{n\to\infty} B(x_n,r_n) = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} B(x_n,r_n).$$

2.2 Dimension and hitting probabilities of random covering sets

It is easy to see that $\mu(E(\mathbf{x})) = 0$ for all $\mathbf{x} \in \Omega$ if $\sum_{n=1}^{\infty}(r_n)^t < \infty$, whereas by the Borel-Cantelli lemma and Fubini's theorem, $\mu(E(\mathbf{x})) = 1$ for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$ if $\sum_{n=1}^{\infty}(r_n)^t = \infty$. There is a concrete formula for the almost sure Hausdorff dimension of the random covering set:

Theorem 2.1. *For $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$,*

$$\dim_{\mathrm{H}} E(\mathbf{x}) = \inf\left\{ s \le t \mid \sum_{n=1}^{\infty}(r_n)^s < \infty \right\} = \limsup_{n\to\infty}\frac{\log n}{-\log r_n},$$

with the convention $\inf\emptyset = t$.

This result is well known if X is the d-dimensional torus $\mathbb{T}^d$, and similar techniques can be used to extend the result to t-regular metric spaces. For instance, applying the mass transference principle [3, Theorem 3] by Beresnevich and Velani, the almost sure value of $\dim_{\mathrm{H}} E(\mathbf{x})$ can be determined exactly as in [30, Proposition 4.7]. We note that Theorem 2.1 follows also as a special case of Corollary 2.5 below.

For later reference, let us define

$$\alpha = \alpha(r_n) = \min\left\{t, \limsup_{n\to\infty} \frac{\log n}{-\log r_n}\right\}.$$

Further, we set

$$\mathcal{N}_k = \{n \in \mathbb{N} \mid 2^{-(k+1)} \le r_n < 2^{-k}\} \text{ and } n_k = \#\mathcal{N}_k, \tag{2.3}$$

where the number of elements in a set A is denoted by $\#A$. Finally, for all analytic sets $F \subset X$, we define

$$\mathcal{H}(F) = \{\mathbf{x} \in \Omega \mid E(\mathbf{x}) \cap F \neq \emptyset\}.$$

In some of our results, we need to assume that there exists an increasing sequence of positive integers $(k_i)_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \frac{k_{i+1}}{k_i} = 1 \text{ and } \lim_{i\to\infty} \frac{\log_2 n_{k_i}}{k_i} = \alpha. \tag{2.4}$$

This condition essentially means that, in a weak asymptotic sense, the sequence $(r_n)_{n\in\mathbb{N}}$ behaves like $n^{-1/\alpha}$, see [40] for various examples.

Now we are ready to state our first main theorem of this section.

Theorem 2.2. *Let $F \subset X$ be an analytic set. With the above notation, we have*

$$\mathbb{P}(\mathcal{H}(F)) = 0 \textit{ if } \dim_{\mathrm{P}} F < t - \alpha, \tag{2.5}$$

$$\mathbb{P}(\mathcal{H}(F)) = 1 \textit{ if } \dim_{\mathrm{H}} F > t - \alpha \textit{ and} \tag{2.6}$$

$$\mathbb{P}(\mathcal{H}(F)) = 1 \textit{ if } \dim_{\mathrm{P}} F > t - \alpha \textit{ and } (2.4) \textit{ holds.} \tag{2.7}$$

In the circle $\mathbb{T}^1$, the analogues of (2.5) and (2.7) were established by Li, Shieh and Xiao [40]. Bugeaud and Durand [6] also recovered these results within the context of Diophantine approximation. Li and Suomala [41] proved the analogue of (2.6) in the torus $\mathbb{T}^d$ and showed that the assumption $\dim_{\mathrm{P}} F > d - \alpha$ alone is not enough to guarantee that $\mathbb{P}(\mathcal{H}(F)) > 0$.

Proof of Theorem 2.2. We start from the claim (2.5). For each $m \in \mathbb{N}$, let Y_m be a (2^{-m})-net of X and $N_m = \#Y_m$. Since X is compact and t-regular, there is a constant $0 < C_1 < \infty$ such that

$$C_1^{-1} 2^{tm} \le N_m \le C_1 2^{tm}$$

for all $m \in \mathbb{N}$. Using the fact that the packing dimension is equal to the modified upper box counting dimension, which is due to Tricot [55] (see also [19, Proposition 3.8]), that is,

$$\dim_{\mathrm{P}} F = \inf\left\{\sup_n \overline{\dim}_{\mathrm{B}} F_n \mid F \subset \bigcup_{n=1}^{\infty} F_n\right\},$$

it suffices to show that $\mathbb{P}(\{\mathbf{x} \in \Omega \mid E(\mathbf{x}) \cap F \neq \emptyset\}) = 0$ whenever $\overline{\dim}_{\mathrm{B}} F < t - \alpha$.

Let $\overline{\dim}_{\mathrm{B}} F < \gamma < t - \alpha$, $\alpha < \beta < t - \gamma$ and

$$M_m = \#\{y \in Y_m \mid F \cap B(y, 2^{-m}) \neq \emptyset\}.$$

Then $M_m < 2^{m\gamma}$ and $n_m < 2^{m\beta}$ for all m large enough, say $m \ge N_0$ (recall (2.3)). Denote by $B_{m,1}, \dots, B_{m,M_m}$ the balls among $\{B(y, 2^{-m})\}_{y\in Y_m}$ which intersect F. For all

$i = 1, \dots, M_m$ and $n \in \mathcal{N}_m$, we have $B(x_n, r_n) \cap B_{m,i} \neq \emptyset$ only if $d(x_n, y) \leq 2^{-m+1}$, where y is the centre of $B_{m,i}$. Since x_n is distributed according to μ, (2.1) implies the existence of $0 < C_2 < \infty$ such that

$$\mathbb{P}(\{\mathbf{x} \in \Omega \mid B(x_n, r_n) \cap B_{m,i} \neq \emptyset\}) \leq C_2 2^{-mt}.$$

Whence,

$$\begin{aligned}&\mathbb{P}\big(\{\mathbf{x} \in \Omega \mid B(x_n, r_n) \cap B_{m,i} \neq \emptyset \text{ for some } i \in \{1, \dots, M_m\}, n \in \mathcal{N}_m\}\big)\\ &\leq C_2 n_m M_m 2^{-mt} \leq C_2 2^{m(\gamma+\beta-t)}.\end{aligned}$$

Since $\gamma + \beta - t < 0$, it follows from the Borel-Cantelli lemma that for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$,

$$\bigcup_{n \in \mathcal{N}_m} B(x_n, r_n) \cap F = \emptyset$$

for all large enough m, implying that $E(\mathbf{x}) \cap F = \emptyset$.

The proof of (2.6) given in [41] for the case $X = \mathbb{T}^d$ can be generalised in a straightforward way to the current setting (replace the dyadic cubes by the generalised dyadic cubes $\mathcal{Q}$ throughout). We will not repeat the details.

To prove (2.7), let $F \subset X$ with $\dim_{\mathrm{P}} F > t - \alpha$. Replacing F by a subset if necessary, we may assume that F is compact and that $\dim_B(V \cap F) > t - \alpha$ whenever V is an open set with $V \cap F \neq \emptyset$ (see [31]). It suffices to show that

$$\mathbb{P}(\{\mathbf{x} \in \Omega \mid U(x_n, r_n) \cap V \cap F \neq \emptyset \text{ for infinitely many } n \in \mathbb{N}\}) = 1, \tag{2.8}$$

where $U(x_n, r_n)$ is the interior of $B(x_n, r_n)$ and $V \subset X$ is an open set such that $V \cap F \neq \emptyset$. Indeed, if this holds, letting V run over a countable base of the topology of X, it follows that for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$, the set

$$F \cap \bigcup_{n=k}^{\infty} U(x_n, r_n)$$

is open and dense in F for all $k \in \mathbb{N}$. Therefore, by the Baire's category theorem,

$$F \cap \limsup_{n \to \infty} B(x_n, r_n) \supset F \cap \limsup_{n \to \infty} U(x_n, r_n) \neq \emptyset.$$

It remains to prove (2.8). Fix $V \subset X$, let $(k_i)_{i \in \mathbb{N}}$ be as in (2.4) and define random variables

$$S_i(\mathbf{x}) = \#\{n \in \mathcal{N}_{k_i} \mid U(x_n, r_n) \cap V \cap F \neq \emptyset\}$$

for all $i \in \mathbb{N}$. Pick γ and β such that $\overline{\dim}_{\mathrm{B}}(V \cap F) > \gamma > t - \beta > t - \alpha$. For all $m \in \mathbb{N}$, let Z_m be a (2^{-m})-net of $V \cap F$. Replacing $(k_i)_{i \in \mathbb{N}}$ by a subsequence, if necessary, we may conclude by the first part of (2.4) that

$$M_{k_i} := \# Z_{k_i} > 2^{\gamma k_i} \text{ for all } i \in \mathbb{N}. \tag{2.9}$$

By (2.1), there exists a constant $0 < C_3 < \infty$ such that, for all $n \in \mathcal{N}_{k_i}$ and $z \in Z_{k_i}$, we have

$$\begin{aligned}C_3^{-1} 2^{-tk_i} \leq \mathbb{P}(\{\mathbf{x} \in \Omega \mid z \in U(x_n, r_n)\}) \text{ and}\\ \mathbb{P}(\{\mathbf{x} \in \Omega \mid z \in U(x_n, 3r_n)\}) \leq C_3 2^{-tk_i}.\end{aligned} \tag{2.10}$$

Since X is t-regular and Z_{k_i} is a (2^{-k_i})-net, there is a constant $0 < C_4 < \infty$ independent of $i \in \mathbb{N}$ such that any ball of radius 2^{-k_i} contains at most C_4 points $z \in Z_{k_i}$. Combining this with (2.10), yields

$$\begin{aligned}C_5^{-1} M_{k_i} 2^{-tk_i} \leq \mathbb{P}(\{\mathbf{x} \in \Omega \mid Z_{k_i} \cap U(x_n, r_n) \neq \emptyset\}) \text{ and}\\ \mathbb{P}(\{\mathbf{x} \in \Omega \mid Z_{k_i} \cap U(x_n, 3r_n) \neq \emptyset\}) \leq C_5 M_{k_i} 2^{-tk_i},\end{aligned} \tag{2.11}$$

where $C_5 = C_3C_4$. Note that $U(x_n,r_n)\cap V\cap F\neq\emptyset$ only if $U(x_n,3r_n)\cap V\cap Z_{k_i}\neq\emptyset$. Applying (2.11) for all $n\in\mathcal{N}_{k_i}$, we conclude that

$$C_5^{-1}n_{k_i}M_{k_i}2^{-tk_i}\leq\mathbb{E}(S_i)\leq C_5n_{k_i}M_{k_i}2^{-tk_i}. \tag{2.12}$$

To estimate the second moment $\mathbb{E}(S_i^2)$, note that the events $Z_{k_i}\cap U(x_n,r_n)\neq\emptyset$ are independent for different $n\in\mathbb{N}$. Thus, by (2.11),

$$\begin{aligned}\mathbb{E}(S_i^2)\leq&\sum_{p\in\mathcal{N}_{k_i}}\mathbb{P}(\{\mathbf{x}\in\Omega\mid Z_{k_i}\cap U(x_p,3r_p)\neq\emptyset\})\\ &+\sum_{\substack{p,q\in\mathcal{N}_{k_i}\\ p\neq q}}\mathbb{P}(\{\mathbf{x}\in\Omega\mid Z_{k_i}\cap U(x_p,3r_p)\neq\emptyset\})\mathbb{P}(\{\mathbf{x}\in\Omega\mid Z_{k_i}\cap U(x_q,3r_q)\neq\emptyset\})\\ &\leq C_5n_{k_i}M_{k_i}2^{-tk_i}+(C_5)^2(n_{k_i})^2(M_{k_i})^22^{-2tk_i}.\end{aligned}\tag{2.13}$$

Since $\beta<\alpha$, we have that $n_{k_i}\geq 2^{k_i\beta}$ for all large $i\in\mathbb{N}$ by (2.4). Recalling (2.9) and $\gamma+\beta>t$, we observe that the second term in the upper bound in (2.13) is dominating. Whence, applying the Paley-Zygmund inequality, we conclude the existence of a constant $0<C_6<\infty$ satisfying

$$\mathbb{P}(\{\mathbf{x}\in\Omega\mid S_i(\mathbf{x})>0\})\geq\frac{\mathbb{E}(S_i)^2}{\mathbb{E}(S_i^2)}\geq C_6>0$$

for all $i\in\mathbb{N}$. Since the random variables $(S_i)_{i\in\mathbb{N}}$ are independent, the Borel-Cantelli lemma implies that for $\mathbb{P}$-almost all $\mathbf{x}\in\Omega$, $S_i(\mathbf{x})>0$ for infinitely many $i\in\mathbb{N}$, completing the proof of (2.8). □

We will next investigate the dimension of the intersection $E(\mathbf{x})\cap F$ aiming to generalise [40, Theorem 2.4] to the metric setting. We will need the following lemma, which is well known in the case $t\in\mathbb{N}$ and $X=[0,1]^t$. In the metric setting, we derive the result for a version of fractal percolation using results of Lyons [44] on tree percolation. Although we need this only when X is t-regular, we note that the proof works for any complete metric space satisfying a mild doubling condition.

Lemma 2.3. *For any $s>0$, there is a probability space $(\widetilde{\Omega},\Gamma,\widetilde{\mathbb{P}})$ and, for each $\omega\in\widetilde{\Omega}$, a compact set $A(\omega)\subset X$ such that, for any analytic set $E\subset X$, we have*

$$\widetilde{\mathbb{P}}(\{\omega\in\widetilde{\Omega}\mid A(\omega)\cap E=\emptyset\})=1$$

provided $\dim_{\mathrm{H}}E<s$, while

$$\|\dim_{\mathrm{H}}(A(\omega)\cap E)\|_{L^\infty(\widetilde{\mathbb{P}})}=\dim_{\mathrm{H}}E-s$$

if $\dim_{\mathrm{H}}E\geq s$.

Proof. The model example of such random family of sets in the case $X=[0,1]^d$ is given by a fractal percolation process and we will prove the lemma by constructing an analogue of the fractal percolation in the space X by using the generalised dyadic cubes $\mathcal{Q}$ as defined in the beginning of this section.

Let $p=2^{-s}$. For each $Q\in\mathcal{Q}$, we attach a Bernoulli random variable $Z(Q)$ taking value 1 with probability p and value 0 with probability $1-p$. Further, we assume that these random variables are independent for different $Q\in\mathcal{Q}$. We define the random fractal percolation set as

$$A=\bigcap_{n\in\mathbb{N}}\bigcup_{\substack{Q\in\mathcal{Q}_n\\ Z(Q)=1}}\overline{Q},$$

where $\overline{Q}$ is the closure of the set Q. Formally, $\widetilde{\Omega} = \{0,1\}^{\mathcal{Q}}$, Γ is the completion of the Borel σ-algebra on $\widetilde{\Omega}$ and $\widetilde{\mathbb{P}}$ is the infinite product over $\mathcal{Q}$ of the measures $(1-p)\delta_0 + p\delta_1$. There is an apparent tree structure behind the definition of A. Label each $Q \in \mathcal{Q}$ with a vertex v_Q and let T be a graph with vertex set $\{v_Q\}_{Q\in\mathcal{Q}}$. Draw an edge between vertices $v_{Q_{n,i}}$ and $v_{Q_{m,j}}$ if and only if $|n-m| = 1$ and $Q_{n,i} \cap Q_{m,i} \neq \emptyset$. Then T is a tree and we distinguish v_X as its root. The boundary of the tree ∂T consists of all infinite paths $v_0v_1v_2\ldots$, where $v_m = v_{Q_m}$ for some $Q \in \mathcal{Q}_m$ and $Q_m \subset Q_n$, if $m \geq n$. Define a projection $\Pi\colon \partial T \to X$ as $\{\Pi(v_0v_1v_2\ldots)\} = \bigcap_{n=0}^{\infty} \overline{Q_n}$. Note that $\Pi(\partial T) = X$. Then the law of A given by $\widetilde{\mathbb{P}}$ is the same as that of $\Pi(\widetilde{A})$, where $\widetilde{A} \subset \partial T$ is determined by the component of the root in the Bernoulli p-percolation on the tree T (see [44]).

For $v = v_0v_1v_2\ldots, u = u_0u_1u_2\ldots \in \partial T$, we define

$$\kappa(v,u) = \begin{cases} 0 & \text{if } v = u \\ 2^{-\min\{i \mid v_i \neq u_i\}} & \text{otherwise.} \end{cases}$$

Then $(\partial T, \kappa)$ becomes a metric space. Using (2.2), we find a constant $0 < C < \infty$ such that $\operatorname{diam}(Q) \leq C2^{-n}$ for all $n \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$. Further, by (2.1), there exists a constant $0 < \widetilde{C} < \infty$ such that for all $2^{-n} \leq r < 2^{-n+1}$ and $x \in X$, the ball $B(x,r)$ may be covered by $\widetilde{C}$ elements $Q \in \mathcal{Q}_n$. From these observations it readily follows that if $E \subset X$ then

$$\dim_{\mathrm{H}} E = \dim_{\mathrm{H}}^{\kappa} \Pi^{-1}(E), \tag{2.14}$$

where $\dim_{\mathrm{H}}^{\kappa}$ is the Hausdorff dimension with respect to the metric κ. Whence, to finish the proof, we only need to verify that, for any analytic set $F \subset \partial T$, we have

$$\begin{aligned} \widetilde{\mathbb{P}}(\{\omega \in \widetilde{\Omega} \mid \widetilde{A} \cap F = \emptyset\}) = 1 \text{ if } \dim_{\mathrm{H}} F < s \text{ and} \\ \|\dim_{\mathrm{H}}(A \cap F)\|_{L^\infty(\widetilde{P})} = \dim_{\mathrm{H}} F - s \text{ if } \dim_{\mathrm{H}} F \geq s. \end{aligned}$$

But these results can be found in [44, §7], so we are done. □

We are now able to generalise the results of [40, Theorem 1.4] and [41, Corollary 1.5] on the Hausdorff dimension of the intersections $E(\mathbf{x}) \cap F$, when $F \subset X$ is a fixed analytic set.

Theorem 2.4. *If $F \subset X$ is analytic, then for $\mathbb{P}$-almost every $\mathbf{x} \in \Omega$,*

$$\max\{0, \alpha + \dim_{\mathrm{H}} F - t\} \leq \dim_{\mathrm{H}}(E(\mathbf{x}) \cap F) \leq \max\{0, \alpha + \dim_{\mathrm{P}} F - t\}.$$

Proof. With Lemma 2.3 and (2.6) at hand, the lower bound follows from a similar co-dimension argument as in [38, Lemma 3.4]. The upper bound, in turn, can be obtained by a direct first-moment estimation (see the proof of Theorem 1.4 in [40]). We omit the details. □

As an immediate consequence of Theorem 2.4, we obtain the following corollary.

Corollary 2.5. *Let $F \subset X$ be an analytic set with $\dim_{\mathrm{H}} F = \dim_{\mathrm{P}} F$. Then for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$,*

$$\dim_{\mathrm{H}}(E(\mathbf{x}) \cap F) = \max\{0, \alpha + \dim_{\mathrm{H}} F - t\}.$$

In particular, $\dim_{\mathrm{H}} E(\mathbf{x}) = \alpha$ for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$.

Remark 2.6. A problem suggested by Mahler on how well can points, say, in the middle third Cantor set K be approximated by rational points, has stemmed a number of results [4, 5, 18, 49, 57], measuring the sizes of the intersection of K with sets $\mathcal{W}(\psi)$ from (1.1) for different values of the approximation speed ψ. These results are in part inconclusive, and lead to conjectures on the size of the set $K \cap \mathcal{W}(q^{-\tau})$ in [6, 39]. In particular, Bugeaud and Durand [6, Conjecture 1] conjectured that

$$\dim_{\mathrm{H}}(K \cap \mathcal{W}(q^{-\tau})) = \max\{\dim_{\mathrm{H}} \mathcal{W}(q^{-\tau}) + \dim_{\mathrm{H}} K - 1, \tfrac{1}{\tau} \dim_{\mathrm{H}} K\}.$$

Following their reasoning, the first term in the maximum is realised for slow approximation speeds, where the intervals giving the limsup set are big, and the latter term appears when τ is large and the proportion of rationals inside the set K start to play a role. Bugeaud and Durand [6] have offered evidence that supports their conjecture, in particular, by building a model of random Diophantine approximation, where the centres of the generating intervals are independent and uniformly distributed, either in the Cantor set or in the whole circle $\mathbb{T}^1$ [6, Section 2]. The random model is based on a conjecture of Broderick, Fishman and Reich [4] on the proportion of rationals in the Cantor set.

In our Corollary 2.5, the size of the intersection of a random covering set with a given analytic set F is measured. This parallels the results of Bugeaud and Durand in the case of slow approximation speed, and could serve as a basis for more general results on random Diophantine approximation. We recall that, while our result concerns a wider class of sets than that of Bugeaud and Durand, the measure used by them to define the random variables is different from ours. Notice that when the generating sets are ball-like, many methods for the fast approximation also apply directly (see [6, Lemma 5]).

3 Covering sets in tori

3.1 Notations

In this section, we study the hitting probabilities of random covering sets in d-dimensional torus $\mathbb{T}^d$. We identify $\mathbb{T}^d$ with $[-\frac{1}{2},\frac{1}{2}[^d \subset \mathbb{R}^d$ and denote by $\Pi\colon \mathbb{R}^d \to \mathbb{T}^d$ the natural covering map. We use the notation $\widetilde{B}$ for the lift of $B \subset \mathbb{T}^d$ and denote the Lebesgue measure on $\mathbb{T}^d$ and $\mathbb{R}^d$ by $\mathcal{L}$. We consider the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ which is the completion of the infinite product of $(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), \mathcal{L})$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of analytic subsets of $[-\frac{1}{2},\frac{1}{2}[^d \subset \mathbb{R}^d$. For all $\mathbf{x} \in \Omega$, define the covering set by

$$E(\mathbf{x}) = \limsup_{n\to\infty}(x_n + \Pi(A_n)).$$

It follows from Lemma 3.1 below that the set $\{\mathbf{x} \in \Omega \mid \dim_{\mathrm{H}} E(\mathbf{x}) \le s\}$ is $\mathbb{P}$-measurable and, clearly, a tail event for all $0 \le s \le d$. Therefore, by Kolmogorov's zero-one law, there exists $s_0 \in [0,d]$ such that $\dim_{\mathrm{H}} E(\mathbf{x}) = s_0$ for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$. The value of s_0 has been calculated for a general class of sets $(A_n)_{n\in\mathbb{N}}$ in [24, Theorem 1.1].

3.2 Upper bounds

We start with a measurability lemma. Recall that a set is called analytic if it is a continuous image of a Borel set, and a set is universally measurable if it is μ-measurable for any σ-finite Borel measure μ. By [37, Theorem 21.10], analytic sets are universally measurable.

Lemma 3.1. *The covering set $E(\mathbf{x})$ is analytic for all $\mathbf{x} \in \Omega$. Furthermore, assuming that $F \subset \mathbb{T}^d$ is analytic and $t \in \mathbb{R}$, the sets*

$$B = \{(\mathbf{x}, z) \in \Omega \times \mathbb{T}^d \mid (E(\mathbf{x}) + z) \cap F = \emptyset\} \text{ and}$$
$$C = \{(\mathbf{x}, z) \in \Omega \times \mathbb{T}^d \mid \dim_{\mathrm{H}}((E(\mathbf{x}) + z) \cap F) \le t\}$$

are universally measurable.

Proof. The analyticity of $E(\mathbf{x})$ follows from [37, Proposition 14.4], which states that the class of analytic sets is closed under countable unions and intersections.

Define

$$A = \{(\mathbf{x}, z, y) \in \Omega \times \mathbb{T}^d \times \mathbb{T}^d \mid y \in E(\mathbf{x}) + z\}.$$

To prove that A is analytic, it is enough to show that

$$D_n = \{(\mathbf{x}, z, y) \in \Omega \times \mathbb{T}^d \times \mathbb{T}^d \mid y \in x_n + A_n + z\}$$

is analytic for all $n \in \mathbb{N}$. Define $f_n \colon \Omega \times \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{T}^d$ by $f_n(\mathbf{x}, z, y) = y - x_n - z$. Note that $D_n = f_n^{-1}(A_n)$. Since f_n is continuous and A_n is analytic, D_n is analytic by [37, Proposition 14.4]. Let $\pi_{12} \colon \Omega \times \mathbb{T}^d \times \mathbb{T}^d \to \Omega \times \mathbb{T}^d$ be the projection $\pi_{12}(\mathbf{x}, z, y) = (\mathbf{x}, z)$. The observation that the complement of B is $B^c = \pi_{12}\big(A \cap (\Omega \times \mathbb{T}^d \times F)\big)$ implies that B^c is analytic and, thus, B belongs to the σ-algebra generated by analytic sets. Therefore, B is universally measurable by [37, Theorem 21.10].

In [10], it is shown that if S and X are compact metric spaces and $G \subset S \times X$ is analytic, the map $H(s) = \dim_{\mathrm{H}}(G \cap (\{s\} \times X)$ is measurable with respect to the σ-algebra generated by analytic sets. Letting $S = \Omega \times \mathbb{T}^d$, $X = \mathbb{T}^d$ and $G = A \cap (S \times F)$, we have that $C = H^{-1}([0, t])$. Therefore, the analyticity of A and F implies that C is universally measurable. □

The following theorem is a counterpart of the upper bound parts of Theorems 2.2 and 2.4.

Theorem 3.2. *Let $F \subset \mathbb{T}^d$ be analytic. Then for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$,*

$$E(\mathbf{x}) \cap F = \emptyset \textit{ if } \dim_{\mathrm{P}} F < d - s_0 \textit{ and}$$
$$\dim_{\mathrm{H}}(E(\mathbf{x}) \cap F) \le s_0 + \dim_{\mathrm{P}} F - d \textit{ if } \dim_{\mathrm{P}} F \ge d - s_0,$$

where s_0 is the $\mathbb{P}$-almost sure Hausdorff dimension of $E(\mathbf{x})$.

Proof. It is known that (cf. [55] or [46, Theorem 8.10])

$$\dim_{\mathrm{H}}(E(\mathbf{x}) \times F) \le \dim_{\mathrm{H}} E(\mathbf{x}) + \dim_{\mathrm{P}} F \tag{3.1}$$

for all $\mathbf{x} \in \Omega$. In [46, Theorem 8.10], the result is stated only for Borel sets but the part of the proof where inequality (3.1) is proven is valid for all sets. Lifting $E(\mathbf{x})$ and F to $\mathbb{R}^d$ as $\widetilde{E}(\mathbf{x})$ and $\widetilde{F}$, we may apply [46, (13.2)], which implies that for all $\mathbf{x} \in \Omega$ and $z \in \mathbb{R}^d$,

$$(\widetilde{E}(\mathbf{x}) + z) \cap \widetilde{F} = \pi_1((\widetilde{E}(\mathbf{x}) \times \widetilde{F}) \cap V_z), \tag{3.2}$$

where $V_z = \{(u, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid u - v = z\}$ and $\pi_1(u, v) = u$.

We study first the case $\dim_{\mathrm{P}} F < d - s_0$. From (3.1) we deduce that the inequality $\dim_{\mathrm{H}}(E(\mathbf{x}) \times F) < d$ holds for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$. Therefore, since the dimension will not increase under the projection onto the orthogonal complement of V_0 (see [46, Theorem 7.5]), $(\widetilde{E}(\mathbf{x}) \times \widetilde{F}) \cap V_z = \emptyset$ for $\mathcal{L}$-almost all $z \in \mathbb{R}^d$. Projecting the sets back to $\mathbb{T}^d$, we have by (3.2) that, for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$, $(E(\mathbf{x}) + z) \cap F = \emptyset$ for $\mathcal{L}$-almost all $z \in \mathbb{T}^d$. By virtue of Lemma 3.1, the set

$$B = \{(\mathbf{x}, z) \in \Omega \times \mathbb{T}^d \mid (E(\mathbf{x}) + z) \cap F = \emptyset\}$$

is universally measurable. Thus, Fubini's theorem implies that for $\mathcal{L}$-almost all $z \in \mathbb{T}^d$, $(E(\mathbf{x}) + z) \cap F = \emptyset$ for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$. In particular, there exists $z_0 \in \mathbb{T}^d$ with

$$\mathbb{P}(\{\mathbf{x} \in \Omega \mid (E(\mathbf{x}) + z_0) \cap F = \emptyset\}) = 1.$$

Set $\mathbf{z} = (z_i)_{i \in \mathbb{N}}$, where $z_i = z_0$ for all $i \in \mathbb{N}$. Since $E(\mathbf{x}) + z_0 = E(\mathbf{x} + \mathbf{z})$ and $\mathbb{P}$ is translation invariant, we have

$$\mathbb{P}(\{\mathbf{x} \in \Omega \mid E(\mathbf{x}) \cap F = \emptyset\}) = 1.$$

Now we consider the case $\dim_{\mathrm{P}} F \ge d - s_0$. As at the beginning of this section, observe that $\{\mathbf{x} \in \Omega \mid \dim_{\mathrm{H}}(E(\mathbf{x}) \times F) \le s\}$ is a tail event for all $0 \le s \le 2d$ and, by

Kolmogorov's zero-one law, there exists $t_0 \in [0, 2d]$ such that $\dim_H(E(\mathbf{x}) \times F) = t_0$ for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$. If $t_0 < d$, the above argument implies that $E(\mathbf{x}) \cap F = \emptyset$ for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$ and, thus, the second claim in the statement of theorem is true. If $t_0 \geq d$, we may apply [46, Theorem 13.12] which implies that, for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$,

$$\dim_H\big((E(\mathbf{x}) + z) \cap F\big) \leq \dim_H(E(\mathbf{x}) \times F) - d$$

for $\mathcal{L}$-almost all $z \in \mathbb{T}^d$. Observe that [46, Theorem 13.12] is stated only for Borel sets but that assumption is not used in the proof. Furthermore, by (3.1), Lemma 3.1 and Fubini's theorem, for $\mathcal{L}$-almost all $z \in \mathbb{T}^d$,

$$\dim_H\big((E(\mathbf{x}) + z) \cap F\big) \leq s_0 + \dim_P F - d$$

for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$. Now the claim follows as above from the translation invariance of $\mathbb{P}$. □

3.3 A counter example for non-trivial lower bounds

In this subsection, we give an application of our results to a specific class of affine random covering sets in $\mathbb{T}^d$ which motivated this work. This class also demonstrates why the analogues of the lower bounds given by Theorems 2.2 and 2.4 are not true in the setting of general generating sets A_n, $n \in \mathbb{N}$.

Let $(r_n)_{n\in\mathbb{N}}$ be a decreasing sequence of numbers between 0 and 1 tending to zero. Further, let $0 < H_d \leq H_{d-1} \leq \cdots \leq H_2 \leq H_1 = 1$. We consider the covering set

$$E(\mathbf{x}) = \limsup_{n\to\infty}(x_n + \Pi(A_n))$$

for $\mathbf{x} \in \Omega$, where

$$A_n = \prod_{i=1}^{d} [-\tfrac{1}{2}, -\tfrac{1}{2} + (r_n)^{(H_i)^{-1}}] \subset \mathbb{R}^d \tag{3.3}$$

are rectangles in $\mathbb{R}^d$ with sides parallel to the coordinate axes and side lengths given by (3.3).

Let $s \in [0, d]$. We denote the integer and fractional parts of s by $\lfloor s \rfloor$ and $\{s\}$, respectively. For all $n \in \mathbb{N}$, we have

$$\Phi^s(A_n) = r_n^{\sum_{i=1}^{\lfloor s \rfloor}(H_i)^{-1} + \{s\}(H_{\lfloor s \rfloor+1})^{-1}},$$

where Φ^s is the singular value function of a rectangle determined by its side lengths (see [30]). Let $k_0 = \max\{k \in \{1, \ldots, d\} \mid \sum_{i=1}^{k}(H_i)^{-1} \leq \alpha\}$ and $\alpha = \min\{d, \limsup_{n\to\infty} \frac{\log n}{-\log r_n}\}$. By [30, Theorem 2.1],

$$\dim_H E(\mathbf{x}) = s_0 = \min\Big\{d, \inf\big\{s \geq 0 \mid \sum_{n=1}^{\infty} \Phi^s(A_n) < \infty\big\}\Big\}.$$

Combining this with the second equality in Theorem 2.1, we conclude that $\lfloor s_0 \rfloor = k_0$ and $\{s_0\} = H_{k_0+1}(\alpha - \sum_{i=1}^{k_0}(H_i)^{-1})$. Therefore, for $\mathbb{P}$-almost all $\mathbf{x} \in \Omega$,

$$\dim_H E(\mathbf{x}) = s_0 = \min\big\{d, \alpha H_{k_0+1} + \sum_{i=1}^{k_0}\big(1 - \frac{H_{k_0+1}}{H_i}\big)\big\}.$$

Thus, from Theorem 3.2, we conclude that almost surely,

$$\begin{aligned} &E(\mathbf{x}) \cap F = \emptyset \text{ if } \dim_P F < d - s_0 \text{ and} \\ &\dim_H(E(\mathbf{x}) \cap F) \leq s_0 + \dim_P F - d \text{ if } \dim_P F \geq d - s_0. \end{aligned} \tag{3.4}$$

The following example shows that we cannot have similar lower bounds in Theorem 3.2 as in Theorems 2.2 and 2.4.

Example 3.3. Fix $0<\varepsilon<1$. Consider $A_n\subset\mathbb{R}^2$, $n\in\mathbb{N}$, as in (3.3), where $H_2=\frac{\varepsilon}{1+\varepsilon}$ and $r_n=n^{-\varepsilon}$ for all $n\in\mathbb{N}$. Then $\alpha=\varepsilon^{-1}$ and, thus, the $\mathbb{P}$-almost sure value of the Hausdorff dimension of the covering set is $s_0=1+\frac{1-\varepsilon}{1+\varepsilon}$. If $-\frac{1}{2}\le b<\frac{1}{2}$ and $F=\Pi([-\frac{1}{2},\frac{1}{2}]\times\{b\})$, then $\dim_{\mathrm{H}}F=\dim_{\mathrm{P}}F=1>2-s_0=\frac{2\varepsilon}{1+\varepsilon}$. However,

$$\mathbb{P}(\{\mathbf{x}\in\Omega\mid E(\mathbf{x})\cap F\neq\emptyset\})=0.$$

Indeed, $E(\mathbf{x})\cap F\neq\emptyset$ only if $\Pi(0,b)\in\limsup_{n\to\infty}\Pi(\pi_2(\widetilde{x_n}+A_n))$, where π_2 denotes the orthogonal projection onto the y-axis. Now the law of the set $\limsup_{n\to\infty}\Pi(\pi_2(\widetilde{x_n}+A_n))$ is that of a random covering set in $\mathbb{T}^1$ with $r_n=n^{-\varepsilon(H_2)^{-1}}=n^{-1-\varepsilon}$. Since $\sum_{n=1}^\infty n^{-1-\varepsilon}<\infty$, the Borel-Cantelli lemma implies

$$\mathbb{P}\big(\Pi(0,b)\in\limsup_{n\to\infty}\Pi(\pi_2(\widetilde{x_n}+A_n))\big)=0.$$

We will next show how the results from Section 2.2 may be used to replace (3.4) by a sharper estimate and also to get an analogy of the lower bounds. To that end, we define a new metric κ on $\mathbb{T}^d$ by ‘snowflaking’ the Euclidean distance by factor H_i in each coordinate direction. More precisely, for all $y,z\in\mathbb{T}^d$, we set

$$\kappa(z,y)=\max_{1\le i\le d}2^{H_i}|z_i-y_i|^{H_i},$$

where the natural distance between points $a,b\in\mathbb{T}^1$ is denoted by $|a-b|$. With this metric, $\mathbb{T}^d$ becomes a t-regular metric space with $t=\sum_{i=1}^d(H_i)^{-1}$ and $\mathcal{L}$ is the t-regular measure satisfying (2.1). Further, in this metric each $\Pi(A_n)$ is a ball of radius r_n. The constants 2^{H_i} appear in the definition of κ to ensure this. Thus, we are in a situation where the results from Section 2.2 may be applied. For instance, for $\mathbb{P}$-almost all $\mathbf{x}\in\Omega$, we have

$$\dim_{\mathrm{H}}^{\kappa}E(\mathbf{x})=\alpha=\min\Big\{t,\limsup_{n\to\infty}\frac{\log n}{-\log r_n}\Big\},$$

where the Hausdorff and packing dimensions with respect to the metric κ are denoted by $\dim_{\mathrm{H}}^{\kappa}$ and $\dim_{\mathrm{P}}^{\kappa}$, respectively. Further, if $F\subset\mathbb{T}^d$ is analytic, then for $\mathbb{P}$-almost all $\mathbf{x}\in\Omega$,

$$\begin{aligned}
&E(\mathbf{x})\cap F=\emptyset\text{ if }\dim_{\mathrm{P}}^{\kappa}F<t-\alpha,\\
&E(\mathbf{x})\cap F\neq\emptyset\text{ if }\dim_{\mathrm{H}}^{\kappa}F>t-\alpha,\\
&E(\mathbf{x})\cap F\neq\emptyset\text{ if }\dim_{\mathrm{P}}^{\kappa}F>t-\alpha\text{ and (2.4) holds, and}\\
&\alpha+\dim_{\mathrm{H}}^{\kappa}F-t\le\dim_{\mathrm{H}}^{\kappa}(E(\mathbf{x})\cap F)\le\alpha+\dim_{\mathrm{P}}^{\kappa}F-t\text{ if }\dim_{\mathrm{P}}^{\kappa}F>t-\alpha.
\end{aligned}\tag{3.5}$$

Remark 3.4. a) In Example 3.3, we have $\dim_{\mathrm{P}}^{\kappa}F=\dim_{\mathrm{H}}^{\kappa}F=1$, $t=(1+2\varepsilon)\varepsilon^{-1}$ and $\alpha=\varepsilon^{-1}$, implying $1=\dim_{\mathrm{P}}^{\kappa}F<t-\alpha=2$ (which is consistent with (3.5)).

b) Example 3.3 shows that there cannot be any non-trivial lower bound in (3.4) depending only on the Hausdorff and packing dimensions of $E(\mathbf{x})$ and F. However, (3.5) demonstrates that there can be some other quantities like $\dim_{\mathrm{H}}^{\kappa}$ and $\dim_{\mathrm{P}}^{\kappa}$ which do imply non-trivial lower bounds.

3.4 Covering sets involving random rotations

As demonstrated in Example 3.3, the analogues of (2.6) and (2.7), and the lower bound in Theorem 2.4 are not always true for a sequence of general sets $(A_n)_{n\in\mathbb{N}}$. In Example 3.3, this is due to the alignment of the rectangles A_n, $n\in\mathbb{N}$. In this subsection, we consider a slightly modified and, perhaps, more natural version of random covering sets in $\mathbb{T}^d$, where the sets A_n are also rotated, and show that under a standard additional assumption on the dimensions, the analogues of (2.6) and Theorem 2.4 remain valid for any sequence of analytic generating sets A_n.

Let $(\widetilde{\Omega},\widetilde{\mathcal{A}},\widetilde{\mathbb{P}})$ be the completion of the infinite product of $(\mathbb{T}^d\times\mathcal{O}(d),\mathcal{B}(\mathbb{T}^d\times\mathcal{O}(d)),\mathcal{L}\times\theta)$, where $\mathcal{O}(d)$ is the orthogonal group on $\mathbb{R}^d$ and θ is the Haar measure on $\mathcal{O}(d)$. Assume that $A_n\subset U(0,\frac{1}{2})\subset\mathbb{R}^d$ are analytic for all $n\in\mathbb{N}$, where $U(x,r)$ is the open ball with radius r centred at x. We consider the covering sets

$$E(\mathbf{x},\mathbf{h})=\limsup_{n\to\infty}\big(x_n+\Pi(h_n(A_n))\big)\subset\mathbb{T}^d\,,$$

for $(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}$.

Remark 3.5. a) Since $A_n\subset U(0,\frac{1}{2})$, the restriction of Π to $h_n(A_n)$ is injective for all $h_n\in\mathcal{O}(d)$ and $n\in\mathbb{N}$.

b) As above, Kolmogorov's zero-one law implies the existence of $s_0^R\in[0,d]$ such that $\dim_{\mathrm{H}}E(\mathbf{x},\mathbf{h})=s_0^R$ for $\widetilde{\mathbb{P}}$-almost all $(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}$. Note that the value s_0 for the Hausdorff dimension of typical random covering sets calculated in [24] depends only on the shapes of the generating sets A_n, that is, the value of s_0 for the generating sequence $(A_n)_{n\in\mathbb{N}}$ is equal to that of the sequence $(h_n(A_n))_{n\in\mathbb{N}}$ for all $\mathbf{h}\in(\mathcal{O}(d))^{\mathbb{N}}$. Thus, for this large class of sets, we have $s_0^R=s_0$, that is, adding the rotations will not change the dimension of typical random covering sets.

c) Notice that Theorem 3.2 holds for $E(\mathbf{x},\mathbf{h})$ too, as the proof is valid for any fixed sequence of rotations $(h_n)_{n\in\mathbb{N}}$.

Before proving our next main theorem, which gives a lower bound for typical intersections, we prove two lemmas. The first one states that the dimension of a typical covering set is the same inside every ball.

Lemma 3.6. *For $\widetilde{\mathbb{P}}$-almost all $(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}$,*

$$\dim_{\mathrm{H}}\big(E(\mathbf{x},\mathbf{h})\cap B(z,r)\big)=s_0^R$$

for all $z\in\mathbb{T}^d$ and $r>0$.

Proof. For all $z\in\mathbb{T}^d$, let $\mathbf{z}\in(\mathbb{T}^d)^{\mathbb{N}}$ be the sequence such that $z_n=z$ for all $n\in\mathbb{N}$. Since $\mathcal{L}$ is translation invariant, we have for all $t\in[0,d]$ and $z\in\mathbb{T}^d$ that

$$\begin{aligned}&\widetilde{\mathbb{P}}\big(\{(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}\mid\dim_{\mathrm{H}}\big(E(\mathbf{x},\mathbf{h})\cap B(z,r)\big)=t\}\big)\\&=\widetilde{\mathbb{P}}\big(\{(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}\mid\dim_{\mathrm{H}}\big(E(\mathbf{x}+\mathbf{z},\mathbf{h})\cap B(z,r)\big)=t\}\big)\\&=\widetilde{\mathbb{P}}\big(\{(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}\mid\dim_{\mathrm{H}}\big(E(\mathbf{x},\mathbf{h})\cap B(0,r)\big)=t\}\big).\end{aligned}\tag{3.6}$$

As above, we see that, for every $r>0$, there exists a unique $t_r\in[0,d]$ such that $\mathbb{P}\big(\big\{\mathbf{x}\in\Omega\mid\dim_{\mathrm{H}}\big(E(\mathbf{x},\mathbf{h})\cap B(z,r)\big)=t_r\big\}\big)=1$. Since $\mathbb{T}^d$ may be covered by a finite number of balls with radius r, (3.6) implies that $t_r=s_0^R$ for all $r>0$. □

Unlike the translation, the rotation is not well defined on $\mathbb{T}^d$. To deal with the technical problems caused by this fact, we need the following lemma.

Lemma 3.7. *Let $0<r<\frac{1}{2}$. There exist $M\in\mathbb{N}$ and $z_i\in U(0,\frac{1}{2})\subset\mathbb{R}^d$, $i=1,\dots,M$, such that, for all $n\in\mathbb{N}$, we may decompose $A_n=\bigcup_{i=1}^M A_n^i$ into Borel sets A_n^i in such a way that $A_n^i\subset B(z_i,r)$ for all $n\in\mathbb{N}$ and, for all $(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}$, there exists $i\in\{1,\dots,M\}$ such that*

$$\dim_{\mathrm{H}}E(\mathbf{x},\mathbf{h})=\dim_{\mathrm{H}}\Big(\limsup_{n\to\infty}\big(x_n+\Pi(h_n(A_n^i))\big)\Big).$$

Further, for all $i\in\{1,\dots,M\}$, there exist sets $\widehat{A}_n^i\subset B(0,r)$, $n\in\mathbb{N}$, and a bijection $F_i\colon\widetilde{\Omega}\to\widetilde{\Omega}$ preserving $\widetilde{\mathbb{P}}$, that is, $(F_i)_*\widetilde{\mathbb{P}}=\widetilde{\mathbb{P}}$, such that

$$\limsup_{n\to\infty}\big(x_n+\Pi(h_n(A_n^i))\big)=\limsup_{n\to\infty}\Big(\big(\pi_1(F_i(\mathbf{x},\mathbf{h}))\big)_n+\Pi\big(\big(\pi_2(F_i(\mathbf{x},\mathbf{h}))\big)_n(\widehat{A}_n^i)\big)\Big)$$

for all $(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}$, where $\pi_1(\mathbf{x},\mathbf{h})=\mathbf{x}$ and $\pi_2(\mathbf{x},\mathbf{h})=\mathbf{h}$.

Proof. The desired decomposition may be obtained by any partitioning of $U(0,\frac{1}{2})$ into Borel subsets of diameter less than r and for any choice of points z_i inside these sets since

$$\limsup_{n\to\infty}(C_n\cup D_n)=\limsup_{n\to\infty}C_n\cup\limsup_{n\to\infty}D_n$$

for all sets $C_n,D_n\subset\mathbb{T}^d$, $n\in\mathbb{N}$. For the second claim, define $\widehat{A}_n^i=A_n^i-z_i\subset B(0,r)$ and $F_i(\mathbf{x},\mathbf{h})=\big(\mathbf{x}+\big(\Pi(h_n(z_i))\big)_{n\in\mathbb{N}},\mathbf{h}\big)$ for all $i\in\{1,\dots,M\}$. The translation invariance of $\mathcal{L}$ yields $(F_i)_*\widetilde{\mathbb{P}}=\widetilde{\mathbb{P}}$. Further, since $z_i\in U(0,\frac{1}{2})$ and $A_n^i\subset U(0,\frac{1}{2})$ for all $n\in\mathbb{N}$ and $i\in\{1,\dots,M\}$,

$$x_n+\Pi(h_n(A_n^i))=x_n+\Pi(h_n(z_i))+\Pi(h_n(\widehat{A}_n^i))$$

for all $x_n\in\mathbb{T}^d$, implying the last claim. □

Now we are ready to prove the second main theorem of this section.

Theorem 3.8. *Let $F\subset\mathbb{T}^d$ be an analytic set with $\dim_{\mathrm{H}}F>d-s_0^R$. Assume that $\max\{s_0^R,\dim_{\mathrm{H}}F\}>\frac{1}{2}(d+1)$. Then*

$$\dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h})\cap F)\ge s_0^R+\dim_{\mathrm{H}}F-d$$

for $\widetilde{\mathbb{P}}$-almost all $(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}$.

Proof. Since $E(\mathbf{x},\mathbf{h})\cap(F-z)=-z+E(\mathbf{x}+\mathbf{z},\mathbf{h})\cap F$ for all $(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}$ and $z\in\mathbb{T}^d$ (where $\mathbf{z}=(z,z,\dots)$) and $\widetilde{\mathbb{P}}$ is translation invariant, we may assume that $\dim_{\mathrm{H}}(\Pi(V)\cap F)=\dim_{\mathrm{H}}F$, where $V=B(0,\frac{1}{10})\subset\mathbb{R}^d$. Fix $t<s_0^R+\dim_{\mathrm{H}}F-d$. By Lemma 3.6, $\dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h})\cap\Pi(V))=s_0^R$ for $\widetilde{\mathbb{P}}$-almost all $(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}$. According to a general intersection result for Hausdorff dimension [46, Theorem 13.11], if $A,B\subset\mathbb{R}^d$ are analytic sets with $\dim_{\mathrm{H}}A+\dim_{\mathrm{H}}B>d$ and $\max\{\dim_{\mathrm{H}}A,\dim_{\mathrm{H}}B\}>\frac{1}{2}(d+1)$ then, for θ-almost all $h\in\mathcal{O}(d)$,

$$\mathcal{L}\big(\{z\in\mathbb{R}^d\mid\dim_{\mathrm{H}}(h(A)+z)\cap B)>\dim_{\mathrm{H}}A-\dim_{\mathrm{H}}B-d-\varepsilon\}\big)>0$$

for any $\varepsilon>0$. Note that [46, Theorem 13.11] is stated for Borel sets but the proof given is valid for analytic sets as well since Frostman's lemma is valid for analytic sets [7]. In [46, Theorem 13.11], the theorem is stated in asymmetric way, but the above symmetric form is also valid by the translation invariance of $\mathcal{L}$ and rotation invariance of θ. Thus, for every realisation of E with $\dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h})\cap\Pi(V))=s_0^R$, we have, for θ-almost all $h\in\mathcal{O}(d)$, that

$$\mathcal{L}\big(\{z\in B(0,\tfrac{1}{5})\mid\dim_{\mathrm{H}}\big((h(\widetilde{E}(\mathbf{x},\mathbf{h}))+z)\cap\widetilde{F}\cap V\big)\ge t\}\big)>0. \tag{3.7}$$

By Lemma 3.1 and Fubini's theorem, for all $\varepsilon>0$, there exist $h_0\in B(\mathrm{Id},\varepsilon)\subset\mathcal{O}(d)$ and $z_0\in B(0,\frac{1}{5})\subset\mathbb{R}^d$ such that

$$\widetilde{\mathbb{P}}\big(\{(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}\mid\dim_{\mathrm{H}}\big((h_0(\widetilde{E}(\mathbf{x},\mathbf{h}))+z_0)\cap\widetilde{F}\cap V\big)\ge t\}\big)>0.$$

Let $\mathbf{w}\in(\mathbb{T}^d)^{\mathbb{N}}$, where $w_n=\Pi(h_0^{-1}(z_0))$ for all $n\in\mathbb{N}$. Using the fact that $z_0\in B(0,\frac{1}{5})$, we may choose small enough $\varepsilon>0$ depending on d only such that

$$(h_0(\widetilde{E}(\mathbf{x},\mathbf{h}))+z_0)\cap V=h_0\big(\widetilde{E}(\mathbf{x},\mathbf{h})+h_0^{-1}(z_0)\big)\cap V=h_0(\widetilde{E}(\mathbf{x}+\mathbf{w},\mathbf{h}))\cap V.$$

Thus, the translation invariance of $\mathbb{P}$ implies

$$\widetilde{\mathbb{P}}\big(\{(\mathbf{x},\mathbf{h})\in\widetilde{\Omega}\mid\dim_{\mathrm{H}}\big(h_0(\widetilde{E}(\mathbf{x},\mathbf{h}))\cap\widetilde{F}\cap V\big)\ge t\}\big)>0. \tag{3.8}$$

For $z\in\mathbb{T}^d$, let $\tilde{z}\in[-\frac{1}{2},\frac{1}{2}[^d$ be the unique element such that $\Pi(\tilde{z})=z$. Define $\hat{h}_0\colon\mathbb{T}^d\to\mathbb{T}^d$ by

$$\hat{h}_0(z)=\begin{cases}\Pi(h_0(\tilde{z})), & \text{if } \tilde{z}\in B(0,\frac{1}{5})\\ z, & \text{if } \tilde{z}\in[-\frac{1}{2},\frac{1}{2}[^d\setminus B(0,\frac{1}{5}).\end{cases}$$

Then $(\hat{h}_0)_*\mathcal{L} = \mathcal{L}$ and $(H_0)_*\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}$, where $H_0(\mathbf{x},\mathbf{h}) = \big((\hat{h}_0(x_n))_{n\in\mathbb{N}},\mathbf{h}\big)$. By Lemma 3.7, we may assume that $A_n \subset V$ for all $n \in \mathbb{N}$. Therefore, decreasing ε if necessary, we have

$$\Pi\big(h_0(\tilde{x}_n + h_n(A_n))\big) \cap \Pi(V) = \Big(\hat{h}_0(x_n) + \Pi\big(h_0(h_n(A_n))\big)\Big) \cap \Pi(V) \tag{3.9}$$

for all $n \in \mathbb{N}$ and $(\mathbf{x},\mathbf{h}) \in \widetilde{\Omega}$. Using $(H_0)_*\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}$ and combining (3.8) with (3.9), we conclude

$$\widetilde{\mathbb{P}}\big(\big\{(\mathbf{x},\mathbf{h}) \in \widetilde{\Omega} \mid \dim_{\mathrm{H}}\big(\widetilde{E}(\mathbf{x},(h_0h_n)_{n\in\mathbb{N}}) \cap \widetilde{F} \cap V\big) \geq t\big\}\big) > 0.$$

Since θ is the Haar measure, $(h_0)_*\theta = \theta$ and, thus,

$$\widetilde{\mathbb{P}}\big(\big\{(\mathbf{x},\mathbf{h}) \in \widetilde{\Omega} \mid \dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h}) \cap F) \geq t\big\}\big) > 0.$$

Since $\{(\mathbf{x},\mathbf{h}) \in \widetilde{\Omega} \mid \dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h}) \cap F) \geq t\}$ is a tail event, $\dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h}) \cap F) \geq t$ for $\widetilde{\mathbb{P}}$-almost all $(\mathbf{x},\mathbf{h}) \in \widetilde{\Omega}$ by Kolmogorov's zero-one law. The proof is completed by letting t tend to $s_0^R + \dim_{\mathrm{H}} F - d$ along a sequence. □

Combining Theorems 3.2 and 3.8 with Remark 3.5.c), gives the following corollary.

Corollary 3.9. *Let $F \subset \mathbb{T}^d$ be an analytic set with $\dim_{\mathrm{H}} F > d - s_0^R$. Assume that $\max\{s_0^R, \dim_{\mathrm{H}} F\} > \frac{1}{2}(d+1)$. Then, for $\widetilde{\mathbb{P}}$-almost all $(\mathbf{x},\mathbf{h}) \in \widetilde{\Omega}$,*

$$s_0^R + \dim_{\mathrm{H}} F - d \leq \dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h}) \cap F) \leq s_0^R + \dim_{\mathrm{P}} F - d. \tag{3.10}$$

In particular, if $\dim_{\mathrm{H}} F = \dim_{\mathrm{P}} F$, then, for $\widetilde{\mathbb{P}}$-almost all $(\mathbf{x},\mathbf{h}) \in \widetilde{\Omega}$,

$$\dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h}) \cap F) = s_0^R + \dim_{\mathrm{H}} F - d.$$

Remark 3.10. a) Observe that $\dim_{\mathrm{H}} F > d - s_0^R$ implies $\max\{s_0^R, \dim_{\mathrm{H}} F\} > \frac{d}{2}$. We do not know whether the assumption $\max\{s_0^R, \dim_{\mathrm{H}} F\} > \frac{1}{2}(d+1)$ is needed in Theorem 3.8. This is a famous open problem in the theory of intersections of general sets. In [41], Li and Suomala constructed examples featuring that the lower and upper bounds in (3.10) may be achieved. Thus one cannot find better bounds than those in (3.10) involving only the Hausdorff and packing dimensions of $E(\mathbf{x},\mathbf{h})$ and F. However, in [41] there is an example where $\dim_{\mathrm{H}}(E(\mathbf{x},\mathbf{h}) \cap F)$ is almost surely strictly between the bounds given in (3.10).

b) The problem of exceptional geometry of a limsup set produced from axes-parallel rectangles such as in Example 3.3 seems to be a prevalent phenomenon. As examples, see [56, Section 6] and [19, Example 9.10]. Theorem 3.8 offers further evidence to support the folklore conjecture that this is caused by the atypical exact alignment of the construction sets, and can be overcome by re-orienting them which, in our case, was done by random rotations.

c) What Corollary 3.9 implies in the framework of Bugeaud and Durand [6] (recall Remark 2.6) is that, under the additional dimension assumptions, the random Diophantine approximation properties for points in a fixed analytic set F are valid for any choice of shapes of the generating sets, as long as the sets are randomly rotated. Notice that, for a large selection of generating sets, the dimension does not vary with the rotations [24].

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