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REVERSE MATHEMATICS, WELL-QUASI-ORDERS, AND NOETHERIAN SPACES

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Abstract. A quasi-order Q induces two natural quasi-orders on $\mathcal{P}(Q)$, but if Q is a well-quasi-order, then these quasi-orders need not necessarily be well-quasi-orders. Nevertheless, Goubault-Larrecq in [GL07] showed that moving from a well-quasi-order Q to the quasi-orders on $\mathcal{P}(Q)$ preserves well-quasi-orderedness in a topological sense. Specifically, Goubault-Larrecq proved that the upper topologies of the induced quasi-orders on $\mathcal{P}(Q)$ are Noetherian, which means that they contain no infinite strictly descending sequences of closed sets. We analyze various theorems of the form "if Q is a well-quasi-order then a certain topology on (a subset of) $\mathcal{P}(Q)$ is Noetherian" in the style of reverse mathematics, proving that these theorems are equivalent to ACA_0 over RCA_0. To state these theorems in RCA_0 we introduce a new framework for dealing with second-countable topological spaces.

1. Introduction

A topological space is *Noetherian* if it satisfies the following equivalent conditions.

- Every subspace is compact.
- Every ascending sequence of open sets stabilizes: for every sequence $(G_n)_{n\in\mathbb{N}}$ of open sets such that $\forall n(G_n \subseteq G_{n+1})$, there is an N such that $(\forall n > N)(G_n = G_N)$.
- Every descending sequence of closed sets stabilizes: for every sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets such that $\forall n(F_n \supseteq F_{n+1})$, there is an N such that $(\forall n > N)(F_n = F_N)$.

The name 'Noetherian space' comes from the typical example of a Noetherian space, which is the Zariski topology on the spectrum of a Noetherian ring. If R is a commutative ring, let $\mathrm{Spec}(R)$, the spectrum of R, denote the set of prime ideals in R. The Zariski topology on $\mathrm{Spec}(R)$ is the topology whose closed sets are the sets of the form $\{P \in \mathrm{Spec}(R) : I \subseteq P\}$, where $I \subseteq R$ is an ideal. If the ring R is Noetherian, then $\mathrm{Spec}(R)$ with the Zariski topology is a Noetherian space.

The present work, however, is not concerned with the connections between Noetherian spaces and algebraic geometry but with the connections between Noetherian spaces and the theory of well-quasi-orders. Goubault-Larrecq in [GL07], motivated by possible applications to verification problems as explained in [GL10], provided several results demonstrating that Noetherian spaces can be thought of as topological versions, or generalizations, of well-quasi-orders. We analyze these theorems in the style of reverse mathematics, proving that they are equivalent to ACA_0 over the base theory RCA_0. As a byproduct of this analysis, we obtain elementary proofs of Goubault-Larrecq's results which are much more direct than the original category-theoretic arguments used in [GL07]. The logical analysis of Noetherian spaces arising from Noetherian rings is ongoing work.

A quasi-order Q induces various quasi-orders on $\mathcal{P}(Q)$, the power set of Q, which in turn induce various topologies on $\mathcal{P}(Q)$. The theorems of [GL07] state that if a quasi-order Q is in fact a

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well-quasi-order, then several of the resulting topologies on $\mathcal{P}(Q)$ are Noetherian. In order to state these results precisely, we must first introduce the relevant definitions.

A *quasi-order* is a pair $(Q, \leq_Q)$, where Q is a set and $\leq_Q$ is a binary relation on Q satisfying the reflexivity axiom $(\forall q \in Q)(q \leq_Q q)$ and the transitivity axiom $(\forall p, q, r \in Q)((p \leq_Q q \wedge q \leq_Q r) \to p \leq_Q r)$. For notational ease, we usually identify $(Q, \leq_Q)$ and Q. We write $p <_Q q$ when we have both $p \leq_Q q$ and $q \not\leq_Q p$, and we write $p \mid_Q q$ when we have both $p \not\leq_Q q$ and $q \not\leq_Q p$.

If $E \in \mathcal{P}(Q)$, then $E{\downarrow} = \{q \in Q : (\exists p \in E)(q \leq_Q p)\}$ denotes the downward closure of E and $E{\uparrow} = \{q \in Q : (\exists p \in E)(p \leq_Q q)\}$ denotes the upward closure of E. For $p \in Q$, we usually write $p{\downarrow}$ for $\{p\}{\downarrow}$ and $p{\uparrow}$ for $\{p\}{\uparrow}$. A quasi-order Q induces the following quasi-orders $\leq_Q^\flat$ and $\leq_Q^\sharp$ on $\mathcal{P}(Q)$. (We follow the notation of [GL07]. In other works, such as [Mar01, Mar05], '$\leq_Q^\flat$' is written as '$\leq_\forall^\exists$' and '$\leq_Q^\sharp$' is written as '$\leq_\exists^\forall$'.)

Definition 1.1. Let Q be a quasi-order. For $A, B \in \mathcal{P}(Q)$, define

- $A \leq_Q^\flat B$ if and only if $(\forall a \in A)(\exists b \in B)(a \leq_Q b)$, and
- $A \leq_Q^\sharp B$ if and only if $(\forall b \in B)(\exists a \in A)(a \leq_Q b)$.

Notice that $A \leq_Q^\flat B$ is equivalent to $A \subseteq B{\downarrow}$ and that $A \leq_Q^\sharp B$ is equivalent to $B \subseteq A{\uparrow}$. We denote $(\mathcal{P}(Q), \leq_Q^\flat)$ by $\mathcal{P}^\flat(Q)$ and $(\mathcal{P}(Q), \leq_Q^\sharp)$ by $\mathcal{P}^\sharp(Q)$: it is easy to check that these are indeed quasi-orders (they are partial orders if and only if Q is an antichain).

Both $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp(Q)$ have been studied for a long time by computer scientists. In this context, $\mathcal{P}^\flat(Q)$ is known as the *Hoare quasi-order*, and $\mathcal{P}^\sharp(Q)$ is known as the *Smyth quasi-order*. For example, these orders can be used to compare the different executions of a non-deterministic computation (see e.g. [Win85] for an early presentation).

A quasi-order can be topologized in several ways. We consider the Alexandroff topology and the upper topology.

Definition 1.2. Let Q be a quasi-order.

- The *Alexandroff topology* of Q is the topology whose open sets are those of the form $E{\uparrow}$ for $E \subseteq Q$. The topological space consisting of Q with its Alexandroff topology is denoted $\mathcal{A}(Q)$.
- The *upper topology* of Q is the topology whose basic open sets are those of the form $Q \setminus (E{\downarrow})$ for $E \subseteq Q$ finite. The topological space consisting of Q with its upper topology is denoted $\mathcal{U}(Q)$.

Notice that, unless Q is a partial order, neither $\mathcal{A}(Q)$ nor $\mathcal{U}(Q)$ are T_0 spaces, and, unless Q is an antichain, neither $\mathcal{A}(Q)$ nor $\mathcal{U}(Q)$ are T_1 spaces. The order-theoretic significances of $\mathcal{A}(Q)$ and $\mathcal{U}(Q)$ are that they are the finest and coarsest topologies on Q from which $\leq_Q$ can be recovered. Every topological space X induces a *specialization quasi-order* on X defined by $x \preceq y$ if and only if every open set that contains x also contains y. If Q is a quasi-order, then $\mathcal{A}(Q)$ (respectively $\mathcal{U}(Q)$) is the finest (respectively coarsest) topology on Q for which $\preceq \; = \; \leq_Q$ (see for example [GL13, Section 4.2]).

Finally, a *well-quasi-order* (*wqo*) is a quasi-order Q that is well-founded and has no infinite antichains. Equivalently, a quasi-order Q is a wqo if for every function $f \colon \mathbb{N} \to Q$, there are $m, n \in \mathbb{N}$ with $m < n$ such that $f(m) \leq_Q f(n)$. It is easy to check that, for a quasi-order Q, Q is a wqo if and only if $\mathcal{A}(Q)$ is Noetherian, in which case $\mathcal{U}(Q)$ is also Noetherian. In fact, in Proposition 3.8 we show that these facts are provable in RCA_0.

For a quasi-order Q, let $\mathcal{P}_{\mathrm{f}}(Q)$ denote the set of finite subsets of Q, and let $\mathcal{P}_{\mathrm{f}}^\flat(Q)$ and $\mathcal{P}_{\mathrm{f}}^\sharp(Q)$ denote the respective restrictions of $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp(Q)$ to $\mathcal{P}_{\mathrm{f}}(Q)$. If Q is a wqo, then $\mathcal{P}_{\mathrm{f}}^\flat(Q)$ is also a wqo (see [ER52]), but $\mathcal{P}^\flat(Q)$, $\mathcal{P}^\sharp(Q)$, and $\mathcal{P}_{\mathrm{f}}^\sharp(Q)$ need not be wqo's. This can be seen by considering Rado's example [Rad54], the well-quasi-order $(R, \leq_R)$ where $R = \{(i, j) \in \mathbb{N} \times$

$\mathbb{N} : i < j\}$ and $(i,j) \leq_R (k,\ell)$ if $(i = k \wedge j \leq \ell) \vee j < k$ (see [Jan99, Mar01] for a complete explanation). Nevertheless, Goubault-Larrecq [GL07] proved that although passing from a wqo Q to the quasi-orders $\mathcal{P}^\flat(Q)$, $\mathcal{P}^\sharp(Q)$, and $\mathcal{P}^\sharp_{\mathrm{f}}(Q)$ does not necessarily preserve well-quasi-orderedness, it does preserve well-foundedness in the sense that the upper topologies of $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp_{\mathrm{f}}(Q)$ are Noetherian.

Theorem 1.3 ([GL07])**.** *If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat(Q))$ and $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ are Noetherian.*

Though Goubault-Larrecq explicitly proved Theorem 1.3 for $\mathcal{U}(\mathcal{P}^\flat(Q))$ and $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ only, it is also true that if Q is a wqo, then $\mathcal{A}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$, $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$, and $\mathcal{U}(\mathcal{P}^\sharp(Q))$ are Noetherian as well. For $\mathcal{A}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ and $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$, this is because if Q is a wqo, then $\mathcal{P}^\flat_{\mathrm{f}}(Q)$ is also a wqo (the $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ case also follows from Theorem 1.3). The $\mathcal{U}(\mathcal{P}^\sharp(Q))$ case follows from Theorem 1.3 because for every $A \in \mathcal{P}(Q)$ there is a $B \in \mathcal{P}_{\mathrm{f}}(Q)$ that is equivalent to A in the sense that $A \leq^\sharp_Q B$ and $B \leq^\sharp_Q A$. Notice, however, that if Q is a wqo, then $\mathcal{A}(\mathcal{P}^\flat(Q))$, $\mathcal{A}(\mathcal{P}^\sharp(Q))$, and $\mathcal{A}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ need not necessarily be Noetherian. This is because $\mathcal{P}^\flat(Q)$, $\mathcal{P}^\sharp(Q)$, and $\mathcal{P}^\sharp_{\mathrm{f}}(Q)$ need not necessarily be wqo's, and a quasi-order's Alexandroff topology is Noetherian if and only if the quasi-order itself is a wqo.

Concerning $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp(Q)$, we wish to remark that Nash-Williams [NW68] strengthened well-quasi-orders to better-quasi-orders (bqo's), an ingenious insight that led to a rich theory, including Laver's proof of Fraïssé's conjecture in [Lav71]. A few years later, Pouzet [Pou72] introduced a hierarchy of notions intermediate between wqo and bqo by defining the α-well-quasi-orders (α-wqo's) for each countable ordinal α. The ω-wqo's are exactly the wqo's, and the larger α is, the closer the notion of α-wqo is to the notion of bqo. Indeed, Q is a bqo if and only if Q is an α-wqo for every $\alpha < \omega_1$. By imposing these stronger conditions on Q, we may ensure that $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp(Q)$ are wqo's.

Theorem 1.4 (see [Mar01] for a complete discussion and further results)**.**

- *If Q is a bqo, then $\mathcal{P}^\flat(Q)$, $\mathcal{P}^\sharp(Q)$, $\mathcal{P}^\flat_{\mathrm{f}}(Q)$, and $\mathcal{P}^\sharp_{\mathrm{f}}(Q)$ are all bqo's.*
- *If Q is a ω^2-wqo, then $\mathcal{P}^\flat(Q)$, $\mathcal{P}^\sharp(Q)$, $\mathcal{P}^\flat_{\mathrm{f}}(Q)$, and $\mathcal{P}^\sharp_{\mathrm{f}}(Q)$ are all wqo's.*

The purpose of this work is to study Theorem 1.3 and related statements from the viewpoint of reverse mathematics. Our main result is the following.

Theorem 4.7. *The following are equivalent over* RCA_0.

(i) ACA_0.
(ii) If Q is a wqo, then $\mathcal{A}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ is Noetherian.
(iii) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ is Noetherian.
(iv) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ is Noetherian.
(v) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat(Q))$ is Noetherian.
(vi) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\sharp(Q))$ is Noetherian.

The following table summarizes the logical strengths of implications such as "if Q is a wqo, then $\mathcal{P}^\flat_{\mathrm{f}}(Q)$ is a wqo" and "if Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat(Q))$ is Noetherian." The entry 'false' indicates that the corresponding implication is false, as witnessed by Rado's example. The entry 'RCA_0' indicates that the corresponding implication is provable in RCA_0. The entry 'ACA_0' indicates that the corresponding implication is equivalent to ACA_0 over RCA_0. The entry '$\leq \mathsf{ACA}_0$' indicates that the corresponding implication is provable in ACA_0 but a reversal is not yet known. The table also provides references for the true implications.

	$\mathcal{P}^\flat(Q)$	$\mathcal{P}^\sharp(Q)$	$\mathcal{P}^\flat_{\mathrm{f}}(Q)$	$\mathcal{P}^\sharp_{\mathrm{f}}(Q)$
Q wqo $\Rightarrow$ $\bullet$ wqo	false	false	ACA_0 [Mar05, 5.10] Thm 2.5	false
Q bqo $\Rightarrow$ $\bullet$ bqo	$\leq \mathsf{ACA}_0$ [Mar05, 5.4]	RCA_0 [Mar05, 5.6]	RCA_0 [Mar05, 5.4]	RCA_0 [Mar05, 5.4]
Q wqo $\Rightarrow$ $\mathcal{A}(\bullet)$ Noeth.	false	false	ACA_0 Thm 4.7	false
Q wqo $\Rightarrow$ $\mathcal{U}(\bullet)$ Noeth.	ACA_0 Thm 4.7	ACA_0 Thm 4.7	ACA_0 Thm 4.7	ACA_0 Thm 4.7

If Q is a countable quasi-order, then $\mathcal{P}_{\mathrm{f}}(Q)$ is also countable and hence easy to manage in second-order arithmetic. The spaces $\mathcal{A}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$, $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$, and $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ fit very nicely into Dorais's framework of countable second-countable spaces in second-order arithmetic [Dor11], and so we consider the equivalence of items (i)–(iv) in Theorem 4.7 as not only contributing to the reverse mathematics of wqo's but also as a proof-of-concept example of the usefulness of Dorais's framework.

On the other hand, if Q is infinite, then $\mathcal{P}(Q)$ is uncountable and hence neither it nor the basic open sets of $\mathcal{U}(\mathcal{P}^\flat(Q))$ and $\mathcal{U}(\mathcal{P}^\sharp(Q))$ exist as sets in second-order arithmetic. Thus for items (v) and (vi) of Theorem 4.7, we code $\mathcal{U}(\mathcal{P}^\flat(Q))$ and $\mathcal{U}(\mathcal{P}^\sharp(Q))$ using a scheme that is broadly similar to the usual coding of complete separable metric spaces in second-order arithmetic (as detailed in [Sim09, Section II.5], for example). This scheme can be adapted to deal with quite general second-countable topological spaces, including the countably based MF spaces of [Mum06].

To prove the reversals of Theorem 4.7, we isolate a way of constructing recursive partial orders such that every sequence witnessing that such a partial order is not a wqo computes $0'$. These partial orders generalize the recursive linear order of type $\omega + \omega^*$ used in [MS11, FM12, FM14] in which every sequence witnessing that the linear order is not a well-order computes $0'$. The construction is introduced in Definition 4.2, and its main property is proved in Lemma 4.4.

The plan of the paper is as follows. In Section 2 we give some background concerning reverse mathematics in general, the reverse mathematics of well-quasi-orders, and Dorais's coding of countable second-countable topological spaces. Section 3 covers the details of expressing the notion of Noetherian space in second-order arithmetic, both in the countable second-countable case and in the uncountable case. In this section we also show that ACA_0 proves statements (ii)-(vi) of Theorem 4.7. The reversals of these implications are proved in Section 4, using the construction mentioned in the previous paragraph.

2. Background

2.1. **Reverse mathematics.** Reverse mathematics is a foundational program introduced by Friedman [Fri75] with the goal of classifying the theorems of ordinary mathematics by their proof-theoretic strengths. Theorem φ is considered stronger than theorem ψ if φ requires stronger axioms to prove than ψ does or, equivalently, if φ implies ψ but not conversely over some fixed weak base theory. The usual setting for reverse mathematics is second-order arithmetic. The language of second-order arithmetic is a two-sorted language, with first-order variables (intended to range over natural numbers) and second-order variables (intended to range over sets of natural numbers), and the membership relation to connect the two sorts. In this setting a remarkable phenomenon is that a natural theorem φ is most often equivalent to some well-known theory T over the base theory B. This means that $T \vdash \varphi$ and that $B + \varphi \vdash \psi$ for every $\psi \in T$. The proofs of the axioms of T from $B + \varphi$ is called a *reversal*, from which 'reverse mathematics' gets its name. This article is only concerned with the standard base theory RCA_0 and the theory ACA_0, so we give only the definitions of these theories and refer the reader to [Sim09] for a comprehensive treatment of the reverse mathematics program.

The axioms of RCA_0 are: a first-order sentence expressing that $\mathbb{N}$ is a discretely ordered commutative semi-ring with identity; the Σ^0_1 *induction scheme*, which consists of the universal closures (by both first- and second-order quantifiers) of all formulas of the form

$$[\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))] \to \forall n \varphi(n),$$

where φ is Σ^0_1; and the Δ^0_1 *comprehension scheme*, which consists of the universal closures (by both first- and second-order quantifiers) of all formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where φ is Σ^0_1, ψ is Π^0_1, and X is not free in φ.

The system RCA_0 is taken as the standard base system in reverse mathematics. The name 'RCA_0' stands for 'recursive comprehension axiom', which refers to the Δ^0_1 comprehension scheme because a set X is Δ^0_1 in a set Y if and only if X is recursive in Y. Thus in RCA_0, to define a set by comprehension, one must compute that set from an existing set. For this reason, we think of RCA_0 as capturing what might be called 'recursive mathematics' or 'effective mathematics'. The subscript '0' in 'RCA_0' refers to the fact that induction in RCA_0 is limited to Σ^0_1 formulas (and to Π^0_1 formulas because RCA_0 proves the Π^0_1 induction scheme; see [Sim09, Corollary II.3.10]). Despite being a weak system, several interesting and familiar theorems are provable in RCA_0, such as the intermediate value theorem and the fact that every field has an algebraic closure (though RCA_0 does *not* suffice to prove that algebraic closures are unique). See [Sim09, Chapter II] for more about RCA_0.

The axioms of ACA_0 are a first-order sentence expressing that $\mathbb{N}$ is a discretely ordered commutative semi-ring with identity; the *induction axiom*

$$\forall X[[0 \in X \wedge \forall n(n \in X \to n+1 \in X)] \to \forall n(n \in X)];$$

and the *arithmetical comprehension scheme*, which consists of the universal closures (by both first- and second-order quantifiers) of all formulas of the form

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where φ is an arithmetical formula in which X is not free. Equivalently, ACA_0 may be obtained by adding the arithmetical comprehension scheme to the axioms of RCA_0.

The name 'ACA_0' stands for 'arithmetical comprehension axiom', which refers to the arithmetical comprehension scheme. The subscript '0' in 'ACA_0' refers to the fact that induction in ACA_0 is essentially limited to arithmetical formulas, which is what can be derived from the induction axiom and the arithmetical comprehension scheme. In terms of computability, ACA_0 can be characterized by adding the statement "for every set X, the Turing jump of X exists" to RCA_0. Many familiar theorems are equivalent to ACA_0 over RCA_0, such as the Bolzano-Weierstraß theorem, the fact that every vector space has a basis, the fact that every commutative ring has a maximal ideal (the existence of prime ideals is weaker), König's lemma, and Ramsey's theorem for k-tuples for any fixed $k \geq 3$ (Ramsey's theorem for pairs is weaker, and Ramsey's theorem for arbitrary tuples is stronger). See [Sim09, Chapter III] for more about ACA_0.

A common strategy for proving that a theorem reverses to ACA_0 over RCA_0 is to take advantage of the following lemma, which states that ACA_0 is equivalent over RCA_0 to the statement that every injection has a range.

Lemma 2.1 ([Sim09, Lemma III.1.3])**.** *The following are equivalent over* RCA_0*.*

(i) ACA_0.
(ii) *If* $f\colon \mathbb{N} \to \mathbb{N}$ *is an injection, then there is a set* X *such that*

$$\forall n(n \in X \leftrightarrow \exists m(f(m) = n)).$$

2.2. **Well-quasi-orders in second-order arithmetic.** The reverse mathematics of wqo and bqo theory is a vibrant area with many results and many open problems, and we refer the reader to [Mar05] for a thorough introduction. Here we simply present the basic information needed for the work at hand.

In RCA_0 we can easily give the definition of quasi-order made in the introduction. In RCA_0, the official definition of a well-quasi-order is the following.

Definition 2.2 (RCA_0)**.** A *well-quasi-order* (*wqo*) is a quasi-order Q such that for every function $f\colon \mathbb{N} \to Q$, there are $m, n \in \mathbb{N}$ with $m < n$ such that $f(m) \leq_Q f(n)$.

We usually think of a function $f\colon \mathbb{N} \to Q$ as a sequence $(q_n)_{n\in\mathbb{N}}$ of elements of Q, in which case Definition 2.2 states that a quasi-order Q is a wqo if for every such sequence there are $m, n \in \mathbb{N}$ with $m < n$ such that $q_m \leq_Q q_n$. Thus Q is not a wqo if and only if there is an infinite so-called *bad sequence* $(q_n)_{n\in\mathbb{N}}$ such that $\forall m \forall n (m < n \to q_m \not\leq_Q q_n)$. For convenience, we also define a finite sequence $(q_n)_{n<k}$ to be bad if $\forall m \forall n (m < n < k \to q_m \not\leq_Q q_n)$.

Several of the well-known classically equivalent definitions of well-quasi-order are not equivalent over RCA_0. For example, RCA_0 proves that if Q is a wqo according to Definition 2.2, then Q has no infinite strictly descending chains and no infinite antichains [Mar05]. However, the reverse implication, that a quasi-order with no infinite strictly descending chains and no infinite antichains is a wqo according to Definition 2.2, is equivalent to CAC over RCA_0, where CAC states that every infinite partial order has an infinite chain or an infinite antichain [CMS04, Fri14]. Thus the equivalence of these two definitions of wqo is provable in $\mathsf{RCA}_0 + \mathsf{CAC}$ but not in RCA_0.

By using the usual coding of finite subsets of $\mathbb{N}$ as elements of $\mathbb{N}$, one readily sees that RCA_0 proves that if Q is a quasi-order, then $\mathcal{P}_{\mathrm{f}}(Q)$,

$$\leq_Q^\flat = \{(\mathbf{a}, \mathbf{b}) \in \mathcal{P}_{\mathrm{f}}(Q) \times \mathcal{P}_{\mathrm{f}}(Q) : (\forall a \in \mathbf{a})(\exists b \in \mathbf{b})(a \leq_Q b)\},$$

and

$$\leq_Q^\sharp = \{(\mathbf{a}, \mathbf{b}) \in \mathcal{P}_{\mathrm{f}}(Q) \times \mathcal{P}_{\mathrm{f}}(Q) : (\forall b \in \mathbf{b})(\exists a \in \mathbf{a})(a \leq_Q b)\}$$

all exist as sets. The proof that if Q is a quasi-order then $\mathcal{P}_{\mathrm{f}}^\flat(Q)$ and $\mathcal{P}_{\mathrm{f}}^\sharp(Q)$ are both quasi-orders is also straightforward in RCA_0.

A little care must be taken to describe $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp(Q)$ in RCA_0. First, if Q is infinite, then $\mathcal{P}(Q)$ is of course too big to exist as a set in any subsystem of second-order arithmetic. Second, if Q is a quasi-order and $E \subseteq Q$, then RCA_0 proves that $E{\downarrow}$ and $E{\uparrow}$ exist as sets when E is finite, but in general ACA_0 is required to prove that $E{\downarrow}$ and $E{\uparrow}$ exist as sets when E is infinite. Thus when working in RCA_0, '$A \leq_Q^\flat B$' and '$A \leq_Q^\sharp B$' must be interpreted by their respective defining formulas '$(\forall a \in A)(\exists b \in B)(a \leq_Q b)$' and '$(\forall b \in B)(\exists a \in A)(a \leq_Q b)$.' Under this interpretation, in RCA_0 one can prove that $A \leq_Q^\flat B \leq_Q^\flat C \to A \leq_Q^\flat C$ for all $A, B, C \subseteq Q$, one can work with sequences $(A_n)_{n\in\mathbb{N}}$ of subsets of Q, and one can consider whether or not there are $m, n \in \mathbb{N}$ with $m < n$ such that $A_m \leq_Q^\flat A_n$. Using this approach, Marcone has shown the following theorem.

Theorem 2.3 ([Mar05, Theorem 5.4 and Theorem 5.6])**.**

- RCA_0 *proves that if Q is a bqo, then $\mathcal{P}_{\mathrm{f}}^\flat(Q)$, $\mathcal{P}_{\mathrm{f}}^\sharp(Q)$, and $\mathcal{P}^\sharp(Q)$ are all bqo's.*
- ACA_0 *proves that if Q is a bqo, then $\mathcal{P}^\flat(Q)$ is a bqo.*

The reversal for the second item in the above theorem remains open.

Theorem 5.10 of [Mar05] states that ACA_0 is equivalent to the statement "if Q is a wqo then $\mathcal{P}_{\mathrm{f}}^\flat(Q)$ is a wqo" over $\mathsf{RCA}_0 + \mathsf{RT}^2_2$, where RT^2_2 is Ramsey's theorem for pairs and two colors. In the proof of the reversal of this theorem, RT^2_2 is only used to prove that $Q \times R$ is a wqo (where $(q_0, r_0) \leq_{Q\times R} (q_1, r_1)$ if and only if $q_0 \leq_Q q_1$ and $r_0 \leq_R r_1$) whenever Q and R are wqo's. Notice that by [CMS04, Corollary 4.7] RCA_0 and even the stronger system WKL_0 do not suffice to prove

this statement. We eliminate RT^2_2 from the reversal, thereby improving the result of [Mar05], via the following lemma.

Lemma 2.4 (RCA_0)**.** *Suppose that $\mathcal{P}^\flat_{\mathrm{f}}(Q)$ is a wqo whenever Q is a wqo. Then $Q \times R$ is a wqo whenever Q and R are wqo's.*

Proof. Let Q and R be wqo's, and let $((q_n, r_n))_{n\in\mathbb{N}}$ be a sequence of elements from $Q \times R$. We need to find m and n in $\mathbb{N}$ with $m < n$ such that $(q_m, r_m) \leq_{Q\times R} (q_n, r_n)$. To this end, let $Q \oplus R$ be the disjoint sum of Q and R, where $Q \oplus R = (Q \times \{0\}) \cup (R \times \{1\})$ and $(x, i) \leq_{Q\oplus R} (y, j)$ if and only if $(i = j = 0 \wedge x \leq_Q y) \vee (i = j = 1 \wedge x \leq_R y)$. It is easy to see that $Q \oplus R$ is a quasi-order, and by [Mar05, Lemma 5.13] it is a wqo. By hypothesis, $\mathcal{P}^\flat_{\mathrm{f}}(Q \oplus R)$ is also a wqo. Consider now the sequence $(\{(q_n, 0), (r_n, 1)\})_{n\in\mathbb{N}}$ of elements of $\mathcal{P}^\flat_{\mathrm{f}}(Q \oplus R)$, and let m and n in $\mathbb{N}$ be such that $m < n$ and $\{(q_m, 0), (r_m, 1)\} \leq^\flat_{Q\oplus R} \{(q_n, 0), (r_n, 1)\}$. It must be that $q_m \leq_Q q_n$ and $r_m \leq_R r_n$, so we have our desired $m < n$ such that $(q_m, r_m) \leq_{Q\times R} (q_n, r_n)$. □

Theorem 2.5. *The following are equivalent over* RCA_0.

(i) ACA_0.
(ii) *If Q is a wqo, then $\mathcal{P}^\flat_{\mathrm{f}}(Q)$ is a wqo.*

Proof. Use the proof of [Mar05, Theorem 5.10], but, in the reversal, prove that $L \times \omega$ is a wqo using Lemma 2.4 instead of RT^2_2. □

Theorem 4.5 below provides a different proof of the (ii)⇒(i) implication in Theorem 2.5.

2.3. **Countable second-countable topological spaces in second-order arithmetic.** A topological space is second-countable if it has a a countable base. Dorais in [Dor11] provides the appropriate definitions for working with countable second-countable spaces in RCA_0. We situate our work in his framework.

Definition 2.6 (RCA_0; [Dor11, Definition 2.1])**.** A *base* for a topology on a set X is an indexed sequence $\mathcal{U} = (U_i)_{i\in I}$ of subsets of X together with a function $k\colon X \times I \times I \to I$ such that the following properties hold.

- If $x \in X$, then $x \in U_i$ for some $i \in I$.
- If $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

Definition 2.7 (RCA_0; [Dor11, Definition 2.2])**.** A *countable second-countable space* is a triple $(X, \mathcal{U}, k)$ where $\mathcal{U} = (U_i)_{i\in I}$ and $k\colon X \times I \times I \to I$ form a base for a topology on the set X.

Subsets of countable second-countable spaces produce subspaces in a natural way.

Definition 2.8 (RCA_0; [Dor11, Definition 2.9])**.** Let $(X, \mathcal{U}, k)$ be a countable second-countable space with $\mathcal{U} = (U_i)_{i\in I}$. If $X' \subseteq X$, then the corresponding *subspace* $(X', \mathcal{U}', k')$ is defined by $U'_i = U_i \cap X'$ for all $i \in I$ and $k' = k \upharpoonright (X' \times I \times I)$.

Let $(X, \mathcal{U}, k)$ be a countable second-countable space, and recall that $\mathcal{P}_{\mathrm{f}}(I)$ denotes the set of finite subsets of I. Every function $h\colon \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ codes a so-called *effectively open* set, the idea being that h enumerates (sets of) indices of basic open sets whose union is the open set being coded. Explicitly, h is a code for the open set $G_h = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in h(n)} U_i$. Of course RCA_0 does not prove that such unions exist in general, so we must interpret the statement "x is in the effectively open set coded by h" as the formula '$(\exists n)(\exists i \in h(n))(x \in U_i)$'. To simplify notation, we abbreviate this formula by '$x \in G_h$' or by '$x \in \bigcup_{n\in\mathbb{N}} \bigcup_{i\in h(n)} U_i$'. Similarly, we also interpret h as coding the *effectively closed* set $F_h = X \setminus G_h = \bigcap_{n\in\mathbb{N}} \bigcap_{i\in h(n)} (X \setminus U_i)$. The reason for coding open sets by functions $\mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ rather than by functions $\mathbb{N} \to I$ is that the coding by functions $\mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ allows for a natural coding of the empty set via the function that is constantly $\emptyset$. Otherwise we would need to enforce that $U_i = \emptyset$ for some $i \in I$ for there to be a code for the empty set as an effectively open set.

Definition 2.9 (RCA_0; [Dor11, Definition 3.1])**.** Let $(X,\mathcal{U},k)$ be a countable second-countable space with $\mathcal{U} = (U_i)_{i\in I}$. We say that $(X,\mathcal{U},k)$ is *compact* if for every $h\colon \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ such that $X = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in h(n)} U_i$ (i.e., such that $(\forall x \in X)(\exists n \in \mathbb{N})(\exists i \in h(n))(x \in U_i)$), there is an $N \in \mathbb{N}$ such that $X = \bigcup_{n<N} \bigcup_{i\in h(n)} U_i$.

Dorais's [Dor11, Proposition 3.2] expresses that this definition of compactness does not depend on the choice of base $(\mathcal{U},k)$ for the topology on X. For matters of convenience, Definition 2.9 defines compactness in terms of covers by basic open sets. It is equivalent to define compactness in terms of covers by arbitrary open sets. Let $(X,\mathcal{U},k)$ be a countable second-countable space. A *sequence of effectively open sets* in $(X,\mathcal{U},k)$ is a function $g\colon \mathbb{N}\times\mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ thought of as coding the sequence $(G_n)_{n\in\mathbb{N}}$, where each G_n is $G_{g(n,\cdot)} = \bigcup_{m\in\mathbb{N}} \bigcup_{i\in g(n,m)} U_i$. Similarly, a *sequence of effectively closed sets* in $(X,\mathcal{U},k)$ is a function $g\colon \mathbb{N}\times\mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ thought of as coding the sequence $(F_n)_{n\in\mathbb{N}}$, where each F_n is $F_{g(n,\cdot)} = \bigcap_{m\in\mathbb{N}} \bigcap_{i\in g(n,m)} X \setminus U_i$. RCA_0 proves that a countable second-countable space $(X,\mathcal{U},k)$ is compact if and only if for every sequence $(G_n)_{n\in\mathbb{N}}$ of effectively open sets such that $X = \bigcup_{n\in\mathbb{N}} G_n$, there is an $N \in \mathbb{N}$ such that $X = \bigcup_{n<N} G_n$.

3. Noetherian spaces in second-order arithmetic

3.1. **Countable second-countable spaces.** Let $(X,\mathcal{U},k)$ be a countable second-countable space, and let G_h be an effectively open set. One is tempted to define compactness for G_h via Definition 2.8 and Definition 2.9 by saying that the subspace corresponding to G_h is compact. However, G_h is a coded object, and RCA_0 need not in general prove that it exists as a set, and so Definition 2.8 need not apply.[1] We simply extend Definition 2.9 as follows.

Definition 3.1 (RCA_0)**.** Let $(X,\mathcal{U},k)$ be a countable second-countable space with $\mathcal{U} = (U_i)_{i\in I}$. An effectively open set G_h is *compact* if for every function $f\colon \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ with $G_h \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{i\in f(n)} U_i$, there is an $N \in \mathbb{N}$ such that $G_h \subseteq \bigcup_{n<N} \bigcup_{i\in f(n)} U_i$.

Note that, *a posteriori*, if G_h is a compact effectively open set, then $G_h = \bigcup_{n<N} \bigcup_{i\in h(n)} U_i$ for some $N \in \mathbb{N}$, so RCA_0 does indeed prove that it exists as a set and that the corresponding subspace is compact.

Now we show that the equivalent definitions of Noetherian space are equivalent over RCA_0.

Proposition 3.2 (RCA_0)**.** *For a countable second-countable space $(X,\mathcal{U},k)$, the following statements are equivalent.*

(i) *Every effectively open set is compact.*
(ii) *For every effectively open set G_h, there is an $N \in \mathbb{N}$ such that $G_h = \bigcup_{n<N} \bigcup_{i\in h(n)} U_i$.*
(iii) *Every subspace is compact.*
(iv) *For every sequence $(G_n)_{n\in\mathbb{N}}$ of effectively open sets such that $\forall n(G_n \subseteq G_{n+1})$, there is an N such that $(\forall n > N)(G_n = G_N)$.*
(v) *For every sequence $(F_n)_{n\in\mathbb{N}}$ of effectively closed sets such that $\forall n(F_n \supseteq F_{n+1})$, there is an N such that $(\forall n > N)(F_n = F_N)$.*

Proof. The proof that (i), (ii), (iv), and (v) are equivalent is a simple exercise in chasing the definitions. Likewise, it is easy to see that each of (i), (ii), (iv), and (v) implies (iii). That (iii) implies the others requires proof. We show that (iii)⇒(ii). Let $(X,\mathcal{U},k)$ be a countable second-countable space, and let G_h be an effectively open set. We would like to apply (iii) to the subspace G_h, but this cannot be done in RCA_0 because G_h need not exist as a set.

[1] However, if Y is a Σ^0_1 subset of X and $f\colon \mathbb{N} \to Y$ is an enumeration of Y, then $(\mathbb{N},\mathcal{V},\ell)$, where $V_i = f^{-1}(U_i)$ and $\ell(n,i,j) = k(f(n),i,j)$ is a countable second-countable space that is essentially a homeomorphic copy of the subspace of $(X,\mathcal{U},k)$ corresponding to Y.

Assume for a contradiction that no N satisfies $G_h = \bigcup_{n<N} \bigcup_{i\in h(n)} U_i$, and fix an enumeration $g\colon \mathbb{N} \to X$ of the elements of G_h (which is possible because G_h has a Σ^0_1 definition). We recursively define an injection $f\colon \mathbb{N} \to X$. Assuming we defined $f(m)$ for $m < n$, let

$$F_n = \{f(m) : m < n\} \cup \bigcup_{m<n} \bigcup_{i\in h(m)} U_i,$$

and define $f(n) = g(k)$ where k is least such that $g(k) \notin F_n$. Such a k exists because otherwise $G_h \subseteq F_n$ is the union of finitely many of the sets $\bigcup_{i\in h(m)} U_i$.

Let $X' \subseteq X$ be an infinite set such that $(\forall x \in X')(\exists n \in \mathbb{N})(f(n) = x)$. (It is well-known and easy to show that RCA_0 proves that if $f\colon \mathbb{N} \to X$ is an injection, then the range of f is infinite and there is an infinite set X' of elements in the range of f. This is a formalization of the fact that every infinite r.e. set contains an infinite recursive subset.) By (iii), the subspace $(X', \mathcal{U}', k')$ (using the notation of Definition 2.8) is compact, so there is an $N \in \mathbb{N}$ such that $X' = \bigcup_{m<N} \bigcup_{i\in h(m)} (U_i \cap X')$. Now pick $n > N$ such that $f(n) \in X'$. We have the contradiction that both $f(n) \in \bigcup_{m<N} \bigcup_{i\in h(m)} U_i$ by the choice of N and $f(n) \notin \bigcup_{m<N} \bigcup_{i\in h(m)} U_i$ by the definition of f. □

Definition 3.3 (RCA_0)**.** A countable second-countable space is *Noetherian* if it satisfies any of the equivalent conditions from Proposition 3.2.

We can make any quasi-order a countable second-countable space by giving it either the Alexandroff topology or the upper topology.

Definition 3.4 (RCA_0)**.** Let Q be a quasi-order.

- A base for the Alexandroff topology on Q is given by $\mathcal{U} = (U_q)_{q\in Q}$, where $U_q = q{\uparrow}$ for each $q \in Q$, and $k(q, p, r) = q$. Let $\mathcal{A}(Q)$ denote the countable second-countable space $(Q, \mathcal{U}, k)$.
- A base for the upper topology on Q is given by $\mathcal{V} = (V_{\mathbf{i}})_{\mathbf{i}\in\mathcal{P}_{\mathrm{f}}(Q)}$, where $V_{\mathbf{i}} = Q \setminus (\mathbf{i}{\downarrow})$ for each $\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)$, and $\ell(q, \mathbf{i}, \mathbf{j}) = \mathbf{i} \cup \mathbf{j}$. Let $\mathcal{U}(Q)$ denote the countable second-countable space $(Q, \mathcal{V}, \ell)$.

That a quasi-order's Alexandroff topology is finer than its upper topology can be made precise via the following definition.

Definition 3.5 (RCA_0)**.** Let X be a set, and let $(\mathcal{U} = (U_i)_{i\in I}, k)$ and $(\mathcal{V} = (V_j)_{j\in J}, \ell)$ be two bases for topologies on X. We say that $(X, \mathcal{U}, k)$ is *effectively finer* than $(X, \mathcal{V}, \ell)$ and that $(X, \mathcal{V}, \ell)$ is *effectively coarser* than $(X, \mathcal{U}, k)$ if there is a function $f\colon J \times \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ such that

$$(\forall j \in J)(\forall x \in X)(x \in V_j \leftrightarrow (\exists n \in \mathbb{N})(\exists i \in f(j, n))(x \in U_i)).$$

Essentially, $(X, \mathcal{U}, k)$ is effectively finer than $(X, \mathcal{V}, \ell)$ if there is a sequence of sets $(G_j)_{j\in J}$ indexed by J and effectively open in $(X, \mathcal{U}, k)$ such that $(\forall j \in J)(G_j = V_j)$. It follows that every effectively open set in $(X, \mathcal{V}, \ell)$ is effectively open in $(X, \mathcal{U}, k)$, which leads to the following proposition.

Proposition 3.6 (RCA_0)**.** *Let $(X, \mathcal{U}, k)$ and $(X, \mathcal{V}, \ell)$ be countable second-countable spaces with $(X, \mathcal{U}, k)$ effectively finer than $(X, \mathcal{V}, \ell)$.*

- *If $(X, \mathcal{U}, k)$ is compact, then $(X, \mathcal{V}, \ell)$ is compact.*
- *If $(X, \mathcal{U}, k)$ is Noetherian, then $(X, \mathcal{V}, \ell)$ is Noetherian.*

Proposition 3.7 (RCA_0)**.** *Let Q be a quasi-order. Then $\mathcal{A}(Q)$ is effectively finer than $\mathcal{U}(Q)$.*

Proof. Let Q be a quasi-order, let $((U_q)_{q\in Q}, k)$ be the base for the Alexandroff topology on Q, and let $((V_{\mathbf{i}})_{\mathbf{i}\in\mathcal{P}_{\mathrm{f}}(Q)}, \ell)$ be the base for the upper topology on Q. Define $f\colon \mathcal{P}_{\mathrm{f}}(Q) \times \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(Q)$ by

$$f(\mathbf{i}, n) = \begin{cases} \{n\} & \text{if } n \in Q \setminus (\mathbf{i}{\downarrow}) \\ \emptyset & \text{otherwise.} \end{cases}$$

Then for each $\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)$, $\bigcup_{n\in\mathbb{N}} \bigcup_{q\in f(\mathbf{i},n)} U_q = \bigcup_{q\in Q\setminus(\mathbf{i}\downarrow)} U_q = Q \setminus (\mathbf{i}\downarrow) = V_{\mathbf{i}}$. So f witnesses that $\mathcal{A}(Q)$ is effectively finer that $\mathcal{U}(Q)$. □

The basic relationships among a quasi-order, its Alexandroff topology, and its upper topology are provable in RCA_0.

Proposition 3.8 (RCA_0)**.** *Let Q be a quasi-order.*

(i) *If $\mathcal{A}(Q)$ Noetherian, then $\mathcal{U}(Q)$ Noetherian.*
(ii) *Q is a wqo if and only if $\mathcal{A}(Q)$ is Noetherian.*

Proof. Item (i) follows from Proposition 3.7 and Proposition 3.6.

For item (ii), first suppose that $\mathcal{A}(Q)$ is not Noetherian, and let $(G_n)_{n\in\mathbb{N}}$ be an ascending sequence of effectively open sets that does not stabilize, meaning that $\forall n(G_n \subseteq G_{n+1})$ and $(\forall n)(\exists m > n)(G_n \subsetneqq G_m)$. We recursively define a bad sequence $(q_i)_{i\in\mathbb{N}}$ of elements of Q together with a sequence of indices $(n_i)_{i\in\mathbb{N}}$ such that $\forall i(q_i \in G_{n_i})$. Suppose that $(q_i)_{i<k}$ and $(n_i)_{i<k}$ have been defined so that $(q_i)_{i<k}$ is a finite bad sequence and that $(\forall i < k)(q_i \in G_{n_i})$. Search for a q_k and n_k such that $q_k \in G_{n_k}$ and $(\forall i < k)(q_i \not\leq_Q q_k)$. Such a pair must exist because there is an m such that $\bigcup_{i<k} G_{n_i} \subsetneqq G_m$, and in such a G_m there must be a q such that $(\forall i < k)(q_i \not\leq_Q q)$.

Conversely, suppose that Q is not a wqo, and let $(q_i)_{i\in\mathbb{N}}$ be a bad sequence of elements of Q. Then the sequence $(G_n)_{n\in\mathbb{N}}$, where $G_n = \{q_i : i < n\}\uparrow$ for each $n \in \mathbb{N}$, is an ascending sequence of effectively open sets that does not stabilize (in fact, $G_n \subsetneqq G_{n+1}$ for each n). □

Our analysis immediately yields the first two forward directions of Theorem 4.7.

Theorem 3.9 (ACA_0)**.** *If Q is a wqo, then $\mathcal{A}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ and $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ are Noetherian.*

Proof. Let Q be a wqo. Then $\mathcal{P}^\flat_{\mathrm{f}}(Q)$ is a wqo by Theorem 2.5, so $\mathcal{A}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ and $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ are Noetherian by Proposition 3.8. □

We defer the proof in ACA_0 of the statement "if Q is a wqo, then $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ is Noetherian" to Corollary 3.22, because a direct proof would essentially repeat the proof of Theorem 3.21, which is the analogous theorem for the more general $\mathcal{P}^\sharp(Q)$ case.

3.2. **Uncountable second-countable spaces.** If Q is an infinite quasi-order, then $\mathcal{P}(Q)$ is uncountable and thus $\mathcal{U}(\mathcal{P}^\flat(Q))$ and $\mathcal{U}(\mathcal{P}^\sharp(Q))$ cannot be coded as countable second-countable spaces. However, in second-order arithmetic we can still express, for example, that the sets $E_0, \dots, E_{n-1}$ code the basic closed set $\{E_0, \dots, E_{n-1}\}\downarrow^\flat$ by defining $A \in \{E_0, \dots, E_{n-1}\}\downarrow^\flat$ to mean that $(\exists i < n)(A \leq^\flat_Q E_i)$. (Notice here that we use the notation '$\downarrow^\flat$' to emphasize that the downward closure is with respect to $\leq^\flat_Q$. We similarly use '$\downarrow^\sharp$' to denote the downward closure with respect to $\leq^\sharp_Q$.) Although not immediately obvious from the definitions, it is the case that the spaces $\mathcal{U}(\mathcal{P}^\flat(Q))$ and $\mathcal{U}(\mathcal{P}^\sharp(Q))$ are second-countable (as Proposition 3.15 and Proposition 3.17 below imply). This situation and Definition 2.7 inspire the following meta-definition, each instance of which is made in RCA_0.

Definition 3.10 (instance-wise in RCA_0)**.** A (general) *second-countable space* is coded by a set $I \subseteq \mathbb{N}$ and formulas $\varphi(X)$, $\Psi_=(X, Y)$, and $\Psi_\in(X, n)$ (possibly with undisplayed parameters) such that the following properties hold.

- If $\varphi(X)$, then $\Psi_\in(X, i)$ for some $i \in I$.
- If $\varphi(X)$, $\Psi_\in(X, i)$, and $\Psi_\in(X, j)$ for some $i, j \in I$, then there is a $k \in I$ such that $\Psi_\in(X, k)$ and $\forall Y[\Psi_\in(Y, k) \to (\Psi_\in(Y, i) \wedge \Psi_\in(Y, j))]$.
- If $\varphi(X)$, $\varphi(Y)$, $\Psi_\in(X, i)$ for an $i \in I$, and $\Psi_=(X, Y)$, then $\Psi_\in(Y, i)$.

The intuition behind Definition 3.10 is that I is a set of codes for open sets, $\varphi(X)$ says "X codes a point", $\Psi_=(X,Y)$ says "X and Y code the same point", and $\Psi_\in(X,i)$ says "the point coded by X is in the open set coded by i". Effectively open sets and effectively closed sets are coded as they are in the countable case. A function $h\colon \mathbb{N}\to\mathcal{P}_\mathrm{f}(I)$ codes the effectively open set $G_h=\bigcup_{n\in\mathbb{N}}\bigcup_{i\in h(n)}\{X:\varphi(X)\wedge\Psi_\in(X,i)\}$ and the effectively closed set $F_h=\bigcap_{n\in\mathbb{N}}\bigcap_{i\in h(n)}\{X:\varphi(X)\wedge\neg\Psi_\in(X,i)\}$. Again, '$X\in G_h$' is an abbreviation for the formula '$\varphi(X)\wedge(\exists n\in\mathbb{N})(\exists i\in h(n))\Psi_\in(X,i)$', and similarly for '$X\in F_h$'. Sequences of effectively open sets and sequences of effectively closed sets are coded by functions $g\colon\mathbb{N}\times\mathbb{N}\to\mathcal{P}_\mathrm{f}(I)$, with $g(n,\cdot)$ coding the n^{th} set in the sequence.

As an example, the typical coding of complete separable metric spaces in RCA_0 (see [Sim09, Section II.5]) fits nicely into this framework. Here we first fix a set A and a metric $d\colon A\times A\to\mathbb{R}$, and we let $I=A\times\mathbb{Q}^+$. Then we let $\varphi(X)$ be a formula expressing that X is a rapidly converging Cauchy sequence of points in A, $\Psi_=(X,Y)$ be a formula expressing that the distance between the point coded by X and the point coded by Y is 0, and $\Psi_\in(X,\langle a,q\rangle)$ be a formula expressing that the distance between X and a is less than q.

Our framework easily accommodates also the countably based MF spaces studied in [Mum06] (by [MS10] these are exactly the second-countable T_1 spaces with the strong Choquet property), although in this case the existence of some X satisfying $\varphi(X)$ in general requires ACA_0, as shown in [LM06].

We also define compact spaces and Noetherian spaces as in the countable case.

Definition 3.11 (RCA_0)**.** A second-countable space coded by I, φ, $\Psi_=$, and $\Psi_\in$ is *compact* if for every $h\colon\mathbb{N}\to\mathcal{P}_\mathrm{f}(I)$ such that $\forall X(\varphi(X)\to(\exists n\in\mathbb{N})(\exists i\in h(n))\Psi_\in(X,i))$, there is an $N\in\mathbb{N}$ such that $\forall X(\varphi(X)\to(\exists n<N)(\exists i\in h(n))\Psi_\in(X,i))$.

Similarly, an effectively open set G_h in a second-countable space is *compact* if for every $f\colon\mathbb{N}\to\mathcal{P}_\mathrm{f}(I)$ such that $\forall X(X\in G_h\to(\exists n\in\mathbb{N})(\exists i\in f(n))\Psi_\in(X,i))$, there is an $N\in\mathbb{N}$ such that $\forall X(X\in G_h\to(\exists n<N)(\exists i\in f(n))\Psi_\in(X,i))$.

The equivalent characterizations of a Noetherian space given in Proposition 3.1 (i), (ii), (iv) and (v) are also equivalent in the uncountable case. We omit the "every subspace is compact" characterization because quantifying over subspaces of an uncountable space is difficult. One could quantify over a parameterized collection of subspaces of an uncountable space via a formula $\theta(X,Y)$ such that $\forall X\forall Y(\theta(X,Y)\to\varphi(X))$, in which case each Y corresponds to a subspace, but this is not useful for our purposes.

Proposition 3.12 (RCA_0)**.** *For a second-countable space, the following statements are equivalent.*

(i) *Every effectively open set is compact.*
(ii) *For every effectively open set G_h, there is an $N\in\mathbb{N}$ such that $\forall X(X\in G_h\leftrightarrow(\exists n<N)(\exists i\in h(n))\Psi_\in(X,i))$*
(iii) *For every sequence $(G_n)_{n\in\mathbb{N}}$ of effectively open sets such that $\forall n(G_n\subseteq G_{n+1})$ there is an N such that $(\forall n>N)(G_n=G_N)$.*
(iv) *For every sequence $(F_n)_{n\in\mathbb{N}}$ of effectively closed sets such that $\forall n(F_n\supseteq F_{n+1})$ there is an N such that $(\forall n>N)(F_n=F_N)$.*

Definition 3.13 (RCA_0)**.** A second-countable space is *Noetherian* if it satisfies any of the equivalent conditions from Proposition 3.12.

We now define $\mathcal{U}(\mathcal{P}^\flat(Q))$ and $\mathcal{U}(\mathcal{P}^\sharp(Q))$ as second-countable spaces.

Definition 3.14 (RCA_0)**.** Let Q be a quasi-order. The second-countable space $\mathcal{U}(\mathcal{P}^\flat(Q))$ is coded by the set $I=\mathcal{P}_\mathrm{f}(Q)$ and the formulas

- $\varphi(X)\coloneqq X\subseteq Q$;
- $\Psi_=(X,Y)\coloneqq X=Y$;
- $\Psi_\in(X,\mathbf{i})\coloneqq\mathbf{i}\subseteq X{\downarrow}$.

Notice that $\mathbf{i} = \emptyset$ codes the whole space and that the code for the intersection of the open sets coded by $\mathbf{i}$ and $\mathbf{j}$ is simply $\mathbf{i} \cup \mathbf{j}$. The idea behind $\Psi_\in(X, \mathbf{i})$ is that $\mathbf{i}$ codes the complement of the basic closed set $\{Q \setminus (q\uparrow) : q \in \mathbf{i}\}\downarrow^\flat$, whence

$$\begin{aligned} X \notin \{Q \setminus (q\uparrow) : q \in \mathbf{i}\}\downarrow^\flat &\Leftrightarrow (\forall q \in \mathbf{i})[X \not\leq^\flat_Q Q \setminus (q\uparrow)] \\ &\Leftrightarrow (\forall q \in \mathbf{i})[X \not\subseteq Q \setminus (q\uparrow)] \\ &\Leftrightarrow (\forall q \in \mathbf{i})[q \in X\downarrow] \Leftrightarrow \mathbf{i} \subseteq X\downarrow. \end{aligned}$$

The basic closed sets of the upper topology on $\mathcal{P}^\flat(Q)$ are those of the form $\{E_0, \dots, E_{n-1}\}\downarrow^\flat$ for arbitrary subsets $E_0, \dots, E_{n-1}$ of Q, whereas in Definition 3.14 we defined the basic closed sets to be those of the form $\{Q \setminus (q_0\uparrow), \dots, Q \setminus (q_{n-1}\uparrow)\}\downarrow^\flat$ for $q_0, \dots, q_{n-1} \in Q$. Thus to show that our definition really captures the upper topology on $\mathcal{P}^\flat(Q)$, we need to show that every $\{E_0, \dots, E_{n-1}\}\downarrow^\flat$ is effectively closed in the topology of Definition 3.14. In fact, it suffices to show that every set $\{E\}\downarrow^\flat$ is effectively closed in that topology because the effectively closed sets are closed under finite unions, and $\{E_0, \dots, E_{n-1}\}\downarrow^\flat = \{E_0\}\downarrow^\flat \cup \dots \cup \{E_{n-1}\}\downarrow^\flat$. Unfortunately, as the next proposition shows, proving that $\{E\}\downarrow^\flat$ is effectively closed in the topology of Definition 3.14 requires ACA_0 in general, even when Q is a well-order.

Proposition 3.15. *The following are equivalent over* RCA_0.

(i) ACA_0.
(ii) *If Q is a quasi-order and $E \subseteq Q$, then $\{E\}\downarrow^\flat$ is effectively closed in $\mathcal{U}(\mathcal{P}^\flat(Q))$.*
(iii) *If W is a well-order and $E \subseteq W$, then $\{E\}\downarrow^\flat$ is effectively closed in $\mathcal{U}(\mathcal{P}^\flat(W))$.*

Proof. For (i)⇒(ii), let Q be a quasi-order and let $E \subseteq Q$. Using ACA_0 to obtain the set $E\downarrow$, we can define a code for the effectively closed set $F = \bigcap_{q \notin E\downarrow} \{Q \setminus (q\uparrow)\}\downarrow^\flat$. Then for any $X \subseteq Q$,

$$X \in \{E\}\downarrow^\flat \;\Leftrightarrow\; X \subseteq E\downarrow \;\Leftrightarrow\; (\forall q \notin E\downarrow)[X \subseteq Q \setminus (q\uparrow)] \;\Leftrightarrow\; X \in F.$$

Thus $F = \{E\}\downarrow^\flat$, and so $\{E\}\downarrow^\flat$ is effectively closed.

The implication (ii)⇒(iii) is clear.

For (iii)⇒(i), let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. By Lemma 2.1, it suffices to show that the range of f exists. Let W be a linear order with the following properties:

- W has order-type $\omega + \omega^*$ (that is, every element of W has either finitely many predecessors or finitely many successors, and there are infinitely many instances of each);
- if W is not a well-order, then the range of f exists; and
- the ω part of W is Σ^0_1 in f.

That such a W can be constructed in RCA_0 is well-known (see for example [MS11, Lemma 4.2]). In fact, our main reversals in the next section are based on the construction of generalizations of such a W, and one may take $W = \Xi_f(\{x\}, x)$, where $\Xi_f(\{x\}, x)$ is the partial order (in this case linear order) from Definition 4.2.

If W is not a well-order, then the range of f exists by the assumptions on W. So suppose that W is a well-order. Let E be an infinite subset of the ω part of W, which exists for the same reason that the X' in the proof of Proposition 3.2 exists because the ω part of W is infinite and Σ^0_1 in f. By (iii), $\{E\}\downarrow^\flat$ is effectively closed, so there is an $h\colon \mathbb{N} \to \mathcal{P}_\mathrm{f}(\mathcal{P}_\mathrm{f}(Q))$ such that $\{E\}\downarrow^\flat = F_h$. For each $w \in W$, $\{w\} \in F_h = \{E\}\downarrow^\flat$ if and only if $w \in E\downarrow$ if and only if w is in the ω part of W. On the other hand, by definition $\{w\} \in F_h$ if and only if $\forall k(\forall \mathbf{i} \in h(k))(\mathbf{i} \not\subseteq w\downarrow)$, which is Π^0_1. Thus we have a Π^0_1 definition of the ω part of W. Thus by Δ^0_1 comprehension, the ω part of W exists. Therefore the ω^* part of W also exists, contradicting that W is a well-order. □

The previous proposition is not merely an artifact of a poorly chosen base in the definition of $\mathcal{U}(\mathcal{P}^\flat(Q))$. In fact, the proof of the reversal goes through whenever $\Psi_\in$ is defined in such a way

that $\Psi_\in(\{b\}, i)$ is Σ^0_1. Thus we can see Proposition 3.15 as expressing that the second-countability of the upper topology on $\mathcal{P}^\flat(Q)$ is equivalent to ACA_0.

Definition 3.16 (RCA_0)**.** Let Q be a quasi-order. The second-countable space $\mathcal{U}(\mathcal{P}^\sharp(Q))$ is coded by the set $I = \mathcal{P}_\mathrm{f}(Q)$ and the formulas

- $\varphi(X) \coloneqq X \subseteq Q$;
- $\Psi_=(X, Y) \coloneqq X = Y$;
- $\Psi_\in(X, \mathbf{i}) \coloneqq \mathbf{i} \cap X{\uparrow} = \emptyset$.

Again, $\mathbf{i} = \emptyset$ codes the whole space, and the code for the intersection of the open sets coded by $\mathbf{i}$ and $\mathbf{j}$ is $\mathbf{i} \cup \mathbf{j}$. The idea behind $\Psi_\in(X, \mathbf{i})$ is that $\mathbf{i}$ codes the complement of the basic closed set $\{\{q\} : q \in \mathbf{i}\}{\downarrow^\sharp}$, whence

$$\begin{aligned} X \notin \{\{q\} : q \in \mathbf{i}\}{\downarrow^\sharp} &\Leftrightarrow (\forall q \in \mathbf{i})(X \not\leq^\sharp_Q \{q\}) \\ &\Leftrightarrow (\forall q \in \mathbf{i})(q \notin X{\uparrow}) \Leftrightarrow \mathbf{i} \cap X{\uparrow} = \emptyset. \end{aligned}$$

Unlike in the $\flat$ case, RCA_0 suffices to show that $\mathcal{U}(\mathcal{P}^\sharp(Q))$ as defined in Definition 3.16 indeed captures the upper topology on $\mathcal{P}^\sharp(Q)$.

Proposition 3.17 (RCA_0)**.** *If Q is a quasi-order and $E \subseteq Q$, then $\{E\}{\downarrow^\sharp}$ is effectively closed in $\mathcal{U}(\mathcal{P}^\sharp(Q))$.*

Proof. Let Q be a quasi-order, and let $E \subseteq Q$. Then $\{E\}{\downarrow^\sharp}$ is exactly the effectively closed set $\bigcap_{e\in E}\{e\}{\downarrow^\sharp}$ because

$$X \in \{E\}{\downarrow^\sharp} \;\Leftrightarrow\; E \subseteq X{\uparrow} \;\Leftrightarrow\; (\forall e \in E)(e \in X{\uparrow}) \;\Leftrightarrow\; X \in \bigcap_{e\in E}\{e\}{\downarrow^\sharp}$$

for any $X \subseteq Q$. □

We now examine the correspondences between the countable spaces and the uncountable spaces. Our goal is to prove, in RCA_0, that if Q is a quasi-order, then $\mathcal{U}(\mathcal{P}^\flat(Q))$ is Noetherian implies that $\mathcal{U}(\mathcal{P}^\flat_\mathrm{f}(Q))$ is Noetherian and likewise with '$\sharp$' in place of '$\flat$.'

We warn the reader that the upper topology on $\mathcal{P}^\flat_\mathrm{f}(Q)$ is in general not the same as the subspace topology on $\mathcal{P}_\mathrm{f}(Q)$ induced by the upper topology on $\mathcal{P}^\flat(Q)$. For example, if Q is an infinite antichain and $q \in Q$, then the basic closed set $\{Q \setminus \{q\}\}{\downarrow^\flat}$ in the upper topology on $\mathcal{P}^\flat(Q)$ induces the closed set $\{\mathbf{x} \in \mathcal{P}_\mathrm{f}(Q) : q \notin \mathbf{x}\}$ in the subspace topology on $\mathcal{P}_\mathrm{f}(Q)$, but this set is not closed in the upper topology on $\mathcal{P}^\flat_\mathrm{f}(Q)$. To see this, observe that a closed set in the upper topology on $\mathcal{P}^\flat_\mathrm{f}(Q)$ that is not the whole space must be contained in a basic closed set of the form $\{\mathbf{e}_i : i < n\}{\downarrow^\flat}$, and if $\mathbf{x} \in \{\mathbf{e}_i : i < n\}{\downarrow^\flat}$, then $|\mathbf{x}| \leq \max\{|\mathbf{e}_i| : i < n\}$. A similar argument shows that these two topologies need not be the same even if Q is a well-order. Let $Q = \omega + 1$. Then the closed set $\omega{\downarrow^\flat}$ in $\mathcal{P}^\flat(Q)$ induces the closed set $\{\mathbf{x} \in \mathcal{P}_\mathrm{f}(Q) : \mathbf{x} \subseteq \omega\}$ in the subspace topology on $\mathcal{P}_\mathrm{f}(Q)$, but the closed sets in the upper topology on $\mathcal{P}^\flat_\mathrm{f}(Q)$ are all of the form $\{\mathbf{x} \in \mathcal{P}_\mathrm{f}(Q) : \mathbf{x} \subseteq q{\downarrow}\}$ for some $q \in Q$.

However, the upper topology on $\mathcal{P}^\sharp_\mathrm{f}(Q)$ is indeed the same as the subspace topology on $\mathcal{P}_\mathrm{f}(Q)$ induced by the upper topology on $\mathcal{P}^\sharp(Q)$. This is easy to see because $\{\{q_i\} : i < n\}{\downarrow^\sharp}$ contains the same finite sets regardless of whether it is interpreted as a basic closed set in the upper topology on $\mathcal{P}^\sharp(Q)$ or as a basic closed set in the upper topology on $\mathcal{P}^\sharp_\mathrm{f}(Q)$.

Lemma 3.18 (RCA_0)**.** *Let Q be a quasi-order.*

(i) *For every effectively closed set F in $\mathcal{U}(\mathcal{P}^\flat_\mathrm{f}(Q))$, there is an effectively closed set $\mathcal{F}$ in $\mathcal{U}(\mathcal{P}^\flat(Q))$ such that $(\forall \mathbf{x} \in \mathcal{P}_\mathrm{f}(Q))(\mathbf{x} \in \mathcal{F} \leftrightarrow \mathbf{x} \in F)$.*

(ii) *For every effectively closed set F in $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$, there is an effectively closed set $\mathcal{F}$ in $\mathcal{U}(\mathcal{P}^\sharp(Q))$ such that $(\forall \mathbf{x} \in \mathcal{P}_{\mathrm{f}}(Q))(\mathbf{x} \in \mathcal{F} \leftrightarrow \mathbf{x} \in F)$.*

Proof. We first prove (i) for basic closed sets. A basic closed set in $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ has the form $E{\downarrow^\flat}$ for some $E \in \mathcal{P}_{\mathrm{f}}(\mathcal{P}_{\mathrm{f}}(Q))$. Suppose that $E = \{\mathbf{e}_0, \dots, \mathbf{e}_{n-1}\}$, and consider the effectively closed set $\mathcal{F}_E$ in $\mathcal{U}(\mathcal{P}^\flat(Q))$ given by

$$\mathcal{F}_E = \bigcap_{\substack{(q_0,\dots,q_{n-1}) \in Q^n \\ (\forall i<n)(q_i \notin \mathbf{e}_i{\downarrow})}} \{Q \setminus (q_i{\uparrow}) : i < n\}{\downarrow^\flat}.$$

We show that $(\forall \mathbf{x} \in \mathcal{P}_{\mathrm{f}}(Q))(\mathbf{x} \in \mathcal{F}_E \leftrightarrow \mathbf{x} \in E{\downarrow^\flat})$. Suppose that $\mathbf{x} \in E{\downarrow^\flat}$. Then there is an $i < n$ such that $\mathbf{x} \leq^\flat_Q \mathbf{e}_i$, so $\mathbf{x} \subseteq \mathbf{e}_i{\downarrow}$, and therefore $(\forall q \notin \mathbf{e}_i{\downarrow})[\mathbf{x} \subseteq Q \setminus (q{\uparrow})]$. Hence $\mathbf{x} \in \mathcal{F}_E$. Conversely, suppose that $x \notin E{\downarrow^\flat}$. Then $(\forall i < n)(\mathbf{x} \not\leq^\flat_Q \mathbf{e}_i)$, so $(\forall i < n)(\mathbf{x} \not\subseteq \mathbf{e}_i{\downarrow})$, and finally $(\forall i < n)(\exists q_i \in \mathbf{x})(q_i \notin \mathbf{e}_i{\downarrow})$. Then $\mathcal{F}_E \subseteq \{Q \setminus (q_i{\uparrow}) : i < n\}{\downarrow^\flat}$ and $\mathbf{x} \notin \{Q \setminus (q_i{\uparrow}) : i < n\}{\downarrow^\flat}$. Thus $\mathbf{x} \notin \mathcal{F}_E$.

To complete the proof of (i), let us now consider the effectively closed set $F_h = \bigcap_{n\in\mathbb{N}} \bigcap_{E\in h(n)} E{\downarrow^\flat}$ in $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ coded by $h\colon \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(\mathcal{P}_{\mathrm{f}}(\mathcal{P}_{\mathrm{f}}(Q)))$. The procedure that produces (the code for) $\mathcal{F}_E$ given $E \in \mathcal{P}_{\mathrm{f}}(\mathcal{P}_{\mathrm{f}}(Q))$ is uniform in E, so from h we can produce $g\colon \mathbb{N} \times \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(\mathcal{P}_{\mathrm{f}}(Q))$ such that, for every $n \in \mathbb{N}$, $\mathcal{F}_{g(n,\cdot)} = \bigcap_{E\in h(n)} \mathcal{F}_E$. The intersection of a sequence of effectively closed sets is also an effectively closed set, so from g we can produce a code for the effectively closed set $\mathcal{F} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_{g(n,\cdot)} = \bigcap_{n\in\mathbb{N}} \bigcap_{E\in h(n)} \mathcal{F}_E$. Then, for any $\mathbf{x} \in \mathcal{P}_{\mathrm{f}}(Q)$,

$$\mathbf{x} \in \mathcal{F} \Leftrightarrow \mathbf{x} \in \bigcap_{n\in\mathbb{N}} \bigcap_{E\in h(n)} \mathcal{F}_E \Leftrightarrow \mathbf{x} \in \bigcap_{n\in\mathbb{N}} \bigcap_{E\in h(n)} E{\downarrow^\flat} \Leftrightarrow \mathbf{x} \in F_h.$$

Now we prove (ii) for basic closed sets. A basic closed set in $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ has the form $E{\downarrow^\sharp}$ for some $E \in \mathcal{P}_{\mathrm{f}}(\mathcal{P}_{\mathrm{f}}(Q))$. Suppose that $E = \{\mathbf{e}_0, \dots, \mathbf{e}_{n-1}\}$, and consider the effectively closed set $\mathcal{F}_E$ in $\mathcal{U}(\mathcal{P}^\sharp(Q))$ given by

$$\mathcal{F}_E = \bigcap_{(q_0,\dots,q_{n-1}) \in \mathbf{e}_0 \times \cdots \times \mathbf{e}_{n-1}} \{\{q_0\}, \dots, \{q_{n-1}\}\}{\downarrow^\sharp}.$$

We show that $(\forall \mathbf{x} \in \mathcal{P}_{\mathrm{f}}(Q))(\mathbf{x} \in \mathcal{F}_E \leftrightarrow \mathbf{x} \in E{\downarrow^\sharp})$. Suppose that $\mathbf{x} \in E{\downarrow^\sharp}$. Then there is an $i < n$ such that $\mathbf{x} \leq^\sharp_Q \mathbf{e}_i$, so $\mathbf{e}_i \subseteq \mathbf{x}{\uparrow}$, and therefore $(\forall q \in \mathbf{e}_i)(q \in \mathbf{x}{\uparrow})$. Hence $\mathbf{x} \in \mathcal{F}_E$. Conversely, suppose that $x \notin E{\downarrow^\sharp}$. Then $(\forall i < n)(\mathbf{x} \not\leq^\sharp_Q \mathbf{e}_i)$, so $(\forall i < n)(\mathbf{e}_i \not\subseteq \mathbf{x}{\uparrow})$, and therefore $(\forall i < n)(\exists q_i \in \mathbf{e}_i)(q_i \notin \mathbf{x}{\uparrow})$. Then $\mathcal{F}_E \subseteq \{\{q_0\}, \dots, \{q_{n-1}\}\}{\downarrow^\sharp}$ and $\mathbf{x} \notin \{\{q_0\}, \dots, \{q_{n-1}\}\}{\downarrow^\sharp}$. Thus $\mathbf{x} \notin \mathcal{F}_E$.

To complete the proof of (ii), given an effectively closed set F in $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$, we can produce an effectively closed set $\mathcal{F}$ in $\mathcal{U}(\mathcal{P}^\sharp(Q))$ such that $(\forall \mathbf{x} \in \mathcal{P}_{\mathrm{f}}(Q))(\mathbf{x} \in \mathcal{F} \leftrightarrow \mathbf{x} \in F)$ just as in the proof of (i). □

Theorem 3.19 (RCA_0)**.** *Let Q be a quasi-order.*

(i) *If $\mathcal{U}(\mathcal{P}^\flat(Q))$ is Noetherian, then $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ is Noetherian.*
(ii) *If $\mathcal{U}(\mathcal{P}^\sharp(Q))$ is Noetherian, then $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ is Noetherian.*

Proof. For (i), suppose that $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ is not Noetherian, and let $(F_n)_{n\in\mathbb{N}}$ be a non-stabilizing descending sequence of effectively closed sets in $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$. The proof of Lemma 3.18 (i) is uniform, so from $(F_n)_{n\in\mathbb{N}}$ we can produce a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of effectively closed sets in $\mathcal{U}(\mathcal{P}^\flat(Q))$ such that $(\forall n \in \mathbb{N})(\forall \mathbf{x} \in \mathcal{P}_{\mathrm{f}}(Q))(\mathbf{x} \in \mathcal{F}_n \leftrightarrow \mathbf{x} \in F_n)$. Define a new sequence $(\mathcal{H}_n)_{n\in\mathbb{N}}$ by $\mathcal{H}_n = \bigcap_{m\leq n} \mathcal{F}_m$ for each $n \in \mathbb{N}$. Then $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a descending sequence of closed sets in $\mathcal{U}(\mathcal{P}^\flat(Q))$ that does not

stabilize because $(F_n)_{n\in\mathbb{N}}$ does not stabilize and $(\forall n \in \mathbb{N})(\forall \mathbf{x} \in \mathcal{P}_{\mathrm{f}}(Q))(\mathbf{x} \in \mathcal{H}_n \leftrightarrow \mathbf{x} \in F_n)$. Hence $\mathcal{U}(\mathcal{P}^\flat(Q))$ is not Noetherian.

The proof of (ii) is the same, except we use Lemma 3.18 (ii) in place of Lemma 3.18 (i). □

Theorem 3.19 tells us that in the forward direction we need only work with the uncountable spaces and that in the reverse direction we need only work with the countable spaces.

Theorem 3.20 (ACA_0)**.** *If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat(Q))$ is Noetherian.*

Proof. We prove the contrapositive. Let Q be a quasi-order, suppose that $\mathcal{U}(\mathcal{P}^\flat(Q))$ is not Noetherian, and let $(F_n)_{n\in\mathbb{N}}$ be a non-stabilizing descending sequence of effectively closed sets. Our goal is to build a bad sequence in $\mathcal{P}^\flat_{\mathrm{f}}(Q)$, thereby proving that $\mathcal{P}^\flat_{\mathrm{f}}(Q)$ is not a wqo and hence, by Theorem 2.5, that Q is not a wqo.

Claim. *If F is an effectively closed set in $\mathcal{U}(\mathcal{P}^\flat(Q))$ and $A \subseteq Q$, then $A \in F$ if and only if $\mathcal{P}_{\mathrm{f}}(A) \subseteq F$.*

Proof of claim. The forward direction is clear because effectively closed sets are closed downward under $\leq^\flat_Q$, and $B \leq^\flat_Q A$ whenever $B \subseteq A$. For the reverse direction, suppose that F is coded by $h\colon \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(\mathcal{P}_{\mathrm{f}}(Q))$. Then $A \notin F$ means that $(\exists n \in \mathbb{N})(\exists \mathbf{i} \in h(n))(\mathbf{i} \subseteq A{\downarrow})$. As the witnessing $\mathbf{i}$ is finite, there is a finite $\mathbf{a} \subseteq A$ such that $\mathbf{i} \subseteq \mathbf{a}{\downarrow}$, and this $\mathbf{a}$ satisfies $\mathbf{a} \notin F$. □

It follows from the claim that if $F_n \setminus F_{n+1} \neq \emptyset$ for some $n \in \mathbb{N}$, then there is a finite $\mathbf{a} \in F_n \setminus F_{n+1}$. Suppose we have constructed a sequence $(\mathbf{a}_i)_{i<n}$ of elements of $\mathcal{P}_{\mathrm{f}}(Q)$ along with an increasing sequence $(m_i)_{i<n}$ such that $(\forall i < n)(\mathbf{a}_i \in F_{m_i} \setminus F_{m_i+1})$. As $(F_n)_{n\in\mathbb{N}}$ is non-stabilizing, we may extend the sequence by finding an $m_n > m_{n-1}$ (or an $m_n \geq 0$ if $n = 0$) and an $\mathbf{a}_n \in \mathcal{P}_{\mathrm{f}}(Q)$ that is in $F_{m_n} \setminus F_{m_n+1}$. In the end, $(\mathbf{a}_n)_{n\in\mathbb{N}}$ is a bad sequence because, for each $n \in \mathbb{N}$, $\mathbf{a}_n \in F_{m_n}$ but $(\forall i < n)(\mathbf{a}_i \notin F_{m_n})$, which means that $(\forall i < n)(\mathbf{a}_i \not\leq^\flat_Q \mathbf{a}_n)$. □

Theorem 3.21 (ACA_0)**.** *If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\sharp(Q))$ is Noetherian.*

Proof. Let Q be a wqo. Suppose for a contradiction that $\mathcal{U}(\mathcal{P}^\sharp(Q))$ is not Noetherian, and let $(F_n)_{n\in\mathbb{N}}$ be a non-stabilizing descending sequence of effectively closed sets. Our goal is to construct a bad sequence $(q_n)_{n\in\mathbb{N}}$ of elements of Q, contradicting that Q is a wqo.

Claim 1. *If F is an effectively closed set in $\mathcal{U}(\mathcal{P}^\sharp(Q))$ and $A \subseteq Q$, then $A \in F$ if and only if $(\exists \mathbf{a} \in \mathcal{P}_{\mathrm{f}}(A))(\mathbf{a} \in F)$*

Proof of claim. The backwards direction is clear because effectively closed sets are closed downward under $\leq^\sharp_Q$, and $A \leq^\sharp_Q B$ whenever $B \subseteq A$.

For the forward direction, the fact that Q is a wqo implies that there is a finite $\mathbf{a} \subseteq A$ such that $\mathbf{a} \leq^\sharp_Q A$, for otherwise it is easy to construct a bad sequence by choosing elements of A (see [Mar05, Lemma 4.8]). □

It follows from Claim 1 that two effectively closed sets are equal if and only if they agree on $\mathcal{P}_{\mathrm{f}}(Q)$. Therefore the equality of two effectively closed sets is an arithmetical property of the sets, and whether or not a descending sequence of effectively closed sets stabilizes is an arithmetical property of the sequence.

Suppose we have constructed a finite bad sequence $(q_i)_{i<k}$ of elements of Q such that the sequence $(F'_n)_{n\in\mathbb{N}}$ given by $F'_n = F_n \cap \bigcap_{i<k}\{q_i\}{\downarrow^\sharp}$ for each $n \in \mathbb{N}$ does not stabilize. Search for an $\mathbf{a} \in \mathcal{P}_{\mathrm{f}}(Q)$ and an ℓ such that $\mathbf{a} \in F'_\ell \setminus F'_{\ell+1}$. As $\mathbf{a} \in \bigcap_{i<k}\{q_i\}{\downarrow^\sharp}$, it must be that $\mathbf{a} \notin F_{\ell+1}$ and hence that $\mathbf{a} \notin \{\{r_j\} : j < m\}{\downarrow^\sharp}$ for some superset $\{\{r_j\} : j < m\}{\downarrow^\sharp}$ of $F_{\ell+1}$. Notice that $(\forall i < k)(\forall j < m)(q_i \not\leq_Q r_j)$ because if $q_i \leq_Q r_j$ for some $i < k$ and $j < m$, then $\mathbf{a} \leq^\sharp_Q \{q_i\} \leq^\sharp_Q \{r_j\}$ would contradict $\mathbf{a} \notin \{\{r_j\} : j < m\}{\downarrow^\sharp}$. Thus we could chose any r_j for $j < m$ to extend our bad sequence. We need to show that at least one such choice allows us to continue the construction.

Claim 2. *There is $j < m$ such that the sequence $(F'_n \cap \{r_j\}\downarrow^\sharp)_{n\in\mathbb{N}}$ does not stabilize.*

Proof of claim. Suppose for a contradiction that the sequence $(F'_n \cap \{r_j\}\downarrow^\sharp)_{n\in\mathbb{N}}$ stabilizes for each $j < m$. Let $N > \ell+1$ be large enough so that $(\forall j < m)(\forall n > N)(F'_n \cap \{r_j\}\downarrow^\sharp = F'_N \cap \{r_j\}\downarrow^\sharp)$. Such an N exists because the stabilization of $(F'_n \cap \{r_j\}\downarrow^\sharp)_{n\in\mathbb{N}}$ is an arithmetical property, and ACA_0 proves the bounding axiom for every arithmetical formula. For all $n \geq N$, we have that

$$\bigcup_{j<m} (F'_n \cap \{r_j\}\downarrow^\sharp) = F'_n \cap \bigcup_{j<m} \{r_j\}\downarrow^\sharp = F'_n \cap \{\{r_j\} : j < m\}\downarrow^\sharp = F'_n,$$

where the last equality holds because $F'_n \subseteq F'_{\ell+1} \subseteq \{\{r_j\} : j < m\}\downarrow^\sharp$, and that

$$\bigcup_{j<m} (F'_n \cap \{r_j\}\downarrow^\sharp) = \bigcup_{j<m} (F'_N \cap \{r_j\}\downarrow^\sharp) = F'_N.$$

Thus $(\forall n > N)(F'_n = F'_N)$, contradicting that the sequence $(F'_n)_{n\in\mathbb{N}}$ does not stabilize. □

Let q_k be r_j for the r_j guaranteed by Claim 2. Again, the procedure for computing q_k is arithmetical because the stabilization of a sequence is an arithmetical property. Then $(q_i)_{i<k+1}$ is a bad sequence and the sequence $(F_n \cap \bigcap_{i<k+1}\{q_i\}\downarrow^\sharp)_{n\in\mathbb{N}}$ does not stabilize, so we may continue the construction and build a contradictory infinite bad sequence. □

Corollary 3.22 (ACA_0)**.** *If Q is a wqo, then the countable second-countable space $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ is Noetherian.*

Proof. Immediate from Theorem 3.19 and Theorem 3.21. □

A similar corollary can be obtained from Theorem 3.19 and Theorem 3.20, providing a new proof that ACA_0 proves that if Q is a wqo, then the countable second-countable space $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ is Noetherian (which we already saw in Theorem 3.9).

Notice also that one could omit the application of Theorem 3.19 and prove directly, in ACA_0, that if Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ (respectively $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$) is Noetherian by implementing the proof of Theorem 3.20 (respectively Theorem 3.21) in the countable second-countable spaces setting. It is also possible to give a direct proof of Theorem 3.20 in which one builds a bad sequence in Q instead of in $\mathcal{P}^\flat_{\mathrm{f}}(Q)$ in the style of the proof of Theorem 3.21. Finally, recall that Proposition 3.15 shows, essentially, that without ACA_0 the definition of $\mathcal{U}(\mathcal{P}^\flat(Q))$ as a second-countable space (Definition 3.14) codes a coarser topology than the upper topology on $\mathcal{P}^\flat(Q)$. Nevertheless, we may still give an *ad hoc* definition of the upper topology on $\mathcal{P}^\flat(Q)$ in RCA_0 by interpreting a sequence $((E^n_i)_{i<m_n})_{n\in\mathbb{N}}$ of finite sequences of subsets of Q as a code for the closed set $\bigcap_{n\in\mathbb{N}}\{E^n_i : i < m_n\}\downarrow^\flat$. Then, by a proof in the style of that of Theorem 3.21, ACA_0 proves that if Q is a wqo, then this topology is Noetherian.

4. The reversals

The strategy for reversing, for example, the statement "if Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ is Noetherian" to ACA_0 is to produce a recursive quasi-order Q such that $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ is not Noetherian as witnessed by some uniformly r.e. descending sequence of closed sets, yet every bad sequence from Q computes $0'$. In [MS11, FM12, FM14] the main reversals to ACA_0 are based on the construction of a recursive linear order of type $\omega+\omega^*$ with the property that every descending sequence computes $0'$ (we used this technique in the proof of Proposition 3.15). We generalize this construction to partial orders. Given a finite partial order P and an $x \in P$, we define a recursive partial order $Q = \Xi(P, x)$ with the property that every bad sequence from Q computes $0'$. The special case $P = \{x\}$ produces a recursive linear order $\Xi(\{x\}, x)$ of type $\omega + \omega^*$ in which every descending sequence computes $0'$. As in the reversals using linear orders of type $\omega + \omega^*$, the notion of *true stage* is crucial.

Definition 4.1 (RCA_0)**.** Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. An $n \in \mathbb{N}$ is *f-true* (or simply *true*) if $(\forall k > n)(f(n) < f(k))$. An $n \in \mathbb{N}$ is *f-true* (or simply *true*) *at stage* $s \in \mathbb{N}$ if $n < s$ and $\forall k(n < k \leq s \to f(n) < f(k))$.

The notion of true stages is not new. Dekker [Dek54] introduced this notion (but he used the term 'minimal') to show that every non-recursive r.e. degree contains a hypersimple set. Indeed, given a recursive enumeration of a non-recursive r.e. set A, the set of non-true stages is hypersimple and Turing equivalent to A (see also [Rog87, Theorem XVI]). An early use of true stages in reverse mathematics is in [Sho93, Section 1]. In recursion theory, true stages are also known as non-deficiency stages (see [Soa87]).

The import of this definition is that the range of an injection $f\colon \mathbb{N} \to \mathbb{N}$ is Δ^0_1 in the join of f and any infinite set T of f-true stages: indeed for any $n \in \mathbb{N}$, $\exists m(f(m) = n)$ if and only if $(\forall m \in T)(f(m) > n \to (\exists k < m)(f(k) = n))$. Thus RCA_0 proves that, for any injection f, if there is an infinite set of f-true stages, then the range of f exists.

For the purposes of the following definition, given an injection $f\colon \mathbb{N} \to \mathbb{N}$, set

$$T_s = \{n < s : n \text{ is } f\text{-true at stage } s\},$$

and note that RCA_0 proves that the sequence $(T_s)_{s\in\mathbb{N}}$ exists.

Definition 4.2 (RCA_0)**.** Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection, let P be a finite partial order, and let $x \in P$. We define the partial order $Q = \Xi_f(P, x)$ as follows. Make $\mathbb{N}$ disjoint copies of P by letting $P_n = \{n\} \times P$ for each $n \in \mathbb{N}$, and let $x_n = (n, x)$ denote the copy of x in P_n. The domain of Q is $\bigcup_{n\in\mathbb{N}} P_n$. Define $\leq_Q$ in stages, where at stage s, $\leq_Q$ is defined on $\bigcup_{n\leq s} P_n$.

- At stage 0, $\leq_Q$ is simply $\leq_{P_0}$ on P_0.
- Suppose $\leq_Q$ is defined on $\bigcup_{n\leq s} P_s$. There are two cases.
  - (i) If $T_{s+1} \subsetneqq T_s \cup \{s\}$, let n_0 be the least element of $(T_s \cup \{s\}) \setminus T_{s+1}$, and place P_{s+1} immediately above x_{n_0}. That is, place the elements of P_{s+1} above all $y \in \bigcup_{n\leq s} P_s$ such that $y \leq_Q x_{n_0}$, below all $y \in \bigcup_{n\leq s} P_s$ such that $y >_Q x_{n_0}$, and incomparable with all $y \in \bigcup_{n\leq s} P_s$ that are incomparable with x_{n_0}.
  - (ii) If $T_{s+1} = T_s \cup \{s\}$, place P_{s+1} immediately below x_s. That is, place the elements of P_{s+1} above all $y \in \bigcup_{n\leq s} P_s$ such that $y <_Q x_s$, below all $y \in \bigcup_{n\leq s} P_s$ such that $y \geq_Q x_s$, and incomparable with all $y \in \bigcup_{n\leq s} P_s$ that are incomparable with x_s.

  In both cases, define $\leq_Q$ to be $\leq_{P_{s+1}}$ on P_{s+1}.

We could extend the construction of Definition 4.2 by starting from any sequence $(P_n)_{n\in\mathbb{N}}$ of finite (or even infinite) quasi-orders and any choice of elements $x_n \in P_n$ for each n, but we have no need for such generality. We just note that if each P_n is allowed to be infinite, then Lemma 4.3 is still provable in RCA_0, but Lemma 4.4 holds only if each P_n is a wqo, and its proof requires the infinite pigeonhole principle for an arbitrary number of colors (i.e., $\forall k\mathsf{RT}^1_k$, which is equivalent to $\mathsf{B}\Sigma^0_2$ over RCA_0 [Hir87]).

For the purposes of the next lemmas, $P_m \leq_Q x_n$ means $(\forall z \in P_m)(z \leq_Q x_n)$, $x_n \leq_Q P_m$ means $(\forall z \in P_m)(x_n \leq_Q z)$, and $P_m \mid_Q y$ means $(\forall z \in P_m)(z \mid_Q y)$.

Lemma 4.3 (RCA_0)**.** *Let $f\colon \mathbb{N} \to \mathbb{N}$, P be a finite partial order, $x \in P$, and $Q = \Xi_f(P, x)$, and consider $m, n \in \mathbb{N}$ with $n < m$.*

(i) If $n \in T_m$, then $P_m \leq_Q x_n$ and $(\forall y \in P_n)(x_n \mid_Q y \to P_m \mid_Q y)$.
(ii) If $n \notin T_m$, then $x_n \leq_Q P_m$.

Proof. We simultaneously prove (i) and (ii) by Σ^0_0 induction on m. The case $m = 0$ is vacuously true.

Consider $m+1$. First suppose that $T_{m+1} \subsetneqq T_m \cup \{m\}$ and thus that P_{m+1} is placed immediately above x_{n_0}, where n_0 is the least element of $(T_m \cup \{m\}) \setminus T_{m+1}$. Now, either $n_0 = m$ or $n_0 \in T_m \setminus T_{m+1}$,

and in both cases it must be that $f(m+1) < f(n_0)$ and $(\forall k \in (n_0, m])(f(n_0) < f(k))$. Notice that the interval $(n_0, m]$ is empty when $n_0 = m$.

For item (i), suppose that $n < m+1$ is such that $n \in T_{m+1}$. First we claim that $n < n_0$. As $n \in T_{m+1}$ and $n_0 \notin T_{m+1}$ we have $n \neq n_0$. Now, if $n_0 < n$, then either $f(n_0) < f(n)$, in which case $f(m+1) < f(n_0) < f(n)$, contradicting $n \in T_{m+1}$, or $f(n) < f(n_0)$, contradicting that $n_0 \notin T_{m+1}$ is only witnessed by $m+1 \neq n$. Hence $n < n_0$ as claimed. This implies that $n \in T_{n_0}$ because $n \in T_{m+1}$ and $n_0 < m+1$. By the induction hypothesis, $P_{n_0} \leq_Q x_n$ and $(\forall y \in P_n)(x_n \mid_Q y \to P_{n_0} \mid_Q y)$. Thus $x_{n_0} \leq_Q x_n$, so $P_{m+1} \leq_Q x_n$ because P_{m+1} is placed immediately above x_{n_0}. Furthermore, every $y \in P_n$ that is incomparable with x_n is incomparable with x_{n_0} and is hence incomparable with every element of P_{m+1}.

For item (ii), suppose that $n < m+1$ is such that $n \notin T_{m+1}$. If $n = n_0$, then P_{m+1} is placed immediately above $x_{n_0} = x_n$, as desired. Suppose $n_0 < n$. Then $n_0 \in T_n$ because $n < m+1$. By the induction hypothesis, $P_n \leq_Q x_{n_0}$, so $x_n \leq_Q x_{n_0}$. P_{m+1} is placed immediately above x_{n_0}, so $x_n \leq P_{m+1}$. If instead $n < n_0$, we claim that $n \notin T_{n_0}$. This is clear if $f(n_0) < f(n)$, so suppose that $f(n) < f(n_0)$. As $n \notin T_{m+1}$, there is a least $k \in (n, m+1]$ such that $f(k) < f(n)$. If $k = m+1$, then $n \in (T_m \cup \{m\}) \setminus T_{m+1}$, contradicting that n_0 was the least such number. If $k \in (n_0, m]$, then $f(k) < f(n) < f(n_0)$, contradicting that only $m+1$ witnesses that $n_0 \notin T_{m+1}$. Thus $k \in (n, n_0]$, which means that k witnesses that $n \notin T_{n_0}$, establishing the claim. By the induction hypothesis, $x_n \leq_Q P_{n_0}$, so $x_n \leq_Q x_{n_0}$. P_{m+1} is placed immediately above x_{n_0}, so $x_n \leq_Q P_{m+1}$. This concludes the proof of (i) and (ii) for $m+1$ in the $T_{m+1} \subsetneqq T_m \cup \{m\}$ case.

Now suppose that $T_{m+1} = T_m \cup \{m\}$, so that P_{m+1} is placed immediately below x_m. For item (i), suppose that $n \in T_{m+1}$. If $n = m$, then $P_{m+1} \leq_Q x_n = x_m$, and every $y \in P_n = P_m$ that is incomparable with $x_n = x_m$ is incomparable with every element of P_{m+1}. If $n < m$, then $n \in T_m$, so by the induction hypothesis $P_m \leq_Q x_n$ and $(\forall y \in P_n)(x_n \mid_Q y \to P_m \mid_Q y)$. Thus $x_m \leq_Q x_n$, and so $P_{m+1} \leq_Q x_n$ because P_{m+1} is placed immediately below x_m. Furthermore, every $y \in P_n$ such that $x_n \mid_Q y$ is incomparable with x_m and is hence incomparable with every element of P_{m+1}.

For item (ii), suppose that $n \notin T_{m+1} = T_m \cup \{m\}$. Then $n < m$ and $n \notin T_m$, so, by the induction hypothesis, $x_n \leq_Q P_m$. Thus $x_n \leq_Q x_m$. So $x_n \leq_Q P_{m+1}$ because P_{m+1} is placed immediately below x_m. This concludes the proof. □

Lemma 4.4 (RCA_0)**.** *Let $f\colon \mathbb{N} \to \mathbb{N}$, P, $x \in P$, and $Q = \Xi_f(P, x)$ be as above. If Q is not a wqo, then the range of f exists.*

Proof. Suppose that Q is not a wqo, and let $(q_i)_{i\in\mathbb{N}}$ be a bad sequence. We show that $n \in \mathbb{N}$ is true if and only if $\exists i(q_i \leq_Q x_n)$. Thus the set of true stages has both a Π^0_1 definition (as in Definition 4.1) and a Σ^0_1 definition, so it exists by Δ^0_1 comprehension. It follows that the range of f exists as explained following Definition 4.1.

Suppose that $n \in \mathbb{N}$ is true. The sequence $(q_i)_{i\in\mathbb{N}}$ is injective and each P_m is of the same finite size, so there must be an i and an m in $\mathbb{N}$ with $m > n$ such that $q_i \in P_m$. As n is a true stage, $n \in T_m$, so $P_m \leq_Q x_n$ by Lemma 4.3 (i). Thus $q_i \leq_Q x_n$ as desired. Conversely, suppose that $n \in \mathbb{N}$ is not true and suppose for a contradiction that $q_i \leq_Q x_n$ for some $i \in \mathbb{N}$. As n is not a true stage, there is some $k > n$ such that $f(k) < f(n)$, and therefore $n \notin T_m$ for all $m \geq k$. Let $m > k$ be such that P_m contains q_j for some $j > i$. Then $x_n \leq_Q P_m$ by Lemma 4.3 (ii), so we have that $q_i \leq_Q x_n \leq_Q q_j$, contradicting that $(q_i)_{i\in\mathbb{N}}$ is a bad sequence. □

We now present our main reversals.

Theorem 4.5. *The statement "if Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ is Noetherian" implies ACA_0 over RCA_0.*

Proof. Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. By Lemma 2.1, it suffices to show that the range of f exists. Let P be the partial order $P = \{x, y, z\}$ with $x <_P z$ and $x, z \mid_P y$, and let $Q = \Xi_f(P, x)$.

We show that $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ is not Noetherian. Then by our hypothesis Q is not a wqo, and the existence of the range of f follows from Lemma 4.4. To witness that $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ is not Noetherian, we define a sequence $(E_s)_{s\in\mathbb{N}}$ of finite subsets of $\mathcal{P}_{\mathrm{f}}(Q)$ so that the corresponding sequence of effectively closed sets $(F_s)_{s\in\mathbb{N}}$, given by $F_s = E_s{\downarrow}^{\flat}$ for each $s \in \mathbb{N}$, is descending but does not stabilize. Notice that in fact $(F_s)_{s\in\mathbb{N}}$ is a sequence of basic closed sets.

For each $s \in \mathbb{N}$, let

$$E_s = \{\mathbf{a}_s, \mathbf{b}_s\} \cup \{\mathbf{b}_n : n \in T_s\},$$

where

$$\mathbf{a}_s = \{x_s, y_s\} \cup \{y_n : n \in T_s\} \quad \text{and} \quad \mathbf{b}_s = \{z_s\} \cup \{y_n : n \in T_s\}.$$

We need to show that $F_s \supsetneqq F_{s+1}$ for each $s \in \mathbb{N}$. As $T_{s+1} \subseteq T_s \cup \{s\}$, by the definition of E_s we always have that $\{\mathbf{b}_n : n \in T_{s+1}\} \subseteq E_s$. Thus to prove the inclusion, we focus on $\mathbf{a}_{s+1}$ and $\mathbf{b}_{s+1}$.

First suppose that $T_{s+1} \subsetneqq T_s \cup \{s\}$, and let n_0 be the least element of $(T_s \cup \{s\}) \setminus T_{s+1}$. By the construction of Q, P_{s+1} is placed between x_{n_0} and z_{n_0}, and therefore $x_{s+1}, y_{s+1}, z_{s+1} <_Q z_{n_0}$. As argued in the proof of Lemma 4.3, it must be that $f(s+1) < f(n_0)$ and $(\forall k \in (n_0, s])(f(n_0) < f(k))$. Therefore $(\forall k \in [n_0, s])(f(s+1) < f(k))$, and $s+1$ witnesses that no element in the interval $[n_0, s]$ is true. This implies that $T_{s+1} \subseteq T_{n_0}$. We now see that $E_s{\downarrow}^{\flat} \supseteq E_{s+1}{\downarrow}^{\flat}$: $\mathbf{a}_{s+1}, \mathbf{b}_{s+1} \leq_Q^{\flat} \mathbf{b}_{n_0}$ because $x_{s+1}, y_{s+1}, z_{s+1} <_Q z_{n_0}$ and $\{y_n : n \in T_{s+1}\} \subseteq \{y_n : n \in T_{n_0}\}$, and $\mathbf{b}_{n_0} \in E_s$ because either $n_0 = s$ or $n_0 \in T_s$.

We now show that $E_s{\downarrow}^{\flat} \supsetneqq E_{s+1}{\downarrow}^{\flat}$ by showing that $\mathbf{b}_{n_0} \notin E_{s+1}{\downarrow}^{\flat}$. This means that we need to show that $\mathbf{b}_{n_0} \not\leq_Q^{\flat} \mathbf{a}_{s+1}$, $\mathbf{b}_{n_0} \not\leq_Q^{\flat} \mathbf{b}_{s+1}$, and $\mathbf{b}_{n_0} \not\leq_Q^{\flat} \mathbf{b}_n$ for each $n \in T_{s+1}$. Notice that $x_{s+1}, y_{s+1}, z_{s+1} <_Q z_{n_0}$, and if $n \in T_{s+1} \subseteq T_{n_0}$, then $z_{n_0} \mid_Q y_n$ by Lemma 4.3 (i). Hence $z_{n_0} \notin \mathbf{a}_{s+1}{\downarrow}$ and $z_{n_0} \notin \mathbf{b}_{s+1}{\downarrow}$. As $z_{n_0} \in \mathbf{b}_{n_0}$, it follows that $\mathbf{b}_{n_0} \not\leq_Q^{\flat} \mathbf{a}_{s+1}$ and $\mathbf{b}_{n_0} \not\leq_Q^{\flat} \mathbf{b}_{s+1}$. Now fix $n \in T_{s+1}$, and note that $y_n \in \mathbf{b}_{n_0}$ because $T_{s+1} \subseteq T_{n_0}$. However, $y_n \notin \mathbf{b}_n{\downarrow}$ because $y_n \mid_Q z_n$ by the definition of P, and $y_n \mid_Q y_\ell$ for all $\ell \in T_n$ by Lemma 4.3 (i). Thus $\mathbf{b}_{n_0} \not\leq_Q^{\flat} \mathbf{b}_n$.

Now suppose that $T_{s+1} = T_s \cup \{s\}$. Then obviously $\{y_n : n \in T_{s+1}\} = \{y_n : n \in T_s\} \cup \{y_s\}$ and, since in this case P_{s+1} is placed immediately below x_s, we have $x_{s+1}, y_{s+1}, z_{s+1} <_Q x_s$. Thus $\mathbf{a}_{s+1}, \mathbf{b}_{s+1} \leq_Q^{\flat} \mathbf{a}_s$, and so $E_s{\downarrow}^{\flat} \supseteq E_{s+1}{\downarrow}^{\flat}$. We show that $E_s{\downarrow}^{\flat} \supsetneqq E_{s+1}{\downarrow}^{\flat}$ by showing that $\mathbf{a}_s \notin E_{s+1}{\downarrow}^{\flat}$. We already noticed that $x_{s+1}, y_{s+1}, z_{s+1} <_Q x_s$. If $n \in T_{s+1}$, then either $n = s$, in which case $x_s \mid_Q y_n$ by the definition of P, or $n \in T_s$, in which case $x_s \mid_Q y_n$ by Lemma 4.3 (i). This shows that $x_s \notin \mathbf{a}_{s+1}{\downarrow} \cup \mathbf{b}_{s+1}{\downarrow}$ and thus (because $x_s \in \mathbf{a}_s$) that $\mathbf{a}_s \not\leq_Q^{\flat} \mathbf{a}_{s+1}$ and $\mathbf{a}_s \not\leq_Q^{\flat} \mathbf{b}_{s+1}$. For $n \in T_{s+1} = T_s \cup \{s\}$, we have that $\mathbf{a}_s \not\leq_Q^{\flat} \mathbf{b}_n$ because $y_n \in \mathbf{a}_s$ but, as explained in the preceding paragraph, $y_n \notin \mathbf{b}_n{\downarrow}$. Thus $\mathbf{a}_s \not\leq_Q^{\flat} \mathbf{b}_n$ for each $n \in T_{s+1}$ and therefore $\mathbf{a}_s \notin E_{s+1}{\downarrow}^{\flat}$.

This completes the proof that $(F_s)_{s\in\mathbb{N}}$ witnesses that $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ is not Noetherian. □

Notice that Theorem 4.5 gives an alternate reversal for Theorem 2.5 because the statement "if Q is a wqo, then $\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ is a wqo" implies the statement "if Q is a wqo, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ is Noetherian" over RCA_0 by Proposition 3.8. Thus we may see Theorem 4.5 as a strengthening of the reversal in Theorem 2.5.

Theorem 4.6. *The statement "if Q is a wqo, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ is Noetherian" implies* ACA_0 *over* RCA_0.

Proof. Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. Let P be the partial order $P = \{x, y\}$ with $x \mid_P y$, and let $Q = \Xi_f(P, x)$. As in the proof of Theorem 4.5, it suffices to show that $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ is not Noetherian, and then appeal to the hypothesis, Lemma 2.1, and Lemma 4.4.

To show that $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ is not Noetherian, we define a sequence $(E_s)_{s\in\mathbb{N}}$ of finite subsets of $\mathcal{P}_{\mathrm{f}}(Q)$ so that the sequence of effectively closed sets $(F_s)_{s\in\mathbb{N}}$, where $F_s = \bigcap_{t\leq s} E_t{\downarrow}^{\sharp}$ for each $s \in \mathbb{N}$, is descending but does not stabilize. We define the sequence $(E_s)_{s\in\mathbb{N}}$ in stages along with sequences

of elements $(\mathbf{a}_s)_{s\in\mathbb{N}}$ and $(\mathbf{b}_s)_{s\in\mathbb{N}}$ from $\mathcal{P}_{\mathrm{f}}(Q)$. We ensure that E_s is always a subset of $\{\mathbf{a}_t : t \leq s\} \cup \{\mathbf{b}_t : t \leq s\}$ and always contains $\mathbf{a}_s$ and $\mathbf{b}_s$. Among the sets in E_s, $\mathbf{a}_s$ is the unique set containing x_s, and $\mathbf{b}_s$ is the unique set containing y_s.

At stage 0, let $\mathbf{a}_0 = \{x_0\}$, let $\mathbf{b}_0 = \{y_0\}$, and let $E_0 = \{\mathbf{a}_0, \mathbf{b}_0\}$. At stage $s+1$, the definition of E_{s+1} proceeds according to the construction of Q.

(i) If $T_{s+1} \subsetneqq T_s \cup \{s\}$ and n_0 is the least element of $(T_s \cup \{s\}) \setminus T_{s+1}$, then set $\mathbf{a}_{s+1} = \mathbf{b}_{n_0} \cup \{x_{s+1}\}$, $\mathbf{b}_{s+1} = \mathbf{b}_{n_0} \cup \{y_{s+1}\}$, and $E_{s+1} = (E_{n_0} \setminus \{\mathbf{a}_{n_0}, \mathbf{b}_{n_0}\}) \cup \{\mathbf{a}_{s+1}, \mathbf{b}_{s+1}\}$.

(ii) If $T_{s+1} = T_s \cup \{s\}$, then set $\mathbf{a}_{s+1} = (\mathbf{a}_s \setminus \{x_s\}) \cup \{x_{s+1}\}$, $\mathbf{b}_{s+1} = (\mathbf{b}_s \setminus \{y_s\}) \cup \{y_{s+1}\}$, and $E_{s+1} = (E_s \setminus \{\mathbf{a}_s\}) \cup \{\mathbf{a}_{s+1}, \mathbf{b}_{s+1}\}$.

The sequence $(F_s)_{s\in\mathbb{N}}$ is clearly descending by definition; we must show that it is strictly descending. To do this, we identify some helpful properties of the sequences $(\mathbf{a}_s)_{s\in\mathbb{N}}$, $(\mathbf{b}_s)_{s\in\mathbb{N}}$, and $(E_s)_{s\in\mathbb{N}}$. First, observe that $\forall s(\mathbf{a}_s \setminus \{x_s\} = \mathbf{b}_s \setminus \{y_s\})$ by an easy induction argument.

Claim 1. *$\forall s(\bigcup E_s$ is an antichain in $Q)$.*

Proof of claim. By Σ^0_0 induction on s. The case $s = 0$ is clear because $x_0 \mid_Q y_0$. Consider $s+1$. First suppose that E_s is defined according to (i) and that n_0 is the least element of $(T_s \cup \{s\}) \setminus T_{s+1}$. By the induction hypothesis, $\bigcup E_{n_0}$ is an antichain. By the construction of Q, x_{s+1} and y_{s+1} are placed immediately above $x_{n_0} \in \bigcup E_{n_0}$ and hence are incomparable with the elements of $\bigcup E_{n_0} \setminus \{x_{n_0}\}$. Thus $\bigcup E_{s+1} \subseteq (\bigcup E_{n_0} \setminus \{x_{n_0}\}) \cup \{x_{s+1}, y_{s+1}\}$ (in fact the reader can check that $\bigcup E_{s+1} = (\bigcup E_{n_0} \setminus \{x_{n_0}\}) \cup \{x_{s+1}, y_{s+1}\}$) is an antichain.

Now suppose that E_{s+1} is defined according to (ii). By the induction hypothesis, $\bigcup E_s$ is an antichain. By the construction of Q, x_{s+1} and y_{s+1} are placed immediately below $x_s \in \bigcup E_s$ and hence are incomparable with the elements of $\bigcup E_s \setminus \{x_s\}$. Thus $\bigcup E_{s+1} \subseteq (\bigcup E_s \setminus \{x_s\}) \cup \{x_{s+1}, y_{s+1}\}$ (again, in fact the reader can check that $\bigcup E_{s+1} = (\bigcup E_s \setminus \{x_s\}) \cup \{x_{s+1}, y_{s+1}\})$ is an antichain. □

Claim 2. *$\forall s(\mathbf{a}_s \notin (E_s \setminus \{\mathbf{a}_s\})\downarrow^\sharp \wedge \mathbf{b}_s \notin (E_s \setminus \{\mathbf{b}_s\})\downarrow^\sharp)$.*

Proof of claim. By Σ^0_0 induction on s. The case $s = 0$ is clear. Consider $s+1$. First suppose that E_{s+1} is defined according to (i), and let n_0 be the least element of $(T_s \cup \{s\}) \setminus T_{s+1}$. Suppose that $\mathbf{a}_{s+1} \leq^\sharp_Q \mathbf{e}$ for some $\mathbf{e} \in E_{n_0} \setminus \{\mathbf{a}_{n_0}, \mathbf{b}_{n_0}\}$. Then $\mathbf{e} \subseteq \mathbf{a}_{s+1}\uparrow$. As $\mathbf{a}_{n_0}$ is the only element of E_{n_0} containing x_{n_0}, $x_{n_0} \notin \mathbf{e}$ and so $\mathbf{e} \cup \{x_{n_0}\}$ is an antichain by Claim 1; in particular, no element of $\mathbf{e}$ is $\geq_Q x_{n_0}$. However, $\mathbf{a}_{s+1}$ is $\mathbf{b}_{n_0} \cup \{x_{s+1}\}$, and $x_{s+1} \geq_Q x_{n_0}$. It follows that $\mathbf{e} \subseteq \mathbf{b}_{n_0}\uparrow$ and so $\mathbf{b}_{n_0} \in (E_{n_0} \setminus \{\mathbf{b}_{n_0}\})\downarrow^\sharp$. This contradiction to the induction hypothesis shows that $\mathbf{a}_{s+1} \notin (E_{s+1} \setminus \{\mathbf{a}_{n_0}, \mathbf{b}_{n_0}\})\downarrow^\sharp$. Finally, $\mathbf{a}_{s+1} \mid^\sharp_Q \mathbf{b}_{s+1}$ because $\mathbf{a}_{s+1} \cup \mathbf{b}_{s+1}$ is an antichain by Claim 1, which means that $x_{s+1} \notin \mathbf{b}_{s+1}\uparrow$ and $y_{s+1} \notin \mathbf{a}_{s+1}\uparrow$. Thus $\mathbf{a}_{s+1} \notin (E_{s+1} \setminus \{\mathbf{a}_{s+1}\})\downarrow^\sharp$. A similar argument shows that $\mathbf{b}_{s+1} \notin (E_{s+1} \setminus \{\mathbf{b}_{s+1}\})\downarrow^\sharp$.

Now consider the case that E_{s+1} is defined according to (ii), and suppose that $\mathbf{a}_{s+1} \leq^\sharp_Q \mathbf{e}$ for some $\mathbf{e} \in E_s \setminus \{\mathbf{a}_s\}$. Then $\mathbf{e} \subseteq \mathbf{a}_{s+1}\uparrow$. However, since $\mathbf{a}_{s+1}$ is given by replacing x_s with x_{s+1} in $\mathbf{a}_s$ and x_{s+1} is placed immediately below x_s, any $z \in \mathbf{e}$ that is $\geq_Q x_{s+1}$ is also $\geq_Q x_s$, and therefore $\mathbf{e} \subseteq \mathbf{a}_s\uparrow$ as well. So $\mathbf{a}_s \in (E_s \setminus \{\mathbf{a}_s\})\downarrow^\sharp$, which contradicts the induction hypothesis. Thus $\mathbf{a}_{s+1} \notin (E_s \setminus \{\mathbf{a}_s\})\downarrow^\sharp$. Since $\mathbf{a}_{s+1} \mid^\sharp_Q \mathbf{b}_{s+1}$, as argued in the previous case, we again have that $\mathbf{a}_{s+1} \notin (E_{s+1} \setminus \{\mathbf{a}_{s+1}\})\downarrow^\sharp$. A similar argument shows that $\mathbf{b}_{s+1} \notin (E_{s+1} \setminus \{\mathbf{b}_{s+1}\})\downarrow^\sharp$. □

Claim 3. *$(\forall s)(\forall i \leq s)(\mathbf{a}_s \in E_i\downarrow^\sharp \wedge \mathbf{b}_s \in E_i\downarrow^\sharp)$.*

Proof of claim. By Σ^0_0 induction on s. The case $s = 0$ is clear. Consider $s+1$. First suppose that E_{s+1} is defined according to (i), and let n_0 be the least element of $(T_s \cup \{s\}) \setminus T_{s+1}$. By the induction hypothesis for n_0, $(\forall i \leq n_0)(\mathbf{b}_{n_0} \in E_i\downarrow^\sharp)$. Since $\mathbf{a}_{s+1} \leq^\sharp_Q \mathbf{b}_{n_0}$ and $\mathbf{b}_{s+1} \leq^\sharp_Q \mathbf{b}_{n_0}$, it suffices to show that $(\forall i \leq s)(\mathbf{b}_{n_0} \in E_i\downarrow^\sharp)$. By definition, $\mathbf{b}_{n_0} \in E_{n_0}$; and if $\mathbf{b}_{n_0} \in E_i$ and E_{i+1} is defined according to (ii), then $\mathbf{b}_{n_0} \in E_{i+1}$. So if $\mathbf{b}_{n_0} \notin E_{i+1}$ for some $i+1 \in (n_0, s]$, it must be

because E_{i+1} is defined according to (i) and the least element n_1 of $(T_i \cup \{i\}) \setminus T_{i+1}$ is less than n_0. Then $f(i+1) < f(n_1) < f(n_0)$ because $i+1$ is the least number witnessing that n_1 is not true, contradicting that $n_0 \in T_s \cup \{s\}$. Hence $(\forall i \leq s)(\mathbf{b}_{n_0} \in E_i{\downarrow^\sharp})$.

Now suppose that E_{s+1} is defined according to (ii). By the induction hypothesis, $(\forall i \leq s)(\mathbf{a}_s \in E_i{\downarrow^\sharp})$. As $\mathbf{a}_{s+1} \leq_Q^\sharp \mathbf{a}_s$ and $\mathbf{b}_{s+1} \leq_Q^\sharp \mathbf{a}_s$ (because in this case $x_{s+1}, y_{s+1} <_Q x_s$), it follows that $(\forall i \leq s+1)(\mathbf{a}_{s+1} \in E_i{\downarrow^\sharp} \wedge \mathbf{b}_{s+1} \in E_i{\downarrow^\sharp})$ as well. □

We can now show that $(F_s)_{s\in\mathbb{N}}$ is strictly descending. Consider $s \in \mathbb{N}$. Suppose that $E_{s+1} = E_{n_0} \setminus (\{\mathbf{a}_{n_0}, \mathbf{b}_{n_0}\}) \cup \{\mathbf{a}_{s+1}, \mathbf{b}_{s+1}\}$ is defined according to (i), where n_0 is the least element of $(T_s \cup \{s\}) \setminus T_{s+1}$. Then $\mathbf{b}_{n_0} \in \bigcap_{t\leq s} E_t{\downarrow^\sharp} = F_s$ as shown in the proof of Claim 3. However, $\mathbf{a}_{s+1} <_Q^\sharp \mathbf{b}_{n_0}$ and $\mathbf{b}_{s+1} <_Q^\sharp \mathbf{b}_{n_0}$ because neither x_{s+1} nor y_{s+1} is above any element of $\mathbf{b}_{n_0}$ by Claim 1, and $\mathbf{b}_{n_0}$ is not $\leq_Q^\sharp$ any element of $E_{n_0} \setminus \{\mathbf{a}_{n_0}, \mathbf{b}_{n_0}\}$ by Claim 2. Thus $\mathbf{b}_{n_0} \notin E_{s+1}{\downarrow^\sharp}$, so $\mathbf{b}_{n_0} \notin F_{s+1}$.

Finally, suppose that $E_{s+1} = (E_s \setminus \{\mathbf{a}_s\}) \cup \{\mathbf{a}_{s+1}, \mathbf{b}_{s+1}\}$ is defined according to (ii). Then by Claim 3, $\mathbf{a}_s \in \bigcap_{t\leq s} E_t{\downarrow^\sharp} = F_s$. On the other hand, since neither x_{s+1} nor y_{s+1} is above any element of $\mathbf{a}_s$, $\mathbf{a}_{s+1} <_Q^\sharp \mathbf{a}_s$ and $\mathbf{b}_{s+1} <_Q^\sharp \mathbf{a}_s$, while $\mathbf{a}_s$ is not $\leq_Q^\sharp$ any element of $E_s \setminus \{\mathbf{a}_s\}$ by Claim 2. Thus $\mathbf{a}_s \notin E_{s+1}{\downarrow^\sharp}$, and so $\mathbf{a}_s \notin F_{s+1}$.

This completes the proof that $(F_s)_{s\in\mathbb{N}}$ witnesses that $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$ is not Noetherian. □

Theorem 4.7. *The following are equivalent over* RCA_0.

(i) ACA_0.
(ii) If Q is a wqo, then $\mathcal{A}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$ is Noetherian.
(iii) If Q is a wqo, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$ is Noetherian.
(iv) If Q is a wqo, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\sharp(Q))$ is Noetherian.
(v) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat(Q))$ is Noetherian.
(vi) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\sharp(Q))$ is Noetherian.

Proof. That (i) implies (ii) and (iii) is Theorem 3.9. That (i) implies (iv) is Corollary 3.22. That (i) implies (v) is Theorem 3.20. That (i) implies (vi) is Theorem 3.21. That (ii), (iii), and (v) imply (i) is Theorem 4.5. For (ii), use also Proposition 3.8, and for (v), use also Theorem 3.19 (i). That (iv) and (vi) imply (i) is Theorem 4.6. For (vi), use also Theorem 3.19 (ii). □

Upon hearing the third author speak about the results contained in this paper, Takashi Sato asked whether the converses of the statements in Theorem 4.7 hold, and, for those that do, what system is needed to prove them. First notice that $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$ can be Noetherian without Q being a wqo. Indeed, if Q is an infinite antichain, then all closed sets in $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$ are finite and thus $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$ is Noetherian. Nevertheless, RCA_0 easily proves that if $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$ is Noetherian, then Q is well-founded. The next proposition shows that the other converses are provable in RCA_0.

Proposition 4.8 (RCA_0)**.** *Let Q be a quasi-order. If $\mathcal{A}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$, $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\sharp(Q))$, $\mathcal{U}(\mathcal{P}^\flat(Q))$, or $\mathcal{U}(\mathcal{P}^\sharp(Q))$ is Noetherian, then Q is a wqo.*

Proof. We prove the contrapositive. The result for $\mathcal{A}(\mathcal{P}_{\mathrm{f}}^\flat(Q))$ follows from the fact that if Q is not wqo then $\mathcal{P}_{\mathrm{f}}^\flat(Q)$ is not wqo (because $q \mapsto \{q\}$ embeds Q into $\mathcal{P}_{\mathrm{f}}^\flat(Q)$) and from the second part of Proposition 3.8.

For the other spaces, first fix a bad sequence $(q_i)_{i\in\mathbb{N}}$ of elements of Q.

To see that $\mathcal{U}(\mathcal{P}^\flat(Q))$ is not Noetherian, let $F_n = \bigcap_{i\leq n}\{Q \setminus (q_i{\uparrow})\}{\downarrow^\flat}$. The sequence $(F_n)_{n\in\mathbb{N}}$ is a non-stabilizing descending sequence of effectively closed sets as witnessed by $\{q_{n+1}\} \in F_n \setminus F_{n+1}$.

To see that $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\sharp(Q))$ is not Noetherian, let $H_n = \{q_i : i \leq n\}{\downarrow^\sharp}$. The sequence $(H_n)_{n\in\mathbb{N}}$ is a non-stabilizing descending sequence of effectively closed sets as witnessed by $\{q_i : i \leq n\} \in H_n \setminus H_{n+1}$.

The result for $\mathcal{U}(\mathcal{P}^\sharp(Q))$ follows easily from the result for $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^\sharp(Q))$ and Theorem 3.19 (ii). □

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