



UNIVERSITY OF LEEDS

This is a repository copy of *The local counting function of operators of Dirac and Laplace type*.

White Rose Research Online URL for this paper:
<http://eprints.whiterose.ac.uk/111662/>

Version: Accepted Version

Article:

Li, L and Strohmaier, A orcid.org/0000-0002-8446-3840 (2016) The local counting function of operators of Dirac and Laplace type. *Journal of Geometry and Physics*, 104. C. pp. 204-228. ISSN 0393-0440

<https://doi.org/10.1016/j.geomphys.2016.02.006>

© 2016. This manuscript version is made available under the CC-BY-NC-ND 4.0 license
<http://creativecommons.org/licenses/by-nc-nd/4.0/>

Reuse

Unless indicated otherwise, fulltext items are protected by copyright with all rights reserved. The copyright exception in section 29 of the Copyright, Designs and Patents Act 1988 allows the making of a single copy solely for the purpose of non-commercial research or private study within the limits of fair dealing. The publisher or other rights-holder may allow further reproduction and re-use of this version - refer to the White Rose Research Online record for this item. Where records identify the publisher as the copyright holder, users can verify any specific terms of use on the publisher's website.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.



eprints@whiterose.ac.uk
<https://eprints.whiterose.ac.uk/>

THE LOCAL COUNTING FUNCTION OF OPERATORS OF DIRAC AND LAPLACE TYPE

LIANGPAN LI AND ALEXANDER STROHMAIER

ABSTRACT. Let P be a non-negative self-adjoint Laplace type operator acting on sections of a hermitian vector bundle over a closed Riemannian manifold. In this paper we review the close relations between various P -related coefficients such as the mollified spectral counting coefficients, the heat trace coefficients, the resolvent trace coefficients, the residues of the spectral zeta function as well as certain Wodzicki residues. We then use the Wodzicki residue to obtain results about the local counting function of operators of Dirac and Laplace type. In particular, we express the second term of the mollified spectral counting function of Dirac type operators in terms of geometric quantities and characterize those Dirac type operators for which this coefficient vanishes.

1. INTRODUCTION

Let M be a closed Riemannian manifold of dimension d and with metric g . Let E be a smooth complex hermitian vector bundle over M . As usual we denote by $C^\infty(M; E)$ the space of smooth sections of E , by $L^2(M; E)$ the Hilbert space of square integrable sections equipped with the natural inner product defined by the hermitian structure on the fibres and the metric measure μ_g on M .

A second order partial differential operator $P : C^\infty(M; E) \rightarrow C^\infty(M; E)$ is said to be of Laplace type if its principal symbol σ_P is of the form $\sigma_P(\xi) = g_x(\xi, \xi)\text{id}_{E_x}$ for all covectors $\xi \in T_x^*M$. In local coordinates this means that P is of the form

$$(1.1) \quad P = -g^{ij}(x)\partial_i\partial_j + a^k(x)\partial_k + b(x),$$

where a^k, b are smooth matrix-valued functions, and we have used Einstein's sum convention. Given a Laplace type operator P , it is well known that there exist a unique connection ∇ on E and a unique bundle endomorphism $V \in C^\infty(M; \text{End}(E))$ such that $P = \nabla^*\nabla + V$. Then P is called a generalized Laplace operator if $V = 0$. We will assume here that P is self-adjoint and non-negative. Thus there exists an orthonormal basis $\{\phi_j\}_{j=1}^\infty$ for $L^2(M; E)$ consisting of smooth eigensections such that $P\phi_j = \lambda_j^2\phi_j$, where λ_j are chosen non-negative and correspond to the eigenvalues of the operator $P^{1/2}$.

2000 *Mathematics Subject Classification.* 35P20 (primary), 35R01, 35Q41, 58J42 (secondary).

Key words and phrases. Dirac and Laplace type operators, spectral zeta and eta functions, heat traces, Wodzicki residues, local counting functions, resolvent powers, Bochner-Weitzenböck technique.

A first order partial differential operator $D : C^\infty(M; E) \rightarrow C^\infty(M; E)$ is said to be of Dirac type if its square is of Laplace type. This means that the principal symbol σ_D of D satisfies the Clifford algebra relations

$$(1.2) \quad \sigma_D(\xi)\sigma_D(\eta) + \sigma_D(\eta)\sigma_D(\xi) = 2g_x(\xi, \eta)\text{id}_{E_x} \quad (\xi, \eta \in T_x^*M).$$

Given a Dirac type operator D , we denote by γ the action of the Clifford algebra bundle on E generated by the principal symbol of D , and suppose ∇ is a connection on E that is compatible with the Clifford action γ (see Section 3 for details). We call the triple (E, γ, ∇) a Dirac bundle, and can express D uniquely as $D = \gamma\nabla + \psi$, where $\psi \in C^\infty(M; \text{End}(E))$ is called the potential of D associated with the Dirac bundle (E, γ, ∇) . In particular, D is called the generalized Dirac operator associated with the Dirac bundle (E, γ, ∇) if $\psi = 0$. We also assume D is self-adjoint, which means there exists a discrete spectral resolution $\{\phi_j, \mu_j\}_{j=1}^\infty$ of D , where $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis for $L^2(M; E)$, and $D\phi_j = \mu_j\phi_j$ for each j . Obviously, ϕ_j will be eigensections of $P = D^2$ with eigenvalues μ_j^2 . Therefore, using the notation from before $\lambda_j = |\mu_j|$.

Given a classical (polyhomogeneous) pseudodifferential operator A of order $m \in \mathbb{R}$ acting on sections of E , the microlocalized spectral counting function $N_A(\mu)$ of D is defined as

$$(1.3) \quad N_A(\mu) = \begin{cases} \sum_{0 \leq \mu_j < \mu} \langle A\phi_j, \phi_j \rangle & \text{if } \mu > 0, \\ \sum_{\mu \leq \mu_j < 0} \langle A\phi_j, \phi_j \rangle & \text{if } \mu \leq 0. \end{cases}$$

Thus, $N_A(\mu)$ is a piecewise constant function on \mathbb{R} such that

$$N'_A(\mu) = \sum_{j=1}^{\infty} \langle A\phi_j, \phi_j \rangle \delta_{\mu_j},$$

where δ_{μ_j} denotes the delta function on \mathbb{R} centered at μ_j . In case A is the operator of multiplication by a function $f(x) \in C^\infty(M)$ then

$$N_A(\mu) = \int_M f(x) N_x(\mu) d\mu_g(x),$$

where

$$(1.4) \quad N_x(\mu) = \begin{cases} \sum_{0 \leq \mu_j < \mu} \|\phi_j(x)\|_{E_x}^2 & \text{if } \mu > 0, \\ \sum_{\mu \leq \mu_j < 0} \|\phi_j(x)\|_{E_x}^2 & \text{if } \mu \leq 0 \end{cases}$$

is the so-called local counting function of D .

Let $\chi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function such that the Fourier transform $\mathcal{F}\chi$ of χ is 1 near the origin and $\text{supp}(\mathcal{F}\chi) \subset (-\delta, \delta)$, where δ is a positive constant smaller than half the radius of injectivity of M . It is well known (see e.g. [10, 30, 31, 42, 43, 52] for various special cases) that

$$(1.5) \quad (\chi * N'_A)(\mu) = \sum_{j=1}^{\infty} \langle A\phi_j, \phi_j \rangle \chi(\mu - \mu_j) \sim \sum_{k=0}^{\infty} \mathcal{A}_k(A, D) \mu^{d+m-k-1} \quad (\mu \rightarrow \infty).$$

This can be derived from studying the Fourier integral operator representation of $A^{\frac{\text{Sign}(D)+\text{Id}_E}{2}}e^{-it|D|}$ via the stationary phase method. The mollified spectral counting coefficients $\mathcal{A}_k(A, D)$ do not depend on the choice of χ , and are locally computable in terms of the total symbols of $A^{\frac{\text{Sign}(D)+\text{Id}_E}{2}}$ and $|D|$. Note also the corresponding expansion for $\mu \rightarrow -\infty$ can be easily obtained from replacing D with $-D$:

$$(1.6) \quad (\chi * N'_A)(\mu) \sim \sum_{k=0}^{\infty} \mathcal{A}_k(A, -D)|\mu|^{d+m-k-1} \quad (\mu \rightarrow -\infty).$$

Therefore, the function $\chi * N'_A$ contains all the information about $\{\mathcal{A}_k(A, \pm D)\}_{k=0}^{\infty}$.

One of the purposes of the paper is to show how to explicitly determine $\mathcal{A}_k(A, D)$. In particular, for any bundle endomorphism F of E , we can express $\mathcal{A}_1(F, D)$ in terms of geometric quantities such as $g, \gamma, \nabla, \psi, F$. To compare, Sandoval ([43]) obtained an explicit expression of $\mathcal{A}_1(\text{Id}_E, D)$, while Branson and Gilkey ([3]) can also do so for $\mathcal{A}_1(F, D)$ whenever F is of the form $f\text{Id}_E$ where f is a smooth function on M . In the case of more general first order systems with $F = \text{Id}_E$ a formula was also more recently obtained by Chervova, Downes and Vassiliev ([7]). In the case of operators of Dirac-type our paper also explains the relation between the results of [7] and the subsequent [8] on one hand and known heat-trace invariants on the other.

The mollified spectral counting coefficients $\mathcal{A}_k(F, D)$ closely relate to the local counting function of D . Recall D is a self-adjoint Dirac type operator with spectral resolution $\{\phi_j, \mu_j\}_{j=1}^{\infty}$. For each j we denote by $\Psi_j = \Psi_j(x, y)$ the Schwartz kernel of the orthogonal projection onto the space spanned by ϕ_j , that is,

$$\langle \phi, \phi_j \rangle \phi_j(x) = \int_M \Psi_j(x, y) \phi(y) d\mu_g(y),$$

where $\phi \in C^\infty(M; E)$, $\Psi_j(x, y)$ maps $\phi(y) \in E_y$ to $\langle \phi(y), \phi_j(y) \rangle_{E_y} \phi_j(x) \in E_x$. We then denote by $\Phi_j = \Phi_j(x) = \Psi_j(x, x) \in C^\infty(M; \text{End}(E))$ the diagonal restriction of Ψ_j . Similar to (1.5) there exists (see e.g. [30]) an asymptotic expansion

$$(1.7) \quad \sum_{j=1}^{\infty} \Phi_j \chi(\mu - \mu_j) \sim \sum_{k=0}^{\infty} \mu^{d-k-1} \mathcal{L}_k(D) \quad (\mu \rightarrow \infty),$$

where $\mathcal{L}_k(D) \in C^\infty(M; \text{End}(E))$. This immediately implies that

$$(1.8) \quad \mathcal{A}_k(F, D) = \int_M \text{Tr}_E(F \mathcal{L}_k(D))$$

for all non-negative integers k . On the other hand, it is possible to recover $\mathcal{L}_k(D)$ from $\mathcal{A}_k(F, D)$ whenever explicit expressions of $\mathcal{A}_k(F, D)$ are known for all F . Note at each $x \in M$, $\text{Tr}_{E_x}(\Phi_j(x)) = \|\phi_j(x)\|_{E_x}^2$. Thus

$$(1.9) \quad (\chi * N'_x)(\mu) \sim \sum_{k=0}^{\infty} \mu^{d-k-1} \text{Tr}_{E_x}(\mathcal{L}_k(D)(x)) \quad (\mu \rightarrow \infty),$$

whose asymptotic expansion coefficients $\text{Tr}_{E_x}(\mathcal{L}_k(D)(x))$ are determined by $\mathcal{L}_k(D)$.

The mollified spectral counting coefficients $\mathcal{A}_k(F, D)$ also closely relate to the small-time asymptotic expansion of the Schwartz kernels of e^{-tD^2} and De^{-tD^2} . Let $Q : C^\infty(M; E) \rightarrow C^\infty(M; E)$ be a partial differential operator of order m . It is well known (see e.g. [14, 18, 48]) that Qe^{-tD^2} is a smoothing operator with smooth kernel $K(t, x, y, Q, D^2)$ in the sense of

$$(1.10) \quad (Qe^{-tD^2}\phi)(x) = \int_M K(t, x, y, Q, D^2)\phi(y)d\mu_g(y),$$

where $K(t, x, y, Q, D^2)$ maps E_y to E_x , and the diagonal values of K admit a uniform small-time asymptotic expansion

$$(1.11) \quad K(t, x, x, Q, D^2) \sim \sum_{k=0}^{\infty} t^{\frac{k-d-m}{2}} \mathcal{H}_k(Q, D^2)(x) \quad (t \rightarrow 0^+),$$

where $\mathcal{H}_k(Q, D^2) \in C^\infty(M; \text{End}(E))$ vanishes if $k+m$ is odd. It is not hard to show for all non-negative integers k that

$$(1.12) \quad \mathcal{L}_k(D) = \frac{\mathcal{H}_k(\text{Id}_E, D^2)}{\Gamma(\frac{d-k}{2})} + \frac{\mathcal{H}_k(D, D^2)}{\Gamma(\frac{d+1-k}{2})},$$

where Γ denotes the Gamma function. This immediately implies that one is able to recover all the $\mathcal{H}_k(D, D^2)$ if the dimension d of M is odd and $\{\mathcal{H}_k(D, D^2)\}_{k=0}^d$ if d is even from all the $\mathcal{L}_k(D)$. To the authors' knowledge, although how to express $\mathcal{H}_k(\text{Id}_E, D^2)$, say for example $k \leq 10$, is a well-studied topic ([19, 20]), it seems that no explicit formula for $\mathcal{H}_k(D, D^2)$ with k odd is stated in the literature.

Now we state the main results of the paper. Let $\sigma_A, \text{Sub}(A)$ denote the principal and sub-principal symbol of A , respectively (See Section 4 for details), and let T_1^*M denote the unit cotangent bundle of M . For any self-adjoint Dirac type operator D of potential ψ associated with the Dirac bundle (E, γ, ∇) , define $\tilde{\psi} = \gamma(e_i)\psi\gamma(e_i)$, where $\{e_i\}_{i=1}^d$ is a local orthonormal frame in T^*M . Obviously, $\tilde{\psi} \in C^\infty(M; \text{End}(E))$ is independent of the choice of local orthonormal frames. Recall that d denotes the dimension of M . If $k=0$, then it is easy to show that

$$(1.13) \quad \mathcal{A}_0(A, D) = \frac{1}{2 \cdot (2\pi)^d} \int_{T_1^*M} \text{Tr}_E(\sigma_A + \sigma_A \cdot \sigma_D).$$

Theorem 1.1. *Let D be a self-adjoint Dirac type operator and let A be a classical pseudodifferential operator of order m on sections of E . Then*

$$\begin{aligned} \mathcal{A}_1(A, D) = & \frac{1}{2 \cdot (2\pi)^d} \int_{T_1^*M} \text{Tr}_E \left(\text{Sub}(A) - \frac{d+m-1}{2} \cdot \sigma_A \cdot \text{Sub}(D^2) \right) + \\ & \frac{1}{2 \cdot (2\pi)^d} \int_{T_1^*M} \text{Tr}_E \left(\text{Sub}(AD) - \frac{d+m}{2} \cdot \sigma_A \cdot \sigma_D \cdot \text{Sub}(D^2) \right). \end{aligned}$$

The subprincipal symbols appearing in the above expressions depend on a choice of local bundle frame. The integrals are however invariantly defined as integration

over the co-sphere at each point yields a well defined density on M (see Section 4). Note that

$$(1.14) \quad \text{Sub}(AB) = \text{Sub}(A) \cdot \sigma_B + \sigma_A \cdot \text{Sub}(B) + \frac{1}{2i} \{\sigma_A, \sigma_B\},$$

where B is any other classical pseudodifferential operator on sections of E , $\{\sigma_A, \sigma_B\}$ denotes the Poisson bracket between σ_A and σ_B . This means that $\mathcal{A}_1(A, D)$ depends only on the principal and sub-principal symbols of A and D . In general, our method of proof can also provide an explicit formula for $\mathcal{A}_k(A, D)$ in terms of the local symbols of A and D for each $k \geq 2$. In the case when A is a bundle endomorphism this specializes to the following theorem.

Theorem 1.2. *Let D be a self-adjoint Dirac type operator of potential ψ associated with the Dirac bundle (E, γ, ∇) and let F be a smooth bundle endomorphism of E . Then*

$$(1.15) \quad \mathcal{A}_1(F, D) = \frac{1}{(4\pi)^{d/2} \cdot \Gamma(\frac{d}{2})} \int_M \text{Tr}_E(F \cdot \frac{\widehat{\psi} - (d-2)\psi}{2}).$$

As immediate consequences of Theorem 1.2, one obtains

$$(1.16) \quad \mathcal{L}_1(D) = \frac{1}{(4\pi)^{d/2} \cdot \Gamma(\frac{d}{2})} \cdot \frac{\widehat{\psi} - (d-2)\psi}{2}$$

and

$$(1.17) \quad \mathcal{H}_1(D, D^2) = \frac{1}{(4\pi)^{d/2}} \cdot \frac{\widehat{\psi} - (d-2)\psi}{2}.$$

Theorem 1.3. *Let D be a self-adjoint Dirac type operator. Then $\mathcal{L}_1(D) = 0$ if and only if D is a generalized Dirac operator.*

Note that in the special case $\text{rk}(E) = 2$ and $d = 3$ formula (1.16) implies that pointwise $\mathcal{L}_1(D)$ is proportional to the identity matrix. Hence it vanishes if and only if the local coefficient $\text{tr}(\mathcal{L}_1(D))$ vanishes (see also Example 5.5). Then Theorem 1.3 directly implies the characterization theorem in [8] (see Example 5.6 for details). Our result can therefore be seen as a generalization of Theorem 1.2 in [8] to higher dimensions and a more general geometric context.

Finally, we explain how to explicitly determine $\mathcal{A}_k(A, D)$. Recall that P is a self-adjoint Laplace type operator with spectral resolution $\{\phi_j, \lambda_j^2\}_{j=1}^\infty$ and A is a classical pseudodifferential operator of order $m \in \mathbb{R}$. Similar to (1.5) one has

$$(1.18) \quad (\chi * N'_A)(\lambda) = \sum_{j=1}^{\infty} \langle A\phi_j, \phi_j \rangle \chi(\lambda - \lambda_j) \sim \sum_{k=0}^{\infty} \mathcal{A}_k(A, P) \lambda^{d+m-k-1} \quad (\lambda \rightarrow \infty),$$

where the microlocalized spectral counting function $N_A(\lambda)$ of P is defined as

$$(1.19) \quad N_A(\lambda) = \sum_{\lambda_j < \lambda} \langle A\phi_j, \phi_j \rangle.$$

Then it is straightforward to show for all non-negative integers k that

$$(1.20) \quad \mathcal{A}_k(A, D) = \mathcal{A}_k\left(A \frac{\text{Sign}(D) + \text{Id}_E}{2}, D^2\right),$$

which means to determine $\mathcal{A}_k(A, D)$ it suffices to do so for $\mathcal{A}_k(A, D^2)$ for arbitrary classical pseudodifferential operators A . Apart from the Fourier integral operator representation method introduced before, there exist several other ways to recover the mollified counting coefficients $\mathcal{A}_k(A, P)$. First, the microlocalized spectral zeta function $\zeta(s, A, P)$ is defined by

$$(1.21) \quad \zeta(s, A, P) = \sum_{\lambda_j > 0} \frac{\langle A\phi_j, \phi_j \rangle}{\lambda_j^s} \quad (\text{Re}(s) > d + m).$$

It is well-known (see e.g. [10, 52]) that $\zeta(s, A, P)$ admits a meromorphic continuation to \mathbb{C} whose only singularities are simple poles at $s = d + m - k$ ($k = 0, 1, 2, \dots$) with residues $\mathcal{A}_k(A, P)$. Second, the Mellin transform of

$$\text{tr}(Ae^{-tP}) - \sum_{\lambda_j=0} \langle A\phi_j, \phi_j \rangle \quad (t \in (0, \infty))$$

admits a meromorphic continuation $\zeta(2s, A, P)\Gamma(s)$ to \mathbb{C} whose singularities can be completely determined from those of $\zeta(s, A, P)$ and $\Gamma(s)$. After establishing suitable vertical decay estimate for $\zeta(2s, A, P)\Gamma(s)$, one can deduce from the inverse Mellin transform theorem the following widely used heat expansion (see e.g. [22, 25, 26, 37, 47])

$$(1.22) \quad \text{tr}(Ae^{-tP}) \sim \sum_{k=0}^{\infty} \left(\mathcal{B}_k(A, P)t^{\frac{k-d-m}{2}} + \mathcal{C}_k(A, P)t^k \log(t) + \mathcal{D}_k(A, P)t^k \right),$$

where $t \rightarrow 0^+$, and the relations between the mollified counting coefficients and some of the heat coefficients can be summarized as follows:

Case 1: If the order m of A is an integer, then

- $\mathcal{B}_k(A, P) = \frac{\Gamma(\frac{d+m-k}{2})}{2} \cdot \mathcal{A}_k(A, P)$ ($d + m - k$ is positive or negative but odd);
- $\mathcal{C}_k(A, P) = 0$ ($d + m + 2k < 0$);
- $\mathcal{C}_k(A, P) = \frac{(-1)^{k+1}}{2 \cdot k!} \cdot \mathcal{A}_{d+m+2k}(A, P)$ ($d + m + 2k \geq 0$).

Case 2: If the order m of A is not an integer, then for all non-negative integers k :

- $\mathcal{B}_k(A, P) = \frac{\Gamma(\frac{d+m-k}{2})}{2} \cdot \mathcal{A}_k(A, P)$;
- $\mathcal{C}_k(A, P) = 0$.

Thus the heat expansion (1.22) contains all the information about $\{\mathcal{A}_k(A, P)\}_{k=0}^{\infty}$. In exactly the same way, the following resolvent trace expansion (see e.g. [23, 25, 47])

$$\text{tr}(A(1 + tP)^{-\frac{N}{2}}) \sim \sum_{k=0}^{\infty} \left(\mathcal{B}_k^{(N)}(A, P)t^{\frac{k-d-m}{2}} + \mathcal{C}_k^{(N)}(A, P)t^k \log(t) + \mathcal{D}_k^{(N)}(A, P)t^k \right)$$

also contains all the information about $\{\mathcal{A}_k(A, P)\}_{k=0}^{\infty}$, where N is any complex number such that $\text{Re}(N) > \max\{d + m, 0\}$. To summarize, there exist at least four

different ways (mollified counting functions, spectral zeta functions, heat expansions, resolvent trace expansions) to determine the mollified counting coefficients $\mathcal{A}_k(A, P)$. For example, using parametrix constructions in any of these methods results in the well known leading term

$$(1.23) \quad \mathcal{A}_0(A, P) = \frac{1}{(2\pi)^d} \int_{T_1^*M} \text{Tr}(\sigma_A).$$

In this paper we will use the Wodzicki residue (see e.g [27, 34, 44, 47, 50, 51]) to study $\mathcal{A}_k(A, P)$. On the algebra $\Psi(M; E)$ of all classical pseudo-differential operators on $C^\infty(M; E)$, there exists a trace called Wodzicki's residue or non-commutative residue which is defined by

$$(1.24) \quad \text{res}(A) = \int_M \text{res}_x(A) dx,$$

where

$$(1.25) \quad \text{res}_x(A) dx := \left(\frac{1}{(2\pi)^d} \int_{|\xi|=1} \text{Tr}(\sigma_{-d}(A)(x, \xi)) dS(\xi) \right) dx$$

is independent of the choice of local coordinates and thus is a global density on M , $\sigma_{-d}(A)(x, \xi)$ denotes the homogeneous part of degree $-d$ of the total symbol of A , $dS(\xi)$ denotes the sphere measure on \mathbb{S}^{d-1} . If M is connected any trace τ on $\Psi(M; E)$ is a multiple of res . It is well-known (see e.g. [1, 16, 22, 33, 34, 44, 47, 52]) that Wodzicki's residues are closely related to the mollified counting coefficients as well as the heat coefficients. The connecting formula is ([27, 51], see also [22, (0.2)], [36, (1.2)], [44, (1.16)])

$$(1.26) \quad \text{res}(A) = -2\mathcal{C}_0(A, P),$$

which means $\mathcal{C}_0(A, P)$ is independent of the choice of P . As for other coefficients, it is known for all non-negative integers k that (see e.g. [16, Thm. 5.2])

$$(1.27) \quad \mathcal{C}_k(A, P) = \frac{(-1)^{k+1}}{2 \cdot k!} \cdot \text{res}(AP^k),$$

and (see e.g. [34, Prop. 4.2], [47, P. 106])

$$(1.28) \quad \mathcal{A}_k(A, P) = \text{res}(AP^{\frac{k-d-m}{2}}).$$

Thus combining (1.20) with (1.28), one gets

$$(1.29) \quad \mathcal{A}_k(A, D) = \text{res}\left(A \frac{D + |D|}{2} |D|^{k-d-m-1}\right),$$

which is the main tool for us to establish Theorem 1.1. We should mention that any of the previous four methods can also be used to derive Theorem 1.1, but it seems that these methods do not provide such a clear interpretation of $\mathcal{A}_k(A, P)$ for $k \geq 2$.

This paper is structured as follows.

Section 2 first reviews the construction of the Fourier integral operator $Ae^{-itP^{1/2}}$ and the stationary phase expansion of $\chi * N'_A$, then provides as consequences complete proofs of the singularity structures of the spectral zeta functions, the heat expansions

as well as the resolvent trace expansions. As applications, (1.27) and (1.28) follow from suitably applying formula (1.26). The local counting function of operators of Dirac and Laplace type is also studied in the last part of this section.

Section 3 first reviews the concept of the Dirac bundle and the associated Dirac type operators, then gives a direct proof of Theorem 1.2 by appealing to the Bochner–Weitzenböck technique in Riemannian geometry.

Section 4 proves Theorem 1.1 and Theorem 1.2 by applying the Wodzicki residue method. The connections between certain Wodzicki residues and the sub-principal symbols of classical pseudodifferential operators are also discussed.

Section 5 characterizes Dirac type operators with vanishing second expansion coefficient. As an application, the so-called massless Dirac operators are studied.

Throughout the paper we assume on smooth sections of E ,

- P is a self-adjoint Laplace type operator with spectral resolution $\{\phi_j, \lambda_j^2\}_{j=1}^\infty$,
- D is a self-adjoint Dirac type operator with spectral resolution $\{\phi_j, \mu_j\}_{j=1}^\infty$ unless otherwise stated in the last section,
- A is a classical (polyhomogeneous) pseudodifferential operator of order m ,
- Q is a partial differential operator of order m ,
- F is a smooth endomorphism.

2. MOLLIFIED COUNTING COEFFICIENTS I: QUALITATIVE THEORY

This section first reviews the construction of the Fourier integral operator $Ae^{-itP^{1/2}}$ and the stationary phase expansion of $\chi * N'_A$, then proves many consequences.

2.1. FIO method. Formula (1.18) essentially is Proposition 2.1 in [10], Corollary 2.2 in [30], Theorem 2.2 in [43] and Proposition 1.1 in [52], except they either consider scalar operators or assume A is of order zero. Recall that $\chi \in \mathcal{S}(\mathbb{R})$ is chosen so that $\mathcal{F}\chi = 1$ near the origin and $\text{supp}(\mathcal{F}\chi) \subset (-\delta, \delta)$, where δ is smaller than half the radius of injectivity of M . If t is sufficiently small, say $|t| < \delta_1 < \delta$, then locally the integral kernel $(Ae^{-itP^{1/2}})(t, x, y)$ of the operator $Ae^{-itP^{1/2}}$ is well known to have the form

$$(Ae^{-itP^{1/2}})(t, x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a(t, x, y, \xi) e^{i\theta(t, x, y, \xi)} d\xi,$$

where a is a classical (matrix-valued) symbol of order m . The scalar-valued phase function $\theta(t, x, y, \xi) = \kappa(x, y, \xi) - t\sigma_{P^{1/2}}(y, \xi)$, where $\kappa(y, y, \xi) = 0$, was introduced by Hörmander ([29]). It is also known that $\text{tr}(Ae^{-itP^{1/2}})$ is smooth in $(-\delta, \delta) \setminus \{0\}$, so we introduce a cut-off function $\varrho \in \mathcal{S}(\mathbb{R})$ satisfying $\varrho(t) = 1$ if $|t| < \frac{\delta_1}{2}$ and

$\text{supp}(\varrho) \subset (-\delta_1, \delta_1)$. Using integration by parts one gets

$$\begin{aligned} (\chi * N'_A)(\lambda) &= \frac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{F}\chi)(t)(\varrho(t) + 1 - \varrho(t)) \text{tr}(Ae^{-itP^{1/2}}) e^{i\lambda t} dt \\ &= \frac{1}{(2\pi)^{d+1}} \int_M \int_{\mathbb{R}^d} \int_{\mathbb{R}} (\mathcal{F}\chi)(t)\varrho(t) \text{Tr}(a(t, y, y, \xi)) e^{-it\sigma_{P^{1/2}}(y, \xi)} e^{i\lambda t} dy d\xi dt \\ &\quad + o(\lambda^{-\infty}) \quad (\lambda \rightarrow \infty), \end{aligned}$$

where $o(\lambda^{-\infty})$ is short for $o(\lambda^{-h})$ for all positive integers h . Now proceed as in [10] via the stationary phase method to get (1.18).

2.2. Finite heat expansions. It is easy to prove that

$$(2.1) \quad \langle T, \varphi_t \rangle = \langle \rho * T, \varphi_t \rangle + o(t^\infty) \quad (t \rightarrow 0^+),$$

where T is a tempered distribution on \mathbb{R} , $\varphi \in \mathcal{S}(\mathbb{R})$, $\varphi_t(\lambda) = \varphi(t\lambda)$, and $\rho \in \mathcal{S}(\mathbb{R})$ is chosen so that $\mathcal{F}\rho = 1$ near the origin. Actually, this formula appears in equivalent forms in [10, 28, 52], so its proof is omitted. As an application, one easily gets

$$(2.2) \quad \langle N'_A, \lambda^h \varphi(t\lambda) \rangle = \langle \chi * N'_A, \lambda^h \varphi(t\lambda) \rangle + o(t^\infty) \quad (t \rightarrow 0^+),$$

where h is an arbitrary non-negative integer. Note the left hand side of (2.2) just is $\text{tr}(AP^{\frac{h}{2}}\varphi(t\sqrt{P}))$. Now we claim the right hand side of (2.2) is of the following form

$$(2.3) \quad \sum_{k < d+m+h} \mathcal{A}_k(A, P) \cdot \int_0^\infty \lambda^{d+m+h-k-1} \varphi(\lambda) d\lambda \cdot t^{k-d-m-h} + o(t^{[m]-m-1}).$$

To this end we first decompose

$$\begin{aligned} \langle \chi * N'_A, \lambda^h \varphi(t\lambda) \rangle &= \int_{-\infty}^0 (\chi * N'_A)(\lambda) \cdot \lambda^h \varphi(t\lambda) d\lambda + \\ &\quad \sum_{k < d+m+h} \int_0^\infty \mathcal{A}_k(A, P) \lambda^{d+m-k-1} \cdot \lambda^h \varphi(t\lambda) d\lambda + \\ &\quad \int_0^\infty ((\chi * N'_A)(\lambda) - \sum_{k < d+m+h} \mathcal{A}_k(A, P) \lambda^{d+m-k-1}) \cdot \lambda^h \varphi(t\lambda) d\lambda \\ &=: \alpha_1(t) + \alpha_2(t) + \alpha_3(t). \end{aligned}$$

Since $\text{supp}(N'_A) \subset [0, \infty)$, it is easy to verify that $(\chi * N'_A)(\lambda) = o(|\lambda|^{-\infty})$ as $\lambda \rightarrow -\infty$. This implies $\alpha_1(t) = O(1)$ as $t \rightarrow 0^+$. Note also

$$\alpha_2(t) = \sum_{k < d+m+h} \mathcal{A}_k(A, P) \cdot \int_0^\infty \lambda^{d+m+h-k-1} \varphi(\lambda) d\lambda \cdot t^{k-d-m-h}.$$

For simplicity we introduce

$$f(\lambda) = ((\chi * N'_A)(\lambda) - \sum_{k < d+m+h} \mathcal{A}_k(A, P) \lambda^{d+m-k-1}) \cdot \lambda^h \quad (\lambda > 0)$$

and note $\alpha_3(t) = \int_0^\infty f(\lambda) \varphi(t\lambda) d\lambda$. To prove the claim we have two cases to consider. Case 1: Suppose $d+m+h > 0$. Let \tilde{k} be the unique non-negative integer such that

$\tilde{k} < d + m + h \leq \tilde{k} + 1$. Let $\beta := d + m + h - \tilde{k} - 1 \in (-1, 0]$, which implies there exists a constant C_1 such that for all $\lambda \in (0, 1)$, $|f(\lambda)| \leq C_1 \lambda^\beta$. According to (1.18), there exists another constant C_2 such that for all $\lambda \geq 1$, $|f(\lambda)| \leq C_2 \lambda^{\beta-1} \leq C_2 \lambda^{-1}$. So as $t \rightarrow 0^+$,

$$\begin{aligned} |\alpha_3(t)| &\leq C_1 \int_0^1 \lambda^\beta |\varphi(t\lambda)| d\lambda + C_2 \int_1^\infty \lambda^{-1} |\varphi(t\lambda)| d\lambda \\ &= C_1 \int_0^t \lambda^\beta |\varphi(\lambda)| d\lambda \cdot t^{-(\beta+1)} + C_2 \int_t^\infty \lambda^{-1} |\varphi(\lambda)| d\lambda \\ &= o(t^{-(\beta+1)}) + O(|\log(t)|) \\ &= o(t^{-(\beta+1)}). \end{aligned}$$

Since $-(\beta + 1) = \tilde{k} - d - m - h = [m] - m - 1 < 0$, we are done in the first case. Case 2: Suppose $d + m + h \leq 0$. Obviously, there is a constant C_3 such that for all $\lambda \in (0, 1)$, $|f(\lambda)| \leq C_3$. According to (1.18), $(\chi * N'_A)(\lambda) = O(\lambda^{d+m-1})$ as $\lambda \rightarrow \infty$. Thus there is a constant C_4 such that for all $\lambda \geq 1$, $|f(\lambda)| \leq C_4 \lambda^{-1}$. So as $t \rightarrow 0^+$,

$$\begin{aligned} |\alpha_3(t)| &\leq C_3 \int_0^1 |\varphi(t\lambda)| d\lambda + C_4 \int_1^\infty \lambda^{-1} |\varphi(t\lambda)| d\lambda \\ &= O(1) + O(|\log(t)|) \\ &= O(|\log(t)|). \end{aligned}$$

But note $[m] - m - 1 < 0$, we are also done in the second case.

To summarize, we have shown for all $\varphi \in \mathcal{S}(\mathbb{R})$ as $t \rightarrow 0^+$ that,

$$(2.4) \quad \text{tr}(AP^{\frac{h}{2}}\varphi(t\sqrt{P})) = \sum_{k < d+m+h} \frac{\mathcal{A}_k(A, P) \int_0^\infty \lambda^{d+m+h-k-1} \varphi(\lambda) d\lambda}{t^{d+m+h-k}} + o(t^{[m]-m-1}).$$

2.3. Mellin transforms. The Mellin transform of a continuous function f on $(0, \infty)$ is the function $(Mf)(s)$ of the complex variable s , given by

$$(Mf)(s) = \int_0^\infty f(t) t^{s-1} dt$$

whenever the integral is well-defined. An open strip $\Pi(\beta_1, \beta_2) = \{s \in \mathbb{C} : \beta_1 < \text{Re}(s) < \beta_2\}$ is called a basic strip of Mf if the integral is absolutely convergent in that strip. Given meromorphic functions u, v defined over an open subset Π of \mathbb{C} , we denote $u \asymp v$ ($s \in \Pi$) to mean $u - v$ is analytic in Π . For example, assume f is a continuous function on $(0, \infty)$ satisfying $f(t) = o(t^{-\infty})$ as $t \rightarrow \infty$ and assume there exist real numbers $\omega_0 < \omega_1 < \dots < \omega_N$ such that

$$(2.5) \quad f(t) = \sum_{k=0}^{N-1} a_k t^{\omega_k} + O(t^{\omega_N}) \quad (t \rightarrow 0^+).$$

Then one may easily check that $\Pi(-\omega_0, \infty)$ is a basic strip of Mf , and Mf admits a meromorphic continuation to $\Pi(-\omega_N, \infty)$ such that

$$(2.6) \quad (Mf)(s) \asymp \sum_{k=0}^{N-1} \frac{a_k}{s + \omega_k} \quad (s \in \Pi(-\omega_N, \infty)).$$

Actually, for any $s \in \Pi(\gamma_1, \gamma_2)$ with $-\omega_N < \gamma_1 < \gamma_2 < \infty$, one has

$$(Mf)(s) = \sum_{k=0}^{N-1} \frac{a_k}{s + \omega_k} + \int_0^1 (f(t) - \sum_{k=0}^{N-1} a_k t^{\omega_k}) t^{s-1} dt + \int_1^\infty f(t) t^{s-1} dt,$$

which immediately implies (2.6) as well as an upper bound estimate:

$$(2.7) \quad (Mf)(s) = O(1) \quad (s \in \Pi(\gamma_1, \gamma_2), |s| \rightarrow \infty).$$

Lemma 2.1 ([15]). *Let f be a continuous function over $(0, \infty)$. Assume there exist real numbers $\beta_1 < \beta_2 < \beta_3 < \beta_4$ such that*

- $\Pi(\beta_3, \beta_4)$ is a basic strip of Mf ,
- Mf admits a meromorphic continuation to $\Pi(\beta_1, \beta_4)$ with finite poles there,
- $Mf \asymp \sum_{(\omega, j)} \frac{C_{\omega, j}}{(s+\omega)^{j+1}}$ ($s \in \Pi(\beta_2, \beta_4)$),
- Mf is analytic on $\text{Re}(s) = \beta_2$,
- $(Mf)(s) = O(|s|^{-2})$ ($s \in \Pi_{\beta_1, \beta_4}, |s| \rightarrow \infty$).

Then

$$f(t) = \sum_{(\omega, j)} C_{\omega, j} \left(\frac{(-1)^j}{j!} t^\omega (\log t)^j \right) + O(t^{-\beta_2}) \quad (t \rightarrow 0^+).$$

2.4. Spectral zeta functions. In this part we deduce the well-known singularity structures of the spectral zeta functions. Let h be a positive integer such that $d + m + h > 1$, and define

$$(2.8) \quad f_h(t) = \text{tr}(AP^{\frac{h}{2}} e^{-tP}) = \sum_{\lambda_j > 0} \langle A\phi_j, \phi_j \rangle \lambda_j^h e^{-t\lambda_j^2} \quad (t > 0).$$

Now we have two ways to study the Mellin transform of f_h . First, we list $\{\lambda_j\}_{j=1}^\infty$ in non-decreasing order and thus by Weyl's law (see e.g. [2]), $\lim_{j \rightarrow \infty} \lambda_j^d / j$ exists and is positive. It is easy to see (see e.g. [4, 48]) that there exists a constant $C = C(A, P)$ such that $|\langle A\phi_j, \phi_j \rangle| \leq C\lambda_j^m$ whenever $\lambda_j > 0$. Consequently,

- f_h is smooth over $(0, \infty)$ with $f_h(t) = o(t^{-\infty})$ as $t \rightarrow \infty$;
- the spectral zeta function $\zeta(s, A, P)$ is analytic in $\Pi(d + m, \infty)$;
- Mf_h has a basic strip $\Pi(\frac{d+m+h}{2}, \infty)$ in which $(Mf_h)(s) = \zeta(2s - h, A, P)\Gamma(s)$.

Second, according to (2.4) f_h is easily seen to have the form (2.5):

$$(2.9) \quad f_h(t) = \sum_{k=0}^{N-1} \frac{\Gamma(\frac{d+m+h-k}{2})}{2} \cdot \mathcal{A}_k(A, P) \cdot t^{\frac{k-d-m-h}{2}} + O(t^{\omega_N}) \quad (t \rightarrow 0^+),$$

where $N = N_h = d+h+\lceil m \rceil - 1 \geq 1$, $\omega_N = \frac{\lceil m \rceil - m - 1}{2} < 0$, $\omega_k = \omega_N - \frac{N-k}{2} = \frac{k-d-m-h}{2}$ ($k = 0, 1, \dots, N-1$). So by (2.6), Mf_h initially analytically defined in $\Pi(\frac{d+m+h}{2}, \infty)$, admits a meromorphic continuation to $\Pi(-\omega_N, \infty)$ in which

$$(2.10) \quad (Mf_h)(s) \asymp \sum_{k=0}^{N-1} \frac{\Gamma(\frac{d+m+h-k}{2})}{2} \cdot \frac{\mathcal{A}_k(A, P)}{s - \frac{d+m+h-k}{2}}.$$

Considering in $\Pi(-\omega_N, \infty)$ the Gamma function Γ is analytic and has no zeros, it is easy to see that $\zeta(s, A, P)$ initially analytically defined in $\Pi(d+m, \infty)$, admits a meromorphic continuation to $\Pi(-2\omega_N - h, \infty)$ in which

$$(2.11) \quad \zeta(s, A, P) = \frac{(Mf_h)(\frac{s+h}{2})}{\Gamma(\frac{s+h}{2})} \asymp \sum_{k=0}^{N_h-1} \frac{\mathcal{A}_k(A, P)}{s - (d+m-k)}.$$

Letting $h \rightarrow \infty$ we get that $\zeta(s, A, P)$ admits a meromorphic continuation to \mathbb{C} whose only singularities are simple poles at $s = d+m-k$ ($k = 0, 1, 2, \dots$) with residues $\mathcal{A}_k(A, P)$.

Remark 2.2. Note that if B is a smoothing operator on sections of E , then $\zeta(s, B, P)$ is an entire function on \mathbb{C} , and consequently, $\mathcal{A}_k(A, P) = \mathcal{A}_k(A+B, P)$ for all k .

For any $q \in \mathbb{R}$, we let $P^q : C^\infty(M; E) \rightarrow C^\infty(M; E)$ be the operator defined by functional calculus of P if q is non-negative and by sending ϕ to $\sum_{\lambda_j > 0} \lambda_j^{2q} \langle \phi, \phi_j \rangle \phi_j$ if q is negative. By a classical result by Seeley ([48]) the operator P^q is a classical pseudodifferential operator of order $2q$.

Proposition 2.3. *For any real number q , $\mathcal{A}_k(A, P) = \mathcal{A}_k(AP^q, P)$ holds for all non-negative integers k .*

Proof. Note AP^q is a classical pseudodifferential operator of order $m+2q$. Thus $\zeta(s, AP^q, P)$ admits a meromorphic continuation to \mathbb{C} whose only singularities are simple poles at $s = d+m+2q-k$ ($k = 0, 1, 2, \dots$) with residues $\mathcal{A}_k(AP^q, P)$. But noting that $\zeta(s, AP^q, P) = \zeta(s-2q, A, P)$, we must have $\mathcal{A}_k(AP^q, P) = \mathcal{A}_k(A, P)$. This finishes the proof of the proposition. \square

Remark 2.4. To determine $\mathcal{A}_k(A, P)$ we can assume without loss of generality that A is of order zero as $\mathcal{A}_k(AP^{-\frac{m}{2}}, P)$, which is $\mathcal{A}_k(A, P)$ by Proposition 2.3.

2.5. Full heat expansions. In this part we provide another approach to the heat expansion (1.22). Define a smooth function f over $(0, \infty)$ by

$$(2.12) \quad f(t) = \sum_{\lambda_j > 0} \langle A\phi_j, \phi_j \rangle e^{-t\lambda_j^2} = \text{tr}(Ae^{-tP}) - \sum_{\lambda_j = 0} \langle A\phi_j, \phi_j \rangle \quad (t > 0).$$

Obviously, $\Pi(\max\{\frac{d+m}{2}, 0\}, \infty)$ is a basic strip of Mf which admits a meromorphic continuation to \mathbb{C} with $(Mf)(s) = \zeta(2s, A, P)\Gamma(s)$. Recall $\zeta(\cdot, A, P)$ is meromorphic on \mathbb{C} whose only singularities are simple poles at $s = d+m-k$ ($k = 0, 1, 2, \dots$) with residues $\mathcal{A}_k(A, P)$, while Γ is meromorphic on \mathbb{C} whose only singularities are simple

poles at $s = -k$ ($k = 0, 1, 2, \dots$) with residues $\frac{(-1)^k}{k!}$. Hence to establish (1.22) via Lemma 2.1 it suffices to prove

$$(2.13) \quad (Mf)(s) = O(|s|^{-2}) \quad (s \in \Pi(-n, n), |s| \rightarrow \infty)$$

for all positive integers n . To this end we first assume $s \in \Pi(-n, n)$ ($n \in \mathbb{N}$), then let $h \geq n + 2$ be a large enough integer so that one can use (2.11) to get

$$(2.14) \quad (Mf)(s) = \zeta(2s, A, P)\Gamma(s) = \frac{(Mf_{2h})(s+h)}{\Gamma(s+h)}\Gamma(s) = (Mf_{2h})(s+h) \prod_{i=0}^{h-1} \frac{1}{s+i}.$$

Finally by considering $s+h \in \Pi(2, n+h)$, (2.13) follows immediately from suitably applying (2.7) to uniformly bound $(Mf_{2h})(s+h)$ above for all $s \in \Pi(-n, n)$ with $|s| \rightarrow \infty$. This finishes the proof of (1.22).

The precise relations between the mollified counting coefficients and some of the heat coefficients as mentioned in the introductory part are left to the interested readers to verify. For completeness we record these relations as a proposition for later use. We also remark that the above proof actually yields

$$(2.15) \quad \zeta(2s, A, P)\Gamma(s) = O(|s|^{-\infty}) \quad (s \in \Pi(-n, n), |s| \rightarrow \infty)$$

for all positive integers n .

Proposition 2.5. *If the order m of A is an integer, then*

- $\mathcal{B}_k(A, P) = \frac{\Gamma(\frac{d+m-k}{2})}{2} \cdot \mathcal{A}_k(A, P)$ ($d+m-k$ is positive or negative but odd);
- $\mathcal{C}_k(A, P) = 0$ ($d+m+2k < 0$);
- $\mathcal{C}_k(A, P) = \frac{(-1)^{k+1}}{2 \cdot k!} \cdot \mathcal{A}_{d+m+2k}(A, P)$ ($d+m+2k \geq 0$).

If the order m of A is not an integer, then for all non-negative integers k :

- $\mathcal{B}_k(A, P) = \frac{\Gamma(\frac{d+m-k}{2})}{2} \cdot \mathcal{A}_k(A, P)$;
- $\mathcal{C}_k(A, P) = 0$.

Remark 2.6. There exists an even stronger estimate than (2.15) in [26]. But to apply the inverse Mellin transform theorem, a rough estimate like (2.15) is sufficient.

2.6. Resolvent trace expansions. In this part we study the short time asymptotic expansion of the resolvent trace $f^{(N)}(t) := \text{tr}(A(1+tP)^{-N/2})$, where $N \in \mathbb{C}$ is such that $\text{Re}(N) > \max\{d+m, 0\}$. Let $M^{(N)}$ denote the Mellin transform of

$$\text{tr}(A(1+tP)^{-N/2}) - \sum_{\lambda_j=0} \langle A\phi_j, \phi_j \rangle \quad (t > 0).$$

It is easy to verify that $M^{(N)}$ has a non-empty basic strip $\Pi(\max\{\frac{d+m}{2}, 0\}, \frac{\text{Re}(N)}{2})$ in which $M^{(N)}(s) = \zeta(2s, A, P)B(s, \frac{N}{2} - s)$, where we recall the Beta function

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

is defined for $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$. Hence $M^{(N)}$ admits a meromorphic continuation to $\Pi(-\infty, \frac{\operatorname{Re}(N)}{2})$ in which

$$(2.16) \quad M^{(N)}(s) = \zeta(2s, A, P) \frac{\Gamma(s)\Gamma(\frac{N}{2} - s)}{\Gamma(\frac{N}{2})}.$$

For any real β in the strip $\Pi(\max\{\frac{d+m}{2}, 0\}, \frac{\operatorname{Re}(N)}{2})$ and any positive integer n , it is easy to see that

$$\sup_{s \in \Pi(-n, \beta)} \left| \Gamma\left(\frac{N}{2} - s\right) \right| \leq \max_{\frac{\operatorname{Re}(N)}{2} - \beta \leq t \leq \frac{\operatorname{Re}(N)}{2} + n} \Gamma(t) < \infty.$$

Consequently by (2.13),

$$(2.17) \quad M^{(N)}(s) = O(|s|^{-2}) \quad (s \in \Pi(-n, \beta), |s| \rightarrow \infty),$$

which immediately implies

$$(2.18) \quad f^{(N)}(t) \sim \sum_{k=0}^{\infty} (\mathcal{B}_k^{(N)}(A, P)t^{\frac{k-d-m}{2}} + \mathcal{C}_k^{(N)}(A, P)t^k \log(t) + \mathcal{D}_k^{(N)}(A, P)t^k)$$

via Lemma 2.1 as the singularities of $\zeta(\cdot, A, P)$ and Γ are known to be completely determined, so is $M^{(N)}$. The precise relations between the mollified counting coefficients and some of the resolvent trace coefficients are summarized in the following proposition:

Proposition 2.7. *If the order m of A is an integer, then*

- $\mathcal{B}_k^{(N)}(A, P) = \frac{\Gamma(\frac{d+m-k}{2})}{\Gamma(\frac{N}{2})} \cdot \frac{\Gamma(\frac{N+k-d-m}{2})}{\Gamma(\frac{N}{2})} \cdot \mathcal{A}_k(A, P)$ ($d+m-k$ is positive or negative but odd);
- $\mathcal{C}_k^{(N)}(A, P) = 0$ ($d+m+2k < 0$);
- $\mathcal{C}_k^{(N)}(A, P) = \frac{(-1)^{k+1}}{2 \cdot k!} \cdot \frac{\Gamma(\frac{N}{2}+k)}{\Gamma(\frac{N}{2})} \cdot \mathcal{A}_{d+m+2k}(A, P)$ ($d+m+2k \geq 0$).

If the order m of A is not an integer, then for all non-negative integers k :

- $\mathcal{B}_k^{(N)}(A, P) = \frac{\Gamma(\frac{d+m-k}{2})}{\Gamma(\frac{N}{2})} \cdot \frac{\Gamma(\frac{N+k-d-m}{2})}{\Gamma(\frac{N}{2})} \cdot \mathcal{A}_k(A, P)$;
- $\mathcal{C}_k^{(N)}(A, P) = 0$.

2.7. Wodzicki residues. This part is devoted to the proofs of (1.27) and (1.28) and their consequences. Both formulae are known and we state them here as two propositions for completeness. Also, we refer the readers to the original references for the proof of the crucial formula (1.26).

Proposition 2.8. $\mathcal{A}_k(A, P) = \operatorname{res}(AP^{\frac{k-d-m}{2}})$.

Proof. Since $AP^{\frac{k-d-m}{2}}$ is of integer order $k-d$ and $d+(k-d)+2 \cdot 0 = k \geq 0$, it follows first from Proposition 2.5 then from Proposition 2.3 that

$$-2\mathcal{C}_0(AP^{\frac{k-d-m}{2}}, P) = \mathcal{A}_k(AP^{\frac{k-d-m}{2}}, P) = \mathcal{A}_k(A, P).$$

But according to (1.26) we also have

$$-2\mathcal{C}_0(AP^{\frac{k-d-m}{2}}, P) = \text{res}(AP^{\frac{k-d-m}{2}}).$$

Combining the above two formulae yields the desired result. \square

Proposition 2.9. $\mathcal{C}_k(A, P) = \frac{(-1)^{k+1}}{2 \cdot k!} \cdot \text{res}(AP^k).$

Proof. If A is of integer order $m \geq -d - 2k$, then it follows first from Proposition 2.5 then from Proposition 2.8 that

$$\mathcal{C}_k(A, P) = \frac{(-1)^{k+1}}{2 \cdot k!} \cdot \mathcal{A}_{d+m+2k}(A, P) = \frac{(-1)^{k+1}}{2 \cdot k!} \cdot \text{res}(AP^k).$$

If A is not of integer order or A is of integer order $m < -d - 2k$, then it follows from definition that $\text{res}(AP^k) = 0$ and from Proposition 2.5 that $\mathcal{C}_k(A, P) = 0$. This finishes the proof. \square

Next we study the property $\mathcal{A}_k(A, P) = 0$ for certain operators $A \in \Psi(M; E)$. We denote by $\Psi_{\text{odd}}^{\mathbb{Z}}(M; E)$ the space of odd-class pseudodifferential operators and by $\Psi_{\text{even}}^{\mathbb{Z}}(M; E)$ the space of even-class pseudodifferential operators on sections of E , that is, $A \in \Psi_{\text{odd}}^{\mathbb{Z}}(M; E)$ if in local coordinates its symbol $\sum_{j \geq 0} \sigma_{m-j}(A)$ ($m \in \mathbb{Z}$) satisfies

$$(2.19) \quad \sigma_{m-j}(A)(x, -\xi) = (-1)^{m-j} \sigma_{m-j}(A)(x, \xi)$$

for all x, ξ and j , while $A \in \Psi_{\text{even}}^{\mathbb{Z}}(M; E)$ if in local coordinates

$$(2.20) \quad \sigma_{m-j}(A)(x, -\xi) = (-1)^{m-j+1} \sigma_{m-j}(A)(x, \xi)$$

for all x, ξ and j . For example, any partial differential operator is odd-class while $P^{1/2}$ is even-class. An operator $A \in \Psi_{\text{odd}}^{\mathbb{Z}}(M; E) \cup \Psi_{\text{even}}^{\mathbb{Z}}(M; E)$ is said of regular parity class if its parity class agrees with that of d . It is easy to verify that $\text{res}(A) = 0$ if A is of regular parity class.

Proposition 2.10. *Let $A \in \Psi_{\text{odd}}^{\mathbb{Z}}(M; E) \cup \Psi_{\text{even}}^{\mathbb{Z}}(M; E)$. If the parity class of A agrees with that of $k - m$, then $\mathcal{A}_k(A, P) = 0$.*

Proof. We define a map τ by sending odd-class pseudodifferential operators to 0 and even-class ones to 1. Then (see [41, Prop. 1.11]) $\tau(AP^{\frac{k-d-m}{2}}) = \tau(A) + k - d - m \pmod{2}$. Thus if the parity class of A agrees with that of $k - m$, or equivalently, the parity class of $\tau(AP^{\frac{k-d-m}{2}})$ does not agree with that of d , or further equivalently, the parity class of $AP^{\frac{k-d-m}{2}}$ agrees with that of d , then $AP^{\frac{k-d-m}{2}}$ is of regular parity class and consequently by Proposition 2.8, $\mathcal{A}_k(A, P) = \text{res}(AP^{\frac{k-d-m}{2}}) = 0$. \square

Remark 2.11. The heat expansion (1.22) can be written in a more rigorous way as

$$(2.21) \quad \text{tr}(Ae^{-tP}) \sim \sum_{j \in \Lambda} \tilde{\mathcal{B}}_j(A, P)t^{\frac{j-d-m}{2}} + \sum_{k=0}^{\infty} (\tilde{\mathcal{C}}_k(A, P)t^k \log(t) + \tilde{\mathcal{D}}_k(A, P)t^k),$$

where Λ denotes the set of non-negative integers j such that $\frac{j-d-m}{2}$ is not a non-negative integer. In this notation system $\tilde{\mathcal{B}}_j(A, P)$, $\tilde{\mathcal{C}}_k(A, P)$ and $\tilde{\mathcal{D}}_k(A, P)$ are

uniquely determined. According to Propositions 2.5, 2.8, 2.9, $\tilde{\mathcal{B}}_j(A, P)$ and $\tilde{\mathcal{C}}_k(A, P)$ are certain Wodzicki residues. In most situations, $\tilde{\mathcal{D}}_k(A, P)$ are not Wodzicki residues but operator traces. For the simplest example that A is smoothing operator, one can (regard A as of any real order and thus) deduce from (2.21) that

$$(2.22) \quad \text{tr}(Ae^{-tP}) \sim \sum_{k=0}^{\infty} \tilde{\mathcal{D}}_k(A, P)t^k.$$

In this situation it is so natural to expect $\text{tr}(A) = \tilde{\mathcal{D}}_0(A, P)$. Actually, if any of the following three cases happens, then $\tilde{\mathcal{D}}_0(A, P)$ just is the Kontsevich-Vishik trace of A which is independent of the choice of P (see e.g. [24, 34, 38]):

- $m < -d$;
- m is not an integer;
- A is of regular parity class.

Since the Kontsevich-Vishik trace agrees with the standard L^2 -operator trace on trace class operators, the previous expectation is confirmed. Furthermore, if $A = Q$ is a partial differential operator and if d is even, then one can express $\tilde{\mathcal{D}}_0(A, P)$ as the Wodzicki residue of certain pseudodifferential operator with log-polyhomogeneous symbol (see [39, 40, 46] for details).

2.8. Dirac operators. This part is devoted to the proofs of (1.5), (1.20) and (1.29). First, we introduce a classical pseudodifferential operator B of order m by defining $B = A \frac{\text{Sign}(D) + \text{Id}_E}{2}$. Since $\text{supp}(N'_A - N'_B) \subset (-\infty, 0]$, we get (see also Subsection 2.2)

$$(\chi * N'_A)(\mu) - (\chi * N'_B)(\mu) = o(\mu^{-\infty}) \quad (\mu \rightarrow \infty).$$

By appealing to (1.18) with A, P, λ, λ_j replaced respectively by $B, D^2, \mu, |\mu_j|$, we obtain as $\mu \rightarrow \infty$ that

$$\begin{aligned} (\chi * N'_B)(\mu) &= \sum_{j=1}^{\infty} \langle B\phi_j, \phi_j \rangle \chi(\mu - \mu_j) \\ &= \sum_{j=1}^{\infty} \langle B\phi_j, \phi_j \rangle \chi(\mu - |\mu_j|) \\ &\sim \sum_{k=0}^{\infty} \mathcal{A}_k(B, D^2) \mu^{d+m-k-1}, \end{aligned}$$

which proves (1.5) as well as $\mathcal{A}_k(A, D) = \mathcal{A}_k(B, D^2)$ for all non-negative integers k . Obviously by setting $A = F$, (1.20) is a special case of $\mathcal{A}_k(A, D) = \mathcal{A}_k(B, D^2)$. Now it follows from Proposition 2.8 that

$$(2.23) \quad \mathcal{A}_k(A, D) = \text{res}\left(A \frac{\text{Sign}(D) + \text{Id}_E}{2} |D|^{k-d-m}\right) = \text{res}\left(A \frac{D + |D|}{2} |D|^{k-d-m-1}\right),$$

which proves (1.29).

Next, we give an equivalent expression for $\mathcal{A}_k(A, D)$. By Proposition 2.3 one gets

$$(2.24) \quad \mathcal{A}_k(A, D) = \mathcal{A}_k(B, D^2) = \frac{\mathcal{A}_k(A, D^2) + \mathcal{A}_k(AD, D^2)}{2}.$$

Similar to the microlocalized spectral zeta function (1.21), one can introduce the microlocalized spectral eta function (see e.g. [3, 21]) by defining

$$(2.25) \quad \eta(s, A, D) = \sum_{\mu_j \neq 0} \frac{\langle A\phi_j, \phi_j \rangle}{|\mu_j|^s} \cdot \text{Sign}(\mu_j) \quad (\text{Re}(s) > d + m).$$

But it is easy to see that

$$(2.26) \quad \eta(s, A, D) = \zeta(s, A \cdot \text{Sign}(D), D^2).$$

So according to Proposition 2.8, $\eta(s, A, D)$ admits a meromorphic continuation to \mathbb{C} whose only singularities are simple poles at $s = d + m - k$ ($k = 0, 1, 2, \dots$) with residues $\text{res}(AD|D|^{k-d-m-1})$.

2.9. Local counting functions. This part is devoted to the proof of (1.12) and its consequences. Recall Q denotes a partial differential operator of order m acting on $C^\infty(M; E)$. Combining (1.10) \sim (1.11) yields

$$(2.27) \quad \text{tr}(Qe^{-tD^2}) \sim \sum_{k=0}^{\infty} \int_M \text{Tr}_E(\mathcal{H}_k(Q, D^2)) \cdot t^{\frac{k-d-m}{2}} \quad (t \rightarrow 0^+),$$

which compared with the heat expansion (1.22) gives

- $\mathcal{B}_k(Q, D^2) = \int_M \text{Tr}_E(\mathcal{H}_k(Q, D^2))$ ($d + m - k$ is positive or negative but odd);
- $\mathcal{C}_k(Q, D^2) = 0$.

According to Proposition 2.5 we also have

- $\mathcal{B}_k(Q, D^2) = \frac{\Gamma(\frac{d+m-k}{2})}{2} \mathcal{A}_k(Q, D^2)$ ($d + m - k$ is positive or negative but odd);
- $\mathcal{C}_k(Q, D^2) = \frac{(-1)^{k+1}}{2 \cdot k!} \mathcal{A}_{d+m+2k}(Q, D^2)$ ($d + m + 2k \geq 0$).

Thus for all non-negative integers k ,

$$(2.28) \quad \mathcal{A}_k(Q, D^2) = \frac{2}{\Gamma(\frac{d+m-k}{2})} \int_M \text{Tr}_E(\mathcal{H}_k(Q, D^2)).$$

By combining (1.8), (2.24), (2.28) and noting $\mathcal{H}_k(FQ, D^2) = F\mathcal{H}_k(Q, D^2)$, we get

$$(2.29) \quad \begin{aligned} \int_M \text{Tr}_E(F\mathcal{L}_k(D)) &= \mathcal{A}_k(F, D) = \frac{\mathcal{A}_k(F, D^2) + \mathcal{A}_k(FD, D^2)}{2} \\ &= \int_M \text{Tr}_E \left(F \left(\frac{\mathcal{H}_k(\text{Id}_E, D^2)}{\Gamma(\frac{d-k}{2})} + \frac{\mathcal{H}_k(D, D^2)}{\Gamma(\frac{d+1-k}{2})} \right) \right), \end{aligned}$$

which proves (1.12) as F can be any smooth endomorphism of E . To be precise, one can set F to be the adjoint of

$$\mathcal{L}_k(D) = \left(\frac{\mathcal{H}_k(\text{Id}_E, D^2)}{\Gamma(\frac{d-k}{2})} + \frac{\mathcal{H}_k(D, D^2)}{\Gamma(\frac{d+1-k}{2})} \right)$$

then apply (2.29) to conclude (1.12).

Next, we study the local counting function of Laplace type operators but first have a look at that of Dirac type operators. According to (1.7),

$$\sum_{j=1}^{\infty} \Phi_j \chi(\mu \mp \mu_j) \sim \sum_{k=0}^{\infty} \mu^{d-k-1} \mathcal{L}_k(\pm D) \quad (\mu \rightarrow \infty),$$

which easily implies (see also [30])

$$\sum_{j=1}^{\infty} \Phi_j \chi(\mu - |\mu_j|) \sim \sum_{k=0}^{\infty} \mu^{d-k-1} (\mathcal{L}_k(D) + \mathcal{L}_k(-D)) \quad (\mu \rightarrow \infty).$$

As an application of (1.12), one gets $\mathcal{L}_k(D) + \mathcal{L}_k(-D) = \frac{2\mathcal{H}_k(\text{Id}_E, D^2)}{\Gamma(\frac{d-k}{2})}$. Thus

$$(2.30) \quad \sum_{j=1}^{\infty} \Phi_j \chi(\mu - |\mu_j|) \sim \sum_{k=0}^{\infty} \mu^{d-k-1} \frac{2\mathcal{H}_k(\text{Id}_E, D^2)}{\Gamma(\frac{d-k}{2})} \quad (\mu \rightarrow \infty).$$

Generally for Laplace type operators P , one can similarly establish that

$$(2.31) \quad \sum_{j=1}^{\infty} \Phi_j \chi(\lambda - \lambda_j) \sim \sum_{k=0}^{\infty} \lambda^{d-k-1} \frac{2\mathcal{H}_k(\text{Id}_E, P)}{\Gamma(\frac{d-k}{2})} \quad (\lambda \rightarrow \infty).$$

3. THE BOCHNER-WEITZENBÖCK TECHNIQUE AND A DIRECT PROOF OF THEOREM 1.2

In this section we will introduce some basic notations about Dirac operators and give a direct proof of Theorem 1.2 by employing the Bochner-Weitzenböck technique in Riemannian geometry (see e.g. [2, 17, 19, 35]).

3.1. Bochner-Weitzenböck technique. We begin by reviewing the concept of the Dirac bundle $(E, \gamma, \tilde{\nabla})$ (see e.g. [35]), where γ is a Clifford module structure on E and $\tilde{\nabla}$ is a connection on E that is compatible with γ .

First, let $\text{Cl}(TM)$ be the universal unital complex algebra bundle generated by the tangent bundle TM subject to the commutation relation $X*Y + Y*X = -2g(X, Y)$, where $X, Y \in C^\infty(M; TM)$ and $*$ denotes the algebra operation. A Clifford module structure on E just is a unital algebra morphism $\gamma : \text{Cl}(TM) \rightarrow \text{End}(E)$. Given a connection $\hat{\nabla}$ on the Clifford module $E = (E, \gamma)$, $\gamma \hat{\nabla} = \gamma(e_k) \hat{\nabla}_{e_k}$ is a well-defined first order partial differential operator acting on $C^\infty(M; E)$, where $\{e_k\}_{k=1}^d$ denotes any orthonormal basis of $T_x M$ at each point $x \in M$. Next, one can naturally extend the Riemannian metric μ_g and connection ∇ on TM to $\text{Cl}(TM)$ with the properties

that: the extended connection denoted also by ∇ preserves the extended metric, and

$$(3.1) \quad \nabla_X(\alpha * \beta) = (\nabla_X\alpha) * \beta + \alpha * (\nabla_X\beta)$$

for all $X \in C^\infty(M; TM)$, $\alpha, \beta \in C^\infty(M; \text{Cl}(TM))$. We then call a connection $\tilde{\nabla}$ on the Clifford module (E, γ) compatible with γ if

$$(3.2) \quad \tilde{\nabla}_X(\gamma(\alpha)\phi) = \gamma(\nabla_X\alpha)\phi + \gamma(\alpha)(\tilde{\nabla}_X\phi)$$

for all $X \in C^\infty(M; TM)$, $\alpha \in C^\infty(M; \text{Cl}(TM))$, $\phi \in C^\infty(M; E)$. It is well-known ([3]) that on a Clifford module there always exists a compatible connection. Such a triple $(E, \gamma, \tilde{\nabla})$ is called a Dirac bundle. With the concept of Dirac bundle available, it is easy to see that any operator D of Dirac type acting on $C^\infty(M; E)$ can always be written as $D = \gamma\tilde{\nabla} + \psi$, where $(E, \gamma, \tilde{\nabla})$ is some Dirac bundle with γ uniquely determined by the principal symbol σ_D of D , $\psi \in C^\infty(M; \text{End}(E))$. We call ψ the potential of D associated with the Dirac bundle $(E, \gamma, \tilde{\nabla})$.

Let $(E, \gamma, \tilde{\nabla})$ be a Dirac bundle and let $D_\psi = \gamma\tilde{\nabla} + \psi$ be a Dirac type operator with potential ψ . The Bochner-Weitzenböck technique permits us to find a unique connection ∇_ψ on E and a unique $V_\psi \in C^\infty(M; \text{End}(E))$ such that $D_\psi^2 = \Delta^{\nabla_\psi} + V_\psi$, where $\Delta^{\nabla_\psi} = \nabla_\psi^* \nabla_\psi$ denotes the connection Laplacian generated by ∇_ψ . In a series of papers by Gilkey and his collaborators (see e.g. [3, 19, 20]), V_ψ is always written as a function explicitly depending on ∇_ψ and a few others. Our feature of proving Theorem 1.2 is to first express V_ψ as a function of $\gamma, \tilde{\nabla}, \psi$, then treat the first two variables as dummy ones.

Given a Dirac bundle $(E, \gamma, \tilde{\nabla})$, we define a connection $\bar{\nabla}$ on $\text{End}(E)$ by

$$(3.3) \quad (\bar{\nabla}_X\omega)(\phi) = \tilde{\nabla}_X(\omega(\phi)) - \omega(\tilde{\nabla}_X\phi)$$

for all $X \in C^\infty(M; TM)$, $\omega \in C^\infty(M; \text{End}(E))$, $\phi \in C^\infty(M; E)$. It is easy to verify that $\bar{\nabla}$ is compatible with γ in the following sense:

$$(3.4) \quad (\bar{\nabla}_X)(\omega\gamma(Y)) = (\bar{\nabla}_X\omega)\gamma(Y) + \omega\gamma(\nabla_X Y)$$

$$(3.5) \quad (\bar{\nabla}_X)(\gamma(Y)\omega) = \gamma(Y)(\bar{\nabla}_X\omega) + \gamma(\nabla_X Y)\omega$$

where X, ω, ϕ remain the same meanings as before, $Y \in C^\infty(M; TM)$.

We also need to fix a few notations. If (x^1, \dots, x^d) is a system of local coordinates on M , then let $\{\partial_i = \frac{\partial}{\partial x^i}\}$ denote the local coordinate frames of TM . We use the metric tensor to identify the tangent and cotangent bundles $TM = T^*M$, which means locally $dx^j \equiv g^{ij}\partial_i$, where (g^{ij}) is the inverse of the matrix $(g_{ij} = g(\partial_i, \partial_j))$. The curvature tensor of any connection $\hat{\nabla}$ on E is denoted by $R^{\hat{\nabla}}$, that is, $R^{\hat{\nabla}}_{XY} = [\hat{\nabla}_X, \hat{\nabla}_Y] - \hat{\nabla}_{[X, Y]}$ for $X, Y \in C^\infty(M; TM)$. Let Γ_{ij}^k denote the Christoffel symbols of the Riemannian connection ∇ , that is, $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k$. The connection Laplacian $\Delta^{\hat{\nabla}}$ locally is of the form $-g^{ij}(\hat{\nabla}_{\partial_i}\hat{\nabla}_{\partial_j} - \Gamma_{ij}^k\hat{\nabla}_{\partial_k})$. Finally, we denote $U^{i_1, \dots, i_n} = U(dx^{i_1}, \dots, dx^{i_n})$ and $U_{i_1, \dots, i_n} = U(\partial_{i_1}, \dots, \partial_{i_n})$ for any map U defined on the n -fold product of TM .

Proposition 3.1. *Let D be a Dirac type operator of potential ψ associated with the Dirac bundle $(E, \gamma, \tilde{\nabla})$. Let $L = L_\psi : TM \rightarrow \text{End}(E)$ denote the map defined by*

$$(3.6) \quad L(X) = \frac{\gamma(X)\psi + \psi\gamma(X)}{2}.$$

Then $\nabla_\psi := \tilde{\nabla} - L$ is a connection on E and $V_\psi := D^2 - \Delta^{\nabla_\psi} \in C^\infty(M; \text{End}(E))$. Locally we have

$$(3.7) \quad V_\psi = \frac{1}{2}\gamma^i\gamma^j R_{ij}^{\tilde{\nabla}} + \frac{1}{2}[\gamma^i, \bar{\nabla}_i\psi] + L^i L_i + \psi^2.$$

Proof. Let L be naturally identified with

$$C^\infty(M; TM) \times C^\infty(M; E) \xrightarrow{L} C^\infty(M; E),$$

which is $C^\infty(M)$ -linear in both variables. Thus $\hat{\nabla} := \tilde{\nabla} - L$ is a connection on E . According to (3.3) we have

$$\begin{aligned} \Delta^{\tilde{\nabla}} - \Delta^{\hat{\nabla}} &= -g^{ij}(\tilde{\nabla}_i\tilde{\nabla}_j - \tilde{\nabla}_i\hat{\nabla}_j + \tilde{\nabla}_i\hat{\nabla}_j - \hat{\nabla}_i\hat{\nabla}_j) + g^{ij}\Gamma_{ij}^k(\tilde{\nabla}_k - \hat{\nabla}_k) \\ &= -g^{ij}\tilde{\nabla}_i L_j - g^{ij}L_i\hat{\nabla}_j + g^{ij}\Gamma_{ij}^k L_k \\ &= -g^{ij}((\bar{\nabla}_i L_j) + L_j\tilde{\nabla}_i) - g^{ij}L_i(\tilde{\nabla}_j - L_j) + g^{ij}\Gamma_{ij}^k L_k \\ &= -2L^i\tilde{\nabla}_i - g^{ij}(\bar{\nabla}_i L_j) + L^i L_i + g^{ij}\Gamma_{ij}^k L_k. \end{aligned}$$

On the other hand, according to (3.2) and (3.3) we have

$$\begin{aligned} D^2 &= (\gamma^i\tilde{\nabla}_i + \psi)(\gamma^j\tilde{\nabla}_j + \psi) \\ &= \gamma^i\gamma^j\tilde{\nabla}_i\tilde{\nabla}_j - \Gamma_{ik}^j\gamma^i\gamma^k\tilde{\nabla}_j + \gamma^i(\bar{\nabla}_i\psi) + \gamma^i\psi\tilde{\nabla}_i + \psi\gamma^j\tilde{\nabla}_j + \psi^2 \\ &= \gamma^i\gamma^j\frac{\tilde{\nabla}_i\tilde{\nabla}_j + \tilde{\nabla}_j\tilde{\nabla}_i}{2} + \gamma^i\gamma^j\frac{R_{ij}^{\tilde{\nabla}}}{2} + \Gamma_{ik}^j g^{ik}\tilde{\nabla}_j + \gamma^i(\bar{\nabla}_i\psi) + (\gamma^i\psi + \psi\gamma^i)\tilde{\nabla}_i + \psi^2 \\ &= -g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j + g^{ij}\Gamma_{ij}^k\tilde{\nabla}_k + \gamma^i\gamma^j\frac{R_{ij}^{\tilde{\nabla}}}{2} + \gamma^i(\bar{\nabla}_i\psi) + (\gamma^i\psi + \psi\gamma^i)\tilde{\nabla}_i + \psi^2 \\ &= \Delta^{\tilde{\nabla}} + (\gamma^i\psi + \psi\gamma^i)\tilde{\nabla}_i + \gamma^i\gamma^j\frac{R_{ij}^{\tilde{\nabla}}}{2} + \gamma^i(\bar{\nabla}_i\psi) + \psi^2. \end{aligned}$$

Consequently, combining the above calculations yields

$$D^2 = \Delta^{\hat{\nabla}} + \gamma^i\gamma^j\frac{R_{ij}^{\tilde{\nabla}}}{2} + \gamma^i(\bar{\nabla}_i\psi) + \psi^2 - g^{ij}(\bar{\nabla}_i L_j) + L^i L_i + g^{ij}\Gamma_{ij}^k L_k =: \Delta^{\hat{\nabla}} + V_\psi.$$

Next let us simplify the expression of V_ψ . According to (3.4) and (3.5) we have

$$\begin{aligned} -g^{ij}(\bar{\nabla}_i L_j) &= -g^{ij}\frac{\gamma_j(\bar{\nabla}_i\psi) + \Gamma_{ij}^k\gamma_k\psi + (\bar{\nabla}_i\psi)\gamma_j + \Gamma_{ij}^k\psi\gamma_k}{2} \\ &= -g^{ij}\frac{\gamma_j(\bar{\nabla}_i\psi) + (\bar{\nabla}_i\psi)\gamma_j}{2} - g^{ij}\Gamma_{ij}^k L_k \\ &= -\frac{\gamma^i(\bar{\nabla}_i\psi) + (\bar{\nabla}_i\psi)\gamma^i}{2} - g^{ij}\Gamma_{ij}^k L_k, \end{aligned}$$

which implies

$$\begin{aligned} V_\psi &= \gamma^i \gamma^j \frac{R_{ij}^{\tilde{\nabla}}}{2} + \gamma^i (\bar{\nabla}_i \psi) + \psi^2 - \frac{\gamma^i (\bar{\nabla}_i \psi) + (\bar{\nabla}_i \psi) \gamma^i}{2} + L^i L_i \\ &= \gamma^i \gamma^j \frac{R_{ij}^{\tilde{\nabla}}}{2} + \frac{1}{2} [\gamma^i, \bar{\nabla}_i \psi] + \psi^2 + L^i L_i. \end{aligned}$$

This finishes the proof of Proposition 3.1. \square

3.2. Direct proof of Theorem 1.2. Let D_ϵ denote the Dirac type operator of potential $\psi_\epsilon = \psi - \epsilon F$ associated with the Dirac bundle $(E, \gamma, \tilde{\nabla})$, where ϵ is a real parameter. In particular $D = D_0$. Let $\nabla_\epsilon = \nabla_{\psi_\epsilon}$, $V_\epsilon = V_{\psi_\epsilon}$ be as determined by Proposition 3.1. Locally,

$$V_\epsilon = \frac{1}{2} \gamma^i \gamma^j R_{ij}^{\tilde{\nabla}} + \frac{1}{2} [\gamma^i, \bar{\nabla}_i \psi_\epsilon] + (L_{\psi_\epsilon})^i (L_{\psi_\epsilon})_i + \psi_\epsilon^2.$$

Thus

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{Tr}(V_\epsilon) = \text{Tr}((L_{-F})^i (L_\psi)_i + (L_\psi)^i (L_{-F})_i) - 2\text{Tr}(F\psi),$$

which following simplification via the facts $\gamma^i \gamma_i = -d$ and $\widehat{\psi} = \gamma^i \psi \gamma_i = \gamma_i \psi \gamma^i$ gives

$$(3.8) \quad \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{Tr}(V_\epsilon) = \text{Tr}(F((d-2)\psi - \widehat{\psi})).$$

But it is known (see e.g. [3, Lemma 2.1], [19, Thm. 3.3.1]) that

$$(3.9) \quad \int_M \text{Tr}(\mathcal{H}_1(FD, D^2)) = \frac{1}{12 \cdot (4\pi)^{d/2}} \cdot \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_M \text{Tr}(\tau \text{Id}_E - 6V_\epsilon),$$

where τ is the scalar curvature of M . By combining (3.8), (3.9) and by considering the fact $\mathcal{H}_1(FD, D^2) = F \mathcal{H}_1(D, D^2)$, one gets

$$\int_M \text{Tr}(F \cdot \mathcal{H}_1(D, D^2)) = \frac{1}{(4\pi)^{d/2}} \cdot \int_M \text{Tr}(F \cdot \frac{\widehat{\psi} - (d-2)\psi}{2}).$$

Consequently,

$$(3.10) \quad \mathcal{H}_1(D, D^2) = \frac{1}{(4\pi)^{d/2}} \cdot \frac{\widehat{\psi} - (d-2)\psi}{2}.$$

as F can be any smooth endomorphism of E . By (1.8) and (1.12) (see also (2.29)),

$$\mathcal{A}_1(F, D) = \int_M \text{Tr}\left(F \left(\frac{\mathcal{H}_1(\text{Id}_E, D^2)}{\Gamma(\frac{d-1}{2})} + \frac{\mathcal{H}_1(D, D^2)}{\Gamma(\frac{d}{2})} \right)\right).$$

But $\mathcal{H}_1(\text{Id}_E, D^2) = 0$ (see e.g. [18]), which completes the proof of Theorem 1.2.

Remark 3.2. It is known (see e.g. [18, 48]) that (1.10) and (1.11) still hold even if without assuming the self-adjointness of D if the integral kernels are appropriately interpreted. In this general situation our direct proof of Theorem 1.2 can also yield (3.10) with no modification at all.

4. MOLLIFIED COUNTING COEFFICIENTS II

In this section we will express $\mathcal{A}_1(A, P)$ in terms of principal and sub-principal symbols of A and P , and prove Theorems 1.1, 1.2. We refer the readers to [7, 8, 10, 12, 30, 31, 42, 43] for some other explicit two-term asymptotic expansions in various situations.

We begin by reviewing the definition of the sub-principal symbol of a classical pseudodifferential operator.

Recall M is a closed manifold over which the space of smooth sections of the half-density bundle is denoted by $C^\infty(M, \Omega_{1/2})$. An operator $H : C^\infty(M, \Omega_{1/2}) \rightarrow C^\infty(M, \Omega_{1/2})$ is called a classical pseudodifferential operator of order m if on every coordinate patch its total symbol admits an asymptotic expansion $\sum_{j=0}^{\infty} \sigma_H^{(j)}(x, \xi)$ with $\sigma_H^{(j)}$ homogeneous of degree $m - j$ in ξ . It is well-known that the sub-principal symbol (see e.g. [11, Section 5.2]) of H defined in local coordinates by

$$(4.1) \quad \text{Sub}(H) = \sigma_H^{(1)} + \frac{i}{2} \cdot \frac{\partial^2 \sigma_H^{(0)}}{\partial x^k \partial \xi_k},$$

transforms like a homogeneous smooth function of degree $m - 1$ on $T^*M \setminus 0$ under change of charts.

Let $A : C^\infty(M) \rightarrow C^\infty(M)$ be a classical pseudodifferential operator of order m . In local coordinates we denote by $\sigma_A^{(j)} = \sigma_A^{(j)}(x, \xi)$ ($j = 0, 1, 2, \dots$) the homogeneous part of degree $m - j$ of the total symbol of A^1 . Given a Riemannian metric g on M one can identify $C^\infty(M)$ with $C^\infty(M, \Omega_{1/2})$ via the map

$$f \in C^\infty(M) \xrightarrow{\Phi} f \sqrt{\mu_g} \in C^\infty(M, \Omega_{1/2}),$$

where μ_g is the Riemannian density (called also metric measure) on (M, g) . Then $H = \Phi \circ A \circ \Phi^{-1}$ is a classical pseudodifferential operator of order m on half-densities. On local coordinates let $G := \det(g_{ij})$. Note $H = G^{1/4} A G^{-1/4}$ can be regarded as the product of three pseudodifferential operators with corresponding total symbols $G(x)^{1/4}$, $\sum_{j=0}^{\infty} \sigma_A^{(j)}(x, \xi)$, $G(x)^{-1/4}$. Thus one can deduce from the product rule of pseudodifferential operators that

$$\text{Sub}(H) = \sigma_A^{(1)} + \frac{i}{2} \cdot \frac{\partial^2 \sigma_A^{(0)}}{\partial x^k \partial \xi_k} + \frac{i}{2} \cdot \frac{\partial \sigma_A^{(0)}}{\partial \xi_k} \cdot \frac{\partial(\log \sqrt{G})}{\partial x^k}$$

on local coordinates. This means the sub-principal symbol of A locally defined by ([42, Remark 2.1.10])

$$(4.2) \quad \text{Sub}(A) = \sigma_A^{(1)} + \frac{i}{2} \cdot \frac{\partial^2 \sigma_A^{(0)}}{\partial x^k \partial \xi_k} + \frac{i}{2} \cdot \frac{\partial \sigma_A^{(0)}}{\partial \xi_k} \cdot \frac{\partial(\log \sqrt{G})}{\partial x^k},$$

is a homogeneous smooth function of degree $m - 1$ on $T^*M \setminus 0$.

¹Note we also denote $\sigma_A^{(j)}$ by $\sigma_{m-j}(A)$ in the previous sections.

Now let $A : C^\infty(M; E) \rightarrow C^\infty(M; E)$ be a classical pseudodifferential operator of order m . For a fixed local bundle trivialization of E over a coordinate neighborhood $U \subset M$, one can naturally identify $A^U : C_c^\infty(U; E) \rightarrow C^\infty(U; E)$, the restriction of A onto U , with a matrix $(A_{\mu\nu}^U)_{1 \leq \mu, \nu \leq r}$ of classical pseudodifferential operators $A_{\mu\nu}^U : C^\infty(M) \rightarrow C^\infty(M)$ of order m , where r denotes the rank of E . This implies that the sub-principal symbol of A defined by (see also [32]) $\text{Sub}(A) = (\text{Sub}(A_{\mu\nu}^U))_{1 \leq \mu, \nu \leq r}$ is a homogeneous smooth $\text{Mat}(r, \mathbb{C})$ -valued function of degree $m - 1$ on $T^*U \setminus 0$. On local coordinates $\text{Sub}(A)$ is still of the form (4.2). We should remember, however, that the definition of the sub-principal symbol depends on the choice of local frames for E and Riemannian metric on M . If the principal symbol is a multiple of the identity then the sub-principal symbol can be invariantly defined as a partial connection along the Hamiltonian vector field generated by the principal symbol (see [32], Section 3.1).

Remark 4.1. Let M be a closed manifold and let $A : C^\infty(M) \rightarrow C^\infty(M)$ be a classical pseudodifferential operator of order m . Some authors ([42]) like to identify $C^\infty(M)$ with $C^\infty(M, \Omega_{1/2})$ by first introducing a smooth positive density ρ on M then identifying functions with half-densities by $f \leftrightarrow f\sqrt{\rho}$. Suppose this is the case then the sub-principal symbol of A locally defined by ([42, Remark 2.1.10])

$$(4.3) \quad \text{Sub}(A) = \sigma_A^{(1)} + \frac{i}{2} \cdot \frac{\partial^2 \sigma_A^{(0)}}{\partial x^k \partial \xi_k} + \frac{i}{2} \cdot \frac{\partial \sigma_A^{(0)}}{\partial \xi_k} \cdot \frac{\partial(\log \rho)}{\partial x^k},$$

is a homogeneous smooth function of degree $m - 1$ on $T^*M \setminus 0$. We point out there is no essential difference between (4.2) and (4.3) as it is always possible to endow M with a Riemannian metric g such that $\mu_g = \rho$. To see this one can first endow M with an arbitrary Riemannian metric \hat{g} , then get a positive smooth function κ on M so that $\rho = \kappa \mu_{\hat{g}}$, and finally define $g = \kappa^{2/d} \hat{g}$, where d is the dimension of M .

Example 4.2. Let $P = \Delta^\nabla$ denote the connection Laplacian generated by the connection ∇ on a vector bundle E of rank r . For a fixed local frame $\{s_\mu\}_{\mu=1}^r$ for $E|_U$ there exists a matrix $\omega = (\omega_{\mu\nu})_{1 \leq \mu, \nu \leq r}$ of one-forms on U such that $\nabla s_\mu = \omega_{\mu\nu} \otimes s_\nu$. Note in any local coordinates system (x^1, \dots, x^d) on U , $P = -g^{ij}(\nabla_i \nabla_j - \Gamma_{ij}^k \nabla_k)$. Letting $\nabla_j = \partial_j + b_j$ one can easily get $\sigma_P^{(0)} = g^{jk}(x) \xi_j \xi_k$, $\sigma_P^{(1)} = -2ig^{jk} b_j \xi_k + ig^{ln} \Gamma_{ln}^k \xi_k$. Considering $\frac{\partial(\log \sqrt{G})}{\partial x^k} = \Gamma_{kn}^n$ (see e.g. [5, Prop. 2.8], [6, Sec. 2.5], [8, Sec. 6]) we have

$$\begin{aligned} \text{Sub}(P) &= \sigma_P^{(1)} + \frac{i}{2} \cdot \frac{\partial^2 \sigma_P^{(0)}}{\partial x^k \partial \xi_k} + \frac{i}{2} \cdot \frac{\partial \sigma_P^{(0)}}{\partial \xi_k} \cdot \frac{\partial(\log \sqrt{G})}{\partial x^k} \\ &= \sigma_P^{(1)} + i \cdot \frac{\partial g^{jk}}{\partial x^k} \cdot \xi_j + i \cdot g^{jk} \xi_j \cdot \Gamma_{kn}^n \\ &= (-2ig^{jk} b_j \xi_k + ig^{ln} \Gamma_{ln}^k \xi_k) + i \cdot (-\Gamma_{kn}^j g^{nk} - \Gamma_{kn}^k g^{nj}) \cdot \xi_j + i \cdot g^{jk} \xi_j \cdot \Gamma_{kn}^n \\ &= -2ig^{jk} b_j \xi_k. \end{aligned}$$

Also, it is easy to check that $b_j = \omega^T(\partial_j)$ where ω^T denotes the transpose of ω . Thus in an invariant manner, $\text{Sub}(P)(x, \xi) = -2ig(dx^j, dx^k) \omega^T(\partial_j) \xi_k = -2ig(\omega^T, \xi)$.

Example 4.3. Let $D = \gamma \nabla$ be the associated generalized Dirac operator of a Dirac bundle (E, γ, ∇) of rank r . We adopt all the notations about ∇ used in the previous

example. Then $\sigma_D^{(0)} = i\gamma^j \xi_j$, $\sigma_D^{(1)} = \gamma^j b_j$, and consequently,

$$\begin{aligned} \text{Sub}(D) &= \sigma_D^{(1)} + \frac{i}{2} \cdot \frac{\partial^2 \sigma_D^{(0)}}{\partial x^k \partial \xi_k} + \frac{i}{2} \cdot \frac{\partial \sigma_D^{(0)}}{\partial \xi_k} \cdot \frac{\partial(\log \sqrt{G})}{\partial x^k} \\ &= \gamma^j b_j - \frac{1}{2} \cdot \frac{\partial \gamma^k}{\partial x^k} - \frac{1}{2} \cdot \gamma^k \cdot \Gamma_{kn}^n \\ &= \gamma^j b_j - \frac{1}{2} \cdot ([\gamma^k, b_k] - \Gamma_{kn}^k \gamma^n) - \frac{1}{2} \cdot \gamma^k \cdot \Gamma_{kn}^n \\ &= \frac{\gamma^k b_k + b_k \gamma^k}{2}, \end{aligned}$$

where the third equality follows from (3.2) (see also the next subsection). On each inner product space $(T_x M, g_x)$, we introduce two bilinear maps J_x, K_x sending $X_x, Y_x \in T_x M$ respectively to $\gamma(X_x) \omega^T(Y_x)$ and $\omega^T(X_x) \gamma(Y_x)$. Then it is easy to see that

$$(4.4) \quad \text{Sub}(D)(x, \xi) = \frac{\text{Tr}(J_x) + \text{Tr}(K_x)}{2},$$

which means $\text{Sub}(D)(x, \xi)$ actually is independent of ξ .

With the sub-principal symbol concept available, we can express certain Wodzicki residues in a more invariant way. A key ingredient (see e.g. [13, 49]) is that if f is a smooth homogeneous function of degree $1-d$ on $\mathbb{R}^d \setminus \{0\}$, then $\int_{|\xi|=1} \frac{\partial f}{\partial \xi_k} dS(\xi) = 0$ for each k . For example, if $A : C^\infty(M; E) \rightarrow C^\infty(M; E)$ is a classical pseudodifferential operator of order $1-d$, then it is easy to see that

$$\int_{|\xi|=1} \text{Tr}(\sigma_A^{(1)}(x, \xi)) dS(\xi) = \int_{|\xi|=1} \text{Tr}(\text{Sub}(A)(x, \xi)) dS(\xi),$$

and consequently, the global density $\text{res}_x(A) dx$ is also of the form

$$\left(\frac{1}{(2\pi)^d} \int_{|\xi|=1} \text{Tr}(\text{Sub}(A)(x, \xi)) dS(\xi) \right) dx,$$

which is independent of the choice of local coordinates of M and local frames of E . For this reason, if A is a classical pseudodifferential operator of order $1-d$, then $\text{res}(A)$ can be written as

$$(4.5) \quad \text{res}(A) = \frac{1}{(2\pi)^d} \int_{T_1^* M} \text{Tr}(\text{Sub}(A)),$$

which is simply short for

$$\int_M \left(\frac{1}{(2\pi)^d} \int_{|\xi|=1} \text{Tr}(\text{Sub}(A)(x, \xi)) dS(\xi) \right) dx.$$

Similarly, any integration over the unit cotangent bundle in the paper is always understood in this manner.

Proposition 4.4. *Let A, B be classical pseudodifferential operators on sections of E such that the sum of the order of A and B is $1 - d$. Then*

$$\text{res}(AB) = \frac{1}{(2\pi)^d} \int_{T_1^*M} \text{Tr}(\text{Sub}(A) \cdot \sigma_B^{(0)} + \sigma_A^{(0)} \cdot \text{Sub}(B)).$$

Proof. It is straightforward to verify that (see also [10, (1.4)])

$$(4.6) \quad \text{Sub}(AB) = \text{Sub}(A) \cdot \sigma_B^{(0)} + \sigma_A^{(0)} \cdot \text{Sub}(B) + \frac{1}{2i} \{\sigma_A^{(0)}, \sigma_B^{(0)}\},$$

where

$$\{\sigma_A^{(0)}, \sigma_B^{(0)}\} = \frac{\partial \sigma_A^{(0)}}{\partial \xi_k} \cdot \frac{\partial \sigma_B^{(0)}}{\partial x^k} - \frac{\partial \sigma_A^{(0)}}{\partial x^k} \cdot \frac{\partial \sigma_B^{(0)}}{\partial \xi_k}.$$

By considering the fact $\text{Tr}(\{\sigma_A^{(0)}, \sigma_B^{(0)}\} + \{\sigma_B^{(0)}, \sigma_A^{(0)}\}) = 0$, one can easily get

$$\frac{\text{Tr}(\text{Sub}(AB) + \text{Sub}(BA))}{2} = \text{Tr}(\text{Sub}(A) \cdot \sigma_B^{(0)} + \sigma_A^{(0)} \cdot \text{Sub}(B)),$$

which proves the proposition via (4.5) as $\text{res}(AB) = \text{res}(BA)$. We are done. \square

According to Proposition 2.8 and Proposition 4.4, one has

$$(4.7) \quad \mathcal{A}_1(A, P) = \frac{1}{(2\pi)^d} \int_{T_1^*M} \text{Tr}(\text{Sub}(A) \cdot \sigma_{P^{\frac{1-d-m}{2}}}^{(0)} + \sigma_A^{(0)} \cdot \text{Sub}(P^{\frac{1-d-m}{2}})).$$

For any real number q , it is not hard to verify that (see also [10, (1.3)] and its proof therein)

$$(4.8) \quad \text{Sub}(P^q) = q \cdot (\sigma_P^{(0)})^{q-1} \cdot \text{Sub}(P).$$

Thus by combining (4.7) \sim (4.8) and by considering $\sigma_{P^q}^{(0)} = (\sigma_P^{(0)})^q$, we get

Theorem 4.5. *Let A be a classical pseudodifferential operator of order m and let P be a non-negative self-adjoint Laplacian on sections of E . Then*

$$(4.9) \quad \mathcal{A}_1(A, P) = \frac{1}{(2\pi)^d} \int_{T_1^*M} \text{Tr}(\text{Sub}(A) + \frac{1-d-m}{2} \cdot \sigma_A^{(0)} \cdot \text{Sub}(P)).$$

Theorem 1.1 is an immediate consequence of Theorem 4.5 as it suffices to first note from (2.24) that

$$(4.10) \quad \mathcal{A}_1(A, D) = \frac{\mathcal{A}_1(A, D^2) + \mathcal{A}_1(AD, D^2)}{2},$$

then apply Theorem 4.5 accordingly.

Theorem 1.2 is also a consequence of Theorem 4.5, which can be seen as follows.

STEP 1: Let F be a smooth bundle endomorphism and let D be a self-adjoint Dirac type operator on sections of E . Obviously, it follows from Theorem 4.5 that $\mathcal{A}_1(F, D^2) = 0$. So according to (4.10), we get $\mathcal{A}_1(F, D) = \frac{\mathcal{A}_1(FD, D^2)}{2}$. By Proposition 2.8, we also have

$$\mathcal{A}_1(FD, D^2) = \text{res}(FD|D|^{-d}) = \text{res}(F|D|^{-d}D) = \text{res}(DF|D|^{-d}) = \mathcal{A}_1(DF, D^2).$$

Consequently,

$$(4.11) \quad \mathcal{A}_1(F, D) = \frac{\mathcal{A}_1(F\tilde{D}, D^2) + \mathcal{A}_1(DF, D^2)}{4}.$$

Application of Theorem 4.5 results in

$$\mathcal{A}_1(F, D) = \frac{1}{(2\pi)^d} \int_{T_1^*M} \text{Tr} \left(F \cdot \left[\frac{\text{Sub}(D)}{2} - \frac{d}{2} \cdot \frac{\sigma_D^{(0)} \cdot \text{Sub}(D^2) + \text{Sub}(D^2) \cdot \sigma_D^{(0)}}{4} \right] \right).$$

STEP 2: Recall $D = \gamma\tilde{\nabla} + \psi$ is a self-adjoint Dirac type operator with potential ψ associated with the Dirac bundle $(E, \gamma, \tilde{\nabla})$. In local coordinates we assume $\tilde{\nabla}_{\partial_j} = \partial_j + b_j$, where b_j are smooth matrix-valued functions. Thus $\sigma_D^{(0)}(x_0, \xi) = i\gamma^j(x_0)\xi_j$, $\sigma_D^{(1)}(x_0, \xi) = \gamma^j(x_0)b_j(x_0) + \psi(x_0)$. The compatible condition (3.2) between γ and $\tilde{\nabla}$ gives $\frac{\partial\gamma^k}{\partial x^j} = [\gamma^k, b_j] - \Gamma_{jn}^k \gamma^n$. We further assume the local coordinate system is a Riemannian normal one centered at $x_0 \in M$. Hence

$$\left(\frac{\partial\gamma^k}{\partial x^j} \right)(x_0) = [\gamma^k(x_0), b_j(x_0)].$$

So by the definition of the sub-principal symbol of D , we get

$$(4.12) \quad \text{Sub}(D)(x_0, \xi) = \gamma^j(x_0)b_j(x_0) + \psi(x_0) - \frac{1}{2} \left(\frac{\partial\gamma^k}{\partial x^k} \right)(x_0) = \left(\frac{\gamma^j b_j + b_j \gamma^j}{2} + \psi \right)(x_0).$$

Similarly, one can show that $\sigma_{D^2}^{(0)}(x_0, \xi) = |\xi|^2$, $\sigma_{D^2}^{(1)}(x_0, \xi) = i(\gamma^k \psi + \psi \gamma^k - 2b_k)(x_0)\xi_k$. So by the definition of the sub-principal symbol of D^2 , we get

$$\text{Sub}(D^2)(x_0, \xi) = i(\gamma^k \psi + \psi \gamma^k - 2b_k)(x_0)\xi_k.$$

Consequently,

$$(4.13) \quad (\sigma_D^{(0)} \cdot \text{Sub}(D^2))(x_0, \xi) = -(\gamma^j \gamma^k \psi + \gamma^j \psi \gamma^k - 2\gamma^j b_k)(x_0)\xi_j \xi_k,$$

$$(4.14) \quad (\text{Sub}(D^2) \cdot \sigma_D^{(0)})(x_0, \xi) = -(\gamma^k \psi \gamma^j + \psi \gamma^k \gamma^j - 2b_k \gamma^j)(x_0)\xi_k \xi_j.$$

STEP 3: By the above formulae we obtain

$$\begin{aligned} \int_{|\xi|=1} \text{Tr}(F \cdot \text{Sub}(D))(x_0, \xi) dS(\xi) &= \text{Tr} \left(F \frac{\gamma^j b_j + b_j \gamma^j}{2} + F\psi \right)(x_0) \cdot \text{Vol}(\mathbb{S}^{d-1}), \\ \int_{|\xi|=1} \text{Tr}(F \cdot \sigma_D^{(0)} \cdot \text{Sub}(D^2))(x_0, \xi) dS(\xi) &= \text{Tr} \left(F\psi - \frac{F\hat{\psi}}{d} + 2\frac{F\gamma^j b_j}{d} \right)(x_0) \cdot \text{Vol}(\mathbb{S}^{d-1}), \\ \int_{|\xi|=1} \text{Tr}(F \cdot \text{Sub}(D^2) \cdot \sigma_D^{(0)})(x_0, \xi) dS(\xi) &= \text{Tr} \left(F\psi - \frac{F\hat{\psi}}{d} + 2\frac{Fb_j \gamma^j}{d} \right)(x_0) \cdot \text{Vol}(\mathbb{S}^{d-1}). \end{aligned}$$

Therefore, by the last formula in Step 1,

$$\mathcal{A}_1(F, D) = \frac{1}{(2\pi)^d} \int_M \text{Tr} \left(F \frac{\hat{\psi} - (d-2)\psi}{4} \right) \cdot \text{Vol}(\mathbb{S}^{d-1}),$$

which proves Theorem 1.2 as $\text{Vol}(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

5. DIRAC TYPE OPERATORS OF VANISHING SECOND COEFFICIENT

Let $D = \gamma\tilde{\nabla} + \psi$ be a Dirac type operator of potential ψ associated with the Dirac bundle $(E, \gamma, \tilde{\nabla})$. In this section we will no longer assume the self-adjointness of D . According to Remark 3.2, $\mathcal{H}_1(D, D^2) \in C^\infty(M; \text{End}(E))$ is well-defined.

Theorem 5.1. *Let D be a Dirac type operator. Then $\mathcal{H}_1(D, D^2) = 0$ if and only if D is a generalized Dirac operator.*

By considering (1.16) and (1.17), we see that Theorem 5.1 implies Theorem 1.3 as a self-adjoint Dirac type operator is a Dirac type operator. But we should remark that Theorem 5.1 and Theorem 1.3 actually are equivalent as for any Dirac bundle one can always introduce a hermitian structure so that the associated generalized Dirac operator is self-adjoint on the Hilbert space defined by this hermitian structure and the metric measure on M (see e.g. [3]).

Now we explain how to prove Theorem 5.1. Let D be a Dirac type operator of potential ψ associated with the Dirac bundle (E, γ, ∇) . According to Remark 3.2, $\mathcal{H}_1(D, D^2) = 0$ if and only if $\hat{\psi} = (d-2)\psi$. Thus to prove Theorem 5.1 it suffices to show that if $\hat{\psi} = (d-2)\psi$, then there exists a connection $\tilde{\nabla}$ on E compatible with γ such that $D = \gamma\tilde{\nabla}$. Note any connection on E must be of the form $\nabla_L = \nabla + L$, where

$$L : C^\infty(M; TM) \rightarrow C^\infty(M; \text{End}(E))$$

is a $C^\infty(M)$ -linear map. It is easy to check that ∇_L is compatible with γ if and only if $L(X)$ commutes with $\gamma(Y)$ for all $X, Y \in C^\infty(M; TM)$. Suppose we do have such a map L such that $D = \gamma\nabla_L$. Letting $\{e_k\}_{k=1}^d$ be a local orthonormal frame in TM , we have $D = \gamma\nabla + \gamma(e_k)L(e_k)$, which means $\psi = \gamma(e_k)L(e_k)$. Consequently, for any fixed $i \in \{1, \dots, d\}$,

$$\frac{\gamma(e_i)\psi + \psi\gamma(e_i)}{2} = L(e_k) \frac{\gamma(e_i)\gamma(e_k) + \gamma(e_k)\gamma(e_i)}{2} = -L(e_i).$$

This means that globally L must be uniquely of the following form

$$(5.1) \quad L(X) = -\frac{\gamma(X)\psi + \psi\gamma(X)}{2}$$

for any $X \in C^\infty(M; TM)$. Thus to prove Theorem 5.1, with assuming $\hat{\psi} = (d-2)\psi$ and defining $\tilde{\nabla} = \nabla + L$ with L given by (5.1), we need only to show that 1) $D = \gamma\tilde{\nabla}$, and 2) $\tilde{\nabla}$ is compatible with γ .

The first property can be verified in the following way. Let $\{e_k\}_{k=1}^d$ be a local orthonormal frame in TM , then

$$\gamma(e_k)L(e_k) = -\gamma(e_k) \frac{\gamma(e_k)\psi + \psi\gamma(e_k)}{2} = \frac{d\psi - \hat{\psi}}{2} = \psi,$$

which gives $D = \gamma\nabla + \psi = \gamma\nabla + \gamma(e_k)L(e_k) = \gamma\tilde{\nabla}$.

To prove the second property we need to show that $L(X)$ commutes with $\gamma(Y)$ for all $X, Y \in C^\infty(M; TM)$. But considering L and γ are $C^\infty(M)$ -linear maps, it

suffices to prove that $L_x(X_x)$ commutes with $\gamma_x(Y_x)$ for all $X_x, Y_x \in T_x M$ at each $x \in M$. This is guaranteed by the next proposition and thus a proof of Theorem 5.1 is achieved.

Proposition 5.2. *Let (W, γ) be a complex $\text{Cl}(\mathbb{R}^d)$ -module. Let $\psi \in \text{End}(W)$ be such that $\widehat{\psi} = (d - 2)\psi$, and let $L : \mathbb{R}^d \rightarrow \text{End}(W)$ be the linear map defined by*

$$L(X) = -\frac{\gamma(X)\psi + \psi\gamma(X)}{2} \quad (X \in \mathbb{R}^d).$$

Then $L(X)$ commutes with $\gamma(Y)$ for all $X, Y \in \mathbb{R}^d$.

The rest of this section is mainly devoted to proving Proposition 5.2. In the first subsection we begin with reviewing a few basic concepts about the representations of irreducible complex Clifford modules (see e.g. [45]), then give a proof of Proposition 5.2. A few examples such as the massless Dirac operators ([8, 9]) are studied in the second part.

5.1. Clifford modules. Let $\text{Cl}(\mathbb{R}^d)$ be the complex Clifford algebra generated by $(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ subject to the commutation relation $X * Y + Y * X = -2\langle X, Y \rangle$, where $X, Y \in \mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean metric on \mathbb{R}^d , and $*$ denotes the Clifford algebra operation. Any complex $\text{Cl}(\mathbb{R}^d)$ -module W with structure γ (unital algebra morphism from $\text{Cl}(\mathbb{R}^d)$ to $\text{End}(W)$) studied in this paper is always assumed to be of dimension in \mathbb{N} . A complex $\text{Cl}(\mathbb{R}^d)$ -module (W, γ) is said to be *irreducible* if for any decomposition $W = W_1 \oplus W_2$ into subspaces invariant under γ one has $W_1 = W$ or $W_2 = W$. It is known that a complex $\text{Cl}(\mathbb{R}^d)$ -module is irreducible if and only if it is of complex dimension $2^{\lfloor \frac{d}{2} \rfloor}$. Two complex $\text{Cl}(\mathbb{R}^d)$ -modules (W_1, γ) and $(W_2, \widetilde{\gamma})$ are said to be *equivalent* if there exists an invertible element $\kappa \in \text{Hom}(W_1, W_2)$ such that $\widetilde{\gamma} = \kappa^* \gamma$, where κ^* is defined by sending $\rho \in \text{End}(W_1)$ to $\kappa \rho \kappa^{-1} \in \text{End}(W_2)$. Up to isomorphism there are exactly $\frac{3-(-1)^d}{2}$ inequivalent irreducible complex $\text{Cl}(\mathbb{R}^d)$ -modules. It is also known that any complex $\text{Cl}(\mathbb{R}^d)$ -module is a sum of irreducible ones.

Let (W, γ) be a complex $\text{Cl}(\mathbb{R}^d)$ -module and define $\widehat{\psi} = \sum_{k=1}^d \gamma(e_k)\psi\gamma(e_k)$ for any $\psi \in \text{End}(W)$, where $\{e_k\}_{k=1}^d$ is an arbitrary orthonormal basis for $(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$. Let

$$\text{End}_k(W) := \text{Span}_{\mathbb{C}}\{\gamma(e_{i_1})\gamma(e_{i_2})\cdots\gamma(e_{i_k}) : 1 \leq i_1 < i_2 < \cdots < i_k \leq d\},$$

for $k > 0$ and let $\text{End}_0(W)$ be the complex span of the identity map. A simple computation using the Clifford algebra relations shows that $\text{End}_k(W)$ is an eigenspace for the map $\psi \mapsto \widehat{\psi}$ with eigenvalue $(-1)^k(2k - d)$. It is clear that Clifford multiplication by the volume element $\gamma(e_1)\gamma(e_2)\cdots\gamma(e_d)$ defines a linear map from $\text{End}_k(W)$ to $\text{End}_{d-k}(W)$. If the module is irreducible the subspaces $\text{End}_k(W)$ generate $\text{End}(W)$ and therefore this gives a decomposition into eigenspaces. Hence, in the irreducible

case we have

$$\begin{aligned} \text{End}(W) &= \bigoplus_{k=0}^d \text{End}_k(W) \quad \text{if } d \text{ is even,} \\ \text{End}(W) &= \bigoplus_{k=0}^{\frac{d-1}{2}} \text{End}_k(W) \quad \text{if } d \text{ is odd.} \end{aligned}$$

In the latter case we have used $\text{End}_{d-k}(W) = \text{End}_k(W)$. This can be seen directly because Clifford multiplication by the volume element commutes with the Clifford action. Therefore, by Schur's lemma, it must be a multiple of the identity. As an immediate consequence of this discussion we get the following Proposition.

Proposition 5.3. *If (W, γ) is irreducible, then the $d - 2$ eigenspace of the map $\psi \mapsto \widehat{\psi}$ equals $\text{End}_1(W) = \text{Span}_{\mathbb{C}} \gamma(\mathbb{R}^d)$. If furthermore d is odd, then $2 - d$ is not an eigenvalue of the map $\psi \mapsto \widehat{\psi}$.*

Now we can start to prove Proposition 5.2. Let (W, γ) be a (finite-dimensional) complex $\text{Cl}(\mathbb{R}^d)$ -module and let $(W_0, \gamma^{(0)})$ be an irreducible complex $\text{Cl}(\mathbb{R}^d)$ -module. It is easy to see that up to isomorphism $(W_0, \gamma^{(0)})$ can represent all irreducible complex $\text{Cl}(\mathbb{R}^d)$ -modules if d is even, and so for $(W_0, \pm\gamma^{(0)})$ if d is odd. We then have two cases to consider.

Case 1: Suppose d is even. In this case there exists a (finite-dimensional) complex vector space V such that (W, γ) is equivalent to $(W_0 \otimes V, \widetilde{\gamma})$, where the Clifford structure $\widetilde{\gamma}$ on $W_0 \otimes V$ is applied only to the first tensor factor W_0 . To be precise, $\widetilde{\gamma}(Y) = \gamma^{(0)}(Y) \otimes \text{Id}_V$ for any $Y \in \mathbb{R}^d$. Without loss of generality we can assume that $(W, \gamma) = (W_0 \otimes V, \widetilde{\gamma})$ since the general cases can be dealt with simply by studying isomorphisms between Clifford modules. Then as an application of Proposition 5.3, we claim that the $d - 2$ eigenspace of the map $\psi \mapsto \widehat{\psi}$ equals $\gamma^{(0)}(\mathbb{R}^d) \otimes \text{End}(V)$. To prove this claim, it suffices to first fix a linear basis $\{K_i\}$ for $\text{End}(V)$, then express ψ with $\widehat{\psi} = (d - 2)\psi$ uniquely as $\psi_i^{(0)} \otimes K_i$ where $\psi_i^{(0)} \in \text{End}(W_0)$, note

$$\widehat{\psi_i^{(0)}} \otimes K_i = \widehat{\psi} = (d - 2)\psi = (d - 2)\psi_i^{(0)} \otimes K_i,$$

and finally apply Proposition 5.3 appropriately. This claim immediately implies that

$$L(\mathbb{R}^d) = \{\text{Id}_{W_0}\} \otimes \text{End}(V),$$

which obviously commutes with $\gamma(\mathbb{R}^d) = \gamma^{(0)}(\mathbb{R}^d) \otimes \{\text{Id}_V\}$.

Case 2: Suppose d is odd. In this case there exist (finite-dimensional) complex vector spaces V_1, V_2 such that (W, γ) is equivalent to $((W_0 \otimes V_1) \oplus (W_0 \otimes V_2), \widetilde{\gamma})$, where the Clifford structure $\widetilde{\gamma}$ on $(W_0 \otimes V_1) \oplus (W_0 \otimes V_2)$ applied only to W_0 , is defined by

$$\widetilde{\gamma}(Y) = (\gamma^{(0)}(Y) \otimes \text{Id}_{V_1}) \bigoplus (-\gamma^{(0)}(Y) \otimes \text{Id}_{V_2})$$

for any $Y \in \mathbb{R}^d$. Similar to the discussions in the previous case, we assume without loss of generality that $(W, \gamma) = ((W_0 \otimes V_1) \oplus (W_0 \otimes V_2), \widetilde{\gamma})$. Then as an application

of Proposition 5.3, we claim that the $d - 2$ eigenspace of the map $\psi \mapsto \widehat{\psi}$ equals $(\gamma^{(0)}(\mathbb{R}^d) \otimes \text{End}(V_1)) \oplus (\gamma^{(0)}(\mathbb{R}^d) \otimes \text{End}(V_2))$. To prove this claim, we first choose a linear basis $\{K_i^{\alpha\beta}\}$ for each $\text{Hom}(V_\alpha, V_\beta)$, then express ψ with $\widehat{\psi} = (d - 2)\psi$ uniquely as

$$(\psi_i^{(11)} \otimes K_i^{(11)}) \bigoplus (\psi_j^{(22)} \otimes K_j^{(22)}) + (\psi_i^{(12)} \otimes K_i^{(12)}) \bigoplus (\psi_j^{(21)} \otimes K_j^{(21)}),$$

where $\psi_i^{(\alpha\beta)} \in \text{End}(W_0)$. Here to be clear, $W = (W_0 \otimes V_1) \oplus (W_0 \otimes V_2)$ is naturally identified with $(W_0 \otimes V_2) \oplus (W_0 \otimes V_1)$ by interchanging the positions, thus any element in W mapped via the second summand is indeed contained in W . Following this convention, it is straightforward to check that

$$\widehat{\psi} = (\widehat{\psi_i^{(11)}} \otimes K_i^{(11)}) \bigoplus (\widehat{\psi_j^{(22)}} \otimes K_j^{(22)}) + (-\widehat{\psi_i^{(12)}} \otimes K_i^{(12)}) \bigoplus (-\widehat{\psi_j^{(21)}} \otimes K_j^{(21)}).$$

Thus the claim is an immediate consequence of Proposition 5.3. This claim implies that

$$L(\mathbb{R}^d) = (\{\text{Id}_{W_0}\} \otimes \text{End}(V_1)) \bigoplus (\{\text{Id}_{W_0}\} \otimes \text{End}(V_2)),$$

which commutes with $\widetilde{\gamma}(Y)$ for any $Y \in \mathbb{R}^d$. This finishes the proof of Proposition 5.2.

5.2. Examples. In this part we study a few examples. Recall the diagonal values of the smooth kernel of De^{-tD^2} admit a uniform small-time asymptotic expansion

$$K(t, x, x, D, D^2) \sim \sum_{k=0}^{\infty} t^{\frac{k-d-1}{2}} \mathcal{H}_k(D, D^2)(x) \quad (t \rightarrow 0^+),$$

where

$$\mathcal{H}_1(D, D^2) = \frac{1}{(4\pi)^{d/2}} \cdot \frac{\widehat{\psi} - (d - 2)\psi}{2}.$$

If D is further self-adjoint, then the mollified local counting function of D has a pointwise asymptotic expansion

$$(\chi * N'_x)(\mu) \sim \sum_{k=0}^{\infty} \mu^{d-k-1} \text{Tr}_{E_x}(\mathcal{L}_k(D)(x)) \quad (\mu \rightarrow \infty),$$

where $\mathcal{L}_1(D) = \frac{\mathcal{H}_1(D, D^2)}{\Gamma(\frac{d}{2})}$.

Example 5.4 (Generalized Dirac operators). Let D be a generalized Dirac operator associated with a Dirac bundle. Obviously, $\mathcal{H}_1(D, D^2) = 0$. To compare, it was shown in [3] that $\text{Tr}(\mathcal{H}_1(D, D^2)) = 0$.

Example 5.5 (Three-dimensional manifolds). Let D be a Dirac type operator of potential ψ associated with a Dirac bundle of rank two over a three-dimensional closed Riemannian manifold M . We claim $\mathcal{H}_1(D, D^2) = 0$ if and only if $\text{Tr}(\psi) = 0$. To prove this claim, we note that if W is an irreducible complex $\text{Cl}(\mathbb{R}^3)$ -module, then W is of complex dimension 2 and $\text{End}(W) = \text{End}_0(W) \oplus \text{End}_1(W)$. Consequently, $\text{End}_1(W)$ is precisely the trace-free part of $\text{End}(W)$.

Example 5.6 (Parallelizable manifolds). In this example we construct generalized Dirac operators acting on $C^\infty(M; E)$, where M is a closed parallelizable Riemannian manifold of dimension d , $E = M \times \mathbb{C}^r$ with rank $r = 2^{\lfloor \frac{d}{2} \rfloor}$. The Riemannian metric on M is denoted by g and the Riemannian connection on TM is denoted by ∇ . Since M is parallelizable, there exist smooth real vector fields $\{X_k\}_{k=1}^d$ such that $\{X_k(x)\}_{k=1}^d$ is an orthonormal basis for $(T_x M, g_x)$ at each $x \in M$. Let $\{R_k\}_{k=1}^d$ be fixed complex matrices of size $r \times r$ with $R_j R_k + R_k R_j = -2\delta_{jk}$ for all $1 \leq j, k \leq d$. Then it is straightforward to verify that $\gamma : TM \rightarrow \text{End}(E)$ defined by

$$(5.2) \quad \gamma(X) = g(X, X_k) R_k$$

is an irreducible $\text{Cl}(TM)$ -module structure. Define a flat connection $\widehat{\nabla}$ on E by

$$(5.3) \quad \widehat{\nabla}_X \phi = \widehat{\nabla}_X \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix} = \begin{pmatrix} X\phi_1 \\ \vdots \\ X\phi_r \end{pmatrix}$$

and define $L : TM \rightarrow \text{End}(E)$ by

$$(5.4) \quad L(X) = \frac{R_k \gamma(\nabla_X X_k)}{4}.$$

Following Branson-Gilkey ([3, Lemma 1.3]), we claim $\widetilde{\nabla} := \widehat{\nabla} + L$ is a compatible connection on (E, γ) . To this end we first let $X, \alpha \in C^\infty(M; TM)$, $\phi \in C^\infty(M; E)$. Obviously, α is of the form $\alpha = \alpha_k X_k$ for some $\alpha_k \in C^\infty(M)$. Thus

$$\begin{aligned} \widetilde{\nabla}_X(\gamma(\alpha)\phi) &= \widetilde{\nabla}_X(\alpha_k R_k \phi) = \widehat{\nabla}_X(\alpha_k R_k \phi) + L(X)\gamma(\alpha)\phi \\ &= (X\alpha_k)(R_k \phi) + \gamma(\alpha)\widehat{\nabla}_X \phi + L(X)\gamma(\alpha)\phi \end{aligned}$$

and

$$\begin{aligned} \gamma(\nabla_X \alpha)\phi + \gamma(\alpha)\widetilde{\nabla}_X \phi &= \gamma((X\alpha_k)X_k + \alpha_k \nabla_X X_k)\phi + \gamma(\alpha)\widehat{\nabla}_X \phi + \gamma(\alpha)L(X)\phi \\ &= (X\alpha_k)(R_k \phi) + \alpha_k \gamma(\nabla_X X_k)\phi + \gamma(\alpha)\widehat{\nabla}_X \phi + \gamma(\alpha)L(X)\phi. \end{aligned}$$

Hence to prove the compatibility condition (3.2) for $\alpha \in C^\infty(M; TM)$ it suffices to show $[L(X), \gamma(\alpha)] = g(\alpha, X_k)\gamma(\nabla_X X_k)$, which is obviously equivalent to

$$(5.5) \quad [L(X_i), \gamma(X_j)] = g(X_j, X_k)\gamma(\nabla_{X_i} X_k) \quad (1 \leq i, j \leq d).$$

Letting $\Gamma_{ij}^k = -\Gamma_{ik}^j$ denote the Christoffel symbols, that is, $\nabla_{X_i} X_j = \Gamma_{ij}^k X_k$, we have

$$\begin{aligned} [L(X_i), \gamma(X_j)] &= \frac{1}{4}(R_k \gamma(\nabla_{X_i} X_k)R_j - R_j R_k \gamma(\nabla_{X_i} X_k)) \\ &= \frac{1}{4}(R_k \Gamma_{ik}^n R_n R_j - R_j R_k \Gamma_{ik}^n R_n) \\ &= \frac{1}{4}(-2R_k \Gamma_{ik}^n \delta_{nj} - R_k \Gamma_{ik}^n R_j R_n - R_j R_k \Gamma_{ik}^n R_n) \\ &= \frac{1}{2}(-R_k \Gamma_{ik}^j + \Gamma_{ij}^n R_n) = \Gamma_{ij}^n R_n \\ &= \gamma(\nabla_{X_i} X_j) = g(X_j, X_k)\gamma(\nabla_{X_i} X_k), \end{aligned}$$

which proves (5.5). Generally, if $\alpha \in C^\infty(M; \text{Cl}(TM))$ is of the form $\alpha = \alpha^{(1)} * \alpha^{(2)} * \dots * \alpha^{(n)}$ where $\alpha^{(k)} \in C^\infty(M; TM)$ ($1 \leq k \leq n$), then according to (3.1),

$$\begin{aligned} \tilde{\nabla}_X(\gamma(\alpha)\phi) &= \tilde{\nabla}_X(\gamma(\alpha^{(1)})[\gamma(\alpha^{(2)}) \dots \gamma(\alpha^{(n)})\phi]) \\ &= \gamma(\nabla_X \alpha^{(1)})[\gamma(\alpha^{(2)}) \dots \gamma(\alpha^{(n)})\phi] + \gamma(\alpha^{(1)})\tilde{\nabla}_X[\gamma(\alpha^{(2)}) \dots \gamma(\alpha^{(n)})\phi] \\ &= \sum_{k=1}^n \gamma(\alpha^{(1)}) \dots \gamma(\alpha^{(k-1)})\gamma(\nabla_X \alpha^{(k)})\gamma(\alpha^{(k+1)}) \dots \gamma(\alpha^{(n)})\phi + \gamma(\alpha)\tilde{\nabla}_X \phi \\ &= \gamma(\nabla_X \alpha)\phi + \gamma(\alpha)\tilde{\nabla}_X \phi, \end{aligned}$$

which suffices to prove (3.2) for arbitrary $\alpha \in C^\infty(M; \text{Cl}(TM))$ by linearity. Thus $(E, \gamma, \tilde{\nabla})$ is a Dirac bundle and

$$(5.6) \quad D = \gamma\tilde{\nabla} = \gamma\hat{\nabla} + \frac{R_i R_j \gamma(\nabla_{X_i} X_j)}{4}$$

is the associated generalized Dirac operator. By Theorem 5.1, we get $\mathcal{H}_1(D, D^2) = 0$. It is well-known ([3]) that one can always furnish $(E, \gamma, \tilde{\nabla})$ with a smooth hermitian fiber metric so that D is self-adjoint on $L^2(M; E)$ defined by this fiber metric and the metric measure on M . So in this context $\mathcal{L}_1(D) = 0$. Next we derive a local expression of D . To this end we first define a flat connection $\ddot{\nabla}$ on $\text{End}(E)$ by

$$(5.7) \quad \ddot{\nabla}_X \begin{pmatrix} \phi_{11} & \cdots & \phi_{1r} \\ \vdots & \ddots & \vdots \\ \phi_{r1} & \cdots & \phi_{rr} \end{pmatrix} = \begin{pmatrix} X\phi_{11} & \cdots & X\phi_{1r} \\ \vdots & \ddots & \vdots \\ X\phi_{r1} & \cdots & X\phi_{rr} \end{pmatrix}$$

then introduce

$$(5.8) \quad \Psi(X, Y, Z, W) = \gamma(X)\gamma(Y)(\gamma(\nabla_Z W) - \ddot{\nabla}_Z(\gamma(W))),$$

where $X, Y, Z, W \in C^\infty(M; TM)$. It is easy to verify that Ψ is $C^\infty(M)$ -linear in all of the four variables. This implies $\text{Tr}_{2,4}(\text{Tr}_{1,3}(\Psi))$ is a smooth endomorphism of E , where $\text{Tr}_{1,3}(\Psi)$ denotes the trace of Ψ with respect to the first and third variables, and $\text{Tr}_{2,4}(\Psi)$ is understood in a similar way. Thus globally we have

$$\text{Tr}_{2,4}(\text{Tr}_{1,3}(\Psi)) = \Psi(X_i, X_j, X_i, X_j) = R_i R_j \gamma(\nabla_{X_i} X_j),$$

which means the generalized Dirac operator D can also be written as

$$(5.9) \quad D = \gamma\hat{\nabla} + \frac{\text{Tr}_{2,4}(\text{Tr}_{1,3}(\Psi))}{4}.$$

Hence in a local coordinate system (x^1, \dots, x^d) , D is of the form

$$(5.10) \quad D = \gamma(dx^k) \frac{\partial}{\partial x^k} + \frac{1}{4} \gamma(dx^i) \gamma\left(\frac{\partial}{\partial x^j}\right) (\gamma(\nabla_{\partial_i} dx^j) - \frac{\partial(\gamma(dx^j))}{\partial x^i}).$$

As an example, we assume that M is a three-dimensional oriented closed Riemannian manifold. By Steenrod's theorem M is parallelizable and we can set

$$R_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Then it is straightforward to verify that D is nothing but the massless Dirac operator W locally defined by (A.3) in [8]. Note $\text{Tr}_E(\mathcal{L}_1(D)) = 0$ was established in [8]. To compare, we have shown that $\mathcal{L}_1(D) = 0$. Moreover, it was proved in [8] that the sub-principal symbol of D is proportional to the identity matrix at each point of the manifold M , where the corresponding bundle trivialization $\{s_1, s_2\}$ of $E = M \times \mathbb{C}^2$ is chosen to be

$$s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the following we will give a short proof of this fact. According to Example 4.3 and our previous discussions, it is easy to see that $\omega = L$, and consequently,

$$\begin{aligned} \text{Sub}(D) &= \frac{1}{2}(\gamma(X_j)L(X_j) + L(X_j)\gamma(X_j)) \\ &= \frac{1}{8}(R_j R_k \Gamma_{jk}^n R_n + R_k \Gamma_{jk}^n R_n R_j), \end{aligned}$$

where we recall the Christoffel symbols Γ_{jk}^n are given by $\nabla_{X_j} X_k = \Gamma_{jk}^n X_n$ with $\Gamma_{jk}^n = -\Gamma_{jn}^k$. Note $\text{Sub}(D)$ is a sum of $27 = 3 \times 3 \times 3$ items. Next we no longer use Einstein's sum convention and decompose $\text{Sub}(D)$ into four parts:

$$\text{Sub}(D) = \sum_{k=n} \star + \sum_{k \neq n, j=n} \star + \sum_{k \neq n, j \neq n, k=j} \star + \sum_{k \neq n, j \neq n, k \neq j} \star =: \text{I} + \text{II} + \text{III} + \text{IV},$$

where \star is short for $\frac{1}{8}(R_j R_k \Gamma_{jk}^n R_n + R_k \Gamma_{jk}^n R_n R_j)$. One can easily deduce from the properties $\Gamma_{jk}^n = -\Gamma_{jn}^k$, $R_j R_k + R_k R_j = -2\delta_{jk}$ that $\text{I} = \text{II} = \text{III} = 0$ and

$$\begin{aligned} \text{IV} &= \sum_{(j,k,n) \in \pi_3} \frac{1}{8}(R_j R_k \Gamma_{jk}^n R_n + R_k \Gamma_{jk}^n R_n R_j) \\ &= \frac{1}{4} \sum_{(j,k,n) \in \pi_3} \Gamma_{jk}^n R_j R_k R_n \\ &= \frac{1}{2} \sum_{(j,k,n) \in \pi_3 \text{ is even}} \Gamma_{jk}^n R_j R_k R_n \\ &= -\frac{\Gamma_{12}^3 + \Gamma_{23}^1 + \Gamma_{31}^2}{2}, \end{aligned}$$

where π_3 denotes the symmetric group of degree 3, and the last equality follows from $R_1 R_2 R_3 = R_2 R_3 R_1 = R_3 R_1 R_2 = -1$. To conclude we get

$$(5.11) \quad \text{Sub}(D) = -\frac{\Gamma_{12}^3 + \Gamma_{23}^1 + \Gamma_{31}^2}{2},$$

which agrees with [8, Lemma 6.1]. Furthermore, it is characterized in [8] that given a self-adjoint Dirac type operator D on sections of $M \times \mathbb{C}^2$, it is a massless one if and only if $\text{Tr}(\mathcal{L}_1(D)) = 0$ and $\text{Sub}(D)$ is pointwise proportional to the identity matrix, where the corresponding bundle trivialization $\{s_1, s_2\}$ of $M \times \mathbb{C}^2$ is chosen as before. We can also give a proof of this fact. Based on the previous discussions, it suffices to prove the necessary part. So let D be a self-adjoint Dirac type operator such that $\text{Tr}(\mathcal{L}_1(D)) = 0$ and $\text{Sub}(D)$ is pointwise proportional to the identity matrix. Since

M is a parallelizable three-dimensional manifold, it is easy to see that there exists a unique massless Dirac operator $\gamma\tilde{\nabla}$ sharing the same principal symbol with D and a unique bundle endomorphism ψ such that $D = \gamma\tilde{\nabla} + \psi$. According to Example 5.5, $\text{Tr}(\psi) = 0$. Note also

$$\text{Sub}(D) = \text{Sub}(\gamma\tilde{\nabla}) + \psi,$$

which implies that ψ is pointwise proportional to the identity matrix. Thus we must have $\psi = 0$, and consequently, $D = \gamma\tilde{\nabla}$ is indeed a massless Dirac operator.

Acknowledgments. Part of the work for this paper was carried out at the programme “Modern Theory of Wave Equations” at the Schrödinger institute and both authors are grateful to the ESI for support and the hospitality during their stay. The second author would also like to thank the *Hausdorff Center of Mathematics* in Bonn for the support for his stay during the trimester program “Non-commutative Geometry and its Applications”. Both authors thank Dmitri Vassiliev for interesting discussions.

REFERENCES

- [1] T. Ackermann, A note on the Wodzicki residue, *J. Geom. Phys.* **20** (1996) 404–406.
- [2] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag Berlin Heidelberg, 2004.
- [3] T. P. Branson, P. B. Gilkey, Residues of the eta function for an operator of Dirac type, *J. Funct. Anal.* **108** (1992) 47–87.
- [4] A. P. Calderón, R. Vaillancourt, On the boundedness of pseudo-differential operators, *J. Math. Soc. Japan* **23** (1971) 374–378.
- [5] I. Chavel, *Riemannian Geometry: A Modern Introduction*, Cambridge Univ. Press, 1993.
- [6] W. Chen, X. Li, *Riemannian Geometry*, Vol. 1, Peking University Press, 2002.
- [7] O. Chervova, R. J. Downes, D. Vassiliev, The spectral function of a first order elliptic system, *J. Spectral Theory* **3** (2013) 317–360.
- [8] O. Chervova, R. J. Downes, D. Vassiliev, Spectral theoretic characterization of the massless Dirac operator, *J. London. Math. Soc.* **89** (2014) 301–320.
- [9] O. Chervova, D. Vassiliev, The stationary Weyl equation and Cosserat elasticity, *J. Phys. A: Math. Theor.* **43** (2010) 335203 (14pp).
- [10] J. J. Duistermaat, V. W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.* **29** (1975) 39–79.
- [11] J. J. Duistermaat, L. Hörmander, Fourier integral operators II, *Acta Math.* **128** (1972) 183–269.
- [12] Y-L. Fang, D. Vassiliev, Analysis of first order systems of partial differential equations, to appear in *Complex Analysis and Dynamical Systems VI*, AMS Contemp. Math. Series.
- [13] B. Fedosov, F. Golse, E. Leichtnam, E. Schrohe, The noncommutative residue for manifolds with boundary, *J. Funct. Anal.* **142** (1996) 1–31.
- [14] H. D. Fegan, P. Gilkey, Invariants of the heat equation, *Pacific J. Math.* **117** (1985) 223–254.
- [15] P. Flajolet, X. Gourdon, P. Dumas, Mellin transforms and asymptotics: harmonic sums, *Theoretical Computer Sciences* **144** (1995) 3–58.
- [16] J. B. Gil, P. A. Loya, On the noncommutative residue and the heat trace expansion on conic manifolds, *Manuscripta Math.* **109** (2002) 309–327.
- [17] P. Gilkey, The spectral geometry of a Riemannian manifold, *J. Diff. Geom.* **10** (1975) 601–618.
- [18] P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, Second edition, CRC Press, 1995.

- [19] P. Gilkey, *Asymptotic Formulae in Spectral Geometry*, CRC Press, 2004.
- [20] P. Gilkey, The spectral geometry of operators of Dirac and Laplace type, “Handbook of Global Analysis”, Elsevier, 2007, 287–324.
- [21] P. Gilkey, L. Smith, The eta invariants for a class of elliptic boundary value problems, *Comm. Pure Appl. Math.* **36** (1983) 85–131.
- [22] G. Grubb, A resolvent approach to traces and zeta Laurent expansions, *AMS Contemp. Math. Proceedings*, vol. 366 “Spectral Geometry of Manifolds with Boundary and Decomposition of Manifolds”, 2005, 67–93.
- [23] G. Grubb, L. Hansen, Complex powers of resolvents of pseudodifferential operators, *Comm. Part. Diff. Equ.* **27** (2002) 2333–2361.
- [24] G. Grubb, E. Schrohe, Traces and quasi-traces on the Boutet de Monvel algebra, *Ann. Inst. Fourier (Grenoble)* **54** (2004) 1641–1696.
- [25] G. Grubb, R. T. Seeley, Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems, *Invent. Math.* **121** (1995) 481–529.
- [26] G. Grubb, R. T. Seeley, Zeta and eta functions for Atiyah-Patodi-Singer operators, *J. Geom. Anal.* **6** (1996) 31–77.
- [27] V. Guillemin, A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues, *Adv. Math.* **55** (1985) 131–160.
- [28] V. Guillemin, Wave-trace invariants, *Duke Math. J.* **83** (1996) 287–352.
- [29] L. Hörmander, The spectral function of an elliptic operator, *Acta Math.* **121** (1968) 193–218.
- [30] V. Ivrii, Accurate spectral asymptotics for elliptic operators that act in vector bundles, *Funktsional Anal. i. Prilozhen.* **16** (1982) 30–38.
- [31] V. Ivrii, *Microlocal Analysis and Precise Spectral Asymptotics*, Springer-Verlag, 1998.
- [32] D. Jakobson, A. Strohmaier, High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows, *Commun. Math. Phys.* **270** (2007) 813–833.
- [33] W. Kalau, M. Walze, Gravity, non-commutative geometry and the Wodzicki residue, *J. Geom. Phys.* **16** (1995) 327–344.
- [34] M. Kontsevich, S. Vishik, Geometry and determinants of elliptic operators, In S. Gindikin et al. (ed), *Functional Analysis on the Eve of the 21st Century, Vol. I. In Honor of the Eightieth Birthday of I. M. Gelfand*. Progress in Mathematics Vol.131, Birkhäuser, Boston, pp. 173–197.
- [35] H. B. Lawson, M. L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, 1989.
- [36] M. Lesch, On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols, *Ann. Global Ana. Geom.* **17** (1999) 151–187.
- [37] P. Loya, The structure of the resolvent of elliptic pseudodifferential operators, *J. Funct. Anal.* **184** (2001) 77–135.
- [38] L. Maniccia, E. Schrohe, J. Seiler, Uniqueness of the Kontsevich-Vishik trace, *Proc. Amer. Math. Soc.* **136** (2007) 747–752.
- [39] J. Mickelsson, S. Paycha, The logarithmic residue density of a generalized Laplacian, *J. Aust. Math. Soc.* **90** (2011) 53–80.
- [40] S. Paycha, *Regularised Integrals, Sums and Traces An Analytic Point of View*, University Lecture Series vol. 59, Amer. Math. Soc., Providence, 2012.
- [41] R. Ponge, Traces and pseudodifferential operators and sums of commutators, *J. Anal. Math.* **110** (2010) 1–30.
- [42] Y. Safarov, D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators*, American Mathematical Society, 1997.
- [43] M. R. Sandoval, Wave-trace asymptotics for operators of Dirac type, *Comm. Part. Diff. Equ.* **24** (1999) 1903–1943.
- [44] E. Schrohe, Noncommutative residues, Dixmier’s traces, and heat trace expansions on manifolds with boundary, in *Geometric aspects of partial differential equations (Roskilde, 1998)*, volume 242 of *Contemp. Math.*, pp. 161–186, Amer. Math. Soc., Providence, RI, 1999.
- [45] H. Schröder, On the definition of geometric Dirac operators, arXiv:math/0005239, 2000.
- [46] S. Scott, The residue determinant, *Comm. Part. Diff. Equations* **30** (2005) 483–507.

- [47] S. Scott, *Traces and Determinants of Pseudodifferential Operators*, Oxford Univ. Press, 2010.
- [48] R. T. Seeley, Complex powers of an elliptic operator, *Amer. Math. Soc. Proc. Symp. Pure Math.* **10** (1966) 288–307.
- [49] A. Sitarz, Wodzicki residue and minimal operators on a noncommutative 4-dimensional torus, *J. Pseudo-Differ. Oper. Appl.* **5** (2014) 305–317.
- [50] M. Wodzicki, Local invariants of spectral asymmetry, *Invent. Math.* **75** (1984) 143–178.
- [51] M. Wodzicki, *Spectral Asymmetry and Noncommutative Residue*, Thesis, Steklov Institute of Mathematics, Moscow, 1984.
- [52] S. Zelditch, Lectures on wave invariants, in *Spectral Theory and Geometry*, edited by B. Davies and Y. Safarov, 1999, pp. 284–328.

DEPARTMENT OF MATHEMATICAL SCIENCES, LOUGHBOROUGH UNIVERSITY, LE11 3TU, UK

E-mail address: `l.li@lboro.ac.uk`, `a.strohmaier@lboro.ac.uk`