AN ALGEBRAIC MODEL FOR RATIONAL TORUS-EQUIVARIANT SPECTRA

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Abstract. We provide a universal de Rham model for rational G-equivariant cohomology theories for an arbitrary torus G. More precisely, we show that the representing category, of rational G-spectra, is Quillen equivalent to an explicit small and calculable algebraic model.

Contents

The authors are grateful to D. Barnes, A. Blumberg, M. Hill, M. Mandell, J.P. May and M. Kedziorek for numerous conversations about this work, to the Isaac Newton Institute (Cambridge) during the early stages of the work, and to MSRI (Berkeley) during recent revisions, for providing excellent working environments. The first author is grateful for support under EPSRC grant number EP/H040692/1. The second author was supported in part by NSF Grants DMS-0417206, DMS-0706877, DMS-1104396, and DMS-1406468 and by a Sloan Research Fellowship.

Part 1. Introduction

1. Overview

1.A. **Preamble.** Cohomology theories are contravariant homotopy functors on topological spaces satisfying the Eilenberg-Steenrod axioms (except for the dimension axiom), and any cohomology theory $E^*(\cdot)$ is represented by a homotopy theoretic spectrum E in the sense that $E^*(X) = [X, E]^*$. Accordingly, the category of spectra gives an embodiment of the category of cohomology theories in which one can do homotopy theory. The complexity of the homotopy theory of spectra is visible even in the homotopy endomorphisms of the unit object: this is the ring of stable homotopy groups of spheres, which is so intricate that we cannot expect a complete analysis of the category of spectra in general. However, most of the complication comes from $\mathbb{Z}$-torsion so we can simplify things by rationalizing. The resulting category of rational spectra represents cohomology theories with values in rational vector spaces. The simplicity of this rationalized category is apparent by Serre's theorem: the rationalization of the stable homotopy groups of spheres simply consists of $\mathbb{Q}$ in degree 0, and it is a small step to see that there is nothing more to the topology of rational cohomology theories than their graded rational vector space of coefficients. On the other hand, de Rham cohomology shows that a large amount of useful geometry remains even when we rationalize. Accordingly, the study of rational cohomology theories and rational spectra is both accessible and useful.

These facts are well-known, and it is natural to ask what happens when we consider spaces with an action of a compact Lie group G. Once again, a G-equivariant cohomology theory is a contravariant homotopy functor on G-spaces satisfying suitable conditions, and each such G-equivariant cohomology theory is represented by a G-spectrum [48]. In the equivariant case, when we rationalize a G-spectrum, considerably more structure remains than in the non-equivariant case. It is natural to expect rational representation theory to play a role in understanding rational equivariant cohomology theories, and when G is finite this is the only ingredient. However in general, the other significant piece of structure is exemplified by the Localization Theorem: for a torus G this states that (for finite complexes) there is no difference between the Borel cohomology of a G-space and its G-fixed points once the Euler classes are inverted. These ingredients can be used to build the algebraic model [24] for rational G-spectra described in Section 2 below.

The archetype for giving an algebraic model for the homotopy theory of topological origin is Quillen's analysis of simply connected rational spaces [55]. To prove the result, he introduced the axiomatic framework of model categories which underly the homotopy category, and the notion of a Quillen equivalence between model categories preserving the homotopy theories. The use of these ideas is now widespread, and we refer to [41] and [40] for details.

Our main result is a Quillen equivalence between the category of rational G-spectra for a torus G and an explicit and calculable algebraic model. In the course of our proof, we introduce a number of techniques of broader interest, in equivariant homotopy theory and in the theory of model categories. In the rest of the introduction, we give a little history, and then describe our results, methods and conventions.

1.B. **Equivariant cohomology theories.** Non-equivariantly, rational stable homotopy theory is very simple: the homotopy category of rational spectra is equivalent to the category of graded rational vector spaces, and all cohomology theories are ordinary in the sense that they are naturally equivalent to ordinary cohomology with coefficients in a graded vector space. The first author has conjectured [23] that for each compact Lie group G, there is an abelian category $\mathcal{A}(G)$, so that the homotopy category of rational G-spectra is equivalent to the homotopy category of differential graded objects of $\mathcal{A}(G)$:

$$\mathrm{Ho}(G\text{-spectra}/\mathbb{Q}) \simeq \mathrm{Ho}(DG - \mathcal{A}(G)).$$

In general terms, the objects of $\mathcal{A}(G)$ are sheaves of graded modules with additional structure over the space of closed subgroups of G, with the fibre over H giving information about the geometric H-fixed points. The conjecture describes various properties of $\mathcal{A}(G)$, and in particular asserts that its injective dimension is equal to the rank of G. According to the conjecture one may therefore expect to make complete calculations in rational equivariant stable homotopy theory, and one can classify cohomology theories. Indeed, one can construct a cohomology theory by writing down a differential graded object in $\mathcal{A}(G)$: this is how $SO(2)$-equivariant elliptic cohomology was constructed in [26], and it is hoped to construct cohomology theories associated to generic curves of higher genus in a similar way using the results of this paper.

The conjecture is elementary for finite groups, where $\mathcal{A}(G) = \prod_{(H)} \mathbb{Q}W_G(H)$-mod [29], where the product is over conjugacy classes of subgroups H and $W_G(H) = N_G(H)/H$. This means that any cohomology theory is again ordinary in the sense that it is a sum over conjugacy classes (H) of ordinary cohomology of the H-fixed points with coefficients in a graded $\mathbb{Q}W_G(H)$-module. The conjecture has been proved for the rank 1 groups $G = SO(2), O(2), SO(3)$ in [21, 20, 22], where $\mathcal{A}(G)$ is more complicated. It is natural to go on to conjecture that the equivalence comes from a Quillen equivalence

$$G\text{-spectra}/\mathbb{Q} \simeq DG - \mathcal{A}(G),$$

for suitable model structures. The second author proved that for $G = SO(2)$ the Quillen equivalence would follow from a triangulated equivalence on the derived categories [61]. It was claimed in [21] that the equivalence of homotopy categories was in fact a triangulated equivalence, but the proof is incomplete, and subsequent work of Patchkoria [53] shows that the method of [21] is insufficient. In any case, there is no prospect of extending the methods of [21] or [61] to higher rank. Even if one only wants an equivalence of triangulated categories, it appears essential to establish the Quillen equivalence when $r \geq 2$. Building on the present work, Barnes [2, 3] has shown how to deduce the Quillen equivalence for $G = O(2)$ from a suitable proof for $G = SO(2)$ (such as the one we use here), and Kedziorek [45] has done so for $G = SO(3)$.

Recently, Barnes, Kedziorek and the present authors have given a separate account of a Quillen equivalence for the $G = SO(2)$ [5]. This has the merit of avoiding the massive complication due to the complexity of the space of connected subgroups for a general torus, and also gives a stronger conclusion than the specialization of our result here, since the equivalence is monoidal.

1.C. **The classification theorem.** The present paper completes the programme begun in [24, 25] and supported by [33, 34, 35, 28]. The purpose of the series is to provide a small and calculable algebraic model for rational G-equivariant cohomology theories for a torus G of rank $r \geq 0$. Such cohomology theories are represented by rational G-spectra, and in this paper we show that the category of rational G-spectra is Quillen equivalent to the small and concrete abelian category $\mathcal{A}(G)$ introduced in [24] (its definition and properties are summarized in Section 2). The category $\mathcal{A}(G)$ is designed as a natural target of a homology theory

$$\pi_*^{\mathcal{A}} : G\text{-spectra} \longrightarrow \mathcal{A}(G);$$

the idea is that $\mathcal{A}(G)$ is a category of sheaves of modules, with the stalk over a closed subgroup H being the Borel cohomology of the geometric H-fixed point set with suitable coefficients. A main theorem of [24] shows that $\mathcal{A}(G)$ is of finite injective dimension (shown in [25] to be r).

The main theorem of the present paper and the culmination of the series is as follows. Model structures will be described in Sections 3 and 12 below.

Theorem 1.1. *For any torus G, there is a Quillen equivalence*

$$G\text{-}spectra/\mathbb{Q} \simeq_Q DG - \mathcal{A}(G)$$

of model categories. In particular their homotopy categories are equivalent

$$Ho(G\text{-}spectra/\mathbb{Q}) \simeq Ho(DG - \mathcal{A}(G))$$

as triangulated categories.

Remark 1.2. The functors involved in these Quillen equivalences are monoidal, but their interaction with the model structures is not straightforward. For this reason, the extension of this result to Quillen equivalences on the associated categories of monoids will be discussed elsewhere (as done in [5] in the rank 1 case).

Because of the nature of the theorem, it is easy to impose restrictions on the isotropy groups occurring in topology and algebra, and one may deduce versions of this theorem for categories of spectra with restricted isotropy groups. For example we recover a special case of the result of [31], which states that if G is any connected compact Lie group there is a Quillen equivalence

$$\text{free-}G\text{-spectra}/\mathbb{Q} \simeq_Q \text{DG-torsion-}H^*(BG)\text{-modules},$$

with a quite different proof. The methods of the present paper are used to extend the result on free G-spectra to disconnected groups G in [32].

1.D. **Applications.** Beyond the obvious structural insight, the type of applications we anticipate may be seen from those already given for the circle group $\mathbb{T}$ (i.e., the case $r = 1$). For example [21] gives a classification of rational $\mathbb{T}$-equivariant cohomology theories, a precise formulation and proof of the rational $\mathbb{T}$-equivariant Segal conjecture, and an algebraic analysis of existing theories, such as K-theory. More significant is the construction in [26] of a rational equivariant cohomology theory associated to an elliptic curve C over a $\mathbb{Q}$-algebra, and the identification of a part of $\mathbb{T}$-equivariant stable homotopy theory modelled on the derived category of sheaves over C. The philosophy in which equivariant cohomology theories correspond to algebraic groups is expounded in [27], and there are encouraging signs suggesting that one may use the model described in the present paper to construct torus-equivariant cohomology theories associated to generic complex curves of higher genus.

1.E. **Outline of strategy.** The proof is made possible by the apparatus of model categories and the existence of a good symmetric monoidal category of spectra, allowing us to talk about commutative ring spectra and modules over them. The argument also requires several more delicate formal properties of the model categories of equivariant spectra and the functors between them, as laid out in Axiom 3.3.

There are two other particular ingredients. The second author's results [62] gives Quillen equivalences between algebras over the Eilenberg-Mac Lane spectrum $H\mathbb{Q}$ and differential graded $\mathbb{Q}$-algebras, and between the module categories of corresponding algebras; this allows us to pass from topology to algebra. Since we are working over $\mathbb{Q}$, we may retain commutativity of the rings in this transition. Finally, the first author's [24] defining the algebraic category $\mathcal{A}(G)$ provides an algebraic model and the Adams spectral sequence based on it gives a means for calculation in the homotopy category.

In outline, what we have to achieve is to move from the category of rational G-spectra to the category of DG objects of the abelian category $\mathcal{A}(G)$. There are five main stages to this, which we first describe and then illustrate on a chain of Quillen equivalences.

(1) Isotropy separation: (Sections 4 to 7) We replace the category of G-spectra, which is the category of modules over the sphere spectrum, by a category of diagrams of modules over commutative equivariant ring spectra. Indeed, the sphere spectrum is shown to be the pullback of a diagram $\widetilde{R}_{top}$ of ring G-spectra, so the equivalence follows by the methods of [35]. The diagram $\widetilde{R}_{top}$ has the shape of a punctured $(r+1)$-cube, which we call the 'formal' punctured cube PC_f. The module category of each individual ring spectrum captures isotropical information about subgroups with a specified dimension and the diagram shows how to reassemble this isotropically local information into a global spectrum.

(2) Removal of equivariance: (Section 8) At each point in the diagram, we replace the commutative ring G-spectrum by a commutative non-equivariant ring spectrum by passage to fixed points, and show that the module categories are equivalent using the general methods described in [34].

(3) Transition to algebra: (Section 9) At each point in the diagram, we apply the second author's machinery [62] to replace all the commutative ring spectra in the diagram by commutative DGAs, and the category of module spectra by the corresponding category of DG modules over the DGAs.

(4) Rigidity: (Section 10) The diagram of commutative DGAs is intrinsically formal in the sense that it is determined up to equivalence by its homology. Accordingly the diagram of commutative DGAs may be replaced by a diagram of commutative algebras.

(5) Simplification: (Sections 12 and 13) At each stage so far, we have used cellularization to pick out the relevant homotopy category as the localizing subcategory built from certain specified 'cells'. The final step is to replace this cellularization of the category of DG-modules over the diagram of commutative rings by a much smaller category of modules with special properties, so that no cellularization is necessary; using apparatus from [28], this category turns out to be $\mathcal{A}(G)$.

These steps correspond to the following sequence of Quillen equivalences, several of which are themselves zig-zags of simple Quillen equivalences. The cellularizations are all with respect to the set of images of the cells G/H_+ as H runs through closed subgroups, and the diagrams of rings are all punctured $(r+1)$-cubes.

$$G\text{-spectra} \overset{(1)}{\simeq} \text{cell-}\widetilde{R}_{top}\text{-mod-}G\text{-spectra} \overset{(2)}{\simeq} \text{cell-}R_{top}\text{-mod-spectra} \overset{(3)}{\simeq} \text{cell-}R_t\text{-mod}$$
$$\overset{(4)}{\simeq} \text{cell-}R_a\text{-mod} \overset{(5)}{\simeq} \text{pqce-}R_a\text{-mod} \overset{(5)}{\simeq} \mathcal{A}(G)$$

It is worth highlighting some of the techniques of more general applicability.

First, we constantly use the Cellularization Principle [33]. The idea is that a Quillen adjunction induces a Quillen equivalence between cellularized model categories, provided we cellularize with respect to cells that are small and correspond under the adjunction. The hypotheses are mild, and it may appear like a tautology, but it has been useful innumerable times in the present paper and deserves emphasis. It can be directly compared to another extremely powerful formality, that a natural transformation of cohomology theories that is an isomorphism on spheres is an equivalence.

Second, we make extensive use of categories of modules over diagrams of rings [35], and prove that up to Quillen equivalence and cellularization, we can replace a category of modules over a diagram of rings by the category of modules over its pullback.

Third, the fact that if A is a ring G-spectrum, passage to Lewis-May K-fixed points establishes a close relationship between the category of A-module G-spectra and the category of A^K-module G/K-spectra [34]. More precisely, we consider a Quillen adjunction

$$A \otimes_{A^K} (\cdot) : A^K\text{-mod-}G/K\text{-spectra} \rightleftarrows A\text{-mod-}G\text{-spectra} : (\cdot)^K .$$

This is especially effective in conjunction with the Cellularization Principle.

Finally, we note that at the centre of the proof is rigidity: any two model categories with suitable specified homotopy level properities are equivalent. The equivariant sphere ring spectrum should be viewed as the sheaf of functions on a non-affine variety; we find a cover by affine varieties which are individually rigid, and the configuration of the cover is also rigid.

In effect, we have used only one basic rigidity result: any two commutative DGAs which have the same polynomial cohomology are quasi-isomorphic. This elementary result has far reaching consequences. Our main use of it here is to patch together local rigidity results (each based on polynomial rings) to give a global rigidity result. In [31] we applied it to prove rigidity of Koszul duals. We also need a rigidity result for modules, that by an Adams spectral sequence argument, the standard cells are determined by their homology [24, 12.1].

1.F. **Relationship to other results.** We should explain the relationship between the strategy implemented here and that used for free spectra in [31]. Both strategies start with a category of G-spectra and end with a purely algebraic category, and the connection in both relies on finding an intermediate category which is visibly rigid in the sense that it is determined by its homotopy category (the archetype of this is the category of modules over a commutative DGA with polynomial cohomology).

The difference comes in the route taken. Roughly speaking, the strategy in [31] is to move to non-equivariant spectra as soon as possible, whereas that adopted here is to keep working in the ambient category of G-spectra for as long as possible.

The advantage of the strategy of [31] is that it is close to commutative algebra, and should be adaptable to proving uniqueness of other algebraic categories. However, it is hard to retain control of the monoidal structure, and adapting the method to deal with many isotropy groups makes the formal framework very complicated. This was our original approach to the result for tori.

The present method appears to have several advantages. It uses fewer steps, and the monoidal structures are visible throughout. Furthermore, it reflects traditional approaches to the homotopy theory of G-spaces in that it displays the category of G-spectra as built from categories of spectra with restricted isotropy group using Borel cohomology.

Finally, we should explain that early versions of the present paper (specifically arXiv:1101.2511 v1, v2, v3 posted in 2011) differed from the present one in two important respects. Firstly, they included in condensed form the parts of [33, 34, 35] that they required; we separated out those papers partly to improve readability and partly because they appeared to be of wider interest. During the process of revising this paper to take advantage of the separation, we found a signficant simplification, and this led to the second main difference. The method for dealing with the equivalence between the category of G-spectra and a category of diagrams is much more technically complicated in the early versions because the diagrams themselves are infinite. In the present version, the manipulations with diagrams are now largely replaced by an equivalence of G-spectra showing how the sphere spectrum $\mathbb{S}$ can be constructed from isotropically simpler pieces. Having made that change, it was necessary to refer to the paper [28] for the behaviour of an algebraic torsion functor. The present account was essentially complete by Summer 2014, but we took the decision to work with a set of foundations that was not fully documented at the time, so we delayed posting this until we could refer to [11] for precise details.

1.G. **Conventions.** Certain conventions are in force throughout the paper. The most important is that *everything is rational*: henceforth all spectra and homology theories are rationalized without comment. For example, the category of rational G-spectra will now be denoted 'G-spectra'. Whenever possible we work in the derived category; for example, most equivalences are verified at this level. We also use the standard conventions that 'DG' abbreviates 'differential graded' and that 'subgroup' means 'closed subgroup'. We attempt to let inclusion of subgroups follow the alphabet, so that $G \supseteq H \supseteq K \supseteq L$.

We focus on homological (lower) degrees, with differentials reducing degrees; for clarity, cohomological (upper) degrees are called *codegrees* and may be converted to degrees by negation in the usual way. Finally, we write $H^*(X)$ for the unreduced cohomology of a space X with rational coefficients.

We have adopted a number of more specific conventions in our choice of notation, and it may help the reader to be alerted to them.

- There are several cases where we need to talk about ring G-spectra $\widetilde{R}$ and their fixed points $R = (\widetilde{R})^G$. The equivariant form is indicated by a tilde on the non-equivariant one.
- We need to discuss rings in various categories of spectra, and then modules over them. Since it often needs to be made explicit, we write, for example, R-module-G-spectra for the category of R-modules in the category of G-spectra.
- We will not usually make explicit the universe over which our spectra are indexed. The default is that a category of G-spectra will be indexed over a complete G-universe, and we only mention the universe when it needs emphasis.
- The purpose of this paper is to give an algebraic model of a topological phenomenon. Accordingly, characters arise in various worlds, and it is useful to know they play

corresponding roles. We sometimes point this out by use of subscripts. For example R_a (with 'a' for 'algebra') might be a (conventional, graded) ring, R_{top} its counterpart in spectra, $\tilde{R}_{top}$ its counterpart in G-spectra, and R_t its counterpart in DG-algebra (a large and poorly understood DGA).

- We often have to discuss diagrams of rings and diagrams of modules over them, but we will usually say that R is a diagram of rings and M is an R-module (leaving the fact that M is also a diagram to be deduced from the context).

1.H. **Organization of the paper.** Section 2 recalls the definition of the algebraic model $\mathcal{A}(G)$. Section 3 discusses some fundamental technical issues determining which models of spectra we use; it considerably simplifies our argument to be able to work in a context in which certain homotopy commutative ring G-spectra have commutative models. Section 4 illustrates the argument by giving a complete outline in the simple case of the circle.

Section 5 introduces the formalism for discussing modules over diagrams of rings.

In Section 6 we explain that the sphere spectrum is the homotopy pullback of a punctured $(r+1)$-cube of isotropically simpler ring spectra, and in Section 7 we explain that it is the homotopy pullback of a punctured $(r+1)$-cube diagram $\widetilde{R}_{top}$ of ring spectra which are formal in the sense that they are determined by their homotopy. This punctured cube is PC_f, and all the subsequent diagrams have this shape. The results of [35] then establishes Equivalence (1), showing that the category of rational G-spectra is equivalent to a category of module G-spectra over the diagram $\widetilde{R}_{top}$ of ring G-spectra. This completes the isotropy separation step of the proof.

Until this point, all arguments and calculations are within the category of G-spectra. The remaining steps change ambient categories. We not only need to recognize the categories of modules, but we also need to recognize the cells we use to cellularize them. The fact that the natural cells G/H_+ are characterized by their homology ([24, 12.1]) means that we do not need to comment further on the cells.

Having shown the category of G-spectra is equivalent to a category of modules over the diagram $\widetilde{R}_{top}$ of ring G-spectra, we can move from G-spectra to non-equivariant spectra in Section 8, using the results of [34] to establish that this category is equivalent to a category of modules over the diagram $R_{top} = (\widetilde{R}_{top})^G$ of ring spectra (i.e., Equivalence (2)). In Section 9 we use the results of [62] to establish that the category of R_{top}-modules is equivalent to a category of modules over the diagram R_t of DGAs (i.e., Equivalence (3)). It is then quite straightforward to establish Equivalence (4), showing in Section 10 that the PC_f-diagram $R_a = H_*(R_t)$ is intrinsically formal, so that the category of modules over R_t and R_a are equivalent.

In Section 11 we recognize our progress by seeing that $\mathcal{A}(G)$ can be viewed as a category of modules over the diagram R_a of graded rings. Finally Sections 12 and 13 establish Equivalence (5), showing that the cellularization is equivalent to the particular category $\mathcal{A}(G)$ of DG-R_a-modules.

2. The algebraic model

In this section we recall relevant results from [24] which constructs an abelian category $\mathcal{A}(G)$ giving an algebraic reflection of the structure of the category of G-spectra and an Adams spectral sequence based on it; the present account is very brief and readers may need to refer to [24] for details. The structures from that analysis will be relevant to much of what we do here.

More precisely, this model is based on pairs of connected subgroups and is denoted $\mathcal{A}^p_c(G)$ in the more precise notation of [28], and we introduce it as the most convenient and practical model. In fact the first output of the topological argument is a model based on flags of dimensions of subgroups which is denoted $\mathcal{A}^f_d(G)$ in [28]. This was introduced and shown to be equivalent to $\mathcal{A}^p_c(G)$ in [28]; building on [28], we show in Section 11 how to move directly from the algebraic model coming from our proof (namely $\mathcal{A}^f_d(G)$) to $\mathcal{A}^p_c(G)$.

2.A. **Definition of the category.** First we must construct the category $\mathcal{A}(G)$, which is a category of modules over a diagram of rings. For a category $\mathbf{D}$ and a diagram of $R : \mathbf{D} \longrightarrow \mathbf{Rings}$ of rings, an R-module is given by a $\mathbf{D}$-diagram M such that $M(x)$ is an $R(x)$-module for each object x in $\mathbf{D}$, and for every morphism $a : x \longrightarrow y$ in $\mathbf{D}$, the map $M(a) : M(x) \longrightarrow M(y)$ is a module map over the ring map $R(a) : R(x) \longrightarrow R(y)$.

The shape of the diagram for $\mathcal{A}(G)$ is given by the partially ordered set $\mathbf{ConnSub(G)}$ of connected subgroups of G. To start with we consider the single graded ring

$$\mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(BG/F),$$

where the product is over the family $\mathcal{F}$ of finite subgroups of G. To specify the value of the ring at a connected subgroup K, we use Euler classes: indeed if V is a representation of G we may define $c(V) \in \mathcal{O}_{\mathcal{F}}$ by specifying its components. In the factor corresponding to the finite subgroup F we take $c(V)(F) := c_{|V^F|}(V^F) \in H^{|V^F|}(BG/F)$ where $c_{|V^F|}(V^F)$ is the classical Euler class of V^H in ordinary rational cohomology.

The diagram of rings $\tilde{\mathcal{O}}_{\mathcal{F}}$ is defined by the following functor on $\mathbf{ConnSub(G)}$

$$\tilde{\mathcal{O}}_{\mathcal{F}}(K) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$$

where $\mathcal{E}_K = \{c(V) \mid V^K = 0\} \subseteq \mathcal{O}_{\mathcal{F}}$ is the multiplicative set of Euler classes of K-essential representations. Each of the Euler classes is a finite sum of mutually orthogonal homogeneous terms, and so this localization is again a graded ring.

Next we consider the category of modules M over the diagram $\tilde{\mathcal{O}}_{\mathcal{F}}$. Thus the value $M(K)$ is a module over $\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$, and if $L \subseteq K$, the structure map

$$M(L) \longrightarrow M(K)$$

is a map of modules over the map

$$\mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$$

of rings. Note this map of rings is a localization since $V^L = 0$ implies $V^K = 0$ so that $\mathcal{E}_L \subseteq \mathcal{E}_K$. The category $\mathcal{A}(G)$ is formed from a subcategory of the category of $\tilde{\mathcal{O}}_{\mathcal{F}}$-modules by adding structure. There are two requirements which we briefly indicate here. We make the necessary extra structure explicit in Section 11. Firstly they must be *quasi-coherent*, in that they are determined by their value at the trivial subgroup 1 by the formula

$$M(K) := \mathcal{E}_K^{-1}M(1).$$

The second condition involves the relation between G and its quotients. Choosing a particular connected subgroup K, we consider the relationship between the group G with the collection $\mathcal{F}$ of its finite subgroups and the quotient group G/K with the collection $\mathcal{F}/K$ of its finite subgroups. For G we have the ring $\mathcal{O}_{\mathcal{F}}$ and for G/K we have the ring

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K})$$

where we have identified finite subgroups of G/K with their inverse images in G, i.e., with subgroups $\tilde{K}$ of G having identity component K. Combining the inflation maps associated to passing to quotients by K for individual groups, there is an inflation map

$$\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}.$$

The second condition is that the object should be *extended*, in the sense that for each connected subgroup K there is a specified isomorphism

$$M(K) \cong \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

for some $\mathcal{O}_{\mathcal{F}/K}$-module $\phi^K M$, which is a given part of the structure. These identifications should be compatible when we have inclusions of connected subgroups. If we choose a subgroup L and then the modules $\phi^K M$ for $K \supseteq L$ fit together to make an object of $\mathcal{A}(G/L)$.

2.B. **Diagrams of quotient pairs.** For some purposes it is useful to have an alternative view of $\mathcal{A}(G)$ as introduced in [25] making more of the structure explicit. Here the values $\phi^H M$ are all displayed in a single diagram indexed by pairs of quotient groups. Pairs of quotient groups are equivalent to pairs of subgroups, but here we will stick with the indexing by quotients G/K as in [25] since it is the quotients that enter most directly into the model. We use the notations $\mathbb{R}^p_c$ for the ring and $\mathcal{A}^p_c(G)$ for the category as in [28], since this is descriptive of the fact that we use **p**airs of **c**onnected subgroups.

Definition 2.1. The diagram of *quotient pairs* of G is the partially ordered set with objects $(G/K)_{G/L}$ for $L \subseteq K \subseteq G$, and with two types of morphisms. The *horizontal* morphisms

$$h^H_K : (G/K)_{G/L} \longrightarrow (G/H)_{G/L} \text{ for } L \subseteq K \subseteq H \subseteq G$$

and the *vertical* morphisms

$$v^K_L : (G/H)_{G/K} \longrightarrow (G/H)_{G/L} \text{ for } L \subseteq K \subseteq H \subseteq G.$$

One particular diagram will be of special significance for us.

Definition 2.2. The structure diagram for G is the diagram of rings $\mathbb{R}^p_c$ defined by

$$\mathbb{R}^p_c(G/K)_{G/L} := \mathcal{E}^{-1}_{K/L}\mathcal{O}_{\mathcal{F}/L}.$$

Since $V^K = 0$ implies $V^H = 0$, we see that $\mathcal{E}_{H/L} \supseteq \mathcal{E}_{K/L}$, so it is legitimate to take the horizontal maps to be localizations

$$h^H_K : \mathcal{E}^{-1}_{K/L}\mathcal{O}_{\mathcal{F}/L} \longrightarrow \mathcal{E}^{-1}_{H/L}\mathcal{O}_{\mathcal{F}/L}.$$

To define the vertical maps, we begin with the inflation map $\inf^{G/L}_{G/K} : \mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}/L}$, and then observe that if V is a representation of G/K with $V^H = 0$, it may be regarded as a representation of G/L, and Euler classes correspond in the sense that $\inf(e_{G/K}(V)) = e_{G/L}(V)$. We therefore obtain a map

$$v_K^L : \mathcal{E}_{H/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L}.$$

Illustrating this for a group G of rank 2 in the usual way, we obtain

$$\begin{array}{ccccc} & & & & \mathcal{O}_{\mathcal{F}/G} \\ & & & & \downarrow \\ & & \mathcal{O}_{\mathcal{F}/K} & \longrightarrow & \mathcal{E}_{G/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \\ & & \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{F}} & \longrightarrow & \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} & \longrightarrow & \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \end{array}$$

At the top right, of course $\mathcal{O}_{\mathcal{F}/G} = \mathbb{Q}$, but clarifies the formalism to use the more complicated notation.

In discussing modules, we need to refer to the structure maps for rings, so for an $\mathbb{R}_c^p$-module M, if $L \subseteq K \subseteq H \subseteq G$, we generically write

$$\alpha_K^L : M(G/H)_{G/K} \longrightarrow M(G/H)_{G/L}$$

for the vertical map, and

$$\tilde{\alpha}_K^L : \mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} M(G/H)_{G/K} = (v_K^L)_* M(G/H)_{G/K} \longrightarrow M(G/H)_{G/L}$$

for the associated map of $\mathcal{O}_{\mathcal{F}/L}$-modules. Similarly, we generically write

$$\beta_K^H : M(G/K)_{G/L} \longrightarrow M(G/H)_{G/L}$$

for the horizontal map, and

$$\tilde{\beta}_K^H : \mathcal{E}_{H/L}^{-1} M(G/K)_{G/L} = (h_K^H)_* M(G/K)_{G/L} \longrightarrow M(G/H)_{G/L}$$

for the associated map of $\mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L}$-modules, which we refer to as the *basing map* after [21].

Definition 2.3. If M is an $\mathbb{R}_c^p$-module, we say that M is *extended* if whenever $L \subseteq K \subseteq H$ the vertical map α_K^L is an extension of scalars along $v_K^L : \mathcal{E}_{H/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L}$, which is to say that

$$\tilde{\alpha}_K^L : \mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} M(G/H)_{G/K} \xrightarrow{\cong} M(G/H)_{G/L}$$

is an isomorphism of $\mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L}$-modules.

If M is an R_{qp}-module, we say that M is *quasi-coherent* if whenever $L \subseteq K \subseteq H$ the horizontal map β_K^H is an extension of scalars along $h_K^H : \mathcal{E}_{K/L}^{-1}\mathcal{O}_{\mathcal{F}/L} \longrightarrow \mathcal{E}_{H/L}^{-1}\mathcal{O}_{\mathcal{F}/L}$, which is to say that

$$\tilde{\beta}_K^H : \mathcal{E}_{H/L}^{-1} M(G/K)_{G/L} \xrightarrow{\cong} M(G/H)_{G/L}$$

is an isomorphism.

We write qc-$\mathbb{R}_c^p$-mod, e-$\mathbb{R}_c^p$-mod and $\mathcal{A}_c^p(G) := qce$-$\mathbb{R}_c^p$-mod. for the full subcategories of $\mathbb{R}_c$-modules with the indicated properties.

Next observe that the most significant part of the information in an extended object is displayed in its restriction to the leading diagonal. For example in our rank 2 example they take the form

$$\begin{array}{ccccc}
 & & & & M(G/G)_{G/G} \\
 & & & & \downarrow \\
 & & M(G/K)_{G/K} & \longrightarrow & \mathcal{E}_{G/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \otimes_{\mathcal{O}_{\mathcal{F}/G}} M(G/G)_{G/G} \\
 & & \downarrow & & \downarrow \\
M(G/1)_{G/1} & \longrightarrow & \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} M(G/K)_{G/K} & \longrightarrow & \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/G}} M(G/G)_{G/G}
\end{array}$$

In effect our description of the category $\mathcal{A}(G)$ abbreviates such a diagram by just writing the final row and taking $\phi^K M = M(G/K)_{G/K}$:

$$\phi^1 M \longrightarrow \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M \longrightarrow \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/G}} \phi^G M,$$

leaving it implicit that the particular decomposition as a tensor product is part of the structure.

Lemma 2.4. [25, 5.5] *The functor*

$$i : \mathcal{A}(G) \longrightarrow \mathcal{A}_c^p(G) = qce\text{-}\mathbb{R}_c^p\text{-}mod$$

defined by

$$i(M)(G/K)_{G/L} := \mathcal{E}_{K/L}^{-1}\phi^L M.$$

gives an equivalence

$$\mathcal{A}(G) \simeq \mathcal{A}_c^p(G).$$

□

Henceforth we will identify the two, thinking of $\mathcal{A}(G)$ as given by the values of $\mathcal{A}_c^p(G)$ on the objects $(G/K)_{G/K}$ with additional structure given by the horizontal and vertical maps.

2.C. **Connection with topology.** The connection between G-spectra and $\mathcal{A}(G)$ is given by a homotopy functor

$$\pi_*^{\mathcal{A}} : G\text{-spectra} \longrightarrow \mathcal{A}(G)$$

with the exactness properties of a homology theory. It is rather easy to write down the value of the functor as a diagram of abelian groups.

Definition 2.5. For a G-spectrum X we define $\pi_*^{\mathcal{A}}(X)$ on K by

$$\pi_*^{\mathcal{A}}(X)(K) = \pi_*^G(DE\mathcal{F}_+ \wedge S^{\infty V(K)} \wedge X).$$

Here $E\mathcal{F}_+$ is the universal space for the family $\mathcal{F}$ of finite subgroups with a disjoint basepoint added and $DE\mathcal{F}_+ = F(E\mathcal{F}_+, S^0)$ is its functional dual (the function G-spectrum of maps from $E\mathcal{F}_+$ to S^0). The G-space $S^{\infty V(K)}$ is defined by

$$S^{\infty V(K)} = \lim_{\substack{\rightarrow \\ V^K=0}} S^V,$$

when $K \subseteq H$, so there is a map $S^{\infty V(K)} \longrightarrow S^{\infty V(H)}$ inducing the map $\pi_*^{\mathcal{A}}(X)(K) \longrightarrow \pi_*^{\mathcal{A}}(X)(H)$. $\square$

The definition of $\pi_*^{\mathcal{A}}(X)$ shows that quasi-coherence for $\pi_*^{\mathcal{A}}(X)$ is just a matter of understanding Euler classes. The extendedness of $\pi_*^{\mathcal{A}}(X)$ is a little more subtle, and will play a significant role later. We take

$$\phi^K \pi_*^{\mathcal{A}}(X) = \pi_*^{G/K}(DE\mathcal{F}/K_+ \wedge \Phi^K(X)),$$

where Φ^K is the geometric fixed point functor, and the extendedness follows from properties of the geometric fixed point functor.

To see that $\pi_*^{\mathcal{A}}(X)$ is a module over $\mathcal{O}$, the key is to understand S^0.

Theorem 2.6. [24, 1.5] *The image of S^0 in $\mathcal{A}(G)$ is the structure functor:*

$$\widetilde{\mathcal{O}}_{\mathcal{F}} = \pi_*^{\mathcal{A}}(S^0),$$

with the canonical structure as an extended module.

Some additional work confirms that $\pi_*^{\mathcal{A}}$ has the appropriate behaviour.

Corollary 2.7. [24, 1.6] *The functor $\pi_*^{\mathcal{A}}$ takes values in the abelian category $\mathcal{A}(G)$.*

2.D. **The Adams spectral sequence.** The homology theory $\pi_*^{\mathcal{A}}$ may be used as the basis of an Adams spectral sequence for calculating maps between rational G-spectra. The main theorem of [24] is as follows.

Theorem 2.8. *(*[24, 9.1]*) For any rational G-spectra X and Y there is a natural Adams spectral sequence*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}(G)}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \Rightarrow [X, Y]^G_*.$$

It is a finite spectral sequence concentrated in rows 0 to r (the rank of G) and strongly convergent for all X and Y. $\square$

This was what led us to attempt to prove the main theorem of the present paper, and many of the methods used to construct the Adams spectral sequence are adapted to the present work. Nonetheless, it appears that the only way we explicitly use the Adams spectral sequence is in the fact that cells are characterized by their homology.

Corollary 2.9. [24, 12.1] *If X is a G-spectrum with $\pi_*^{\mathcal{A}}(X) \cong \pi_*^{\mathcal{A}}(G/H_+)$ then $X \simeq G/H_+$.*

The proof proceeds by giving an explicit resolution of $\pi_*^{\mathcal{A}}(G/H_+)$ in $\mathcal{A}(G)$, and then observing that this gives appropriate vanishing at the E_2-page so as to ensure an isomorphism $\pi_*^{\mathcal{A}}(X) \cong \pi_*^{\mathcal{A}}(G/H_+)$ lifts to a homotopy class of maps $G/H_+ \longrightarrow X$. Since $\pi_*^{\mathcal{A}}$ detects weak equivalences, this suffices. Evidently, this argument applies in any model category with a similar Adams spectral sequence.

In the present paper, we often need to know how our chosen cells behave under functors between model categories. We will apply the corollary repeatedly to see that each cell maps to the obvious object up to equivalence.

3. Cochain ring spectra

The purpose of this section is twofold. First we explain our choice of coefficients in cochains at a homotopical level in Subsections 3.A and 3.B. In Subsections 3.C and 3.D we turn to the question of which model category of spectra we use. In fact our argument is not particularly sensitive to the choice, but it is essential to be clear about commutative rings.

3.A. **The sphere spectrum.** Just as abelian groups are $\mathbb{Z}$-modules, giving $\mathbb{Z}$ a special role, so too spectra are modules over the sphere spectrum $\mathbb{S}$. Although $\mathbb{S}$ is the suspension spectrum of S^0, we will generally use the special notation $\mathbb{S}$ to emphasize its special role. Since we are working rationally, $\mathbb{S}$ will denote the rational sphere spectrum.

3.B. **Choice of coefficients.** Central to our formalism is that we consider 'rings of functions' on certain spaces, and then consider modules over these. In effect we take a suitable model for cochains on the space with coefficients in a ring. The purpose of the present subsection is to describe the options, and explain why we end up simply using the functional dual $DX = F(X, \mathbb{S})$ rather than one of the natural alternatives.

If X is a G-space and k is a ring G-spectrum then we may write

$$C^*(X; k) := D_k X_+ := F_{\mathbb{S}}(X_+, k)$$

for the G-spectrum of functions from X to k. The first notation comes from the special case of an Eilenberg-Mac Lane spectrum, which gives a model for cohomology. The second notation comes from the special case $k = \mathbb{S}$ of the functional dual. This spectrum has a ring structure using the multiplication on k and the diagonal map of X. If k is a commutative ring spectrum then so is $C^*(X; k)$.

There are a number of related ring spectra of this form associated to different choices of k and we briefly discuss their properties before explaining which is most relevant to us.

First, we could take k to be the rational sphere G-spectrum $\mathbb{S}$, alternatively, we could take it to be one of two Eilenberg-Mac Lane G-spectra associated to Green functors. The first Green functor is the Burnside functor $\mathbb{A}$, whose value on G/H is the Burnside ring of H, and the second Green functor is the constant functor $\mathbb{Q}$.

To start with we observe that there are maps

$$\mathbb{S} \longrightarrow H\mathbb{A} \longrightarrow H\mathbb{Q}$$

of commutative ring G-spectra where the first map kills higher homotopy groups and the second kills the augmentation ideal. This induces maps

$$D_{\mathbb{S}} X_+ \longrightarrow D_{H\mathbb{A}} X_+ \longrightarrow D_{H\mathbb{Q}} X_+.$$

These are very far from being equivalences in general. For the second map that is clear since $\mathbb{A}(G/H) \neq \mathbb{Q}$ if H is a non-trivial finite subgroup. For the first, it is clear from the fact that $\mathbb{S}$ has non-trivial higher homotopy (even rationally) when G is not finite.

Lemma 3.1. *(i) If X is free, the above maps induce equivalences*

$$D_{\mathbb{S}} X_+ \simeq D_{H\mathbb{A}} X_+ \simeq D_{H\mathbb{Q}} X_+.$$

(ii) If X has only finite isotropy, then the first map is an equivalence

$$D_{\mathbb{S}} X_+ \simeq D_{H\mathbb{A}} X_+.$$

Proof: For Part (i) we note that $\mathbb{S}, H\mathbb{A}$ and $H\mathbb{Q}$ all have non-equivariant homotopy $\mathbb{Q}$ in degree 0.

For Part (ii), $\mathbb{S}$ is (rationally) an Eilenberg-Mac Lane spectrum for any finite group of equivariance. □

The functor $D_{H\mathbb{Q}}$ has the convenient property that

$$(D_{H\mathbb{Q}}Y)^G = D_{H\mathbb{Q}}(Y/G)$$

for any based space Y. On the other hand, this lets us calculate values which show the functor is not the one we want to use (specifically, the homotopical analysis of [24] makes clear that the homotopy groups of the cochains on $E\mathcal{F}_+$ should be those of $D_{\mathbb{S}}E\mathcal{F}_+$). Since we will in fact only apply the duality functor to spaces with finite isotropy, we could equally well apply the functor $D_{H\mathbb{A}}$.

3.C. **Some commutative ring spectra.** Our arguments use ideas from commutative algebra, so we want to work in a context where certain G-spectra R behave like commutative rings. What we need is a symmetric monoidal category of R-modules with a well behaved homotopy category, and which behaves well under various change of groups constructions. It is conceptually simplest if we work in a category of G-spectra where the relevant rings R actually are commutative monoids, and we will describe such a context in Subsection 3.D. For now we introduce the G-spectra R in question.

The starting point is the function spectrum $DE\mathcal{F}_+$. This is a commutative ring since it consists of maps from a G-space of the form X_+ (which has a strictly cocommutative diagonal) into a commutative ring spectrum. We then wish to consider the spectra $DE\mathcal{F}_+ \wedge S^{\infty V(H)}$ for subgroups H, where

$$S^{\infty V(H)} = \bigcup_{V^H=0} S^V.$$

These can be obtained as a smash product as written, or as the Bousfield localization of $DE\mathcal{F}_+$ with respect to $S^{\infty V(H)}$. The importance of $S^{\infty V(H)}$ is firstly that it has geometric isotropy conisting of precisely the subgroups containing H, and secondly it is built from spheres.

However $S^{\infty V(H)}$ also has excellent multiplicative properties. It is clear that it is a commutative ring up to homotopy. Moreover, for the reasons pointed out by McClure [50] for the Tate spectrum, $S^{\infty V(H)}$ admits an action of the linear isometries operad based on a G-fixed universe. One may be rather explicit about this action, or alternatively one may apply [37] to see that Bousfield localization preserves the existence of an action by a non-equivariant E_∞ operad. This will serve our purposes because we can do the two things that are required. Firstly, passage to H-fixed points gives a G/H-spectrum which is also an algebra over a non-equivariant E_∞ operad. Secondly, there is an operadic smash product on the category of G-spectra based on this operad, making the category a symmetric monoidal model category, and so that algebras over the operad correspond precisely to commutative monoids in this monoidal model category. We can thus form a symmetric monoidal model category of modules over a naively commutative ring spectrum.

Remark 3.2. McClure also points out that spectra like this do not admit an action of the equivariant linear isometries operad on a complete universe. The analogous statement for finite groups of equivariance, is clear, as pointed out by Hill and Hopkins [37]: a non-equivariantly contractible ring spectrum with multiplicative norms is equivariantly contractible (the norm from 1 to G takes the non-equivariant unit to the equivariant unit).

3.D. **Requirements for categories of equivariant spectra.** We lay out in this section what we require of the category of spectra we work in. We then observe in Subsection 3.E that the category constructed by Blumberg-Hill in [11] satisfies all of these requirements. We have adopted the axiomatic approach because the Blumberg-Hill categories have not yet been proved to have all of the properties we would need for certain monoidal extensions of our results, and partly because we believe that the properties we require may eventually be proved for other categories as well.

The properties we require of our category of G-spectra are as follows. The point is that we want a symmetric monoidal model category in which the commutative monoids (a) include the examples we need (as laid out in Subsection 3.C) and (b) have a symmetric monoidal category of modules.

Axiom 3.3. *The category of G-spectra has the following properties.*

1. *It is a weakly symmetric monoidal proper G-topological model category with weak equivalences detected by a functor U to orthogonal spectra GSp^O.*
2. *The functor U induces an equivalence of homotopy categories.*
3. *The smash product is the usual one in the homotopy category, and in the non-equivariant setting is monoidally equivalent to the usual smash product of orthogonal spectra.*
4. *The monoids in the category are non-Σ algebras over the linear isometries operad $\mathcal{L}$*
5. *The commutative monoids in the category are algebras over the linear isometries operad $\mathcal{L}$ for a G-fixed universe $\mathcal{V}$.*
6. *The rational sphere spectrum $\mathbb{S}$ is a commutative monoid.*
7. *Commutative monoids are cotensored over unbased spaces.*
8. *All localizations preserve commutative monoids over the linear isometries operad for a G-fixed universe $\mathcal{V}$.*

The equivariant categories for G and its quotients are related as follows.

9. *If $L \subseteq K$ then inflation from G/L-spectra to G/K-spectra takes commutative rings to commutative rings.*
10. *The H-fixed point functor is lax symmetric monoidal and hence preserves commutative monoids.*
11. *There is a zig-zag of Quillen equivalences between commutative monoids in* 1*-spectra and in symmetric spectra. Denote the derived functor of commutative monoids from* 1*-spectra to symmetric spectra by $\mathbb{F}$.*
12. *For a commutative monoid A in* 1*-spectra, there is a Quillen equivalence between the categories of A-modules over* 1*-spectra and $\mathbb{F}A$-modules over symmetric spectra.*

3.E. **The category of orthogonal $\mathcal{L}$-spectra.** The category is constructed by following the approach of Elmendorf-Kriz-Mandell-May [16] applied to orthogonal G-spectra. Details are given in Blumberg-Hill [11]; their main concern is to understand the homotopy theory of different types of norm and different degrees of commutativity. Since we are only concerned with the simplest type of commutativity and not with norms at all, we only need the more formal parts of their argument. In particular, although their paper is written for finite groups, the relevant part of it applies to all compact Lie groups. We will comment on this in more detail below when we have sketched the construction. We are very grateful to Blumberg and Hill for discussions about their category, and for explicitly including statements from which the properties we require are apparent. We would also like to thank Blumberg for suggestions which led to the current approaches to Axiom 3.3 (11) and (12).

The construction starts with the category $G\mathrm{Sp}^O$ of orthogonal G-spectra based (additively) on a complete orthogonal G-universe as usual. For the *multiplicative* properties, we now choose a G-fixed universe $\mathcal{V}$ (i.e., infinite dimensional but with trivial G-action) with a view to constructing an operadic smash product based on the $\mathcal{V}$-linear isometries operad.

More precisely, we let $\mathcal{L}$ denote the non-equivariant linear isometries operad defined by

$$\mathcal{L}(n) = \mathrm{Isom}(\mathcal{V}^n, \mathcal{V}).$$

There is an associated monad $\mathbb{L}$ given by smashing with $\mathcal{L}(1)_+$ and we consider the category $G\mathrm{Sp}^O[\mathbb{L}]$ of $\mathbb{L}$-algebras in orthogonal G-spectra. Applying $\mathbb{L}$ is left adjoint to the forgetful functor relating orthogonal G-spectra to those with an $\mathbb{L}$ action. Since $\mathcal{L}(1)$ is contractible, the functors relate objects with the same underlying homotopy type. Precisely as in [16], the category of $\mathbb{L}$-spectra has a symmetric monoidal smash product $\wedge_{\mathcal{L}}$ and we restrict to those which are unital in the sense that the unit map $S \wedge_{\mathcal{L}} X \longrightarrow X$ is an isomorphism. This category of unital $\mathbb{L}$-orthogonal G-spectra, denoted $G\mathcal{S}_U$ in [11], is analagous to the category referred to as S-modules in [16], and is a monoidal model category satisfying the monoid axiom.

Proposition 3.4. *The category of orthogonal equivariant $\mathcal{L}$-spectra, $G\mathcal{S}_U$, satisfies Axiom 3.3 (1) through (12).*

Proof: Axiom 3.3 (1) is [11, 4.2, 4.8].

Axiom 3.3 (2) is [11, 4.3, 4.10].

Axiom 3.3 (3) is [11, 4.3, 4.11].

Axioms 3.3 (4) and (5) are [11, 3.16].

Axiom 3.3 (6) follows [38, 6.4] and the fact that the sphere spectrum is a commutative monoid.

Axiom 3.3 (7) is [11, 3.17].

Axiom 3.3 (8) is [38, 6.4].

Axiom 3.3 (9) and (10) are [11, 3.24].

For Axiom 3.3 (11): First, [11, 3.16], referring to [16, II.4.6], shows that commutative monoids in non-equivariant $\mathcal{L}$-spectra are isomorphic to the category of E_∞-algebras in orthogonal spectra. Then [52, 0.14] shows that E_∞-algebras in orthogonal spectra and in symmetric spectra are Quillen equivalent and both are Quillen equivalent to the respective categories of commutative monoids.

For Axiom 3.3 (12): The category of modules over a commutative monoid A in non-equivariant $\mathcal{L}$-spectra is isomorphic to the category of operadic modules over the associated E_∞ orthogonal spectrum $\mathbb{U}A$ from [11, 3.16]. As above, this follows by an analogue of the argument in [16, II.5.1]. See also [17] for a careful definition of operadic modules via multicategories.

Next we use the monoidal Quillen equivalence between orthogonal spectra and symmetric spectra from [52, 0.4] to show that the category of operadic modules over an E_∞-algebra in orthogonal spectra is Quillen equivalent to the category of operadic modules over an associated E_∞-algebra in symmetric spectra by [8, 2.14]. Finally, [17, 1.4] shows that this category of modules over an E_∞-algebra is Quillen equivalent to the category of modules over a commutative monoid in symmetric spectra. The commutative monoid here may differ from the image of $\mathbb{F}$, but the two will be weakly equivalent. The statement then follows by [43, 5.4.5]. □

Remark 3.5. As mentioned above, the paper [11] is written for finite groups. We explain here that the same proofs show that results we need are true for all compact Lie groups.

First of all, we only need the results when the multiplicative universe is trivial and therefore fixed by G. This is much simpler than the general case, since at many points G-equivariance becomes entirely separate from the $\mathcal{L}$-structures. With this in mind, the first three sections of [11] are purely categorical, and apply directly to compact Lie groups. Section 4 considers the homotopy theory and the relevant part consists of Subsections 4.1 and 4.2, with Theorems 4.12 and 4.14 being most delicate. Working through it systematically, one sees that the results we require (as listed in the above proof), involve quoting [51] (which is explicitly written for compact Lie groups) and especially [39, Appendix B] (there are also alternative references to [16] (which states that it applies to compact Lie groups)). The main concern of [39, Appendix B] is again the multiplicative norm for finite groups, so [39] also restricts attention to finite groups, so that once again we need to check that the relevant facts apply to compact Lie groups. The first main ingredient is the fact that cofibrant objects admit a filtration with well behaved subquotients. There are two aspects to this. One relates to G-equivariance; the relevant subquotients are wedges of induced spectra, so that the equivariant properties required follow from a change of groups to reduce directly to nonequivariant facts. The second and more subtle condition is the key one. It relates to the commuting actions of G and the symmetric group acting on a smash power, essentially saying that the $\mathcal{L}$-structures ensure enough freeness. Specifically, the results are [39, B.117, B.130]; the core of the proof of B.117 already uses equivariance for compact Lie groups (because of the role of orthogonal groups in orthogonal spectra), and the proof applies equally well when G is an arbitrary compact Lie group.

4. The circle group

Our overall strategy is to assemble a model for all G-spectra from models for G-spectra with geometric isotropy K as K ranges over all closed subgroups. In fact we will collect together information from all the isotropy groups with the same identity component, so the pieces to be assembled are indexed by the connected subgroups of G. This is especially simple when G is the circle because there are only two connected subgroups 1 and G. In this section we sketch the argument in this case, since it lets us introduce the techniques in a simple context. Full details will be given when we turn to the general case.

4.A. **Formal context.** Suppose given a commutative square of rings

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R^l \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ R^c & \xrightarrow[\delta]{} & R^t. \end{array}$$

To start with these can be genuine rings, but later we will want to consider squares of DGAs, ring spectra or ring G-spectra.

Delete R and consider the diagram

$$R^{\lrcorner} = \left(\begin{array}{ccc} & & R^l \\ & & \downarrow{\scriptstyle g} \\ R^c & \xrightarrow[d]{} & R^t \end{array} \right)$$

with three objects. We may form the category $R^{\lrcorner}$-mod of diagrams

$$\begin{array}{ccc} & & M^l \\ & & \downarrow{\scriptstyle g} \\ M^c & \xrightarrow[d]{} & M^t \end{array}$$

where M^l is an R^l-module, M^c is an R^c-module, M^t is an R^t-module and the maps g and d are module maps over the corresponding maps of rings.

Since $R^{\lrcorner}$ is a diagram of R-algebras, termwise tensor product gives a functor

$$R^{\lrcorner}\otimes_R : R\text{-mod} \longrightarrow R^{\lrcorner}\text{-mod}.$$

Similarly, since R maps to the pullback $PR^{\lrcorner}$, pullback gives a functor

$$P : R^{\lrcorner}\text{-mod} \longrightarrow R\text{-mod}.$$

It is easily verified that these give an adjoint pair

$$R^{\lrcorner}\otimes_R : R\text{-mod} \rightleftarrows R^{\lrcorner}\text{-mod} : P\ .$$

We may then consider the unit

$$\eta : M \longrightarrow P(R^{\lrcorner} \otimes_R M),$$

and the first condition for it to be a natural isomorphism is that it should be so when $M = R$, which is to say the original diagram of rings is a pullback. It is quite easy to identify sufficient conditions for η to be an isomorphism in general. First we require that the diagram is a pushout of modules, so that there is a long exact Tor sequence, and second that R^t is a flat R-module so that the sequence is actually short.

On the other hand, we cannot expect the counit of the adjunction to be an equivalence since we can add any module to M^t without changing $PM^{\lrcorner}$. Accordingly, we need to find a way to focus attention on diagrams arising from actual R-modules.

4.B. **Model structures.** We now suppose that the commutative diagram given above is a diagram of DGAs and use R-mod to denote the category of DG R-modules. We give it the algebraically projective model structure, with quasi-isomorphisms as weak equivalences and fibrations the surjections. The cofibrations are retracts of relative cell complexes, where the spheres are shifted copies of R. The diagram category $R^{\lrcorner}$-mod gets the diagram-injective model structure in which cofibrations and weak equivalences are maps which have this property objectwise; the fibrant objects have γ and δ surjective. This diagram-injective model structure is shown to exist over ring spectra in [35, Theorem 3.1], and the same proof works for DGAs.

Since extension of scalars is a left Quillen functor for the algebraically projective model structure for any map of DGAs, $R^{\lrcorner} \otimes_R -$ preserves objectwise cofibrations and weak equivalences and is therefore also a left Quillen functor to the diagram-injective model structure. We then apply the Cellularization Principle [33] to obtain the following result.

Proposition 4.1. *Assume given a commutative square of surjections (fibrations) of DGAs which is a pullback square. The adjunction induces a Quillen equivalence*

$$R\text{-}mod \stackrel{\simeq}{\longrightarrow} cell\text{-}R^{\lrcorner}\text{-}mod,$$

where cellularization is with respect to the image, $R^{\lrcorner}$, of the generating R-module R.

Proof: We apply the Cellularization Principle [33], which states that if we cellularize the model categories with respect to corresponding sets of small objects for which unit and counit are equivalences, we obtain a Quillen equivalence.

In the present case, we cellularize with respect to the single R-module R on the left, and the corresponding diagram $R^{\lrcorner}$ on the right. It is clear that R is a small R-module, and it is not hard to show that $R^{\lrcorner}$ is a small $R^{\lrcorner}$-module. Since the original diagram of rings is a pullback of a fibrant diagram (both as rings and as modules), the unit of the adjunction is an equivalence for R, and we see that the generator R and the generator $R^{\lrcorner}$ correspond under the equivalence, as required in the hypothesis in Part (2) of [33, Theorem 2.7].

Since R is cofibrant and generates R-mod, cellularization with respect to R has no effect on R-mod and we obtain the stated equivalence with the cellularization of $R^{\lrcorner}$-mod with respect to the diagram coming from R. □

4.C. **Rational G-spectra.** We can now outline the proof of our general theorem in the special case of the circle group $G = T$. Indeed, the diagram analogous to the starting diagram of rings is

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & \widetilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}. \end{array}$$

The square of G-spectra is a homotopy pullback square because the map between the fibres of the two horizontals is $S^0 \wedge E\mathcal{F}_+ \longrightarrow DE\mathcal{F}_+ \wedge E\mathcal{F}_+$, which is an equivalence since $E\mathcal{F}_+ \longrightarrow S^0$ is an $\mathcal{F}$-equivalence. We may replace this diagram by a suitably fibrant diagram and we write $\widetilde{R}^{\lrcorner}_{top}$ for the resulting diagram

$$\begin{array}{ccc} & & \widetilde{E}\mathcal{F} \\ & & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}. \end{array}$$

An $\widetilde{R}^{\lrcorner}_{top}$-module is a diagram

$$\begin{array}{ccc} & & M^l \\ & & \downarrow \\ M^c & \longrightarrow & M^t. \end{array}$$

where M^l is an $\widetilde{E}\mathcal{F}$-module, M^c is a $DE\mathcal{F}_+$-module, M^t is a $DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}$-module, and $M^c \longrightarrow M^t$ and $M^l \longrightarrow M^t$ are maps over the corresponding rings.

The existence of a diagram-injective type model structure on this category of modules is established in [35, Theorem 3.1]. The discussion proceeds exactly as in the algebraic case, but when we cellularize we must use a generating set, so we use the cells T/H_+ as H runs through all closed subgroups of T.

Proposition 4.2. *The adjunction induces a Quillen equivalence*

$$T\text{-}spectra \stackrel{\simeq}{\longrightarrow} cell\text{-}\widetilde{R}^{\lrcorner}_{top}\text{-}mod\text{-}G\text{-}spectra.$$

Proof: The proof precisely follows the algebraic case (Proposition 4.1). To see that the unit is an equivalence for all cells G/H_+ (and not just for $S^0 = G/G_+$), we observe that smashing with G/H_+ preserves homotopy pullback squares. □

This sets the scene for the rest of the argument, which we sketch only very briefly here. It proceeds by observing that the categories of $\widetilde{E}\mathcal{F}$-modules, $DE\mathcal{F}_+$-modules and $DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}$-modules are each easy to understand; in fact they are formal in the sense that they are equivalent to the category of modules over their homotopy rings. This formality is also true of the diagram as a whole, so that the category of modules over $\widetilde{R}^{\lrcorner}_{top}$-modules in G-spectra is equivalent to the category of $\pi^G_*(\widetilde{R}^{\lrcorner}_{top})$-modules in $\mathbb{Q}$-modules. It then remains to show that $\mathcal{A}(G)$ gives an economical model of the cellularization of the category of $\pi^G_*(\widetilde{R}^{\lrcorner}_{top})$-modules.

Part 2. Formality of the sphere spectrum

5. Diagrams of rings and modules

Throughout this paper we consider categories of modules over diagrams of rings in two contexts: differential graded modules over DGAs and module spectra over ring spectra. In this section we describe the context and the basic Quillen equivalences arising from a pullback diagram of rings. These and related results are discussed more fully and proved in [35].

5.A. **The archetype.** Given a diagram shape $\mathbf{D}$, consider a diagram of rings $R : \mathbf{D} \longrightarrow \mathbb{C}$ in a symmetric monoidal category $\mathbb{C}$. Each map $R(a) : R(s) \longrightarrow R(t)$ gives rise to an extension of scalars functor

$$R(s)\text{-mod} \xrightarrow{a_*} R(t)\text{-mod}$$

defined by $a_*(X) = R(t) \otimes_{R(s)} X$, with right adjoint the restriction of scalars functor

$$R(s)\text{-mod} \xleftarrow{a^*} R(t)\text{-mod}.$$

Now consider a category of *R-modules*; the objects are diagrams $X : \mathbf{D} \longrightarrow \mathbb{C}$ for which $X(s)$ is an $R(s)$-module for each object s, and for every morphism $a : s \longrightarrow t$ in $\mathbf{D}$, the map $X(a) : X(s) \longrightarrow X(t)$ is a *module map over the ring map* $R(a) : R(s) \longrightarrow R(t)$. More precisely, there is a map $X(s) \longrightarrow a^*X(t)$ of $R(s)$-modules (the *restriction* form) or, equivalently, there is a map

$$R(t) \otimes_{R(s)} X(s) = a_*X(s) \longrightarrow X(t)$$

of modules over the ring $R(t)$ (the *extension of scalars* form). Although restriction of scalars seems very simple, we view the left adjoint a_* as the primary one, following the convention that the left Quillen functor dicates the direction of a Quillen pair.

5.B. **Model structures.** We say that $\mathbb{M}$ is *a diagram of model categories* if each category $\mathbb{M}(s)$ has a model structure, the functors a_* all have right adjoints and the adjoint pair $a_* \dashv a^*$ of functors relating the model categories form a Quillen pair.

For instance, the motivating example of a diagram of ring spectra (or DGAs) gives a diagram of model categories if we use the projective model structure on the category $\mathbb{M}(s)$ of $R(s)$-modules.

When $\mathbb{M}$ is a diagram of model categories, there are two ways to attempt to put a model structure on the category of $\mathbb{M}$-diagrams $\{X(s)\}_{s\in\mathbf{D}}$. The *diagram-projective* model structure (if it exists) has its fibrations and weak equivalences defined objectwise. The *diagram-injective* model structure (if it exists) has its cofibrations and weak equivalences defined objectwise. It must be checked in each particular case whether or not these specifications determine a model structure. When both model structures exist, it is clear that the identity functors define a Quillen equivalence between them.

We will apply [35, Theorem 3.1] to show that the diagram-projective and diagram-injective model structures exist in the cases of interest to us.

5.C. **Pullback diagrams of rings.** The basic input from the diagrams of model categories from [35] is as follows.

Proposition 5.1. [35, Proposition 4.1] *For $\mathbf{D}$ a finite, inverse category with at most one morphism in each $\mathbf{D}(s,t)$ and R a $\mathbf{D}$-diagram of ring spectra with homotopy inverse limit $\overline{R}$, there is a zig-zag of Quillen equivalences between the category of $\overline{R}$-modules and the cellularization with respect to R of R-modules (with the diagram-injective model structure):*

$$\overline{R}\text{-}mod \simeq_Q R\text{-}cell\text{-}R\text{-}mod$$

We will be applying this when $\mathbf{D}$ is a punctured cube, and $\overline{R} = \mathbb{S}$ is the sphere spectrum. Indeed the category of G-spectra is equivalent to the category of module-G-spectra over the sphere spectrum $\mathbb{S}$. By Proposition 5.1, this is in turn equivalent to the cellularization of a category of modules over a diagram of ring G-spectra. The rest of the work will be based on diagrams of this punctured cube shape. The argument proceeds by replacing the diagram of ring G-spectra successively by diagrams of nonequivariant ring spectra, DGAs and finally graded commutative rings.

Our next task is the core of the paper. We show that the sphere is the pullback of a diagram of spectra which are both isotropically simpler and very rigid.

6. The sphere as an isotropic pullback

Our analysis is based on expressing the sphere spectrum as the homotopy pullback of an $(r+1)$-cube of ring G-spectra as we described for the circle in Subsection 4.C. More precisely, we will construct a diagram $\widetilde{R}_{top} : C \longrightarrow$ Ring-G-spectra where the cube C is the poset of subsets of $\{0, \dots, r\}$, where $\mathbb{S}$ is the value on the initial vertex (the empty set) and so that this is equivalent to the homotopy pullback of the restriction of $\widetilde{R}_{top}$ to the punctured cube PC of non-empty subsets.

6.A. **Strategy.** In the course of the proof, we will need to consider an extension of $\widetilde{R}_{top}$ to a bigger diagram, and we introduce this extended diagram as we go along. The cube C above will appear as $C = C_f$ in due course. The letter f stands for 'formal' though the word 'affine' or the word 'rigid' would be sensible alternatives. Corollary 7.2 will show that the sphere spectrum is the homotopy pullback of $\widetilde{R}_{top}$ restricted to the punctured cube PC_f, so that the results of [35] (as quoted in Proposition 5.1) show that the category of modules over the sphere spectrum is equivalent to the cellularization of the category of modules over the PC_f-diagram of ring G-spectra. The reason this is useful is that the ring G-spectra A at the vertices of the punctured cube PC_f have two very special rigidity properties. Firstly, passage to G-fixed points as in [34] gives an equivalence between categories of A-module-G-spectra and categories of A^G-module-spectra. This means we can reduce from considering $\widetilde{R}_{top}$-modules in G-spectra to considering modules over the PC_f-diagram $R_{top} = (\widetilde{R}_{top})^G$ of non-equivariant ring spectra. We can then use the second author's results to move to considering modules over a PC_f-diagram R_t of DGAs. The second feature of the spectra A is that $\pi^G_*(A) = \pi_*(A^G)$ is intrinsically formal in that any commutative DGA with this homology is equivalent to $\pi^G_*(A)$ with zero differential. As shown in Section 10, the proof of this is compatible with the PC_f-diagram, so we are reduced to considering DG-modules over a PC_f-diagram R_a of graded rings. We may then show the cellularized category of R_a-modules is equivalent to the category $\mathcal{A}(G)$ of [24].

Our first task (Sections 6 and 7) is to describe the $(r+1)$-cube C_f of ring spectra with the sphere spectrum at the initial vertex and to show it is a homotopy pullback. We will do this in steps: we identify C_f inside a larger diagram C_{if} containing a second cube C_i, giving inclusions of diagrams

$$C_i \subseteq C_{if} \supseteq C_f,$$

and we will prove equivalences

$$S^0 \overset{1}{\simeq} \underset{\leftarrow \; v \in PC_i}{\operatorname{holim}} \widetilde{R}_{top}(v) \overset{2}{\simeq} \underset{\leftarrow \; v \in PC_{if}}{\operatorname{holim}} \widetilde{R}_{top}(v) \overset{3}{\simeq} \underset{\leftarrow \; v \in PC_f}{\operatorname{holim}} \widetilde{R}_{top}(v).$$

Of these, Equivalence 2 is elementary, since PC_i is cofinal in PC_{if}. Equivalence 1 (Proposition 6.6) is the essential one, since in fact Equivalence 3 (Proposition 7.1) is essentially given by using Equivalence 1 for quotient groups of lower rank.

For this reason we will begin with the cube C_i of ring spectra constructed purely on isotropical principles, and Equivalence 1. Since the ring G-spectra at the vertices of the punctured cube PC_i do not have the rigidity properties we need, we will then take the further step of reducing to the diagram on the punctured cube PC_f.

For the rest of Sections 6 and 7 we simpify notation and write $\tilde{R} = \widetilde{R}_{top}$.

6.B. **The isotropic cube.** We consider the coordinates $(a_0, a_1, \ldots, a_r)$ where each coordinate a_i can take the value 0 or 1. For $0 \leq c \leq r-1$ the cth coordinate refers to subgroups of codimension c. The rth coordinate also refers to codimension r (i.e., to finite subgroups), but these must be treated differently, and in effect it refers to whether or not the ring is complete (roughly speaking, whether it is S^0 or $DE\mathcal{F}_+$).

To a first approximation, the idea is that the cube is obtained by smashing together $r+1$ maps of rings, with $S^0 = A_i(0) \longrightarrow A_i(1)$ in the ith position, so that $\tilde{R}(a_0, \ldots, a_r) = \bigwedge_{i=0}^r A_i(a_i)$. However we need to refine this, so as to assemble information from individual subgroups, and reflect containments of subgroups.

The simplest coordinates are the 0th and rth, where we have $A_0(1) = S^{\infty V(G)}$ and $A_r(1) = DE\mathcal{F}_+$. In the rank 1 case, this is everything, so we obtain the usual diagram

$$\begin{array}{ccc} S^0 \wedge S^0 & \longrightarrow & S^0 \wedge S^{\infty V(G)} \\ \downarrow & & \downarrow \\ DE\mathcal{F}_+ \wedge S^0 & \longrightarrow & DE\mathcal{F}_+ \wedge S^{\infty V(G)}. \end{array}$$

Supposing $r \geq 2$ we now move on to the other coordinates. For $1 \leq i \leq r-1$, for each connected subgroup H, we take

$$A^K(1) = S^{\infty V(K)},$$

and then

$$A_i(1, \subset H) = \prod_{\mathrm{codim}(K)=i, K\subset H} A^K(1) = \prod_{\mathrm{codim}(H)=i, K\subset H} S^{\infty V(K)}.$$

The formula for codimensions 0 and r fits the same pattern, although there is only one term in the product and containment imposes no restrictions. To make the formulae typographically manageable we need to introduce some more notation. Indeed, in the ith spot we need to have index sets $I(i,0)$ and $I(i,1)$ for certain products. The index set $I(i,0)$ is a singleton and

$$I(i,1) = \{H \mid H \text{ is connected and } \mathrm{codim}(H) = i\}. \tag{6.1}$$

Now we can define the ring spectrum to be placed at the $(a_r, \ldots, a_0)$ vertex.

$$\begin{aligned} \tilde{R}(a_0, \ldots, a_r) = A_0(a_0) \wedge \prod_{H_1 \in I(1,a_1)} \big[A^{H_1}(a_1) \wedge \prod_{H_2 \in I(2,a_2), H_2 \subset H_{<2}} \big[A^{H_2}(a_2) \wedge \cdots \\ \cdots \wedge \prod_{H_{r-1} \in I(r-1, a_{r-1}), H_{r-1} \subset H_{<r-1}} \big[A^{H_{r-1}}(a_{r-1}) \wedge A_r(a_r)\big] \cdots \big]\big] \end{aligned}$$

Remark 6.1. (a) To help parse this, note that in the sth term we have S^0 if $a_s = 0$ and otherwise it is the product of copies of $S^{\infty V(H)}$ as H runs through codimension s subgroups contained in the earlier subgroups (the notation $H_s \subset H_{<s}$ allows for the fact that only terms with $a_t = 1$ correspond to actual subgroups).

(b) The convention is that the products include everything to the right of them so the ordering of the vertices is important. From now on, we will often omit parentheses, relying on this convention.

(c) This notation shows all structure maps clearly, but the formula is easier to digest if we pick out just those indices with $a_i \neq 0$, say $i_{c_0} < i_{c_1} < \ldots < i_{c_s}$. In this case if $c_s < r$ we have

$$\tilde{R}(a_0, \ldots, a_r) = \prod_{\mathrm{codim} H_0 = c_0} \big[S^{\infty V(H_0)} \wedge \prod_{\mathrm{codim} H_1 = c_1, H_1 \subset H_0} \big[S^{\infty V(H_1)} \wedge \prod_{\mathrm{codim} H_2 = c_2, H_2 \subset H_1} \big[S^{\infty V(H_2)} \wedge \cdots \wedge \prod_{\mathrm{codim} H_s = c_s, H_s \subset H_{s-1}} \big[S^{\infty V(H_s)} \big] \cdots \big]\big]\big]$$

and if $c_s = r$ we have

$$\tilde{R}(a_0, \ldots, a_r) = \prod_{\mathrm{codim} H_0 = c_0} \big[S^{\infty V(H_0)} \wedge \prod_{\mathrm{codim} H_1 = c_1, H_1 \subset H_0} \big[S^{\infty V(H_1)} \wedge \prod_{\mathrm{codim} H_2 = c_2, H_2 \subset H_1} \big[S^{\infty V(H_2)} \wedge \cdots \wedge \prod_{\mathrm{codim} H_{s-1} = c_{s-1}, H_s \subset H_{s-2}} \big[S^{\infty V(H_{s-1})} \wedge DE\mathcal{F}_+ \big] \cdots \big]\big]\big]$$

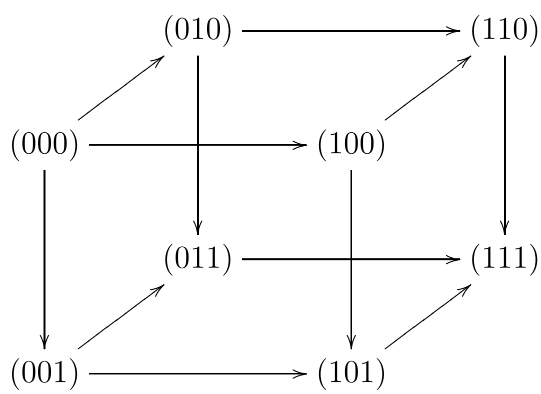
The notation somewhat obscures the simplicity of this construction. Thus in rank 2, we have

$$R(a_0, 0, a_2) = A_0(a_0) \wedge S^0 \wedge A_2(a_2)$$

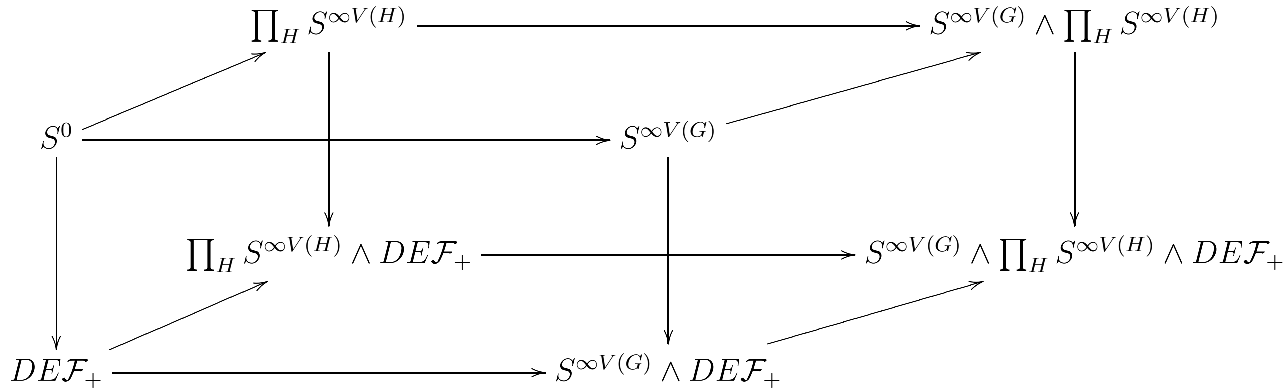
and

$$R(a_0, 1, a_2) = A_0(a_0) \wedge \prod_{\dim(H) = 1} \big[A^H(1) \wedge A_2(a_2) \big].$$

It is worth writing the diagram completely in this case. The layout is



and the diagram of ring spectra is as follows:



One r dimensional face will play a preferred role in our proof that this cube is a homotopy pullback, so we give a special name to the $a_0 = 0$ face (the left hand face in the above illustration). The r-cube diagram R' is defined by

$$R'(a_1, \ldots, a_r) = R(0, a_1, \ldots, a_r).$$

We note that

$$R = (S^0 \longrightarrow S^{\infty V(G)}) \wedge R'.$$

This notation will be even more convenient when we refine the filtration $S^0 \longrightarrow S^{\infty V(G)}$.

6.C. **Observations about isotropy.** It is natural to consider a filtration of all subgroups by dimension, so we let

$$\mathcal{F}^{\leq i} = \{H \mid \dim(H) \leq i\} \text{ and } \mathcal{C}_{\geq i} = \{H \mid \dim(H) \geq i\}$$

The first is a family and the second is a cofamily. We also need to consider the subgroups above and below a fixed group K:

$$\Lambda(K) = \{H \mid H \subseteq K\} \text{ and } V(K) = \{H \mid K \subseteq H\}.$$

Again, the first is a family and the second is a cofamily.

The point is that the category of spectra with geometric isotropy in the cofamiliy $V(K)$ of subgroups (the spectra "over K") is equivalent to the category of G/K-spectra. To obtain good inductive arguments we want to express the naturally occurring sets of isotropy in terms of those of the form $V(K)$.

We are burdened with a standard notation in which the geometric isotropy is given by $\mathcal{GI}(\widetilde{E}\mathcal{F}) = All \setminus \mathcal{F}$, so we adopt the convention that for any cofamily $\mathcal{C}$

$$X\mathcal{C} := X \wedge \widetilde{E}(All \setminus \mathcal{C}),$$

so that

$$\mathcal{GI}(X\mathcal{C}) = \mathcal{GI}(X) \cap \mathcal{C}$$

and in particular

$$\mathbb{S}\mathcal{C} = \widetilde{E}(All \setminus \mathcal{C}), \text{ giving } \quad \mathcal{GI}(\mathbb{S}\mathcal{C}) = \mathcal{C}.$$

This notation extends naturally to families, and indeed to any collection of subgroups which can be expressed as an intersection between a family and a cofamily.

We abbreviate further, taking

$$\mathbb{S}_{\geq i} = \mathbb{S}\mathcal{C}_{\geq i}$$

and consider the filtration

$$\mathbb{S} = \mathbb{S}_{\geq 0} \longrightarrow \mathbb{S}_{\geq 1} \longrightarrow \mathbb{S}_{\geq 2} \longrightarrow \cdots \longrightarrow \mathbb{S}_{\geq r} = S^{\infty V(G)}.$$

More precisely we realise this filtration in the category of commutative ring spectra by a process of localization; this is possible by Axiom 3.3 (8)

Lemma 6.2. *For any map $f : X \longrightarrow Y$ which is an $\mathcal{F}^{\leq i}$-equivalence, there is a homotopy pullback square*

$$\begin{array}{ccc}
\mathbb{S}_{\geq i} \wedge X & \longrightarrow & \mathbb{S}_{\geq i+1} \wedge X \\
\downarrow & & \downarrow \\
\mathbb{S}_{\geq i} \wedge Y & \longrightarrow & \mathbb{S}_{\geq i+1} \wedge Y.
\end{array}$$

Proof: The space $\mathbb{S}_{\geq i+1}/\mathbb{S}_{\geq i}$ has geometric isotropy concentrated on subgroups H of dimension exactly i. This means it can be built from cells G/K_+ where K has dimension $\leq i$. □

We will apply this to a large number of slightly different maps, but it is worth highlighting one which embodies the philosophy.

Corollary 6.3. *For any spectrum X, there is a homotopy pullback square*

$$\begin{array}{ccc}
\mathbb{S}_{\geq i} \wedge X & \longrightarrow & \mathbb{S}_{\geq i+1} \wedge X \\
\downarrow & & \downarrow \\
\mathbb{S}_{\geq i} \wedge \prod_{\dim(H)=i} S^{\infty V(H)} \wedge X & \longrightarrow & \mathbb{S}_{\geq i+1} \wedge \prod_{\dim(H)=i} S^{\infty V(H)} \wedge X
\end{array}$$

Remark 6.4. (i) The bottom left hand entry is equivalent to $\prod_{\dim(H)=i} S^{\infty V(H)} \wedge X$ since all terms are $\mathcal{F}^{\leq i-1}$-contractible.

(ii) The essence of the corollary is that we can start with $\mathbb{S}_{\geq r} \wedge X = S^{\infty V(G)} \wedge X$, and build $X = \mathbb{S}_{\geq 0} \wedge X$ in steps. At each stage $\mathbb{S}_{\geq i} \wedge X$ can be constructed as a homotopy pullback from $\mathbb{S}_{\geq i+1} \wedge X$ by using only spectra of the form $S^{\infty V(H)} \wedge X$ for subgroups H of dimension i.

Since the category of module G-spectra over $S^{\infty V(H)}$ is equivalent to the category of G/H-spectra, this establishes an inductive scheme.

Proof of Proposition 6.3: If K is of dimension less than i then all terms are K-contractible. If K is of dimension i, there is precisely one factor in the product which is not K-contractible, and $X \longrightarrow S^{\infty V(K)} \wedge X$ is a K-equivalence. □

The variant that we will apply is obtained by adapting a special case of this corollary.

Corollary 6.5. *For $X = \tilde{R}(a_0, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_r)$ and $Y = \tilde{R}(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_r)$, there is a homotopy pullback square*

$$\begin{array}{ccc}
\mathbb{S}_{\geq i} \wedge X & \longrightarrow & \mathbb{S}_{\geq i+1} \wedge X \\
\downarrow & & \downarrow \\
\mathbb{S}_{\geq i} \wedge Y & \longrightarrow & \mathbb{S}_{\geq i+1} \wedge Y
\end{array}$$

Proof: First note that if $a_s \neq 0$ for some $s < i$, this is immediate, since if H is of dimension $s > i$ then $S^{\infty V(H)}$ is $\mathcal{F}^{\leq i}$-contractible.

Now consider the case $a_0 = \cdots = a_{i-1} = 0$. This is very close to the special case of Corollary 6.3 in which

$$X = \prod_{H_{i+1} \in I(i+1, a_{i+1})} \big[A^{H_{i+1}}(a_{i+1}) \wedge \prod_{H_{i+2} \in I(i+2, a_{i+2}), H_{i+2} \subset H_{<i+2}} \big[A^{H_{i+2}}(a_{i+2}) \wedge \cdots$$
$$\cdots \wedge \prod_{H_{r-1} \in I(r-1, a_{r-1}), H_{r-1} \subset H_{<r-1}} \big[A_{r-1}^{H_{r-1}}(a_{r-1}) \wedge A_r(a_r)\big] \cdots\big]\big]$$

This is the value of X in the present corollary. The main difference is that instead of taking Y to be given as $\prod_H S^{\infty V(H)} \wedge X$, the products in the Hth factor are now restricted to subgroups of H.

To see that this does not alter the fact that we have a pullback square, we need only observe that the omitted factors in the products in the Hth factor are $\mathcal{F}/H$-contractible. Indeed, if K is a connected subgroup with $K \not\subseteq H$, and $\tilde{H}$ has identity subgroup H then $K \not\subseteq \tilde{H}$ and so $S^{\infty V(K)}$ is $\tilde{H}$-contractible. □

6.D. **The isotropic cube is a homotopy pullback.** We are ready to prove that the isotropic cube is a homotopy pullback.

Proposition 6.6. *The C_i-diagram $\tilde{R}$ is a homotopy pullback, which is to say that the sphere spectrum $\mathbb{S}$ is the homotopy pullback of R restricted to the punctured cube PC_i:*

$$\mathbb{S} \simeq \underset{\leftarrow}{\operatorname{holim}}_{v \in PC_i} \tilde{R}(v).$$

Remark 6.7. The corresponding statement is also true for the diagram in which the products in the definition of the ring spectrum are over *all* $S^{\infty V(H)}$ with H of a fixed codimension (the proof is the same, except that one applies Corollary 6.3 instead of Corollary 6.5). The reason for restricting to products over decreasing flags is to obtain an algebraically tractable result.

Proof of Proposition 6.6: Some readers may find it helpful to refer to the case of Rank 2 made explicit in Subsection 6.E whilst reading this proof.

The method is to use a succession of intermediate homotopy pullbacks inside the cube. We place the terms of the intermediate homotopy pullbacks along the a_0 edges of PC_i. It is helpful to describe first the basic filtration we are using.

The general reconstruction process works by enlarging the diagram to permit the 0th coordinate to run through the entire filtration

$$\mathbb{S} = \mathbb{S}_{\geq 0} \longrightarrow \mathbb{S}_{\geq 1} \longrightarrow \mathbb{S}_{\geq 2} \longrightarrow \cdots \longrightarrow \mathbb{S}_{\geq r} = S^{\infty V(G)}.$$

We do this by letting a_0 take on the fractional values $0 = 0/r, 1/r, \ldots, r/r = 1$ and take

$$\tilde{R}(i/r, a_1, \ldots, a_r) = \mathbb{S}_{\geq i} \wedge \tilde{R}'(a_1, \ldots, a_r);$$

For brevity we write

$$\tilde{R}'(i/r) = \mathbb{S}_{\geq i} \wedge \tilde{R}'$$

for these r-cube diagrams.

The idea is to imagine filling in the values of the diagram from scratch. To start with, we are given the values at PC_i (this includes all entries with $a_0 = 1 = r/r$). We then show successively for $a_0 = (r-1)/r, (r-2)/r, \ldots, 1/r, 0/r = 0$ that the entries in the diagrams of ring spectra $\tilde{R}'(a_0)$ can be filled in (using only homotopy equivalences and homotopy pullbacks) from values already filled in. The only value of real importance is $\mathbb{S} = \mathbb{S}_{\geq 0} = \tilde{R}(0, \ldots, 0)$, but it is easier to describe a uniform procedure which fills in other entries on the way.

At the start, we are given the ring spectra $\tilde{R}(a_0, \ldots, a_r)$ for vertices of PC_i. This means all vertices with $a_r \in \{0, 1\}$ and not all entries a_i zero. We observe first that the entries at many other points are equivalent to these.

Lemma 6.8. *Provided $a_j = 1$ for some $j \leq i$ we have an equivalence*

$$\mathbb{S}_{\geq i} \wedge \tilde{R}'(a_1, \ldots, a_r) \simeq \tilde{R}'(a_1, \ldots, a_r).$$

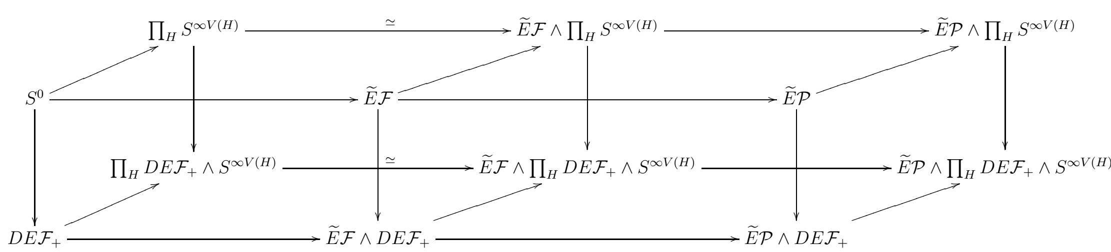
Proof: The mapping cone of the comparison map is $E\mathcal{F}_+^{\leq i-1} \wedge \tilde{R}'(a_1, \ldots, a_r)$. If $\dim(H) = j$ then $S^{\infty V(j)}$ is $\mathcal{F}^{\leq i-1}$-contractible so the mapping cone is contractible. □

Now suppose that that the entries of $\tilde{R}'((i+1)/r)$ are filled in. To fill in the entries of $\tilde{R}'(i/r)$ with $a_i = 1$ we use Lemma 6.8, and for the points with $a_i = 0$ we apply Corollary 6.5, with $X = \tilde{R}'(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_r)$ and $Y = \tilde{R}'(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_r)$.

□

6.E. **The case of rank 2.** The above inductive scheme is sufficiently complicated that it seems worth making one case explicit.

Consider the following diagram.



We have used traditional names $S^0 = \mathbb{S}_{\geq 0}$, $\widetilde{E}\mathcal{F} = \mathbb{S}_{\geq 1}$ and $\widetilde{E}\mathcal{P} = \mathbb{S}_{\geq 2}$, where $\mathcal{F}$ is the family of finite subgroups and $\mathcal{P}$ is the family of proper subgroups. The zeroth coordinate is horizontal (left to right on the printed page), the first coordinate is into the paper (diagonally on the printed page) and the rth coordinate is vertical (downwards on the printed page). The left hand square is $\tilde{R}'(0/2)$, the central square is $\tilde{R}'(1/2)$ and the right hand square is $\tilde{R}'(2/2)$.

Thus the left and right hand end (except for S^0) are in the PC_i-diagram of which we want to identify the homotopy limit. The back central entries can be filled in by the equivalences illustrated on the two left hand horizontals without affecting the homotopy limit. Now the top and bottom faces of the right hand cube are homotopy pullbacks. This means that we can fill in the two central entries on the front face without affecting the whole homotopy limit. Finally the front face of the left hand cube is a homotopy pullback, so that S^0 is the homotopy limit of the original PC_i-diagram.

7. The sphere as a formal pullback

We now move towards introducing the formal cube. As described above, we will define this by extending C_i to a larger diagram C_{if} and then finding C_f inside it. We briefly explain the motivation.

The C_i-diagram does not do what we require, since the terms $S^{\infty V(K)}$ are not formal unless $K = G$. However a strategy is already apparent from our work on the isotropic cube in lower ranks. To see the idea, we may imagine that we have already completed the proof for lower ranks, and constructed the G/K-sphere S^0 from formal ring G/K-spectra B. Accordingly, we can construct $S^{\infty V(K)} = S^{\infty V(K)} \wedge S^0$ from formal G-spectra $S^{\infty V(K)} \wedge B$. Most of the spectra B that occur are products of those of the form $S^{\infty V(H/K)}$ (and since $S^{\infty V(K)} \wedge S^{\infty V(H/K)} \simeq S^{\infty V(H)}$ these correspond to ring spectra in our diagram) which have already been constructed in lower rank. There is only one other spectrum B, namely $DE\mathcal{F}/K_+$, and it is the most important one. This outlines why the sphere can be constructed from spectra of the form $S^{\infty V(K)} \wedge DE\mathcal{F}/K_+$, and we will give a detailed proof below.

We will extend the C_i-cube to a larger poset C_{if} also containing the formal cube C_f, and we will extend $\tilde{R}$ to C_{if}. Now the $a_r = 1$ face of the C_i cube is the $a_r = 1$ cube of C_f and $\tilde{R}$ already takes formal values on that face. The values of $\tilde{R}$ on the $a_r = 0$ face of C_i are not formal, and for each point we give a new value at the corresponding point of C_f. The new formal ring is obtained by identifying the smallest codimension c for which $S^{\infty V(K)}$ (rather than S^0) occurs with $\mathrm{codim}(K) = c$ and then smashes with $DE\mathcal{F}/K_+$.

We flesh out this sketch in the course of the next few subsections, starting by describing the larger diagram C_{if} and then identify C_f inside it.

7.A. **A subdivision of the isotropic cube.** The diagram C_{if} is obtained from C_i by inserting new layers in the a_r direction.

Altogether we have $r + 1$ layers placed at $a_r = i/r$ for $i = 0, 1, \ldots, r$, so that the $a_r = 0$ and $a_r = 1$ layers are just as before.

We will be using maps to relate the various ring spectra $DE\mathcal{F}/K_+$ as K varies. Indeed, $DE\mathcal{F}/K_+$ is a commutative ring G/K-spectrum by Axiom 3.3 (9) and if $L \subseteq K$ there is a map

$$\mathrm{inf}_{G/K}^{G/L} DE\mathcal{F}/K_+ \longrightarrow DE\mathcal{F}/L_+$$

of ring G/L-spectra. To see where this comes from, we observe that its adjunct

$$E\mathcal{F}/L_+ \wedge \mathrm{inf}_{G/K}^{G/L} DE\mathcal{F}/K_+ \longrightarrow S^0$$

is obtained by composing the G/L-map $E\mathcal{F}/L_+ \longrightarrow E\mathcal{F}/K_+$ with evaluation.

If we have any decreasing sequence

$$G = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{r-1} \supseteq H_r = 1$$

of connected subgroups with $\mathrm{codim}(H_i) = i$, then, omitting notation for inflation, we have a sequence of maps of ring G-spectra

$$S^0 = DE(\mathcal{F}/G)_+ \longrightarrow D(E\mathcal{F}/H_1)_+ \longrightarrow \ldots \longrightarrow D(E\mathcal{F}/H_{r-1})_+ \longrightarrow D(E\mathcal{F}/1)_+ = DE\mathcal{F}_+$$

To define the C_{if} diagram of rings we use the same formula as before except that the range of values of a_r is extended to the fractional values and the rth entry becomes dependent on other coordinates. More briefly, $A_r(a_r)$ is replaced by $A_r^{i_1,\ldots,i_r}(a_0,\ldots,a_r)$. Thus, with $I(i,0)$ a singleton and $I(i,1)=\{H \mid H \text{ is connected and } \mathrm{codim}(H)=i\}$ as before, we define the ring G-spectrum to be placed at the $(a_0,\ldots,a_r)$ vertex:

$$\tilde{R}(a_0,\ldots,a_r) = A_0(a_0) \wedge \prod_{H_1\in I(1,a_1)} \Big[A^{H_1}(a_1) \wedge \prod_{H_2\in I(2,a_2),H_2\subset H_{<2}} \Big[A^{H_2}(a_2) \wedge \cdots$$
$$\cdots \wedge \prod_{H_{r-1}\in I(r-1,a_{r-1}),H_{r-1}\subset H_{<r-1}} \Big[A^{H_{r-1}}(a_{r-1},) \wedge A_r^{i_0,\ldots,i_r}(a_0,\cdots,a_r)\Big]\cdots\Big]\Big]$$

For the last term, we take

$$A_r^{i_1,\ldots,i_r}(a_0,\cdots,a_r) = \mathrm{inf}_{G/H}^{G} DE\mathcal{F}/H_+$$

where the subgroup $H=H(i_1,\ldots,i_r;a_0,\ldots,a_r)$ is determined as follows. When $a_r=s/r$, we consider the sequence $a_0,\ldots,a_s$; if it is zero we take $H=H_{i_0}=G$, and otherwise we find the last nonzero term a_t and take the codimension t subgroup H_t:

$$H(i_1,\ldots,i_r;a_0,\ldots,a_{r-1},s/r) := H_{lnz(a_0,\ldots,a_s)}.$$

where

$$lnz(a_0,\ldots,a_s) = \max(\{t \mid a_t \neq 0\} \cup \{0\}).$$

Note that since $lnz(a_0,\ldots,a_s) \leq lnz(a_0,\ldots,a_s,a_{s+1})$ we have an inclusion

$$H(i_0,\ldots,i_{r-1};a_0,\ldots,a_{r-1},s/r) \supseteq H(i_0,\ldots,i_{r-1};a_0,\ldots,a_{r-1},(s+1)/r)$$

so that we do have the appropriate comparison maps.

The diagram C_{if} is not a cube, so we should state explicitly that the punctured diagram PC_{if} is obtained by omitting the r points $(0,\ldots,0,a_r)$ with $a_r \neq 1$, which are the points where $\tilde{R}$ takes the value $\mathbb{S}$.

7.B. **Selecting the formal cube.** The formal cube C_f consists of the $a_r=1$ face together with an opposite face that we need to describe. First, the initial vertex is the point $(0,\ldots,0)$. Next, the point in the opposite face corresponding to a non-zero $(a_0,\ldots,a_{r-1})$ can be found by looking for the least value of a_r for which the entry at $(a_0,\ldots,a_{r-1},a_r)$ is formal. The formal entries in the diagram are those with a term $DE\mathcal{F}/K_+$ for some K, where we take this to include the terms $DE\mathcal{F}/G_+=S^0$ when $a_0=1$. Thus the least value of a_r with a formal entry is $a_r=lnz(a_0,\ldots,a_{r-1})/r$. Continuing with the convention that $lnz(0,\ldots,0)=0$,

$$C_f = \{(a_0,\ldots,a_r) \mid a_r = 1 \text{ or } a_r = \frac{lnz(a_0,\ldots,a_{r-1})}{r}\}.$$

As a poset, these vertices form a cube. To see this, we identify the vertex $(a_0,\ldots,a_r)$ of C_f with the subset

$$S(a_0,\ldots,a_r) = \{i \mid a_i = 1\}.$$

To see that the morphisms correspond to containment of subsets (so that C_f is a cube) we note that if $(a_0,\ldots,a_{r-1})$ and $(b_0,\ldots,b_{r-1})$ differ only by changing some entries $a_i=0$ to $b_i=1$ (so $S(a)\subseteq S(b)$) then $a_r := lnz(a_0,\ldots,a_{r-1}) \leq lnz(b_0,\ldots,b_{r-1}) =: b_r$, so that there is a path from $(a_0,\ldots,a_{r-1},a_r)$ to $(b_0,\ldots,b_{r-1},b_r)$ in C_{if}.

Proposition 7.1. *The inclusion $C_f \subseteq C_{if}$ induces an equivalence*

$$\underset{\leftarrow}{\text{holim}}_{v \in PC_f} \tilde{R}(v) \simeq \underset{\leftarrow}{\text{holim}}_{v \in PC_{if}} \tilde{R}(v).$$

Before proving this we note that in view of Proposition 6.6 and the fact that PC_i is cofinal in PC_{if} we immediately see that $\mathbb{S}$ is the pullback of the formal ring spectra.

Corollary 7.2. *The C_f-diagram $\tilde{R}$ is a homotopy pullback, which is to say that $\mathbb{S}$ is the homotopy pullback of the PC_f-diagram $\tilde{R}$:*

$$\mathbb{S} \simeq \underset{\leftarrow}{\text{holim}}_{v \in PC_f} \tilde{R}(v). \quad \square$$

It then follows from Proposition 5.1 that we have the desired Quillen equivalence. For this statement we revert to the full notation $\widetilde{R}_{top} = \tilde{R}$.

Corollary 7.3. *There is a Quillen equivalence between equivariant G-spectra, modelled by the category of $\mathbb{S}$-modules, and the cellularization of the diagram-injective model structure on $\widetilde{R}_{top}$-modules.*

$$G\text{-}spectra \simeq cell\text{-}\widetilde{R}_{top}\text{-}mod\text{-}G\text{-}spectra$$

It remains to give the proof comparing the limits over PC_f and PC_{if}.

Proof of Proposition 7.1: Some readers may find it helpful to refer to the case of Rank 2 made explicit in Subsection 7.C whilst reading this proof.

We will work in the diagram C_{if} (i.e., permitting $a_r \in \{0/r, 1/r, \ldots, r/r\}$). In Subsection 7.A we defined $\tilde{R}(v)$ for all vertices v. The proof here consists of showing how we could recover all of them from the entries in PC_f alone, using homotopy pullbacks. This will show in particular that the entry $\tilde{R}((0, \ldots, 0)) = \mathbb{S}$ at the initial vertex is the homotopy pullback of the PC_f-diagram $\tilde{R}$.

We view this as starting with an empty slate, adding the entries at points of PC_f and steadily filling in the values at different vertices by using homotopy pullbacks of entries filled in previously.

First, we fill in all the points of PC_{if} which admit a map from an entry of PC_f; this does not change the homotopy pullback, since PC_f remains cofinal. For example, since $(1, 0, \ldots, 0)$ is in PC_f, we may fill in all vertices $(1, 0, \ldots, 0, a_r)$ with $a_r \neq 1$, which all have value $S^{\infty V(G)} \wedge DE\mathcal{F}/G_+ = S^{\infty V(G)}$.

The C_{if}-diagram $\tilde{R}$ takes the value $\mathbb{S}$ at $(0, 0, \ldots, 0, a_r)$ for $a_r \neq 1$. The rest of the diagram is called PC_{if} and has $r+1$ initial points, namely the vertices $v_c = (0, \ldots, 0, 1, 0, \ldots, 0)$ (where the 1 in the cth position) for $0 \leq c \leq r$. The entries at $v_r = (0, \ldots, 0, 1)$ (viz $DE\mathcal{F}_+$) and $v_0 = (1, 0, \ldots, 0)$ (viz $S^{\infty V(G)}$) lie in PC_f and are therefore already filled in. The entry when $0 < c < r$ is $S^{\infty V(c)} := \prod_{\text{codim}(H)=c} S^{\infty V(H)}$, and we need to explain how this is filled in by homotopy pullbacks.

Note first that $S^{\infty V(c)}$ is also the entry at the points $(0, \ldots, 0, 1, 0, \ldots, a_r)$ for $a_r = 0/r, 1/r, \ldots (c-1)/r$. The point with $a_r = c/r$ lies in PC_f, and the entry there is therefore filled in at the start. To fill in the entry at the initial vertex $v_c = (0, \ldots 0, 1, 0, \ldots 0)$ we consider a $(c+1)$-cube $C_f(c)$ with initial vertex at $(0,\ldots,0,1,0,\ldots,0)$. More precisely

$$C_f(c) = \{(a_0, a_1, \ldots, a_{c-1}, 1, 0, \ldots, 0, a_r) \mid a_r = 0 \text{ or } c/r\}.$$

We note that entries at $PC_f(c)$ are already filled in, and the following lemma shows that the entry $S^{\infty V(c)}$ can be filled in as a homotopy pullback of entries on $PC_f(c)$.

Lemma 7.4. *The $C_f(c)$-diagram $\tilde{R}$ is a homotopy pullback, which is to say that $S^{\infty V(c)}$ is the homotopy pullback of the $PC_f(c)$-diagram $\tilde{R}$.*

Proof: The proof follows precisely the same pattern as Proposition 6.6 above. The cube is rather similar to a product of copies of the isotropic pullback diagrams for the rank c quotients, but it is slightly different, so we provide some reference points for the proof.

We first note that $S^{\infty V(c)} = \mathbb{S}_{\geq c} \wedge S^{\infty V(c)}$ and then filter the 0th coordinate by

$$\mathbb{S}_{\geq r-c} \longrightarrow \mathbb{S}_{\geq r-c+1} \longrightarrow \ldots \longrightarrow \mathbb{S}_{\geq r} = S^{\infty V(G)}.$$

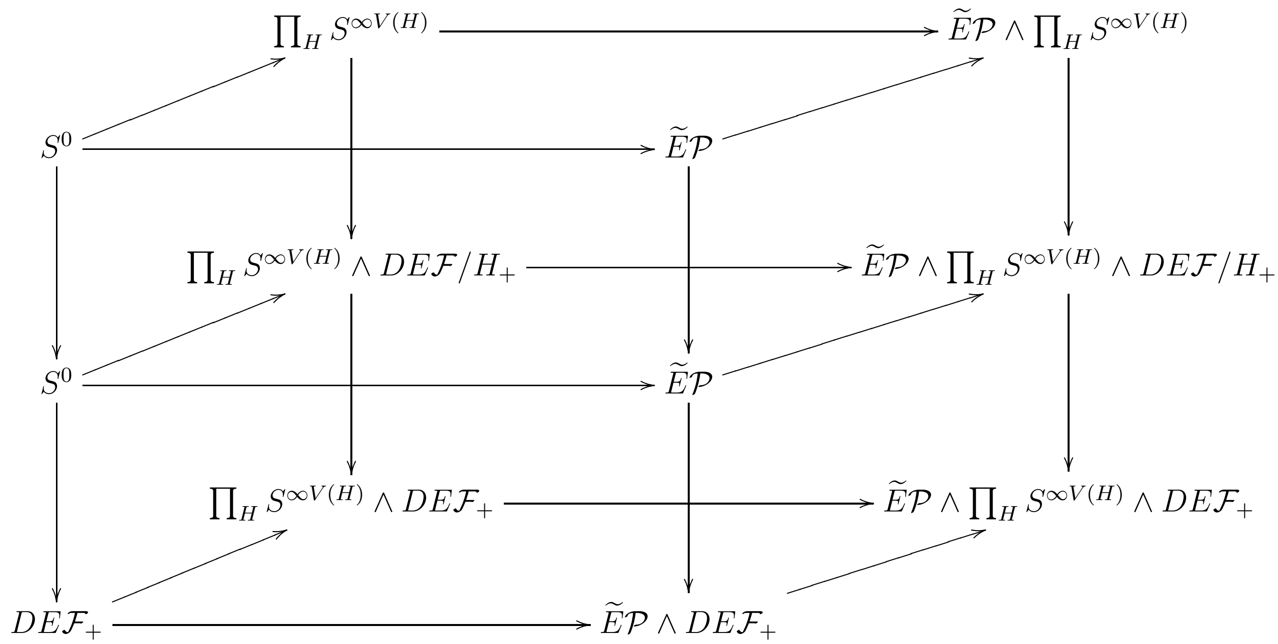
We refine the map from $a_0 = 0$ to $a_0 = 1$ into c steps. The structure of the proof is precisely like that of Proposition 6.6. The only difference is that our application of Corollary 6.5 is in the special case $X = \tilde{R}(a_0, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_{c-1}, 1, 0, \ldots, 0, a_r)$ and $Y = \tilde{R}(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_{c-1}, 1, 0, \ldots, 0, a_r)$.

□

Since we have now filled in the initial points of PC_{if}, we may fill in the remaining vertices without changing the homotopy pullback. Accordingly the homotopy pullback over PC_f agrees with that over PC_{if} as required. □

7.C. **The case of rank 2.** The above account is again sufficiently complicated that it is worth making one case explicit. For typographical reasons we have only illustrated the case $r = 2$, though in fact some features only appear at rank 3. As before, we have used traditional names $S^0 = \mathbb{S}_{\geq 0}$, $\widetilde{E}\mathcal{F} = \mathbb{S}_{\geq 1}$ and $\widetilde{E}\mathcal{P} = \mathbb{S}_{\geq 2}$, where $\mathcal{F}$ is the family of finite subgroups and $\mathcal{P}$ is the family of proper subgroups.

Consider the diagram



The whole diagram is C_{if}. The top square has $a_2 = 0/2$ the middle square has $a_2 = 1/2$ and the bottom square has $a_2 = 2/2$. The cube C_f consists of the bottom square, the middle horizontal on the back face and the top front edge.

Wiping the slate clean, and starting with the entries in PC_f we describe how to fill in the other entries. First, we may fill in $\widetilde{E}\mathcal{P} \wedge \prod_H S^{\infty V(H)}$ at the top right back position without changing the homotopy pullback since it admits a map from $\widetilde{E}\mathcal{P}$ at the top right front. Now Lemma 7.4 with $c = 1$ states that the top back square is a homotopy pullback so that we have filled in $\prod_H S^{\infty V(H)}$ at the top, back left. This gives all vertices of PC_i from those of PC_f, and S^0 is the homotopy pullback of PC_i by Proposition 6.6.

7.D. **Diagrams.** Now that we have a PC_f-diagram $\widetilde{R}_{top}$ of ring G-spectra we should explicitly introduce the corresponding diagrams in other context.

Definition 7.5. From the PC_f diagram $\widetilde{R}_{top}$ of commutative ring G-spectra we form

(1) the PC_f diagram $R_{top} = (\widetilde{R}_{top})^G$ of commutative ring spectra.
(2) the PC_f diagram R_t of commutative DGAs obtained from R_{top} using the fact [62] that the category of commutative $H\mathbb{Q}$-algebras is equivalent to commutative DGAs over $\mathbb{Q}$ (see Section 9)
(3) the PC_f diagram $R_a = \pi_*^G(\widetilde{R}_{top}) = \pi_*(R_{top}) = H_*(R_t)$ of graded rings.

Part 3. From G-spectra, through spectra to algebra

8. Fixed point equivalences for module categories

The category of G-spectra is modelled by $\mathbb{S}$-modules in G-spectra, and since $\mathbb{S}$ is a homotopy pullback of the PC_f-diagram $\widetilde{R}_{top}$ of ring G-spectra, G-spectra is also modelled by a category of $\widetilde{R}_{top}$-modules in G-spectra. Our next step is to remove equivariance and find a model in terms of a category of non-equivariant module spectra over a PC_f-diagram of non-equivariant ring spectra.

8.A. **The fixed point adjunction for module spectra.** We briefly recall some results of [34] for an individual ring G-spectrum. The context is that when we are given a fibrant ring G-spectrum, $\tilde{A}$ with fixed point spectrum $A = \tilde{A}^G$ there is a Quillen adjoint pair

$$\Psi^G : \tilde{A}\text{-mod-}G\text{-spectra} \rightleftarrows A\text{-mod-spectra} : \inf_1^G .$$

Here Ψ^G takes Lewis-May fixed points and then uses the fact that the fixed point functor is lax monoidal by Axiom 3.3 (10) to view the result as a module over A. The inflation functor views a non-equivariant spectrum as a G-spectrum by pullback along the quotient and then extends scalars along $\inf A \longrightarrow \tilde{A}$ to give an $\tilde{A}$-module.

Since the category A-mod-spectra is generated by A, the Cellularization Principle gives a Quillen equivalence

$$\tilde{A}\text{-cell-}\tilde{A}\text{-mod-}G\text{-spectra} \simeq A\text{-mod-spectra}.$$

Surprisingly often (in particular [34, 4.4] when G is a torus and A has Thom isomorphisms), the category $\tilde{A}$-mod-G-spectra is generated by $\tilde{A}$, so that we obtain a Quillen equivalence

$$\tilde{A}\text{-mod-}G\text{-spectra} \simeq A\text{-mod-spectra}$$

showing that a category of equivariant module spectra is equivalent to a category of non-equivariant module spectra.

Before turning to our applications it will be helpful to mention three special cases.

Example 8.1. *(Eilenberg-Moore Theorem* [34, 8.1]*)* We take $\tilde{A} = DEG_+$, so that $A = DBG_+$ and obtain a version of the Eilenberg-Moore theorem: when G is a torus, there is a Quillen equivalence

$$DEG_+\text{-mod-}G\text{-spectra} \simeq DBG_+\text{-mod-spectra}.$$

We emphasize that no cellularization is necessary here for a torus.

Example 8.2. *(Spectra over G* [51, VI.5.3], [34, 3.3] *)* We take $\tilde{A} = S^{\infty V(G)}$ so that $A = S^0$ and note that the category modules over $S^{\infty V(G)}$ is a model for spectra over G (i.e., for spectra with geometric isotropy in $\{G\}$), whilst the category of S^0-modules is the category of spectra. Thus we recover the well known result that there is a Quillen equivalence

$$G\text{-spectra}/G \simeq \text{spectra}.$$

The variant of the first example with all finite isotropy collected together is directly relevant to us.

Example 8.3. *(Almost free spectra* [34, Corollary 9.2]*)* Taking $\tilde{A} = DE\mathcal{F}/K_+$ we obtain

$$DE\mathcal{F}/K_+\text{-mod-}G/K\text{-spectra} \simeq D(E\mathcal{F}/K_+)^{G/K}\text{-mod-spectra}.$$

8.B. **Fixed point adjunctions for diagrams of ring G-spectra.** We now move to the case of *diagrams* of ring spectra. Suppose $\tilde{R}$ is a diagram of ring G-spectra, fibrant in the diagram-injective model structure and consider the corresponding diagram $R = \tilde{R}^G$ of spectra where fixed points are applied objectwise. We may again consider the diagram-injective model categories of $\tilde{R}$-module G-spectra and R-module spectra and once again form the Quillen pair

$$\Psi^G : \tilde{R}\text{-mod-}G\text{-spectra} \rightleftarrows R\text{-mod-spectra} : \inf_1^G .$$

Lemma 8.4. *The Quillen adjunction on diagrams with the diagram-injective model structure is a Quillen equivalence provided it is a Quillen equivalence objectwise.*

Proof: We note that unit and counit when evaluated at any vertex give the unit and counit of the adjunction for a single ring G-spectrum. We claim this is also true for the derived unit and counit. Since weak equivalences are detected objectwise, this will suffice.

To see that the statement about the derived unit and counit follows from that about the underived ones, we need to consider fibrant and cofibrant replacement. For fibrant replacement the implication is clear since fibrancy is defined objectwise. For cofibrant replacement, we note that cofibrant diagrams are objectwise cofibrant. Finally, since fixed points preserve all weak equivalences, we see that the derived unit and counit of the Quillen pair on diagram categories are objectwise the derived unit and counit. □

8.C. **The fixed point adjunction for $\widetilde{R}_{top}$.** We consider the special case $\tilde{R} = \widetilde{R}_{top}$ of the above discussion. The category of spectra is generated by the cells G/H_+ as H varies over closed subgroups of G and the cellularization in the following statement is with respect to the images of these generating cells.

Theorem 8.5. *There is a Quillen equivalence*

$$\Psi^G : \widetilde{R}_{top}\text{-}mod\text{-}G\text{-}spectra \rightleftarrows R_{top}\text{-}mod\text{-}spectra : \inf_1^G .$$

It follows by cellularizing both categories that there is a Quillen equivalence

$$\Psi^G : cell\text{-}\widetilde{R}_{top}\text{-}mod\text{-}G\text{-}spectra \rightleftarrows cell\text{-}R_{top}\text{-}mod\text{-}spectra : \inf_1^G .$$

Proof of 8.5: Without changing notation, we take the fibrant replacement of $\widetilde{R}_{top}$ in the diagram-injective model category of PC_f-diagrams of commutative ring G-spectra [41, 5.1.3]. By [35, Lemma 4.2] the category of modules over this fibrant replacement is Quillen equivalent to the original category $\widetilde{R}_{top}$-mod-G-spectra.

By Lemma 8.4 it suffices to deal with the individual G-spectra at a particular vertex v of PC_f, so we take $\tilde{A} = \tilde{R}(v)$ for some vertex v.

For any ring G-spectrum $\tilde{A}$ we get the equivalence

$$\tilde{A}\text{-cell-}\tilde{A}\text{-mod-}G\text{-spectra} \simeq A\text{-cell-}A\text{-mod-spectra}.$$

It is clear that A generates the category of A-modules so that the A-cellularization on the right is a Quillen equivalence. It remains only to show that the cellularization on the left has no effect.

To establish that the $\tilde{A}$-cellularization on the right is also a Quillen equivalence, it suffices to show that $\tilde{A}$ generates the category of $\tilde{A}$-modules. The argument (as in [34, 4.4]) is to show that cells G/H_+ are all built from complex representation spheres.

If $\tilde{A}$ has Thom isomorphisms this is exactly as in [34, 4.4], but we need the slightly more general argument from [34, Section 9]. We show that for each complex representation W we may express $\tilde{A}$ as a finite product $\tilde{A} \simeq \prod_i \tilde{A}_i$ of factors $\tilde{A}_i$ so that $\tilde{A}_i \wedge S^W$ is a G-fixed suspension of $\tilde{A}_i$. This shows that $\tilde{A} \wedge S^W$ is in the thick category generated by $\tilde{A}$ as required.

Now $\tilde{A} = \widetilde{R}_{top}(v)$ and suppose that the last non-zero entry of v is of codimension c. Then $\tilde{A}$ takes the form

$$A_0(a_0) \wedge \cdots \wedge \prod_{\mathrm{codim} H = c} S^{\infty V(H)} \wedge \cdots \wedge DE\mathcal{F}/H_+$$

Furthermore $DE\mathcal{F}/H_+ \simeq \prod_{\tilde{H}} DE\langle \tilde{H} \rangle$, where the product is indexed by closed subgroups $\tilde{H}$ with identity component H. First note that S^W is a finite complex, and therefore can be moved inside all the products. For each H, we have $W = W^H \oplus W'(H)$ and $S^{\infty V(H)} \wedge S^W \simeq S^{\infty V(H)} \wedge S^{W^H}$ so that

$$\tilde{A} \wedge S^V = A_0(a_0) \wedge \cdots \wedge \prod_{\mathrm{codim} H = c} S^{\infty V(H)} \wedge \cdots \wedge DE\mathcal{F}/H_+ \wedge S^{V^H}$$

Now we use the Thom isomorphism

$$DE\langle \tilde{H} \rangle \wedge S^{V^H} \simeq DE\langle \tilde{H} \rangle \wedge S^{|V^{\tilde{H}}|}.$$

Collecting together all the factors with the same suspension:

$$\Sigma_i = \{\tilde{H} \mid \mathrm{codim}(\tilde{H}) = c \text{ and } \dim(V^{\tilde{H}}) = i\}$$

we obtain a decomposition $\tilde{A} \simeq \prod_i \tilde{A}_i$ as required.

□

8.D. **Modules over product rings.** We are repeatedly working with infinite products $R = \prod_i R_i$ of ring spectra R_i, and we let e_i be the idempotent projecting onto the ith factor. Even in algebra, such infinite products are poorly behaved (for example infinite products of Noetherian rings need not be Noetherian). If M is a module over $\prod_i R_i$ and we take $M_i = e_i M$ then we have maps

$$\bigoplus_i M_i \longrightarrow M \longrightarrow \prod_i M_i.$$

The first is a monomorphism, but typically neither will be an isomorphism (for example if we take $M = \prod_i R_i / \bigoplus_i R_i$ then $M_i = 0$ for all i).

It seems worth observing that from the point of view of model categories we may rather generally apply the Cellularization Principle [33] to recover the more familiar product category from the by suitable cellularization.

Lemma 8.6. *We have a Quillen equivalence*

$$\{R_s\}_s\text{-}cell\text{-}(\textstyle\prod_s R_s)\text{-}modules \simeq \prod_i [R_i\text{-}modules].$$

Proof: For each s we have projection $\pi_s : R \longrightarrow R_s$ inducing a restriction on module categories. This has both a left and a right adjoint, and they agree up to equivalence. Combining these we obtain

$$p : R\text{-mod} \longrightarrow \prod_s [R_s\text{-modules}]$$

whose right adjoint p^R takes the product of the terms and whose left adjoint p^L takes the sum.

The adjoint pair (p^L, p) is a Quillen pair if the categories have the injective model structures. The adjoint pair (p, p^R) is a Quillen pair if the categories are given the projective model structures.

In the second case, the objects R_s are small generators in $\prod_s [R_s\text{-modules}]$. Since both p and p^R preserve all weak equivalences, the unit and counit are equivalences on the generators R_s and we may apply the Cellularization Principle to give the desired conclusion. □

9. From spectra to DGAs

In this section we use the results from [62] to show that the category of module spectra over the diagram R_{top} of commutative ring spectra is Quillen equivalent to a category of differential graded modules over a diagram R_t of commutative DGAs. It then follows that the cellularizations of these model categories are also Quillen equivalent. Since [62] is based on symmetric spectra, we use Axiom 3.3 (12) to show that there is a Quillen equiavlence between the respective categories of modules over R_{top} and $\mathbb{F}R_{top}$.

We next apply the functors from [62] to move from symmetric spectra to differential graded modules. In more detail, in [62, 1.1] a composite functor Θ is defined which produces a Quillen equivalence between $H\mathbb{Z}$-algebra spectra and DGAs. Given an $H\mathbb{Z}$-algebra spectrum, B, it is shown in [62, 2.15] that the category of module spectra over B is Quillen equivalent to the category of differential graded modules over ΘB, a DGA. Furthermore, rationally there is a second functor Θ' which is symmetric monoidal, so that it takes rational commutative rings spectra to rational commutative DGAs. Finally, over the rationals the two functors are naturally equivalent, so that by [62, 1.2], if B is a commutative $H\mathbb{Q}$-algebra then ΘB is naturally weakly equivalent to a commutative DGA $\Theta' B$.

Definition 9.1. Applying functors to the PC_f-diagram of commutative rational ring spectra R_{top}, we define R_t to be the PC_f-diagram $\Theta'(H\mathbb{Q} \wedge \mathbb{F}R_{top})$ of commutative DGAs.

Note, throughout this section we are implicitly considering the standard (or diagram projective) model structures from [35, 3.1(i)] on modules over diagrams of rings.

Proposition 9.2. *There is a zig-zag of Quillen equivalences between the category of module spectra R_{top}-mod and the category of differential graded modules R_t-mod.*

Proof: As mentioned above, the first step is a Quillen equivalence between R_{top}-mod over 1-spectra and $\mathbb{F}R_{top}$-mod over symmetric spectra by Axiom 3.3 (12) extended to diagrams of rings. Since R_{top} is rational, the unit map $\mathbb{F}R_{top} \to H\mathbb{Q} \wedge \mathbb{F}R_{top}$ is a weak equivalence which induces a Quillen equivalence on the associated module categories by extension and restriction of scalars, [35, 4.2] and [43, 5.4.5].

Combining these steps with [62, 2.15] produces a Quillen equivalence between R_{top}-mod and $\Theta(H\mathbb{Q} \wedge \mathbb{F}R_{top})$-mod. Since $H\mathbb{Q} \wedge \mathbb{F}R_{top}$ is a diagram of commutative $H\mathbb{Q}$-algebras, it follows from the proof of [62, 1.2] that $\Theta'(H\mathbb{Q} \wedge \mathbb{F}R_{top})$ is a diagram of commutative rational DGAs which is weakly equivalent to the diagram $\Theta(H\mathbb{Q} \wedge \mathbb{F}R_{top})$.

By [35, 4.2] and [43, 5.4.5], extension and restriction of scalars over these weak equivalences produce the last steps in the stated zig-zag of Quillen equivalences. □

The Cellularization Principle, [33, Corollary 2.8] shows that cellularization preserves zig-zags of Quillen equivalences as long as the cells in the target category are taken to be the images under the relevant derived functors of the cells in the source category. Here we begin with the cellularization of R_{top}-mod with respect to the images of G/H_+ as H runs through closed subgroups. Then, at each of the next steps, the cells are the images of G/H_+ under the appropriate derived functor.

Corollary 9.3. *There is a zig-zag of Quillen equivalences between the cellularizations of the model categories in Proposition 9.2; that is, cell-R_{top}-mod-spectra and cell-R_t-mod-spectra are Quillen equivalent.*

10. Formality

We have shown that the category of rational G-spectra is equivalent to the cellularization of modules over a suitable PC_f diagram of commutative DGAs. On the other hand, we know very little about the diagram except the fact that the terms are commutative and we know the homology. The purpose of this section is to show that in fact this determines the diagram up to equivalence.

10.A. **Terminology.** A map $f : \tilde{R} \longrightarrow \tilde{R}'$ of commutative DGAs inducing an isomorphism is called a *homology isomorphism*. Two commutative DGAs related by a zig-zag of homology isomorphisms of commutative DGAs are said to be *quasi-isomorphic*. Of course isomorphisms of homology need not be induced by maps of DGAs and DGAs with isomorphic homology need not be quasi-isomorphic.

A commutative DGA which is quasi-isomorphic to its homology is said to be *formal*. A graded commutative ring R is said to be *intrinsically formal* if every commutative DGA $\tilde{R}$ with $H_*(\tilde{R}) \cong R$ is formal. We say that $\tilde{R}$ is *strongly formal* if there is a homology isomorphism $H_*(\tilde{R}) \longrightarrow \tilde{R}$. A commutative graded ring is *strongly intrinsically formal* if every commutative DGA with homology R is strongly formal.

All of these notions apply similarly to diagrams of commutative DGAs, and it is our purpose to show that the PC_f-diagram $R_a = \pi_*^G(\widetilde{R}_{top})$ is intrinsically formal. This is based on the fact that polynomial rings are strongly intrinsically formal as commutative rings. This single fact is extended in generality in both the algebraic and diagrammatic senses.

10.B. **Constructing new formal objects from old.** The general form of the results is not surprising, but care is necessary in their formulation.

Lemma 10.1. *(i) The k-algebra $k[x_1, \ldots, x_r]$ is strongly intrinsically formal for commutative DG k-algebras.*

(ii) If R_i is intrinsically formal for all i then $\prod_i R_i$ is intrinsically formal.

(iii) If R is strongly intrinsically formal and $\mathcal{E}$ is a multiplicatively closed subset of R then $\mathcal{E}^{-1}R$ is intrinsically formal relative to R in the sense that if $\tilde{R} \longrightarrow \tilde{R}_{\mathcal{E}^{-1}}$ is a map of DGAs inducing $R \longrightarrow \mathcal{E}^{-1}R$ in homology, then there exists a homology isomorphism $\tilde{R}_{\mathcal{E}^{-1}} \to \tilde{R}'_{\mathcal{E}^{-1}}$ such that the diagram

$$\begin{array}{ccc}
\tilde{R} & \longrightarrow & \tilde{R}_{\mathcal{E}^{-1}} \\
\uparrow & & \downarrow \simeq \\
 & & \tilde{R}'_{\mathcal{E}^{-1}} \\
 & & \uparrow \\
R & \longrightarrow & \mathcal{E}^{-1}R
\end{array}$$

can be completed by a dotted arrow which is a homology isomorphism.

Proof: (i) If $H_*(R) = k[x_1, \ldots, x_r]$ then we may pick representative cycles $\tilde{x}_1, \ldots, \tilde{x}_r$ for $x_1, \ldots, x_r$ in R and then since $k[x_1, \ldots, x_r]$ is free as a commutative ring, there is a map $k[x_1, \ldots, x_r] \longrightarrow R$ taking x_i to $\tilde{x}_i$, and this induces an isomorphism in homology.

(ii) Suppose $H_*(\tilde{R}) = \prod_i R_i$. First, we replace $\tilde{R}$ by a DGA which is actually a product. Indeed, we may choose cycles $\tilde{e}_i$ representing the idempotents for the factors. Now form $\tilde{R}_i = \underrightarrow{\mathrm{holim}}(\tilde{R}, \tilde{e}_i)$, so that $H_*(\tilde{R}_i) = R_i$. We therefore have a quasi-isomorphism $\tilde{R} \longrightarrow \prod_i \tilde{R}_i$, and then we may take the product of the individual zig zags of quasi-isomorphisms connecting $\tilde{R}_i$ and R_i.

(iii) Since R is strongly intrinsically formal, we have a map $R \to \tilde{R}$; let $\tilde{\mathcal{E}}$ denote the image of the multiplicatively closed subset $\mathcal{E}$ in $\tilde{R}$. Then the map $\tilde{R}_{\mathcal{E}^{-1}} \longrightarrow \tilde{\mathcal{E}}^{-1}\tilde{R}_{\mathcal{E}^{-1}}$ is a quasi-isomorphism and by the universal property of localization we may extend $R \longrightarrow \tilde{\mathcal{E}}^{-1}\tilde{R}_{\mathcal{E}^{-1}}$ to a quasi-isomorphim $\mathcal{E}^{-1}R \stackrel{\cong}{\longrightarrow} \tilde{\mathcal{E}}^{-1}\tilde{R}_{\mathcal{E}^{-1}}$. □

We now need a tool for using these facts in diagrams.

Lemma 10.2. *Suppose given a partially ordered set A, a subset $B \subseteq A$ with no maps out of it, and a diagram $R : A \longrightarrow$ DGAs. If we have a B-diagram $R' : B \longrightarrow$ DGAs and a map $\theta_B : R|_B \longrightarrow R'$, we may extend R' to an A-diagram $\hat{R}'$ (taking $\hat{R}'(a) = R(a)$ if $a \notin B$) and extend θ_B to a map $\theta : R \longrightarrow \hat{R}'$. If θ_B is a homology isomorphism, so is θ.* □

Example 10.3. *(Extending a diagram of rings along a map at a vertex v.)* Suppose v is a vertex in a poset A and we have a map $R(v) \longrightarrow R'(v)$. We may take B to be the set of vertices with a map from v, and define R' on B by taking

$$R'(b) = R'(v) \otimes_{R(v)} R(b).$$

We obtain a map $R|_B \longrightarrow R'$ by identifying $R|_B(b)$ as $R(v) \otimes_{R(v)} R|_B(b)$ and using the map $R(v) \longrightarrow R'(v)$ at each point.

Applying Lemma 10.2 we obtain a map of A diagrams $R \longrightarrow \hat{R}'$. This is a pointwise homology isomorphism provided $R(v) \longrightarrow R'(v)$ is an isomorphism in homology and all the rings $R(b)$ are flat over $R(v)$.

10.C. **The intrinsic formality of the diagram R_a.** We are now prepared to prove the intrinsic formality of the PC_f-diagram $R_a = \pi^G_*(\widetilde{R}_{top})$ of graded rings.

The reader may find it helpful to refer to Subsections 10.D and 10.E where the rank 1 and rank 2 cases are made rather explicit.

Proposition 10.4. *The PC_f-diagram R_a is intrinsically formal, and in particular R_t is formal.*

Proof: The punctured cube PC_f is a poset (indeed, it is the barycentric subdivision of the r-simplex Δ^r; we may identify each vertex v of PC_f with the non-empty subset $S(v) = \{i \mid a_i = 1\}$ of $\{0, \ldots, r\}$). The collection of vertices is ordered by the size of $S(v)$, and we will work in order of increasing size.

More precisely, we let $PC_f^{(d)}$

denote the $(d-1)$-skeleton of the subdivided r-simplex. In other words, it contains all vertices v with $|S(v)| \leq d$.

Given a PC_f diagram $\tilde{R}$ with homology isomorphic to R_a, we replace it by an equivalent cofibrant diagram without change in notation, and then procede to construct a succession of homology isomorphisms

$$\tilde{R} = \tilde{R}_0 \xrightarrow{i_0} \tilde{R}_1 \xrightarrow{i_1} \cdots \xrightarrow{i_r} \tilde{R}_{r+1} = \tilde{R}$$

of PC_f-diagrams of DGAs, where $i_{d-1} : \tilde{R}_{d-1} \longrightarrow \tilde{R}_d$ is constant on $PC_f^{(d-1)}$. As we do this, we construct maps

$$\theta_d : R_a|_{PC_f^{(d)}} \longrightarrow \tilde{R}_d|_{PC_f^{(d)}}$$

for $d \geq 1$ which are homology isomorphisms on the diagram on which they are defined. For $d \geq 2$, the map θ_d extends $i_{d-1} \circ \theta_{d-1}$.

After $r+1$ steps we obtain a homology isomorphism

$$R^a = R^a|_{PC_f^{(r+1)}} \longrightarrow \tilde{R}_{r+1}|_{PC_f^{(r+1)}} = \tilde{R}.$$

To start with, we construct $\tilde{R}_1$. Note first that for each of the $r+1$ vertices v of Δ^r the DGA $R_a(v)$ is a product of polynomial rings indexed by i (if the vertex corresponds to connected subgroups of codimension c, then we take a product of all the $\mathcal{O}_{\mathcal{F}/H}$ with H connected of codimension c, each of which is a product of the cohomology rings $H^*(BG/\tilde{H})$ as $\tilde{H}$ runs through the subgroups with identity component H. Altogether, i will run through all subgroups of codimension c, connected or not).

As in Lemma 10.1 (ii) we construct DGAs $\tilde{R}(v)_i$ with homology $R_a(v)_i$ and a quasi-isomorphism

$$\tilde{R}(v) \longrightarrow \prod_i \tilde{R}(v)_i.$$

Choosing some ordering of the vertices, we extend $\tilde{R}_0$

along each of these quasi-isomorphisms (as in Example 10.3) in turn to obtain $\tilde{R}_1$. We note that since there are no maps from one vertex to another, all $r+1$ vertices end up with a product of DGAs. Now using Lemma 10.1 (i) at each vertex we obtain a map

$$\theta_1 : R^a|_{PC_f^{(1)}} \longrightarrow \tilde{R}_1|_{PC_f^{(1)}}.$$

We continue inductively, supposing that after d steps we have defined $\tilde{R}_s$ for $s \leq d$, and

$$\theta_d : R^a|_{PC_f^{(d)}} \longrightarrow \tilde{R}_d|_{PC_f^{(d)}}.$$

Once again we will form $\tilde{R}_{d+1}$ from $\tilde{R}_d$ by extending the diagram of rings along ring maps at the $\binom{r+1}{d+1}$ vertices v with $|S(v)| = d+1$ in turn. When it comes to the turn of v, since there are no maps between these vertices, we still have $\tilde{R}_d(v)$ at v. This has homology

$$H_*(\tilde{R}_d(v)) = H_*(\tilde{R}(v)) = R_a(v)$$

and this is obtained from polynomial rings by alternately taking products and localizing with respect to sets of Euler classes. Furthermore, we note that the Euler classes concerned come from the vertices w with $|S(w)| \leq d$, so that θ_d gives their images in the DGAs. We now form a new PC_f-diagram of DGAs by extending $\tilde{R}_d(v)$ along the alternate products and localizations

using Lemma 10.1. When we have extended along all these vertices we have obtained $\tilde{R}_{d+1}$ from $\tilde{R}_d$, and the products and localizations let us extend θ_d to θ_{d+1}. $\square$

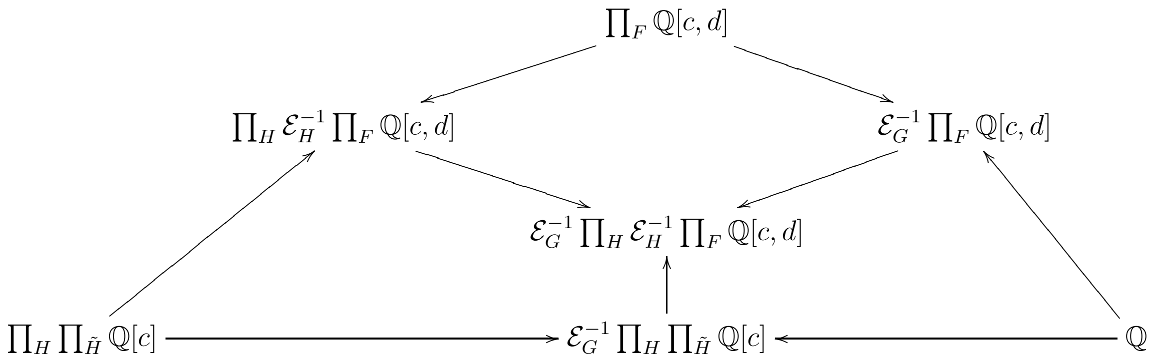
10.D. **The example of rank 1.** The argument proceeds as follows. We start with the cofibrant PC_f-diagram $\tilde{R}$ as in the top row. Extending along the top left hand vertical we form the second row. The map from the two outer vertices of R_a on the bottom row can then be defined. The Euler classes are defined by the image of $R_a(0,1)$, and those are inverted to form the third row, after which the middle vertical can be filled in.

$$\begin{array}{ccccccc}
\tilde{R} & & \tilde{R}(0,1) & \longrightarrow & \tilde{R}(1,1) & \longleftarrow & \tilde{R}(1,0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{R}_1 & & \prod_i \tilde{R}(0,1)_i & \longrightarrow & \prod_i \tilde{R}(0,1)_i \otimes_{\tilde{R}(0,1)} \tilde{R}(1,1) & \longleftarrow & \tilde{R}(1,0) \\
\downarrow & & {=}\downarrow & & \downarrow & & \downarrow{=} \\
\tilde{R}_2 & & \prod_i \tilde{R}(0,1)_i & \longrightarrow & \mathcal{E}_G^{-1} \prod_i \tilde{R}(0,1)_i \otimes_{\tilde{R}(0,1)} \tilde{R}(1,1) & \longleftarrow & \tilde{R}(1,0) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
R_a & & \mathcal{O}_{\mathcal{F}} & \longrightarrow & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} & \longleftarrow & \mathbb{Q}
\end{array}$$

$$\begin{array}{ccccccc}
R_{top} & & (DE\mathcal{F}_+)^G & \longrightarrow & (S^{\infty V(G)} \wedge DE\mathcal{F}_+)^G & \longleftarrow & (S^{\infty V(G)})^G
\end{array}$$

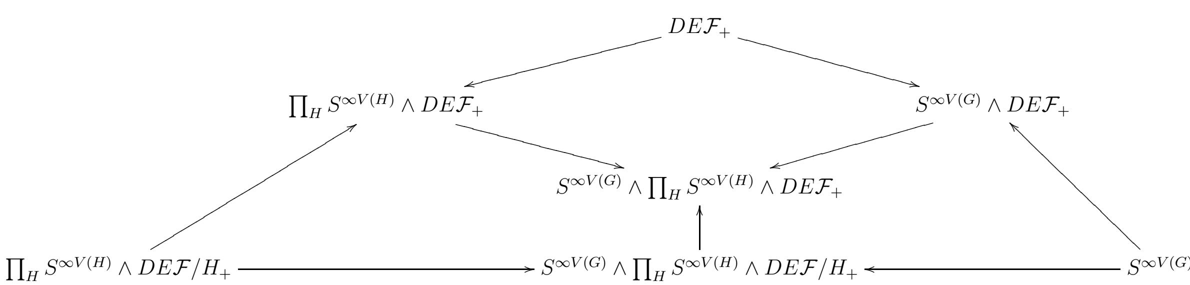
$$\begin{array}{ccccccc}
\widetilde{R}_{top} & & DE\mathcal{F}_+ & \longrightarrow & S^{\infty V(G)} \wedge DE\mathcal{F}_+ & \longleftarrow & S^{\infty V(G)}
\end{array}$$

10.E. **The example of rank 2.** It is too typographically complicated to display the full argument in the way we did for rank 1, but it still seems worth displaying R_a and $\widetilde{R}_{top}$. This lets one see the way that extending along (say) a map of rings at the top vertex only affects the three other points not on the bottom face, and then extending along (say) the middle vertex on the bottom face only affects the central vertex.



The subgroups F run through finite subgroups, the subgroups H run through circle subgroups, and the subgroups $\tilde{H}$ run through subgroups with identity component H. The polynomial rings $\mathbb{Q}[c,d]$ are the cohomology rings of $B(G/F)$ (all different but isomorphic), and the polynomial rings $\mathbb{Q}[c]$ are the cohomology rings of $B(G/\tilde{H})$ (the polynomial ring $\mathbb{Q}$ is the cohomology ring of $B(G/G)$!).

The above diagram is obtained by taking homotopy groups of the following diagram $\widetilde{R}_{top}$ of ring G-spectra.



Part 4. **Algebra**

11. Modules over R_a and the standard model $\mathcal{A}_c^p(G)$

We have now established that the category of G-spectra is equivalent to the cellularization of the category of DG-R_a-modules, where R_a is a PC_f-diagram of rings. We want to show this is equivalent to the category of DG objects in $\mathcal{A}(G)$.

11.A. **Strategy.** We will use the algebraic machinery and terminology set up in [28]. As described in Section 2 above, $\mathcal{A}(G) = \mathcal{A}_c^p(G)$ is a category of modules over the diagram $\mathbb{R}_c^p$ of rings based on **p**airs of **c**onnected subgroups. However the topological argument delivers a category of modules over the diagram R_a based on subsets of $[0,r] = \{0,1,\ldots,r\}$ which are the dimensions of subgroups. For the totally ordered poset $[0,r]$ there is no distinction between subsets and flags. Taking a subset of $[0,r]$ with s elements in decreasing order, we obtain a flag $d_0 > d_1 > \cdots > d_s$. We will make the diagram R_a explicit in Subsection 11.B), and observe that $R_a = \mathbb{R}^f_d$ in the notation of [28].

It is shown in [28] that there is a subcategory $\mathcal{A}^f_d(G)$ of $\mathbb{R}^f_d$-modules equivalent to $\mathcal{A}^p_c(G)$, namely *pqce*-modules, which is to say that satisfy a quasi-coherence condition (qc) an extendedness condition (e) and whose values on vertices are products (p). There is in fact a diagram of categories and adjoint pairs

$$\begin{array}{ccccc}
\mathcal{A}^p_c(G) & & \mathcal{A}^f_c(G) & & \mathcal{A}^f_d(G) \\
\| & & \| & & \| \\
qce\text{-}\mathbb{R}^p_c\text{-mod} & \underset{p,f}{\overset{\simeq}{\longleftrightarrow}} & qce\text{-}\mathbb{R}^f_c\text{-mod} & \overset{\simeq}{\longleftrightarrow} & pqce\text{-}\mathbb{R}^f_d\text{-mod} \\
 & & i \downarrow \;\; \uparrow \Gamma^f_c & & \downarrow \;\; \uparrow \Gamma^f_d \\
 & & \mathbb{R}^f_c\text{-mod} & \underset{e}{\overset{d_*}{\rightleftarrows}} & \mathbb{R}^f_d\text{-mod}
\end{array}$$

The absence of a label on the functor left adjoint to Γ^f_d is intentional: the functor is obtained by following round the other three sides of the square, and is not the inclusion (the inclusion does not preserve sums). In fact, there is no need to give further details of *pqce* $\mathbb{R}^f_d$-modules here, since we will proceed directly between $\mathbb{R}^f_d$-modules and qce-$\mathbb{R}^f_c$-modules. The relevant result from [28] is as follows.

Proposition 11.1. [28, Subsection 11.C] *There is an adjoint pair*

$$l : qce\text{-}\mathbb{R}^p_c\text{-}modules \rightleftarrows \mathbb{R}^f_d\text{-}modules : \Gamma$$

where $l = d_* i f$ *and* $\Gamma = p\Gamma^f_c e$. □

We will briefly describe the functors in Subsection 11.F below.

11.B. **The diagram R_a.** We will make explicit the diagram $R_a = \pi^G_*(\widetilde{R}_{top})$ of homotopy rings of our PC_f-diagram $\widetilde{R}_{top}$ of ring spectra as in Definition 7.5. It will appear that this is a special case of the machinery of [28], so that $R_a = \mathbb{R}^f_d$ in the notation of [28].

Since $\pi^G_*(S^{\infty V(H)} \wedge DE\mathcal{F}/H_+) = \mathcal{O}_{\mathcal{F}/H}$ and since the map $S^0 \longrightarrow S^V$ induces multiplication by the Euler class $c(V)$ in $\pi^G_*(DE\mathcal{F}_+) = \mathcal{O}_\mathcal{F}$, it is straighfroward to read off from the definition of $\widetilde{R}_{top}$ in Subsection 7.A an explicit and totally algebraic account.

At the point $(a_0, \ldots, a_s, 0, \ldots, 0)$ with $a_s = 1$, we form a ring from the product

$$\prod_{\mathrm{codim}(H)=s} \mathcal{O}_{\mathcal{F}/H}$$

by taking retracts and alternating products and localizations. To write this down, we recall from Equation 6.1 the indexing set $I(t, a_t)$ which is a singleton if $a_t = 0$ or all codimension t connected subgroups otherwise. We also recall that $\mathcal{E}_K$ consists of Euler classes of all representations W with $W^K = 0$, and adopt a convention to let us refer to a vacuous localization in a similar notation: we take $\mathcal{E}_{K,1} = \mathcal{E}_K$ and $\mathcal{E}_{K,0} = \{1\}$. Now we may write

$$R_a(a_0, \ldots, a_s, 0, \ldots, 0) = \mathcal{E}^{-1}_{G,a_0} \prod_{H_1 \in I(1,a_1)} \mathcal{E}^{-1}_{H_1,a_1} \prod_{H_2 \in I(2,a_2)} \mathcal{E}^{-1}_{H_2,a_2} \cdots \mathcal{E}^{-1}_{H_{s-1},a_{s-1}} \prod_{H_s \in I(s,a_s)} \mathcal{O}_{\mathcal{F}/H_s}.$$

To save on the notation required to say we have nested subgroups, we use the convention that inverting $\mathcal{E}_H$ is deemed to annihilate factors corresponding to lower dimensional subgroups K not contained in H.

We will say more about what is meant by inverting Euler classes in Subsection 11.C, but first it is helpful illustrate the definition in low ranks to show its simplicity.

Example 11.2. *(The diagram R_a in rank 1.)* In rank 1, if the objects of PC_f are layed out as

$$v_1 = (01) \longrightarrow (11) \longleftarrow (10) = v_0$$

the rings are

$$\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \longleftarrow \mathcal{O}_{\mathcal{F}/G} = \mathbb{Q}$$

Example 11.3. *(The diagram R_a in rank 2.)* In rank 2, if the objects are layed out as

$$\begin{array}{ccccc}
 & & v_2 = (001) & & \\
 & \swarrow & & \searrow & \\
(011) & & & & (101) \\
\nearrow & \searrow & & \swarrow & \nwarrow \\
 & & (111) & & \\
 & & \uparrow & & \\
v_1 = (010) & \longrightarrow & (110) & \longleftarrow & (100) = v_0
\end{array}$$

the diagram of rings is

$$\begin{array}{ccccc}
 & & \mathcal{O}_{\mathcal{F}} & & \\
 & \swarrow & & \searrow & \\
\prod_H \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} & & & & \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \\
\nearrow & \searrow & & \swarrow & \nwarrow \\
 & & \mathcal{E}_G^{-1}\prod_H \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} & & \\
 & & \uparrow & & \\
\prod_H \mathcal{O}_{\mathcal{F}/H} & \longrightarrow & \mathcal{E}_G^{-1}\prod_H \mathcal{O}_{\mathcal{F}/H} & \longleftarrow & \mathcal{O}_{\mathcal{F}/G} = \mathbb{Q}
\end{array}$$

Example 11.4. *(The diagram R_a in rank 3.)* The diagram in rank 3 is that of a subdivided 3-simplex, and a little too complicated to display in print. However we note that a new phenomenon occurs in rank 3 since not every circle subgroup is contained in every

2-torus subgroup (in lower ranks, containment of connected subgroups was determined by dimension). This means that at points of the form $(a_0 1 1 a_3)$, we have

$$R_a = \cdots \prod_H \mathcal{E}_H^{-1} \prod_K \cdots .$$

where H is of codimension 1 and K of codimension 2. In view of our convention about inverting $\mathcal{E}_H$, the second product is in fact over circle subgroups K contained in H (and not over all circle subgroups).

11.C. **Internal and external Euler classes.** The G-equivariant homotopy of $S^{\infty V(H)} \wedge X$ is always the G/H-equivariant homotopy of the geometric fixed point spectrum $\Phi^H X$. Sometimes this is calculated from geometric knowledge of X, but if X has Thom isomorphisms for representations V with $V^H = 0$ it can also be calculated from $\pi_*^G(X)$ by inverting Euler classes. This leads to the use of algebraic notation for inverting Euler classes is some cases requiring some explanation.

The issue first arises at (110) in rank 2. A brief explanation of this special case will make plain the general meaning.

The notation suggests we are inverting G-equivariant Euler classes on something (viz $\prod_H \mathcal{O}_{\mathcal{F}/H}$), but the object in question is not an $\mathcal{O}_{\mathcal{F}}$-module. Considering the geometry of the situation we see that what is really happening is passage to a direct limit along maps $S^{W_1} \longrightarrow S^{W_2}$ coming from inclusions $W_1 \subseteq W_2$ with $W_1^G = W_2^G = 0$. Since the spheres are finite complexes this passes inside the product. To see what happens on the Hth factor we write $W = W^H \oplus W'$, and note that $S^W \wedge S^{\infty V(H)} \simeq S^{W^H} \wedge S^{\infty V(H)}$. Thus when we write $\mathcal{E}_G^{-1} \prod_H \mathcal{O}_{\mathcal{F}/H}$, this means a colimit over multiplication by the the product elements $\prod_H c(W_2^H/W_1^H)$, which is the Euler class of the inclusion $W_1^H \longrightarrow W_2^H$, as an element of $\mathcal{O}_{\mathcal{F}/H}$.

One might note that this discussion extends one stage further to explain for example why the $S^{\infty V(G)}$ does not lead to any algebraic inversion at (100).

11.D. **Structure maps for rings.** Next we describe the structure maps in R_a more precisely. Once again, the only real complication is notational.

If we have an inclusion $i_\sigma^\tau : \sigma \longrightarrow \tau$ of subsets of $\{0, \ldots, r\}$ then we have a structure map

$$R_a(i_\sigma^\tau) : R_a(\sigma) \longrightarrow R_a(\tau).$$

Suppose s is the last non-zero term in σ. We start by describing the case when τ has exactly one more element than σ, say $\tau = \sigma \cup \{t\}$. There are two cases.

Case 1: $t > s$**.** In this case t is the last non-zero term in τ and we may concentrate on the contribution of the last two non-trivial terms, namely the sth and tth. Thus we must describe

$$j_s^t : \prod_{H_s \in I(s,1)} \mathcal{O}_{\mathcal{F}/H_s} \longrightarrow \prod_{H_s \in I(s,1)} \mathcal{E}_{H_s}^{-1} \prod_{H_t \in I(t,1)} \mathcal{O}_{\mathcal{F}/H_t}$$

in the sense that the map is obtained from this by applying alternating products and localizations for the 0th to the $(s-1)$st terms. Now j_s^t is itself a product over $I(s,1)$ of terms given as the composite

$$\mathcal{O}_{\mathcal{F}/H_s} \longrightarrow \prod_{H_t \in I(t,1)} \mathcal{O}_{\mathcal{F}/H_t} \longrightarrow \mathcal{E}_{H_s}^{-1} \prod_{H_t \in I(t,1)} \mathcal{O}_{\mathcal{F}/H_t}.$$

The first map has components given by inflations for $H_s \supseteq H_t$ and the second is localization.

Case 2: $t < s$**.** In this case s is the last non-zero term in both σ and τ and the only change is to replace the expression $\prod_{H_t \in I(t,0)} \mathcal{E}^{-1}_{H_t,0}$ (which actually means take the product over a singleton of a localization doing nothing!) with $\prod_{H_t \in I(t,1)} \mathcal{E}^{-1}_{H_t}$, and here a diagonal map is used.

More precisely if

$$R_a(a_{t+1}, \ldots, a_s, 0, \ldots, 0) = \prod_{H_{t+1} \in I(t+1, a_{t+1})} \mathcal{E}^{-1}_{H_{t+1}, a_{t+1}} \prod_{H_{t+2} \in I(t+2, a_{t+2})} \mathcal{E}^{-1}_{H_{t+2}, a_{t+2}} \cdots \mathcal{E}^{-1}_{H_{s-1}, a_{s-1}} \prod_{H_s \in I(s, a_s)} \mathcal{O}_{\mathcal{F}/H_s}$$

we take the map into the product whose components are localizations

$$\{l_{i_t}\} : R_a(a_{t+1}, \ldots, a_s, 0, \ldots, 0) \longrightarrow \prod_{H_t \in I(t,1)} \mathcal{E}^{-1}_{H_t} R_a(a_{t+1}, \ldots, a_s, 0, \ldots, 0)$$

and then apply alternate products and localizations to incorporate the terms from the 0th to the $(t-1)$st.

When τ has more than one extra vertex than σ the map $R_a(i^\tau_\sigma)$ is the composite of the maps adding one vertex at a time. It is apparent from the description above that the order in which this is done makes no difference.

11.E. **The algebraic diagram R_a is the diagram $\mathbb{R}^f_d$ from** [28]**.** We briefly recall the framework of [28], so that we may observe that R_a is precisely the diagram of rings appearing there as $\mathbb{R}^f_d$.

The diagram $\mathbb{R}_c$ is the contravariant functor on the poset $\mathbf{ConnSub(G)}$ of connected subgroups of G with value $\mathcal{O}_{\mathcal{F}/K}$ at K, and with inflation maps between them. The dimension function $d : \mathbf{ConnSub(G)} \longrightarrow [0, r]$ gives rise to a dimension function on the posets of flags. The general construction $d^e_!$ described there collects together the subgroups of the same dimension, and extends to flags using localizations and products. This specializes precisely to the description of R_a, so that $\mathbb{R}^f_d = R_a$.

11.F. **Description of the functors.** We now briefly recall from [28] the functors appearing in the diagram from Subsection 11.A above.

The left hand horizontal translates between indexing over pairs and indexing over flags. For *qce*-modules the value of a module on a flag only depends on the largest and smallest subgroup in the flag, so this translation is nugatory; the letter p is for the translation to pairs and the letter f for the translation to flags.

The vertical i is the inclusion of *qce*-modules in all $\mathbb{R}^f_c$-modules, and the functor Γ^f_c is the right adjoint to i constructed in [28, Section 11] following the pattern of [25]; we will not need to use the exact construction.

The functor e is obtained by taking idempotent pieces. Indeed, if M is an $\mathbb{R}^f_d$-module and $F = (K_0 \supset K_1 \supset \cdots \supset K_s)$ is a flag of connected subgroups with dimension $dF = (d_0 > d_1 > \cdots > d_s)$ there is an idempotent $e_F \in \mathbb{R}^f_d(dF)$ picking out the flag F; we take $(eM)(F) = e_F(M(dF))$ (see [28, Section 6] for further details).

The functor d_* is left adjoint to e. The natural idea is to take direct sums: if N is an $\mathbb{R}^f_c$-module then $(d_*N)(d) = \bigoplus_{dF=d} N(F)$. However this is not compatible with structure maps and one must take the submodule of the product it generates. There is a little work to be done to check this makes sense, and the construction is described in detail in [28, Section 6].

12. Model structures and equivalences on the algebraic categories

The output of the work above is a Quillen equivalence between the category of rational G-spectra and an algebraic category cell-R_a-mod, the cellularization of the category of modules over the diagram R_a of rings. The purpose of this section and the next is to simplify the model by avoiding the need for cellularization: we show that the cellularization of the category of R_a-modules is Quillen equivalent to the smaller category of objects in the category of qce-$\mathbb{R}_c$-modules, $\mathcal{A}^p_c(G)$.

12.A. Two examples.

Before turning to general results we give two examples of this phenomenon in a simpler context.

The first example is for modules over a single polynomial ring.

Example 12.1. *(Torsion modules over a polynomial ring.)* If G is a connected compact Lie group, the category of free rational G-spectra is Quillen equivalent to the category of torsion modules over the polynomial ring $H^*(BG)$ [31].

The topology gives a Quillen equivalence with the model category cell-$H^*(BG)$-mod_p: the category of DG-modules over $H^*(BG)$ with the algebraically projective model structure cellularized with respect to the residue field $\mathbb{Q}$. This in turn is Quillen equivalent to the model category cell-$H^*(BG)$-mod_i, which has the same underlying category of DG-modules over $H^*(BG)$, but now endowed with the algebraically injective model structure and again cellularized with respect to the residue field $\mathbb{Q}$.

Finally, if $\mathfrak{m}$ is the ideal of positive codegree elements in $H^*(BG)$, we consider the adjunction

$$i : \text{tors-}H^*(BG)\text{-mod} \rightleftarrows H^*(BG)\text{-mod}_i : \Gamma_{\mathfrak{m}}$$

where $\Gamma_{\mathfrak{m}}$ is the $\mathfrak{m}$-power torsion functor. The category of torsion modules has an injective model structure (weak equivalences are homology isomorphisms and cofibrations are monomorphisms). Accordingly, i preserves cofibrations and acyclic cofibrations and the adjunction is a Quillen adjunction. Finally, the Koszul complex is a small generator of the torsion modules, so the cellularization principle shows this induces a Quillen equivalence

$$\text{tors-}H^*(BG)\text{-mod} \simeq \text{cell-}H^*(BG)\text{-mod}_i.$$

This example is directly relevant to the algebraic model $\mathcal{A}(G)$ for a torus G. Indeed, if we consider objects of $\mathcal{A}(G)$ which are concentrated at the connected subgroup 1, and for which there is no contribution from other finite subgroups, the quasicoherence condition on $\mathbb{R}_c$-modules in $\mathcal{A}(G)$ implies that objects concentrated at the subgroup 1 are precisely the torsion $H^*(BG)$-modules.

The second example works with a rather small diagram of rings, with each of the rings Noetherian.

Example 12.2. *(Semifree $\mathbb{T}$-spectra.)* For the circle group $\mathbb{T}$, our models are over a punctured square of rings. If we simplify the category by restricting attention to semifree spectra, the rings that occur are much smaller and we can see the issues introduced by diagrams without having the infinite number of subgroups to complicate matters.

The diagram of rings for semifree $\mathbb{T}$-spectra is

$$R_a = \begin{pmatrix} & & R^v \\ & & \downarrow \\ R^n & \longrightarrow & R^t \end{pmatrix} = \begin{pmatrix} & & \mathbb{Q} \\ & & \downarrow \\ \mathbb{Q}[c] & \longrightarrow & \mathbb{Q}[c, c^{-1}] \end{pmatrix}$$

An R_a module M consists of a diagram

$$M = \begin{pmatrix} & & M^v \\ & & \downarrow \\ M^n & \longrightarrow & M^t \end{pmatrix} = \begin{pmatrix} & & V \\ & & \downarrow \\ N & \longrightarrow & P \end{pmatrix}$$

There are four relevevant model categories. To start with, on each of the three objectwise module categories we can choose either the algebraically projective model structure or the algebraically injective model structure. We need to make the same choice at each vertex so that the maps in the diagram respect the model structures. Secondly, having made that choice, we may choose either the diagram theoretically projective or injective model. Since the diagrams are both direct and inverse, the results of [35] show these models all exist, and it is clear there are Quillen equivalences between either of the two binary choices by using the identity functors. In fact, we only need three of the four possibilities; a diagram-projective, algebraically-injective model structure does not appear.

Having made a choice, we cellularize with respect to the two modules corresponding to basic geometric generators

$$\mathbb{S} = R_a = \begin{pmatrix} & & \mathbb{Q} \\ & & \downarrow \\ \mathbb{Q}[c] & \longrightarrow & \mathbb{Q}[c, c^{-1}] \end{pmatrix} \text{ and } G_+ = \begin{pmatrix} & & 0 \\ & & \downarrow \\ \mathbb{Q} & \longrightarrow & 0 \end{pmatrix}$$

By [33, Corollary 2.8], cellularization preserves the Quillen equivalences mentioned above.

Finally, for qce-R-mod, the underlying category consists of quasi-coherent extended modules. The quasi-coherence condition is that the horizontal is localization in the sense that

$$M^t \cong M^n[1/c].$$

The extendedness is the condition that the vertical is induction in the sense that

$$M^t \cong \mathbb{Q}[c, c^{-1}] \otimes V.$$

The inclusion of this category of modules has a right adjoint, and we may argue as in the previous example. We will give the category of *qce*-modules a model structure so that it is Quillen equivalent to the cellularization of the doubly injective model structure.

12.B. **Construction of model structures.** In the remainder of this section we turn to the full PC_f-diagram R_a of rings. We saw in Section 11 that $R_a = \mathbb{R}^f_d$ in the notation of [28], and we outline here the proof that the cellularization of the doubly projective model category of R_a-modules is equivalent to the category of DG qce-$\mathbb{R}^p_c$-modules $\mathcal{A}^p_c(G)$ as in Section 11.

We begin by formally introducing the algebraic model structures we use.

These are model structures on diagrams of modules over diagrams of DGAs. For each individual DGA there is an algebraically projective model structure [58], which is constructed from free modules in the usual way; the proof may be obtained by adapting [41, 2.3]. Similarly, for an individual DGA with 0 differential there is an algebraically injective model structure [31].

Making a choice of algebraically projective or injective model structures at all points in the diagram we may then seek to define a diagram-theoretically projective model structure (in which weak equivalences and fibrations are given pointwise) or a diagram-theoretically injective model structure (in which weak equivalences and cofibrations are given pointwise). Since the finite diagram shapes we are interested in here are both direct and inverse, both diagram-projective and diagram-injective model structures exist by [35, Proposition 3.1] for either of the algebraic choices (made consistently throughout the diagram). Only three of the four choices appear in our work here, the doubly-projective case (which also follows from [60, 6.1]), the doubly-injective case, and the diagram-injective, algebraically-projective case.

12.C. **A model structure on torsion modules.** We consider the category $\mathcal{A}^p_c(G)$ of *qce*-$\mathbb{R}^p_c$-modules and show the associated category of DG objects admits a model structure with quasi-isomorphisms as the weak equivalences.

Proposition 12.3. *The category $DG-\mathcal{A}^p_c(G)$ of DG qce-$\mathbb{R}^p_c$-modules admits a model structure with weak equivalences the quasi-isomorphisms and cofibrations the monomorphisms at each object. The fibrant objects are injective if the differential is forgotten, and fibrations are surjective maps with fibrant kernel.*

Proof: We use the method of [21, Appendix B], where it is shown that one can often construct a model structure using a type of fibrant generation argument provided one has a suitable finiteness of injective dimension.

We have an abelian category $\mathcal{A} = \mathcal{A}^p_c(G)$ and we aim to put a model structure on the category of DG objects of $\mathcal{A}$. We will specify a set $\mathcal{BI}$ of *basic injectives* containing sufficiently many injectives (i.e., any object of $\mathcal{A}$ embeds in a product of basic injectives). An injective I is viewed as an object $K(I)$ of $DG-\mathcal{A}$ with zero differential. The notation is chosen to suggest an Eilenberg-Mac Lane object (or cosphere). Next, we let $P(I) = \mathrm{fibre}(1 : K(I) \longrightarrow K(I))$, with the notation chosen to suggest a path object (or codisc). The set $\mathcal{L}$ of generating fibrations consists of the maps $P(I) \longrightarrow K(I)$ for I in $\mathcal{BI}$. The set $\mathcal{M}$ of generating acyclic fibrations consists of the maps $P(I) \longrightarrow 0$ for I in $\mathcal{BI}$.

We now take **we** to consist of quasi-isomorphisms, **cof** to be the maps with the left lifting property with respect to $\mathcal{M}$ and **fib** to be the maps with the right lifting property with respect to (**we** $\cap$ **cof**), and prove this forms the model structure of the lemma. We outline the four main steps and then turn to proving they can be completed in our current situation.

Step 1: Show that **cof** consists of objectwise monomorphisms.

Step 2: Show that for any X there is an objectwise monomorphism $\alpha : X \longrightarrow P(I)$ for some injective I.

Step 3: Show that the maps $P(I) \longrightarrow K(I)$ and $P(I) \longrightarrow 0$ in $\mathcal{L}$ and $\mathcal{M}$ respectively are in **fib**.

Note that since any injective is a retract of a product of basic injectives, it follows that $P(I) \longrightarrow K(I)$ and $P(I) \longrightarrow 0$ are fibrations for any injective I. Since we have chosen $\mathcal{BI}$ to contain enough injectives, one of the factorization axioms follows immediately, since we may factorize $f : X \longrightarrow Y$ as

$$X \stackrel{\{f,\alpha\}}{\longrightarrow} Y \times P(I) \stackrel{\simeq}{\longrightarrow} Y,$$

with α as in Step 2.

Step 4: Prove the second factorization axiom using only fibrations formed from those named in Step 3.

More precisely, given $f : X \longrightarrow Y$, we form a factorization $X \longrightarrow X' \longrightarrow Y$ with $X \longrightarrow X'$ a quasi-isomorphism and $X' \longrightarrow Y$ a fibration formed by iterated pullbacks of fibrations $P(I) \longrightarrow K(I)$. This is precisely dual to the usual argument attaching cells to make a map of spaces into a weak equivalence, but because the dual of the small object arguement does not apply, we use the finiteness of injective dimension of $\mathcal{A}$ to see that only finitely many steps are involved in the process (details below). The map $X \longrightarrow X'$ can be made into a cofibration by taking the product of X' with a suitable $P(I)$ as in the proof of the first factorization argument. It follows using the defining right lifting property that an arbitrary fibration is a retract of one formed by iterated pullbacks of fibrations $P(I) \longrightarrow K(I)$ or $P(I) \longrightarrow 0$.

It remains to verify the four steps can be completed. We follow the pattern from the case of the circle group in [21, Appendix B]. We note that for each connected subgroup H of G there is an evaluation functor

$$ev_H : R_a\text{-modules} \longrightarrow \mathcal{O}_{\mathcal{F}/H}\text{-modules}$$

with right adjoint f_H. In particular, if N is a torsion module, $f_H(N)$ lies in $\mathcal{A}(G)$ and

$$\mathrm{Hom}_{\mathcal{A}(G)}(X, f_H(N)) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/H}}(\phi^H X, N).$$

We take the basic injectives to be those of the form

$$\mathbb{I}_{\tilde{H}} = f_H(H_*(BG/\tilde{H}))$$

where $\tilde{H}$ is any subgroup with identity component H. It is shown in [24, 2.20] that this set contains sufficiently many injectives.

The following elementary lemma lets us reduce verifications to statements about modules with zero differential over a (single object) ring. We write Hom for the differential graded object of graded $\mathcal{A}$-morphisms and let $DG-\mathrm{Hom}$ denote the group of morphisms commuting with the differential. The differential on Hom is defined so that the DG-morphisms are the 0-cycles in Hom.

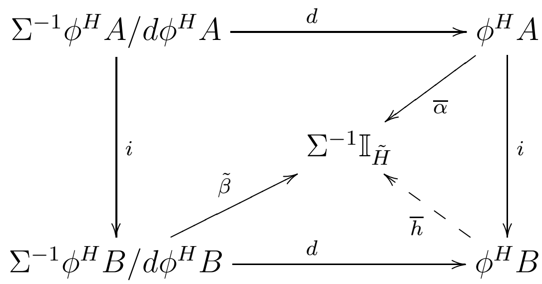
Lemma 12.4. *(i)* $\mathrm{Hom}_{\mathcal{A}}(X, K(f_H(M))) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/H}}(\phi^H X, M)$
(ii) $DG - \mathrm{Hom}(X, K(f_H(M))) = \mathrm{Hom}(\phi^H X/d\phi^H X, M)$
(iii) $DG - \mathrm{Hom}(X, P(f_H(M))) = \mathrm{Hom}(\Sigma\phi^H X, M)$

It follows from this lemma by the left lifting property that **cof** consists of objectwise monomorphisms (Step 1), see also [21, Lemma B.2], and that we may find a monomorphism α in the first factorization argument (Step 2): for this we first embed all $\phi^H X$ in some injective $I_H(X)$ ignoring the differential and use Lemma 12.4(iii) to obtain a map to $P(f_H(I_H(X)))$, and take the product of these over all H to to obtain $P(I)$.

This lemma also makes it straightforward to verify that objects of $\mathcal{L}$ and $\mathcal{M}$ are fibrations. The case of $P(I) \longrightarrow 0$ is simply the defining property of an injective. The problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & P(f_H(\mathbb{I}_{\tilde{H}})) \\ \downarrow i & \overset{h}{\nearrow} & \downarrow \\ B & \xrightarrow{\beta} & K(f_H(\mathbb{I}_{\tilde{H}}))) \end{array}$$

is equivalent to



To find a solution we use a standard diagram chase. We first use the fact that i is a homology epimorphism to deduce that $\tilde{\beta}$ vanishes on cycles and the fact that it is a homology monomorphism to see that this means that $\tilde{h}$ is consistently defined on $\phi^H A + d\phi^H B$. Finally, we use the defining property of injectives to extend it over $\phi^H B$.

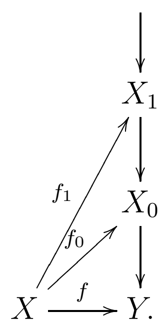
This leaves Step 4. Here we start by forming an exact sequence

$$0 \longrightarrow H_*(X) \longrightarrow H_*(Y) \oplus I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_N \longrightarrow 0$$

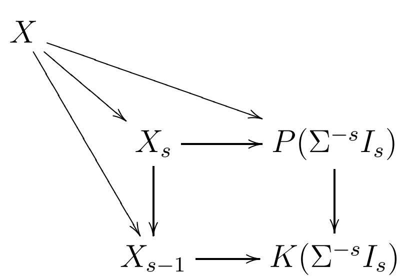
in $\mathcal{A}^s(G)$, where the I_s are injective. The finite injective dimension of $\mathcal{A}^s(G)$ ensures such an exact sequence exists. We now realize this by a tower of fibrations

$$Y \longleftarrow X_0 \longleftarrow \cdots \longleftarrow X_N = X',$$

together with lifts



We take $X_0 = Y \oplus K(I_0)$, and the subsequent objects and maps are constructed using the diagram



where the lower horizontal is chosen to realize the inclusion of $\mathrm{im}(I_{s-1} \longrightarrow I_s)$ in I_s. The map $f_N : X \longrightarrow X_N$ is necessarily a quasi-isomorphism, and can be made into a monomorphism by taking a product with a suitable $P(I)$.

This completes the sketch proof of the proposition. $\square$

12.D. **Equivalence of models of torsion modules.** We recall that $R_a = \mathbb{R}^f_d$, and work with the adjunction of Proposition 11.1.

Proposition 12.5. *The adjunction*

$$l : \mathcal{A}^p_c(G) = qce\text{-}\mathbb{R}^p_c\text{-}modules \rightleftarrows R_a\text{-}mod_{ii} : \Gamma \ .$$

is a Quillen adjunction, where the subscript ii refers to the use of the doubly injective model structure on R_a-modules (i.e., injective in both the module theoretic and diagram theoretic sense) and where $l = d_* if$ and $\Gamma = p\Gamma^f_c e$

Cellularizing with respect to the images of the topological cells induces a Quillen equivalence

$$\mathcal{A}(G) = \mathcal{A}^p_c(G) = qce\text{-}\mathbb{R}^p_c\text{-}modules \simeq cell\text{-}R_a\text{-}mod_{ii}.$$

Proof: First we need to check that $l = d_* if$ preserves cofibrations and acyclic cofibrations so that we have a Quillen adjunction.

The cofibrations in $\mathcal{A}^p_c(G)$ are the monomorphisms, which are the objectwise monomorphisms. Similarly, the cofibrations in an algebraically injective model structure are precisely the monomorphisms. The cofibrations in the doubly injective R_a-module category are precisely the morphisms which are objectwise cofibrations, namely the objectwise monomorphisms. It is obvious that f and i preserve monomorphisms. It is also clear that the functor $d_!$ (given by taking the product of the values) preserves monomorphisms. Since $d_*N \subseteq d_!N$, it follows that d_* also preserves momomorphisms.

The weak equivalences in both categories are objectwise quasi-isomorphisms, and we will show l preserves all homology isomorphisms. Since l is defined at the level of abelian categories, it takes mapping cones to mapping cones. It therefore suffices to show that if X is a *qce*-module with $H_*(X) = 0$ then $H_*(lX) = 0$. For this we use a filtration described in [28, Section 6] (the map $d : \Sigma_c \longrightarrow [0, r]$ and the diagram $\mathbb{R}^f_c$ take the roles of the map $\pi : \Sigma \longrightarrow \overline{\Sigma}$ and the ring R^f). To avoid complicating the notation we will omit the notation if since ifX takes the same values as X on pairs.

For each flag $f = (f_0 > \cdots > f_s)$ of dimensions we consider the value $(d_*X)(f)$ at f. Inside this we have the generating submodules M_{f_i} for $i = 0, 1, \ldots, s$ (this is the submodule generated by the image of $(d_*X)(f_i) = \bigoplus_{\dim K = f_i} X(K)$). There is an associated Mayer-Vietoris spectral sequence for these, showing that it suffices to show that for each face $e = (e_0 > e_1 > \cdots > e_t) \subset (f_0 > f_1 > \cdots > f_s) = f$ the intersection

$$M_e = \bigcap_j M_{e_j}$$

is acyclic. A combinatorial lemma [28, Lemma 6.7] shows that M_e is generated by the image of the diagonals including e in f. Furthermore

$$M_e = \sum_{\dim E = e} M_E = \bigoplus_{\dim E = e} M_E$$

so it suffices to show that M_E is acyclic.

Now consider the diagram

$$\begin{array}{ccc} \mathbb{R}_d(f)\otimes_{\mathbb{R}_d(e)} X(E) & \longrightarrow & (d_!M)(f) \\ \downarrow & & \downarrow \cong \\ d_!e[\mathbb{R}_d(f)\otimes_{\mathbb{R}_d(e)} X(E)] & \longrightarrow & (d_!M)(f). \end{array}$$

in which M_E is the image of the top horizontal. We argue that the top horizontal is in fact a monomorphism, and it then follows since $\mathbb{R}_d(f)$ is flat over $\mathbb{R}_d(e)$ that M_E is acyclic.

In fact the bottom horizontal is an isomorphism since X is *qce*; indeed the Fth idempotent piece is the map $\mathbb{R}_d(F)\otimes_{\mathbb{R}_d(E)} X(E)\longrightarrow X(F)$. The left hand vertical is a monomorphism since it can be viewed as a composite

$$\mathbb{R}_d(f)\otimes_{\mathbb{R}_d(e)} X(E)\longrightarrow \mathbb{R}_d(f)\otimes_{\mathbb{R}_d(e)} \prod X(E)\longrightarrow d_!e\mathbb{R}_d(f)\otimes_{\mathbb{R}_d(e)} X(E);$$

the first is a monomorphism since the diagonal is and $\mathbb{R}_d(f)$ is flat over $\mathbb{R}_d(e)$, and the second map is an isomorphism. It follows that the top horizontal is a monomorphism as required.

This shows that we have a Quillen pair, and we now cellularize with respect to the images of the cells G/H_+. By the Cellularization Principle [33] this induces a Quillen equivalence of cellularizations since the cells are small and lie in $\mathcal{A}^p_c(G)$.

Finally, it remains to check that cellularization is the identity on $\mathcal{A}^p_c(G)$. This will be completed by Proposition 13.8 which states that cellular equivalences for qce modules are precisely the quasi-isomorphisms. Thus,

$$\mathcal{A}^p_c(G) = qce\text{-}\mathbb{R}^p_c\text{-modules} = \text{cell-}qce\text{-}\mathbb{R}^p_c\text{-modules}.$$

□

Remark 12.6. In order to obtain monoidal equivalences, we will need a monoidal model structure on $\mathcal{A}(G)$. For this we need to extend Barnes's flat model structures from the rank 1 case to the arbitrary case, using [25].

13. Cellular equivalences in $\mathcal{A}(G)$

The main purpose of this section is to show that cellular equivalences coincide with quasi-isomorphisms for $\mathcal{A}(G)$. This comes in Subsection 13.E. Since cells are determined by their homology (by [24, 12.1], quoted as Corollary 2.9), we need not choose particular models. Nonetheless, we begin by describing some models for the algebraic cells, since this gives us an opportunity to introduce some essential properties in a concrete fashion.

13.A. **Cohomology of subgroups and homotopy of cells.** To guide our construction, we calculate the homology of the natural cells G/K_+ as objects of $\mathcal{A}(G)$, and then give the appropriate adaption using Koszul complex constructions. First we need a little more background. We have already discussed the relationship between G and its quotient groups G/K, together with the associated inflation map $\mathcal{O}_{\mathcal{F}/K}\longrightarrow\mathcal{O}_{\mathcal{F}}$. We now need to discuss the cohomology of subgroups and the associated restriction maps.

To start with note that rational cohomology of a subgroup K of the torus depends only on the identity component: the restriction map induces an isomorphism $H^*(BK)\xrightarrow{\cong} H^*(BK_1)$, since the component group K/K_1 necessarily acts trivially on $H^*(BK_1)$.

Next, write $\mathcal{F}(K)$ for the set of finite subgroups of K when emphasis is required, and similarly

$$\mathcal{O}_{\mathcal{F}}^K = \prod_{F \in \mathcal{F}(K)} H^*(BK/F),$$

noting that this makes sense whether or not K is connected. The restriction map is the composite

$$\mathcal{O}_{\mathcal{F}}^G = \prod_{F \in \mathcal{F}(G)} H^*(BG/F) \longrightarrow \prod_{F \in \mathcal{F}(K)} H^*(BG/F) \longrightarrow \prod_{F \in \mathcal{F}(K)} H^*(BK/F) = \mathcal{O}_{\mathcal{F}}^K$$

where the first map is projection onto the direct summand corresponding to the factors with $F \in \mathcal{F}(K)$.

Lemma 13.1. *If K is a subgroup of G of codimension $c(K)$ then as an R_a-module $\pi_*^{\mathcal{A}}(G/K_+)$ is concentrated on the connected subgroups of K and is given by*

$$\pi_*^{\mathcal{A}}(G/K_+)(L) = \begin{cases} \mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/L}} \Sigma^{c(K)}\mathcal{O}_{\mathcal{F}/L}^{\overline{K}} & \textit{if } L \textit{ is a connected subgroup of } K \\ 0 & \textit{otherwise.} \end{cases}$$

where bars indicate images in $\overline{G} = G/L$.

Proof: We must calculate

$$\pi_*^G(DE\mathcal{F}_+ \wedge S^{\infty V(L)} \wedge G/K_+) = \mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/L}} \pi_*^{G/L}(DE\mathcal{F}/L_+ \wedge \Phi^L G/K_+).$$

This is evidently zero unless $L \subseteq K$, and if $L \subseteq K$, then $\Phi^L G/K_+ = (G/K)_+^L = \overline{G}/\overline{K}_+$, where bars indicate the image in G/L. Accordingly,

$$\pi_*^{\overline{G}}(DE\mathcal{F}/L_+ \wedge \overline{G}/\overline{K}_+) \cong \Sigma^{c(K)}\pi_*^{\overline{K}}(DE\mathcal{F}_+/L) = \Sigma^{c(K)} \prod_{L \subseteq \tilde{L} \subseteq K} H^*(B(K/\tilde{L})).$$

□

13.B. **Koszul complexes.** To form suitable models, we use a standard construction from commutative algebra. Given a graded commutative ring B and elements $x_1, \ldots, x_n$ we may form the Koszul complex

$$K(x_1, \ldots, x_n) = (\Sigma^{|x_1|}B \xrightarrow{x_1} B) \otimes_B \cdots \otimes_B (\Sigma^{|x_n|}B \xrightarrow{x_n} B),$$

which is finitely generated and free as a B-module. If B is a polynomial ring $B = k[x_1, \ldots, x_n]$ we write just Kos_B for the complex; this is independent of the choice of homogeneous generators up to isomorphism, and the natural map $\mathrm{Kos}_B \longrightarrow k$ is a quasi-isomorphism. If M is a B-module we write $\mathrm{Kos}_B(M) = \mathrm{Kos}_B \otimes_B M$.

We say that L is *cotoral* in K if L is a normal subgroup of K and K/L is a torus. We note that if L is cotoral in K then we may choose a map $G/L \longrightarrow K/L$ giving an isomorphism $G/L = G/K \times K/L$ and hence

$$H^*(BG/L) = H^*(BG/K) \otimes H^*(BK/L).$$

We may therefore form a version of $H^*(BG/K)$ which is flat over $H^*(BG/L)$ by tensoring with the Koszul complex model for $H^*(BK/L)$ based on a set of generators for

$\ker(H^*(BG/L) \longrightarrow H^*(BK/L))$. However it is not necessary to make this choice, since (even if K/L is not a torus), the exact sequence

$$K/L \longrightarrow G/L \longrightarrow G/K$$

induces an exact sequence

$$H^*(BK/L) \longleftarrow H^*(BG/L) \longleftarrow H^*(BG/K)$$

of Hopf algebras, so that

$$H^*(BK/L) = H^*(BG/L) \otimes_{H^*(BG/K)} k.$$

Now replace k by the complex $\mathrm{Kos}_{H^*(BG/K)}$ of projective $H^*(BG/K)$-modules, independent of L, and the tensor product will be a complex of projective $H^*(BG/L)$-modules,

$$\mathrm{Kos}_{H^*(BG/K)}(H^*(BG/L)) \simeq H^*(BK/L).$$

If $H^*(BG/K)$ is replaced by a DGA $A(K)$ with the same cohomology, the complexes $\Sigma^{|x|}B \longrightarrow B$ are replaced by the fibres of the maps $\Sigma^{|x|}A(K) \longrightarrow A(K)$; up to equivalence, this only depends on the cohomology classes. The full Koszul complex $\mathrm{Kos}_{A(K)}$ is obtained as before by tensoring together the DG modules for a chosen set of polynomial generators.

13.C. **The flat form of the natural cells.** To apply the Koszul complexes in our case we choose generators for $H^*(BG/K)$ and form the associated Koszul complex, $\mathrm{Kos}_{H^*(BG/K)}$. Note that we have a diagonal map

$$\Delta : H^*(BG/K) \longrightarrow \prod_{\tilde{L}\in\mathcal{F}/L} H^*(BG/L) \cong \prod_{\tilde{L}\in\mathcal{F}/L} H^*(BG/\tilde{L}) = \mathcal{O}^G_{\mathcal{F}/L},$$

so we may form $\mathrm{Kos}_{H^*(BG/K)}(\mathcal{O}^G_{\mathcal{F}/L})$, which is free as a $\mathcal{O}^G_{\mathcal{F}/L}$-module.

Lemma 13.2. *The Koszul complex gives good approximations of quotient groups in the sense that*

$$H_*(e_{\overline{K}}\mathcal{O}^{G/L}_{\mathcal{F}/L} \otimes_{\mathcal{O}^G_{\mathcal{F}/L}} \mathrm{Kos}_{H^*(BG/K)}(\mathcal{O}^G_{\mathcal{F}/L})) \cong \mathcal{O}^{\overline{K}}_{\mathcal{F}/L},$$

where $e_{\overline{K}}$ is the idempotent corresponding to the finite subgroups of G/L contained in $\overline{K}$.

Proof: We have remarked that the short exact sequence

$$0 \longrightarrow H^*(BG/K) \longrightarrow H^*(BG/L) \longrightarrow H^*(BK/L) \longrightarrow 0$$

of Hopf algebras shows that

$$H_*(H^*(BG/L) \otimes_{H^*(BG/K)} \mathrm{Kos}_{H^*(BG/K)}) \cong H^*(BK/L).$$

We are just taking a product of instances of this indexed by finite subgroups of G/L contained in K/L. □

This motivates the following definition.

Definition 13.3. The flat form of model for the cell G/K_+ is defined by

$$\underline{\sigma}_K(L) = \begin{cases} \mathcal{E}_L^{-1} e_K \mathcal{O}^G_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \Sigma^{c(K)}\mathrm{Kos}_{H^*(BG/K)}(\mathcal{O}_{\mathcal{F}/K}) & \text{if } L \text{ is cotoral in } K_1 \\ 0 & \text{otherwise} \end{cases}$$

We may now prove that this is indeed a model.

Lemma 13.4. *The flat model $\underline{\sigma}_K$ is a model for G/K_+ in $\mathcal{A}(G)$ and any other model is weakly equivalent to it.*

Proof: The uniqueness theorem [24, 12.1] states that cells are characterized by their homology, so it suffices to show that

$$H_*(\underline{\sigma}_K) \cong \pi_*^{\mathcal{A}}(G/K_+).$$

Lemmas 13.1 and 13.2 show that the value at L should be

$$\mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}}^G \otimes_{\mathcal{O}_{\mathcal{F}/L}} e_{\overline{K}}\mathcal{O}_{\mathcal{F}/L}^{\overline{G}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \Sigma^{c(K)}\mathrm{Kos}_{H^*(BG/K)}(\mathcal{O}_{\mathcal{F}/K}),$$

and we calculate

$$\mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}}^G \otimes_{\mathcal{O}_{\mathcal{F}/L}} e_{\overline{K}}\mathcal{O}_{\mathcal{F}/L}^{\overline{G}} \cong \mathcal{E}_L^{-1}e_K\mathcal{O}_{\mathcal{F}}^G.$$

□

It is worth recording the following immediate consequence of the definition.

Lemma 13.5. *The flat model $\underline{\sigma}_K$ is built from the model $\underline{\sigma}_G = \mathcal{O}_{\mathcal{F}}^G$ of the sphere by taking a retract and then using finitely many fibre sequences.* □

13.D. **Properties of the flat model $\underline{\sigma}_K$.** By construction the cells themselves have torsion homology, which gives one of the properties we require.

Lemma 13.6. *In the model category cell-R_a-mod the cellular objects have torsion homology.*

Proof: By construction, $H_*(\sigma_K) = \pi_*^{\mathcal{A}}(G/K_+)$, which lies in $\mathcal{A}(G)$. Since the subcategory of torsion modules is an abelian subcategory closed under sums, any object built from cells has torsion homology. □

Next, there is a finiteness requirement if we are to use these as the generating objects for our cofibrantly generated model structure.

Lemma 13.7. *For any subgroup K, the flat model σ_K for the cell G/K_+ is small in the abelian category in the sense that*

$$\mathrm{Hom}(\sigma_K, \bigoplus_i N_i) = \bigoplus_i \mathrm{Hom}(\sigma_K, N_i).$$

Proof: In view of Lemma 13.5, it suffices to prove the special case $K = G$.

The value of a map $f : \sigma_G \longrightarrow M$ is determined by its value at $L = 1$. On the other hand

$$f(1) : \sigma_G(1) = \mathcal{O}_{\mathcal{F}} \longrightarrow M(1)$$

is determined by the image of the identity. □

13.E. **Algebraic cells and quasi-isomorphisms.** We need to understand weak equivalences between objects of $\mathcal{A}(G)$. For clarity, we refer to maps $X \longrightarrow Y$ inducing an isomorphism of $H_*(\mathrm{Hom}(\underline{\sigma}_K, \cdot))$ for all cells σ_K as *cellular equivalences*, and for brevity we write

$$\underline{\pi}^G_*(X) = H_*(\mathrm{Hom}(\underline{\sigma}_\bullet, \cdot))$$

for the resulting Mackey-functor.

We begin with a warning: the flat models are not cofibrant.

For example if G is a circle, we have

$$\mathrm{Hom}(S^0, X) = PB(X),$$

where X and its pullback are as displayed:

$$\begin{array}{ccc} PB(X) & \longrightarrow & V \\ \downarrow & & \downarrow \\ N & \longrightarrow & \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes V. \end{array}$$

It is not hard to construct examples of acyclic X for which $PB(X)$ is not acyclic. This simply means that we retreat from being so explicit at the level of models. Our remaining work takes place at the level of homotopy categories.

The key to removing the cellularization process in the formation of $\mathcal{A}(G)$ is to show that there are enough cells in the sense that cellular equivalences of torsion modules are quasi-isomorphisms.

Proposition 13.8. *For objects of $\mathcal{A}(G)$, cellular equivalences are homology isomorphisms.*

Remark 13.9. Some may find it helpful to consider the corresponding result for spectra where the isotropic filtration is more familiar.

Taking mapping cones, it suffices to show that cellularly trivial objects of are acyclic. Suppose then that X is cellularly trivial. Since S^0 is built from objects $E(\mathcal{F}/K)_+ \wedge S^{\infty V(K)}$ by finitely many cofibre sequences, X is built from $X\langle \mathcal{F}/K\rangle = X \wedge E(\mathcal{F}/K)_+ \wedge S^{\infty V(K)}$ by finitely many cofibre sequences, and it suffices to show that the homology of $X\langle \mathcal{F}/K\rangle$ is zero. However, for spectra of this form, cellular triviality and acyclicity are equivalent. Finally, by duality of cells, cellular triviality of X implies that of $X \wedge T$ for any T.

Before we begin, we need two lemmas.

Lemma 13.10. *If $\underline{\pi}^G_*(X) = 0$ then $\underline{\pi}^G_*(X \otimes C) = 0$ for any cellular object C.*

Proof: The lemma follows from the special case in which C is a cell $\underline{\sigma}_L$. In fact, we have a homotopy equivalence of DGAs

$$\mathrm{Hom}(\underline{\sigma}_K, X \otimes \underline{\sigma}_L) \simeq \Sigma^{c(L)}\mathrm{Hom}(\underline{\sigma}_{K\cap L}, X) \otimes H^*(G/(KL))$$

where $H^*(G/(KL))$ has zero differential and $c(L)$ is the codimension of L.

In fact both sides can be formed from $e_{K\cap L}\mathrm{Hom}(S^0, X)$ by taking iterated fibres of maps which are multiplication by some element of $\mathcal{O}_{\mathcal{F}}$ obtained by inflation. On the left we take a set of polynomial generators for $H^*(BG/K)$ and a set $y_1, \ldots, y_e$ of polynomial generators for $H^*(BG/L)$ and inflate them. It is convenient to choose generators $x_1, \ldots, x_d$ of $H^*(BG/(KL))$ and extend the collection by $y_1, \ldots, y_e$ to give generators of $H^*(BG/K)$ and by $z_1, \ldots, z_f$ to give generators of $H^*(BG/L)$. On the left we may then choose $x_1, \ldots, x_d, y_1, \ldots, y_d, z_1, \ldots, z_e$ as our generators of $H^*(BG/(K \cap L))$. Since the map

x of a Koszul complex $K(x)$ is nullhomotopic, the effect of the second set of generators $x_1, \ldots, x_e$ simply increases multiplicity. □

We need to use certain standard objects of $\mathcal{A}(G)$ constructed from modules. There is a right adjoint f_K to evaluation at G/K. Roughly speaking, for suitable $\mathcal{O}_{\mathcal{F}/K}$-modules M, the object $f_K(M)$ is obtained by putting M at G/K, and filling in other values accordingly (see [24, Section 4] for further details).

Lemma 13.11. *Suppose X is a torsion object with $H_*(X) = f_K(M)$ for some connected subgroup K and some $\mathcal{O}_{\mathcal{F}/K}$-module M. If X is cellularly trivial, it is acyclic: if $\underline{\pi}^G_*(X) = 0$ then $H_*(X) = 0$.*

Proof: Since f_K is right adjoint to evaluation at K, and since this is compatible with resolutions, the Adams spectral sequence for $[T, X]^G$ takes the simple form

$$E_2^{s,t} = \mathrm{Ext}^{*,*}_{\mathcal{O}_{\mathcal{F}/K}}(\phi^K H_*(T), M) \Rightarrow [T, X]^G_*.$$

In particular, taking $T = S^0$, we see that $H_*(X)$ is one of the entries in $\underline{\pi}^G_*(X)$. □

Proof of Proposition 13.8: Taking mapping cones, it suffices to show that cellularly trivial objects of $\mathcal{A}(G)$ are acyclic. Suppose then that X is cellularly trivial.

We argue by induction on the dimension of support of $H_*(X)$ that cellularly trivial objects of $\mathcal{A}(G)$ are acyclic. There is nothing to prove if the support is in dimension < 0 (i.e., if $H_*(X) = 0$).

Suppose then that the result is proved for objects with support in dimension $< d$ and that X is supported in dimension $\leq d$. We then define X' using the fibre sequence

$$X' \longrightarrow X \longrightarrow \bigoplus_{\dim(K)=d} f_K(\phi^K H_*(X)).$$

Next, note that

$$f_K(\phi^K H_*(X)) \simeq X \otimes S^{\infty V(K)}.$$

This is cellularly trivial by Lemma 13.10, and hence also acyclic by Lemma 13.11.

Finally, we claim there is an equivalence

$$X' \simeq X \otimes F$$

for suitable F. Indeed, since X is supported in dimension $\leq d$ we have $X \simeq X \otimes E\mathcal{F}(d)_+$, where $\mathcal{F}(d)$ is the family of subgroups of dimension $\leq d$. We may then take F to be defined by the fibre sequence

$$F \longrightarrow E\mathcal{F}(d)_+ \longrightarrow \bigvee_{\dim(K)=d} E\mathcal{F}(d)_+ \otimes S^{\infty V(K)}.$$

Thus X' is cellularly trivial by Lemma 13.10 and it is thus acyclic by induction. It follows that X is acyclic, which completes the inductive step. The general case follows in $r+1$ steps.

This completes the proof of Proposition 13.8 □

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