Note No. 93

THE COLLEGE OF AERONAUTICS

CRANFIELD

A SIMPLIFIED THEORY OF SKIN FRICTION AND HEAT TRANSFER FOR A COMPRESSIBLE LAMINAR BOUNDARY LAYER

by

G. M. LILLEY

NOTE NO. 93.

January, 1959.

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A simplified theory of skin friction and heat transfer for a compressible laminar boundary layer

- by -

G. M. Lilley, M.Sc., D.I.C.
of the
Department of Aerodynamics

SUMMARY

The compressible laminar boundary layer equations for a perfect gas in steady flow at arbitrary external Mach number and wall temperature distribution are solved approximately by the combined use of the Stewartson-Illingworth transformation and application of Lighthill's method to yield the skin friction and rate of heat transfer.

Appendices are added which give the necessary modifications to the method for the separate cases of very low Prandtl number and for the flow near a separation point. A further appendix describes Spalding's method for improving the accuracy of the wall value of shear stress and rate of heat transfer distributions along a wall having a non-uniform temperature distribution.

This paper was first written in January, 1959. A few minor alterations have been done during proof reading January, 1960.

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Appendices

List of Symbols

a	speed of sound
$\bar{a}_m$	constant (see equ. 42)
b	constant (see equ.40)
c	constant (see equ. 37)
C_p, C_v	Specific heats at constant pressure and constant volume respectively
C	$\frac{\mu T_o}{\mu_o T}$
$f(\bar{x})$, $g(\bar{x})$	functions of $\bar{x}$ (see equ. 44")
$F(x)$, $G(x)$, $H(x)$, $J(x)$	functions of x (see equ. 34 and 35)
$G(X, \psi) = Z - \int_0^X S(z, \psi)\, dU_1^2(z)$	
h	stagnation enthalpy
i	enthalpy
$\bar{J}(x)$	function of x (see equ. D.5)
k	thermal conductivity
k_h	Stanton heat transfer coefficient
$K(x)$, $\bar{K}(x)$	functions of x (see equ. D.11 and D.12)
l	body reference length
m	external velocity gradient index
M	Mach number
n	wall temperature gradient index
p	pressure; Heaviside operator
$\dot{q}_w$	rate of heat transfer per unit area
$\dot{Q}_w$	overall rate of heat transfer $(\frac{1}{l} \int_0^l \dot{q}_w(x)\, dx)$
R_x	local value of Reynolds number

List of Symbols (Continued)

$s_w(x)$	=	$\dfrac{\mathring{q}_w(x)}{(h_w(+0) - h_{wo}(+0))} \sqrt{\dfrac{x}{\rho_a \; \mu_a \; u_a}}$
S	=	$1 - {}^{h}/{}_{h_1}$
t_w	=	$\tau_w(x) \sqrt{\dfrac{x}{\rho_a \; \mu_a \; u_a^3}}$
t		independent variable (see equ.22 and 51)
T		Temperature
(u,v)		velocity components in compressible flow
(U,V)		velocity components in transformed (incompressible) flow
(x,y)		coordinates in compressible flow
(X,Y)		coordinates in transformed (incompressible) flow
z		dummy variable of integration
Z	=	$U_1^2 - U^2$
γ		ratio of specific heats
η	=	$x^{\frac{3}{4}}$
μ		viscosity
ν		kinematic viscosity
ρ		density
σ		Prandtl number
ψ		stream function
τ_w		wall shear stress
χ	=	$\tau_w(x) \sqrt{\dfrac{x}{\rho_1 \mu_1 u_1^3}}$
ω		viscosity - temperature index

List of Symbols (Continued)

Subscripts

o	stagnation value
1	value outside the boundary layer
w	value at the wall
a	reference condition

1. Introduction

For bodies travelling at very high speeds through the atmosphere there exists near the body nose extensive regions of laminar boundary layer flow. There is, therefore, considerable interest in finding rapid accurate methods for estimating the skin friction and heat transfer in a laminar boundary layer of a perfect gas at high speeds under conditions of arbitrary velocity and wall temperature distributions. Although in practical problems real gas effects are likely to be of importance in the higher speed ranges, yet under certain conditions, solutions obtained assuming the fluid in the boundary layer is a perfect gas, may be used in preliminary calculations, provided they are interpreted correctly.

It is generally accepted that the rapid accurate estimation of the overall characteristics of a compressible laminar boundary layer, for arbitrary distributions of external velocity and wall temperature (or heat transfer), is best performed by the use of the momentum and energy integral equations. For the case when the Prandtl number (σ) equals unity and the viscosity - temperature index (ω) equals unity Curle (1958b) describes a modified Pohlhausen method, analogous to Thwaite's method in incompressible flow, by which the skin friction can be evaluated for the case of heat transfer with uniform wall temperature, and in Curle (1958a) for non-uniform wall temperature. An earlier paper by Curle (1957a) describes a similar method for the case of zero heat transfer. For the latter case other methods exist including those of Young (1949) and Tani (1954). Young's method has been extended by Luxton and Young (1958) to deal with the effect of heat transfer. For the special cases $\omega = 1$ and $\sigma = 1$, 0.7 Levy (1954), and when $\omega = \sigma = 1$ Cohen and Reshotko (1956a, 1956b), give results for arbitrary pressure distribution and constant wall temperature. The work of Levy stems from the Illingworth (1949) whilst that of Cohen and Reshotko stems from the Stewartson (1949) transformation of the compressible boundary layer equations whereby, for $\sigma = \omega = 1$ and zero heat transfer, the compressible flow equations are transformed exactly into the incompressible flow equations. Thus known solutions of the latter equations can be used to solve corresponding compressible flow problems. A modified form of this transformation for arbitrary σ has been described by Rott (1953) and was used by Tani (1954).

An alternative approach has been used by Lighthill (1950) for finding the skin friction and heat transfer from a wall of non-uniform temperature. The method, which makes use of Von Mises' form of the boundary layer equations and uses a linear approximation to the velocity distribution near the wall, is applicable to low Mach number flows of variable external velocity and to high Mach number flows of uniform external velocity. Lighthill has applied this method to the problem of the wall temperature distribution on a flat plate at high Mach numbers which is losing heat by radiation only. Tifford (1951), and Tribus and Klein (1955), have modified Lighthill's method to include a better approximation to the velocity distribution near the wall and so provide a more accurate method for retarded flows, in which the linear approximation to the velocity distribution is inadequate. A modified and improved from of Tifford's correction has been given by Spalding (1958), who shows that with this new correction the error in the Lighthill method for heat transfer can be reduced to less than 2.5% regardless of pressure gradient. Liepmann (1958) has rederived Lighthill's formula for the rate of heat transfer using an energy integral approach. Other methods, such as Chapman and Rubesin (1949), Schuh (1953) and Imai (1958), for solving the heat transfer in compressible flow in which the wall temperature distribution is expressed as a polynomial in x (the distance along the surface) do not have the range of application of Lighthill's method.

Illingworth (1954) has extended Lighthill's method to deal with variable freestream and wall temperature distribution in a compressible flow when $\sigma = \omega = 1$. However in applications he considered only the case of constant wall temperature and mainstream velocity distributions expressed as polynomials in x.

The aim of the present paper is to produce an approximate rapid method for solving the compressible flow boundary layer equations for arbitrary external Mach number and wall temperature distribution. The Prandtl number (σ) will be taken as arbitrary, though not small compared with unity, and although it is assumed $\mu \alpha$ T across the boundary layer a more accurate viscosity-temperature dependence for the wall viscosity will be taken as suggested by Chapman and Rubesin (1949). The proposed method makes use of the Stewartson-Illingworth transformation, but not restricting its use to zero heat transfer, whereby the transformed equations are solved by the method of Lighthill (1950). A few examples of its application are given and the results are compared with other known solutions. It is found that even in the severe test of the application of the method to the case of a small adverse pressure gradient the accuracy is probably adequate for engineering purposes. The accuracy of the method can be improved by the addition of Spalding's correction and a brief account of this has been included in an appendix to the present paper.

In other appendices the rate of heat transfer is evaluated using different approximations to the velocity distribution close to the wall for the two cases,

(a) very small Prandtl number

and (b) at a separation point.

2. Basic equations

The steady two-dimensional boundary layer equations of continuity, motion and energy for a perfect gas are respectively

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad (1)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \rho_1 u_1 \frac{du_1}{dx} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad (2)$$

$$\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} + u \rho_1 u_1 \frac{du_1}{dx} = \frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\frac{\partial i}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad (3)$$

where i is the enthalpy and σ is the Prandtl number. Suffix (1) denotes the local conditions outside the boundary layer.

If the flow external to the boundary layer is isentropic (and only this case will be considered in this paper)

$$i_1 + u_1^2/2 = \text{const.} \qquad (4)$$

or if a_1 is the local speed of sound and γ is the ratio of the specific heats

$$a_1^2 + \frac{\gamma - 1}{2} u_1^2 = \text{const.} \qquad (4')$$

The stagnation enthalpy (or as it is sometimes called the specific total energy) equation is found by multiplying (2) by u and adding it to equation (3). If the stagnation enthalpy

$$h = i + u^2/2$$

the result is

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{\mu}{\sigma}\frac{\partial}{\partial y}\left(h - (1-\sigma)\frac{u^2}{2}\right)\right\} \qquad (5)$$

The boundary conditions to be applied are at

$$y = 0 \qquad u = v = 0 \quad ; \quad T_W \equiv T_W(x)$$

$$-\left(k\frac{\partial T}{\partial y}\right)_W = \mathring{q}_W(x) \; ; \; \left(\mu\frac{\partial u}{\partial y}\right)_W = \tau_W(x)$$

and at

$$y = \infty \qquad u = u_1(x) \qquad T = T_1(x)$$

$$\frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0 ,$$

where suffix (x) denotes the wall value ($y = 0$). τ_W is the wall shear stress and $\dot{q}_W$ is the rate of heat transfer per unit area from the wall to the fluid in the boundary layer.

The viscosity-temperature relation is assumed to be given by a law of the form

$\mu \sim T^{\omega}$ where (ω) is a constant, chosen so that in the range of temperatures considered the viscosity agrees with that obtained from the more accurate Sutherland relation*. The Prandtl number (σ) is assumed to be constant.

3. Stewartson - Illingworth transformation

In the compressible flow a stream function (ψ) can be introduced which satisfies the equation of continuity (1). Thus

$$\frac{\rho u}{\rho_o} = \frac{\partial \psi}{\partial y} \; ; \quad \frac{\rho v}{\rho_o} = - \frac{\partial \psi}{\partial x} \tag{6}$$

where ρ_o is the density at some constant reference condition.

Now Stewartson (1949) and Illingworth (1949) have shown that by means of the transformation from (x,y) coordinates in the compressible flow to (X,Y) coordinates, where

$$\left.\begin{aligned} X &= \int_o^x \left(\frac{a_1(x)}{a_o}\right)^{\frac{3\gamma-1}{\gamma-1}} dx \\ \text{and} \quad Y &= \frac{a_1(x)}{a_o} \int_o^y \frac{\rho(x,y)}{\rho_o}\, dy \end{aligned}\right\} \tag{7}$$

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* If $\frac{T}{\mu}\frac{d\mu}{dT}$ is equated between the Sutherland and the approximate viscosity relation it is found that $\omega = \frac{(3 + T/C_{Suth.})}{2(1 + T/C_{Suth.})}$ so that ω varies between the values 1.5 to 0.5 depending on the range of temperature considered.

$$\left(\text{Note} \quad \left(\frac{a_1}{a_o}\right)^{\frac{3\gamma-1}{\gamma-1}} = \frac{a_1 \, p_1}{a_o \, p_o} \right)$$

the equations of motion and energy become respectively

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{h}{h_1} U_1 \frac{dU_1}{dX} + \frac{p_o}{p_1} \nu_o \frac{\partial}{\partial Y}\left(\frac{\rho\mu}{\rho_o \mu_o} \frac{\partial U}{\partial Y} \right) \qquad (8)$$

and $$U \frac{\partial h}{\partial X} + V \frac{\partial h}{\partial Y} = \frac{p_o \, \nu_o}{p_1 \, \sigma} \frac{\partial}{\partial Y} \left\{ \frac{\rho \, \mu}{\rho_o \mu_o} \frac{\partial}{\partial Y} \left(h - (1-\sigma) \frac{U^2 a_1^2}{2 \, a_o^2} \right) \right\} \qquad (9)$$

where $$U(X,Y) = \frac{\partial \psi}{\partial Y} = \frac{u(x,y) a_o}{a_1(x)}$$

and $U_1(X) = \dfrac{u_1(x) \, a_o}{a_1(x)}$. We further note that

$$a_1^2 + \frac{\gamma-1}{2} u_1^2 = a_1^2 + \frac{\gamma-1}{2} U_1^2 \frac{a_1^2}{a_o^2} = \text{const.},$$

or $$\frac{a_1^2}{a_o^2} = \frac{\text{const}}{\left(1 + \frac{\gamma-1}{2} \cdot \frac{U_1^2}{a_o^2}\right)} \qquad (10),$$

and if (a_o) is chosen as the stagnation speed of sound in the external flow (i.e. suffix (o) refers to stagnation conditions in the isentropic external flow)

$$\frac{a_1^2}{a_o^2} = \frac{1}{1 + \frac{\gamma-1}{2 \, a_o^2} U_1^2} \qquad (10')$$

Thus in the transformed flow the constant fluid properties outside the boundary layer are taken at the stagnation values of the given compressible flow.

When $\sigma = 1$ and $\frac{\rho\mu}{\rho_o \mu_o} = \frac{p_1}{p_o}$ $(\mu \sim T)$

equations (8) and (9) are exactly the equations for an incompressible flow having velocity components (U,V) in coordinates (X,Y), provided the heat transfer to the wall is zero. For in that case only $h = \text{const} = h_1$. It follows that known boundary layer solutions in incompressible flow (X,Y) can be used to find the solution of corresponding compressible flows (x,y). However this method of attack only applies to the case of zero heat transfer and when $\sigma = \omega = 1$. It cannot be used when heat transfer is present and when σ and ω are not equal to unity.

In the general case equ. (8) and (9) can be written

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = (1 - S)\, U_1 \frac{dU_1}{dX} + \nu_o \frac{\partial}{\partial Y}\left(C \frac{\partial U}{\partial Y}\right) \tag{11}$$

and $$U \frac{\partial S}{\partial X} + V \frac{\partial S}{\partial Y} = \frac{\nu_o}{\sigma} \frac{\partial}{\partial Y}\left\{ C \frac{\partial}{\partial Y}\left(S + \frac{(1-\sigma)}{2} \frac{U^2 a_1^2}{a_o^2 h_1} \right) \right\} \tag{12}$$

where $$\frac{\mu T_o}{\mu_o T} = C\,(X,\, Y), \text{ and}$$

$$S = 1 - {}^{h}/_{h_1}.$$

Since $\mu \sim T^{\omega}$

$$C = \left(\frac{T}{T_o}\right)^{\omega - 1}$$ but in finding solutions to (11) and (12)

we will assume C is independent of Y, although not necessarily of X. In the final answer we will choose a value of C which gives a best fit with known exact solutions.

Equations (11) and (12) can now be modified, so that C(X) is eliminated, by changing equation (7) to read

$$\tilde{X} = \int_o^X C(z) \; \frac{a_1 \, p_1}{a_o \, p_o} \; dz.$$

However no advantages are gained as will be seen in the next section.

The equations in this form were obtained by Cohen and Reshotko (1956b). For the case $\sigma = 1$, constant wall temperature, and $U_1 \sim X^m$ the equations were solved numerically.

Similar equations have been obtained by Hayes (1956) for the case of imperfect gases, where the transformation formulae between (X,Y) and (x,y) include a general term $a(x)$ in place of a_1/a_o and a method for finding $a(x)$ is given. For a perfect gas $a(x)$ becomes $\sqrt{(h_1/h_o)}$, which is equal to a_1/a_o when the specific heats of the gas are constant.

4. Von Mises' transformation

Equations (11) and (12) in terms of the independent variables (X,Y) can be transformed into equations in (X, ψ) by means of Von Mises' transformation. This gives, respectively,

$$\frac{\partial Z}{\partial X} = S \frac{dU_1^2}{dX} + \nu_o UC \frac{\partial^2 Z}{\partial \psi^2} \tag{13}$$

and

$$-\frac{S}{X} = \frac{\nu_o}{\sigma} C \frac{\partial}{\partial \psi} \left\{ U \frac{\partial}{\partial \psi} \left(S + \frac{1-\sigma}{2} \frac{U^2 a_1^2}{a_o^2 h_1} \right) \right\} \tag{14}$$

since C is a function of X only, and where $Z = U_1^2 (X) - U^2(X, \psi)$.

Equations (13) and (14) are the transformed boundary layer equations for a pseudo-incompressible flow having a density and kinematic viscosity of ρ_o and ν_o respectively. Along and across the boundary layer a property S, analogous to temperature is convected and diffused.

Illingworth (1954) gives similar equations to (13) and (14) above, except that in his case he uses (x, ψ) as independent variables and puts $\sigma = C = 1$. Illingworth uses g in place of S and $h_1 f$ in place of Z.

Now

$$S (X, \psi) \frac{dU_1^2(X)}{dX} = \frac{\partial}{\partial X} \int_o^X S(z, \psi) \frac{dU_1^2 (z)}{dz} dz$$

so that equation (13) can be written

$$\frac{\partial}{\partial X} \left(Z - \int_o^X S(z, \psi)\, d\, U_1^2 (z) \right) = \nu_o U C \frac{\partial^2}{\partial \psi^2} \left(Z - \int_o^X S(z,0)\, d\, U_1^2 (z)\right) \tag{15}$$

where for compactness the integrals are written in Stieltje's form.

But at $\psi = 0$

$$\int_o^X S(z, \psi)\, d\, U_1^2 (z) = \int_o^X S(z,0)\, d\, U_1^2 (z)$$

and at $\psi = \infty$, since $S(X,\infty) = 0$

$$\int_o^X S(z,\psi)\, d\, U_1^2 (z) = 0.$$ It can also be shown that

at $\psi = \infty$

$$\frac{\partial^2}{\partial \psi^2} \int_0^X S(z,\psi)\, d\, U_1^2\,(z) = 0 \,.$$

It follows that near $\psi = 0$ we can write equation (15) approximately as

$$\frac{\partial\, G(X,\psi)}{\partial X} = \nu_o\, U\, C\, \frac{\partial^2}{\partial \psi^2}\, G(X,\psi) \tag{16}$$

where $G(X,\psi) = Z - \int_0^X S(z,0)\, d\, U_1^2\,(z)$

whilst for large values of ψ it has a similar form with

$$G(X,\psi) = Z - \int_0^X S(z,\psi)\, d\, U_1^2\,(z).$$

The boundary conditions for $G(X,\psi)$, for which (16) is to be solved, are therefore

$$\psi = 0 \qquad G(X,\psi) = U_1^2\,(X) - \int_0^X S(z,o)\, d\, U_1^2\,(z)$$

$$\psi = \infty \qquad G(X,\psi) = 0$$

$$X \to 0 \qquad G(X,\psi) \to 0\,,$$

whilst near $\psi = 0$

$$G(X,\psi) = U_1^2\,(X) - U^2\,(X,\psi) - \int_0^X (1 - h_W(z)/h_1)\, d\, U_1^2\,(z)$$

$$= U_1^2\,(+0) - U^2\,(X,\psi) + \int_0^X \frac{h_W(z)}{h_1}\, d\, U_1^2\,(z).$$

Equation (16), with the above boundary conditions, is only approximately equal to (13) for all values of ψ, although it is exact at $\psi = 0$ and $\psi = \infty$. Since the solution of the equation of motion near $y = 0$ ($\psi = 0$) is only required (see paragraph 5 below) it will be assumed that, for this purpose, equation (16) will be found adequate. (See first footnote on page 15.

A simplified form for equation (14) will now be obtained. Since

$$a_1^2\,/a_o^2 = \frac{1}{1 + \frac{\gamma-1}{2\,a_o^2}\,U_1^2} \quad ; \quad h_1 = \left(1 + \frac{\gamma-1}{2\,a_o^2}\,U_1^2\right) \frac{a_1^2}{(\gamma-1)}$$

it can be written

$$\frac{\partial S}{\partial X} - \frac{\nu_o}{\sigma}\, C\, \frac{\partial}{\partial \psi}\left(U\, \frac{\partial S}{\partial \psi}\right) = -\frac{(1-\sigma)}{\sigma\, U_1^2}\, \nu_o\, C \left(\frac{\frac{\gamma-1}{2\,a_o^2}\,U_1^2}{1 + \frac{\gamma-1}{2\,a_o^2}\,U_1^2}\right) \frac{\partial}{\partial \psi}\left(U\, \frac{\partial Z}{\partial \psi}\right) \tag{17}$$

When $\sigma = 1$ the right hand side of equation (17) vanishes. For other values of σ, since U,Z are known functions of (X,ψ), having been found from (16), equation (17) can be solved by the 'method of variation of parameters'. In the case of an incompressible flow, U is independent of i_w and the right hand side of (17) then gives the heat transfer correction term to allow for the recovery enthalpy. It will be assumed throughout this paper that the recovery enthalpy is independent of the wall temperature distribution and that the rate of heat transfer at the wall can be obtained from the solution to (17) with the right hand side put equal to zero, or*

$$\frac{\partial S}{\partial X} = \frac{\nu_o}{\sigma} C \frac{\partial}{\partial \psi} \left(U \frac{\partial S}{\partial \psi}\right) \tag{18}$$

The above discussion however only applies to the case when heat transfer to or from the wall is present. When the wall heat transfer is zero the term on the right hand side of equation (17) contributes significantly to the value of S near the wall. From the works of Pohlhausen (1921) for incompressible flow, and Brainerd and Emmons (1941) for compressible flow both for the flat plate in zero pressure gradient, and from the work of Tifford and Chu (1952) in compressible flow with a pressure gradient, we find that the value of S_w is approximately given by

$$S_{w_o} = (1 - \sigma^{\frac{1}{2}}) \frac{\frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_1^2}$$

at least for values of the Prandtl number near unity. Thus when $\sigma \neq 1$ the wall temperature varies according to the external velocity even when the heat transfer is zero.

Footnote 1. If in equ.(11) $S(X,\psi)$ and $C(X,\psi)$ are replaced by constant mean values $S^*(X)$ and $C^*(X)$, evaluated at some 'intermediate' enthalpy, an equation similar to (16) can be derived. However such an equation cannot have the same boundary conditions listed above. In the later sections $h_w(X)$ will be replaced by an intermediate enthalpy consistent with, but not equal to, the intermediate enthalpy at which C^* is evaluated, but the boundary condition $G(X,\infty)=0$ will always be used. In this way the value of $G(X,0)$ can be employed for all values of $h_w(X)$ including $h_w(X)=0$.

Footnote 2.* In section 11 below it is argued that S in equ.(18) should be replaced by the difference between the actual S and that for zero heat transfer. This will be an adequate approximation in many problems. However it must not be overlooked that (17) can be solved exactly if $U(X,\psi)$ is obtained from (16).

5. Approximate solution of the transformed equation of motion

The approximate form of the equation of motion in terms of Von Mises' variables was found above (equation 16) to be

$$\frac{\partial G(X,\psi)}{\partial X} = \nu_o \, U \, C \, \frac{\partial^2}{\partial \psi^2} G(X, \psi) \tag{16}$$

where $G(X,\infty) = 0$ and near $\psi = 0$

$$G(X,\psi) = U_1^2(0) - U^2(X,\psi) + \int_0^X \frac{h_w(z)}{h_1} \, d\, U_1^2(z) .$$

This equation is similar to Von Mises' equation for the velocity field in an incompressible laminar boundary layer. It is identical with it if G is replaced by $Z = U_1^2 - U^2$ and $C = 1$.

If only the wall shear stress (τ_w) is required as a function of $U_1^2(X)$ and $h_w(X)$, and not the velocity profile over the entire boundary layer, an approximate solution of (16) can be obtained by replacing U by its approximate form near the wall. This method of approach was used by Fage and Falkner (1931) in obtaining approximate solutions of the incompressible energy equation in the case of variable wall temperature, and by Lighthill (1950) for approximate solutions of both the equations of motion and energy in incompressible flow.* If we then follow Lighthill's method of solution we find on using the approximate form for $U(X,\psi)$ near the surface, namely,

$$U = \sqrt{\frac{2\,\tau_w(X)}{\mu_o}} \; \psi^{\frac{1}{2}} \tag{19}$$

(since $U = \dfrac{\tau_w(X)\, Y}{\mu_o}$ and $\psi = \int_0^Y U\, dY$)

that

$$\frac{\partial G}{\partial X} = \sqrt{\frac{2\,\mu_o}{\rho_o^2} \; \tau_w(X) \; C(X)^2} \;\; \psi^{\frac{1}{2}} \; \frac{\partial^2 G}{\partial \psi^2} \tag{20}$$

with the boundary conditions $G \to 0$ as $\psi \to \infty$, and as $X \to 0$, and

$$G = U_1^2(0) + \int_0^X \frac{h_w(z)}{h_1} \, d\, U_1^2(z) - \frac{2\tau_w(X)}{\mu_o} \, \psi + O(\psi^{3/2}) \tag{21}$$

as $\psi \to 0$.

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* A similar method was used by Illingworth (1954), who used this approximation in the compressible flow equations in Von Mises form.

If $$t = \int_0^X \sqrt{\frac{2\mu_o}{\rho_o^2} \tau_w(z)\, C^2(z)}\, dz \tag{22}$$

and p is the Heaviside operator[**] corresponding to $\partial/\partial t$, then equation (20) becomes

$$pG = \psi^{\frac{1}{2}} \frac{\partial^2 G}{\partial \psi^2} \tag{23}$$

This equation is similar to equation (66) in the paper by Lighthill (1950) and satisfies similar boundary conditions. The solution of (23) satisfying (21) is therefore (see Lighthill)[*]

$$G = \left(\tfrac{2}{3} p^{\frac{1}{2}}\right)^{2/3} \psi^{\frac{1}{2}} (-\tfrac{2}{3})!\; I_{-\frac{2}{3}} \left(\tfrac{4}{3} p^{\frac{1}{2}} \psi^{\frac{3}{4}}\right)\left[U_1^2(o) + \int_0^X \frac{h_w(z)}{h_1} dz\right]$$

$$+ \left(\tfrac{2}{3} p^{\frac{1}{2}}\right)^{2/3} \psi^{\frac{1}{2}} (\tfrac{2}{3})!\; I_{\frac{2}{3}} \left(\tfrac{4}{3} p^{\frac{1}{2}} \psi^{\frac{3}{4}}\right)\left[-\frac{2\tau_w(X)}{\mu_o}\right] \tag{24}$$

Since $G \to 0$ as $\psi \to \infty$ the coefficients of $I_{-\frac{2}{3}}$ and $I_{\frac{2}{3}}$ must be equal and opposite and therefore

$$U_1(o)^2 + \int_0^X \frac{h_w(z)}{h_1} d\, U_1^2(z) = \frac{3^{4/3} (\frac{2}{3})!}{2^{\frac{1}{3}} (-\frac{2}{3})!\, \mu_o} p^{-\frac{2}{3}} \tau_w(X) \tag{25}$$

$$= \frac{3^{\frac{1}{3}} 2^{\frac{2}{3}}}{(-\frac{2}{3})!\, \mu_o} \int_0^X \frac{\tau_w(X_1)}{(t - t_1)^{\frac{1}{3}}} \frac{dt_1}{dX_1} dX_1$$

$$= \frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})!\, (\rho_o \mu_o)^{\frac{2}{3}}} \int_0^X C(X_1)\, \tau_w(X_1)^{3/2} \left(\int_{X_1}^X \sqrt{\tau_w(z)\, C(z)^2}\, dz\right)^{-\frac{1}{3}} dX_1 \tag{26}$$

---

* $I_{\frac{2}{3}}$ and $I_{-\frac{2}{3}}$ are Bessel functions.

** The operational form of a function f(t) will be denoted by f(p), where $f(p) = p \int_0^\infty e^{-pt} f(t)\, dt.$

This integral equation for the wall shear stress is identical with Lighthill's equation (69) if $C = 1$ and $h_W = h_1$. (In Lighthill's equation the $\frac{2}{3}$ power of $\rho_o \mu_o$ was omitted).

If we put $U_1 \sim X^m$ and h_W = const. in equation (26) we can compare the results with those of Cohen and Reshotko (1956). It can be shown that errors of less than 10% in the value of τ_W are obtained for cases of accelerated flow and wall enthalpies (i_W) of the same order, or greater than the mainstream stagnation enthalpy. In cases of retarded flow or very cool surfaces the errors increase and therefore a correction term must be added to improve the accuracy as outlined in section 9 and in appendices 4 and 5.

In the present section the analysis will be continued without any attempt being made to improve the accuracy.

The compressible flow solution is now obtained by application of the Stewartson-Illingworth transformation (7) to equation (26). If we put, for convenience, $\gamma = 1.4$

then the necessary transformation relations are

$$\frac{dX}{dx} = \frac{1}{(1 + M_1^2(x)/5)^4} \tag{27}$$

$$U_1(X) = a_o M_1(x) \tag{28}$$

and

$$\tau_W(X) = \frac{\tau_W(x)\left(1 + \frac{M_1^2(x)}{5}\right)^{9/2}}{C_W(x)} \qquad (29)^{*}$$

where, consistent with the other approximations, C_W is identical to C.

---

* In this equation $\frac{9}{2} \equiv \frac{2\gamma - 1}{\gamma - 1}$ and in the equations below

$$\frac{11}{4} \equiv \frac{3\gamma - 2}{2(\gamma - 1)} \; ; \quad \frac{7}{4} \equiv \frac{\gamma}{2(\gamma - 1)} \; ; \quad \frac{1}{5} \equiv \frac{\gamma - 1}{2}$$

Thus equation (26) becomes

$$a_o^2 \left(M_1(o)^2 + \int_0^x \frac{h_w(z)}{h_1} \, d\, M_1^2(z) \right)$$

$$= \frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})!\;(\rho_o \mu_o)^{\frac{2}{3}}} \int_0^x \frac{\tau_w(x_1)^{3/2} \left(1 + \frac{M_1^2(x_1)}{5}\right)^{11/4}}{C(x_1)^{\frac{1}{2}}} \left(\int_{x_1}^{x} \frac{\sqrt{C(z)\;\tau_w(z)}\;dz}{\left(1 + \frac{M_1^2(z)}{5}\right)^{7/4}} \right)^{-\frac{1}{3}} dx_1 \qquad \ldots\ldots \quad (30)$$

If suffix (a) denotes an arbitrary constant reference condition and

$$t_w(x) = \tau_w(x) \sqrt{\frac{x}{\rho_a \; \mu_a \; u_a^3}}$$

then equation (28) can be written

$$M_1^2(o)^2 + \int_0^x \frac{h_w(z)}{h_1} \; d\, M_1^2(z) = M_a^2 \left(\frac{a_a}{a_o}\right)^2 .$$

$$\left(\frac{\rho_a \mu_a}{\rho_o \mu_o}\right)^{\frac{2}{3}} \frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})!} \int_0^x \frac{t_w(x_1)^{3/2} \left(1 + \frac{M_1^2(x)}{5}\right)^{11/4}}{\left(\frac{T_o \mu_w}{T_w \mu_o}\right)^{\frac{1}{2}} \quad x_1^{\frac{3}{4}}} \left(\int_{x_1}^{x} \frac{\left(\frac{T_o \mu_w}{T_w \mu_o} \; t_w(z)\right)^{\frac{1}{2}} dz}{z^{\frac{1}{4}} \left(1 + \frac{M_1^2(z)}{5}\right)^{7/4}} \right)^{-\frac{1}{3}} dx_1 \qquad \ldots\ldots \quad (30')$$

where $\left(\frac{a_a}{a_o}\right)^2 \left(\frac{\rho_a \;\mu_a}{\rho_o \;\mu_o}\right)^{2/3} = \left(\frac{\rho_a \mu_a a_a^3}{\rho_o \mu_o a_o^3}\right)^{2/3} = \left(\frac{T_o \mu_a}{T_a \mu_o}\right)^{2/3} \Big/ \left(1 + M_a^2/5\right)^{10/3}.$

If further we now replace

$\frac{T_o \; \mu_w}{T_w \; \mu_o}$ by $\left(\frac{T_o}{T_w}\right)^{1-\omega}$ etc. and noting that

$$\frac{T_o}{T_w} = \frac{h_1}{h_w} = \frac{h_1}{i_w} \quad \text{and} \quad \frac{T_o}{T_a} = 1 + M_a^2/5$$

equation (30') becomes

$$\left(\frac{M_1(o)}{M_a}\right)^2 + \int_o^x \frac{i_w(z)}{h_1} \, d\left(\frac{M_1^2(z)}{M_a^2}\right)$$

$$= \frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})!} \int_o^x \frac{t_w(x_1)^{3/2}}{x_1^{\frac{3}{4}}} \left(\frac{i_w(x_1)}{i_a}\right)^{\frac{1-\omega}{2}} \left(\frac{1 + M_1^2(x_1)/5}{1 + M_a^2/5}\right)^{11/4} \cdot$$

$$\left(\int_{x_1}^x \frac{t_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} \left(\frac{i_a}{i_w(z)}\right)^{\frac{1-\omega}{2}} \left(\frac{1 + M_a^2/5}{1 + M_1^2(z)/5}\right)^{7/4} dz\right)^{-\frac{1}{3}} dx_1 \qquad (31)$$

which is a convenient non-dimensional form of the integral equation for $t_w(x)$ in terms of $M_1(x)$ and $h_w(x)$.

$$\left(\frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})!} = \frac{1.4422 \cdot 2}{2.678} = 1.078\right)$$

When $\gamma = 1$ equation (31) reduces to a form similar to that in a heated or cooled incompressible flow. In this case since $a_1 = a_o = a_a$ and i_1 = const.,

$$\left(\frac{u_1(+o)}{u_a}\right)^2 + \int_o^x \frac{i_w(z)}{i_1} \, d\left(\frac{u_a(z)}{u_a}\right)^2$$

$$= \frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})!} \int_o^x \frac{t_w(x_1)^{3/2}}{x_1^{\frac{3}{4}}} \; \frac{i_w(x_1)}{i_1}^{\frac{1-\omega}{2}} \left(\int_{x_1}^x \frac{t_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} \left(\frac{i_1}{i_w(z)}\right)^{\frac{1-\omega}{2}} dz\right)^{-\frac{1}{3}} dx_1$$

..... (31')

Thus finally, in this section, when $\gamma = 1$, $\omega = 1$ we find that

$$\left(\frac{u_1(+0)}{u_a}\right)^2 + \int_0^x \frac{i_w(z)}{i_1} \, d\left(\frac{u_1(z)}{u_a}\right)^2$$

$$= \frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})!} \int_0^x \frac{t_w(x_1)^{3/2}}{x_1^{\frac{3}{4}}} \left(\int_{x_1}^x \frac{t_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} dz\right)^{-\frac{1}{3}} dx_1 \qquad (31'')$$

which gives the approximate extension to Lighthill's equation (7)) to allow for variable wall temperature. Equation (31") shows, as noted by many workers, that in an incompressible uniform flow (u_1 = const) the skin friction parameter, t_w, is independent of the wall temperature.

6. Flat plate at zero pressure gradient and constant wall temperature

If $M_a = M_1(+0)$ is the constant external Mach number to the boundary layer on a flat plate whose constant wall enthalpy is i_w, then from equation (31) we find that

$$t_w = \frac{3^{\frac{1}{4}}}{\left[(-\frac{1}{3})!\right]^{\frac{3}{4}} 2^{7/4}} \left(\frac{i_a}{i_w}\right)^{\frac{1-\omega}{2}}$$

$$= 0.312 \left(\frac{i_a}{i_w}\right)^{\frac{1-\omega}{2}} \qquad (32)$$

This relation is similar to that given by Young (1948) except that

0.312 is replaced by 0.332 (Blasius' value)

and $\frac{i_w}{i_a}$ is replaced by $\left(0.45 + 0.55 \frac{i_w}{i_a} + 0.036 M_a^2 \sigma^{\frac{1}{2}}\right)$,

to give the best fit with Crocco's exact results.

This comparison suggests that a more accurate form of the integral equation (31) can be obtained if

$\frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})!}$ is replaced by unity, and $\frac{i_w}{i_a}$ on the right hand side is replaced by $\left(0.45 + 0.55 \frac{i_w(x)}{i_a} + 0.036 M_a^2 \sigma^{\frac{1}{2}}\right)$.

For the unheated incompressible flow case ($M_a \to 0$ and $h_w \to h_1$) Lighthill (1950) showed that when $u_1(x) = cx^m$ errors of less than about 1% in the value of t_w when $m \geq 0$ would be obtained if the constant in equation (31) was suitably modified.*

7. Improved relation between wall shear stress and Mach number for variable wall temperature

In view of the comparison between results obtained from equation (31) and the exact results for the flat plate the following improved form for (31) is proposed ($\gamma = 1.4$)

$$\left(\frac{M_1(o)}{M_a}\right)^2 + \int_o^x \frac{h_w(z)}{h_1}\, d\left(\frac{M_1(z)}{M_a}\right)^2 =$$

$$\int_o^x \frac{t_w(x_1)^{3/2}}{x_1^{\frac{3}{4}}} \left(0.45 + 0.55\,\frac{i_w(x_1)}{i_a} + 0.036\, M_a^2\, \sigma^{\frac{1}{2}}\right)^{\frac{1-\omega}{2}} \left(\frac{1 + M_1^2(x_1)/5}{1 + M_a^2/5}\right)^{11/4} .$$

$$\left(\int_{x_1}^x \frac{t_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} \left(0.45 + 0.55\,\frac{i_w(z)}{i_a} + 0.036\, M_a^2\, \sigma^{\frac{1}{2}}\right)^{-\left(\frac{1-\omega}{2}\right)} \left(\frac{1 + M_a^2/5}{1 + M_1^2(z)/5}\right)^{7/4} dz\right)^{-\frac{1}{3}} dx_1 \qquad \ldots\ldots \quad (33)$$

Results for values of γ other than 1.4 can similarly be obtained. For instant when $\gamma = 1$ the terms in $(1 + M^2/5)$ vanish as well as the term involving the Prandtl number.

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* Lighthill quoted the modified value of the constant as 1.157 but a better value would be 0.98.

If we put

$$F(x) = \left(\frac{M_1(o)}{M_a}\right)^2 + \int_o^x \frac{h_w(z)}{h_1} \, d\left(\frac{M_1(z)}{M_a}\right)^2$$

$$^{*}G(x) = \left(0.45 + 0.55\,\frac{i_w(x)}{i_a} + 0.036\, M_a^2\, \sigma^{\frac{1}{2}}\right)^{\frac{1-\omega}{2}} \left(\frac{1 + M_1^2(x)/5}{1 + M_a^2/5}\right)^{11/4}$$

$$H(x) = \left(0.45 + 0.55\,\frac{i_w(x)}{i_a} + 0.036\, M_a^2\, \sigma^{\frac{1}{2}}\right)^{-\left(\frac{1-\omega}{2}\right)} \left(\frac{1 + M_a^2/5}{1 + M_1^2(x)/5}\right)^{7/4}$$

then (33) becomes

$$F(x) = \int_o^x \frac{t_w(x_1)^{3/2}}{x_1^{\frac{3}{4}}} \, G(x_1) \left(\int_{x_1}^{x} \frac{t_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} \, H(z)\, dz\right)^{-\frac{1}{3}} dx_1 \qquad (34)$$

where, in general, $F(x)$, $G(x)$ and $H(x)$ will be known functions of x and equation (34) is to be inverted to find $t_w(x)$.

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* Not to be confused with $G(X, \psi)$ defined in equation 16.

8. Approximate inversion of the wall shear stress integral equation

Lighthill (1950) has shown how the incompressible form of (33) or (34) can be inverted if as an approximation

$$\int_{x_1}^{x} \frac{t_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} \quad H(z)\ dz \quad \text{is replaced by}$$

$$(x - x_1)\ \frac{t_w(x_1)^{\frac{1}{2}}\ H(x_1)}{x_1^{\frac{1}{4}}}\ .$$ The resulting solution for $t_w(x)$ in the case $u_1(x) = cx^m$ differed from the exact solution by ± 10% when $m \geqslant 0$. If then errors of that order of magnitude are acceptable we can replace equation (33) or (34) (but retaining the constant term $\left.\frac{3^{\frac{1}{3}}.\ 2}{(-\frac{2}{3})!}\right)$ by

$$F(x) = \frac{3^{\frac{1}{3}}.\ 2}{(-\frac{2}{3})!} \int_0^x \frac{t_w(x_1)^{4/3}\ J(x_1)}{x_1^{\frac{2}{3}}\ (x - x_1)^{\frac{1}{3}}}\ dx_1 \qquad (35)$$

where $$J(x) = \left(0.45 + 0.55\,\frac{i_w(x)}{i_a} + 0.036\,M_a^2\,\sigma^{\frac{1}{2}}\right)^{\frac{2}{3}(1-\omega)} \left(\frac{1 + M_1^2(x)/5}{1 + M_a^2/5}\right)^{10/3}$$

which is invertible as

$$\left(\frac{t_w(x)\ J(x)^{\frac{3}{4}}}{x^{\frac{1}{2}}}\right)^{4/3} = \frac{x^{-\frac{2}{3}}}{(2.3^{\frac{1}{3}}\ (-\frac{1}{3})!)} \left[\left(\frac{M_1(o)}{M_a}\right)^2 + \frac{x^{\frac{2}{3}}}{M_a^2}\int_0^x \frac{h_w(z)}{h_1}\ \frac{d\,M_1^2(z)}{(x-z)^{\frac{2}{3}}}\right]$$

or $$t_w(x) = \frac{J(x)^{-\frac{3}{4}}}{(2.3^{\frac{1}{3}}.(-\frac{1}{3})!)^{\frac{3}{4}}} \left[\left(\frac{M_1(o)}{M_a}\right)^2 + x^{\frac{2}{3}}\int_0^x \frac{h_w(z)h_1}{(x-z)^{\frac{2}{3}}}\ d\left(\frac{M_1(z)}{M_a}\right)^2\right]^{\frac{3}{4}}$$

..... (36)

(An alternative method of inverting equation (33) is given in Appendix 3).

Now $\left(\frac{1}{2.3^{\frac{1}{3}} \cdot (-\frac{1}{3})!}\right)^{\frac{3}{4}} = 0.360$ so that equation (36) becomes

$$t_w(x) = \frac{0.360}{\left[0.45 + 0.55 \frac{i_w(x)}{i_a} + 0.036 M_a^2 \sigma^{\frac{1}{2}}\right]^{\frac{1-\omega}{2}}} \left(\frac{1 + M_a^2/5}{1 + M_1^2(x)/5}\right)^{5/2} \quad \ldots\ldots \quad (36')$$

$$\left[\left(\frac{M_1(o)}{M_a}\right)^2 + x^{\frac{2}{3}} \int_o^x \frac{h_w(z)/h_1}{(x-z)^{\frac{2}{3}}} \, d\left(\frac{M_1(z)}{M_a}\right)^2\right]^{\frac{3}{4}}$$

On comparison of equation (36') with exact solutions for the flat plate at zero pressure gradient we see that the constant 0.360 should be replaced by 0.332.

However Lighthill (1950) has shown that in incompressible flow with $u_1(x) = cx^m$ the error introduced by equation (36') varies from + 8.4% when m = 0 to - 10.6% when m = ∞ . Also the accuracy is poor when m is negative. From these results it would appear that little accuracy will be gained by a change in the value of the constant 0.360.

For the special case when $M_a = M_1(+o)$ and

$$\frac{1 + M_1^2(x)/5}{1 + M_a^2/5} = 1 + cx^m \qquad (37)$$

, where m > 0,

then $$\frac{M_1^2(x)}{M_a^2} = 1 + \frac{5c}{M_a^2}(1 + M_a^2/5)x^m \qquad (38)$$

and $$\frac{d}{dx}\left(\frac{M_1^2(x)}{M_a^2}\right) = c\,(1 + 5/M_a^2)\; m\, x^{m-1} \qquad (39)$$

If in addition $\dfrac{h_w(+o) - h_w(x)}{h_1} = b\,x^n$ (40)

, where $n > 0$,

then equation (36') becomes

$$\frac{t_w(x)}{t_w(o)} = \frac{1}{\left(0.45 + 0.55\,\dfrac{i_w(x)}{i_a} + 0.036\,M_a^2\,\sigma^{\frac{1}{2}}\right)^{\frac{1-\omega}{2}}} \left(\frac{1 + M_a^2/5}{1 + M_1^2(x)/5}\right)^{5/2} \cdot$$

$$\left[1 + \left(\frac{M_1^2(x)}{M_a^2} - 1\right)\left\{\frac{h_w(+o)}{h_1}\;\frac{m!\,(-\frac{2}{3})!}{(m - \frac{2}{3})!} - \left(\frac{h_w(+o) - h_w(x)}{h_1}\right)\frac{m.\,(n + m - 1)!\,(-\frac{2}{3})!}{(n + m - \frac{2}{3})!}\right\}\right]^{\frac{3}{4}} \quad \ldots\ldots \quad (41)$$

The values of the constants for various values of m and n are given in table 1. (Similar results for other forms of external velocity distribution and wall enthalpy distribution can easily be obtained).

As an example we have taken $m = 1$ in both an accelerated and a retarded flow and, $n = 1$ and 10. These results are plotted in figures 6 and 7 respectively. Since equation (41) does not contain the correction terms, discussed in the next paragraph, it is unlikely that the numerical accuracy of these results will be good. However the results do show some interesting trends. In accelerated flow, when the wall temperature is roughly constant except near $x = 1$, ($n = 10$), the skin friction is greater at a certain distance from the origin than for the case where the wall temperature falls linearly from the origin, $n = 1$. On the other hand in retarded flow we find that separation is earlier when the wall temperature is roughly constant. These calculations do not in fact predict separation for the case $n = 1$ although beyond $x/l = 0.4$ the value of t_w is very small. However the separation point is not well predicted in the above analysis, as will be shown in the next paragraph, and a small correction term must be introduced in order to improve the accuracy. But the prediction that the wall must be cooled significantly, immediately downstream of the origin, in order to delay separation is an important conclusion.

9. Approximate relation between the wall shear stress and Mach number when the wall temperature is constant

When the wall temperature is constant and*

$$\frac{d}{dx}\left(\frac{M_1(x)}{M_a}\right)^2 = \sum_{m=0}^{m_1} \bar{a}_m \; x^m \qquad (42)$$

equation (36') becomes (if $M_1(+o) = M_a$)

$$t_w(x) = \frac{0.360}{\left(0.45 + 0.55\,\frac{i_w}{i_a} + 0.036\, M_a^2\, \sigma^{\frac{1}{2}}\right)\frac{1-\omega}{2}} \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{5/2}$$

$$\left|\, 1 + \frac{h_w}{h_1}\left(3\,\bar{a}_o\, x + 2.25\,\bar{a}_1\, x^2 + 1.9286\,\bar{a}_2\, x^3 + 1.7357\,\bar{a}_3\, x^4 + 1.6022\,\bar{a}_4\, x^5\right)\right|^{\frac{3}{4}} \quad \ldots\ldots \quad (43)$$

where 0.360 must be replaced by 0.332 if agreement with the exact solution is desired when $M_1 = M_a = \text{const.}$

A more accurate solution can be obtained following the method outlined in appendix 4. In the special case of zero heat transfer when $M\ (+o) = M_a$ and $\sigma = \omega = 1$ we find that

$$2\, t_w(x) = \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{5/2} \left[\frac{x\,\frac{M_1^2}{M_a^2}\,\frac{d(M_1/M_a)}{dx} \pm \sqrt{x^2\left(\frac{M_1}{M_a}\right)^4\left(\frac{d\,(M_1/M_a)}{dx}\right)^2}}{+\,16\left(\frac{3}{2^7\cdot(-\frac{1}{3})!^3}\right)\left(1 + x^{\frac{1}{2}}\int_0^x \frac{d\, M_1^2/M_a^2}{(x^{\frac{3}{4}} - z^{\frac{3}{4}})^{\frac{2}{3}}}\right)^3}\right]^{\frac{1}{2}} \quad \ldots(43')$$

---

* This form of Mach number distribution is chosen to facilitate the evaluation of the integral in equ. (36'). However any other Mach number distribution can be equally well be used.

and when M_1/M_a is substituted from equation (42)

$$2\, t_w(x) = \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{5/2} \left[\left(\sum_{m=0} \frac{\bar{a}_m x^{m+1}}{2}\right)\left(1 + \sum_{m=0} \frac{\bar{a}_m x^{m+1}}{m+1}\right)\right.$$

$$\pm\sqrt{\left\{\left(\sum_{m=0} \frac{\bar{a}_m x^{m+1}}{2}\right)^2\left(1 + \sum_{m=0}\frac{\bar{a}_m x^{m+1}}{m+1}\right) + 16\left(\frac{3}{128.(-\frac{1}{3})!^3}\right)\left(1 + \right.\right.}$$

$$\left.\left.\left. + \frac{4}{3}\sum_{m=0} \bar{a}_m \frac{(\frac{4m+1}{3})!\ (-\frac{2}{3})!}{\left(\frac{2(2m+1)}{3}\right)!} x^{m+1}\right)^3\right\}\right]^{\frac{1}{2}} \qquad (43'')$$

where $16\left(\frac{3}{128.(-\frac{1}{3})!^3}\right) = 0.1511$

and values of $\dfrac{(\frac{4m+1}{3})!\ (-\frac{2}{3})!}{(\frac{2(2m+1)}{3})!}$ are given in the following table.

m	0	1	2	3
$\frac{4}{3}\cdot\dfrac{(\frac{4m+1}{3})!(-\frac{2}{3})!}{(\frac{2(2m+1)}{3})!}$	3.533	2.687	2.314	

Our results above can now be compared with those of Luxton and Young (1958) and others for various distributions of Mach number. Thus when $\sigma = \omega = 1$ and

$$\frac{M_1}{M_a} = 1 + x/l \;,$$ where l is a reference length, and

the wall temperature is constant we obtain from equation (43)

$$2\, t_w(x) = 0.720\left(\frac{1 + M_a^2/5}{1 + M\ /5}\right)^{5/2}\left(1 + \frac{h_w}{h_1}\,(6\bar{x} + 4.5\,\bar{x}^2)\right)^{\frac{3}{4}} \qquad (44)$$

where $\bar{x} = x/l$.

Similarly for zero heat transfer ($h_w = h_1$), we obtain from equation (43")

$$2\, t_w(x) = \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{5/2} \left[\bar{x}(1+\bar{x})^3 \pm \sqrt{\left\{ \bar{x}^2(1+\bar{x})^6 + \frac{}{0.1511\,(1 + 7.066\,\bar{x} + 5.374\,\bar{x}^2)^3} \right\}} \right]^{\frac{1}{2}} \qquad (44')$$

which can be written approximately

$$2\, t_w(x) \approx 0.624\,(1 + f(\bar{x}))\,(1 + 7.066\,\bar{x} + 5.374\,\bar{x}^2)^{\frac{3}{4}} \qquad (44")$$

where $f(\bar{x}) = \left[1 + \frac{g(\bar{x})}{2}\left(1 + \frac{g(\bar{x})}{2}\right) \right]$

and $g^2(\bar{x}) = \frac{1}{0.1511} \cdot \frac{\bar{x}^2(1+\bar{x})^6}{(1 + 7.066\,\bar{x} + 5.374\,\bar{x}^2)^{3/2}}$

From equations (44') and (44") we see that an improved form of (44) is, if $2\, t_w(o) = 0.664$,

$$2\, t_w(x) \approx 0.664 \left[1 + \frac{6}{5}\,(6\,\bar{x} + 4.5\,\bar{x}^2) \right]^{\frac{3}{4}} \qquad (44''')$$

for <u>zero heat transfer,</u> while for the case of heat transfer it is suggested that $6/5$ is replaced by

$$\left(\frac{i_w}{h_1}\right)^{*} = \frac{6}{5}\,\frac{i_{wo}}{h_1} - \frac{5}{7}\left(\frac{i_{wo} - i_w}{h_1}\right) \quad \text{(see appendix 5).}$$

These modified relations are plotted in figure 1 together with Luxton and Young's results. The agreement is very good. The conclusion from both sets of results is that marked reductions in skin friction are obtained by cooling of the wall.

In a retarded flow, $M_1 = M_a(1 - {}^x/l)$, results can be obtained in a similar way. Thus from equation (43) with the above correction term added, we obtain the results plotted in figure 2. Separation is delayed by cooling the wall and it is also noted that the wall shear stress multiplied by the square root of the Reynolds number, initially increases slightly in the cooled wall case so that for a certain distance $c_f \approx 1/\sqrt{x}$ as in the case of the flat plate in zero pressure gradient.

Luxton and Young (1958) and others have considered the following retarded flow case, $u_1 = u_a(1 - {}^x/l)$, which is not so amenable to treatment, by the present method, as is the case of $M_1 = M_a\ (1 - {}^x/l)$. The reason for this difference in application lies in the fact that for the linear velocity gradient equation $(42)_x$ becomes an infinite series, which is only slowly convergent even when ${}^x/l \ll 1$.
When $\sigma = \omega = 1$ (the case treated by Luxton and Young) the modified form of equation 36') becomes

$$2\, t_w(x) = 0.664 \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{5/2} \left[1 - 2\,\bar{x} \left(\frac{h_w}{i_a}\right)^{*} . \int_0^1 \frac{(1 - \bar{x}\,\bar{z})\, dz}{(1 - \bar{z})^{2/3} \left(1 + \frac{2M_a^2}{5}\,\bar{x}\,\bar{z}\,(1 - \frac{\bar{x}\bar{z}}{2})\right)^2} \right]^{\frac{3}{4}} \tag{45}$$

For this form of external velocity distribution it is not convenient to use equ. (43) since a very large number of terms are required for even small values of $\bar{x}$. When $\bar{x} \ll 1$ and $M_a = 4$ the integral in equ. (45) reduces to

$$\frac{6}{40.96\,\bar{x}} \left[\frac{7.4}{6\,\alpha} + \frac{2}{9} \left(\frac{2.7 - 6.4\,\bar{x}}{1 + 6.4\,\bar{x}}\right) \frac{1}{\alpha^{\frac{2}{3}}} \left\{ l_n\left(\frac{\alpha^{\frac{1}{3}}}{\alpha^{\frac{1}{3}} - 1}\right) + \frac{1}{2}\, l_n \left(\frac{1 + \alpha^{\frac{1}{3}} + \alpha^{\frac{2}{3}}}{\alpha^{\frac{2}{3}}}\right) + \sqrt{3}\ \text{arc tan} \left(\frac{\sqrt{3}}{1 + 2\,\alpha^{\frac{1}{3}}}\right) \right\} \right]$$

where $\alpha = \dfrac{1 + 6.4\,\bar{x}}{6.4\,\bar{x}}$. The exact value of the integral is given in Appendix 6.

The results for the cases of zero heat transfer and the cooled wall with $i_w = i_a$ are plotted in figures 3a and 3b respectively and are compared with the results of Luxton and Young, Curle (1958b) and, Cohen and Reshotko (1956). The agreement with Luxton and Young's results is good for zero heat transfer but not so good for the cooled wall cases. The results are lower, for small values of $\bar{x}$, than those of the 'exact' solution obtained by N.P.L. for the cooled wall case, but agreement could be obtained if a slightly different value of $\left(i_w/h_1\right)^{*}$ were used.

When the pressure distribution, in place of the velocity distribution, is defined as a function of ${}^x/l$ the above method needs only small modification.

Results are given in Appendix 5 for the case of a linear adverse pressure gradient where it is shown that a relatively simple result is obtained, in closed form, when $\gamma = 1.5$ and the wall temperature is constant. Figures 4 and 5 show the results obtained both from the unmodified and the modified formulae* for the cases of zero heat transfer and the cooled wall respectively, together with the results obtained by other workers for $\gamma = 1.4$. In both cases it is found that only relatively minor differences exist between these results and those obtained using the modified formula. (In making this comparison it is assumed that changing γ from 1.4 to 1.5 does not seriously modify the results).

In the light of these comparisons with other known accurate results for the case of constant wall temperature it is proposed that in the general case a more accurate form of equation (36') is when $\gamma = 1.4$,

$$2\, t_w(x) = \frac{0.664}{\left(0.45 + 0.55 \frac{i_w}{i_a} + 0.036\, M_a^2\, \sigma^{\frac{1}{2}}\right)^{\frac{1-\omega}{2}}} \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{5/2} \cdot$$

$$\left[\left(\frac{M_1(+0)}{M_a}\right)^2 + x^{\frac{1}{2}} \int_0^x \frac{J(z)}{(x^{\frac{3}{4}} - z^{\frac{3}{4}})^{\frac{2}{3}}}\, d\left(\frac{M_1(z)}{M_a}\right)^2\right]^{\frac{3}{4}} \qquad (46)$$

$$\text{where} \quad J(z) = \frac{6}{5}\, \frac{i_{wo}(z)}{h_1} - \frac{5}{7}\left(\frac{i_{wo}(z) - i_w(z)}{h_1}\right) \qquad (47)$$

It is noted that equation (46) can be applied to the cases of accelerated and retarded flows as well as to cases of constant and variable wall temperature. The effect of variations of σ and ω from unity are also approximately included. It might also be noted that the result above can be used for a dissociated gas in equilibrium, provided the Lewis number for the gas is equal to unity.

Now from equation (46) it is seen that separation occurs when the terms inside the square bracket equal zero, and therefore for constant wall temperature the distance to separation will, in general, be a function both of i_w/h_1 and M_a, as shown by Gadd (1957b). When $M_1 = M_a(1 - x/l)$, however, we see from Fig. 8 that the distance to separation is independent of M_a for a constant value of i_w/h_1. The trends are similar to those shown by Gadd for $u_1 = u_a(1 - x/l)$ apart from the latter result.

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* The correction term is modified slightly to allow for the difference in γ between these results and the value of $\gamma = 1.4$ used previously.

A direct comparison of Gadd's results with those obtained from equ.(45) has not been made, although clearly in this case the distance to separation will be a function of both M_a and i_w/h_1. The increase in the distance to separation as the wall temperature is lowered is in qualitative agreement with the results of Illingworth (1954) and Gadd (1957b). In incompressible flow the distance to separation, as a function of wall temperature, for the external velocity distribution $u_1 = u_a(1 - x/l)$ is plotted in Fig. 9 together with the results of Illingworth(1954) and the tentative results of Gadd (1957b). It is seen that the present method predicts separation distances greatly in excess of the latter results for the cooled wall whereas for the heated wall the agreement is better.

10. Approximate solution of the transformed stagnation enthalpy equation

The approximate form for the transformed stagnation enthalpy equation in terms of Von Mises' variables was found above (equation 18) to be

$$\frac{\partial S}{\partial X} = \frac{\nu_o}{\sigma} C \frac{\partial}{\partial \psi}\left(U \frac{\partial S}{\partial \psi}\right)$$

where $S(X,\infty) = 0$, and $S \to 0$ as $X \to 0$, and

$$S = 1 - \frac{h_w(X)}{h_1} + \sigma \frac{\mathring{q}_w(X)}{h_1} \sqrt{\frac{2\psi}{\mu_o \tau_w(X)}} - \ldots\ldots \qquad (48)$$

as $\psi \to 0$. The rate of heat transfer from the wall to the fluid is

$$\mathring{q}_w(X) = -k_o \left(\frac{\partial T}{\partial Y}\right)_{Y=0}$$

and the Prandtl number (σ) is given by

$$\sigma = \frac{\mu_o C_p}{k_o} .$$

In the case of zero heat transfer we must use the full equation (17) and we write the solution of this equation as $S_o(X, \psi)$.

If in a first approximation, the changes in the velocity distribution $U(X, \psi)$ and $C(X)$ are neglected between the cases of heat transfer and zero heat transfer,* we see that a solution of the complete equation (17) is $S_1 = S - S_o$, where $S_1(X, \psi)$ satisfies the following equation, (which is an improved form of (18),

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* This is tantamount to saying that the wall shear stress is approximately independent of heat transfer. This is true in an incompressible flow, since the velocity distribution is then independent of the temperature distribution. It is however not true in the case of the pseudo-incompressible flow, whose equation of motion is (13), on account of the term in S. The error will be greatest when the wall is cooled. But because we are going to assume that a good approximation to the rate of heat transfer can be obtained from a crude approximation to the velocity distribution, we will conclude, without proof, that the errors due to the one approximation are no greater than the errors due to the other.

$$\frac{\partial S_1}{\partial X} = \frac{\nu_o}{\sigma} C \frac{\partial}{\partial \psi}\left(U \frac{\partial S_1}{\partial \psi}\right) \qquad (49)$$

with the boundary conditions $S_1 \to 0$ as $\psi \to \infty$, and as $X \to 0$, and

$$S_1 = \frac{h_{wo}(X) - h_w(X)}{h_1} + \sigma \frac{\dot{q}_w(X)}{h_1} \sqrt{\frac{2\psi}{\mu_o \tau_w(X)}} - \ldots \qquad (50)$$

as $\psi \to 0$. h_{wo} is the wall enthalpy at zero heat transfer.

If, following Lighthill, we assume that an approximate solution of (49) is found by using an approximate form for $U(X, \psi)$, such as equation (19), and put

$$t = \int_o^X \frac{C(z)}{\rho_o \nu_o} \sqrt{2 \mu_o \tau_w(z)}\, dz \qquad (51)$$

equation 49 becomes

$$p S_1 = \frac{\partial}{\partial \psi}\left(\psi^{\frac{1}{2}} \frac{\partial S_1}{\partial \psi}\right) \qquad (52)$$

where p is the Heaviside operator corresponding to $\frac{\partial}{\partial t}$. This equation for S_1 and its boundary conditions are similar to Lighthill's equation (21) for the temperature distribution in an incompressible flow. The solution of (52) satisfying (50) leads to

$$\sigma \frac{\dot{q}_w(X)}{h_1} \sqrt{\frac{2}{\mu_o \tau_w(X)}} = -\left(\tfrac{2}{3}\right)^{\frac{2}{3}} \frac{(-\frac{1}{3})!}{(\frac{1}{3})!} p^{\frac{1}{3}} \left(\frac{h_{wo}(X) - h_w(X)}{h_1}\right)$$

or

$$\dot{q}_w(X) = \frac{(\rho_o \mu_o)^{\frac{1}{3}} \sqrt{\tau_w(X)}}{(3\sigma)^{\frac{2}{3}} (\frac{1}{3})!} \left[\frac{h_w(+o) - h_{wo}(+o)}{\left(\int_o^X C(z)\sqrt{\tau_w(z)}\, dz\right)^{\frac{1}{3}}} \right.$$

$$\left. + \int_o^X \left(\int_o^X C(z) \sqrt{\tau_w(z)}\, dz\right)^{-\frac{1}{3}} d\Big(h_w(X_1) - h_{wo}(X_1)\Big) \right] \qquad (53)$$

where the latter integral is a Stieljes integral.

If we now transform equation (53) back into the compressible flow coordinates (x,y) then for $\gamma = 1.4$ we find that

$$\frac{q_w(x)}{(h_w(+o)-h_{wo}(+o))}\sqrt{\frac{x}{\rho_a\mu_a u_a}} = \frac{\sqrt{t_w(x)}}{(\frac{1}{3})!\;3^{\frac{2}{3}}\;\sigma^{\frac{2}{3}}}\left(\frac{i_a}{i_w(x)}\right)^{\frac{1-\omega}{2}}\left(\frac{1+M_a^2/5}{1+M_1^2(x)/5}\right)^{7/4}$$

$$\cdot\left[\left(\frac{1}{x^{\frac{3}{4}}}\int_0^x\left(\frac{i_a}{i_w(z)}\right)^{\frac{1-\omega}{2}}\frac{\sqrt{t_w(z)}}{z^{\frac{1}{4}}}\left(\frac{1+M_a^2/5}{1+M_1^2(z)/5}\right)^{7/4}dz\right)^{-\frac{1}{3}}\right.$$

$$+\int_0^x\left(\frac{1}{x^{\frac{3}{4}}}\int_{x_1}^x\frac{\sqrt{t_w(z)}}{z^{\frac{1}{4}}}\left(\frac{i_a}{i_w(z)}\right)^{\frac{1-\omega}{2}}\left(\frac{1+M_a^2/5}{1+M_1^2(z)/5}\right)^{7/4}dz\right)^{-\frac{1}{3}}$$

$$\left.\frac{d(h_w(x_1)-h_{wo}(x_1))}{(h_w(+o)-h_{wo}(+o))}\right]\quad\ldots\ldots\ldots\quad(54)$$

and can be evaluated when t_w, M_1, i_w are given as functions of x.

When M_1 = const. = M_a we found from equation (36') that

$$t_w = \frac{0.360}{\left[0.45+0.55\frac{i_w(x)}{i_a}+0.036\,M_a^2\;\sigma^{\frac{1}{2}}\right]^{\frac{1-\omega}{2}}}$$

where the term in the denominator replaced $\frac{h_w(x)\;(1+M_a^2/5)}{h_1} = \frac{i_w(x)}{i_a}$.

If we then assume that t_w and $h_1/h_w(1+M_a^2/5)$

can be taken as constants during integration we find from (54) that

$$\frac{\dot{q}_w(x)}{(h_w(+o)-h_{wo}(+o))}\sqrt{\frac{x}{\rho_a\mu_a u_a}} = \frac{(0.360)^{\frac{1}{3}}\;(\frac{3}{4})^{\frac{1}{3}}}{\sigma^{\frac{2}{3}}\;3^{\frac{2}{3}}\;(\frac{1}{3})!\left(0.45+0.55\frac{i_w(x)}{i_a}+0.036\,M_a^2\;\sigma^{\frac{1}{2}}\right)^{\frac{1-\omega}{2}}}$$

$$\cdot\left[1+\int_0^x\frac{1}{\left(1-(x_1/x)^{\frac{3}{4}}\right)^{\frac{1}{3}}}\;\frac{d\left[h_w(x_1)-h_{wo}(x_1)\right]}{(h_w(+o)-h_{wo}(+o))}\right]\quad\ldots\ldots\quad(55)$$

where $\left(\frac{0.360\;.\;3}{9.4}\right)^{\frac{1}{3}}\frac{1}{(\frac{1}{3})!} = \quad 0.348$.

The constant 0.348 differs from the value 0.339 given by Lighthill (1950) for he used the more accurate value 0.332 in the expression for t_w in place of 0.360 used here.

As previously stated the value of h_w for zero heat transfer (h_{wo}) is given approximately by the Pohlhausen relation

$$\frac{h_{wo}}{i_1} = 1 + \frac{\gamma-1}{2} M_1^2 \sigma^{\frac{1}{2}} \qquad (56)$$

or,

$$\frac{h_{wo}(x)}{h_1} = 1 - \frac{(1-\sigma^{\frac{1}{2}})\frac{\gamma-1}{2} M_1^2(x)}{1 + \frac{\gamma-1}{2} M_1^2(x)} \qquad (56')$$

for σ new unity.

If further the wall temperature is constant then the Stanton heat transfer coefficient (k_h) is given by

$$k_h(x) = \frac{\dot{q}_w(x)}{\rho_a u_a (h_w - h_{wo})} = \frac{0.348\,\sigma^{-\frac{2}{3}}}{R_x^{\frac{1}{2}}} \left(\frac{\rho_w \mu_w}{\rho_a \mu_a}\right)^{0.5} \qquad (57)$$

where $\left(\frac{\rho_w \mu_w}{\rho_a \mu_a}\right)^{\frac{1}{2}} = \left(\frac{i_a}{i_w}\right)^{\frac{1-\omega}{2}}$ is the unmodified value and

$$R_x = \frac{\rho_a u_a x}{\mu_a} .$$

In the case of a flow commencing from a stagnation point the use of the reference Mach number (M_a) is not convenient unless it is, say, the freestream Mach number.

An alternative form of (54) which is suitable in this case is

$$\frac{\dot{q}_w(x)}{(h_w(+0) - h_{wo}(+0))} \sqrt{\frac{x}{\rho_1 \mu_1 u_1}} = \frac{1}{(\frac{1}{3})!\,(3\sigma)^{\frac{2}{3}}} \left(\frac{i}{i_w}\right)^{\frac{1-\omega}{2}} \frac{\sqrt{x(x)\, M_1(x)^{\frac{1}{2}}}}{(1 + M_1(x)^2/5)^{\frac{3+\omega}{4}}}$$

$$\left[\left(\frac{1}{x^{\frac{3}{4}}} \int_0^x \frac{\left(\frac{i_1}{i_w}\right)^{\frac{1-\omega}{2}} \sqrt{x(z)\, M_1(z)^{3/2}}}{(1 + M_1^2(z)/5)^{\frac{3(3+\omega)}{4}}} \frac{dz}{z^{\frac{1}{4}}} \right)^{-\frac{1}{3}} \right.$$

$$+ \int_0^x \left(\frac{1}{x^{\frac{3}{4}}} \int_{x_1}^{x} \frac{\left(\frac{i_1}{i_w}\right)^{\frac{1-\omega}{2}} \sqrt{\chi(z)\, M_1(z)}^{\,3/2}}{\left(1 + M_1^2(z)/5\right)^{\frac{3(3+\omega)}{4}}} \frac{dz}{z^{\frac{1}{4}}} \right)^{-\frac{1}{3}} \frac{d(h_w(x_1) - h_{wo}(x_1))}{(h_w(+o) - h_{wo}(+o))} \Bigg] \qquad \ldots\ldots \quad (58)$$

where $\chi(x) = \tau_w(x) \sqrt{\dfrac{x}{\rho_1 \mu_1 u_1^3}}$

But in paragraph 8 we have found that when the wall temperature is constant and $M_1(+c) = 0$,

$$\chi(x) = 0.360 \left(\frac{i_1}{i_w}\right)^{\frac{1-\omega}{2}} \left[x^{\frac{2}{3}} \int_0^x \frac{1}{(x-z)^{\frac{2}{3}}} \frac{d\,M_1^2(z)}{M_1^2(x)} \right]^{\frac{3}{4}} \left(\frac{i_w}{h_1}\right)^{\frac{3}{4}}$$

provided that i_w does not approach zero. ...... (59)

Thus when $M_1(x) = cx^m$ $(m > 0)$

$$\chi(x) = 0.360 \left(\frac{i_w}{h_1}\right)^{\frac{3}{4}} \left(\frac{i_1}{i_w}\right)^{\frac{1-\omega}{2}} \left[\frac{(2m)!\ (-\frac{2}{3})!}{(2m - \frac{2}{3})!}\right]^{\frac{3}{4}} \qquad \ldots\ldots \quad (60)$$

and when $m = 1$

$$\chi(x) = 1.112 \left(\frac{i_w}{h}\right)^{\frac{3}{4}} \left(\frac{i_1}{i_w}\right)^{\frac{1-\omega}{2}} \qquad \ldots\ldots \quad (61)$$

If in equation (58) we omit the terms in $1 + M_1^2/5$ and replace them by unity, then for constant wall temperature (noting that χ is approximately independent of x) and assuming that h_{wo} is a constant,

$$\frac{\dot{q}_w(x)}{h_w - h_{wo}} \sqrt{\frac{x}{\rho_1 \mu_1 u_1}} = \frac{1}{(\frac{1}{3})!\ \sigma^{\frac{2}{3}}} \left(\frac{0.360(m+1)}{12}\right)^{\frac{1}{3}} \left(\frac{i_1}{i_w}\right)^{\frac{1-\omega}{2}} \left[\frac{(2m)!\ (-\frac{2}{3})!}{(2m - \frac{2}{3})!}\right]^{\frac{1}{4}} \qquad \ldots\ldots \quad (62)$$

But the left hand side equals $k_h(x)\sqrt{R_x}$, where $R_x = \dfrac{\rho_1 u_1 x}{\mu_1}$,

so that when $m = 1$

$$k_h(x) = \frac{0.637}{R_x^{\frac{1}{2}}\ \sigma^{\frac{2}{3}}} \left(\frac{i_w}{h_1}\right)^{\frac{1}{4}} \left(\frac{i_1}{i_w}\right)^{\frac{1-\omega}{2}}$$

$$= \frac{0.637}{R_x^{\frac{1}{2}}\ \sigma^{\frac{2}{3}}} \left(\frac{\rho_w \mu_w}{\rho_1 \mu_1}\right)^{\frac{1}{2}} \left(\frac{i_w}{h_1}\right)^{\frac{1}{4}} \qquad \ldots\ldots \quad (63)$$

Now for the case $u_1 \sim x$ Cohen and Reshotko (1956) find for $\sigma = \omega = 1$ (two-dimensional body)

$$k_h(x) = \frac{0.54}{\sigma^{0.6}\sqrt{R_x}}$$

while Fay and Riddell (1958) deduce that

$$k_h(x) = \frac{0.54}{\sigma^{0.6}\sqrt{R_x}}\left(\frac{\rho_w \mu_w}{\rho_1 \mu_1}\right)^{0.1}$$

for air under equilibrium dissociation conditions with the Lewis number equal to unity.

The difference between the values of the constants 0.637 and 0.54 is partly the result of using the value of 1.112 as the constant in the shear stress parameter term. Of greater importance is the difference in the powers of $\left(\frac{\rho_w \mu_w}{\rho_1 \mu_1}\right)$. The relations are only similar when ω has a value near 0.4. However if the viscosity is evaluated from the Sutherland formula it is found, on comparison with the reation $\mu \sim T^{\omega}$ that ω varies from 1.5 at very low temperatures to 0.5 at very high temperatures. Thus the relation $\left(\frac{\rho_{w1} \mu_w}{\rho_1 \mu_1}\right)^{\frac{1}{2}} \left(\frac{i_w}{i_1}\right)^{\frac{1}{4}}$ may not in fact differ markedly from $\left(\frac{\rho_w \mu_w}{\rho_1 \mu_1}\right)^{0.1}$ at high temperatures.

In appendix 4 Spalding's method is described whereby the accuracy of the above calculations can be improved.

11. Acknowledgement

The calculations were performed by Miss R. Fuller and Mrs. Lennard.

12. Conclusions

The compressible flow laminar boundary layer equations have been solved approximately for arbitrary pressure gradient and wall temperature distribution by

(a) reducing the compressible flow equations to equations similar to the incompressible flow equations using the Swetartson-Illingworth transformation, and

(b) solving the resulting equations by Lighthill's method to obtain the skin friction and rate of heat transfer.

The method is probably sufficiently accurate for engineering purposes in regions of negative pressure gradient and for small adverse pressure gradients, provided separation is not approached. The method is however improved in accuracy in this region by using the modification to Lighthill's method introduced by Spalding (1958).

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APPENDIX 1

Heat transfer for fluids of very low Prandtl number

When $\sigma << 1$ the temperature boundary layer thickness is small compared with the velocity boundary layer thickness and therefore over a large part of the temperature boundary layer the velocity will be constant as suggested by Liepmann (1958). Hence in equation (17) putting $U = U_1$ and noting that the term on the righthand side vanishes since $\frac{\partial U}{\partial Y} = 0$,

$$\frac{\partial S}{\partial X} = \nu_o \frac{C U_1}{\sigma} \frac{\partial^2 S}{\partial \psi^2} \tag{A.1}$$

But if S_o is a solution of the complete equation when the wall rate of heat transfer is zero we must have approximately

$$\frac{\partial}{\partial X} S_1 = \frac{\nu_o C U_1}{\sigma} \frac{\partial^2 S_1}{\partial \psi^2} \tag{A.2}$$

where $S_1 = S - S_o$, and $S_1 \to 0$ as $\psi \to \infty$, and as $X \to 0$.

If we put $t = \int_0^X \frac{\nu_o C U_1}{\sigma} dz$

and p is the Heaviside operator $\frac{\partial}{\partial t}$ then

$$p S_1 = \frac{\partial^2 S_1}{\partial \psi^2} \tag{A.3}$$

having the operational solution

$$S_1 = \exp(-p^{\frac{1}{2}} \psi) \; S_{1w}(X) \tag{A.4}$$

(where the functions of p 'operate on' the function of X). But near $\psi = 0$, S_1 must satisfy the relation (since $U \sim U_1$)

$$S_1 = \frac{h_{wo}(X) - h_w(X)}{h_1} + \frac{\sigma \dot{q}_w(X)\psi}{\mu_o U_1(X)} \quad \cdots\cdots\cdots \tag{A.5}$$

On expanding $\exp(-p^{\frac{1}{2}}\psi)$ in (A.4) in ascending powers of ψ and comparing coefficients with (A.5) we find that

$$\frac{\dot{q}_w(X)}{\mu_o U_1(X)} = -p^{\frac{1}{2}} S_{1w}(X) \tag{A.6}$$

which on interpreting gives

$$\dot{q}_w(X) = - \frac{\mu_o U_1(X)}{\sigma \sqrt{\pi}} \left[\frac{S_{1w}(+o)}{t^{\frac{1}{2}}} + \int_o^t \frac{(d S_{1w}/dt_1) dt_1}{(t - t_1)^{\frac{1}{2}}} \right] \qquad (A.7)$$

$$= \frac{-(\rho_o \mu_o)^{\frac{1}{2}} U_1(X)}{\sigma^{\frac{1}{2}} \sqrt{\pi}} \left[\frac{S_{1w}(+o)}{\left(\int_o^X C(z) U_1(z) dz \right)^{\frac{1}{2}}} + \int_o^X \left(\int_{X_1}^X C(z) U_1(z) dz \right)^{-\frac{1}{2}} dS_w(X_1) \right]$$

$$= \left(\frac{\rho_o \mu_o}{\pi \sigma} \right)^{\frac{1}{2}} U_1(X) \left[\frac{h_w(+o) - h_{wo}(+o)}{\left(\int_o^X C(z) U_1(z) dz \right)^{\frac{1}{2}}} + \int_o^X \left(\int_{X_1}^X C(z) U_1(z) dz \right)^{-\frac{1}{2}} d\left(h_w(X_1) - h_{wo}(X_1) \right) \right] \qquad (A.8)$$

If we now transform equation A.8 back into the compressible flow coordinates then for $\gamma = 1.4$ we find that

$$\frac{\dot{q}_w(x)}{(h_w(+o) - h_{wo}(+o))} \sqrt{\frac{x}{\rho_a \mu_a u_a}} = \frac{M_1}{M_a} \left(\frac{i_a}{i_w} \right)^{1-\omega} \frac{(1 + M_a^2/5)^4}{(1 + M^2/5)^4} \frac{1}{\sqrt{\pi\sigma}}$$

$$\left[\left(\frac{1}{x} \int_o^x \frac{M_1}{M_a} \left(\frac{i_a}{i_w} \right)^{1-\omega} \frac{dz}{\left(\frac{1 + M_1^2/5}{1 + M_a^2/5} \right)^4} \right)^{-\frac{1}{2}} \right. \qquad (A.9)$$

$$\left. + x^{\frac{1}{2}} \int_o^x \left(\int_{x_1}^x \frac{M_1 (1 + M_a^2/5)^4}{M_a (1 + M_1^2/5)^4} \left(\frac{i_a}{i_w} \right)^{1-\omega} dz \right)^{-\frac{1}{2}} \frac{d\left(h_w(x_1) - h_{wo}(x_1) \right)}{(h_w(+o) - h_{wo}+o)} \right]$$

When $M_1 = M_a$ = const and $(h_w - h_{wo})$ = const we have approximately

$$\frac{q_w(x)}{(h_w - h_{wo})} \sqrt{\frac{x}{\rho_a \mu_a u_a}} = \frac{1}{\sqrt{\pi\sigma}} \left(\frac{i_a}{i_w} \right)^{\frac{1-\omega}{2}} = \frac{1}{\sqrt{\pi\sigma}} \left(\frac{\rho_w \mu_w}{i_a \mu_a} \right)^{\frac{1}{2}} \qquad (A.10)$$

and shows a $\sigma^{-\frac{1}{2}}$ relation compared with $\sigma^{-\frac{1}{3}}$ at higher values of the Prandtl number.

Apart from the term in $\frac{\rho_w \mu_w}{i_a \mu_a}$, and assuming the flow is incompressible, equation (A.10) is identical with the first term of the relation given by Morgan, Pipkin and Warner (1958). It might be noted in passing that from the work of the latter authors it would appear that for the case of zero heat transfer, the Pohlhausen result is approximately correct even at low values of the Prandtl number. Morgan, et al find that

$$\frac{i_w - i_1}{u_1^2} = 0.462 \, \sigma^{\frac{1}{2}} + 0.097\sigma + 0(\sigma^{3/2})$$

compared with a right hand side of $0.5\,\sigma^{\frac{1}{2}}$ in Pohlhausen's relation.

In low Mach number flows at constant wall temperature and $u_1 = c\,x^m$ we find that

$$\frac{q_w(x)}{(h_w - h_{wo})} \sqrt{\frac{x}{\rho_o u_1(x)\,\mu_o}} = \sqrt{\frac{m+1}{\pi\,\sigma}} \qquad (A.11)$$

and is the zeroth order approximation given by Morgan et al. The case when $T_w - T_{wo}$ varies as x^n can similarly be found.

APPENDIX 2

Heat transfer near a separation point

We have shown above that in regions of retarded flow outside the boundary layer our approximate formula for the rate of heat transfer is not in good agreement with known exact solutions. The fault in the approximate solution lies clearly in the poor approximation made to the velocity distribution close to the wall. If however the flow is just separating from the wall a better approximation to the velocity distribution is (following Liepmann(1958))

$$U = U''(o)\, Y^2/2 \qquad (B.1)$$

where $$U''(o) = \left(\frac{\partial^2 U}{\partial Y^2}\right)_{Y=0} = -\frac{h_w(X)}{2\nu_o C(X) h_1} \frac{dU_1^2}{dX}$$

(from equ. 11)

or $$U = -\frac{h_w}{4h_1 \nu_o C} \frac{d\,U_1^2}{dX} Y^2 \qquad (B.2)$$

and since $\psi = \int_0^Y U\, dY$ we obtain approximately

$$U = \left(\frac{-9h_w\, dU_1^2/dX}{4\, h_1 \nu_o\, C}\right)^{\frac{1}{3}} \psi^{\frac{2}{3}} \qquad (B.3)$$

Thus equ. becomes

$$\frac{\partial S_1}{\partial X} = \frac{\nu_o\, C}{\sigma} \left(\frac{-9h_w\, dU_1^2/\partial X}{4\, h_1 \nu_o\, C}\right)^{\frac{1}{3}} \frac{\partial}{\partial\psi}\left(\psi^{\frac{2}{3}} \frac{\partial S_1}{\partial\psi}\right) \qquad (B.4)$$

If $$t = \int_0^X \frac{\nu_o\, C}{\sigma} \left(\frac{-9h_w\, dU_1^2/dX}{4\, h_1 \nu_o\, C}\right)^{\frac{1}{3}} dX \qquad (B.5)$$

then $$p\, S_1 = \frac{\partial}{\partial\psi}\left(\psi^{\frac{2}{3}} \frac{\partial S_1}{\partial\psi}\right) \qquad (B.6)$$

where $S_1 \to 0$ as $\psi \to \infty$, and as $t \to 0$, and

$$S_1 = \frac{h_{wo}(X) - h_w(X)}{h_1} + \frac{\sigma \dot{q}_w(X)}{\mu_o\, h_1} \left(- \frac{12\, h_1\, \nu_o\, C}{h_w\, dU_1^2/dX}\, \psi \right)^{\frac{1}{3}} \qquad (B.7)$$

as $\psi \to 0$.

Two linearly independent power series solutions of (B.6) are easily obtained, namely,

$$S_1 = 1 + \frac{9}{3.4}\, p\, \psi^{4/3} + \frac{(9p\,\psi^{4/3})^2}{3.4\ .\ 7.8} + \ldots.$$

$$= (\tfrac{3}{4}\, p^{\frac{1}{2}}\, \psi^{\frac{2}{3}})^{\frac{1}{4}}\ (-\tfrac{1}{4})!\ \ I_{-\frac{1}{4}}\ (\tfrac{3}{2}\, p^{\frac{1}{2}}\, \psi^{\frac{2}{3}}) \qquad (B.8)$$

and

$$S_1 = \psi^{\frac{1}{3}} \left(1 + \frac{9\, p\, \psi^{4/3}}{4.5} + \frac{(9\, p\ \psi^{4/3})}{4.5\ .\ 8.9} + \ldots. \right)$$

$$= \psi^{\frac{1}{3}}\ (\tfrac{3}{4}\, p^{\frac{1}{2}}\ \psi^{\frac{2}{3}})^{-\frac{1}{4}}\ (\tfrac{1}{4})!\ \ I_{\frac{1}{4}}\ (\tfrac{3}{2}\, p^{\frac{1}{2}}\ \psi^{\frac{2}{3}}) \qquad (B.9)$$

In order to satisfy (B.7) the following combination of solutions (B.8) and (B.9) is required

$$S_1 = (\tfrac{3}{4}\, p^{\frac{1}{2}}\ \psi^{\frac{2}{3}})^{\frac{1}{4}}\ (-\tfrac{1}{4})!\ \ I_{-\frac{1}{4}}\, (\tfrac{3}{2}\, p^{\frac{1}{2}}\ \psi^{\frac{2}{3}}) \left(\frac{h_{wo}(X) - h_w(X)}{h_1} \right)$$

$$+ \ \psi^{\frac{1}{3}}\, (\tfrac{3}{4}\, p^{\frac{1}{2}}\ \psi^{\frac{2}{3}})^{-\frac{1}{4}}\ (\tfrac{1}{4})!\ \ I_{\frac{1}{4}}\ \ (\tfrac{3}{2}\, p^{\frac{1}{2}}\ \ \psi^{\frac{2}{3}})\ \frac{\sigma \dot{q}_w(X)}{\mu_o\, h_1} \left(\frac{-12\, h_1\ \nu_o C(X)}{h_w(X)\ \ \frac{dU_1^2}{dX}} \right)^{\frac{1}{3}} \qquad (B.10)$$

Since $S_1 \to 0$ as $\psi \to \infty$ the coefficients of $I_{-\frac{1}{4}}$ and $I_{\frac{1}{4}}$ must be equal and opposite.

Hence

$$\frac{\sigma \dot{q}_w(X)}{\mu_o\, h_1} \left(\frac{-12\, h_1\, \nu_o C(X)}{h_w(X)\ dU_1^2/dX} \right)^{\frac{1}{3}} = - \ (\tfrac{3}{4})^{\frac{1}{2}}\ \frac{(-\frac{1}{4})!\, p^{\frac{1}{4}}}{(\frac{1}{4})!} \left(\frac{h_{wo}(X) - h_w(X)}{h_1} \right) \quad \ldots.. \qquad (B.11)$$

Now the operational form of

$p^{\frac{1}{4}} S_w(X)$ may be interpreted as

$$\frac{1}{(-\frac{1}{4})!}\left[\frac{S_w(+o)}{t^{\frac{1}{4}}} + \int_o^t \frac{d\,S_w/dt_1 \quad dt_1}{(t-t_1)^{\frac{1}{4}}}\right] = \frac{1}{(-\frac{1}{4})!}\int_o^X \frac{d\,S_w(X_1)}{(t-t_1)^{\frac{1}{4}}} \qquad \text{(B.12)}$$

for short.

Hence equation (B.11) becomes

$$\dot{q}_w(X) = \frac{1}{4^{\frac{3}{4}}\,(\frac{1}{4})!}\;\frac{(\rho_o\mu_o)^{\frac{1}{2}}\left(-\frac{h_w(X)}{h_1}\;\frac{dU_1^2}{dX}\right)^{\frac{1}{3}}}{\sigma^{\frac{3}{4}}\quad C\,(X)^{\frac{1}{3}}}\;\int_o^X \frac{d(h_w(X_1) - h_{wo}(X_1))}{\left(\int_{X_1}^{X}\left(-\frac{h_w(z)}{h_1}\;C^2(z)\;\frac{d\,U_1^2}{dz}\right)^{\frac{1}{3}}dz\right)^{\frac{1}{4}}}$$

$$\ldots\ldots \qquad \text{(B.13)}$$

If we now transform back into compressible flow coordinates then for = 1.4 we obtain

$$\frac{\dot{q}_w(x)}{(h_w(+o) - h_{wo}(+o))}\sqrt{\frac{x}{\rho_a\mu_a\,u_a}} = \frac{1}{(4\sigma)^{\frac{3}{4}}\,(\frac{1}{4})!}\left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{8/3}$$

$$\cdot\left(-\frac{h_w}{h_1 M_a^2}\;\frac{d\,M_1^2}{dx}\right)^{\frac{1}{3}}\left(\frac{i_a}{i_w}\right)^{\frac{2}{3}(1-\omega)}\cdot x^{\frac{1}{2}}\int_o^x \frac{d\left(\frac{h_w(x_1) - h_{wo}(x_1)}{h_w(+o) - h_{wo}(+o)}\right)}{\left[\int_{x_1}^{x}\left(-\frac{h_w}{h_1 M_a^2}\;\frac{d\,M_1^2}{dz}\right)^{\frac{1}{3}}\left(\frac{i_a}{i_w}\right)^{\frac{2}{3}(1-\omega)}\cdot\left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{8/3} dz\right]^{\frac{1}{4}}} \qquad \text{(B.14)}$$

for flow at a separation point.

If $h_w - h_{wo}$ = const and the Mach number is small and we put

$$\frac{M_1}{M_a} = cx^m, \text{ where } m < 0, \text{ then}$$

$$\frac{\dot{q}_w(x)}{h_w - h_{wo}} \sqrt{\frac{x}{\rho_a \mu_a u_a}} = \frac{1}{2\,\sigma^{\frac{3}{4}}(\frac{1}{4})!} \left(\frac{-m(1+m)}{3}\right)^{\frac{1}{4}} \left(\frac{M_1}{M_a}\right)^{\frac{1}{2}}$$

or,

$$\frac{\dot{q}_w(x)}{(h_w - h_{wo})} \sqrt{\frac{x}{\rho_o \mu_o u_1(x)}} = \frac{1}{2\,\sigma^{\frac{3}{4}}(\frac{1}{4})!} \left(\frac{-m(1+m)}{3}\right)^{\frac{1}{4}} \qquad \text{(B.15)}$$

These results agree fairly well with those of Liepmann (1958). In equation (B.13) Liepmann finds in place of

$$\frac{1}{4^{\frac{3}{4}}(\frac{1}{4})!} \quad \text{the constant} \quad \left[\frac{3}{16}\left(\frac{\pi}{4} - \frac{2}{3}\right)\right]^{\frac{1}{4}}$$

or 0.39 against 0.386 respectively. The value of

$$\frac{\dot{q}_w(k)}{(h_w - h_{wo})} \sqrt{\frac{l}{\rho_o \mu_o u(l)}} = \frac{\frac{1}{l}\int_0^l \dot{q}_w(x)dx}{(h_w - h_{wo})} \sqrt{\frac{l}{\rho_o \mu_o u_1(l)}} = 0.452$$

obtained by integrating (B.15) with $\sigma = 0.7$ and $m = -0.0904$

(the exact value found by Hartree (1937))

compares with 0.438 which is the exact value found in incompressible flow at a separation point. Liepmann (1958) obtained the value 0.448.

Equation (B.14) shows the effect of Mach number and wall temperature variation on the rate of heat transfer near a separation point. It shows that the Stanton heat transfer coefficient varies as $\sigma^{-\frac{3}{4}}$ in this region compared with $\sigma^{-\frac{2}{3}}$ for an accelerated flow.

APPENDIX 3

Approximate inversion of the wall shear stress integral equation

An alternative method of performing the inversion described in paragraph 8 is as follows. Since, in general, t_w and H will be slowly varying functions of x a good approximation to

$$\int_{x_1}^{x} \frac{t_w(z)^{\frac{1}{2}} H(z)}{z^{\frac{1}{4}}} dz \quad \text{is} \quad \frac{4}{3} t_w(x_1)^{\frac{1}{2}} H(x_1) (x^{\frac{3}{4}} - x_1^{\frac{3}{4}}) .$$

More generally if we put

$$\frac{t_w(z)^{\frac{1}{2}} H(z)}{z^{\frac{1}{4}}} \approx z^{\beta} \quad \text{then}$$

$$\int_{x_1}^{x} \frac{t_w(z)^{\frac{1}{2}} H(z)}{z^{\frac{1}{4}}} dz \approx \frac{t_w(x_1)^{\frac{1}{2}} H(x_1)}{x_1^{\frac{4\beta+1}{4}}} \left(\frac{x^{\beta+1} - x_1^{\beta+1}}{\beta+1} \right).$$

Equation (34) then approximates to (retaining the constant term $\frac{3^{\frac{1}{3}} . 2}{(-\frac{2}{3})!}$)

$$F(x) = \frac{3^{\frac{1}{3}} . 2}{(-\frac{2}{3})!} (\beta + 1)^{\frac{1}{3}} \int_0^x \frac{t_w(x_1)^{4/3} J(x_1) dx_1}{x_1^{\frac{2-\beta}{3}} (x^{\beta+1} - x_1^{\beta+1})^{\frac{1}{3}}} \qquad (C.1)$$

or
$$F(\eta) = \frac{3^{\frac{1}{3}} . 2}{(-\frac{2}{3})! (\beta+1)^{\frac{2}{3}}} \int_0^{\eta} \frac{t_w(\eta_1)^{4/3} J(\eta_1) d\eta_1}{\eta_1^{\frac{2}{3}} (\eta - \eta_1)^{\frac{1}{3}}}$$

where $\eta = x^{\beta+1}$. This equation is invertible as

$$\left(\frac{t_w(\eta) J(\eta)^{\frac{3}{4}}}{\eta^{\frac{1}{2}}} \right)^{4/3} = \frac{(\beta+1)^{\frac{2}{3}} \eta^{-\frac{2}{3}}}{3^{\frac{1}{3}} . 2 . (-\frac{1}{3})!} \left[\frac{M_1(+0)^2}{M_a^2} + \eta^{\frac{2}{3}} \int_0^{\eta} \frac{h_w(z)/h_1}{(\eta - z)^{\frac{2}{3}}} d\left(\frac{M_1(z)}{M_a}\right)^2 \right]$$

or
$$t_w(x) = \frac{J(x)^{-\frac{3}{4}} \ (\beta + 1)^{\frac{1}{2}}}{\left[3^{\frac{1}{3}} \ . \ 2 \ . \ (-\frac{1}{3})!\right]^{\frac{3}{4}}} \left[\left(\frac{M_1(+0)}{M_a}\right)^2 + x^{\frac{2(\beta+1)}{3}} \int_0^x \frac{h_w(z)/h_1}{(x^{\beta+1} - z^{\beta+1})^{\frac{2}{3}}} \ d\left(\frac{M_1(z)}{M_a}\right)^2 \right]^{\frac{3}{4}} \qquad (C.2)$$

If $\beta = 0$ this equation is equal to equation (36'). In incompressible flow with $h_w = h_1$ and $u_1 \sim x^m$ we find

$$\tau_w(x) \sqrt{\frac{x}{\rho_o \mu_o u_1^3}} = \frac{(\beta + 1)^{\frac{1}{2}}}{(3^{\frac{1}{3}} \ . \ 2)^{\frac{3}{4}}} \left[\frac{\left(\frac{2m}{\beta + 1}\right)! \ (-\frac{2}{3})!}{\left(\frac{2m}{\beta+1} - \frac{2}{3}\right)! \ (-\frac{1}{3})!} \right]^{\frac{3}{4}} \qquad (C.3)$$

But in this case it can easily be shown that

$$\beta + 1 = \frac{3(m + 1)}{4} \quad \text{and so}$$

$$\tau_w(x) \sqrt{\frac{x}{\rho_o \mu_o \ u_1^3}} = \frac{3^{\frac{1}{4}} \ (m + 1)^{\frac{1}{2}}}{2^{7/4}} \left[\frac{\left(\frac{8m}{3m + 3}\right)! \ \ (-\frac{2}{3})!}{\left(\frac{6m - 2}{3m + 3}\right)! \ \ (-\frac{1}{3})!} \right]^{\frac{3}{4}} \qquad (C.4)$$

which is exactly equation 70 of Lighthill's paper. In his double approximation method Lighthill puts $\beta = 0$ in equation (C.3). We note that β changes from a value of about -0.5 near separation to -0.25 for zero pressure gradient, to 0.5 near a stagnation point. Thus $\beta = 0$ is a suitable average value for an accelerated flow. In the table below the right hand side of (C.3) is given for various values of β and a comparison made with the exact values of Hartree.

$\tau_w \sqrt{\frac{x}{\rho_o \mu_o U_1^3}}$ \ m	$-\frac{1}{6}$	$-\frac{1}{8}$	$-\frac{1}{9}$	-0.0904	-0.0654
Exact (Hartree)	-	-	-	0	0.157
$(\beta + 1) = \frac{3(m+1)}{4}$ (Lighthill)	-	-	0	0.101	0.175
$\beta = 0$ (Lighthill)	0	-	-	0.214	0.260
$\beta = -0.25$	-	0	-	0.130	0.191
$\beta = -0.25$ (x 1.09)		0	-	0.142	0.208

$\tau_w \sqrt{\frac{x}{\rho_o \mu_o U_1^3}}$ \ m	0	$\frac{1}{9}$	$\frac{1}{3}$	1	4	∞
Exact (Hartree)	0.332	0.512	0.757	1.233	2.405	$\sim 1.193\ m^{\frac{1}{2}}$
$\beta + 1 = \frac{3(m+1)}{4}$	0.312	0.473	0.698	1.136	2.218	$\sim 1.100\ m^{\frac{1}{2}}$
* $\beta = 0$	0.360	0.495	0.698	1.112	2.158	$\sim 1.066\ m^{\frac{1}{2}}$
$\beta = -0.25$	0.312	0.462	0.676	1.10	2.15	$\sim 1.066\ m^{\frac{1}{2}}$
$\beta = -0.25$ (x 1.09)	0.340	0.503	0.736	1.199	2.342	$\sim 1.162\ m^{\frac{1}{2}}$

We see that with $\beta = -0.25$ and the formula multiplied by 1.09 the error for $m \geq 0$ is less than 2.5%.

In applying equation (C.2) to a flow having an arbitrary pressure gradient the value of β must be suitably chosen. However if we put $\beta = -0.25$ and include the factor 1.09 our equation should be suitable for engineering purposes in accelerated flow.

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* The asymptotic values for $\tau_w \sqrt{\frac{x}{\rho_o \mu_o u_1^3}}$ for large m are independent of β if β is assumed independent of m. In this case

$$\tau_w \sqrt{\frac{x}{\rho_o \mu_o u_1^3}} = m^{\frac{1}{2}} \left[\frac{(-\frac{2}{3})!}{(-\frac{1}{3})!}\right]^{\frac{3}{4}} 6^{-\frac{1}{4}} \quad \text{as } m \to \infty .$$

APPENDIX 4

Improved relations for skin friction and heat transfer

Spalding (1958) has introduced a correction into Lighthill's method for estimating the rate of heat transfer, which takes the form of an improved velocity distribution near the wall. This correction is an improvement on that introduced by Tifford (1951) and Tribus and Klein (1955), since it takes account of differences between the thickness of the velocity and temperature boundary layers.

If in place of (19) we write

$$U = \frac{\tau_W Y}{\mu_o} + \frac{U''(o)\,Y^2}{2}$$

and noting that $U''(o)$ is found from equation (11) we find that

$$U = \frac{\tau_W Y}{\mu_o}\left(1 - \frac{\rho_o U_1 h_W \, dU_1/dX}{2\,C\,\tau_W h_1}\,Y\right)$$

Now in order to solve the skin friction problem approximately by Lighthill's method we require first a reasonable approximation to the velocity distribution in the velocity layer, (and not in the temperature layer as is required in solving the heat transfer problem).

If therefore we replace Y in the brackets by $\dfrac{U_1 \mu_o}{\tau_W}$ then

$$U \sim \sqrt{\frac{2\,\tau_W}{\mu_o}\left(1 - \frac{\rho_o \mu_o h_W U_1^2\, dU_1/dX}{2\,C\,h_1\,\tau_W^2}\right)}\;\psi^{\frac{1}{2}} \qquad (D.1)$$

and in place of equation (20) we have

$$\frac{\partial G}{\partial X} = \sqrt{\frac{2\,\mu_o\,\tau_W\,C^2}{\rho_o^2}\left(1 - \frac{\rho_o \mu_o h_W U_1^2\, dU_1/dX}{2\,C\,h_1\,\tau_W^2}\right)}\;\psi^{\frac{1}{2}}\,\frac{\partial^2 G}{\partial \psi^2} \qquad (D.2)$$

which has the solution (if we retain the exact boundary condition equation (21))

$$U_1(o)^2 + \int_o^X \frac{h_w(z)}{h_1} \, d\,U_1^2(z) = \frac{3^{\frac{1}{3}} \cdot 2}{(-\frac{2}{3})! \; (\rho_o \mu_o)^{\frac{2}{3}}} \int_o^X C\,(X_1)\tau_w(X_1)^{3/2} \cdot$$

$$\left(1 - \frac{\rho_o \mu_o \, h_w(X_1) \; U_1(X_1)^2 dU_1/dX_1}{2\,h_1 C(X_1) \;\; \tau_w(X_1)^2}\right)^{\frac{1}{2}} \left(\int_{X_1}^X \sqrt{\tau_w(z)\; C^2(z) \left(1 - \frac{\rho_o \mu_o h_w(z) U_1(z)^2 dU_1/dz}{2h_1 \; C(z) \;\; \tau_w(z)^2}\right)} \cdot dz \right)^{-\frac{1}{3}} dX_1 \;. \qquad (D.3)$$

If

$$F(x) = M_1(o)^2 + \int_o^x \frac{h_w(z)}{h_1} \; d\,M_1^2(z)$$

$$J(x) = \frac{(1 + M_1(x)^2/5)^{10/3}}{C^{\frac{2}{3}}} \left(1 - \frac{\rho_o \mu_o \; a_o^3 \; h_w(x) C(x) \; M_1(x)^2 \frac{dM_1}{dx}}{2h_1 \; \tau_w(x)^2 \; (1 + M_1(x)^2/5)^5}\right)^{\frac{1}{3}}$$

and $\tau_w(x) \; \sqrt{x} = \lambda(x)$

then the corresponding compressible flow solution is (using the approximation of Appendix 3 with $\beta = -0.25$)

$$F(\eta) \approx \frac{2^{7/3}}{(-\frac{2}{3})! \; 3^{\frac{1}{3}} \; (\rho_o \mu_o \; a_o^3)^{\frac{2}{3}}} \int_o^\eta \frac{\lambda(\eta_1)^{4/3} \;\; J(\eta_1)}{\eta_1^{\frac{2}{3}} \; (\eta - \eta_1)^{\frac{1}{3}}} \; d\eta_1 \qquad (D.4)$$

where $\eta = x^{\frac{3}{4}}$, and is invertible as

$$t_w(x) = \frac{\bar{J}(x)^{-\frac{3}{4}} \; 3^{\frac{1}{4}}}{(-\frac{1}{3})! \; \frac{3}{4} \; 2^{7/4}} \cdot \left[\left(\frac{M_1(+o)}{M_a}\right)^2 + x^{\frac{1}{2}} \int_o^x \frac{h_w(z)/h_1}{(x^{\frac{3}{4}} - z^{\frac{3}{4}})^{\frac{2}{3}}} \; d\left(\frac{M_1(z)}{M_a}\right)^2\right]^{\frac{3}{4}} \qquad \ldots\ldots \; (D.5)$$

where

$$\bar{J}(x) = \left(\frac{1 + M_1^2/5}{1 + M_a^2/5}\right)^{10/3} \left(\frac{i_w}{i_a}\right)^{\frac{2(1-\omega)}{3}} \left(1 - \frac{h_w}{2h_1}\left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^5 \left(\frac{i_a}{i_w}\right)^{1-\omega} \cdot \left(\frac{M_1}{M_a}\right)^2 x \; \frac{d}{dx}\left(\frac{M_1}{M_a}\right) \; \frac{1}{t_w^2}\right)^{\frac{1}{3}}$$

In incompressible flow with $h_w = h_1$ and $u_1 \sim x^m$ we find that

$$\dot{x}(x)\left(1 - \frac{m}{2X(x)^2}\right)^{\frac{1}{4}} = \frac{3^{\frac{1}{4}}}{2^{7/4}}\left[\frac{(\frac{8m}{3})!\ (-\frac{2}{3})!}{(\frac{8m-2}{3})!\ (-\frac{1}{3})!}\right]^{\frac{3}{4}} \qquad (D.6)$$

where $\quad x(x) = \tau_w(x)\sqrt{\dfrac{x}{\rho_o \mu_o u_1^3}}$.

If F(m) is put equal to the right hand side then

$$x^4 - \frac{m}{2}x^2 - F(m)^4 = 0 \qquad (D.7)$$

having the solution

$$x(x) = \frac{(m \pm \sqrt{m^2 + 16F^4})^{\frac{1}{2}}}{2} \qquad (D.8)$$

x \ m	$-\frac{1}{8}$	-0.0904	-0.0654	0
Exact (Hartree)	-	0	0.157	0.332
Equ. D.8	0	0.075	0.154	0.312

x \ m	$\frac{1}{9}$	$\frac{1}{3}$	1	4	∞
Exact (Hartree)	0.512	0.757	1.233	2.405	$1.193\ m^{\frac{1}{2}}$
Equ. D.8	0.493	0.740	1.218	2.395	$1.185\ m^{\frac{1}{2}}$

It is seen from the table above that except for m = 0 the error is reduced by using equation D.8. However the separation point is still given by $m = -\frac{1}{8}$ compared with the exact value of m = -0.0904. Hence except near separation and at zero pressure gradient, the quadratic correction term in the velocity distribution improves the accuracy of Lighthill's formula for the skin friction. It can easily be shown that the overall error is not reduced if we choose a value for β (-0.458) such that separation is predicted exactly by equation D.8.

If we use a similar method to improve the accuracy of Lighthill's formula for the rate of heat transfer then, following Spalding, we must choose a reasonable approximation to the velocity distribution in the temperature layer. We therefore put

$$U \approx \frac{\tau_w Y}{\mu_o}\left[1 - \frac{(i_w - i_1)\,\rho_o\,\mu_o\,h_w\,U_1\,dU_1/dX}{2\,\sigma\,C\,h_1\,\tau_w\,\mathring{q}_w}\right]$$

where Y in the square bracket has been replaced by

$$\frac{k_o\,(i_w - i_1)}{C_p\,\mathring{q}_w}\,.$$

Hence

$$U \sim \sqrt{\frac{2\tau_w}{\mu_o}\left(1 - \frac{(i_w - i_1)\,\rho_o\,\mu_o\,h_w\,U_1\,dU_1/dX}{2\,\sigma\,C\,h_1\,\tau_w\,\mathring{q}_w}\right)}\;\psi^{\frac{1}{2}} \qquad (D.9)$$

and equation (49) becomes

$$\frac{\partial S_1}{\partial X} = \frac{C}{\rho_o\sigma}\left\{2\,\mu_o\,\tau_w\left(1 - \frac{(i_w - i_1)\,\rho_o\mu_o\,h_w\,U_1\,dU_1/dX}{2\,\sigma\,C\,h_1\,\tau_w\,\mathring{q}_w}\right)\right\}^{\frac{1}{2}}\frac{\partial}{\partial\psi}\left(\psi^{\frac{1}{2}}\,\frac{\partial S_1}{\partial\psi}\right)$$

$$\ldots\ldots \qquad (D.10)$$

If we retain the exact boundary condition equ.(50) the solution of D.10 is

$$\mathring{q}_w(X) = \frac{(\rho_o\mu_o)^{\frac{1}{3}}\sqrt{\tau_w(X)}}{(3\sigma)^{\frac{2}{3}}\;(\frac{1}{3})!}\left[\frac{h_w(+o) - h_{wo}(+o)}{\left(\int_o^X K(z)\,dz\right)^{\frac{1}{3}}}\right.$$

$$\left. + \int_o^X\left(\int_{X_1}^X K(z)\,dz\right)^{-\frac{1}{3}} d\left(h_w(X_1) - h_{wo}(X_1)\right)\right] \qquad (D.11)$$

where

$$K(X) = \sqrt{C^2\;\tau_w\left(1 - \frac{(i_w - i_1)\,\rho_o\mu_o\,h_w\,U_1\,dU_1/dX}{2\,\sigma\,C\,h_1\,\tau_w\,\mathring{q}_w}\right)}$$

After transformation into the compressible flow coordinates (x,y) we obtain for $\gamma = 1.4$

$$s_w(x) = \frac{\sqrt{t_w(x)}\left(\frac{i_a}{i_w}\right)^{\frac{1-\omega}{2}}}{(3\sigma)^{\frac{2}{3}}\ (\frac{1}{3})!} \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{7/4} \left[\left(\frac{1}{x^{\frac{3}{4}}} \int_0^x \bar{K}(z)\, dz\right)^{-\frac{1}{3}} + \int_0^x \left(\frac{1}{x^{\frac{3}{4}}} \int_{x_1}^x \bar{K}(z)\, dz\right)^{-\frac{1}{3}} \frac{d\,(h_w - h_{wo})}{(h_w(+o) - h_{wo}(+o))}\right] \qquad (D.12)$$

where

$$s_w(x) = \frac{\dot{q}_w}{(h_w(+o)-h_{wo}(+o))} \sqrt{\frac{x}{\rho_a\ \mu_a\ u_a}}$$

$$t_w(x) = \tau_w \sqrt{\frac{x}{\rho_a \mu_a\ u_a^3}}$$

and

$$\bar{K}(x) = \frac{\sqrt{t_w}\left(\frac{i_a}{i_w}\right)^{\frac{1-\omega}{2}}}{x^{\frac{1}{4}}} \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{7/4} \left(1 - \frac{(i_w - i_1)\ {}^{h_w/h_1}\left(i_a/i_w\right)^{1-\omega}}{4\sigma\ (h_w(+o) - h_{wo}(+o)\ )\ t_w\ s_w} \cdot \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{9/2} \cdot x \frac{d}{dx}\left(\frac{M_1}{M_a}\right)^2\right)^{\frac{1}{2}}$$

In incompressible flow we find that

$$s_w(x) = \frac{\sqrt{t_w}}{(3\sigma)^{\frac{2}{3}}\ (\frac{1}{3})!} \left[\left(\frac{1}{x^{\frac{3}{4}}} \int_0^x \bar{K}(z)\, dz\right)^{-\frac{1}{3}} + \int_0^x \left(\frac{1}{x^{\frac{3}{4}}} \int_{x_1}^x \bar{K}(z)\, dz\right)^{-\frac{1}{3}} d(\Delta i_w)\right]$$

..... (D.13)

where

$$\Delta i_w = \frac{i_w - i_1}{i_w(+o)-i_1}$$

and

$$\bar{K}(z) = \frac{\sqrt{t_w}}{x^{\frac{1}{4}}} \left(1 - \frac{\Delta i_w\ \ x \frac{d}{dx}\left(U_1/U_a\right)^2}{4\,\sigma\ t_w\ s_w}\right)^{\frac{1}{2}}$$

Following Liepmann (1958) we can write equation D.13 in the equivalent form

$$s_w(x) = \sqrt{\frac{t_w \ \Delta i_w^3}{(3\sigma)^{\frac{2}{3}} \ (\frac{1}{3})!}} \left[\frac{1}{x^{\frac{3}{4}}} \int_0^x \Delta i_w^{3/2} \ \bar{K}(z) \ dz \right]^{\frac{1}{3}} \qquad \text{(D.14)}$$

and its corresponding differential form is

$$\left(\frac{x^{\frac{1}{2}}}{t_w \ \Delta i_w^3} \right)^{\frac{1}{2}} \frac{d}{dx} \left(\frac{x^{\frac{1}{2}} \ t_w \ \Delta i_w^3}{s_w^2} \right)^{3/2} = \frac{9\sigma^2 \ (\frac{1}{3})!^3}{\left(1 - \frac{\Delta i_w \ x \ \frac{d}{dx} \left(u_1/u_a \right)}{4\sigma \ t_w \ s_w} \right)^{\frac{1}{2}}} \qquad \text{(D.15)}$$

where $9.(\frac{1}{3})!^3 \doteq 6.41$. This relation is similar to the expression given by Spalding (1958) for isothermal surfaces, but differs from the latter, by the inclusion of the Δi_w terms, for surfaces having non-uniform temperature. For the case of non-uniform wall temperature Spalding suggests a superposition method which is equivalent to finding a solution to equation (D.15). Presumably Spalding's method of solution could be used to solve equation (D.15) for s_w if t_w, Δi_w and du_1/dx are known.

APPENDIX 5

Skin friction in a linear adverse pressure gradient

When $\sigma = \omega = 1$ equation 36' can be written for arbitrary γ

$$t_w(x) = 0.360 \left(\frac{p_1(x)}{p_a}\right)^{1/\gamma} \left[\left(\frac{M_1(+o)}{M_a}\right)^2 - \frac{2\, x^{\frac{2}{3}}}{\gamma M_a^2} \cdot \right. \qquad \ldots\ldots \text{(E.1)}$$

$$\left. \int_o^x \frac{i_w(z)/i_a}{(x-z)^{\frac{2}{3}}} \quad \frac{d\, p_1(z)/p_a}{\left(\frac{p_1(z)}{p_a}\right)^{\frac{2\gamma-1}{\gamma}}} \right]^{\frac{3}{4}}$$

If $M_1(+o) = M_a = 2$, $\frac{p_1}{p_a} = 1 + {}^x/l$ and we put $\frac{i_w}{i_a} = \text{const}$ with $\gamma = 1.5$, then

$$t_w(x) = \frac{0.360}{(1 + {}^x/l)^{1/12}} \left[1 - \frac{x}{l}\left(\frac{i_w}{i_a} - 1\right)\right]^{\frac{3}{4}} \qquad \text{(E.2)}$$

or if 0.360 is replaced by 0.332

$$2t_w = \frac{0.664}{(1+{}^x/l)^{1/12}} \left[1 - \frac{x}{l}\left(\frac{i_w}{i_a} - 1\right)\right]^{\frac{3}{4}} \qquad \text{(E.2')}$$

In the case of zero heat transfer with $M_a = 2$ and $\gamma = 1.5$

$$i_w \approx \text{const} = 2.0\, i_a,$$

and equation (E.2') becomes (writing t_{wo} for t_w)

$$2t_{wo}(x) = \frac{0.664}{(1 + {}^x/l)^{1/12}} \left[1 - {}^x/l\right]^{\frac{3}{4}} \qquad \text{(E.3)}$$

This is plotted in fig.(4) and compared with the results of Luxton and Young (1958) and Gadd (1957a). The agreement between the three sets of results is poor especially approaching separation. If however we modify the term in i_w/i_a in equation (E.1) and replace it by (noting we have taken $\gamma = 1.5$),

$$\frac{5}{4} \quad \frac{i_{wo}}{i_a} - \frac{2}{3}\left(\frac{i_{wo} - i_w}{i_a}\right)$$

(where i_{wo} is the wall enthalpy for zero heat transfer) we obtain in place of (E.3)

$$2t_{wo}(x) = \frac{0.664}{(1+{}^x/l)^{1/12}}\left[1 - \frac{3x}{2l}\right]^{\frac{3}{4}} \qquad (E.3')$$

which gives separation at $\frac{x}{l} = 0.667$ compared with Gadd's value of ${}^x/l = 0.65$.

Similarly, for the case of heat transfer with $i_w = i_a$ we obtain

$$2t_w(x) = \frac{0.664}{(1 + {}^x/l)^{1/12}}\left[1 - \frac{5x}{6l}\right]^{\frac{3}{4}} \qquad (E.4)$$

which is plotted in fig.(5) and compared with the 'exact' N.P.L. results* and those of Luxton and Young (1958), and Cohen and Reshotko (1956a). The modified term has in fact been chosen to give good agreement with the N.P.L. results. It is not surprising in view of the approximations made that some form of correction term is necessary in order to improve the accuracy of the present results.** What is surprising is that a relatively minor modification is all that is necessary, to give good accuracy, in such extreme cases of an adverse pressure gradient and a very cool surface, and the corresponding case with zero heat transfer.

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* Unpublished results referred to in the paper by Luxton and Young.

** The form of the modified ${}^h w/h_1$ term implies that in an approximate solution of equ.(13) the value of S should be chosen not at its wall value but at some average value away from the wall.

A further problem of interest is that of the magnitude of an abrupt pressure gradient which will provoke separation without any pressure increase. This problem has been considered by Morduchow and Grape (1955) and Gadd (1957b). In our analysis, using the modified form for i_w/i_a, we see that separation occurs in a constant adverse pressure gradient, which follows a region of constant pressure, when

$$1 \leq \frac{2}{\gamma M_a^2}\left(1 + \frac{\gamma-1}{2} M_a^2\right)\int_0^x \frac{J(z)}{\left(1 - \left(\frac{z}{x}\right)^{\frac{3}{2}}\right)^{2/3}} \frac{dz/x}{\left(1 + c\,\frac{x}{l}\,z/x\right)^{\frac{2\gamma-1}{\gamma}}} \cdot x\,\frac{d\,p_1/p_a}{dx} \qquad \text{.....} \quad (E.5)$$

where $J(z) = \left[\frac{5}{4}\ i_{wo} - \frac{2}{3}(i_{wo} - i_w)\right] \Big/ h_1$ (for $\gamma = 1.5$)

and $p_1/p_a = 1 + c.\ x/l$.

If $C_p = \dfrac{p_1 - p_a}{\frac{1}{2}\rho_a U_a^2} = \dfrac{2\,(p_1/p_a - 1)}{\gamma M_a^2}$ and J is a constant,

then the relation (E.5) can be written, when $\gamma = 1.5$,

$$x\,\frac{d\,C_p}{dx} \geq \frac{1}{3\left(1 + \frac{\gamma-1}{2} M_a^2\right) J} \quad \text{as} \quad x \to 0 \qquad (E.6)$$

and when $\sigma = 1$

$$x\,\frac{dC_p}{dx} \geq \frac{4}{\left(1 + \frac{\gamma-1}{2} M_a^2\right)\left(7 + 8i_w/h_1\right)} \qquad (E.6')$$

This compares qualitatively with the relation given by Morduchow and Grape (1955)

$$x\,\frac{dC_p}{dx} \geq \frac{6.41}{\left(1 + \frac{\gamma-1}{2} M_a^2\right)\left(11\,\frac{i_w}{h_1} + 4\right)}$$

and the experimental results of Gadd (1956a and b).

APPENDIX 6

Evaluation of a certain integral

When the wall temperature is constant the value of the shear stress parameter, t_W, is determined from a modified form of equation (36'). Thus for the case $\sigma = \omega = 1$

$$2t_W = 0.664 \left(\frac{1 + M_a^2/5}{1 + M_1^2/5}\right)^{5/2} .$$

$$\left[1 + \left(\frac{i_W}{h_1}\right)^{*} x^{\frac{2}{3}} \int_0^x \frac{d\,(M_1/M_a)^2}{(x - z)^{\frac{2}{3}}} \right]^{\frac{3}{4}} \qquad \ldots\ldots \quad (F.1)$$

When $u_1 = u_a\,(1 - x/l)$ it is found that

$$\left(\frac{M_1}{M_a}\right)^2 = \frac{(1 - \bar{x})^2}{1 + \frac{M_a^2}{5}\,(2\bar{x} - \bar{x}^2)} \qquad (F.2)$$

where $\bar{x} = x/l$

and

$$\left(\frac{i_W}{h_1}\right)^{*} x^{\frac{2}{3}} \int_0^x \frac{d(M_1/M_a)^2}{(x - z)^{\frac{2}{3}}} = -2\left(\frac{i_W}{i_a}\right)^{*} \bar{x} \int_0^1 \frac{(1 - \bar{x}z)\,dz}{(1-z)^{2/3}\left[1 + \frac{M_a^2}{5}\,(2\bar{x}z - (\bar{x}z)^2)\right]^2}$$

$$= -6\left(\frac{i_W}{i_a}\right)^{*} \bar{x} \int_0^1 \frac{\left[(1 - \bar{x}) + \bar{x}\,v^3\right]\,dv}{\left\{\left[1 + \frac{M_a^2}{5}\,(2\bar{x} - \bar{x})\right] - \frac{2M_a^2}{5}\,(\bar{x} - \bar{x}^2)v^3 - \frac{M_a^2}{5}\,\bar{x}^2\,v^6\right\}^2}$$

$$\ldots\ldots \quad (F.3)$$

on putting $z = 1 - v^3$.

Hence from (F.1) and (F.3) we find that

$$2\,t_w = 0.664\left[1 + \frac{M_a^2}{5}(2\bar{x} - \bar{x}^2)\right]^{5/2} \cdot \left[1 + 6\left(\frac{i_w}{i_a}\right)^{\bar{x}}\bar{x}\left\{(1-\bar{x})\,I_1 + \bar{x}\,I_2\right\}\right]^{\frac{3}{4}} \qquad (F.4)$$

where

$$I_1 = \int_0^1 \frac{d\,v}{\left\{\left[1 + \frac{M_a^2}{5}(2\bar{x} - \bar{x}^2)\right] - \frac{2M_a^2}{5}(\bar{x} - \bar{x}^2)v^3 - \frac{M_a^2}{5}\bar{x}^2 v^6\right\}^2}$$

and

$$I_2 = \int_0^1 \frac{v^3\,dv}{\left\{1 + \frac{M_a^2}{5}(2\bar{x} - \bar{x}^2) - \frac{2M_a^2}{5}(\bar{x} - \bar{x}^2)v^3 - \frac{M_a^2}{5}\bar{x}^2 v^6\right\}^2}$$

Now

$$I_1 = \frac{25}{M_a^4}\int_0^1 \frac{dv}{(\bar{x}^2 v^6 + A\bar{x}\,v^3 - B)^2} \qquad (F.5)$$

and

$$I_2 = \frac{25}{M_a^4}\int_0^1 \frac{v^3\,dv}{(\bar{x}^2 v^6 + A\bar{x}\,v^3 - B)^2} \qquad (F.6)$$

where

$$A = 2(1 - \bar{x})$$

$$B = \frac{5}{M_a^2} + \bar{x}(2 - \bar{x}) = (1 + 5/M_a^2) - (1 - \bar{x})^2$$

But

$$\frac{1}{(\bar{x}^2 v^6 + A\bar{x}v^3 - B)^2} = -\frac{1}{3B(4B + A^2)}\left\{\frac{d}{dv}\left[\frac{(2B + A^2)v + A\bar{x}\,v^4}{\bar{x}^2 v^6 + A\bar{x}v^3 - B}\right] + 2\left[\frac{(5B + A^2) + A\bar{x}v^3}{\bar{x}^2 v^6 + A\bar{x}v^3 - B}\right]\right\}$$

so that

$$I_1 = -\frac{25}{3B(4B + A^2)M_a^4}\left\{\frac{2B + A^2 + A\bar{x}}{\bar{x}^2 + A\bar{x} - B} + 2(5B + A^2)\int_0^1 \frac{dv}{(\bar{x}^2 v^6 + A\bar{x}v^3 - B)} + 2A\bar{x}\int_0^1 \frac{v^3\,dv}{(\bar{x}^2 v^6 + A\bar{x}v^3 - B)}\right\}$$

..... (F.5')

Let

$$I_3 = \int_0^1 \frac{dv}{\bar{x}^2 v^6 + A \bar{x} v^3 - B}$$

$$= \frac{1}{(C + D)} \left[\int_0^1 \frac{dv}{\bar{x} v^3 - C} - \int_0^1 \frac{dv}{\bar{x} v^3 + D} \right] \qquad \text{(F.7)}$$

where $\bar{x}^2 v^6 + A \bar{x} v^3 - B = (\bar{x} v^3 - C)(\bar{x} v^3 + D)$

and $C = \sqrt{\dfrac{1 + M_a^2/5}{M_a^2/5}} - (1 - \bar{x})$

$D = \sqrt{\dfrac{1 + M_a^2/5}{M_a^2/5}} + (1 - \bar{x})$

Since

$$\int_0^1 \frac{dv}{\bar{x} v - C} = \frac{1}{3C} \left(\frac{C}{\bar{x}}\right)^{\frac{1}{3}} \left\{ \ln. \frac{1 - (\bar{x}/C)^{\frac{1}{3}}}{\sqrt{\left(\frac{\bar{x}}{C}\right)^{\frac{2}{3}} + \left(\frac{\bar{x}}{C}\right)^{\frac{1}{3}} + 1}} - \sqrt{3} \tan^{-1} \left[\frac{\sqrt{3} (\bar{x}/C)^{\frac{1}{3}}}{2 + (\bar{x}/C)^{\frac{1}{3}}} \right] \right\} = \frac{I_4}{3}$$

, say, ... (F.8)

and

$$\int_0^1 \frac{dv}{\bar{x} v^3 + D} = \frac{1}{3D} \left(\frac{D}{\bar{x}}\right)^{\frac{1}{3}} \left\{ \ln. \frac{1 + (\bar{x}/D)^{\frac{1}{3}}}{\sqrt{\left(\frac{\bar{x}}{D}\right)^{\frac{2}{3}} - \left(\frac{\bar{x}}{D}\right)^{\frac{1}{3}} + 1}} + \sqrt{3} \,.\, \tan^{-1} \left[\frac{\sqrt{3} (\bar{x}/D)^{\frac{1}{3}}}{2 - (\bar{x}/D)^{\frac{1}{3}}} \right] \right\} = \frac{I_5}{3}$$

..... (F.9)

we have

$$I_3 = \frac{1}{6}\sqrt{\frac{M_a^2}{5 + M_a^2}}\left[I_4 - I_5\right] \qquad (F.7')$$

which can be evaluated for known values of M_a and $\bar{x}$.

Similarly if we put

$$I_6 = \int_0^1 \frac{v^3\,dv}{\bar{x}^2 v^6 + A\bar{x}v^3 - B}$$

$$= \frac{1}{(C + D)}\left[\int_0^1 \frac{v^3\,dv}{(\bar{x}\,v^3 - C)} - \int_0^1 \frac{v^3\,dv}{(\bar{x}\,v^3 + D)}\right] \qquad (F10)$$

and noting that

$$\int_0^1 \frac{v^3\,dv}{\bar{x}\,v^3 - C} = \frac{1}{\bar{x}} + \frac{C}{\bar{x}}\int_0^1 \frac{dv}{\bar{x}\,v^3 - C} - \frac{1}{4C}$$

$$= \frac{1}{\bar{x}} + \frac{C}{3\bar{x}} I_4 - \frac{1}{4C} \qquad (F.11)$$

and

$$\int_0^1 \frac{v^3\,dv}{\bar{x}\,v^3 + D} = \frac{1}{\bar{x}} - \frac{D}{3\bar{x}} I_5 + \frac{1}{4D} \qquad (F.12)$$

we find that

$$I_6 = \frac{1}{3\bar{x}(C + D)}\left(C\,I_4 + D\,I_5\right) - \frac{1}{4CD} \qquad (F.10')$$

If the relations (F.7') and (F.10') are substituted into (F.5')

$$I_1 = -\frac{25}{3\bar{x}(4B + A^2)M_a^4}\left\{\frac{2B + A^2 + A\bar{x}}{\bar{x}^2 + A\bar{x} - B} + \frac{1}{3}\sqrt{\frac{M_a^2}{5 + M_a^2}}\,(5B + A^2)(I_4 - I_5)\right.$$

$$\left. + \frac{1}{3}\sqrt{\frac{M_a^2}{5 + M_a^2}}\;A(C\,I_4 + D\,I_5) - \frac{A\bar{x}}{2CD}\right\}$$

$$= \frac{1}{12} \frac{1}{(1 + M_a^2/5)\left(1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1-\bar{x})^2\right)} \left\{ 2\left(1 + M_a^2/5 + M_a^2/5 \cdot (1-\bar{x})\right)\right.$$

$$- \frac{1}{3}\left(5\frac{(5+M_a^2)}{M_a^2} - (1-\bar{x})^2\right)(I_4 - I_5)\sqrt{\frac{M_a^2}{5+M_a^2}} + \frac{\bar{x}(1-\bar{x})\ M_a^2/5}{\left(1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1-\bar{x})^2\right)}$$

$$\left. - \frac{2}{3}(1-\bar{x})\sqrt{\frac{M_a^2}{5+M_a^2}}\left[\sqrt{\frac{5+M_a^2}{M_a}}(I_4 + I_5) - (1-\bar{x})(I_4 - I_5)\right]\right\}$$

$$= \frac{1}{12(1 + M_a^2/5)\left(1 + M_a^2/5 - \frac{M_a^2}{5}(1-\bar{x})^2\right)} \left\{ 2\left(1 + \frac{M_a^2}{5} + \frac{M_a^2}{5}(1-\bar{x})\right)\right.$$

$$- \frac{5}{3}\sqrt{\frac{5+M_a^2}{M_a^2}}(I_4 - I_5) + (1-\bar{x})^2\sqrt{\frac{M_a^2}{5+M_a^2}}(I_4 - I_5) + \frac{\bar{x}(1-\bar{x})\ M_a^2/5}{\left(1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1-\bar{x})^2\right)}$$

$$- \frac{2}{3}(1-\bar{x})(I_4 + I_5) \qquad \ldots\ldots \quad (F.13)$$

Since

$$\frac{v^3}{(\bar{x}^2 v^6 + A\bar{x}v^3 - B)^2} = - \frac{1}{3\bar{x}(4B + A^2)} \left\{ \frac{d}{dv}\left[\frac{Av + 2\bar{x}v^4}{\bar{x}^2 v^6 + A\bar{x}v^3 - B}\right] - \frac{A - 4\bar{x}v^3}{\bar{x}^2 v^6 + A\bar{x}v^3 - B}\right\}$$

we find that

$$I_2 = - \frac{25}{3\bar{x}(4B + A^2)M_a^4}\left\{\frac{A + 2\bar{x}}{\bar{x}^2 + A\bar{x} - B} - A I_3 + 4\bar{x} I_6\right\}$$

$$= \frac{5}{12\,\bar{x}\;M_a^2(1 + M_a^2/5)} \left[\frac{2\,M_a^2}{5} + (1-\bar{x})\sqrt{\frac{M_a^2}{5 + M_a^2}}\;(I_4 - I_5) \right.$$

$$\left. + \frac{\bar{x}\;M_a^2/5}{\left(1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1 - \bar{x})^2\right)} - \frac{2}{3}\;(I_4 + I_5) \right]$$

$$= \frac{1}{6\,\bar{x}\,(1 + M_a^2/5)} \left[1 + \frac{5(1 - \bar{x})}{2\,M_a^2} \sqrt{\frac{M_a^2}{5 + M_a^2}}\;(I_4 - I_5) \right.$$

$$\left. + \frac{\bar{x}}{2\left(1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1-\bar{x})^2\right)} - \frac{5}{3M_a^2}\;(I_4 + I_5) \right] \quad \ldots\ldots \quad (F.14)$$

Finally from equations (F.4), (F.13) and (F.14) the shear stress parameter

$$2\,t_W = 0.664 \left[1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1 - \bar{x})^2 \right]^{5/2} \left[1 - \left(\frac{i_W}{h_1}\right)^{\bar{x}} \bar{x} \left(\right.\right.$$

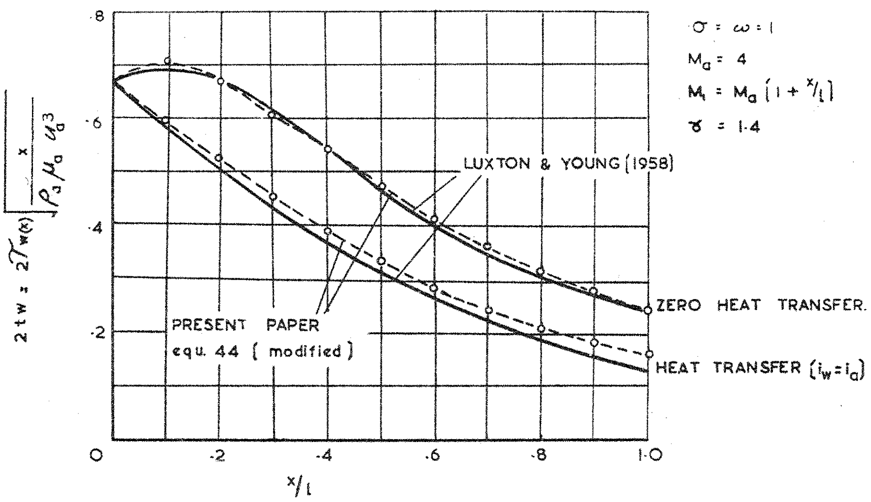
$$\frac{(1 - \bar{x})}{2\left(1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1-\bar{x})^2\right)} \left\{ 2\left(1 + \frac{M_a^2}{5} + \frac{M_a^2}{5}(1 - \bar{x})\right) - \frac{5}{3}\sqrt{\frac{5 + M_a^2}{M_a^2}}\;(I_4 - I_5) \right.$$

$$- \frac{(1 - \bar{x})^2}{3}\sqrt{\frac{M_a^2}{5 + M_a^2}}\;(I_4 - I_5) + \frac{\bar{x}(1 - \bar{x})\;M_a^2/5}{\left(1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1-\bar{x})^2\right)}$$

$$\left. - \frac{2(1 - \bar{x})}{3}\;(I_4 + I_5) \right\} + \left\{ 1 + \frac{5(1 - \bar{x})}{2\,M_a^2}\sqrt{\frac{M_a^2}{5 + M_a^2}}\;(I_4 - I_5) \right.$$

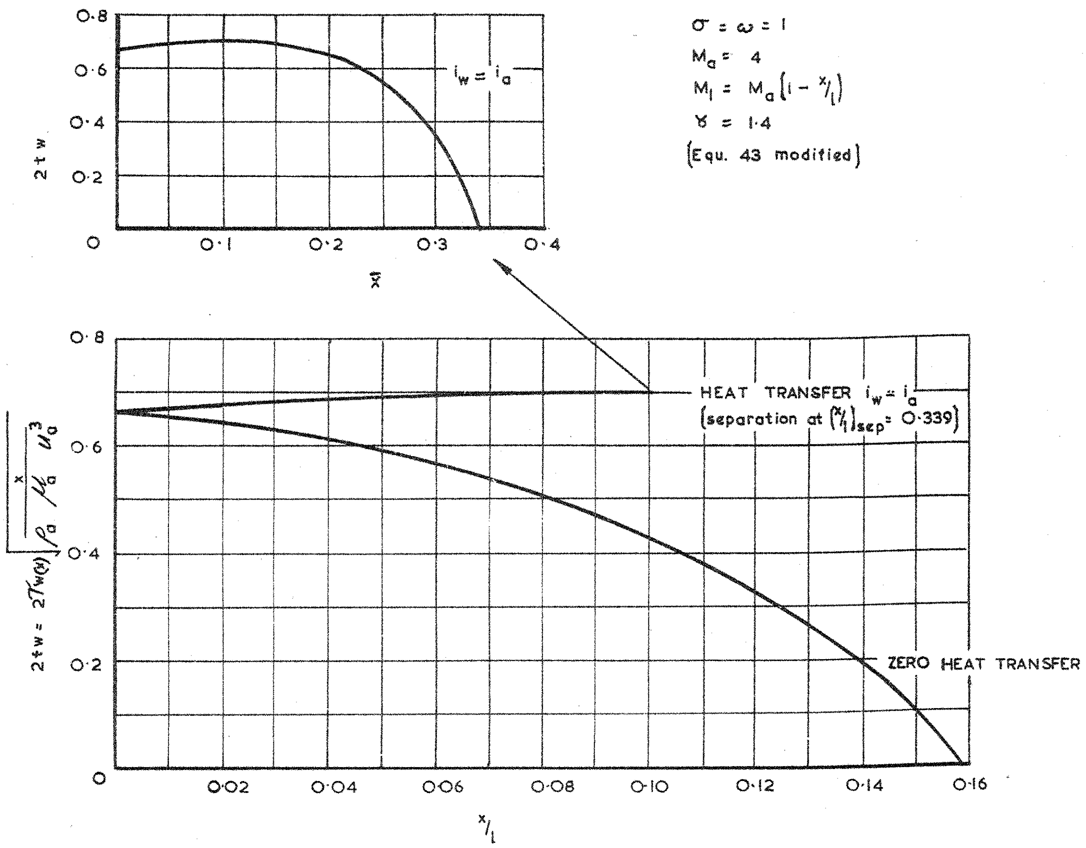
$$\left.\left.\left. + \frac{\bar{x}}{2\left(1 + \frac{M_a^2}{5} - \frac{M_a^2}{5}(1 - \bar{x})^2\right)} - \frac{5}{3M_a^2}\;(I_4 + I_5) \right\}\right) \right]^{\frac{3}{4}} \quad \ldots\ldots \quad (F.15)$$

Numerical results for the cases of zero heat transfer and $T_W = T_a$ when $M_a = 4$ are given in table 4 and are plotted in figures (3a) and (3b).



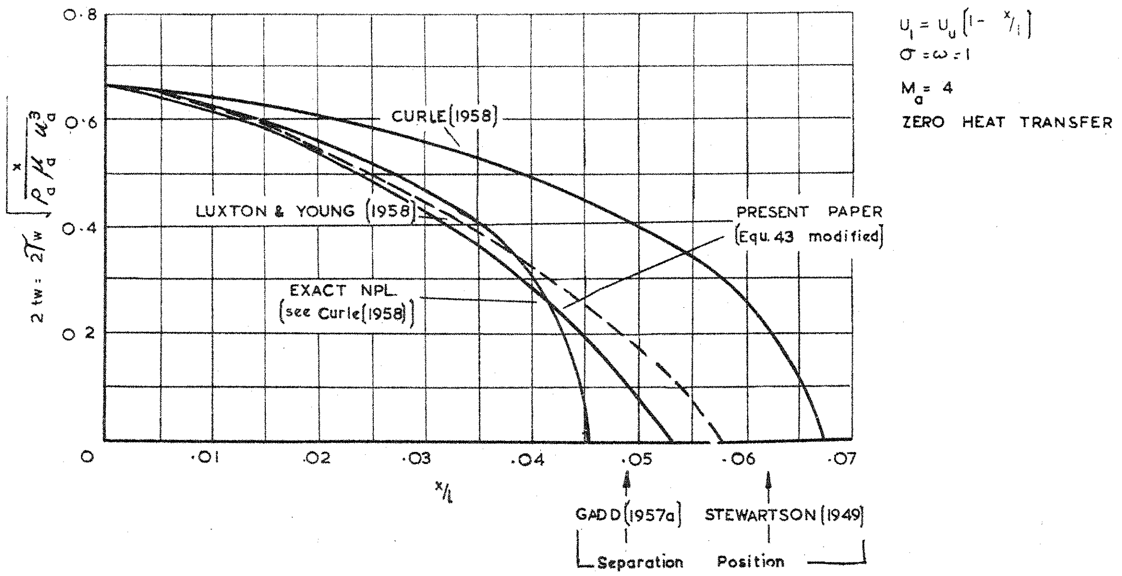
EFFECT OF HEAT TRANSFER ON SKIN FRICTION IN ACCELERATED FLOW

FIG. 1.



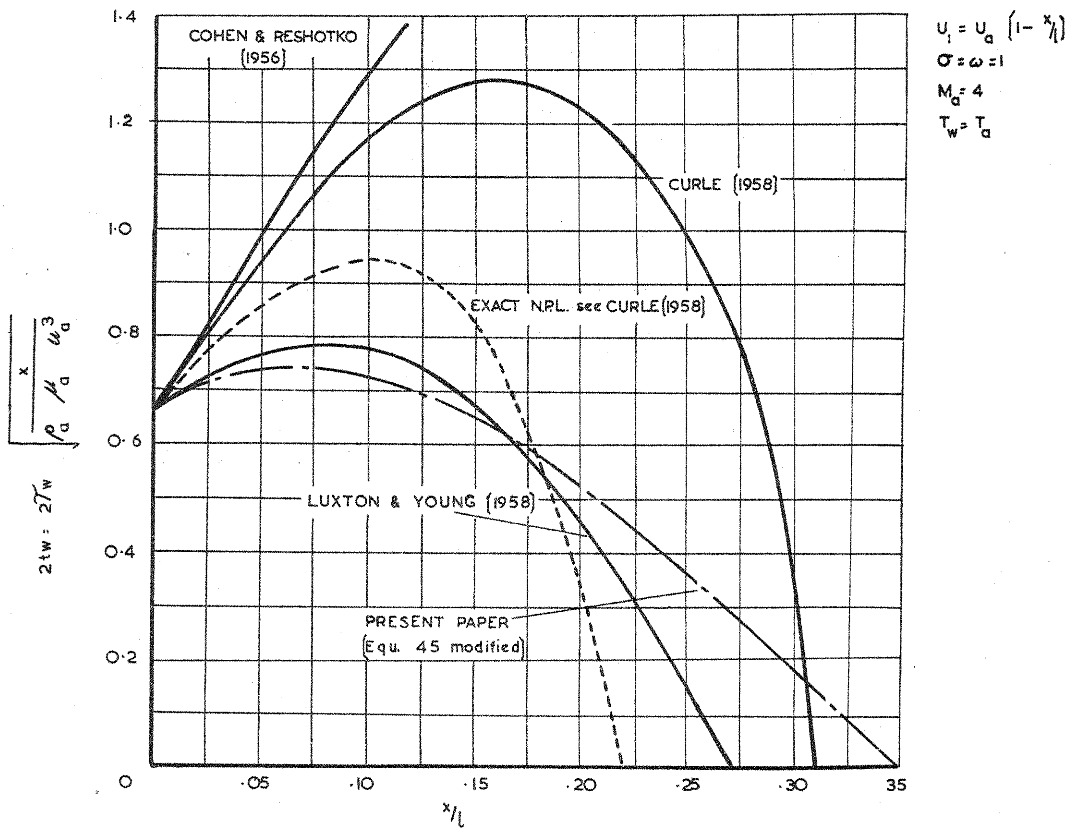
EFFECT OF HEAT TRANSFER ON SKIN FRICTION IN RETARDED FLOW

FIG. 2.



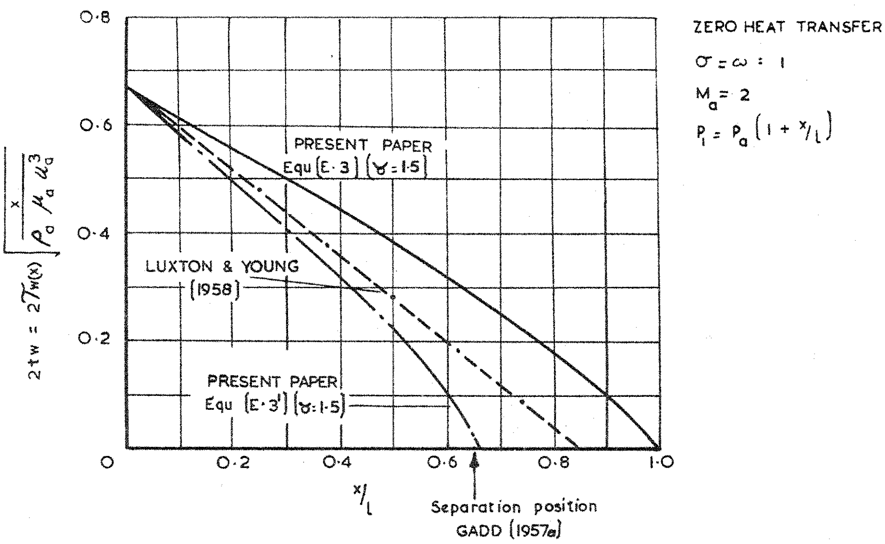
EFFECT OF HEAT TRANSFER ON SKIN FRICTION IN A LINEARLY RETARDED VELOCITY FLOW

FIG 3a.



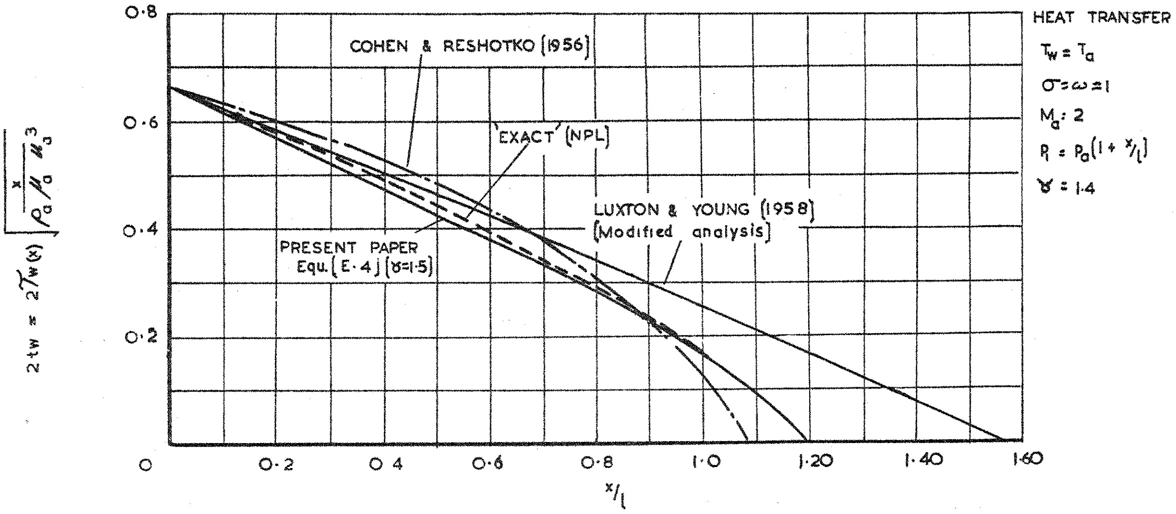
EFFECT OF HEAT TRANSFER ON SKIN FRICTION IN A LINEARLY RETARDED VELOCITY FLOW

FIG. 3b



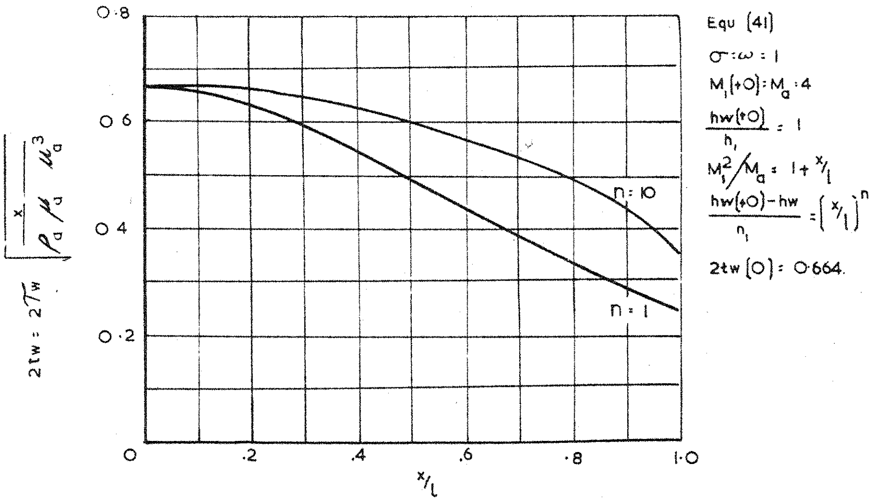
EFFECT OF ZERO HEAT TRANSFER ON SKIN FRICTION IN RETARDED FLOW

FIG. 4



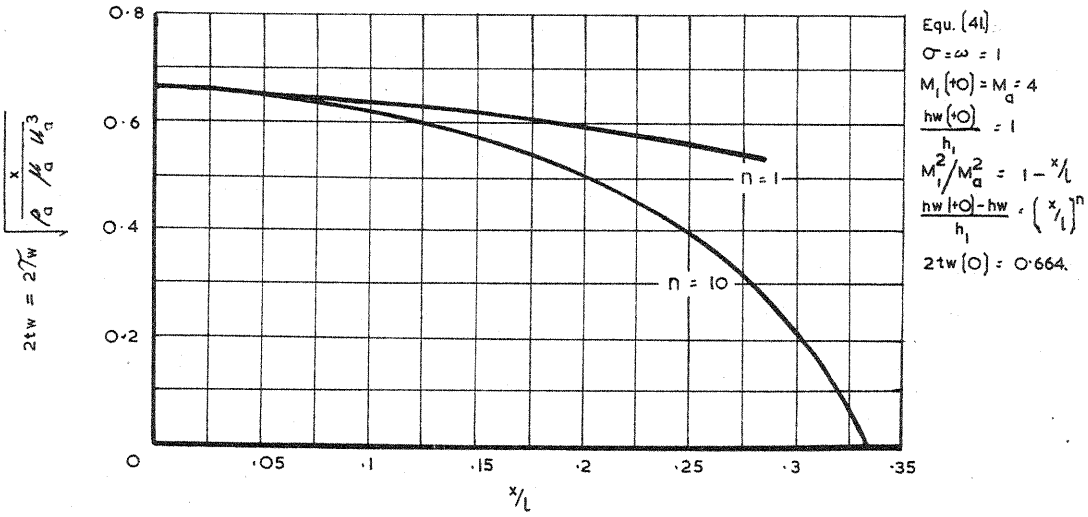
EFFECT OF HEAT TRANSFER ON SKIN FRICTION IN RETARDED FLOW.

FIG. 5.



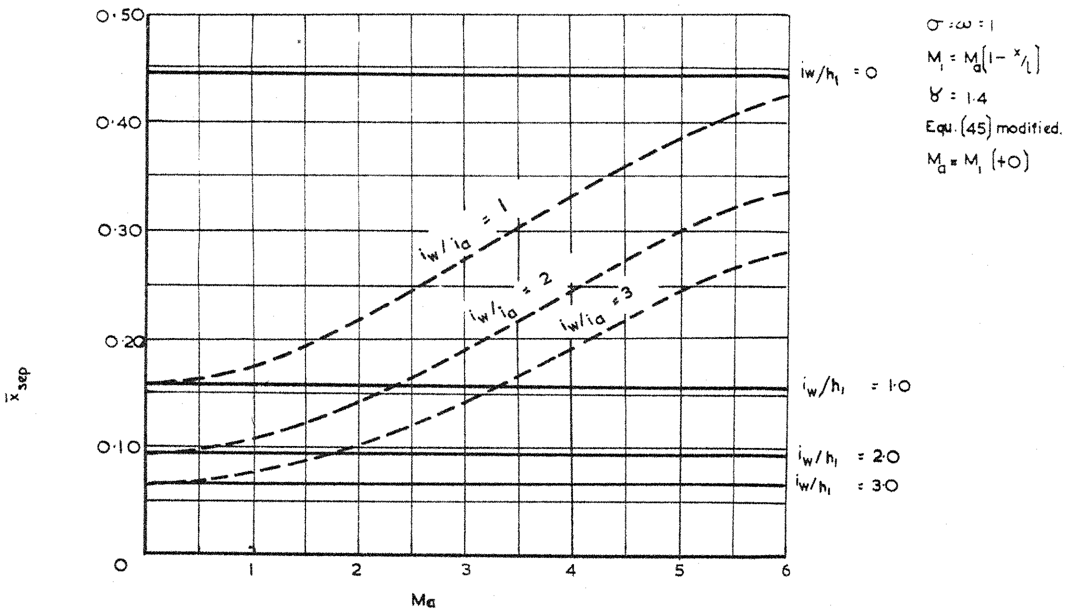
EFFECT OF WALL TEMPERATURE DISTRIBUTION ON SKIN FRICTION IN AN ACCELERATED FLOW

FIG. 6.



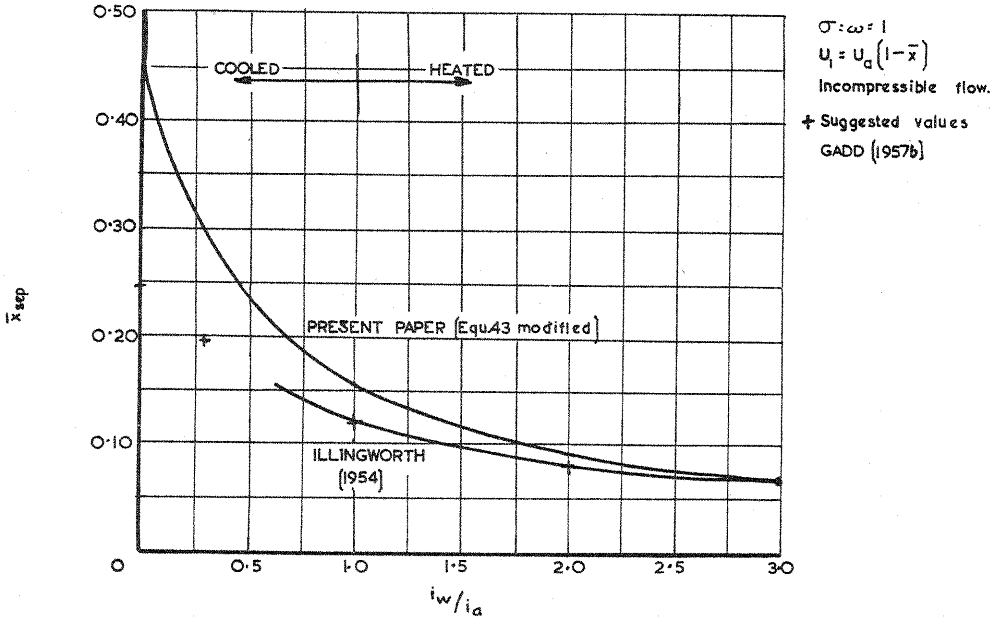
EFFECT OF WALL TEMPERATURE DISTRIBUTION ON SKIN FRICTION IN RETARDED FLOW

FIG. 7.



DISTANCE TO SEPARATION IN A RETARDED FLOW

FIG. 8.



DISTANCE TO SEPARATION IN A RETARDED FLOW AS A FUNCTION OF WALL TEMPERATURE

FIG. 9.

TABLE 1

Values of $m\left[\frac{(n+m-1)!\,(-\frac{2}{3})!}{(n+m-\frac{2}{3})!}\right]$

n \ m	$\frac{1}{9}$	$\frac{1}{3}$	1	4
0	1.2728	1.7667	3	6.9432
1	0.3182	0.8833	2.2499	6.4089
2	0.2448	0.7066	1.9286	6.0083
3	0.2114	0.6183	1.7358	5.6921
4	0.1910	0.5621	1.6022	5.4334
5	0.1766	0.5220	1.5021	5.2161
10	0.1392	0.4144	1.2168	4.4817

TABLE 2

Skin friction parameter in accelerated flow

$M_1 = M_a (1 + \bar{x}) \quad \sigma = \omega = 1 \quad M_a = 4 \quad \gamma = 1.4$

$\bar{x}$	$2\ t_w$ zero heat transfer	$2\ t_w$ $i_w = i_a$
0	0.664	0.664
.1	0.704	0.600
.2	0.671	0.524
.3	0.609	0.453
.4	0.541	0.389
.5	0.475	0.334
.6	0.415	0.287
.7	0.361	0.247
.8	0.315	0.213
.9	0.275	0.184
1.0	0.240	0.160

TABLE 3

Skin friction parameter in retarded flow

$M_1 = M_a\,(1 - \bar{x}) \quad \sigma = \omega = 1 \quad M_a = 4 \quad \gamma = 1.4$

$\bar{x}$	$2\,t_w$	$\bar{x}$	$2\,t_w$
	zero heat transfer		$i_w = i_a$
0	0.664	0	0.664
.02	0.639	.02	0.674
.04	0.606	.04	0.683
.06	0.562	.06	0.691
.08	0.505	.08	0.695
.1	0.423	.10	0.697
.12	0.334	.12	0.695
.14	0.202	.14	0.690
		.16	0.680
		.18	0.663
		.20	0.640
		.22	0.607
		.24	0.565
		.26	0.508
		.28	0.460
		.30	0.339
		.32	0.209
		.339	0

TABLE 4

Skin friction parameter in retarded flow

$u_1 = u_a(1 - \bar{x}) \quad \sigma = \omega = 1 \quad M_a = 4 \quad \gamma = 1.4$

$2\,t_w$ — zero heat transfer

$\bar{x}$	This paper	Curle (1958)	Exact (N.P.L)
0	0.664	0.670	0.664
0.01	0.609	0.636	0.622
.02	0.533	0.602	0.560
.03	0.428	0.554	0.466
.04	0.288	0.488	0.320
0.042			0.264
0.044			0.168
0.045			0.050
.05	0.078	0.400	
.06		0.260	
.067		0	

$2\,t_w$ — $i_w = i_a$

$\bar{x}$	This paper	Curle (1958)	Exact (N.P.L)
0	0.664	0.670	0.664
.02	0.704		
.04	0.731		
.06	0.743		
.08	0.742		
.10	0.730	1.176	0.946
.12	0.709		
.14	0.672		
.16	0.629		
.18	0.578		
.20	0.521	1.226	0.360
.22	0.459		
.24	0.392		
.26	0.322		
.28	0.251		
.30	0.179	0.360	
.311		0	
.32	0.108		
.34	0.033		
.349	0 (Extrapolated)		

TABLE 5

Skin friction parameter in retarded flow

$p_1 = p_a (1 + \bar{x}) \quad \sigma = \omega = 1 \quad M_a = 2 \quad \gamma = 1.4$

$\bar{x}$	$2\,t_w$ zero heat transfer: This paper ($\gamma = 1.5$)
0	0.664
0.1	0.583
0.2	0.500
0.3	0.415
0.4	0.325
0.5	0.227
0.6	0.114
0.667	0

$\bar{x}$	$2\,t_w$, $i_w = i_a$: This paper ($\gamma = 1.5$)	'Exact' N.P.L.	Curle (1958)
0	0.664	0.664	0.670
0.1	0.617	0.630	0.632
0.2	0.570	0.584	0.604
0.3	0.524	0.538	0.576
0.4	0.476	0.494	0.542
0.5	0.428	0.452	0.510
0.6	0.380	0.400	0.474
0.7	0.329	0.348	0.436
0.8	0.277	0.296	0.396
0.9	0.223	0.232	0.350
1.0	0.163	0.170	0.300
1.1	0.097		0.244
1.2	0		0.158
1.3			0.036
1.32			0