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DISCONTINUITY STRESSES AT THE JUNCTION OF A PRESSURISED SPHERICAL SHELL AND A CYLINDER

by

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A note on the discontinuity stresses at the junction of a pressurised spherical shell and a cylinder

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SUMMARY

An analysis has been made of the forces and moments occurring at the junction of a pressurised spherical shell with an intersecting cylinder. The additional effects of having a temperature gradient along the length of the cylinder and the effect of a jointing ring have been considered.

CONTENTS

1. Introduction

This note is a study of the discontinuity stresses which arise when a pressurised spherical shell is constrained by a hollow cylinder whose axis passes through the centre of the sphere, (Fig. 1). The wall of the sphere is considered to be thin - less than 1/10th of the radius.

When a hollow thin walled sphere is under pressure, its shell is in a state of pure membrame stress which is uniform throughout, and is in complete equilibrium with the applied pressure. There is a uniform expansion of the wall.

The presence of the cylinder will prevent free expansion of the sphere at and near the joint, inducing discontinuity stresses both in the sphere and in the cylinder. Since there is no extra external force applied, the force system in the sphere thus produced must be self balancing, without any resultant force whatever. The same is also true with the cylinder. Therefore, by St. Venant's Principle, the effect of the constraint must be entirely local, and the discontinuity stresses diminish at small distances away from the structural discontinuity.

In the following analysis, five cases have been considered. (See Contents). In all the cases the joint between the cylinder and the sphere is assumed to be rigid. That is, the angle between the tangent to the sphere and the wall of the cylinder remains unchanged after deformation.

It is further assumed that the sphere and the cylinder, and later on the jointing ring, are of the same material thus having the same physical constants. It will only be a very simple matter to extend the analysis to allow for the use of different materials. The physical properties are also assumed to be unaltered through the range of temperature concerned.

2. Case 1.

Cylinder joining on the outside of a sphere

When a sphere of radius R and skin thickness t is subjected to a pressure p, the radius is increased by the amount

$$(\Delta R)_p = \frac{pR^2}{2Et}(1-\mu)$$

where μ is the Poisson's Ratio
and E is the Young's Modulus of Elasticity.

Now, if a cylinder of radius r is joined on to the sphere, the expansion at the joint normal to the axis of the cylinder, had the cylinder offered no resistance, would be equal to (see Fig.2).

$$\delta_p = (\Delta R)_p \sin\alpha,$$

and since $\sin\alpha = r/_R$,

$$\delta_p = \frac{prR}{2Et}(1-\mu) \quad . \tag{1}$$

However, the presence of the cylinder will restrict the movement, and the deformation at the joint will finally take the shape as shown in Fig.2.

If the sphere is considered to be cut along the joint and free from the cylinder, (Fig.3), there will be a distributed moment m, and a distributed load q normal to the axis of the cylinder, around the edge of the cut on each of the three free bodies.
The force system on each free body is therefore self-equilibrating. There can be no load in the direction of the axis of the cylinder (due to the restraint of the joint) because there is no extra externally applied load to balance it. Hence the forces must die away hyperbolically as the distance from the joint is increased, and the effect of the disturbance is therefore purely local.

The relation between the deformation and the forces described above are now examined.

(a) For the cylinder (subscript o).

The deflection δ_o and rotation θ_o at the end of the cylinder due to moment m_o and force q_o (for sign convention see Fig.3) are given in Ref.1 (pp. 392-393) as

$$\delta_o = \frac{1}{2\beta^3 D}(\beta m_o + q_o), \tag{2a}$$

and $$\theta_o = \frac{1}{2\beta^2 D} \quad (2\beta m_o + q_o), \tag{2b}$$

where
$$\left.\begin{aligned} \beta^4 &= \frac{3(1-\mu^2)}{r^2 t_o^2}, \\ \text{and} \quad D &= \frac{E t_o^3}{12(1-\mu^2)}. \end{aligned}\right\} \tag{3}$$

t_o = thickness of the cylinder.

(b) For the portion of spherical shell outside the circumference of the cylinder (subscript 1).

Ref.1 (pp.470-471) gives the deflection and rotation at the edge of the shell as

$$\delta_1 = \frac{2\lambda R \sin^2(\pi-\alpha)}{Et} \quad q_1 - \frac{2\lambda^2 \sin(\pi-\alpha)}{Et} m_1 , \tag{4a}$$

and $$\theta_1 = \frac{4\lambda^3}{ERt} m_1 - \frac{2\lambda^2 \sin(\pi-\alpha)}{Et} q_1 , \tag{4b}$$

where $$\lambda^4 = 3(1-\mu^2) \left(\frac{R}{t}\right)^2 . \tag{5}$$

(c) Similarly for the portion of spherical shell inside the circumference of the cylinder (subscript 2).

$$\delta_2 = \frac{2\lambda R \sin^2\alpha}{Et} q_2 + \frac{2\lambda^2 \sin\alpha}{Et} m_2 , \tag{6a}$$

and $$\theta_2 = \frac{4\lambda^3}{ERt} m_2 + \frac{2\lambda^2 \sin\alpha}{Et} q_2 . \tag{6b}$$

Now, assuming that the joint is rigid, then the rotations in all the three portions must be identical, which gives,

$$\theta_o = \theta_1 = \theta_2 .$$

Hence:

$$\frac{4\lambda^3}{ERt} m_1 - \frac{2\lambda^2 \sin\alpha}{Et} q_1 - \frac{1}{2\beta^2 D} (2\beta m_o + q_o) = 0, \tag{7}$$

and $$\frac{2\lambda^2 \sin\alpha}{Et}(q_1 + q_2) + \frac{4\lambda^3}{ERt}(m_2 - m_1) = 0. \tag{8}$$

For continuity the deflection of the two portions of the sphere must also be the same, that is $\delta_1 = \delta_2$.

Hence

$$\frac{2\lambda R \sin^2\alpha}{Et}(q_2 - q_1) + \frac{2\lambda^2 \sin\alpha}{Et}(m_1 + m_2) = 0 \tag{9}$$

From Fig.2, the deflection of the sphere and that of the cylinder must add up to the deflection δ_p of Eq.1. Alternatively looking at it another way, the total inward deflection of the sphere, which is equal to $\delta_1 - \delta_p$, must be the same as that of the cylinder, which is δ_o.

Hence $\delta_1 - \delta_o = \delta_p$.

and $$\frac{2\lambda \sin^2\alpha}{Et} q_1 - \frac{2\lambda^2 \sin\alpha}{Et} m_1 - \frac{1}{2\beta^3 D}(\beta m_o + q_o) = \frac{prR}{2Et}(1-\mu). \tag{10}$$

And finally, for equilibrium at the joint,

$$q_o + q_1 + q_2 = 0 \tag{11}$$

and $$m_o + m_1 + m_2 = 0. \tag{12}$$

The 6 unknown quantities q_o, q_1, q_2, m_o, m_1, and m_2 can now be found by solving the 6 equations 7 to 12 simultaneously. The solution of the equations are:

$$\left.\begin{array}{lll} m_o = \dfrac{M_o}{N}, & m_1 = \dfrac{M_1}{N}, & m_2 = \dfrac{M_2}{N}, \\ q_o = \dfrac{Q_o}{N}, & q_1 = \dfrac{Q_1}{N}, & q_2 = \dfrac{Q_2}{N}. \end{array}\right\} \tag{13}$$

where

$$\left.\begin{aligned}
N &= a^4(1+\frac{4d}{b}) + 4\beta^2 a^2 bd\ (1+\frac{2d}{b}),\\
M_o &= 8\beta a^2 d.e,\\
M_1 &= a(a^2 + 4\beta^2 bd - 4\beta ad)e = -\frac{M_o}{2} - \frac{a}{2c}\ Q_o\ ,\\
M_2 &= -a(a^2 + 4\beta^2 bd + 4\beta ad)e = -\frac{M_o}{2} + \frac{a}{2c}\ Q_o\ ,\\
Q_o &= -2c(a^2 + 4\beta^2 bd)e,\\
Q_1 &= c(a^2 + 4\beta^2 db - 2\beta ad)e = -\frac{Q_o}{2} - \frac{c}{4a} M_o\ ,\\
Q_2 &= c(a^2 + 4\beta^2 bd + 2\beta ad)e = -\frac{Q_o}{2} + \frac{c}{4a} M_o\ .
\end{aligned}\right\} \quad (13a)$$

and

$$\left.\begin{aligned}
\beta^4 &= \frac{3(1-\mu^2)}{r^2 t_o^{\ 2}}\ ,\\
\lambda^4 &= 3(1-\mu^2)\ (\frac{R}{t})^2\ ,\\
a &= \frac{2\lambda^2 r}{R}\ ,\\
b &= \frac{2\lambda r^2}{R}\ ,\\
c &= \frac{4\lambda^3}{R}\ ,\\
d &= \frac{6(1-\mu^2)t}{\beta^3 t_o^{\ 3}}\ ,\ = 2\beta r^2 \frac{t}{t_o}\\
e &= \frac{prR}{2}(1-\mu)\ ,
\end{aligned}\right\} \quad (14)$$

(Note that $bc = 2a^2$).

3. Case 2.

Cylinder extending into the sphere.

If the cylinder is extended into the sphere (Fig.4), and providing that the length inside is sufficiently long, then it can be treated in the same manner as discussed above. The cylinder can also be considered as cut along the joint into two portions. For the portion of cylinder inside the sphere (subscript 3), the relation between the deflection and rotation at the edge of the cut in terms of the distributed force and moment, with the sign convention the same as shown in Fig.3. can be written as

$$\delta_3 = \frac{1}{2\beta^3 D} \quad (-\beta m_3 + q_3), \tag{15a}$$

and

$$\theta_3 = \frac{1}{2\beta^2 D} \quad (2\beta m_3 - q_3). \tag{15b}$$

For continuity of the cylinder,

$$\delta_o = \delta_3 \text{ and } \theta_o = \theta_3 .$$

Hence from equations 2 and 15,

$$\beta(m_o + m_3) + (q_o - q_3) = 0, \tag{16}$$

$$\text{and } 2\beta(m_o - m_3) + (q_o + q_3) = 0. \tag{17}$$

With the addition of m_3 and q_3 at the joint, the equations of equilibrium 11 and 12 become:

$$q_o + q_1 + q_2 + q_3 = 0 , \tag{18}$$

$$\text{and } m_o + m_1 + m_2 + m_3 = 0 . \tag{19}$$

The arguments from which equations 7,8,9 and 10 are derived still apply to this case. Hence using these four equations, together with equations 16, 17, 18 and 19, the eight unknown forces and moments at the joint can be obtained.

The solution to the set of equations is:

$$\left.\begin{aligned} -q_o = q_1 = q_2 = -q_3 &= \frac{ce}{\Psi} , \\ m_1 = -m_2 &= \frac{ae}{\Psi} , \\ m_o = -m_3 &= \frac{ce}{2\beta\Psi} , \\ \text{where} \quad \Psi &= bc - a^2 + \frac{dc}{2} = a^2 + \frac{dc}{2} . \end{aligned}\right\} \tag{20}$$

The coefficients a, b, c, d, e, and β are the same as given in equation 14.

4. Case 3.

Cylinder joining on the outside of a sphere with a temperature change in the sphere and a temperature gradient in the cylinder.

Consider the geometry to be the same as that of Case 1, and let the initial temperature of the whole system be T_o. The sphere is then heated (or cooled) to a temperature T_1. By conduction, the temperature at the end of the cylinder joining on the sphere is also T_1, whilst at the far end the temperature remains at T_o.

4a. It is now convenient to detach the cylinder from the sphere at the joint (Fig.5) and consider the effect of the temperature on each item.

(a) For the sphere.

Let the change of temperature $T_1 - T_o = \Delta T$.

The increase in radius of the sphere due to this temperature change is

$$(\Delta R)_t = \gamma R. \Delta T ,$$

where γ = coefficient of expansion.

The component of expansion at the joint normal to the axis of the cylinder is

$$\delta_t = (\Delta R)_t \sin\alpha = \gamma r. \Delta T. \qquad (21)$$

(b) For the cylinder

For convenience, the temperature gradient along the length of the cylinder is here assumed to be linear. In fact it may be taken to be any other function without affecting the following reasoning.

The change of temperature at any point x from the far end of the cylinder is then given by

$$\Delta T(x) = (T_1 - T_o)\frac{x}{\ell} = \Delta T.\frac{x}{\ell} .$$

Using the same argument as given in Ref.1 (pp.423-425), we can now assess the deformation of the isolated cylinder due to this change of temperature.

Consider a small ring element dx separated from the rest of the cylinder. The expansion due to the change of temperature $\Delta T(x)$ is equal to $\gamma r . \Delta T(x)$. It produces no stress in the ring.

Imagine now an external pressure p' which is applied on the ring element to restore it to its original diameter. The contraction due to p' must be equal to the expansion due to $\Delta T(x)$.

Hence $$\frac{p' r^2}{Et} = \gamma r . \Delta T(x) ,$$

from which $$p' = \frac{E\gamma t}{r} \Delta T(x).$$

The pressure p' produces a hoop stress in the ring.

In fact, there is no actual applied pressure on the cylinder. Therefore a pressure -p' must be applied to the now un-distorted cylinder. This latter pressure produces longitudinal bending stress along the cylinder axis as well as hoop stress. The hoop stresses of the two pressure systems of course cancel each other, leaving the longitudinal bending stress produced by pressure -p' on the straight cylinder as the sole equivalent of the temperature gradient effect. This stress should be included in the final stress system in the cylinder.

The stresses produced by -p' can be obtained if the deflection curve along the cylinder is known, and the skin deflection w (positive inwards towards the axis) is given by the differential equation (Ref. 1, p.392, eq.230. See also p.424, eq.h.)

$$\frac{d^4 w}{dx^4} + 4\beta^4 w = -\frac{p'}{D} .$$

(β and D are the same as given in eq. 3).

We are here interested not so much in the bending stress as the deflection and rotation at the joining end (that is when $x = \ell$) of the detached cylinder due to the temperature gradient. These can be obtained easily by solving the above equation with the appropriate end conditions and substituting into the solution the end value.

After substituting for $-p'$, the equation above becomes

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{E\gamma t_o}{Dr} \quad \Delta T(x).$$

The general solution of this equation, assuming $T(x)$ to be linear is:

$$w = c_1 \cosh\beta x \cos\beta x + c_2 \cosh\beta x \sin\beta x + c_3 \sinh\beta x \cos\beta x + c_4 \sinh\beta x \sin\beta x - c'x, \qquad (22)$$

where $c' = \frac{E\gamma t_o}{4\beta^4 Dr} \cdot \frac{\Delta T}{\ell} = \frac{\gamma r}{\ell} \cdot \Delta T$.

The unknown constants $c_1 \ldots c_4$ will depend on the condition at the far end of the cylinder ($x=o$) where the temperature is kept constant at T_o.

(i) If the cylinder is free at both ends

then $\frac{d^2w}{dx^2} = 0$ and $\frac{d^3w}{dx^3} = 0$

when $x = 0$ or $x = \ell$.

These conditions give

$$c_1 = c_2 = c_3 = c_4 = 0,$$

and eq. 22 becomes simply

$$w = -\gamma r . \Delta T . \frac{x}{\ell} \quad . \qquad (22a)$$

This infers a linear deflection with a rate proportional to ΔT.

The deflection and rotation at the joint ($x = \ell$) are given by:

$$w_\ell = -\gamma r . \Delta T, \qquad (23a)$$

and $$\theta_\ell = \left(\frac{dw}{dx}\right)_{x=\ell} = -\gamma . \Delta T . \frac{r}{\ell} \quad . \qquad (23b)$$

Note that for this case. $-w_\ell = \delta_t$ (see eq. 21), i.e. both the cylinder and sphere are displaced by the same amount in the direction normal to the axis of the cylinder. It can also be seen that there is no longitudinal bending stress anywhere on the cylinder due to this temperature gradient.

(ii) If the far end is fixed, then the boundary conditions are

$$\text{at } x = 0: \quad w = 0 \quad \text{and} \quad \frac{dw}{dx} = 0,$$

$$\text{and} \quad \text{at } x = \ell: \quad \frac{d^2w}{dx^2} = 0 \quad \text{and} \quad \frac{d^3w}{dx^3} = 0.$$

These conditions give the values of the constants as follows:

$$c_1 = 0 \,,$$

$$c_2 = c\,\frac{\cosh^2\beta\ell}{\cos^2\beta\ell + \cosh^2\beta\ell}\,,$$

$$c_3 = c\,\frac{\cos^2\beta\ell}{\cos^2\beta\ell + \cosh^2\beta\ell}\,,$$

$$\text{and} \quad c_4 = c\,\frac{\sin\beta\ell\cos\beta\ell - \sinh\beta\ell\cosh\beta\ell}{\cos^2\beta\ell + \cosh^2\beta\ell}\,,$$

$$\text{where} \quad c = \frac{c'}{\beta} = \frac{\nu r}{\beta\ell}\cdot\Delta T\,. \tag{24}$$

Putting these constants into eq.22, the deflection and rotation at the joint $x = \ell$ are given by:

$$w_\ell = -c\left(\beta\ell - \frac{\cosh\beta\ell\sin\beta\ell + \sinh\beta\ell\cos\beta\ell}{\cos^2\beta\ell + \cosh^2\beta\ell}\right), \tag{25a}$$

$$\theta_\ell = \left(\frac{dw}{dx}\right)_{x=\ell} = -c'\,\frac{(\cosh\beta\ell - \cos\beta\ell)^2}{\cos^2\beta\ell + \cosh^2\beta\ell}\,, \tag{25b}$$

4B. The above calculation for the deflections and rotations of the sphere and cylinder due to temperature variation assumes that the joint is detached and that no moment or force exist at the joint. The fact that the cylinder and sphere are actually attached will induce forces and moments on the edges of the cylinder and the two portions of sphere at the joint. The magnitudes of these forces are such that the deformations of all the parts at the joint are consistent. The treatment of the problem is then exactly the same as that of Case 1.

The relations given in eqs. 2, 4 and 6 still hold, and if the joint is assumed rigid, the deflection and rotation of the two portions of sphere cut at the joint must be the same. Hence eqs. 8 and 9, which give $\theta_1 = \theta_2$ and $\delta_1 = \delta_2$ must still apply.

If a pressure p is also applied in the sphere the sum of the deflection of the sphere in the direction normal to the axis of the cylinder is $\delta_p + \delta_t$ as given in eqs. 1 and 21. Adding to this the deflection of the cylinder (see Fig.5), the total 'gap' developed over the detached joint is $(\delta_p + \delta_t + w_\ell)$.

Therefore, the right hand side of eq. 10 instead of being δ_p, now becomes $(\delta_p + \delta_t + w_\ell)$ and the equation reads

$$\delta_1 - \delta_o = \delta_p + \delta_t + w_\ell \, .$$

This means that the total inward deflection of the sphere $\delta_1 - \delta_p - \delta_t$ must be the same as that of the cylinder $\delta_o + w_\ell$,

$$\text{Hence } \frac{2\lambda R \sin^2\alpha}{Et} q_1 - \frac{2\lambda^2 \sin\alpha}{Et} m_1 - \frac{1}{2\beta^3 D}(\beta m_o + q_o)$$

$$= \frac{prR}{2Et}(1-\mu) + \chi r . \Delta T + w_\ell \, . \qquad (26)$$

Similarly, since we assume that the joint is rigid, the rotation of the sphere θ_1 must be equal to the total rotation of the cylinder $\theta_1 + \theta_\ell$ (see also Fig.6).

$$\text{Hence } \theta_1 - \theta_o = \theta_\ell \quad ,$$

$$\text{or } \frac{4\lambda^3}{ERt} m_1 - \frac{2\lambda^2 \sin(\pi-\alpha)}{Et} q_1 - \frac{1}{2\beta^2 D}(2\beta m_o + q_o) = \theta_\ell \, . \qquad (27)$$

w_ℓ and θ_ℓ in eqs. 26 and 27 are given from either eq. 23 or eq.25, according to the condition at the far end of the cylinder.

The conditions of equilibrium at the joint as given by eq.11 and 12 are still true. Therefore, we have now a set of six equations:

$$\begin{cases} c(m_2 - m_1) + a(q_1 + q_2) = 0, & (8) \\ a(m_1 + m_2) + b(q_2 - q_1) = 0, & (9) \\ q_o + q_1 + q_2 = 0, & (11) \\ m_o + m_1 + m_2 = 0, & (12) \\ bq_1 - am_1 - d(\beta m_o + q_o) = e + e' & (26) \\ cm_1 - aq_1 - \beta d(2\beta m_o + q_o) = f. & (27) \end{cases}$$

where a, b, c, d, and β are the same as given in eq.14,

and

$$\left.\begin{aligned} e' &= +(\gamma r.\Delta T + w_\ell)Et, \\ f &= \theta_\ell . Et. \end{aligned}\right\} \qquad (28)$$

It can be seen that the first four equations are identical with those used in Case 1. Eq. 26 comes from eq.10, but with e replaced by e + e' ; and eq.27 differs from eq.7 only in the right hand side where it is now equal to f instead of zero. When there is no thermal effect, e' = 0 and f = 0, this set of equations is then reduced identically to that of Case 1.

The solution of the equations is:

$$\left.\begin{aligned} m_o &= \frac{M_o+M_o'}{N}, \quad m_1 = \frac{M_1+M_1'}{N}, \quad m_2 = \frac{M_2+M_2'}{N}, \\ q_o &= \frac{Q_o+Q_o'}{N}, \quad q_1 = \frac{Q_1+Q_1'}{N}, \quad q_2 = \frac{Q_2+Q_2'}{N} \quad . \end{aligned}\right\} \qquad (29)$$

where N, M_o, M_1, M_2, Q_o, Q_1, Q_2, are given in eq.13a with the important difference that e is replaced by e + e' in the expressions. The other quantities M_o' ... Q_2' are:

$$\left.\begin{aligned}
M_o' &= -2b(a^2 + 2cd)f, \\
M_1' &= b(a^2 + 2cd - 2\beta ad)f = -\frac{M_o'}{2} - \frac{b}{4a}\,Q_o', \\
M_2' &= b(a^2 + 2cd + 2\beta ad)f = -\frac{M_o'}{2} + \frac{b}{4a}\,Q_o', \\
Q_o' &= 8\beta a^2 df, \\
Q_1' &= \left[-4\beta a^2 d + a(a^2 + 2cd)\right] f = -\frac{Q_o'}{2} - \frac{a}{2b}\,M_o', \\
Q_2' &= \left[-4\beta a^2 d - a(a^2 + 2cd)\right] f = -\frac{Q_o'}{2} + \frac{a}{2b}\,M_o'.
\end{aligned}\right\} \quad (29a)$$

5. Case 4.

Cylinder extending into the sphere, with a temperature change in both the sphere and the portion of cylinder inside it, and a temperature gradient in the portion of cylinder outside.

The geometry of this case is the same as that of Case 2, and the same thermal changes as that of Case 3. The change of temperature from T_o to T_1, applied also to the portion of cylinder inside the sphere. The arguments that lead to equations 8, 9, 18 and 19 for Case 2 are still valid. By a similar consideration as that employed in Case 3 from which yields eqs. 26 and 27 (which still hold true), we obtain the relations between the deflection and rotation of the two portions of cylinders at the joint.

$$\delta_3 - \delta_o = \delta_t + w_\ell . \quad (30)$$

$$\theta_3 - \theta_o = \theta_\ell . \quad (31)$$

The set of equations for this case is therefore

$$\left\{\begin{aligned}
& c(m_2 - m_1) + a(q_1 + q_2) = 0 && (8) \\
& a(m_1 + m_2) + b(q_2 - q_1) = 0 && (9) \\
& q_o + q_1 + q_2 + q_3 = 0 && (18) \\
& m_o + m_1 + m_2 + m_3 = 0 && (19) \\
& bq_1 - am_1 - d(\beta m_o + q_o) = e + e' && (26)
\end{aligned}\right.$$

$$\begin{cases} cm_1 - aq_1 - \beta d(2\beta m_o + q_o) = f & (27) \\ \beta(m_o + m_3) + (q_o - q_3) = -{}^{e'}/d & (30a) \\ 2\beta(m_o - m_3) + (q_o + q_3) = -{}^{f}/\beta d & (31a) \end{cases}$$

where e' and f are given in eq.28 and the other coefficients in eq.14.

The left hand side of the equations is of course identical with that of Case 2, but e' and f are newly introduced into the right hand side as a result of the temperature change.

The solutions of the equations are:

$$\left.\begin{aligned} m_o &= \frac{cu}{4\beta\Psi} + \frac{bv}{2\phi} - \frac{f}{4\beta^2 d}, \\ m_1 &= \frac{au}{2\Psi} - \frac{bv}{2\phi}, \\ m_2 &= -\frac{au}{2\Psi} - \frac{bv}{2\phi}, \\ m_3 &= -\frac{cu}{4\beta\Psi} + \frac{bv}{2\phi} + \frac{f}{4\beta^2 d}, \\ q_o &= -\frac{cu}{2\Psi} - \frac{\beta bv}{2\phi} - \frac{e'}{2d}, \\ q_1 &= \frac{cu}{2\Psi} - \frac{av}{2\phi}, \\ q_2 &= \frac{cu}{2\Psi} + \frac{av}{2\phi}, \\ q_3 &= -\frac{cu}{2\Psi} + \frac{\beta bv}{2\phi} + \frac{e'}{2d}, \end{aligned}\right\} \qquad (32)$$

$$\left.\begin{aligned} \text{where } u &= 2e + e' - {}^{f}/2\beta \\ v &= \beta e' - f \\ \Psi &= a^2 + \frac{dc}{2} \\ \phi &= bc - a^2 + \beta^2 bd = a^2 + \beta^2 bd. \end{aligned}\right\} \qquad (32a)$$

6. Case 5.

Effect of a heavy ring at the joint

The presence of a ring can be dealt with in exactly the same manner as the preceding cases.

Isolating the ring from the rest of the structure and letting the force and moment on it be q_4 and m_4, inducing radial deflection δ_4 and rotation θ_4 (see Fig. 7).

The relation between δ_4 and q_4 is easily seen to be

$$\delta_4 = \frac{r^2}{EA} q_4 \quad , \tag{33}$$

where A = area of ring cross section, which is assumed constant.

Assuming that q_4 is applied on the centroid of the ring and therefore inducing no rotation, the relation between θ_4 and m_4 is

$$\theta_4 = \frac{r^2}{E\left(I_x - \frac{I_{xy}^2}{I_y}\right)} m_4 \quad , \tag{34}$$

where I_x = moment of inertia of the ring cross-section about an axis in the plane of the ring centroid.

I_y = moment of inertia of the ring cross-section about an axis perpendicular to the plane of the ring centroid.

I_{xy} = product of inertia referring to these two axes.

It is now simply a matter of equating these displacements to the rest of the structure at the joint, two extra equations, most conveniently

$$\left.\begin{aligned} \delta_4 &= \delta_o, \\ \text{and} \quad \theta_4 &= \theta_o \end{aligned}\right\} \text{(when there is no thermal effect).}$$

are derived. Also, extra terms q_4 and m_4 must be added to the equations of equilibrium of force and moment. Thus, two new variables q_4 and m_4 are added to the original equations, but the two extra equations will enable the new set of equations to be solved.

If the temperature of the ring is also changed from T_o to T_1, then the radius of the ring, when isolated, is changed by an amount $\gamma r. \Delta T$. i.e., there is no relative movement between the ring and the wall of the sphere due to the temperature effect. The deflection and rotation can more conveniently be equated with those of the sphere to give

$$\delta_4 = \delta_1 - \delta_p ,$$

$$\text{and} \quad \theta_4 = \theta_1 ,$$

which together with the original equations will enable all the unknowns to be found.

REFERENCE

1. Timoshenko — Theory of Plates and Shells. McGraw Hill Book Co. 1940.

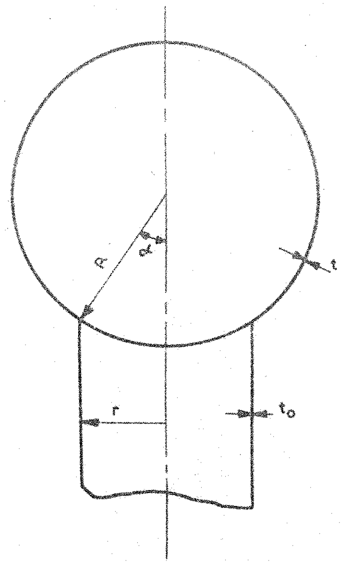

FIG. 1 CYLINDER JOINING ON TO A SPHERE

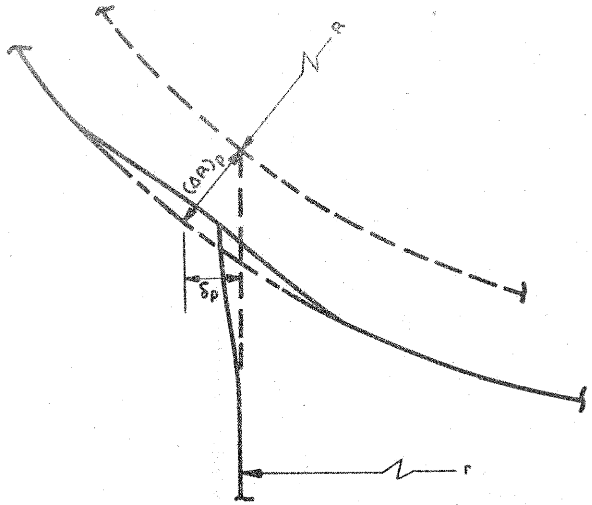

FIG. 2 DEFORMATION AT JOINT WHEN THE SPHERE IS UNDER PRESSURE.

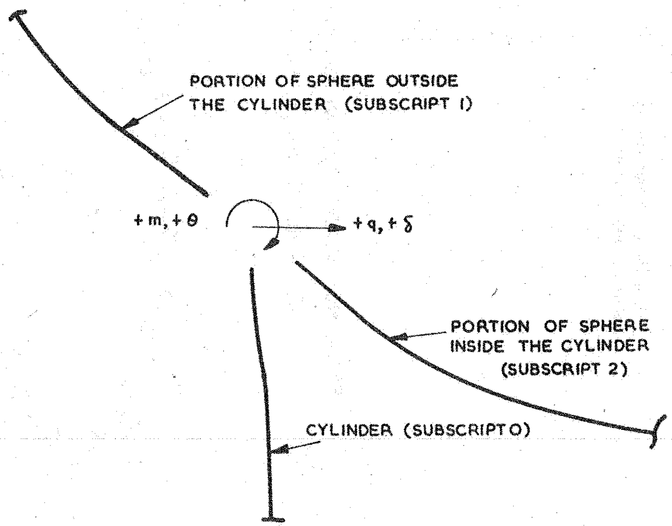

FIG. 3 SIGN CONVENTION OF FORCES AND DISPLACEMENTS AT THE JOINT.

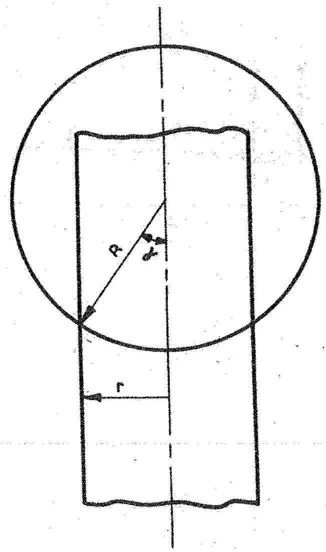

FIG. 4 CYLINDER EXTENDING INTO THE SPHERE.

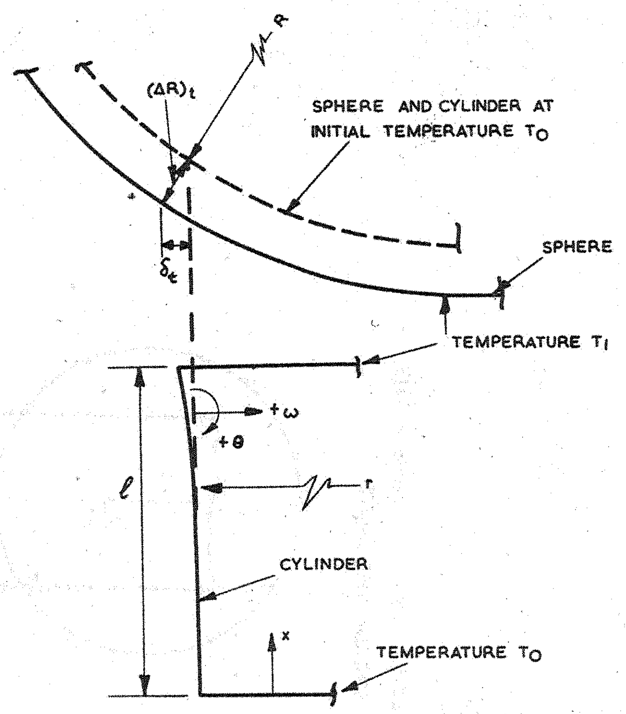

FIG. 5 EFFECT OF TEMPERATURE ON DETACHED SPHERE AND CYLINDER.

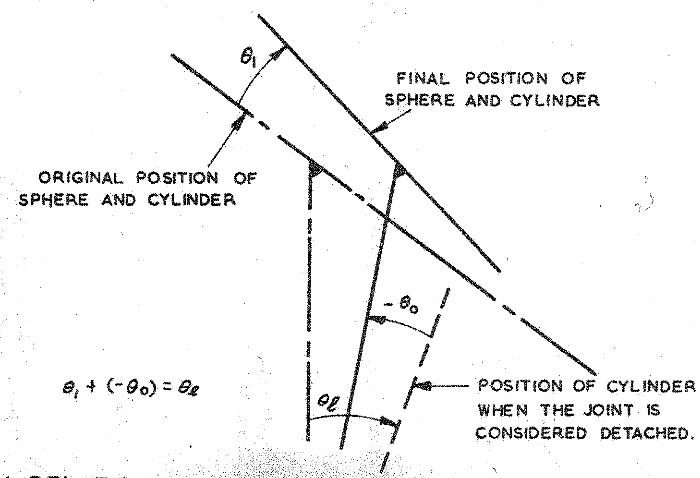

FIG. 6 RELATION BETWEEN ROTATIONS OF THE WALLS OF SPHERE AND CYLINDER.

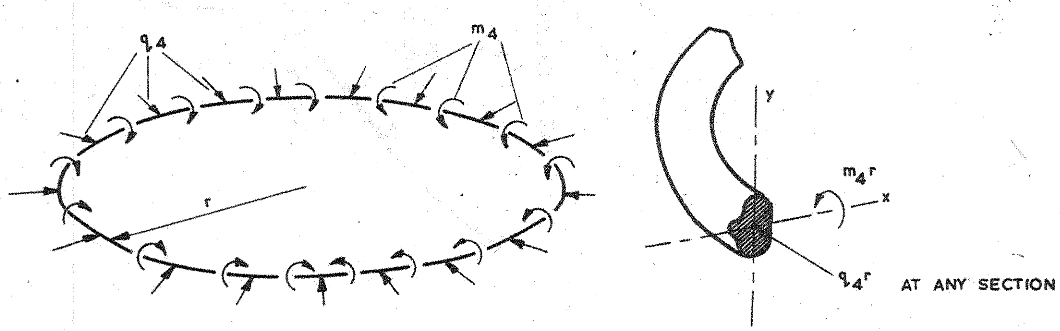

FIG. 7 FORCE AND MOMENT ON A RING AT THE JOINT.