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Plastic Buckling of a Plate in Shear

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This note derives the mathematical equations for the analysis of the shear buckling of a plate, in the case where the initial stresses exceed the elastic limit of the material. It is hoped at a later stage to apply this theory to test results, which are being obtained using rectangular torsion boxes.

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PLASTIC BUCKLING OF A PLATE IN SHEAR

Statement of the Problem

Consider a flat plate referred to Rectangular Cartesian Coordinates $O(x_p)$* in such a way that it occupies the region $(0 \leq x_1 \leq a \ , \ 0 \leq x_2 \leq b \ , \ -h \leq x_3 \leq h)$. $O(x_\alpha)$ are axes in the middle surface and Ox_3 is normal to the plate. The dimensions of the plate are 'a' and 'b' in plan and the thickness is '2h'. Let the plate be loaded in pure shear by stress resultants S applied to its edges in such a way that initially the stress components f_{pq} are given by,

$$f_{pq} = \begin{pmatrix} 0 & , & S/2h & , & 0 \\ S/2h & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 \end{pmatrix} \qquad \ldots (1)$$

It is further assumed that the shear stress S/2h is in excess of the elastic limit for the plate material in shear. The problem to be solved is the calculation of the critical value of S for buckling of the plate by lateral deflection away from its initial plane form.

/ Kinematics ....

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* Indicial notation is used in this Report. Latin indices take the values 1,2 and 3; Greek indices the values 1 and 2. Twice repeated indices are taken to imply summation over the appropriate range.

Kinematics

To test the stability of our initial state we impose a small displacement V_p at the middle surface. This gives rise, according to the Theory of Plates, to strain components $e_{\alpha\beta}$ given by

$$e_{\alpha\beta} = \gamma_{\alpha\beta} - x_3\kappa_{\alpha\beta} \qquad \ldots (2)$$

where $\gamma_{\alpha\beta}$ are the middle surface strains and $\kappa_{\alpha\beta}$ are given by,

$$\kappa_{\alpha\beta} = \frac{\partial^2 V_3}{\partial x_\alpha \partial x_\beta} \qquad \ldots (3)$$

Stress-Strain Relations

Equation (1) shows that in the initial state the 'stress deviator' f'_{pq} given by,

$$f'_{pq} = f_{pq} - \tfrac{1}{3} f_{rr}\delta_{pq} \qquad \ldots (4)$$

where δ_{pq} is Kronecker's Delta, is equal to f_{pq} and so is given by (1).

Following the stress-strain law of Reuss (Ref.1) we then see that the only plastic strain increment for a small change in stress must be confined to the strain component of type e_{12} .
The relations between direct stress and strain increments must be purely 'elastic' and so for a state of 'loading' ($e_{12} \geqslant 0$) we may calculate the stresses $f_{\alpha\beta}$ from the strains of (2) by the equations

$$\left.\begin{aligned} f_{11} = \frac{E}{(1-\sigma^2)}(e_{11} + \sigma e_{22}) \;, \quad & f_{22} = \frac{E}{(1-\sigma^2)}(e_{22} + \sigma e_{11}) \\ f_{12} = S/2h + \mu_T e_{12} \;, & \quad (e_{12} \geqslant 0) \end{aligned}\right\} (5)$$

where E is Young's Modulus, σ = Poisson's Ratio and μ_T is the 'tangent shear modulus' corresponding to the shear stress S/2h.

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Stress Resultants and Couples

Substituting from (2) in (5) and integrating through the plate thickness we find the following formulae for the membrane stress resultants $S_{\alpha\beta}$,

$$\left.\begin{array}{l} S_{11} = \frac{2Eh}{(1-\sigma^2)} (\gamma_{11} + \sigma\gamma_{22}) \quad , \quad S_{22} = \frac{2Eh}{(1-\sigma^2)} (\gamma_{22} + \sigma\gamma_{11}) \\ S_{12} = S_{21} = S + 4h\mu_T\gamma_{12} \end{array}\right\} \quad (6)$$

Similarly, multiplying (5) by x_3 and integrating through the thickness, we find for the stress couples $M_{\alpha\beta}$,

$$\left.\begin{array}{l} M_{11} = -D(\kappa_{11} + \sigma\kappa_{22}) \quad , \quad M_{22} = -D(\kappa_{22} + \sigma\kappa_{11}) \\ M_{12} = M_{21} = -\frac{4}{3} h^3\mu_T\kappa_{12} \\ \text{where, } D = \frac{2}{3} \frac{Eh^3}{(1-\sigma^2)} \end{array}\right\} \quad \ldots (7)$$

Restriction on the Stress Resultants

Equations (6) and (7) are derived from (5) and so are only valid for points where $e_{12} \geqslant 0$. We follow Shanley in assuming that our test displacement is such that this restriction is valid everywhere. By equation (2) this means that

$$\gamma_{12} \geqslant h \left| \kappa_{12} \right| \quad \ldots (8)$$

and by the last of (6), condition (8) is ensured if, at buckling, we chose V_α in such a way that S increases to $S + \Delta S$ where,

$$\Delta S = 4h^2\mu_T \left| \kappa_{12} \right|_{max} \quad \ldots (9)$$

/ Conditions of ....

Conditions of Equilibrium

Equilibrium in the buckled state is maintained if,

$$\frac{\partial^2 M_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} + \kappa_{\alpha\beta} S_{\alpha\beta} = 0 \qquad \ldots (10)$$

Substituting from (6), (7) and (3) and retaining only those terms which are of first order in the test deformation defined by V_p, we find,

$$D\left[\frac{\partial^4 V_3}{\partial x_1^4} + \frac{\partial^4 V_3}{\partial x_2^4} + 2\left\{\frac{2\mu_T(1-\sigma^2)}{E} + \sigma\right\}\frac{\partial^4 V_3}{\partial x_1^2 \partial x_2^2}\right] = 2S\,\frac{\partial^2 V_3}{\partial x_1 \partial x_2} \qquad (11)$$

This is the governing equation for plastic shear buckling.

Variational Principle

Mathematical difficulties associated with the solution of (11), suggest that it would be of value to set up a variational equation, so that methods of approximation can be used to obtain numerical answers. Confining ourselves to the boundary conditions,

$$\left.\begin{array}{lll} V_3 = 0 & \text{on all edges} & \\ \dfrac{\partial V_3}{\partial x_1} = 0 \quad \text{or} \quad M_{11} = 0 & \text{on } x_1 = 0,\ a & \\ \dfrac{\partial V_3}{\partial x_2} = 0 \quad \text{or} \quad M_{22} = 0 & \text{on } x_2 = 0,\ b & \end{array}\right\} \qquad \ldots (12)$$

/ we multiply .....

we multiply (11) by a virtual displacement δV_3 and integrate over the whole plate. Integrating by parts and using (12) to eliminate the integrated terms we find,

$$\delta S = 0 \qquad \ldots (13)$$

when,

$$S = -\frac{1}{2}D \frac{\int_0^a\int_0^b \left\{ \left(\frac{\partial^2 V_3}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 V_3}{\partial x_2^2}\right)^2 + 2\sigma \frac{\partial^2 V_3}{\partial x_1^2}\frac{\partial^2 V_3}{\partial x_2^2} + \frac{4\mu_T(1-\sigma^2)}{E}\left(\frac{\partial^2 V_3}{\partial x_1 \partial x_2}\right)^2 \right\} dx_1 dx_2}{\int_0^a\int_0^b \left(\frac{\partial V_3}{\partial x_1}\frac{\partial V_3}{\partial x_2}\right) dx_1\, dx_2}$$

$$\ldots (14)$$

Equations (13), (14) correspond to the usual equations for the 'Energy Method' of buckling analysis in the elastic case.

Reference

1. Hill Mathematical Theory of Plasticity Oxford University Press