Numerical solution of a stationary 3-dimensional Cauchy problem by the alternating method and boundary integral equations

Ihor Borachok[*], Roman Chapko [†], B. Tomas Johansson [‡]

April 2, 2015

Abstract

We consider the Cauchy problem for the Laplace equation. Substance of them it's reconstruction of harmonic function from knowledge of the value of the function and it's normal derivative given on an external boundary of the solution domain. The solution domain it's a double connected domain in $\mathbb{R}^3$. This problem we will solve the alternating method is an iterative method, and in each iteration we solve two mixed problem. The solution of the mixed problem is represented as a sum of two single-layer potentials defined on each of the two boundary curves and in which both densities are unknown. Integral equation will be solved by Galerkin project method.

1 Introduction

The alternating iterative method was introduced in 1989 by Kozlov and Maz'ya [29] to solve some inverse ill-posed problems notably the Cauchy problem for self-adjoint strongly elliptic operators. Since then, there has been many works on the numerical implementation of their method for such Cauchy problems both with boundary element methods and boundary integral techniques; for references to some of these publications see the introduction in [3] (for references to other methods for Cauchy problems both direct and iterative see the introduction to [13], where, moreover references to applications of the Cauchy problem in cardiology, corrosion detection, electrostatics, geophysics, leak identification, non-destructive testing and plasma physics are given). However, numerical results for the alternating method have largely been obtained for 2-dimensional planar regions. Recently, see [12, 13], integral equation techniques, based on [39], have been developed for some direct and inverse problems in 3-dimensions. We shall build on these

[*]Faculty of Applied Mathematics and Informatics, Ivan Franko National University of Lviv, 79000

[†]Faculty of Applied Mathematics and Informatics, Ivan Franko National University of Lviv, 79000 Lviv, Ukraine

[‡]ITN, Campus Norrköping, Linköping University, Sweden

results and undertake the laborious task of implementing the alternating method for 3-dimensional domains.

Let us formulate the problem to be studied. Let $D_1 \subset \mathbb{R}^3$ be a simply connected smooth bounded domain with boundary surface Γ_1 and let D_2 be a simply connected bounded domain with smooth boundary surface Γ_2, such that $\overline{D}_1 \subset D_2$. We define $D = D_2 \backslash \overline{D}_1$ and let $\nu = (\nu_1, \nu_2, \nu_3)$ be the outward unit normal to the boundary of D, $\partial D = \Gamma_1 \cup \Gamma_2$; an example of the configuration is given in Fig. 1 (only a part of Γ_2 is shown to see the interior surface Γ_1).

Figure 1: A solution domain D with boundary part Γ_1 contained within the outer boundary surface Γ_2

We consider the Cauchy problem of finding a function $u \in C^2(D) \bigcap C^1(\overline{D})$ such that

$$\Delta u = 0 \quad \text{in} \quad D \tag{1.1}$$

with the boundary conditions

$$u = f \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \Gamma_2. \tag{1.2}$$

This problem is ill-posed and we assume that data is given such that there exists a solution. The alternating iterative method is a regularizing procedure for the stable determination of this solution. In each iteration step, mixed boundary value problems are solved. In this method, mixed boundary value problems are solved at each iteration step. To solve these mixed problems, we employ Weinert's method [39]. This method has been applied in some recent works, see [12, 11, 26, 27, 28]. Following [2], where the alternating method was implemented in 2-dimensions, we represent the solution to each mixed problem as a suitable boundary-layer operator leading, via matching of the given boundary data, to a system of boundary integral equations. The discretisation in the method in [39] involves rewriting these boundary integral equations over the unit sphere under the assumption that the surface of the inclusion can be mapped one-to-one to the unit sphere. The densities to be solved for in the system of integral equations are represented in terms of linear combinations of spherical harmonics, and this generates a linear system to solve for the coefficients in this representation.

An advantage with the proposed implementation is that only data on the boundary need to be discretised and not the full 3-dimensional region. An alternative with the similar advantage is to use the boundary element method, however, then the boundary surfaces need to be discretised, a non-trivial task in itself.

A limitation of our approach is the assumption that the given boundary surfaces can each be mapped onto the unit sphere. However, there is a sufficiently large class of domains relevant for applications that can be mapped in this way to the unit sphere. Moreover, for more general boundary surfaces, one can approximate these with surfaces of the requested kind, or even only construct the map numerically.

For the outline of this work, in Chapter 2 we review some results on the alternating method. In Chapter 3, we give the boundary integral solution of the mixed problems, and in Chapter 4 it is discussed how to discretised the obtained boundary integral equations. Some numerical results are given in Chapter 5.

2 Alternating method

We consider two mixed boundary value problems

$$\Delta u = 0 \quad \text{in} \quad D, \tag{2.3}$$

$$\frac{\partial u}{\partial \nu} = h \text{ on } \Gamma_1, \;\; u = f \text{ on } \Gamma_2 \tag{2.4}$$

and

$$\Delta u = 0 \quad \text{in} \quad D, \tag{2.5}$$

$$u = w \text{ on } \Gamma_1, \; \frac{\partial u}{\partial \nu} = g \text{ on } \Gamma_2, \tag{2.6}$$

The alternating iterative procedure for constructing the solution to (1.1), (1.2) runs as follows:

- The first approximation to the solution u of (1.1), (1.2) is obtained by solving (2.3), (2.4) with $h = h_0$, where h_0 is an arbitrary initial guess.
- Having constructed u_{2k}, we find u_{2k+1} by solving problem (2.5), (2.6) with $w = u_{2k}|_{\Gamma_1}$.
- Then we find the element u_{2k+1} by solving problem (2.3), (2.4) with $h = \left.\dfrac{\partial u_{2k+1}}{\partial \nu}\right|_{\Gamma_1}$.

3 An integral equation method for the mixed problems

3.1 Reduction to boundary integral equations

Solutions of mixed problems will be represented as a sum of two single-layer potentials:

$$u(x) = \int_{\Gamma_1} \phi_1(y)\Phi(x,y)ds(y) + \int_{\Gamma_2} \phi_2(y)\Phi(x,y)ds(y), \; x \in D, \tag{3.7}$$

with $\Phi(x,y) = \dfrac{1}{4\pi}\dfrac{1}{|x-y|}$ being fundamental solution of the Laplace equation in $\mathbb{R}^3$ and $\phi_i \in C(\Gamma_i)$, $i = 1,2$ being unknown densities. We introduce boundary integral operators

$$\left(S_{ij}\mu\right)(x) = \int_{\Gamma_j} \mu(y)\Phi(x,y)ds(y), \; x \in \Gamma_i$$

and

$$\left(K_{ij}\mu\right)(x) = \int_{\Gamma_j} \mu(y)\frac{\partial \Phi}{\partial \nu(x)}(x,y)ds(y), \; x \in \Gamma_i$$

for $i,j = 1,2$.

Taking into account properties of the single-layer potential we can reduce the mixed boundary value problem (2.3), (2.4) to the following system of integral equations

$$\begin{cases} \left(S_{21}\phi_1\right)(x) + \left(S_{22}\phi_2\right)(x) = f(x), & x \in \Gamma_2 \\ -\dfrac{1}{2}\phi_1(x) + \left(K_{11}\phi_1\right)(x) + \left(K_{12}\phi_2\right)(x) = h(x), & x \in \Gamma_1 \end{cases}, \tag{3.8}$$

and for the mixed boundary value problem (2.5), (2.6)

$$\begin{cases} \dfrac{1}{2}\phi_2(x) + \left(K_{21}\phi_1\right)(x) + \left(K_{22}\phi_2\right)(x) = g(x), & x \in \Gamma_2 \\ \left(S_{11}\phi_1\right)(x) + \left(S_{12}\phi_2\right)(x) = w(x), & x \in \Gamma_1 \end{cases}. \tag{3.9}$$

3.2 Rewriting the integral equations over the unit sphere

Assume that boundary surfaces Γ_1 and Γ_2 can be bijectively mapped onto the unit sphere $\mathbb{S}^2 = \{\widehat{x} \in \mathbb{R}^3 : \|\widehat{x}\| = 1\}$, i.e. there exist one-to-one mappings $q_\ell = (q_{1\ell}, q_{2\ell}, q_{3\ell}) : \mathbb{S}^2 \to \Gamma_\ell$, $\ell = 1,2$ having a smooth Jacobian J_{q_ℓ}, $\ell = 1,2$. We can rewrite the system of integral equations from the previous section over the unit sphere.

The system (3.9) can be transformed as follows

$$\begin{cases} \dfrac{1}{2}\psi_2(\widehat{x}) + \left(\widetilde{K}_{21}\psi_1\right)(\widehat{x}) + \left(\widetilde{K}_{22}\psi_2\right)(\widehat{x}) = \widetilde{g}(\widehat{x}), & \widehat{x} \in \mathbb{S}^2 \\ \left(\widetilde{S}_{11}\psi_1\right)(\widehat{x}) + \left(\widetilde{S}_{12}\psi_2\right)(\widehat{x}) = \widetilde{w}(\widehat{x}), & \widehat{x} \in \mathbb{S}^2 \end{cases}, \tag{3.11}$$

We introduced here the following functions $\psi_\ell(\widehat{x}) = \phi_\ell(q_\ell(\widehat{x}))$, $\ell = 1, 2$, $\widetilde{f}(\widehat{x}) = f(q_2(\widehat{x}))$, $\widetilde{g}(\widehat{x}) = g_(q_2(\widehat{x}))$, $\widetilde{h}(\widehat{x}) = h(q_1(\widehat{x}))$, $\widetilde{w}(\widehat{x}) = w(q_1(\widehat{x}))$ for $\widehat{x} \in \mathbb{S}^2$. Parametrized integral operators have the form

$$\left(\widetilde{S}_{ij}\mu\right)(\widehat{x}) = \int_{\mathbb{S}^2} \mu(\widehat{y}) Lij(\widehat{x}, \widehat{y}) ds(y), \quad , \widehat{x} \in \mathbb{S}^2 \tag{3.12}$$

and

$$\left(\widetilde{K}_{ij}\mu\right)(\widehat{x}) = \int_{\mathbb{S}^2} \mu(\widehat{y}) Mij(\widehat{x}, \widehat{y}) ds(y), \quad , \widehat{x} \in \mathbb{S}^2, \tag{3.13}$$

$i, j = 1, 2$ with kernels

$$L_{ij}(\widehat{x}, \widehat{y}) = J_{q_j}(\widehat{y}) \begin{cases} \Phi(q_i(\widehat{x}), q_j(\widehat{y})), & i \neq j \\ \dfrac{R_i(\widehat{x}, \widehat{y})}{|\widehat{x} - \widehat{y}|}, & i = j \end{cases}$$

and

$$M_{ij}(\widehat{x}, \widehat{y}) = J_{q_j}(\widehat{y}) \begin{cases} -\dfrac{\langle q_i(\widehat{x}) - q_j(\widehat{y}), \nu(q_i(\widehat{x}))\rangle}{4\pi\, |q_i(\widehat{x}) - q_j(\widehat{y})|^3}, & i \neq j \\ \dfrac{\widetilde{R}_i(\widehat{x}, \widehat{y})}{|\widehat{x} - \widehat{y}|}, & i = j \end{cases}$$

where

$$R_i(\widehat{x}, \widehat{y}) = \begin{cases} \dfrac{1}{4\pi} \dfrac{|\widehat{x} - \widehat{y}|}{|q_i(\widehat{x}) - q_i(\widehat{y})|}, & \widehat{x} \neq \widehat{y} \\ \dfrac{1}{4\pi} \dfrac{1}{J_{q_i}(\widehat{y})}, & \widehat{x} = \widehat{y} \end{cases}$$

4 Numerical solution of integral equations

We shall describe how to discretise the equations (3.10), (3.11).

4.1 Quadrature rules

The following quadrature is used for integrals with continuous integrands

$$\int\limits_{\mathbb{S}^2} f(\widehat{y})ds(\widehat{y}) \approx \sum_{p'=0}^{2n'+1}\sum_{s'=1}^{n'+1} \widetilde{\mu}_{p'}\widetilde{a}_{s'} f(p(\theta_{s'},\varphi_{p'})), \tag{4.14}$$

where $\widehat{y} = p(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$ - unit sphere parametrisation, $\varphi_{p'} = \dfrac{p'\pi}{n'+1}$, $\theta_{s'} = \arccos(z_{s'})$, $z_{s'}$ - zeros of the Legendre polynomials $P_{n'+1}$, $\widetilde{a}_{s'} = \dfrac{2(1-z_{s'}^2)}{((n'+1)P_{n'}(z_{s'}))^2}$ - weights of the Gauss-Legendre quadratures and $\widetilde{\mu}_{p'} = \dfrac{\pi}{n'+1}$.

For the case of weak singularity, we have the quadrature rule

$$\int\limits_{\mathbb{S}^2} \frac{f(\widehat{y})}{|\widehat{n}-\widetilde{y}|} ds(y) \approx \sum_{p'=0}^{2n'+1}\sum_{s'=1}^{n'+1} \widetilde{\mu}_{p'}\widetilde{b}_{s'} f(p(\theta_{s'},\varphi_{p'})), \tag{4.15}$$

where $\widetilde{b}_{s'} = \dfrac{\pi\widetilde{a}_{s'}}{n'+1}\displaystyle\sum_{l=0}^{n'} P_l(z_{s'})$, P_l - Legendre polynomial order of l and $\widehat{n} = (0,0,1)^T$ - noth pole of the unit sphere $\mathbb{S}^2$.

Both quadratures are obtained by approximation of the regular part of the integral via spherical harmonics and then employing exact integration. These quadrature rules have super-algebraic convergence order.

For further discretisation of the system linear integral equations (3.10), (3.11), we shall move the weak singularity in the corresponding integrals to the north pole $\widehat{n} = (0,0,1)$. To do this, we consider the orthogonal transformation $T_{\widehat{x}}$

$$T_{\widehat{x}}\widehat{x} = \widehat{n}, \ \forall \widehat{x} \in \mathbb{S}^2, \tag{4.16}$$

where $T_{\widehat{x}} = D_F(\varphi)D_T(\theta)D_F(-\varphi)$:

$$D_F(\psi)\begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D_T(\psi)\begin{pmatrix} \cos(\psi) & 0 & -\sin(\psi) \\ 0 & 1 & 0 \\ \sin(\psi) & 0 & \cos(\psi) \end{pmatrix}.$$

Take into account the last transformation, (4.14), (4.15) then we can rewrite the system of linear integral equations (3.10) as follows

$$\left\{ \begin{array}{ll} \left(\widetilde{S}_{21}\psi_1\right)(\widehat{x}) + \left(\overline{S}_{22}\psi_2\right)(\widehat{x}) = \widetilde{f}(\widehat{x}), & \widehat{x} \in \mathbb{S}^2 \\ -\frac{1}{2}\psi_1(\widehat{x}) + \left(\overline{K}_{11}\psi_1\right)(\widehat{x}) + \left(\widetilde{K}_{12}\psi_2\right)(\widehat{x}) = \widetilde{h}(\widehat{x}), & \widehat{x} \in \mathbb{S}^2 \end{array} \right., \tag{4.17}$$

where integral operators are as follows

$$(\overline{S}_{\ell\ell}\psi)(\widehat{x}) = \int\limits_{\mathbb{S}^2} \psi(T_{\widehat{x}}^{-1}\widehat{\eta}) L_{\ell\ell}(\widehat{x}, T_{\widehat{x}}^{-1}\widehat{\eta}) ds(\widehat{\eta}), \quad \widehat{x} \in \mathbb{S}^2,$$

and

$$(\overline{K}_{\ell\ell}\psi)(\widehat{x}) = \int\limits_{\mathbb{S}^2} \psi(T_{\widehat{x}}^{-1}\widehat{\eta}) M_{\ell\ell}(\widehat{x}, T_{\widehat{x}}^{-1}\widehat{\eta}) ds(\widehat{\eta}), \quad \widehat{x} \in \mathbb{S}^2$$

for $\ell = 1, 2$. Here we used that $|\widehat{x} - \widehat{y}| = \left|T_{\widehat{x}}^{-1}(\widehat{n} - \widehat{\eta})\right| = |\widehat{n} - \widehat{\eta}|$.

For discretization of systems (4.17) and (4.18) we will use projection Galerkin method. The approximations of the densities ψ_ℓ, $\ell = 1, 2$ can be represented as a linear combination of the spherical harmonics

$$\widetilde{\psi}_i \approx \sum_{k=0}^{n} \sum_{m=-k}^{k} \psi_{k,m}^{i} Y_{k,m}^{R},\ i = 1, 2, \tag{4.19}$$

where real-value spherical harmonics are follows

$$Y_{k,m}^{R} = \begin{cases} \operatorname{Im} Y_{k,|m|}, & 0 < m \leq k \\ \operatorname{Re} Y_{k,|m|}, & -k \leq m \leq 0 \end{cases}.$$

Here $Y_{k,m}$ - spherical function:

$$Y_{k,m}(\theta, \varphi) = c_k^m P_k^{|m|}(\cos\theta) e^{im\varphi},\ m = -k, ..., k,\ k = 0, 1, ...,$$

$c_k^m = (-1)^{\frac{|m|-m}{2}} \sqrt{\dfrac{2k+1}{4\pi} \dfrac{(k-|m|)!}{(k+|m|)!}}$, P_k^m - Legendre functions.

We shall consider scalar product based on the quadrature formula

$$(v, w) = \sum_{p=0}^{2n+1} \sum_{s=1}^{n+1} \mu_p a_s v(\widehat{y}_{sp}) w(\widehat{y}_{sp}),\ v, w \in C(\mathbb{S}^2). \tag{4.20}$$

Here the coefficients a_s and μ_p are same as in (4.14) but they depended from the parameter $n \in \mathbb{N}$. Next our step will be employing the scalar product to systems of integral equations (4.17), (4.18) with spherical harmonics $Y_{k,m}^{R}$. Including representation of integral operators (**??**), and also the density approximation (4.19) we will get the next linear systems

$$\begin{cases} \sum\limits_{k=0}^{n} \sum\limits_{m=-k}^{k} \left(\psi_{k,m}^{1} \hat{A}_{kk'mm'}^{11} + \psi_{k,m}^{2} \hat{A}_{kk'mm'}^{12}\right) = \sum\limits_{p=0}^{2n+1} \sum\limits_{s=1}^{n+1} \mu_p a_s \widetilde{f}(\widehat{x}_{sp}) Y_{k,m}^{R}(\widehat{x}_{sp}) \\ \sum\limits_{k=0}^{n} \sum\limits_{m=-k}^{k} \left(\psi_{k,m}^{1} \check{A}_{kk'mm'}^{21} + \psi_{k,m}^{2} \check{A}_{kk'mm'}^{22}\right) = \sum\limits_{p=0}^{2n+1} \sum\limits_{s=1}^{n+1} \mu_p a_s \widetilde{h}(\widehat{x}_{sp}) Y_{k,m}^{R}(\widehat{x}_{sp}) \end{cases}, \tag{4.21}$$

$k^{'} = 0, ..., n^{'}$, $m = -k, ..., k$, $n = 0, 1, ...$ with coefficients

$$\hat{A}^{ij}_{kk'mm'} = \sum_{p=0}^{2n+1} \sum_{s=1}^{n+1} \sum_{p'=0}^{2n'+1} \sum_{s'=1}^{n'+1} \mu_{p'} \mu_p a_s Y^R_{k',m'}(\widehat{x}_{sp}) \begin{cases} \widetilde{a}'_s L_{ij}(\widehat{x}_{sp}, \widehat{y}_{s'p'}) Y^R_{k,m}(\widehat{y}_{s'p'}), & i \neq j \\ \widetilde{b}'_s L_{ii}(\widehat{x}_{sp}, \widehat{y}^{s'p'}_{sp}) Y^R_{k,m}(\widehat{y}^{s'p'}_{sp}), & i = j \end{cases}$$

and

$$\breve{A}^{ij}_{kk'mm'} = \sum_{p=0}^{2n+1} \sum_{s=1}^{n+1} \sum_{p'=0}^{2n'+1} \sum_{s'=1}^{n'+1} \mu_{p'} \mu_p a_s \left(\begin{cases} \widetilde{a}'_s M_{ij}(\widehat{x}_{sp}, \widehat{y}_{s'p'}) Y^R_{k,m}(\widehat{y}_{s'p'}), & i \neq j \\ \widetilde{b}'_s M_{ii}(\widehat{x}_{sp}, \widehat{y}^{s'p'}_{sp}) Y^R_{k,m}(\widehat{y}^{s'p'}_{sp}), & i = j \end{cases} + \begin{cases} \frac{1}{2}(-1)^{i+1} Y^R_{k,m}(\widehat{x}_{sp}), & i = j \\ 0, & \text{otherwise} \end{cases} \right) \times Y^R_{k',m'}(\widehat{x}_{sp}),$$

$i, j = 1, 2$ and $\widehat{y}^{s'p'}_{sp} = T^{-1}_{\widehat{x}_{sp}} \widehat{y}_{s'p'}$. Thus we can find the numerical solution of each of mixed problems

$$u_{nn'}(x) = \sum_{i=1}^{2} \sum_{s'=1}^{n'+1} \sum_{p'=0}^{2n'+1} \sum_{k=0}^{n} \sum_{m=-k}^{k} \mu_{p'} a_{s'} \psi^i_{k,m} Y^R_{k,m}(\widehat{y}_{s'p'}) \Phi(x, q_i(\widehat{y}_{s'p'})) J_{q_i}(\widehat{y}_{s'p'}). \tag{4.23}$$

4.2 Implementation

For the effective implementation of the algorithm we must reduce the amount of computation coefficients $\hat{A}^{\ell j}_{kk'mm'}$ and $\breve{A}^{\ell j}_{kk'mm'}$. We shall consider real-valued spherical harmonics

$$Y^R_{k,m}(\theta, \varphi) = c^m_k P^{|m|}_k(\cos\theta) \begin{cases} \cos(|m|\varphi), & m < 0 \\ 1, & m = 0 \\ \sin(|m|\varphi), & m > 0 \end{cases}.$$

We use the representation of the rotated spherical harmonics

$$Y^R_{k,m}(\widehat{y}^{s'p'}_{sp}) = \sum_{|\widetilde{m}| \leq k} Y_{k,\widetilde{m}}(\widehat{y}_{s'p'}) e^{-i\widetilde{m}\varphi_p} \times \begin{cases} \frac{1}{2i}\left(F_{sk\widetilde{m}|m|} e^{i|m|\varphi_p} - (-1)^{|m|} F_{sk\widetilde{m}-|m|} e^{-i|m|\varphi_p}\right), & m > 0 \\ F_{sk\widetilde{m}|m|}, & m = 0 \\ \frac{1}{2i}\left(F_{sk\widetilde{m}|m|} e^{i|m|\varphi_p} + (-1)^{|m|} F_{sk\widetilde{m}-|m|} e^{-i|m|\varphi_p}\right), & m < 0 \end{cases},$$

where $F_{sk\widetilde{m}m} = e^{i(m-\widetilde{m})\frac{\pi}{2}} \sum_{|l| \leq k} d^{(k)}_{\widetilde{m}l}\left(\frac{\pi}{2}\right) d^{(k)}_{ml}\left(\frac{\pi}{2}\right) e^{il\theta_s}$ and

$d^{(k)}_{ml}\left(\frac{\pi}{2}\right) = 2^m \sqrt{\frac{(k+m)!(k-m)!}{(k+l)!(k-l)!}} P^{(l-m,-l-m)}_{k+m}(0)$, $P^{(\alpha,\beta)}_n$ - normalized Jacobi polynomial, given by

$$P^{(\alpha,\beta)}_n(0) = 2^{-n} \sum_{t=0}^{n} (-1)^t \binom{n+a}{n-t} \binom{n+b}{t}, \quad a \geq 0, b \geq 0.$$

When $l-m$, $-l-m$ are negative we can calculated $d_{ml}^{(k)}\left(\frac{\pi}{2}\right)$ by using symmetry relation

$$d_{ml}^{(k)}(\varphi)=(-1)^{m-l}d_{lm}^{(k)}(\varphi)=d_{-l-m}^{(k)}(\varphi)=d_{ml}^{(k)}(-\varphi)$$

Elements $\hat{A}_{kk'mm'}^{\ell j}$ can be represented as follows:

$$\hat{A}_{kk'mm'}^{\ell j}=\sum_{s=1}^{n+1}a_s c_{k'}^{m'}P_{k'}^{|m'|}(\cos\theta_s)\sum_{p=0}^{2n+1}\mu_p\begin{cases}\cos(|m'|\varphi_p), & m'<0\\ 1, & m'=0\\ \sin(|m'|\varphi_p), & m'>0\end{cases}$$
$$\times\sum_{p'=0}^{2n'+1}\widetilde{\mu}_p\sum_{s'=1}^{n'+1}\begin{cases}\widetilde{a}_{s'}L_{ij}(\widehat{x}_{sp},\widehat{y}_{s'p'}), & \ell\neq j\\ \widetilde{b}_{s'}L_{ii}(\widehat{x}_{sp},\widehat{y}_{sp}^{s'p'}), & \ell=j\end{cases}$$
$$\times\begin{cases}\sum\limits_{\widetilde{m}\le k}c_k^{\widetilde{m}}P_k^{\widetilde{m}}(\cos\theta_{s'})e^{i\widetilde{m}(\varphi_{p'}-\varphi_p)}, & \ell=j\\ c_k^m P_k^{|m|}(\cos\theta_{s'})\begin{cases}\cos(|m|\varphi_{p'}), & m<0\\ 1, & m=0\\ \sin(|m|\varphi_{p'}), & m>0\end{cases}, & \ell\neq j\end{cases}$$
$$\times\begin{cases}\begin{cases}\frac{1}{2i}\left(F_{sk\widetilde{m}|m|}e^{i|m|\varphi_p}-(-1)^{|m|}F_{sk\widetilde{m}-|m|}e^{-i|m|\varphi_p}\right), & m>0\\ F_{sk\widetilde{m}|m|}, & m=0\\ \frac{1}{2i}\left(F_{sk\widetilde{m}|m|}e^{i|m|\varphi_p}+(-1)^{|m|}F_{sk\widetilde{m}-|m|}e^{-i|m|\varphi_p}\right), & m<0\end{cases}, & \ell=j\\ 1, & \ell\neq j\end{cases}.$$

and the calculation is carried out through the consistent calculation of matrices

$$H_{mp}=\begin{cases}\cos(|m|\varphi_p), & m<0\\ 1, & m=0\\ \sin(|m|\varphi_p), & m>0\end{cases},\quad G_{kms}=c_k^m P_k^{|m|}(\cos\theta_s),$$
$$E_{sp\widetilde{m}s'}^{\ell j}=\sum_{p'=0}^{2n'+1}\mu_{p'}\begin{cases}e^{i\widetilde{m}\varphi_{p'}}L_{\ell\ell}(\widehat{x}_{sp},\widehat{y}_{sp}^{s'p'}), & \ell=j\\ H_{\widetilde{m}p'}L_{\ell j}(\widehat{x}_{sp},\widehat{y}_{s'p'}), & \ell\neq j\end{cases},$$
$$D_{ksp\widetilde{m}}^{\ell j}=\sum_{s'=1}^{n'+1}G_{k\widetilde{m}s'}E_{sp\widetilde{m}s'}^{\ell j}\begin{cases}\widetilde{a}_{s'}, & \ell=j\\ \widetilde{b}_{s'}, & \ell\neq j\end{cases},$$

$$C_{kmsp}^{\ell j}=\begin{cases}\sum\limits_{|\widetilde{m}|\le k}e^{-i\widetilde{m}\varphi_p}D_{ksp\widetilde{m}}^{(kern)ij}, & \ell=j\\ D_{kspm}^{\ell j}, & \ell\neq j\end{cases}$$
$$\times\begin{cases}\begin{cases}\frac{1}{2i}\left(F_{sk\widetilde{m}|m|}e^{i|m|\varphi_p}-(-1)^{|m|}F_{sk\widetilde{m}-|m|}e^{-i|m|\varphi_p}\right), & m>0\\ F_{sk\widetilde{m}|m|}, & m=0\\ \frac{1}{2i}\left(F_{sk\widetilde{m}|m|}e^{i|m|\varphi_p}+(-1)^{|m|}F_{sk\widetilde{m}-|m|}e^{-i|m|\varphi_p}\right), & m<0\end{cases}, & \ell=j\\ 1, & \ell\neq j\end{cases},$$

$$B_{kmm's}^{\ell j} = \sum_{p=0}^{2n+1} \mu_p H_{m'p} \times \left(C_{kmsp}^{\ell j} + \begin{cases} \frac{1}{2}(-1)^{(i+1)} G_{kms} H_{mp}, & kern = 2 \text{ and } \ell = j \\ 0, & \text{otherwise} \end{cases} \right)$$

$$\hat{A}_{kk'mm'}^{\ell j} = \sum_{s=1}^{n+1} a_s G_{k'm's} B_{kmm's}^{\ell j}.$$

The calculation of the coefficients $\check{A}_{kk'mm'}^{\ell j}$ can be handle in similar way.

5 Numerical examples

In this section we will illustrate the robustness of the proposed method for the reconstruction of the harmonic function, for both exact and noisy data. In case of the noisy data, random point wise errors have been added to the values of f with the percentage given in terms of the L^2-norm.

5.1 Example 1

The double-connected solution domain D is given in Fig. **??** and two boundary surfaces are follows:
$\Gamma_1 = \{x(\theta,\varphi) = 3(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}$ - sphere, $\Gamma_2 = \{x(\theta,\varphi) = (\sin\theta\cos\varphi, 2\sin\theta\sin\varphi, 2\cos\theta),\ 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}$ - ellipsoid. The Cauchy data on the Γ_1 is next:

$f_1(x) = \cos x_1 e^{x_2}\ x \in \Gamma_1,\ \ g_1(x) = \langle(-\sin x_1, \cos x_1, 0)e^{x_2}, \nu(x)\rangle$

In the Table 1 we can see errors for this example.

5.2 Example 2

The double-connected solution domain D is given in Fig. **??** and two boundary surfaces are follows:

$\Gamma_1 = \{x(\theta,\varphi) = r(\theta,\varphi)(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}$, $r(\theta,\varphi) = \sqrt{0.8 + 0.2(\cos(2\varphi) - 1)(\cos(4\theta) - 1)}$ - cushion,
$\Gamma_2 = \{x(\theta,\varphi) = 0.5(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}$ - sphere. The Cauchy data on the Γ_1 is next:

$f_1(x) = x_1^2 - x_2^2 + x_3\ x \in \Gamma_1,\ \ g_1(x) = \langle(2x_1, -2x_2, 1), \nu(x)\rangle$ In the Table 2 we can see errors for this example.

Table 2: aaa

	exact data			data with noisy(0.001)		
N	k	$\|u - u_n\|_{L_2}$	$\left\|\frac{\partial u}{\partial\nu} - \frac{\partial u_n}{\partial\nu}\right\|_{L_2}$	k	$\|u - u_n\|_{L_2}$	$\left\|\frac{\partial u}{\partial\nu} - \frac{\partial u_n}{\partial\nu}\right\|_{L_2}$
4	6	$1.98E-001$	$8.44E-001$	6	$1.98E-001$	$8.45E-001$
6	20	$4.09E-002$	$3.07E-001$	19	$4.16E-002$	$3.09E-001$
8	37	$9.81E-003$	$9.46E-002$	36	$1.05E-002$	$9.74E-002$
10	55	$5.95E-003$	$6.37E-002$	66	$6.20E-003$	$6.83E-002$
12	169	$2.51E-003$	$3.17E-002$	87	$4.44E-003$	$4.97E-002$

Acknowledgment

References

[1] K. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge: Cambridge University Press, 1997.

[2] Babenko, C., Chapko, R. and Johansson, B. T., Numerical solution of the Cauchy problem for the Laplace equation in simply connected domains via the alternating method and boundary integrals, Advances in Boundary Integral Methods - Proceedings of the Tenth UK Conference on Boundary Integral Methods, (Ed. P. Harris), University of Brightin, UK, (2015), ??–??.

[3] Baranger, T. N., Johansson, B. T. and Rischette, R., On the alternating method for Cauchy problems and its finite element discretisation, Springer Proceedings in Mathematics & Statistics, (Ed. L. Beilina), 183–197, (2013).

[4] J. Bremer and V. Rokhlin, Efficient discretization of Laplace boundary integral equations on polygonal domains *J. Comput. Physics* **229** 2507–2525, (2010).

[5] F. Cakoni and R. Kress, Integral equations for inverse problems in corrosion detection from partial Cauchy data, *Inverse Problems and Imaging* **1**, 229–245, (2007).

[6] F. Cakoni, R. Kress and C. Schuft, Integral equations for shape and impedance reconstruction in corrosion detection, *Inverse Problems* **26**, 095012 (2010).

[7] F. Cakoni, R. Kress and C. Schuft, Simultaneous reconstruction of shape and impedance in corrosion detection, *Methods and Applications of Analysis* **17**, 357–378, (2010).

[8] J.-R. Chang, W. Yeih and M.-H. Shieh, On the modified Tikhonov's regularization method for the Cauchy problem of the Laplace equation, *Journal of Marine Science and Tekhnology* **9**, 113–121, (2001).

[9] R. Chapko, An integral equation method for the numerical analysis of gravity waves in a channel with free boundary. *Applied Mathematics and Computing* **159**, 247–266, (2004).

[10] R. Chapko and B. T. Johansson, An alternating potential based approach to the Cauchy problem for the Laplace equation in a planar domain with a cut. *Computational Methods in Applied Mathematics* **8**, 315–335, (2008).

[11] R. Chapko and B. T. Johansson, On the numerical solution of a Cauchy problem for the Laplace equation via a direct integral equation approach. *Inverse Problems and Imaging* **6**, 25–36, (2012).

[12] Chapko, R., Johansson, B. T. and Protsyuk, O., A direct boundary integral equation method for the numerical construction of harmonic functions in three-dimensional layered domains containing a cavity, *Int. J. Comput. Math.* **89** (2012), 1448–1462.

[13] Chapko, R. and Johansson, B. T., A direct integral equation method for a Cauchy problem for the Laplace equation in 3-dimensional semi-infinite domains, *CMES Comput. Model. Eng. Sci.* **85** (2012), 105–128.

[14] R. Chapko, B. T. Johansson and Y. Savka, Integral equation method for the numerical solution of the Cauchy problem for the Laplace equation in a double connected planar domain. *Journal of Inverse Problems in Science and Engineering* (submitted).

[15] H. Cao, M. V. Klibanov, and S. V. Pereverzev, A Carleman estimate and the balancing principle in the quasi-reversibility method for solving the Cauchy problem for the Laplace equation, *Inverse Problems* **25**, 1–21, (2009).

[16] M. Costabel and E. P. Stephan, Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation. *Mathematical Models in Mechanics*, Banach Center Publications, PWN-Polish Scientific Publishers, Warsaw, **15**, 175–251, (1985).

[17] A.M. Denisov, E.V. Zakharov, A.V. Kalinin and V.V. Kalinin, Numerical solution of the inverse electrocardiography problem with the use of the Tikhonov regularization method, *Moscow University Computational Mathematics and Cybernetics* **32**, 61-68, (2008).

[18] D.G. Duffy, Mixed Boundary Value Problems. Chapman & Hall / CRC Boca Raton, 2008.

[19] P. Grisvard, Elliptic problems in nonsmooth domains. Pitman, 1985.

[20] J. Hadamard, Lectures on Cauchy's problem in linear partial differential equations. Yale Univ. Press, New Haven, 1923.

[21] P.C. Hansen, The L-curve and its use in the numerical treatment of inverse problems. In Computational Inverse Problems in Electrocardiology, Johnston P (ed.). WIT Press: Southampton, 119–142, (2001).

[22] J. Helsing and R. Ojala, Corner singularities for elliptic problems: integral equations, graded meshes, quadrature, and compressed inverse preconditioning, *J. Comput. Physics* **227** (2008), 8820–8840.

[23] G. Hsiao and W. Wendland, On the integral equation method for the plane mixed boundary value problems of the laplacian. *Math. Methods Appl. Sci.* **1**, 265-321, (1979).

[24] J. Elschner and I. G. Graham, Quadrature methods for Symm's integral equation on polygons. *IMA J. Num. Anal.* **17**, 643–664, (1997).

[25] J. Elschner, Y. Jeon, I. H. Sloan and E. P. Stephan, The collocation method for mixed boundary value problems on domains with curved polygonal boundaries. *Numer. Math.* **76**, 355–381, (1997).

[26] M. Ganesh and I. G. Graham, *A high-order algorithm for obstacle scattering in three dimensions*, J. Comput. Phys. 198 (2004), pp. 211–242.

[27] M. Ganesh, I. G. Graham, and J. A. Sivaloganathan, *New spectral boundary integral collocation method for three-dimensional potential problems*, SIAM J. Numer. Anal. 35 (1998), pp. 778–805.

[28] I. G. Graham and I. H. Sloan, *Fully discrete spectral boundary integral methods for Helmholtz problems on smooth closed surfaces in* $\mathbf{R}^3$, Numer. Math. 92 (2002), pp. 289–323.

[29] V. A. Kozlov, and V. G. Maz'ya, On iterative procedures for solving ill-posed boundary value problems that preserve differential equations, *Algebra i Analiz* **1** , 144–170, (1989) English transl.: *Leningrad Math. J.* **1**, 1207–1228, (1990).

[30] R. Kress, Linear Integral Equations, 2nd. ed. Berlin: Springer-Verlag, 1999.

[31] R. Kress, A Nyström method for boundary integral equations in domains with corners, *Numer. Math.* **58**, 145–161 (1990).

[32] R. Kress and T. Tran, Inverse scattering for a locally perturbed half-plane. *Inverse Problems* **16**, 1541–1559, (2000).

[33] P. Labin, Optimal order collocation for the mixed boundary value problem on polygons. *Mathematics of Computation* **70**, 607–636, (2000).

[34] J.-Y. Lee and J.-R. Yoon, A numerical method for Cauchy problem using singular value decomposition, *Comm. Korean Math. Soc.* **16**, 487–508, (2001)

[35] W. McLean, Strongly Elliptic Systems and Boundary Integral Operators, Cambridge: Cambridge University Press, 2000.

[36] L. E.Payne, Improperly Posed Problems in Partial Differential Equations, Regional Conference Series in Applied Mathematics, No. 22, SIAM, Philadelphia, Pa., 1975.

[37] L. Scuderi, A Chebyshev polinomial collocation BIEM for mixed boundary value problems on nonsmooth boundaries, *Journal of Integral Equations and Applications* **14**, 179–221, (2002).

[38] Y. Sun, D. Zhang and F. Ma, A potential function method for the Cauchy problem of elliptic operators, *J. Math. Anal. Appl.* **395**, 164-174, (2012).

[39] L. Wienert, *Die Numerische Approximation von Randintegraloperatoren für die Helmholtzgleichung im* $\mathbf{R}^3$, Ph.D. thesis, University of Göttingen, 1990.