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A branch and efficiency algorithm for the optimal design of supply chain networks

Konstantinos Petridis[1] · **Prasanta Kumar Dey**[2] · **Ali Emrouznejad**[2]

Abstract Supply chain operations directly affect service levels. Decision on amendment of facilities is generally decided based on overall cost, leaving out the efficiency of each unit. Decomposing the supply chain superstructure, efficiency analysis of the facilities (warehouses or distribution centers) that serve customers can be easily implemented. With the proposed algorithm, the selection of a facility is based on service level maximization and not just cost minimization as this analysis filters all the feasible solutions utilizing Data Envelopment Analysis (DEA) technique. Through multiple iterations, solutions are filtered via DEA and only the efficient ones are selected leading to cost minimization. In this work, the problem of optimal supply chain networks design is addressed based on a DEA based algorithm. A Branch and Efficiency (B&E) algorithm is deployed for the solution of this problem. Based on this DEA approach, each solution (potentially installed warehouse, plant etc) is treated as a Decision Making Unit, thus is characterized by inputs and outputs. The algorithm through additional constraints named "efficiency cuts", selects only efficient solutions providing better objective function values. The applicability of the proposed algorithm is demonstrated through illustrative examples.

Keywords Integer programming · Branch and bound · DEA · Supply chain management · Mixed integer linear programming (MILP)

✉ Ali Emrouznejad
a.emrouznejad@aston.ac.uk

Konstantinos Petridis
k.petridis@uom.edu.gr

Prasanta Kumar Dey
p.k.dey@aston.ac.uk

1 Department of Applied Informatics, University of Macedonia, 156 Egnatia str., 54006 Thessaloniki, Greece

2 Operations and Information Management Group, Aston Business School, Aston University, Birmingham B4 7ET, UK

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1 Introduction

Productivity measurement and efficiency in particular is a common term among the discipline of production economics. Each organization, firm, enterprise, business bank, can extract the efficiency of its branches, units if the latter are described by inputs and outputs. Efficiency is generally defined as the ratio of the amount of output produced given the unit's inputs or resources. Thus, in order to measure the efficiency of a unit, certain inputs and outputs of a unit should be provided a priori. Deploying Data Envelopment Analysis (DEA) technique, the efficiency of each unit can be calculated (Charnes et al. 1981, 1985).

In the past decades there have been advances in Integer Programming (IP); one of which is Mixed Logical Linear Programming (MLLP) or Mixed Integer Linear (MILP) Programming models, where binary variables provide information regarding the installation of a plant or the selection of a route or a procedure, depending on the nature of the model (Hooker and Osorio 1999). This "family" of problems is generally solved with IP techniques, one of which is Branch and Bound (B&B) (Ross and Soland 1975). Based on this approach, a tree representation of the problem is provided; given a minimization direction to the problem (a transformation can be deployed if there is a maximization direction in the objective function) each node is examined separately for feasibility and for objective value improvement.

The optimal design of supply chain network problem is an extension of a transportation problem. Based on this type of problem, multiple echelons are considered representing the different stages of transportation or manufacturing process that products are subject to. The aim of these type of problems includes the minimization of total operational cost, lead time, stock out instances, or the maximization of profit, or customer satisfaction that the firm or enterprise will gain. Assuming that a supply chain is designed with respect to warehouse selection and installation, then each potentially installed warehouse has inputs and outputs which are schematically presented in Fig. 1.

Due to the existence of binary variables B&B solution approach can be easily implemented providing solutions that are subjected to a single criterion. However, as can be seen in Fig. 1, considering each potentially installed warehouse as an entity then another conflicting objective is added to the problem. Based on this new objective facilities are not selected only based on cost minimization (or profit maximization) but also on whether these solutions are efficient. If more than one sites are selected then a possible approach is to add a single sourcing constraint which will select a single facility that reduces greatly the overall cost, but it may also cause the problem to become infeasible. In this case, other methodologies must be employed in order to provide solution to the problem.

The proposed algorithm is formulated in order to reduce the number of facilities (warehouses) in a supply network design problem. When designing the supply chain network, the Decision Maker (DM) seeks for less warehouses (facilities in general) as possible in order
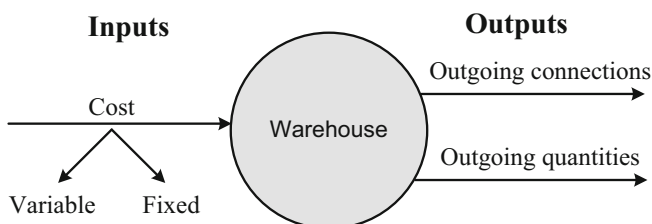

Fig. 1 Inputs and outputs of a warehouse

to reduce cost. This is obtained through economies of scale, however, reduction in facilities leads to less customers' satisfaction. Generally the models that are used in order to design supply chain network, use single sourcing constraint in order to reduce the number of facilities by imposing the model to select one facility to accommodate a cluster of demand zones. This constraint is hard and often leads to infeasibility. In case of infeasibility, the model has to be solved without single sourcing constraint by letting the model to select as much facilities (warehouses) as needed in order to satisfy demand of the customers. Also, the cost is significantly increasing, depending on the demand pattern. On the contrary, the presented algorithm selects the warehouses based on their efficiency, leading to better results, both by reducing cost and by increasing efficiency. The algorithm filters all the optimal results derived from the initial MILP model and iteratively selects those warehouses that are fully technically efficient. Some of the methods that are employed for solving supply chain network design problems, in case of infeasibility, are Lagrangean Relaxation and Benders Decomposition, however, these methods use relaxation of "hard" constraints in order to solve the problem. The presented algorithm solves the problem based on two dimensions: cost and efficiency. Additional constraints remove inefficient solutions. In case that this new problem is now infeasible, different thresholds of efficiency are considered in the constraint, where the lower level of that percentage is left to DM to decide.

In this work, which is an extension of the work of Grigoroudis et al. (2014), a Branch and Efficiency (B&E) algorithm is proposed for the optimal design of supply chain networks. Through an iterative procedure, an initial vector of solutions is provided along with the inputs and outputs of each solution. Each solution is filtered in the following stages through constraints (efficiency cuts). The algorithm stops if the number of the non-zero solutions of the final vector is less than a certain pre-determined level, or there is no change in objective function's value. An approach that is integrating DEA technique in the selection of solutions, providing a Multi-Objective Programming model, as solutions are not only subjected to constraints to the problem, but also to the "efficiency cuts" that are posed by B&E approach, has not yet been proposed in the supply chain literature.

2 Literature review

In the recent years supply chain design literature has expanded to consider the rapidly changing economic environment in which a supply chain network should be designed. There has been proposed a plethora of mathematical programming models that have been applied to supply chain network design problems of which Mixed Integer Linear Programming (MILP) and Mixed Integer Non Linear Programming (MINLP) models have been widely used, providing generic frameworks for managerial use.

Optimal supply chain network design models are divided into two categories; the steady state and the multi-period ones. In the first case, time is absent from the analysis, and this type of formulation provides average levels of decisions, while in the multi-period models, the decisions are made with respect to the planning horizon.

The key point in modeling supply chain networks is the demand uncertainty. A number of studies have captured the stochasticity with distribution functions that best describe demand.

Tsiakis et al. (2001) proposed a steady state model for the optimal supply chain network design, with decisions that regard the installation of facilities (distribution centers and warehouses). Demand uncertainty is modeled through different demand and capacity parameter scenarios.

The optimal design of supply chain networks has been also proposed by Petridis (2015). Demand stochasticity is proposed by Normal Distribution, while probabilistic constraint are integrated in a single framework for the optimal supply chain network providing decisions about the occurrence of stock out instances.

In their work, Rodriguez et al. (2014) employed a multi-period model for the optimal design of supply chain taking into consideration demand uncertainty providing a production plan that integrates tactical and strategic decisions.

Supply chain management in global scale integrates decisions that take into account the factors that contribute to the sustainability of a supply chain network. A holistic approach towards the optimization of sustainability subject to the economic, ecological and social objectives has been proposed in the work of Kannegiesser and Günther (2014).

In the context of supply chain network design model, DEA formulations have been proposed in order to assess the efficiency of supply chain networks (Network DEA, Two stage DEA formulation etc). The following works demonstrate the use of DEA to supply chain area. However, most applications of DEA to supply chain systems is performed in an entirely different context to the one presented in this paper.

A non linear programming model has been proposed by Liang et al. (2006) for the evaluation of supply chain efficiency under intermediated performance evaluation.

A multi-phase supply chain network design model has been proposed by Talluri and Baker (2002) utilizing DEA technique along with a game theoretic approach for pairwise evaluation of performance, designing the supply chain and providing optimal routing decisions.

Besides the evaluation of supply chain efficiency, sub operations conducted among the nodes of supply chain is also of major importance. In their work, Cheung and Hausman (2000), propose an exact measure efficiency measurement of (Q, R) policy of a two-echelon, multiple retailers system. Evaluation of supply chain performance has been proposed in the work of Forker et al. (1997) where through the combination of non linear DEA and regression analysis, Total Quality Management measures were provided.

Frota Neto et al. (2008) applied DEA and utilized DEA technique's ability for efficiency extraction integrating into a unified framework with a multi-objective optimization model.

In their work Chen and Yan (2011) proposed a special network DEA approach in order to provide exact modeling with respect to the internal interactions of the supply chain. Similar works have been also proposed by Prieto and Zofío (2007), Huang et al. (2010) and Färe and Grosskopf (2000) proposed Network DEA models that can eventually be applied to the supply chain network evaluation framework.

Yang et al. (2011), have proposed an exact production possibility set for evaluating the performance different forms of supply chain models, while a game theoretic DEA model has been proposed by Chen et al. (2006), for analyzing the efficiency game between two supply chain parts. The model is proposed to explain bargaining supplier and manufacturer's behavior for decision process. The internal supply chain performance, has been examined by Wong and Wong (2007) using technical and cost efficiency models. Using this DEA to measure the internal operations, inefficiencies in supply chain operations can be identified.

Generally, DEA technique has been used in order to provide efficiency of whole supply chain network system or to measure the performance of specific critical sub-systems, like suppliers, plants etc. Yet, the data for the application of DEA technique are provided a-priori, while the results are not taken into account during the optimization technique.

In this paper a DEA based algorithm is proposed for the optimal design of supply chain networks design. The algorithm utilizes the properties of DEA technique to provide productivity scores based on multiple inputs and outputs for each examined unit. It is assumed that in the proposed algorithm, except for the maximization of profit or revenue (minimization of

cost), there is another objective based on which the selection of solutions is conducted, the maximization of selected solutions. The proposed algorithm is called Branch and Efficiency (B&E) as in each iteration the algorithm adds "efficiency cuts", which are constraints to filter only the feasible and efficient solutions to be accepted. None of the existing papers in the literature have proposed such an algorithm until now.

3 B&E algorithm

3.1 Introduction to B&E

In general, a Mixed Integer Linear Programming (MILP) model is expressed with mathematical formulation as in (1) whereas $\mathbf{x}$ represents the vector of continuous variables while $\mathbf{y}$ the vector of binary variables. In MILP formulation (1), $\mathbf{c^T}$ and $\mathbf{d^T}$ are the coefficients of continuous and binary variables correspondingly. Finally, $\mathbf{f}$ represents the vector of right-hand side of constraints while $\mathbf{x^L}$ and $\mathbf{x^U}$ are vectors representing the upper and lower bounds placed on non-negative variables $\mathbf{x}$.

$$P_0 \min \mathbf{c^T} \cdot \mathbf{x} + \mathbf{d^T} \cdot \mathbf{y}$$
$$s.t.$$
$$\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{f}$$
$$\mathbf{x^L} \cdot \mathbf{y} \leq \mathbf{x} \leq \mathbf{x^U} \cdot \mathbf{y}$$
$$\mathbf{x} \geq \mathbf{0}, \quad \mathbf{y} \in \{0, 1\} \tag{1}$$

Continuous variables represent upstream or downstream flows towards the different nodes of the supply chain while binary variables represent the decision of connections or the selected facilities. Setting in advance what are the characteristics that the DM would like to measure in each of the selected binary variables (facilities), inputs and outputs should be provided. Assuming that $\mathbf{h}$ and $\mathbf{k}$ are sub-matrices of $\mathbf{x}$ and $\mathbf{y}$, that contain nonzero continuous or integer solutions that will be introduced as inputs and outputs correspondingly. The following Linear Programming (LP) model represents an output oriented DEA model in a matrix form. The sub-matrices $\mathbf{h}$ and $\mathbf{k}$ of $\mathbf{x}$ and $\mathbf{y}$ are used in order to extract the efficiency for each binary solution set to 1.

$$\mathbf{DEA} \max \mathbf{z}$$
$$s.t.$$
$$\mathbf{\Lambda^T} \cdot \mathbf{h} \leq \mathbf{h'}$$
$$\mathbf{\Lambda^T} \cdot \mathbf{k} \geq \mathbf{k'} \cdot \mathbf{z}$$
$$\mathbf{\Lambda} \geq \mathbf{0} \tag{2}$$

The aim of this approach is the minimization of total cost by selecting the decisions that concentrate maximum efficiency. For this purpose, a new MILP model is formulated with constraints with respect to efficiency filtering of solutions. In the following MILP model, efficiency cut constraints are introduced for the filtering of solutions that concentrate highest efficiency. Binary variables $\mathbf{y}$ are now replaced by binary variables $\boldsymbol{\xi}$ that are used to select the efficient solutions.

$$P_0^{\varepsilon} \min \mathbf{c^T} \cdot \mathbf{x} + \mathbf{d^T} \cdot \boldsymbol{\xi}$$
$$s.t.$$

$$
\begin{aligned}
&\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \boldsymbol{\xi} \leq \mathbf{f} \\
&\mathbf{x^L} \cdot \boldsymbol{\xi} \leq \mathbf{x} \leq \mathbf{x^U} \cdot \boldsymbol{\xi} \\
&\mathbf{E} \cdot \boldsymbol{\xi} \geq \boldsymbol{\alpha} \\
&\mathbf{x} \geq \mathbf{0}, \quad \boldsymbol{\xi} \in \{0, 1\}
\end{aligned}
\tag{3}
$$

As it can be seen in (3), the formulation is the same as in (1) but due to the fact that only the efficient solutions are selected with addition of efficiency constraint $\mathbf{E} \cdot \boldsymbol{\xi} \geq \boldsymbol{\alpha}$. In formulation (3), $\boldsymbol{\alpha}$ is a vector of efficiency where the solutions with efficiency score greater or equal than this threshold, are selected with variables $\boldsymbol{\xi}$. The solution approach is shown in Fig. 2. Initially the iterations counter (ε) is set to 0 while the set of selected solutions N is empty. Solving MILP model (1) an upper bound for the objective value is obtained; this value will be used as an indicator for the next steps of the B&E algorithm. The solutions provided by (1) will be used as inputs and outputs for efficiency evaluation via DEA technique. The inputs and outputs must be determined in advance by the DM so that would characterize each potential's solution efficiency. It must be noted that if the number of DMUs is less than an acceptable threshold (T), where DEA cannot provide reliable results, B&E algorithm stops. Here it is assumed that $T = \max\{m \cdot n, 3 \cdot (m + n)\}$. This can be modeled with the cardinality of the positive solutions of problem P_0 and is defined as $S = \left\{ X^* \in F \middle/ X^* > 0 \right\}$. This means that each problem is unique and a special customization should be performed in advance.

After efficiency calculation, Technical Efficiency (*TE*) is calculated as an efficiency measure, and is defined as the reciprocal of φ (Andersen and Petersen 1993). Additional constraints are introduced in order to allow only solutions that gather efficiency equal to 1. Correspondingly, new binary variables ($\boldsymbol{\xi}$) are introduced replacing those that concerned the selection of a solution. These variables are triggered only if the solution is efficient.

In case where optimal solutions for binary variables ($\boldsymbol{\xi}^*$) yield infeasibility, the range of efficiency becomes wider in order to incorporate the minimum number of solutions that make the problem to be solved to optimality and reduce the objective function value. The algorithm stops when the number of DMUs drops below a certain level or there is no change in the objective function value.

Throughout this procedure, the initially empty set of solutions N is filled with solutions that satisfy the conditions of iterative set J^γ.

3.2 Notation

Index

i	Plant
j	Warehouse
k	Customer
γ	Iteration

Parameters

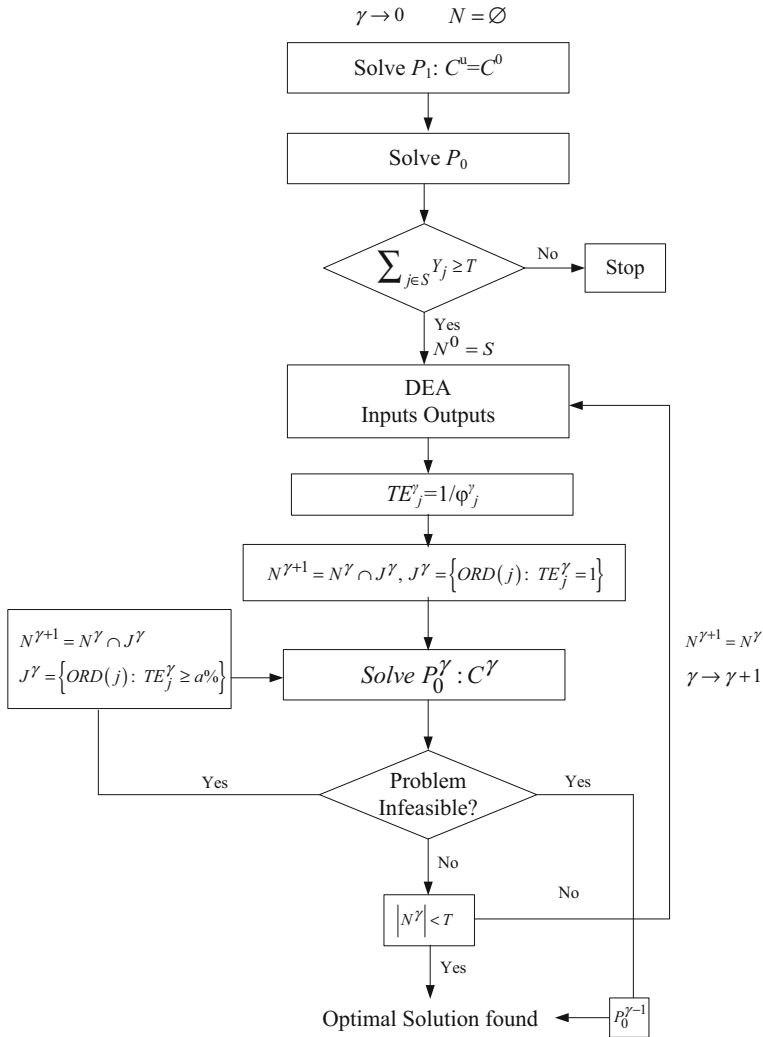|	
P_i^U	Upper bound of produced quantities at plant i
P_i^L	Lower bound of produced quantities at plant i
Q_{ij}^U	Upper bound of transported quantities from plant i to warehouse j
Q_{jk}^U	Upper bound of transported quantities from warehouse j customer k
W_j^U	Upper capacity of warehouse j

Fig. 2 The proposed B&E algorithm in a flowchart presentation

β_j Coefficient relating quantity at capacity at warehouse j
I_j Inventory level stored warehouse j

Monetary parameters

c_i^P Production cost at plant i
c_{ij}^V Unit transportation cost of products transported from plant i to warehouse j
c_{ij}^F Rout transportation cost of products transported from plant i to warehouse j
c_{jk}^V Unit transportation cost of products transported from warehouse j to customer k
c_{jk}^F Route transportation cost of products transported from warehouse j to customer k
c_k^{PEN} Penalty cost assigned to uncovered demand of customer k
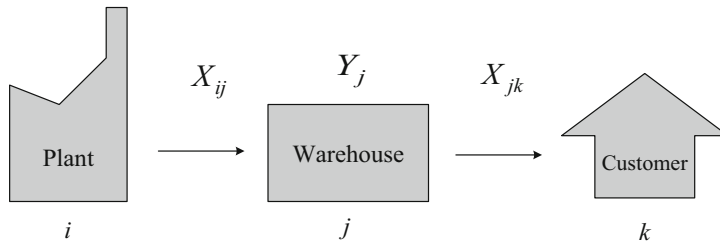F_j Installation cost of warehouse j

Fig. 3 The examined supply chain network

Continuous variables

P_i — Production quantity at plant i
Q_{ij} — Transported quantity from plant i to warehouse j
Q_{jk} — Transported quantity from warehouse j to customer k
W_j — Capacity of warehouse j
λ_j — Lambda (peers)
g_k — Variable modeling deficit in demand of customer k
TC — Total cost
φ_j — Efficiency of each

Binary variables

X_{ij} — 1 if the connection between plant i and warehouse j exists, 0 otherwise
X_{jk} — 1 if the connection between warehouse j exists and customer k exists, 0 otherwise
Y_j — 1 if warehouse j will be installed, 0 otherwise
ξ_j — 1 if warehouse j will be installed under efficiency level $a\,\%$, 0 otherwise

3.3 Introduction through Supply Chain Network Design (SCDN) problem

In order to demostrate the applicability of the B&E algorithm, an introduction to the proposed mathematical programming algorithm is provided, via an application of a simple SCDN problem.

In Fig. 3, the network of the supply chain is presented. In the supply chain network proposed here it is assumed that only a single product is manufactured, stored and transported throughout the channels of the network. The application of B&E algorithm is intensively demonstrated in a simple SCDN model, so that it can be analytically described, however, the algorithm can be applied in any SCDN formulation independently of whether it is theoretical or real.

The supply chain network that is examined in this paper, consists of three nodes; the plants, the warehouses and customers. In the first link of the supply chain, the plants which manufacture the product for the customers, are assumed to be already located. These produced quantities are subjected to certain constraints. Once the products are produced, they are eventually stored in warehouses (the location of which is to be determined). From the warehouses, the quantities are delivered to the final link of the chain (the customer), according to corresponding demand.

In Fig. 3, the binary variables that correspond to the supply chains arcs (connections) and nodes (warehouses) are shown. The present model can be extended in order to take into account, more than one products, while there are many stages and echelons, the problem is extended to a multi-product, multi-stage multi-echelon SCDN problem.

The levels of decision the supply chain may provide are the following:

a) The quantities produced, transported and stored
b) Capacity of the installed facilities and their location.

3.4 SCDN model

Due to the presence of binary variables, the following is a Mixed Integer Linear Programming (MILP) model with objective function (4) and constraints (5)–(15) and models a typical supply chain. Objective function (4) represent overall Total Cost which consists of the production cost (a), variable transportation cost (b), fixed transportation cost (c) among nodes and finally the installation cost (d). The last term represents penalty cost of uncovered demand.

$$P_0 : \min \ TC \ = \overbrace{\sum_i c_i^P \cdot P_i}^{(a)} + \overbrace{\sum_i \sum_j c_{ij}^V \cdot Q_{ij}}^{(b)} + \overbrace{\sum_i \sum_j c_{ij}^F \cdot X_{ij}}^{(c)} + \overbrace{\sum_j \sum_k c_{jk}^V \cdot Q_{jk}}^{(b)}$$

$$+ \overbrace{\sum_j \sum_k c_{jk}^F \cdot X_{jk}}^{(c)} + \overbrace{\sum_j F_j \cdot Y_j}^{(d)} + \sum_k c_k^{PEN} \cdot g_k \tag{4}$$

s.t.

$$P_i = \sum_j Q_{ij}, \quad \forall i \tag{5}$$

$$P_i \le P_i^U, \quad \forall i \tag{6}$$

$$P_i \ge P_i^L, \quad \forall i \tag{7}$$

$$\sum_i Q_{ij} = \sum_k Q_{jk}, \quad \forall j \tag{8}$$

$$Q_{ij} \le Q_{ij}^U \cdot X_{ij}, \quad \forall i, j \tag{9}$$

$$Q_{jk} \le Q_{jk}^U \cdot X_{jk}, \quad \forall j, k \tag{10}$$

$$X_{ij} \le Y_j, \quad \forall i, j \tag{11}$$

$$X_{jk} \le Y_j, \quad \forall j, k \tag{12}$$

$$W_j \ge \beta_j \cdot \left(\sum_i Q_{ij} + I_j \right), \quad \forall j \tag{13}$$

$$W_j \le W_j^U \cdot Y_j, \quad \forall j \tag{14}$$

$$\sum_j Q_{jk} + g_k = d_k, \quad \forall k \tag{15}$$

$$Y_j, X_{ij}, X_{jk} \in \{0, 1\}$$

$$P_i, Q_{ij}, Q_{jk}, W_j, g_k \ge 0, \quad \forall i, j, k \tag{16}$$

In the above problem, constraints (6), (7) suggest that the quantities produced in plant i should not exceed and upper (P_i^U) and lower (P_i^L) bound production correspondingly. Constraint (5) is a mass balance constraint representing that product flow from warehouse j to customer k should be equal. Constraint (15) models the quantity that ends to the final node, the customer, and is assumed to cover demand of customer k. An additional variable is added in order to provide the magnitude of any shortfalls in demand. The introduction of slack variable g_k in

constraint (15), guarantees that the problem is not infeasible, in case that the demand is high and cannot be covered by supply. The following term $\sum_k c_k^{PEN} \cdot g_k$ is introduced in objective function (4) (Erdirik-Dogan and Grossmann 2008).

Besides constraints that regard to continuous variables, there are also logical constraints that are introduced in order to model the logical conditions. Constraints (9) and (10) suggest that the transported quantities from plant i to warehouse j and from warehouse j to customer k are upper bounded if—f the corresponding connection exists. Constraints (11) and (12) are introduced for the supply chain design network. In the two constraints is stated that if warehouse j is installed then the corresponding connection from warehouse j to plant i and from warehouse j to customer k exists. Constraint (13) states that warehouse's capacity should be more than product's quantities that will be transported to warehouse j plus the inventory level stored at warehouse j. The aforementioned quantities are multiplied by coefficient β_j expressing the amount of warehousing capacity required to hold a unit amount of the examined product at warehouse j. Constraint (14) is introduced to model the upper bound of warehouse capacity j.

Non-negativity constraints are imposed to production, transportation and warehousing variables as seen in (16).

4 B&E formulation

In this section the B&E algorithm will be deployed on the SCDN model that was previously described. Based on this approach, each node acts as an unit being described by inputs and outputs. The efficiency of each unit is extracted based on their level of inputs and outputs. Thus, the incurring efficiency can be measured with DEA. The data that will be fed to DEA are acquired by solving the initial problem, while the inputs and outputs are a-priori determined. In Fig. 4, the inputs and outputs of the warehouse facility is provided. If it assumed that a warehouse was a branch of an enterprise, then the most productive one would be the one that would minimize its operational cost and would provide more services.

Integrating DEA model into the SCND model, non-linearity terms will arise that may lead to local optima. In order to obtain global optima and to maintain the linearity of the model and extract the efficiency of each potential solution, B&E algorithm is applied.

4.1 Data

The data used for the efficiency extraction are generally provided in advance after statistical, qualitative or techno economic analysis. As mentioned in the previous section and can be seen in B&E flowchart, DEA technique is applied for the caclulation of DMUs efficiency. Here DMUs are considered to be the potentially installed warehouses. The data that are provided to DEA technique, are not externally provided, but come within the solution of the problem. The only parameter that should be predetermined is the inputs and outputs that the DM considers that capture the productivity of each warehouse.

In this work, an output oriented DEA model is applied in order to extract the efficiency of each DMU (warehouse) with inputs and outputs as seen in Table 1. The choice of these data to serve as inputs and outputs are done on the basis of selecting the warehouses that can provide the maximum services at the minimum cost. In Table 1 the inputs and the outputs that will be provided to DEA technique are presented. The inputs of the study are: (a) the cost which corresponds to the transportation of quantities from plant to warehouse $\left(C_j^{1,V} \right)$ and
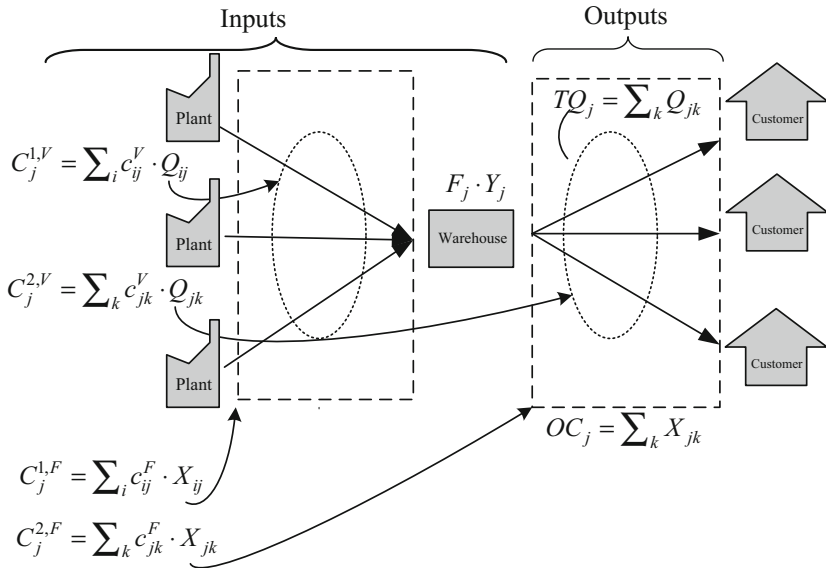
Fig. 4 Inputs and outputs of warehouse node

Table 1 Inputs and Outputs of the proposed supply chain

Inputs	
	$C_j^{1,V} = \sum_i c_{ij}^V \cdot Q_{ij}^*$
	$C_j^{2,V} = \sum_k c_{jk}^V \cdot Q_{jk}^*$
	$C_j^{1,F} = \sum_i c_{ij}^F \cdot X_{ij}^*$
	$C_j^{2,F} = \sum_k c_{jk}^F \cdot X_{jk}^*$
	$C_j^{IN} = \begin{cases} F_j \cdot Y_j^*, & \gamma = 1 \\ F_j \cdot \xi_j^*, & \gamma > 1 \end{cases}$
Output	$OC_j = \sum_k X_{jk}^*$
	$TQ = \sum_k Q_{jk}^*$

from warehouse to customer $\left(C_j^{2,V}\right)$, (b) the fixed transportation cost for routes done from plant to warehouse $\left(C_j^{1,F}\right)$ and from warehouse to customer $\left(C_j^{2,F}\right)$ and c) the installation cost of warehouse j $\left(C_j^{IN}\right)$. The outputs of the study are depicted in order to capture the magnitude of service level each warehouse can provide. Thus as outputs are considered (a) the total quantity that a warehouse can send to each customer (TQ_j) and (b) the out coming connections, that is, how many customers a warehouse is connected to and therefore can serve (OC_j). Thus the higher the efficiency the highest the service level that each warehouse can provide to each customer.

4.2 Measuring DMUs efficiency

As in this model, the efficiency is derived through solutions of a MILP model, inputs and outputs must be selected so that the efficiency of each warehouse is captured. As it can be

seen from Table 1, incoming quantities have not been taken into account because due to the mass balance constraint, incoming quantities are equal to out coming quantities of a node.

In the following LP model, output oriented DEA model is presented. The data provided for the evaluation of efficiency of each warehouse are the solutions from the initial MILP model P_0.

$$LP \max \varphi$$

$$s.t.$$

$$\sum_j \lambda_j \cdot C_j^{1,V} \leq C_o^{1,V}$$

$$\sum_j \lambda_j \cdot C_j^{2,V} \leq C_o^{2,V}$$

$$\sum_j \lambda_j \cdot C_j^{1,F} \leq C_o^{1,F}$$

$$\sum_j \lambda_j \cdot C_j^{2,F} \leq C_o^{2,F}$$

$$\sum_j \lambda_j \cdot F_j \leq F_o$$

$$\sum_j \lambda_j \cdot TQ_j \geq TQ_o \cdot \varphi$$

$$\sum_j \lambda_j \cdot OC_j \geq OC_o \cdot \varphi$$

$$\lambda_j \geq 0, \quad \forall j$$

$$\varphi \text{ free} \tag{17}$$

The extraction of each DMU's efficiency is preformed after solving the LP model (17) for each of the examined DMUs. In order to provide conclusions for the efficiency of each DMU, as φ is a free variable and can obtain any value, Technical Efficiency (*TE*) index is used. This index is defined as the reciprocal of φ (18) and is bounded in the range [0, 1].

$$TE = 1/\varphi \tag{18}$$

The next step in B&E algorithm is to add those DMUs with $TE = 1$ in J^γ set, where all efficient solutions are stored. In order to investigate which DMUs hold an efficiency of 1, the next binary variable is introduced:

$$\xi_j = \begin{cases} 1, & TE_j \geq \alpha \\ 0, & o/w \end{cases} \quad \forall j \tag{19}$$

As it can be seen in (19), binary variable gets the value of 1, if *TE* is greater than or equal to an efficiency threshold. Initially this value is 1, but if the problem yields infeasibility then this value is reduced. Binary variable is triggered when TE of a warehouse is greater or equal to a, using the following constraint.

$$TE_j \geq a \cdot \xi_j, \quad \forall j \tag{20}$$

The set of efficient solutions at each iteration of the algorithm is defined as follows:

$$J^\gamma = \{ORD(j) : TE_j \geq a \cdot \xi_j\} \tag{21}$$

Assuming that N is the set of all non-zero solutions after ε iterations of the algorithm, then DMUs with technical efficiency less than a will be included in the following set:

$$\bar{E} = N/J^{\gamma} \tag{22}$$

In the previous context, the set of inefficient solutions is defined as the set subtraction of the set of all solutions after ε iterations of the algorithm minus the set of inefficient solutions.

4.3 Solving problem P_0^{γ}

In order to design the supply chain network, using binary variables for efficient solutions, constraints that are introduced for the design of the network (logical) are reformulated. A DMU is selected if—f satisfies constraint (20). Thus, the binary variable that corresponds to the selection of warehouses $\left(Y_j\right)$ is replaced with binary variable ξ_j. Constraints (11)–(14) are reformulated as follows:

$$X_{ij} \leq \xi_j, \quad \forall i, j \tag{23}$$

$$X_{jk} \leq \xi_j, \quad \forall j, k \tag{24}$$

$$W_j \geq \beta_j \cdot \left(\sum_i Q_{ij} + \xi_j \cdot I_j \right), \quad \forall j \tag{25}$$

$$W_j \leq W_j^U \cdot \xi_j, \quad \forall j \tag{26}$$

In constraints (23), (24) and (26), binary variable Y_j is replaced with variable ξ_j and as it can be seen (25) has been modified introducing it to the inventory parameter in order to avoid any infeasibilities that may occur. As Q_{ij} is controlled by binary variables that correspond to connections, namely X_{ij}, through constraint (9) and binary variables are connected with constraint (11), such that if a warehouse is installed then the corresponding connection is created, thus Q_{ij} has an indirect relation with the variable that correspond to quantities transferred. For this reason, binary variable is also introduced to the lower bound and especially on the inventory parameter which is not controlled by any logical or continuous variable.

Except for the supply chain network and logical constraints, the objective function of the problem is modified as follows:

$$\min TC = \sum_i c_i^P \cdot P_i + \sum_i \sum_j c_{ij}^V \cdot Q_{ij} + \sum_i \sum_j c_{ij}^F \cdot X_{ij} + \sum_j \sum_k c_{jk}^V \cdot Q_{jk}$$
$$+ \sum_j \sum_k c_{jk}^F \cdot X_{jk} + \sum_j F_j \cdot \xi_j + \sum_k c_k^{PEN} \cdot g_k \tag{27}$$

In objective function (27), the installation cost is now computed upon the product of installation cost and binary variable ξ. The procedure is graphically represented in the following figure (Fig. 5). In Fig. 5, the reduction of cost is performed through consecutive efficiency cuts. The proposed B&E algorithm is not the logic cut type technique. At the beginning the value of the objective function is higher than in any of the iterations of the algorithm. The basis for this argument is that the binary variable that corresponds to supply chain network design is replaced by a binary variable that measures efficiency of solutions and is subjected to an additional constraint, the efficiency cut. Thus in each iteration through the selection of reduced number of facilities, the overall cost is also minimized. The thick black line indicates the point where the data are insufficient for the application of DEA technique.

The new formulated problem P_0^{γ} is described by objective function (27) and constraints (5)–(10), (23)–(26), (15) and (16). The initial value of the objective function will be greater
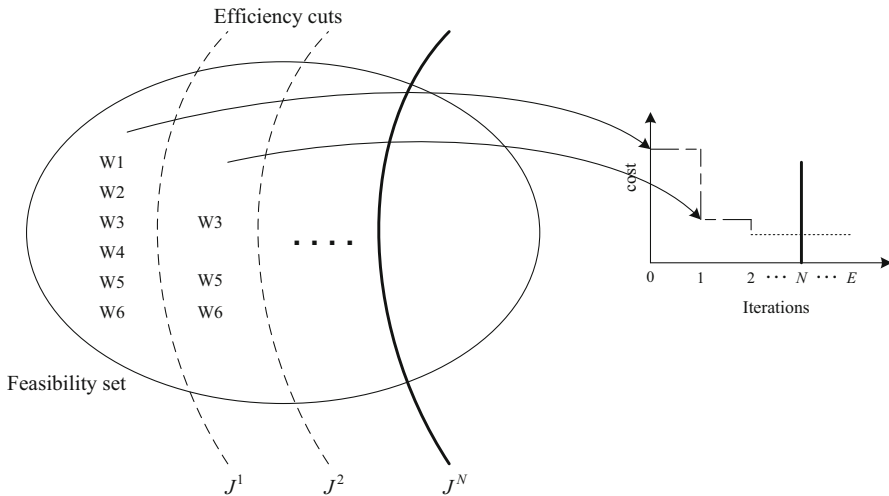
Fig. 5 Minimization of cost through iterative efficiency cuts

than the value of any consecutive iteration. This is attributed to the reduction of solutions (by zeroing) for certain indices as stated in (28).

$$\sum_i \sum_{j \in \bar{E}} c_{ij}^V \cdot Q_{ij} + \sum_i \sum_{j \in \bar{E}} c_{ij}^F \cdot X_{ij} + \sum_{j \in \bar{E}} \sum_k c_{jk}^V \cdot Q_{jk}$$

$$+ \sum_{j \in \bar{E}} \sum_k c_{jk}^F \cdot X_{jk} + \sum_{j \in \bar{E}} F_j \cdot \xi_j = 0 \qquad (28)$$

For non-efficient solutions, when $\xi_{j \in \bar{E}} = 0$ the decisions that concern to the installation of a facility are also 0. The aforementioned state is derived from the following inequalities.

$$X_{ij} \leq 0 \Leftrightarrow$$
$$X_{ij} = 0 \quad \forall i, \quad j \in \bar{E} \qquad (29)$$
$$X_{jk} \leq 0 \Leftrightarrow$$
$$X_{jk} = 0 \quad \forall j \in \bar{E}, \quad k \qquad (30)$$

Finally, the variables that regard to the quantities become also 0 when a solution is not efficient. Given the fact that the system should be in a balanced form, quantities are channeled through the selected facilities, which are the efficient ones. The procedure stops at the point where objective function value has no improvement in two consecutive iterations or when the number of DMUs in each iteration is less than the minimum amount of DMUs needed for the functionality of DEA.

5 Results

5.1 Description of case study

In this section, the applicability of the model is presented through a case study. The proposed B&E algorithm can be applied in any supply chain, reducing overall cost. In the supply chain

Table 2 Data of the proposed case study

Description	Parameter	Value
Upper bound of produced quantities of plant i	P_i^U	8000
Lower bound of produced quantities of plant i	P_i^L	5000
Upper bound of transported quantities from plant i to warehouse j	Q_{ij}^U	500
Upper bound of transported quantities warehouse j to customer k	Q_{jk}^U	500
Unit production cost at plant i	c_i^P	$U[0, 200]$
Fixed route cost from plant i to warehouse j	c_{ij}^F	$U[50, 100]$
Unit transportation cost from plant i to warehouse j	c_{ij}^V	$U[0, 20]$
Fixed route cost from warehouse j to customer k	c_{jk}^F	$U[50, 100]$
Unit transportation cost from warehouse j to customer k	c_{ij}^V	$U[0, 20]$
Penalty cost assigned to uncovered demand of customer k	c_k^{PEN}	10^6
Fixed installation cost of warehouse j	F_j	$U[50, 100] \times 10^3$
Coefficient relating quantity at capacity warehouse j	β_j	$U[0.0001, 0.01]$
Inventory kept at warehouse j	I_j	$U[0, 100]$
Demand of customer k	d_k	$U[500, 1000] \times 10^3$

network presented in the previous section, a single homogeneous product is manufactured and transported throughout the links of the supply chain.

In the following table (Table 2) the data of the case study are presented. Simulated data have been used drawn from uniform distribution which is considered as fair due to the fact that all observation have the same probability of occurrence. The unit production and transportation cost is measured in relative money units/units (r.m.u/u) while parameters with respect to capacity and demand are measured in units (u).

5.2 Numerical results

In this section the application of B&E is demonstrated through the results of the case study. The case study was model and solved in GAMS optimization software using CPLEX as LP and MIP solver on an Intel Pentium, 2.3 GHz, 2 GB RAM laptop computer. Even if the instance is medium to large with $|I| = 50$ plants, $|J| = 50$ plants and $|K| = 50$ customers, the problem was solved in 10 CPU seconds to optimality.

5.2.1 Initialization

First MILP model P_0 is solved to optimality providing decisions about the number of warehouses. A summary of the data (inputs and outputs) that are derived after solving MILP model P_0 are shown in Fig. 6. Regarding the outputs, the average value of the total quantity (TQ_j) (Fig. 6a) that is sent from the selected warehouses to the customers is 5000 r.m.u while the average value of the total outgoing connections is 10 (OC_j) (Fig. 6b). Regarding the inputs, the average value of the variable transportation cost from plant to warehouse (Fig. 6c) ($C_j^{1,V}$)

(a)

(b)

(c)

(d)

(e)

(f)
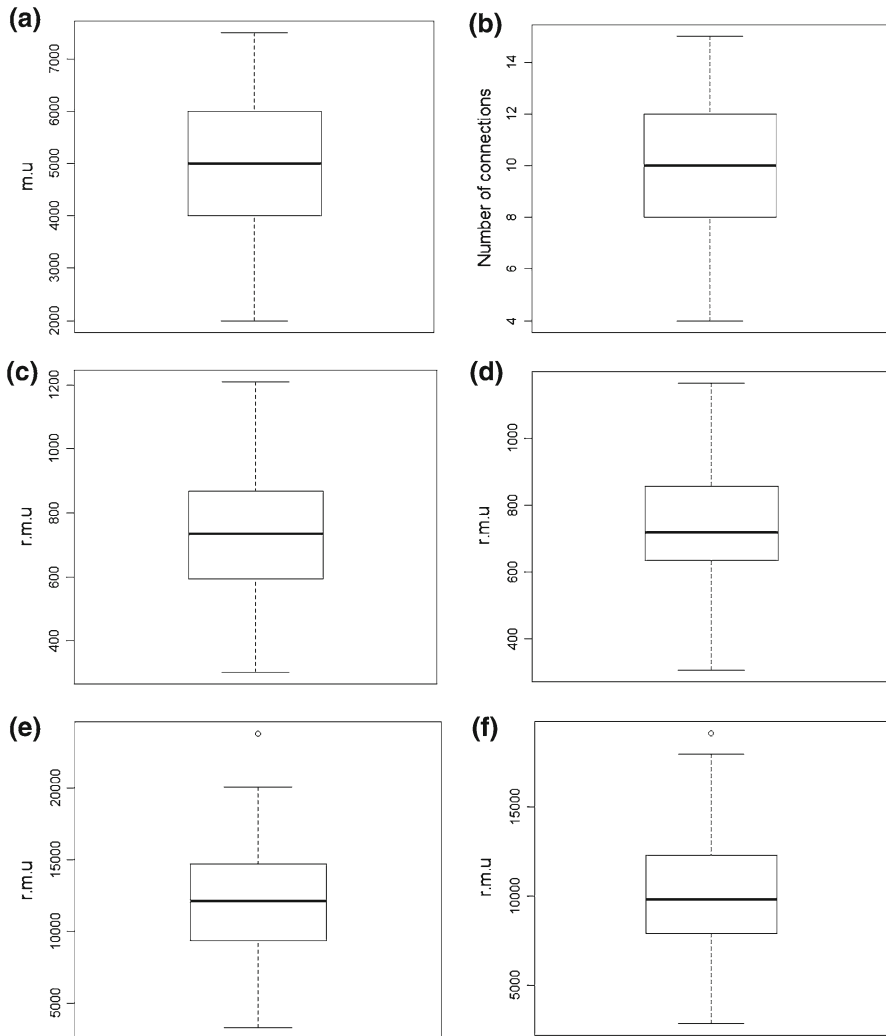
Fig. 6 *Boxplot* for outputs: **a** total quantity **b** outgoing connections and inputs: **c** variable transportation cost from plant to warehouse, **d** variable transportation cost from warehouse to customer, **e** fixed transportation cost from plant to warehouse, **f** fixed transportation cost from warehouse to customer, after solving P_0

is almost 800 r.m.u and almost the same conclusions can be extracted for variable transportation cost from warehouse to customer ($C_j^{2,V}$) (Fig. 6d). Finally, the average values of fixed transportation cost from plant to warehouse ($C_j^{1,F}$) (Fig. 6e) and fixed transportation cost from warehouse to customer ($C_j^{2,F}$) (Fig. 6f) are 12,500 and 10,000 r.m.u correspondingly.

After solving model P_0 50 warehouses are selected providing a total cost of 27,352,117.18 r.m.u. The number of warehouses (viz. the decision variables that correspond to the installation of a warehouse) is larger than the minimum DMU requirements of $T = \max\{5 \cdot 2, 3 \cdot (5+2)\} = 21$ needed in order for the DEA results to have validity, so the procedure continues.

Table 3 Technical efficiency for all selected DMUs (warehouses)

Warehouse	Technical efficiency	Warehouse	Technical efficiency
1	**1**	26	0.845
2	0.978	**27**	**1**
3	0.788	28	0.992
4	0.937	29	0.914
5	**1**	30	0.957
6	**1**	**31**	**1**
7	**1**	32	0.962
8	0.912	33	0.923
9	0.949	**34**	**1**
10	0.957	35	0.999
11	0.983	36	0.962
12	0.967	37	0.937
13	0.874	38	0.743
14	0.954	39	0.898
15	0.896	40	0.972
16	0.965	41	0.998
17	**1**	42	0.954
18	0.999	**43**	**1**
19	0.989	**44**	**1**
20	0.941	**45**	**1**
21	0.945	**46**	**1**
22	0.932	47	0.967
23	**1**	48	0.914
24	0.965	49	0.997
25	0.950	50	0.988

Bold values indicate the point at which the algorithm terminates and the optimal value

5.2.2 DEA

Each DMU's (warehouse) efficiency is evaluated through LP model (17), with the use of pre-formulated inputs and outputs. The inputs and the outputs (as have been defined in Table 1) are calculated from the optimal solutions of $P_0(Q_{ij}^*, Q_{jk}^*, X_{ij}^*, X_{jk}^*, Y_j^*)$ as can be seen in Fig. 4. Solving LP model (17), the efficiency of each warehouse is extracted and the initial set of solutions is the following:

$$N = \{1, 2, 3, \ldots, 50\} \tag{31}$$

The Technical Efficiency of each of the selected DMUs (warehouses) is presented in the next table (Table 3) and the efficient warehouses are shown with bold. As it can be seen in Table 3, the majority of DMUs selected has a technical efficiency of 1 and are most likely to remain in N. Yet, as the problem has indirect two objectives, namely the maximization of efficiency of the final supply chain network and the minimization of cost, if a warehouse may have a large cost comparing to the other, it may be excluded.

Table 4 Results from application of B&E algorithm to larger instances

Instance	Objective (r.m.u)	No. facilities	Iterations
$\|I\| = 100$ $\|J\| = 100$	51,969,399.72	100	P_0
$\|K\| = 100$	**50,682,282.31**	**23**	$\gamma = 1$
$\|I\| = 150$ $\|J\| = 150$	76,899,574.02	150	P_0
$\|K\| = 150$	**73,652,608.31**	**19**	$\gamma = 1$
$\|I\| = 200$ $\|J\| = 200$	102,015,663.39	200	P_0
$\|K\| = 200$	**95,210,734.52**	**35**	$\gamma = 1$

Bold values indicate the point at which the algorithm terminates and the optimal value

5.2.3 Efficiency cuts

As aim of the proposed algorithm is to reduce the number of facilities in order to reduce overall cost but at the same time increase the service level of the customers, the efficiency cuts are applied to the second MILP model $P_0^{\gamma=1}$. Model $P_0^{\gamma=1}$ presented in Sect. 4.3 ($\gamma = 1$ denotes first iteration) receives the initial solutions from model P_0. After solving $P_0^{\gamma=1}$ the number of selected facilities has dropped from 50 to 13 which are the following: 1, 5, 6, 7, 17, 23, 27, 31, 34, 43, 44, 45, and 46. It must be noticed that from all DMUs, the model found and excluded those with Technical Efficiency equal to 1. The problem was solved to optimality and without any infeasibility occurrence. The results of model $P_0^{\gamma=1}$ can be easily confirmed from Table 3 as all the selected facilities have a Technical Efficiency equal to 1; if any infeasibility instance would occur, based on the flowchart of B&E algorithm (Fig. 2), wider Technical Efficiency bounds would be used. The cost after the first iteration of B&E algorithm is 27,059,894.34 r.m.u. As the number of facilities that are selected from $P_0^{\gamma=1}$ is less than the threshold set for DEA functionality (terminating criterion $\left|N^{\gamma=1}\right| < 21$), the algorithm stops. The optimal solutions are the one derived from $P_0^{\gamma=1}$. The number of selected facilities that are eventually selected are 13 (from 50 that have been selected after solving P_0) leading to cost reduction from 27,352,117.18 r.m.u. to 27,059,894.34 r.m.u.

5.2.4 Larger instances

In this section the application of the algorithm to larger instances is demonstrated. In Table 4 the instance characteristics (size of the problem), the results (objective function value and number of selected facilities) and the iterations for final solutions are presented.

As it can be seen from Table 4, in the proposed problems initially all the warehouses are selected. The initial cost is considered as an upper bound for the iterations of B&E algorithm and at each iteration overall cost decreases. In all instances, the algorithm terminated after the first iteration ($\gamma = 1$) either because the number of facilities is less than the threshold or because after selecting the efficient warehouses of first iteration, the MILP model P_0^{γ} was infeasible.

6 Conclusions

Efficiency measurement is applied to firms or to units when the data (inputs and outputs) are known a priori. Using DEA technique the efficiency is extracted for each DMU under

examination. Considering each solution as a DMU, it is possible to evaluate the solutions efficiency using endogenous data. When modeling the supply chain network design, single sourcing constraints are used in order to provide better results in terms of minimizing the overall cost (objective function value) and economies of scale through facilities concentration. However, this type of constraint is a "hard" constraint and may eventually leads to infeasibility.

In this paper a Branch and Efficiency (B&E) algorithm has been deployed for the optimal design of supply chain networks. The algorithm integrates DEA technique in the design of supply chain network, through an iterative process and takes into advantage the strengths of DEA to provide efficiency scores for multiple inputs and outputs. Up to now, DEA technique has been applied in order to measure performance of a system based on exogenous data. The proposed B&E algorithm takes into the data provided within the optimization procedure in each iteration. Based on this approach, the problem is initially solved providing non-zero solutions for the initialization of the algorithm. The initialized values are fed to DEA to measure the efficiency of the unit under examination (in this case warehouses). Through the addition of efficiency cuts, the algorithm selects only the efficient solutions, which minimize overall cost (or maximize profit/revenue).

For illustrative purposes, a two-stage supply chain model is proposed. Production units form the first link of this supply chain network while customer's site form the last link, both of which are assumed to be already installed. The model is designed in such a way so as to provide decisions about the potential installation of warehouses. Setting a-priori the characteristics that could capture the efficiency of each facility (warehouse), the data are provided to the algorithm. The proposed algorithm has two characteristics; heuristic and evaluative. The first comes from the fact that even if initial solutions are provided during the initialization process, the algorithm searches among the "efficient neighborhoods" and would accept or exclude solutions based on efficiency cuts, after the evaluation process. The algorithm is generic and can be applied in any type of supply chain, regardless the level of complexity, making it a valuable tool for long and short term managerial decisions.

Appendix—an illustrative example

The proposed methodology is shown through a toy illustrative example so that it can be reproducible by researchers in supply chain network design and practitioners. Assuming that there are 5 plants ($i = 1, \ldots, 5$), 20 possible warehouses ($j = 1, \ldots, 20$) to be installed and 5 demand zones (customers) ($k = 1, \ldots, 5$). Analytically the model P_0 for the examined instance is shown in mathematical formulation (32)–(44). The production cost for each plant i is presented in Table 5. The data regarding fixed and variable transportation cost ($c_{ij}^F, c_{jk}^F, c_{ij}^V, c_{jk}^V$) are given in Tables 6 and 7. Warehouse installation cost (F_j), capacity coefficient (a_j) and holding inventory (I_j) are given in Table 8. Upper and lower

Table 5 Production cost for each plant i

i	c_i^P
1	34.35
2	168.65
3	110.08
4	60.23
5	58.44

Table 6 Fixed and variable transportation cost from plant i to warehouse j

j	c_{ij}^F					c_{ij}^V				
	i									
	1	2	3	4	5	1	2	3	4	5
1	61.203	91.545	59.105	51.707	58.708	4.515	11.931	4.025	1.742	12.033
2	67.492	61.541	82.286	79.257	66.532	7.922	10.229	5.943	10.808	0.540
3	92.814	83.287	78.037	81.061	65.845	5.520	0.901	3.945	2.537	3.922
4	53.356	88.793	88.498	69.468	66.104	3.047	15.662	4.927	14.680	19.014
5	75.011	65.183	64.890	67.936	98.199	18.726	18.915	12.930	2.265	6.711
6	99.906	55.525	83.055	62.152	99.680	8.453	11.929	14.699	9.767	11.885
7	78.937	75.119	87.791	62.321	68.495	2.693	12.147	1.709	15.912	5.184
8	99.557	58.009	81.372	56.525	68.644	7.721	7.250	3.007	9.841	12.813
9	88.113	93.623	64.193	96.672	88.599	7.493	11.881	8.684	10.671	3.105
10	56.535	63.256	54.321	68.997	69.834	5.370	13.597	3.739	0.212	9.200
11	81.986	64.291	55.126	89.170	95.655	18.967	10.132	13.854	10.877	7.867
12	57.976	79.698	82.063	65.002	55.979	3.779	3.185	15.259	9.023	16.109
13	62.504	86.136	77.265	56.274	86.774	5.950	13.138	3.096	19.507	10.820
14	83.446	81.412	51.576	87.444	52.771	1.491	10.478	7.788	3.677	7.814
15	71.768	73.190	89.618	53.462	78.815	8.027	2.488	13.909	3.271	11.156
16	67.985	70.665	53.638	60.101	52.570	2.034	19.734	16.916	0.493	18.655
17	67.572	55.885	58.783	50.253	50.300	7.678	4.562	12.254	3.556	6.975
18	56.575	65.711	76.282	63.481	70.061	6.482	13.513	19.519	1.226	0.166
19	57.505	52.328	87.510	74.993	75.994	3.843	15.536	0.538	0.333	18.977
20	79.456	66.928	58.906	57.564	81.444	2.247	18.649	3.749	16.713	11.438

bound for produced quantities (P_i^L, P_i^U) are set to be 5000 and 8000 correspondingly. The upper bound of flow from plant to warehouse and from warehouse to customer is set to be $Q_{ij}^U = Q_{jk}^U = 500$. The upper bound of the warehouse capacity (W_j^U) is assumed to be 1000. Demand for each customer k is given in Table 9. In this instance algorithm terminates if the number of possible facilities is less than 10 (Fig. 2, $|N^\gamma| < 10$) or the P_0^γ yields infeasibility in γ iteration for any a value in the range of [0.7, 1]. The penalty cost for uncovered demand is assumed to be 10^6.

Table 7 Fixed and variable transportation cost from warehouse j to customer k

j	c_{jk}^F					c_{jk}^V				
	k									
	1	2	3	4	5	1	2	3	4	5
1	66.68	99.19	88.32	55.5	99.74	11.86	13.68	3.175	6.636	6.317
2	79.02	58.32	82.17	67.22	95.62	10.4	7.276	3.355	13.66	10.11
3	95	50.81	68.43	83.22	79.67	11.52	14.4	13.67	0.397	16.8
4	51.73	92.09	96.6	75.4	64.98	14.2	3.11	12.21	13.23	3.887
5	74.83	52.25	88.69	76.65	87.34	7.27	12.48	14.63	8.279	3.15
6	86	81.58	55.75	98.56	85.34	0.25	0.203	19.04	19.53	19.33
7	99.31	92.74	81.07	85.07	85.04	17.13	2.832	0.995	11.06	3.681
8	89.54	80.51	52.72	74.26	52.63	19.88	16.18	6.124	1.748	8.61
9	84.93	59.74	61.3	90.68	99.59	6.994	2.347	11.72	8.911	8.246
10	87.53	85.92	50.03	63.19	91.19	18.29	4.276	4.483	10.85	12.62
11	90.98	93.02	60.63	72.84	51.92	6.549	2.976	18.58	5.021	1.252
12	66.15	71.99	65.77	56.74	90.55	6.203	0.804	16.42	4.619	8.201
13	70.84	57.09	73.28	64.15	94.78	6.052	8.898	14.32	11.86	2.624
14	53.22	70.73	67.08	73.41	82.13	3.225	6.313	11.44	5.374	0.728
15	82.18	66.88	55.04	95.29	60.87	13.73	13.49	6.643	15.2	3.536
16	95.94	72.59	54.5	68.71	70.75	13.65	13.46	16.62	10.3	5.661
17	70.21	55.58	87.56	90.17	51.18	11.11	8.28	1.468	16.12	6.654
18	74.04	63.93	95.08	50.88	84.05	1.694	11.44	0.441	14.84	18.1
19	97.55	95.01	94.94	93.72	69.55	11.22	9.457	14.35	10.26	17.74
20	75.21	91.56	80.11	54.11	78.89	15.43	2.802	5.29	13.65	8.996

$$P_0 : \min TC = \sum_{i=1}^{5} c_i^P \cdot P_i + \sum_{i=1}^{5} \sum_{j=1}^{20} c_{ij}^V \cdot Q_{ij} + \sum_{i=1}^{5} \sum_{j=1}^{20} c_{ij}^F \cdot X_{ij} + \sum_{j=1}^{20} \sum_{k=1}^{5} c_{jk}^V \cdot Q_{jk}$$

$$+ \sum_{j=1}^{20} \sum_{k=1}^{5} c_{jk}^F \cdot X_{jk} + \sum_{j=1}^{5} F_j \cdot Y_j + \sum_{k=1}^{5} c_k^{PEN} \cdot g_k$$

$$s.t. \tag{32}$$

$$P_i = \sum_{j=1}^{20} Q_{ij}, \quad i = 1, \ldots, 5 \tag{33}$$

$$P_i \leq P_i^U, \quad i = 1, \ldots, 5 \tag{34}$$

$$P_i \geq P_i^L, \quad i = 1, \ldots, 5 \tag{35}$$

$$\sum_{i=1}^{5} Q_{ij} = \sum_{k=1}^{5} Q_{jk}, \quad j = 1, \ldots, 20 \tag{36}$$

$$Q_{ij} \leq Q_{ij}^U \cdot X_{ij}, \quad i = 1, \ldots, 5, \quad j = 1, \ldots, 20 \tag{37}$$

$$Q_{jk} \leq Q_{kj}^U \cdot X_{jk}, \quad j = 1, \ldots, 20, \quad k = 1, \ldots, 5 \tag{38}$$

Table 8 Installation cost, capacity coefficient and holding inventory of warehouse j

j	F_j	β_j	I_j
1	98,276.24	0.009795	70.49
2	97,894.74	0.003893	41.59
3	94,961.33	0.007867	54.98
4	66,377.28	0.009663	34.50
5	72,854.95	0.009541	69.96
6	79,809.01	0.003303	93.35
7	93,931.18	0.003925	46.93
8	58,533.63	0.002933	21.36
9	81,680.11	0.002566	51.08
10	88,579.48	0.007581	36.57
11	78,472.3	0.003431	93.54
12	51,383.89	0.007826	6.80
13	90,549.69	0.006557	50.39
14	63,946.48	0.003619	39.24
15	71,667.46	0.007666	20.49
16	66,813.11	0.00107	52.95
17	79,432.13	0.008799	58.91
18	78,719.58	0.001136	34.58
19	77,171.07	0.004855	25.29
20	78,908.08	0.004228	54.77

Table 9 Demand for each customer k

k	d_k
1	773,740.9
2	529,133.4
3	688,861.1
4	987,033.5
5	689,909.4

$$X_{ij} \leq Y_j, \quad i = 1, \ldots, 5, \quad j = 1, \ldots, 20 \tag{39}$$

$$X_{jk} \leq Y_j, \quad j = 1, \ldots, 20, \quad k = 1, \ldots, 5 \tag{40}$$

$$W_j \geq \beta_j \cdot \left(\sum_{i=1}^{5} Q_{ij} + I_j \right), \quad j = 1, \ldots, 20 \tag{41}$$

$$W_j \leq W_j^U \cdot Y_j, \quad j = 1, \ldots, 20 \tag{42}$$

$$\sum_{j=1}^{20} Q_{jk} + g_k = d_k, \quad k = 1, \ldots, 5 \tag{43}$$

$$Y_j, X_{ij}, X_{jk} \in \{0, 1\}$$

$$P_i, Q_{ij}, Q_{jk}, W_j \geq 0, \quad i = 1, \ldots, 5, \quad j = 1, \ldots, 20, \quad k = 1, \ldots, 5 \tag{44}$$

Table 10 Inputs and outputs after solving model P_0

j	Outputs		Inputs				
	Outgoing connections	Total quantity sent	Installation cost	Fixed transportation cost from plant to warehouse	Fixed transportation cost from warehouse to customer	Variable transportation cost from plant to warehouse	Variable transportation cost from warehouse to customer
1	3	1500	98,276.24	172.01	243.57	5141.10	8064.04
2	3	1500	97,894.74	210.36	236.10	8356.02	10,369.37
3	2	1000	94,961.33	149.13	178.22	2411.60	5960.84
4	2	1000	66,377.28	141.85	157.07	3987.18	3498.76
5	2	1000	72,854.95	166.13	162.17	4487.74	5210.13
6	2	1000	79,809.01	155.20	167.58	11,907.25	226.91
7	3	1500	93,931.18	235.22	258.86	4792.91	3753.73
8	2	1000	58,533.63	139.38	126.97	5128.57	3936.09
9	3	1500	81,680.11	246.42	244.25	11,835.05	8793.43
10	3	1500	88,579.48	179.85	199.14	4660.43	9802.90
11	2	1000	78,472.30	159.95	144.94	8999.28	2113.72
12	3	1500	51,383.89	202.68	194.88	7993.23	5813.02
13	2	1000	90,549.69	139.77	165.62	4523.16	4337.75
14	4	2000	63,946.48	305.07	279.50	11,730.01	7819.90
15	2	1000	71,667.46	126.65	115.91	2879.29	5089.09
16	2	1000	66,813.11	128.09	139.46	1263.24	7982.02
17	3	1500	79,432.13	156.44	194.32	7547.11	8201.16
18	2	1000	78,719.58	133.54	169.12	696.06	1067.45
19	3	1500	77,171.07	220.01	286.28	2356.68	15,466.53
20	2	1000	78,908.08	138.36	171.67	2998.17	4046.40

Solving model P_0 for the specific parameters presented above, the following optimal solutions are derived. The data in Table 10 represent the inputs and outputs that will be used in the following step. The outputs that are used are: (a) outgoing connections ($\sum_{k=1}^{5} X_{jk}^*$), and (b) total quantity sent ($\sum_{k=1}^{5} Q_{jk}^*$). Inputs are (a) installation cost ($F_j \cdot Y_j^*$), (b) fixed transportation cost from plant to warehouse ($\sum_{i=1}^{5} c_{ij}^F \cdot X_{ij}^*$), (c) fixed transportation cost from warehouse to customer ($\sum_{k=1}^{5} c_{jk}^F \cdot X_{jk}^*$), (d) variable transportation cost from plant to warehouse ($\sum_{i=1}^{5} c_{ij}^V \cdot Q_{ij}^*$), (e) variable transportation cost from warehouse to customer ($\sum_{k=1}^{5} c_{jk}^V \cdot Q_{jk}^*$). The cost that is derived is 3,971,290.09 r.m.u.

The DEA formulation that is used in order to assess the efficiency is presented below:

$$LP \max \varphi$$

$$s.t.$$

$$\sum_{j=1}^{20} \lambda_j \cdot \left(\sum_{i=1}^{5} c_{ij}^V \cdot Q_{ij}^* \right) \leq \sum_{i=1}^{5} c_{io}^V \cdot Q_{io}^*$$

Table 11 Efficiency scores per each warehouse j

j	$1/\varphi$	j	$1/\varphi$
1	1	11	1
2	0.814942	12	1
3	0.831777	13	0.857106
4	0.909887	14	1
5	0.792112	15	1
6	1	16	1
7	0.928171	17	1
8	1	18	1
9	0.784875	19	1
10	1	20	0.871074

$$\sum_{j=1}^{20} \lambda_j \cdot \left(\sum_{k=1}^{5} c_{jk}^V \cdot Q_{jk}^* \right) \leq \sum_{k=1}^{5} c_{jo}^V \cdot Q_{jo}^*$$

$$\sum_{j=1}^{20} \lambda_j \cdot \left(\sum_{i=1}^{5} c_{ij}^F \cdot X_{ij}^* \right) \leq \sum_{i=1}^{5} c_{io}^F \cdot X_{io}^*$$

$$\sum_{j=1}^{20} \lambda_j \cdot \left(\sum_{k=1}^{5} c_{jk}^F \cdot X_{jk}^* \right) \leq \sum_{k=1}^{5} c_{jo}^F \cdot X_{jo}^*$$

$$\sum_{j=1}^{20} \lambda_j \cdot \left(F_j \cdot Y_j^* \right) \leq F_o \cdot Y_o^*$$

$$\sum_{j=1}^{20} \lambda_j \cdot \left(\sum_{k=1}^{5} Q_{jk}^* \right) \geq \varphi \cdot \left(\sum_{k=1}^{5} Q_{jo}^* \right)$$

$$\sum_{j=1}^{20} \lambda_j \cdot \left(\sum_{k=1}^{5} X_{jk}^* \right) \geq \varphi \cdot \left(\sum_{k=1}^{5} X_{jo}^* \right)$$

$$\lambda_j \geq 0, \quad j = 1, \ldots, 20$$

$$\varphi \, free \tag{45}$$

After solving output oriented DEA model (45), the efficiency of warehouses are shown in Table 11.

Based on constraint (51), for $\alpha = 1$ facilities with efficiency score less than 1 are excluded, therefore the following MILP model ($P_0^{\gamma=1}$), (γ represents the iteration) is solved:

$$P_0^{\gamma=1} : \min TC = \sum_{i=1}^{5} c_i^P \cdot P_i + \sum_{i=1}^{5} \sum_{j=1}^{20} c_{ij}^V \cdot Q_{ij} + \sum_{i=1}^{5} \sum_{j=1}^{20} c_{ij}^F \cdot X_{ij} + \sum_{j=1}^{20} \sum_{k=1}^{5} c_{jk}^V \cdot Q_{jk}$$

$$+ \sum_{j=1}^{20} \sum_{k=1}^{5} c_{jk}^F \cdot X_{jk} + \sum_{j=1}^{20} F_j \cdot \xi_j + \sum_{k=1}^{5} c_k^{PEN} \cdot q_k \tag{46}$$

$$s.t.$$

$$(33)-(38), (43)$$

$$X_{ij} \leq \xi_j, \quad i = 1, \ldots, 5, \quad j = 1, \ldots, 20 \tag{47}$$

$$X_{jk} \leq \xi_j, \quad j = 1, \ldots, 20, \quad k = 1, \ldots, 5, \tag{48}$$

$$W_j \geq \beta_j \cdot \left(\sum_{i=1}^{5} Q_{ij} + \xi_j \cdot I_j \right), \quad j = 1, \ldots, 20 \tag{49}$$

$$W_j \leq W_j^U \cdot \xi_j, \quad j = 1, \ldots, 20 \tag{50}$$

$$TE_j \geq a \cdot \xi_j, \quad j = 1, \ldots, 20 \tag{51}$$

$$\xi_j, X_{ij}, X_{jk} \in \{0, 1\}$$

$$P_i, Q_{ij}, Q_{jk}, W_j \geq 0, \quad i = 1, \ldots, 5, \quad j = 1, \ldots, 20, \quad k = 1, \ldots, 5 \tag{52}$$

According to constraint (51) warehouses 2–5, 7, 9, 13 and 20 are not selected, thus the new table of data after solving the model $P_0^{\gamma=1}$ are presented in Table 12. The selected warehouses are 1, 6, 8, 10, 11, 12, 14, 15, 16, 17, 18 and 19. The total cost after solving model $P_0^{\gamma=1}$ is 3,433,156.64 r.m.u. The following set of selected warehouses, is constructed $J^1 = \{1, 6, 8, 10, 11, 12, 14, 15, 16, 17, 18, 19\}$.

The results (inputs and outputs) of $P_0^{\gamma=1}$ is shown in Table 11. The warehouses that are selected (J^1) constitute a subset of the set of initially selected facilities. The inputs and outputs of the facilities have changed as a result of some warehouses not being selected. Due to mass balance constraints, new connections are created and through these channels the customers' demand is satisfied. For this reason, just filtering the solutions derived from P_0, is wrong as when a warehouse is not selected the optimal solutions of the variables (quantities,

Table 12 Inputs and outputs after solving model $P_0^{\gamma=1}$

j	Outputs		Inputs				
	Outgoing connections	Total quantity sent	Installation cost	Fixed transportation cost from plant to warehouse	Fixed transportation cost from warehouse to customer	Variable transportation cost from plant to warehouse	Variable transportation cost from warehouse to customer
1	5	2500	98276.24	322.27	409.44	17123.27	20833.52
6	2	1000	79809.01	155.20	167.58	11907.25	226.91
8	4	2000	58533.63	307.58	260.11	15395.47	16331.99
10	4	2000	88579.48	249.69	290.33	9260.60	16113.47
11	4	2000	78472.30	304.24	308.75	21364.91	7898.38
12	4	2000	51383.89	284.74	285.43	15622.97	9913.30
14	5	2500	63946.48	356.65	346.58	15623.80	13540.50
15	5	2500	71667.46	366.85	360.26	19425.22	26298.70
16	4	2000	66813.11	234.29	307.99	19048.96	21536.72
17	5	2500	79432.13	282.79	354.70	17513.22	21815.43
18	4	2000	78719.58	255.83	283.93	10693.55	14209.49
19	4	2000	77171.07	272.34	381.22	10124.45	22642.17

Table 13 Efficiency scores for each warehouse j ($j \in J^1$)

j	$1/\varphi$	j	$1/\varphi$
1	0.935953	14	1
6	1	15	0.953885
8	1	16	0.9656
10	1	17	1
11	0.976109	18	1
12	1	19	1

connections etc) are re-adjusted due to mass balance constraints in order to satisfy customers' demand.

In order to further reduce the number of selected facilities and therefore reduce the cost of the supply chain network, the efficiency of warehouses derived after solving model $P_0^{\gamma=1}$ is calculated. The DEA model used is the following:

$$LP \max \varphi$$

$$s.t.$$

$$\sum_{j \in J^1} \lambda_j \cdot x_{in,j} \leq x_{in,o}, \qquad \forall in$$

$$\sum_{j \in J^1} \lambda_j \cdot y_{out,j} \leq \varphi \cdot y_{out,o}, \qquad \forall out$$

$$\lambda_j, \geq 0, \quad j \in J^1$$

$$\varphi \, free \tag{53}$$

The results for the efficiency scores are presented in Table 13. It can be seen that only 5 warehouses are now efficient. Model $P_0^{\gamma=2}$ is solved while the efficient warehouses that are now will be selected are 6, 8, 10, 12, 14, 17, 18, and 19.

This constraint makes the problem infeasible, and at that point the efficiency score (a), based on which the warehouses are selected, is relaxed. The model is infeasible for all values less than 1 ($0.7 \leq a < 1$). In this case, the algorithm terminates (Fig. 2) and the selected warehouses are 1, 6, 8, 10, 11, 12, 14, 15, 16, 17, 18 and 19 which correspond to the optimal solutions of $P_0^{\gamma=1}$ with total cost of 3,433,156.64 r.m.u.

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