

# On the Theory of the Modulation Instability in Optical Fiber and Laser Amplifiers

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## Abstract

The modulation instability (MI) in optical fiber amplifiers and lasers with anomalous dispersion leads to CW beam breakup and the growth of multiple pulses. This can be both a detrimental effect, limiting the performance of amplifiers, and also an underlying physical mechanism in the operation of MI-based devices. Here we revisit the analytical theory of MI in fiber optical amplifiers. The results of the exact theory are compared with the previously used adiabatic approximation model, and the range of applicability of the latter is determined. The same technique is applicable to the study of spatial MI in solid state laser amplifiers and MI in non-uniform media.

**Keywords:** fibers, modulation instability, soliton laser, optical amplifier

Modulation instability (MI) is a fundamental nonlinear effect [1-3] that manifest itself in optics, for example, as the spontaneous breakup of a continuous wave (CW) beam with sufficiently high power. In optical fibers, MI occurs as a result of the interplay between the effects of the anomalous group-velocity dispersion (GVD) and self-phase modulation. The spatial modulation instability of stationary CW propagated in nonlinear material produces wave transversal modulations and filamentation. Usually, MI is a detrimental effect that degrades the beam quality. However, MI can also be exploited in a constructive way, for instance, as a technique to generate an optical pulse train or as a passive mode-locking mechanism in fiber lasers [4-10]. In this context, MI is a passive nonlinear effect that has economic advantage over schemes using ultrafast modulators. An important feature of this technique is that the generation of continuous streams of short-pulses via MI can be realized at high repetition rates. As a nonlinear fiber effect sensitive to dispersion, MI is also very attractive for various measurement techniques [11, 12]. Recent progress in micro-structured optical fibers offers new opportunities for the control of dispersive properties and, thus, to new potential applications of MI across a broad spectral range.

The growth rate of the instability and of the most unstable scale (temporal or spatial) are determined by the field amplitude. Here we consider MI in amplifiers, where the intensity continuously increases, changing the instability growth and modulation scale. The usual way to study this problem is to assume that, locally, we have MI of a constant field amplitude which adiabatically changes with propagation (adiabatic approximation [3, 15]), but the accuracy of this approximation is not clear. Fortunately, the problem can be solved analytically [13, 15, 16, 17]. Below we consider the exact solution and compare it with the results of adiabatic approximation.

Over a wide range of physical parameters, the propagation of the optical field down a fiber amplifier at leading order is described by the nonlinear Schrödinger equation (NLSE) with gain terms:

$$i \frac{\partial \Psi}{\partial z} - \frac{\beta_2}{2} \Psi_{tt} + \gamma |\Psi|^2 \Psi = i \frac{g_0}{2} \Psi + i \frac{g_0 T_2^2}{2} \Psi_{tt} \quad (1)$$

Here  $\beta_2$  is the group velocity dispersion. The nonlinear parameter is  $\gamma = 2\pi n_2 / (\lambda_0 A_{eff})$  where  $\lambda_0$  is the operational wavelength,  $n_2$  is the nonlinear refractive index, and  $A_{eff}$  is the effective area of the fiber. Finally,  $g_0$  is

the small signal gain of the amplifier. The parameter  $T_2$  characterizes the gain bandwidth of an amplifier (or the effect of external filtering). We consider here an optical field propagating from  $z = 0$  to  $z = L$ . Consider the modulation instability of the CW field:

$$\Psi(z, t) = (\sqrt{P_0} + a + ib) \times \exp\left[\frac{g_0 z}{2} + iP_0 \int \gamma(z') dz'\right],$$

where  $\gamma(z) = \gamma(0) \exp[g_0 z]$ . A perturbation to the power evolution can then be found as:

$$|\Psi(z, t)|^2 = (P_0 + 2a(z, t)\sqrt{P_0} + a^2 + b^2) \times \exp[g_0 z] \quad (2)$$

Assuming  $a, b \ll \sqrt{P_0}$  and expressing the fields  $a, b$  through the corresponding Fourier modes  $a_\omega, b_\omega \propto \exp[-i\omega t]$  (for notational simplicity, we henceforth omit the index  $\omega$ ) yields the standard linear evolution equations (4) for the spectral modes of perturbations with initial conditions appropriate to the Cauchy problem.

Assuming  $\gamma = \text{const}$ ,  $T_2 = 0$ ,  $a \propto \exp[ik_z z]$  we obtain the standard MI relation [1]:

$$k_z^2 = \frac{\beta_2 \omega^2}{2} \left[ \frac{\beta_2 \omega^2}{2} + 2\gamma P_0 \right] \quad (3)$$

with  $k_z$  increasing for small values of  $\omega$ , reaching its maximum at  $\omega_{\text{max}}^2 = -2\gamma P_0 / \beta_2$ , and approaching zero at  $\omega_0^2 = -4\gamma P_0 / \beta_2$ . In amplifiers, however, where the field power grows as  $P(z) = P_0 e^{g_0 z}$ , the most unstable perturbation frequency increases during propagation due to the power exponential growth. To estimate the growth due to MI in an amplifying medium, one can use the expression for the uniform MI, but replace constant power with a growing power,  $P_0 \rightarrow P_0 e^{g_0 z}$ . This corresponds to the so-called adiabatic approximation (see e.g. [3, 15]).

Equation (2) can be solved analytically. Introducing  $\eta = \frac{2\omega \sqrt{-\beta_2 \gamma_0 P_0}}{g_0}$ ,  $2s = g_0 T_2^2 \omega^2$ ,  $\mu = \frac{-\beta_2 \omega^2}{g_0}$ ,

$$p^2 = \frac{\mu^2}{\eta^2} = \frac{-\beta_2 \omega^2}{4\gamma_0 P_0},$$

the equations for  $a(z)$  and  $b(z)$  take the form:

$$\begin{aligned} \frac{da}{dz} &= \frac{g_0 \mu}{2} b(z) - s a(z), \\ \frac{db}{dz} &= \frac{\eta^2 g_0}{2\mu} (-p^2 + \exp[g_0 z]) a(z) - s b(z). \end{aligned} \quad (4)$$

The solution to (4) can be obtained in terms of the Bessel functions  $I_{i\mu}(x)$  and  $K_{i\mu}(x)$  (compare to approaches used in [16] in context of short-scale self-focusing and in [17] for analysis of modulation instability in lossy fibers):

$$\begin{aligned} a(z) &= A I_{i\mu}(\eta e^{g_0 z/2}) + B K_{i\mu}(\eta e^{g_0 z/2}), \\ b(z) &= -\frac{\eta e^{g_0 z/2}}{2\mu} \{C (I_{i\mu-1} + I_{-i\mu+1}) + D (K_{i\mu-1} + K_{i\mu+1})\}. \end{aligned} \quad (5)$$

Here

$$A = [-a(0)\eta K'_{i\mu}(\eta) + b(0)\mu K_{i\mu}(\eta)]e^{-sz}, \quad B = [a(0)\eta I'_{i\mu}(\eta) - b(0)\mu I_{i\mu}(\eta)]e^{-sz}, \quad C = -\eta\mu^{-1}e^{g_0z/2}A, \\ D = \eta\mu^{-1}e^{g_0z/2}B.$$

The solutions (5) are functions of the real and imaginary parts of the initial perturbations and three dimensionless parameters:  $g_0z$ ,  $\mu$ ,  $\eta$ . The dependence of the solution on nonzero bandwidth  $s$  is trivial, and later we mostly consider  $s = 0$ .

The nonlinear Schrödinger equation (NLSE) governs the propagation of a high-power beam through an amplifier medium according to

$$i\frac{\partial\Psi}{\partial z} + \frac{1}{2n_0k_0}\Delta_{\perp}\Psi + k_0n_2|\Psi|^2\Psi = i\frac{g_0}{2}\Psi. \quad (6)$$

Here  $k_0$  is the propagation vector in vacuum;  $n_0$  and  $n_2$  are the linear and nonlinear refractive indices, respectively; and  $g_0$  is the amplifier gain. Similar to (1), this has a uniformly growing solution. Consider the following perturbation of this solution:

$$\Psi(z, r_{\perp}) = (\sqrt{P_0} + a + ib) \times \exp\left[\frac{g_0z}{2} + iP_0 \int n_2(z')dz'\right]$$

with  $a, b \propto e^{ik_{\perp}r_{\perp}}$ . For the perturbations  $a$  and  $b$  we have Eq. (4) with  $s = 0$  and  $\mu = \frac{k_{\perp}^2}{k_0n_0g_0}$ ;  $\eta = \frac{2k_{\perp}\sqrt{n_2P_0}}{g_0\sqrt{n_0}}$ .

Similar equations can be derived for the evaluation of the instability of an intense wave propagating in a non-uniform atmosphere with  $n_2(z) = n_2(0)e^{-\frac{z}{h}}$  [14]. For perturbations  $a, b \propto e^{ik_{\perp}r_{\perp}}$  we obtain (4) with  $g_0 = 1/h$ .

The initially stable perturbations propagating in an amplifier can eventually become unstable, growing ones. The Sturmian theory [18] guarantees for the Sturm–Liouville problem (4) that the solutions (5) are growing with  $z$  under the condition  $\mu < \eta e^{g_0z/2}$ . For  $\eta e^{g_0z/2} > \mu$  and  $\mu \gg 1$  the leading term in the expansion of the exact solution reads:

$$I_{i\mu}(\eta e^{g_0z/2}) \approx \frac{\exp[\sqrt{\eta^2 e^{g_0z} - \mu^2} + \mu \arcsin(\frac{\mu e^{-g_0z/2}}{\eta})]}{\sqrt{2\pi} \sqrt[4]{(\eta^2 e^{g_0z} - \mu^2)}} + \dots \\ K_{i\mu}(\eta e^{g_0z/2}) \approx \frac{\sqrt{\pi} \times \exp[-\sqrt{\eta^2 e^{g_0z} - \mu^2} - \mu \arcsin(\frac{\mu e^{-g_0z/2}}{\eta})]}{\sqrt[4]{4(\eta^2 e^{g_0z} - \mu^2)}} + \dots \quad (7)$$

In this limit, it is seen that  $K_{i\mu}$  is decaying and  $I_{i\mu}$  is growing, and the solutions (5) we used in the unstable case are similar to the growing and decaying exponents without amplifications. The growth of perturbations in the amplifier is super-exponential  $a \propto e^{\eta e^{g_0z}}$ . In the opposite limit  $\mu > \eta e^{g_0z/2}$ , both  $K_{i\mu}$  and  $I_{i\mu}$  are oscillating.

In most of the MI studies, it was assumed that the perturbations grow from the thermal level, and that to become noticeable the initial perturbations must grow by a few orders of magnitude. This means that in (5) only the

terms with the growing function  $I_{i\mu}$  must be taken into account. The power growth of the initial perturbations can be characterized by an increment factor (similar to the homogeneous case, making comparison more convenient) defined as:  $\Gamma = 2 \ln[a(L)/a(0)]$ ,  $\mu < \eta$ ;  $\Gamma = 2 \times \ln[a(z)/a(z^*)]$ ,  $\mu > \eta$ ,  $z^* = \frac{1}{g_0} \ln[\frac{\mu^2}{\eta^2}]$ .

Here we assume  $a(0), a(z^*) \neq 0$ . For large  $a(L)/a(0)$ ,  $\Gamma$  is practically independent of boundary conditions.

In the adiabatic approximation (AA), we have  $\Gamma = 2 \int_{z^*}^L \text{Im} k_z dz$  where  $k_z$  is given by (3) with  $P_0$  replaced by the local intensity  $P = P_0 e^{g_0 z}$ . After integration we have

$$\Gamma = 2f(\mu, \eta, g_0, L) - 2sL,$$

with  $f(\mu, \eta, g_0, L)$  defined as:

$$f = \left(\eta^2 e^{g_0 L} - \mu^2\right)^{1/2} + \mu \arcsin \frac{\mu e^{\frac{g_0 L}{2}}}{\eta} - \left(\eta^2 - \mu^2\right)^{1/2} - \mu \arcsin \frac{\mu}{\eta}; \mu < \eta$$

$$f = \left(\eta^2 e^{g_0 L} - \mu^2\right)^{1/2} + \mu \arcsin \frac{\mu e^{\frac{g_0 L}{2}}}{\eta} - \mu \arcsin 1; \mu > \eta$$
(8)

The second formula means that the growth starts not at the entrance of the amplifier but later, at  $z=z^*$ . This is a most practical case, and later we will focus our attention on it. Formally, AA is applicable when  $\Gamma \gg 1$ . It means that the expression for the growth rate  $\Gamma$  is accurate only at  $z \gg z^*$ , and by keeping the constants in (8) one is exceeding the accuracy of the approximation.

Comparing (8) with (7), we see that the asymptotic form of the exact solution coincides closely to the adiabatic approximation [3,15], but that there are two important differences. First is the presence of the pre-exponential factor, which can be included as a next expansion term in AA. Also the term  $-\mu\pi/2$  in the exponent of the asymptotic (7) is absent in (8). This means that the asymptotic growth, as calculated via (8), will be smaller than exact value by the factor  $e^{-\frac{\mu\pi}{2}}$ . This is a small effect for small  $\mu$  but can be important factor for large  $\mu$  values. The accuracy of the approximation at finite  $z$  is not clear. To clarify it we plot on Fig.1 the asymptotic value of  $I_{i\mu}(x)$  given by (7) (green line) and the result of exact calculations for few values of  $\mu$  (red line). Also we included the standard AA expression [3, 15] (black line) and an improved AA including the pre-exponential factor (blue line).

We note the fast convergence of the exact solution and the asymptotic one (7). However, the AA results are noticeably different from the exact solution. We can see that for small  $\mu$ , AA overestimates the exact solution, and for large  $\mu$ , AA overestimates it.

These general features are manifested in calculations of integrated growth. It is convenient to plot the integrated growth as a function of  $p$ . Without amplification the maximum growth rate takes place at  $p^2 = 1/2$  and for adiabatically growing intensity  $P$  the point of maximal growth rate increases as  $P/2P_0$ .

In Fig. 2 we plot a comparison of the exact solution with AA results given by (8). We use parameters from the paper [3] and plot the integrated growth for a few amplifier lengths. The vertical dotted lines indicate the maximum growth rate for the intensity at the amplifier exit.

We see that the adiabatic approximation can be inaccurate in many real situations.

Small  $p$  corresponds, for fixed gain, to the small values of  $\mu$ , and according to Fig.1 AA overestimates the growth. For large  $p$  and correspondingly large  $\mu$ , AA underestimates the growth. Improvement of AA by inclusion of the pre-exponential factor does not help much.

In general, the increment factor  $\Gamma(\mu, p, g_0, s, L)$  is a multi-parametric function of the parameters  $p, \mu, g_0, s$  and  $L$ . Therefore, the existence of the analytical solution provides the useful tool for design analysis, and the use of simple AA can produce noticeable errors. For fixed values of other parameters we have to determine the maximum value of the increment growth  $\Gamma$  as a function of  $\omega$ . The  $s$  dependence is trivial and later we put  $s = 0$ . It is convenient to use  $p$  as a parameter. In a uniform media ( $g_0 L = 0$ ), the most unstable mode corresponds to  $p^2 = 1/2$  and cutoff at  $p^2 = 1$ . In contrast, in the amplifier, the most unstable value of  $\omega$  increases during propagation, and the value of corresponding to the most unstable mode increases. We see that maximal unstable frequency (transversal perturbation wave-number) increases with amplifier length but remains smaller than the values corresponds to the final power. For a sufficiently long amplifier, the most unstable modes were stable initially. The effect of this sliding of the most unstable frequency with the development of MI in the optical fiber amplifier has a direct impact on the operation of MI-based fiber laser and the generation of pulse trains using MI. For instance, in fiber lasers where MI triggers passive mode-locking, the instability frequency should be in resonance with the resonator frequency and this sliding of the maximum of instability should be taken into account.

Frequently, MI growth is initiated by finite perturbations. In fiber amplifiers these are the deviations of the pulse shape from a flat top. For spatial instability in amplifiers, these can be material defects or misalignments.

When perturbations must grow only a few times to be noticeable, the initial conditions become important. Initial conditions also become important near the cutoff instability as the growth is not large near such points. From (3) one can see that in the linear stage of instability the intensity variations are proportional to the real part of the perturbation  $a$ , and are determined by the coefficient  $A$  in (5). The value of  $A$  is related to the amplitude, phase and scale of the perturbations in a nontrivial way. The contours of  $A$  are presented in Fig.3, where the relative impact of the initial phase  $b(0)$  and amplitude  $a(0)$  perturbations on the growing solution are shown. Here  $a^2(0) + b^2(0) = 1$  and  $\phi = \tan^{-1}[b(0)/a(0)]$ .

We see that the values of  $A$  for different phases can vary by a factor of 10. This result can be used for the optimization of soliton laser design, in that optimization of the initial perturbation can reduce the laser size. For the spatial instability we can find the most dangerous type of optical defects producing the beam perturbations.

The above results indicate the usefulness of the exact solution (5). In a situation when the most unstable mode does not grow at the amplifier entrance, (5) must be used to calculate the values of  $a$  and  $b$  at the moment the growth started, which can be different from the initial conditions.

In addition, a real system frequently has several elements and amplifiers. Using the analytical result (5) we can find the values of  $a$  and  $b$  after an amplifier and propagate the perturbations through the next optical element. Thus we are now able to provide complete modeling of MI through all complex optical systems. The modeling of nonlinear propagation in a powerful laser can now be upgraded to the level of modeling of nonlinear effects in passive optical systems.

We have revisited the theory of modulation instability in fiber amplifiers. We have found the complete analytical solution of the linear growth. This allows us to find the most unstable mode and to calculate the power growth exactly, without restricting considerations to the asymptotically growing mode as in most previous works. We have demonstrated that for practical situations the growth of the perturbation is sensitive to the initial perturbation and to its phases. In many applications, the initial perturbation fields are different from a plane wave and are amplified from some distribution other than noise. Our results indicate how to modulate the signal in order to accelerate the breakup into shorter pulses, and thus to optimize the design of the soliton laser. While our results are directly relevant to the modulation instability in optical fiber amplifiers and lasers, the underlying theory is quite general and has a variety of physical applications beyond fiber optics.

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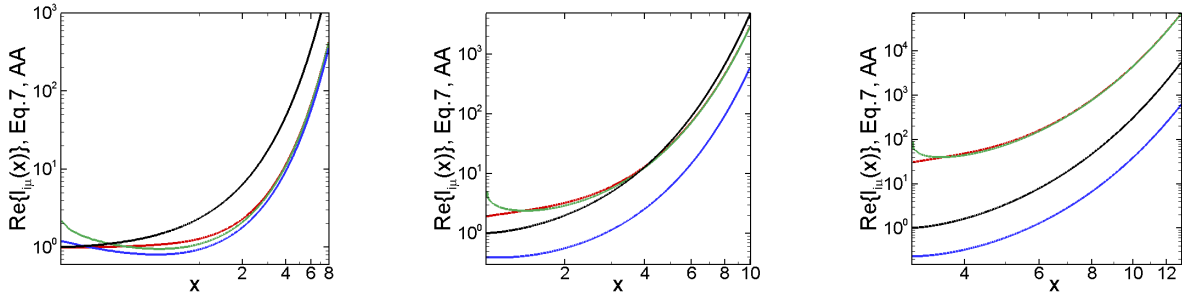


Fig.1 Red lines,  $\text{Re } I_{i\mu}(x)$ , asymptotic value of  $I_{i\mu}(x)$  given by (7); green, AA results; black, improved AA; blue for few values of  $\mu$  for  $x > \mu$ ,  $\mu=0.1$ ,  $\mu=1$ , and  $\mu=3$ , respectively.

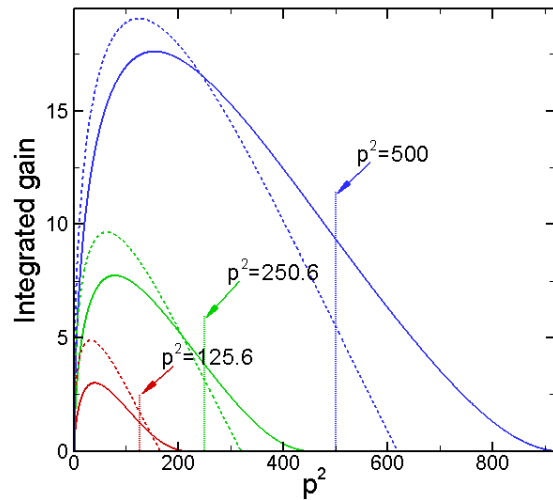


Fig.2 Integrated gain versus  $p^2$  for few amplifiers length  $L$ . Solid lines – exact solution; dashed – AA; red line –  $L=80$  m; green – 90 m; blue – 100 m.  $\beta_2 = -20 \text{ ps}^2/\text{km}$ ,  $\gamma = 10 \text{ 1/Wkm}$ ,  $P_0 = 100 \text{ mW}$ ,  $g_0 = 0.3 \text{ dB/m}$

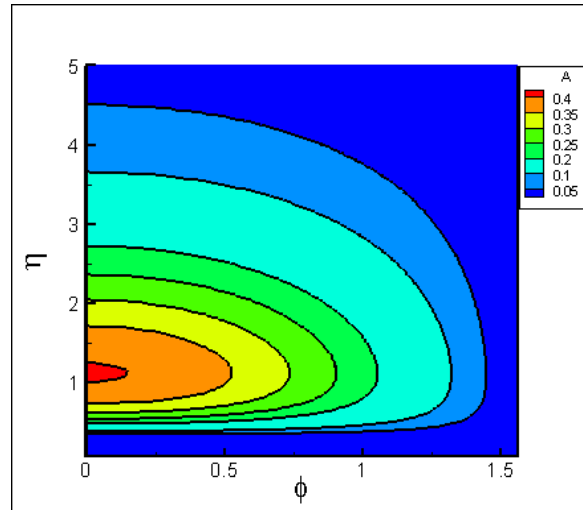


Fig.3 Contour plot of the coefficient  $A = [-a(0)\eta K'_{i\mu}(\eta) + b(0)\mu K_{i\mu}(\eta)]$  before the growing solution in the plane  $(\eta, \phi)$  with  $\mu = 1$ .

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