AN ALTERNATING POTENTIAL-BASED APPROACH TO THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN A PLANAR DOMAIN WITH A CUT

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Abstract — We consider a Cauchy problem for the Laplace equation in a bounded region containing a cut, where the region is formed by removing a sufficiently smooth arc (the cut) from a bounded simply connected domain D. The aim is to reconstruct the solution on the cut from the values of the solution and its normal derivative on the boundary of the domain D. We propose an alternating iterative method which involves solving direct mixed problems for the Laplace operator in the same region. These mixed problems have either a Dirichlet or a Neumann boundary condition imposed on the cut and are solved by a potential approach. Each of these mixed problems is reduced to a system of integral equations of the first kind with logarithmic and hypersingular kernels and at most a square root singularity in the densities at the endpoints of the cut. The full discretization of the direct problems is realized by a trigonometric quadrature method which has super-algebraic convergence. The numerical examples presented illustrate the feasibility of the proposed method.

2000 Mathematics Subject Classification: 65M15.

Keywords: Laplace equation, Cauchy problem, alternating method, mixed boundary value problems, single- and double layer potentials, integral equation of the first kind, logarithmic- and hypersingularities, cosine- transformation, trigonometrical quadrature method.

1. Introduction

Models leading to elliptic partial differential equations to be solved outside open curves occur frequently in engineering problems. For example, the elliptic operator can model stationary heat conduction and the open curves can represent cuts, screens or wings in physical bodies (see, for example, [10] and [16]). Moreover, part of the boundary of the body might be inaccessible due to a hostile environment, but on the accessible part of the boundary measurements of both solution and its normal derivative (the heat flux) can be obtained leading to a so-called Cauchy problem. In this paper we investigate such a situation for bounded planar domains containing a cut.

To formulate this Cauchy problem, we assume that $D \subset \mathbb{R}^2$ is a simply connected bounded domain with boundary $\Gamma_2 \in C^{p+2}$, $p\in\mathbb{N}$, and let ν be the outward unit normal

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on Γ_2. Furthermore, let $\Gamma_1 \in C^{p+2}$ be a cut in D (see Fig. 1.1) with endpoints x^*_{-1} and x^*_1, and assume that Γ_1 has orientation from x^*_{-1} to x^*_1. By Γ_1^- and Γ_1^+ we denote the left-hand and right-hand sides of Γ_1, respectively, and by υ the unit normal vector to Γ_1 directed towards Γ_1^+.

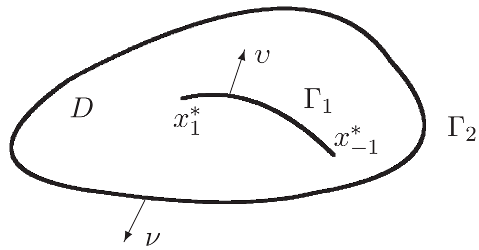


Fig. 1.1. A bounded domain D with a cut Γ_1

Given the (bounded) sufficiently smooth continuous functions f_1 and f_2 on Γ_2, we consider the Cauchy problem of finding a function $u \in C^2(D\setminus\Gamma_1)\bigcap C^1(\bar{D}\setminus\Gamma_1)$ satisfying the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad D\setminus\Gamma_1 \tag{1.1}$$

and the boundary conditions

$$u = f_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = f_2 \quad \text{on} \quad \Gamma_2. \tag{1.2}$$

Uniqueness of a solution to the Cauchy problem (1.1), (1.2) is well established (see, for example, [2] and [3]). We shall assume that data are chosen such that there exists a solution. However, the solution does not depend continuously on the data, i.e., the problem is ill-posed in the sense of Hadamard making it inaccessible by classical methods.

There exist various methods for solving Cauchy problems for elliptic equations and a common approach is to use a Tikhonov regularization which often leads to a change of the operator of the problem (see Chapter 4 in Lattès and Lions [18]). Another possibility is to use iterative methods. Kozlov and Maz'ya [11] proposed an alternating iterative method for solving Cauchy problems for formally self-adjoint elliptic equations in domains without cuts. One of the advantages of this method is that it preserves the original equation and that the regularizing character is achieved by an appropriate change in the boundary conditions. The alternating method has successfully been applied to several engineering problems (see, for example, [19]).

The aim of this paper is to extend the alternating method to the Cauchy problem for the Laplace operator in domains containing cuts. In Section 2, we introduce some further notation and then formulate the alternating iterative method (see Section 2.1). At each iteration step, direct mixed problems with either Dirichlet or Neumann condition imposed on the cut are solved, and a sequence of approximations to the solution of (1.1), (1.2) is obtained. A proof of convergence in the case of exact data is given in Section 2.2. In the case of noisy data a discrepancy stopping criterion can be used. In Section 3, we undertake a full theoretical and numerical investigation of these direct problems. We prove that each problem is well-posed (see Theorem 3.2 and Theorem 3.8), using a potential approach reducing each direct problem to a system of integral equations of the first kind. Furthermore, the full numerical discretisation of the direct problems is realised by a trigonometric quadrature method which has a super-algebraic convergence (see Theorem 3.4 and Theorem 3.8). In Section 4, numerical investigations are presented showing that the proposed numerical discretisation gives accurate approximations for the mixed problems with few collocation points. Also included in Section 4 are numerical examples for the proposed alternating method for solving (1.1), (1.2). Accurate reconstructions are obtained not only for the solution itself but also for the normal derivative on Γ_1.

2. An alternating method for the Cauchy problem (1.1), (1.2)

Throughout this paper we only work with classical solutions to the different boundary value problems that occur. However, in order to prove convergence of the iterative procedure for the Cauchy problem (1.1), (1.2), it is more straightforward to work with Sobolev norms. Locally, we prove that there is convergence to the solution u and all of its derivatives, hence, using the Sobolev imbedding theorem, convergence in the classical norms can be obtained. To settle the notation, let Ω be a bounded domain in $\mathbb{R}^2$. We denote, as usual, by $C^k(\Omega)$, where k is a nonnegative integer, the space which consists of all functions having continuous derivatives up to order k on Ω and this is a Banach space under the norm $\|f\|_{C^k(\Omega)} = \sup_{0\leqslant|\ell|\leqslant k, x\in\Omega} |\partial^\ell f(x)|$. Similar spaces can be introduced on the boundary of Ω. The space $L^2(\Omega)$ is the standard L^2-space, and, as usual, $H^1(\Omega)$ denotes the Sobolev space of real-valued functions in Ω with finite norm

$$\|u\|_{H^1(\Omega)} = \Big(\int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx \Big)^{1/2}.$$

2.1. The alternating procedure. To formulate the alternating iterative procedure for solving (1.1), (1.2), we introduce the following mixed boundary value problem:

$$\Delta u = 0 \quad \text{in} \quad D \setminus \Gamma_1, \tag{2.1}$$

$$\frac{\partial u^\pm}{\partial \upsilon} = h^\pm \quad \text{on} \quad \Gamma_1, \qquad u = f_1 \quad \text{on} \quad \Gamma_2, \tag{2.2}$$

and also the following one

$$\Delta u = 0 \quad \text{in} \quad D \setminus \Gamma_1, \tag{2.3}$$

$$u^\pm = g^\pm \quad \text{on} \quad \Gamma_1, \qquad \frac{\partial u}{\partial \nu} = f_2 \quad \text{on} \quad \Gamma_2. \tag{2.4}$$

Here, we use the notation

$$\frac{\partial u^\pm}{\partial \upsilon}(x) = \lim_{h\to+0} \upsilon(x)\cdot \operatorname{grad} u(x \pm h\upsilon(x))$$

and similar

$$u^\pm(x) = \lim_{h\to+0} u(x \pm h\upsilon(x))$$

for $x \in \Gamma_1$. Assume that f_1 and f_2 are the same as in (1.2). The alternating iterative procedure for constructing the solution to (1.1), (1.2) runs as follows:

- the first approximation u_0 to the solution u of (1.1), (1.2) is obtained by solving (2.1), (2.2) with $h^\pm = h_0^\pm$, where h_0^- and h_0^+ are arbitrary initial guesses;

- having constructed u_{2k}, we find u_{2k+1} by solving problem (2.3), (2.4) with $g^\pm = u_{2k}^\pm$ on Γ_1;
- then we find the element u_{2k+2} by solving problem (2.1), (2.2) with $h^\pm = \frac{\partial u_{2k+1}}{\partial \upsilon}^\pm$ on Γ_1.

In the next section we prove the following,

Theorem 2.1. *Let u be the solution to* (1.1), (1.2) *and let u_k be the k-th approximate solution in the alternating procedure. Then*

$$\lim_{k\to\infty} \|u - u_k\|_{H^1(D\setminus\Gamma_1)} = 0 \tag{2.5}$$

for any sufficiently smooth initial data elements $h_0^\pm$ which start the procedure.

Using local estimates for harmonic functions, we obtain

Corollary 2.1. *Let the assumptions of* Theorem 2.1 *be fulfilled and let D' be a domain such that $\overline{D'} \subset (D \setminus \Gamma_1)$. Then*

$$\lim_{k\to\infty} \|u - u_k\|_{H^l(D')} = 0 \tag{2.6}$$

for $l = 1, 2, \ldots,$ and any sufficiently smooth initial data elements $h_0^\pm$.

In the case of noisy data, a discrepancy principle can be applied and such a stopping rule was introduced in [1, Section 7.5]. To present it assume, for simplicity, that $f_1 = 0$ in (1.2). Let u_{f_2} be the solution to (2.3), (2.4) with $g^\pm = 0$, and let $u_{f_2^\delta}$ be the solution to (2.3), (2.4) with $f_2 = f_2^\delta$ and $g^\pm = 0$. Let u_k^δ be constructed from the alternating method given above, with $f_1 = 0$ and $f_2 = f_2^\delta$. If

$$\|u_{f_2} - u_{f_2^\delta}\|_{H^1(D\setminus\Gamma_1)} \leqslant \delta$$

for some $\delta > 0$, the iterations should be terminated for the first $k = k(\delta)$ with

$$\int_{D\setminus\Gamma_1} |\nabla(u_{2k+2}^\delta - u_{2k}^\delta)|^2 \, dx \leqslant b^2\delta^2,$$

where $b > 1$ is a given constant.

2.2. Proof of Theorem 2.1. If we start this alternating procedure with $h_0^\pm = \frac{\partial u}{\partial \upsilon}^\pm$ on Γ_1, where u is the (smooth) solution to (1.1), (1.2), one can check that $u_k = u$, thus it is enough to prove the theorem when f_1 and f_2 are both zero. Let u_0 be the solution to (2.1), (2.2), with given (smooth) functions $h^\pm$ and $f_1 = 0$. Let then u_1 be the solution to (2.3), (2.4) with $f_2 = 0$ and $u_1^\pm = u_0^\pm$ on Γ_1. We introduce the operator B by

$$op_d(Bh^\pm)(x) = \frac{\partial u_1^\pm}{\partial \upsilon}(x) \quad \text{for} \quad x \in \Gamma_1.$$

In the next section, we shall prove that both the problems (2.1), (2.2) and (2.3), (2.4) are well-posed. Moreover, the gradient of u is continuous except near the endpoints of the cut Γ_1 where we have the estimate An alternating potential based approach to the Cauchy problem for the Laplace equation in a planar domain with a cut

$$|\nabla u(x)| \leqslant C|x - x_j^*|^\alpha \tag{2.7}$$

for $j = -1, 1$, and $\alpha > -1/2$. Therefore, the operator B is well-defined (in a suitable Sobolev trace space), and, clearly, B is a linear operator. We introduce the following inner product

$$(f^\pm, g^\pm) = \int_{D\setminus\Gamma_1} \nabla u \cdot \nabla v \, dx,$$

where u solves (2.1), (2.2) with $h^\pm = f^\pm$ and $f_1 = 0$, and similarly v solves (2.1), (2.2) with $h^\pm = g^\pm$ and $f_1 = 0$; the corresponding norm is denoted by $\|\cdot\|$.

We note that we have the Green formula

$$\int_{D\setminus\Gamma_1} \nabla u_k \cdot \nabla u_l \, dx = \int_{\Gamma_2} u_k \frac{\partial u_l}{\partial \nu} \, ds + \int_{\Gamma_1} u_k^+ \frac{\partial u_l}{\partial \upsilon}^+ ds - \int_{\Gamma_1} u_k^- \frac{\partial u_l}{\partial \upsilon}^- ds, \tag{2.8}$$

where $k, l = 0, 1, 2$. This follows by an approximation argument using the regularity of the solution and its gradient and by enclosing the cut Γ_1 with a closed curve which shrinks to the cut (see, for example, [6] and [22]).

It is possible to prove that the operator B is injective, self-adjoint, positive, non-expansive and unity is not an eigenvalue of B. We show that the kernel of B consists of zero only. Assume that $Bh^\pm = 0$, i.e., $\partial u_1/\partial \upsilon^\pm$ are zero on Γ_1. The above Green formula implies

$$\int_{D\setminus\Gamma_1} \nabla u_1 \cdot \nabla w \, dx = 0$$

for every sufficiently smooth $w \in H^1(D \setminus \Gamma_1)$, hence u_1 is zero in $D \setminus \Gamma_1$. In particular, $u_1^\pm = 0$ on Γ_1 which implies that also $u_0^\pm = 0$ on Γ_1. Thus, u_0 solves the Laplace equation in $D \setminus \Gamma_1$ with a homogeneous Dirichlet boundary condition. However, the Dirichlet problem for the Laplace equation in $D \setminus \Gamma_1$ has a unique solution, therefore $u = 0$ in $D \setminus \Gamma_1$, i.e., $h^\pm = 0$. Thus, the kernel of B consists of zero only.

Let us show that B is non-expansive, the other properties of B can be deduced in a similar way using the above Green formula (see [12]). First, since $u_k|_{\Gamma_2} = 0$ or $\frac{\partial u_k}{\partial \nu}|_{\Gamma_2} = 0$, for $k = 0, 1, 2$, we have

$$\int_{D\setminus\Gamma_1} \nabla u_k \cdot \nabla u_k \, dx = \int_{\Gamma_1} u_k^+ \frac{\partial u_k}{\partial \upsilon}^+ ds - \int_{\Gamma_1} u_k^- \frac{\partial u_k}{\partial \upsilon}^- ds, \quad k = 0, 1, 2. \tag{2.9}$$

In the same way, since $u_2 = 0$ on Γ_2,

$$\int_{D\setminus\Gamma_1} \nabla u_1 \cdot \nabla u_2 \, dx = \int_{\Gamma_1} u_2^+ \frac{\partial u_1}{\partial \upsilon}^+ ds - \int_{\Gamma_1} u_2^- \frac{\partial u_1}{\partial \upsilon}^- ds. \tag{2.10}$$

Using the fact that $\frac{\partial u_1}{\partial \upsilon}^\pm = \frac{\partial u_2}{\partial \upsilon}^\pm$ on Γ_1 in combination with (2.9), we obtain from (2.10)

$$\int_{D\setminus\Gamma_1} \nabla u_1 \cdot \nabla u_2 \, dx = \int_{D\setminus\Gamma_1} \nabla u_2 \cdot \nabla u_2 \, dx.$$

This implies

$$\int_{D\setminus\Gamma_1} \nabla(u_2 - u_1) \cdot \nabla(u_2 - u_1) \, dx = \int_{D\setminus\Gamma_1} \nabla u_1 \cdot \nabla u_1 \, dx - \int_{D\setminus\Gamma_1} \nabla u_2 \cdot \nabla u_2 \, dx. \tag{2.11}$$

In a similar way

$$\int_{D\setminus\Gamma_1} \nabla(u_1 - u_0)\cdot\nabla(u_1 - u_0)\,dx = \int_{D\setminus\Gamma_1} \nabla u_0\cdot\nabla u_0\,dx - \int_{D\setminus\Gamma_1} \nabla u_1\cdot\nabla u_1\,dx. \tag{2.12}$$

Combining (2.11) and (2.12) we have

$$(Bh^{\pm}, Bh^{\pm}) \leqslant \|h^{\pm}\|^2,$$

thus B is non-expansive.

We can now complete the proof of Theorem 2.1. One can check that

$$\int_{D\setminus\Gamma_1} \nabla u_{2k}\cdot\nabla u_{2k}\,dx = (B^k h_0^{\pm}, B^k h_0^{\pm}). \tag{2.13}$$

Moreover, as mentioned above, the operator B is self-adjoint, non-expansive and has no eigenvalue equal to one. This and equality (2.13) imply that u_{2k} tends to zero when k tends to infinity. Similar to (2.12) we have

$$\int_{D\setminus\Gamma_1} \nabla(u_{2k+1}-u_{2k})\cdot\nabla(u_{2k+1}-u_{2k})\,dx = \int_{D\setminus\Gamma_1} \nabla u_{2k}\cdot\nabla u_{2k}\,dx - \int_{D\setminus\Gamma_1} \nabla u_{2k+1}\cdot\nabla u_{2k+1}\,dx. \tag{2.14}$$

It follows that also u_{2k+1} tends to zero and Theorem 2.1 is proved.

3. Numerical solution of the mixed problems by the layer potential approach

From the previous section the proposed iterative method for the Cauchy problem (1.1), (1.2) involves two direct mixed boundary value problems for the Laplace equation. For the numerical implementation of this procedure it is therefore of importance to have effective numerical solvers for these mixed problems. Since the problems are considered for smooth boundaries the most efficient numerical approximation consists in using an integral equation approach with trigonometrical quadrature for the full discretization. To reduce each of the mixed problems to a boundary integral equation, we apply the logarithmic potentials, i.e., an indirect variant of the integral equation method.

3.1. Boundary value problem with a Dirichlet condition on the cut. We start with the mixed Dirichlet — Neumann boundary value problem, i.e., the Dirichlet boundary value condition is given on the cut Γ_1 and the Neumann condition on the exterior closed boundary Γ_2. To simplify our presentation, we first consider the case where the boundary data on the cut satisfy $g^+ = g^- = g$. Later on, in Section 3.1.4, the general case is discussed.

3.1.1. The boundary layer potential approach. We seek the solution $u \in C^2(D\setminus\Gamma_1)\bigcap C(\bar{D}\setminus\Gamma_1)$ which satisfies the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad D\setminus\Gamma_1, \tag{3.1}$$

the Dirichlet boundary value condition

$$u^{\pm} = g \quad \text{on} \quad \Gamma_1 \tag{3.2}$$

and the Neumann boundary value condition

$$\frac{\partial u}{\partial \nu} = f_2 \quad \text{on} \quad \Gamma_2. \tag{3.3}$$

The gradient of the solution has to satisfy estimate (2.7). Both the existence and the uniqueness of this problem have been investigated in [17]. However, in proving the existence of a solution other types of integral equations were used in [17] that are not suitable for our numerical implementation. We shall therefore give an alternative proof of the existence based on the integral equations we use for the numerical investigations. First, the uniqueness of a solution follows in the standard fashion by applying Green formula (2.8) (with u_k and u_l both equal to u), and we obtain:

Theorem 3.1. *The direct mixed Dirichlet — Neumann boundary value problem* (3.1)–(3.3) *has at most one solution.*

To construct a solution to (3.1)–(3.3), we define u as a combination of a single- and a double-layer potential

$$u(x) = \int_{\Gamma_1} \varphi_1(y)\Phi(x,y)\,ds(y) + \int_{\Gamma_2} \varphi_2(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\,ds(y), \quad x \in D \setminus \Gamma_1,$$

where

$$\Phi(x,y) := \frac{1}{2\pi} \ln \frac{1}{|x-y|}$$

and φ_ℓ, $\ell = 1, 2$, are unknown densities. The density φ_2 belong to the class $C^1(\Gamma_2)$ and to model possible singularities at the endpoints of the cut Γ_1 the density φ_1 is assumed to be of the form

$$\varphi_1(x) = \frac{\tilde{\varphi}_1(x)}{\sqrt{|x-x^*_{-1}||x-x^*_1|}}, \quad x \in \Gamma_1 \setminus \{x^*_{-1}, x^*_1\}, \quad \tilde{\varphi}_1 \in C(\Gamma_1).$$

From the continuity of the single-layer potential and the normal derivative of the double-layer potential we obtain for problem (3.1)–(3.3) the following system of integral equations of the first kind:

$$\begin{cases} \displaystyle\int_{\Gamma_1} \varphi_1(y)\Phi(x,y)\,ds(y) + \int_{\Gamma_2} \varphi_2(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\,ds(y) = g(x), \quad x \in \Gamma_1, \\ \displaystyle\int_{\Gamma_1} \varphi_1(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}\,ds(y) + \frac{\partial}{\partial \nu(x)}\int_{\Gamma_2} \varphi_2(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\,ds(y) = f_2(x), \quad x \in \Gamma_2. \end{cases} \tag{3.4}$$

Thus, we obtain a system of integral equations of the first kind involving kernels with a logarithmic singularity as well as kernels with a hypersingularity, and with at most a square root singularity in the density φ_1. From the jump relations for the double-layer potential and normal derivative of the single-layer potential in combination with the uniqueness in Theorem 3.1, we deduce that under the above regularity assumptions on φ_1 and φ_2 the integral equation system (3.4) has at most one solution.

3.1.2. Parametrization and treatment of the singularities. To establish the existence of a solution and for the future numerical implementation we consider a parametrization of system (3.4). This will be done by employing the cosine-substitution proposed in [24]. We assume that the boundaries Γ_ℓ, $\ell = 1, 2$ have the parametric representations

$$\begin{cases} \Gamma_1 := \{x_1(t) = (x_{11}(t), x_{12}(t)) : -1 \leqslant t \leqslant 1\}, \\ \\ \Gamma_2 := \{x_2(t) = (x_{21}(t), x_{22}(t)) : 0 \leqslant t \leqslant 2\pi\}. \end{cases} \tag{3.5}$$

Let $\theta(x)$ be the unit tangential vector for $x \in \Gamma_2$ corresponding to the given orientation of the boundary. To handle the hypersingularity, we use a Maue type transformation [13, 15], i.e.,

$$\frac{\partial}{\partial \nu(x)} \int_{\Gamma_2} \varphi_2(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, ds(y) = \int_{\Gamma_2} \frac{\partial \varphi_2}{\partial \theta}(y) \frac{\partial \Phi(x, y)}{\partial \theta(x)}\, ds(y), \quad x \in \Gamma_2.$$

Then the parametrization of (3.4) leads to the system

$$\begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{-1}^{1} \mu_1(\tau) H_{11}(t, \tau)\, d\tau + \dfrac{1}{2\pi} \displaystyle\int_{0}^{2\pi} \tilde{\mu}_2(\tau) H_{12}(t, \tau)\, d\tau = g(t), \quad t \in [-1, 1], \\ \\ \dfrac{1}{2\pi} \displaystyle\int_{-1}^{1} \mu_1(\tau) H_{21}(t, \tau)\, d\tau + \dfrac{1}{2\pi} \displaystyle\int_{0}^{2\pi} \tilde{\mu}_2'(\tau) H_{22}(t, \tau)\, d\tau = f_2(t), \quad t \in [0, 2\pi], \end{cases} \tag{3.6}$$

where $\mu_1(t) := \varphi_1(x_1(t))|x_1'(t)|$, $\tilde{\mu}_2(t) := \varphi_2(x_2(t))$, $g(t) := g(x_1(t))$, $f_2(t) := f_2(x_2(t))$ and the kernels have the form

$$H_{11}(t, \tau) := \ln \frac{1}{|x_1(t) - x_1(\tau)|}, \quad t \neq \tau,$$

$$H_{12}(t, \tau) := \frac{(x_1(t) - x_2(\tau)) \cdot x_2'(\tau)^\perp}{|x_1(t) - x_2(\tau)|^2}, \quad H_{21}(t, \tau) := \frac{(x_1(\tau) - x_2(t)) \cdot \nu(x_2(t))}{|x_1(t) - x_2(\tau)|^2}$$

and

$$H_{22}(t, \tau) := \frac{(x_2(\tau) - x_2(t)) \cdot \theta(x_2(t))}{|x_2(t) - x_2(\tau)|^2}, \quad t \neq \tau.$$

Here we used the notation $a^\perp$ defined for the vector $a = (a_1, a_2)^\top$ as

$$a^\perp := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} a.$$

To remove the square root singularity in the density μ_1, we use the cosine-substitution in the corresponding integrals and to manage the logarithmic- and hyper-singularities in the kernels, we use suitable transformations to be able to the apply trigonometrical quadrature rules. Thus, after substituting $t = \cos s$ in the first integral equation and $\tau = \cos \sigma$ in the integrals containing the function μ_1 and some additional transformations (see, for example, [4, 13, 14]) we obtain from (3.6) the following equivalent system:

$$\begin{cases} \frac{1}{2\pi}\int_0^{2\pi}\tilde{\mu}_1(\sigma)\left\{\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)+\tilde{H}_{11}(s,\sigma)\right\}d\sigma+\frac{1}{2\pi}\int_0^{2\pi}\tilde{\mu}_2(\tau)\tilde{H}_{12}(s,\tau)d\tau=\tilde{g}(s), \\ \frac{1}{2\pi}\int_0^{2\pi}\tilde{\mu}_1(\sigma)\tilde{H}_{21}(t,\sigma)\,d\sigma+\frac{1}{2\pi}\int_0^{2\pi}\left\{\tilde{\mu}_2'(\tau)\cot\frac{\tau-t}{2}+\tilde{\mu}_2(\tau)\tilde{H}_{22}(t,\tau)\right\}d\tau=\tilde{f}_2(t). \end{cases} \tag{3.7}$$

Here $s,t\in[0,2\pi]$, $\tilde{\mu}_1(s):=\tilde{\varphi}_1(\tilde{x}_1(s))|\tilde{x}_1'(s)|$, $\tilde{x}_1^\ell(s):=x_1^\ell(\cos s)$, $\ell=0,1$, $\tilde{g}(s):=-2g(\cos s)$, $\tilde{f}_2(t):=2f_2(t)|x_2'(t)|$ and the kernels obtained are given as

$$\tilde{H}_{12}(s,\tau):=-2H_{12}(\cos s,\tau),\quad (s,\tau)\in[0,2\pi]\times[0,2\pi],$$

$$\tilde{H}_{21}(t,\sigma):=H_{21}(t,\cos\sigma)|x_2'(t)|,\quad (t,\sigma)\in[0,2\pi]\times[0,2\pi],$$

$$\tilde{H}_{11}(s,\sigma):=\begin{cases}\ln\dfrac{|\tilde{x}_1(s)-\tilde{x}_1(\sigma)|}{2|\cos s-\cos\sigma|/e} & \text{for}\quad s\neq\sigma,\\[2ex] \ln\dfrac{e|\tilde{x}_1'(s)|}{2} & \text{for}\quad s=\sigma,\end{cases}$$

and

$$\tilde{H}_{22}(t,\tau):=\begin{cases}\dfrac{4[x_2'(t)(x_2(\tau)-x_2(t))][x_2'(\tau)(x_2(t)-x_2(\tau))]}{|x_2(t)-x_2(\tau)|^4}-\\ \dfrac{2x_2'(t)x_2'(\tau)}{|x_2(t)-x_2(\tau)|^2}-\dfrac{1}{2\sin^2\frac{t-\tau}{2}}, & \text{for}\quad t\neq\tau,\\[2ex] -\dfrac{1}{6}+\dfrac{1}{3}\dfrac{x_2'(t)x_2'''(t)}{|x_2'(t)|^2}+\dfrac{1}{2}\dfrac{x_2''^2(t)}{|x_2'(t)|^2}-\dfrac{(x_2'(t)x_2''(t))^2}{|x_2'(t)|^4} & \text{for}\quad t=\tau.\end{cases}$$

As we can see, the density $\tilde{\mu}_1$ is even, the kernel $\tilde{H}_{11}$ is even with respect to both variables, and the kernels $\tilde{H}_{12}$ and $\tilde{H}_{12}$ are even in one of the variables. To rewrite system (3.7) in the operator form, we define the integral operators

$$(S\mu)(s):=\frac{1}{2\pi}\int_0^{2\pi}\mu(\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)d\sigma,\quad s\in[0,2\pi],$$

$$(T\mu)(s):=\frac{1}{2\pi}\int_0^{2\pi}\mu'(\sigma)\cot\frac{\sigma-s}{2}\,d\sigma,\quad s\in[0,2\pi],$$

$$(B_{ij}\mu)(s):=\frac{1}{2\pi}\int_0^{2\pi}\mu(\sigma)\tilde{H}_{ij}(s,\sigma)\,d\sigma,\quad s\in[0,2\pi],\quad i,j=1,2.$$

Thus, we have the operator equation

$$(\mathcal{U}+\mathcal{B})\tilde{\mu}=\tilde{h}, \tag{3.8}$$

where we introduced the vectors $\tilde{\mu} := (\tilde{\mu}_1, \tilde{\mu}_2)^\top$, $\tilde{h} := (\tilde{g}, \tilde{f}_2)^\top$ and the operator matrices

$$\mathcal{U} := \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}, \qquad \mathcal{B} := \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Now we can assure unique solvability of the integral equation system (3.7). For $0 < \alpha < 1$ and $m \in \mathbb{N} \cup \{0\}$, by $C^{m,\alpha}[0, 2\pi]$ we denote the Hölder spaces of 2π-periodical functions and by $C_e^{m,\alpha}[0, 2\pi]$ the subspaces of even functions from $C^{m,\alpha}[0, 2\pi]$.

Theorem 3.2. *For $m \in \mathbb{N}$, $m \leqslant p$, where p describes the smoothness of Γ_1 and Γ_2, $\tilde{g} \in C_e^{m,\alpha}[0, 2\pi]$, $\tilde{f}_2 \in C^{m-1,\alpha}[0, 2\pi]$, the integral equation system* (3.7) *has exactly one solution $\tilde{\mu}_1 \in C_e^{m-1,\alpha}[0, 2\pi]$, $\tilde{\mu}_2 \in C^{m,\alpha}[0, 2\pi]$.*

Proof. Since the operators $S : C_e^{m-1,\alpha}[0, 2\pi] \to C_e^{m,\alpha}[0, 2\pi]$ and $T : C^{m,\alpha}[0, 2\pi] \to C^{m-1,\alpha}[0, 2\pi]$ are bounded and have bounded inverses [15], we can rewrite system (3.8) in the following equivalent form:

$$\left[\begin{pmatrix} I_{C_e^{m,\alpha}} & 0 \\ 0 & I_{C^{m-1,\alpha}} \end{pmatrix} + \begin{pmatrix} S^{-1}B_{11} & S^{-1}B_{12} \\ T^{-1}B_{21} & T^{-1}B_{22} \end{pmatrix} \right] \begin{pmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \end{pmatrix} = \begin{pmatrix} S^{-1}\tilde{g} \\ T^{-1}\tilde{f}_2 \end{pmatrix}, \tag{3.9}$$

where I_X is the identity operator in the space X. From the smooth properties of kernels $\tilde{H}_{ij}$, $i, j = 1, 2$, the integral operators $B_{11} : C_e^{m-1,\alpha}[0, 2\pi] \to C_e^{m,\alpha}[0, 2\pi]$, $B_{12} : C^{m,\alpha}[0, 2\pi] \to C_e^{m-1,\alpha}[0, 2\pi]$, $B_{21} : C_e^{m-1,\alpha}[0, 2\pi] \to C_e^{m,\alpha}[0, 2\pi]$ and $B_{22} : C^{m,\alpha}[0, 2\pi] \to C^{m-1,\alpha}[0, 2\pi]$ are compact. Hence, by the Riesz theory applied to the operator equation of the second kind (3.9) and the uniqueness for the integral equations mentioned above, we can assert the existence of a unique solution of (3.7). □

From the equivalence of the integral equation systems (3.4) and (3.7), we deduce the existence result for the mixed problem (3.1)–(3.3).

Theorem 3.3. *For each $g \in C^{1,\alpha}(\Gamma_1)$ and $f_2 \in C^{0,\alpha}(\Gamma_2)$ the Dirichlet-Neumann boundary value problem* (3.1)–(3.3) *has a unique solution, which depends continuously on the boundary data.*

Remark 3.1. Near the endpoints of the cut Γ_1 the gradient of the solution has the behaviour $\mathcal{O}(|x - x_j^*|^{-1/2}) + \mathcal{O}(\ln |x - x_j^*|^{-1})$, $j = -1, 1$, but if the density φ_1 vanishes in the neighborhood of these points, then $\operatorname{grad} u$ will be bounded and continuous.

3.1.3. Full discretization. Now we apply the quadrature method which was originally developed in [4] for the case of a closed boundary and successfully used for other cases. This quadrature method is based on the trigonometric interpolation with equidistant nodal points

$$s_i = \frac{i\pi}{n_1},\ i = 0, \dots, 2n_1 - 1 \quad \text{and} \quad t_i = \frac{i\pi}{n_2},\ i = 0, \dots, 2n_2 - 1, \quad n_1, n_2 \in \mathbb{N}.$$

The following interpolation quadrature rules are used:

$$\frac{1}{2\pi} \int_0^{2\pi} f'(\tau) \cot \frac{t_i - \tau}{2}\, d\tau \approx \sum_{k=0}^{2n_2-1} T_{|k-i|} f(t_k),$$

$$\frac{1}{2\pi} \int_0^{2\pi} f(\sigma) \ln \left(\frac{4}{e} \sin^2 \frac{\sigma - s_i}{2} \right) d\sigma \approx \sum_{k=0}^{2n_1-1} R_{|k-i|} f(s_k),$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\,d\sigma \approx \frac{1}{2n_1}\sum_{k=0}^{2n_1-1} f(s_k), \quad \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{1}{2n_2}\sum_{k=0}^{2n_2-1} f(t_k), \tag{3.10}$$

with known weights R_k and T_k (see [15]). These quadrature formulas are obtained by replacing the function f with its trigonometric interpolation polynomial and then integrating exactly. After using these quadratures for the corresponding integrals in (3.7) together with collocation, we obtain the $(n_1+2n_2+1)\times(n_1+2n_2+1)$ linear system

$$\begin{cases} \sum_{j=0}^{n_1} \tilde{\mu}_{1j} A_{ij}^{11} + \sum_{j=0}^{2n_2-1} \tilde{\mu}_{2j} A_{ij}^{12} = \tilde{g}_i, \quad i=0,\dots,n_1, \\ \sum_{j=0}^{n_1} \tilde{\mu}_{1j} A_{ij}^{21} + \sum_{j=0}^{2n_2-1} \tilde{\mu}_{2j} A_{ij}^{22} = \tilde{f}_{2i}, \quad i=0,\dots,2n_2-1, \end{cases} \tag{3.11}$$

where $\tilde{\mu}_{1k} \approx \tilde{\mu}_1(s_k)$, $\tilde{g}_k = \tilde{g}(s_k)$, $k=0,\dots,n_1$, $\tilde{\mu}_{2k} \approx \tilde{\mu}_2(t_k)$, $\tilde{f}_{2k} = \tilde{f}_2(t_k)$, $k=0,\dots,2n_2-1$, and the matrix coefficients have the form

$$A_{i0}^{11} = R_i + \frac{1}{2n_1}\tilde{H}_{11}(s_i,0), \quad A_{in_1}^{11} = R_{n_1-i} + \frac{1}{2n_1}\tilde{H}_{11}(s_i,\pi), \quad i=0,\dots,n_1,$$

$$A_{ij}^{11} = R_{|i-j|} + R_{i+j} + \frac{1}{n_1}\tilde{H}_{11}(s_i,s_j), \quad i=0,\dots,n_1, \quad j=1,\dots,n_1-1,$$

$$A_{ij}^{12} = \frac{1}{2n_2}\tilde{H}_{12}(s_i,t_j), \quad i=0,\dots,n_1, \quad j=0,\dots,2n_2-1,$$

$$A_{i0}^{21} = \frac{1}{2n_1}\tilde{H}_{21}(t_i,0), \quad A_{in_1}^{21} = \frac{1}{2n_1}\tilde{H}_{21}(t_i,\pi), \quad i=0,\dots,2n_2-1,$$

$$A_{ij}^{21} = \frac{1}{n_1}\tilde{H}_{21}(t_i,s_j), \quad i=0,\dots,2n_2-1, \quad j=1,\dots,n_1-1,$$

$$A_{ij}^{22} = T_{|i-j|} + \frac{1}{2n_2}\tilde{H}_{22}(t_i,t_j), \quad i,j=0,\dots,2n_2-1.$$

To write the linear system in operator form, we consider the trigonometric interpolation operators

$$P_{n_\ell} : C[0,2\pi] \to \mathcal{T}_{n_\ell}, \quad \ell=1,2,$$

where $\mathcal{T}_{n_1}$ and $\mathcal{T}_{n_2}$ are the spaces of trigonometric polynomials of degrees n_1 and n_2, respectively. Then we can rewrite system (3.11) in the equivalent operator form

$$(\mathcal{U} + \mathcal{P}_{n_1,n_2}\mathcal{B}_{n_1,n_2})\,\tilde{\mu}_{n_1,n_2} = \mathcal{P}_{n_1,n_2}\tilde{h}. \tag{3.12}$$

Here $\tilde{\mu}_{n_1,n_2} := \left(\tilde{\mu}_1^{(n_1)}, \tilde{\mu}_2^{(n_2)}\right)^\top$ and

$$\mathcal{P}_{n_1,n_2} := \begin{pmatrix} P_{n_1} & 0 \\ 0 & P_{n_2} \end{pmatrix}, \qquad \mathcal{B}_{n_1,n_2} := \begin{pmatrix} B_{11}^{n_1} & B_{12}^{n_2} \\ B_{21}^{n_1} & B_{22}^{n_2} \end{pmatrix},$$

where $B_{ij}^{n_\ell}$ are the corresponding approximate quadrature operators for the above integral operators B_{ij}.

The convergence and error analysis for this quadrature method can be established on the basis of the collective compact operator theory (see [4]) or on the basis of some estimate of the trigonometric interpolation in Hölder spaces (see, for example, [5, 21]). In the latter case, this analysis is based on the estimate

$$\|P_{n_k}\mu - \mu\|_{m,\alpha} \leqslant c\,\frac{\ln n_k}{n_k^{\ell-m+\beta-\alpha}}\,\|\mu\|_{\ell,\beta}, \quad k = 1,2, \tag{3.13}$$

for the trigonometric interpolation which is valid for $0 \leqslant m \leqslant \ell$, $0 < \alpha \leqslant \beta < 1$, and some constant c depending only on m, ℓ, α and β.

Theorem 3.4. *For $\tilde{g} \in C_e^{\ell,\beta}[0,2\pi]$, $\tilde{f}_2 \in C^{\ell-1,\beta}[0,2\pi]$ and for sufficiently large n_1 and n_2 the system of approximate equations (3.12) has a unique solution $\tilde{\mu}_1^{(n_1)} \in \mathcal{T}_{n_1}$ and $\tilde{\mu}_2^{(n_2)} \in \mathcal{T}_{n_2}$. For the exact solution $\tilde{\mu}_1$ and $\tilde{\mu}_2$ of (3.7) we have the error estimates*

$$\|\tilde{\mu}_k - \tilde{\mu}_k^{(n_k)}\|_{m,\alpha} \leqslant C_k \frac{\ln n_k}{n_k^{\ell-m+\beta-\alpha}}\|\tilde{\mu}_k\|_{\ell,\beta}, \quad k = 1,2, \tag{3.14}$$

for $0 \leqslant m < \ell$, $0 < \alpha \leqslant \beta < 1$ and some constants C_1 and C_2 depending only on α, β, m, ℓ.

Proof. Let $X = C_e^{m-1,\alpha}[0,2\pi] \times C^{m,\alpha}[0,2\pi]$ and $Y = C_e^{m,\alpha}[0,2\pi] \times C^{m-1,\alpha}[0,2\pi]$. By the smoothness properties of the kernels in the operators B_{ij} and by estimate (3.13) it can be shown that

$$\|P_{n_k} B_{ik}^{n_k}\mu - B_{ik}\mu\|_{m,\alpha} \leqslant c\,\frac{\ln n_k}{n_k^{\ell-m+\beta-\alpha}}\,\|\mu\|_{\ell,\beta}, \quad k,i = 1,2.$$

This implies, in particular, for $\ell = m$ the norm convergence

$$\|\mathcal{P}_{n_1,n_2}\mathcal{B}_{n_1,n_2} - \mathcal{B}\|_{X\to Y} \to 0, \quad n_1, n_2 \to \infty.$$

Therefore, from the Neumann series, we can conclude that, for sufficiently large n_1 and n_2, the operators $\mathcal{U}+\mathcal{P}_{n_1,n_2}\mathcal{B}_{n_1,n_2} : X \to Y$ are invertible and the inverse operators are uniformly bounded. Then the error estimate (3.14) follows from the identity

$$\tilde{\mu}_{n_1,n_2} - \tilde{\mu} = (\mathcal{U} + \mathcal{P}_{n_1,n_2}\mathcal{B}_{n_1,n_2})^{-1}\{(\mathcal{P}_{n_1,n_2}\tilde{h} - \tilde{h}) + (\mathcal{B} - \mathcal{P}_{n_1,n_2}\mathcal{B}_{n_1,n_2})\tilde{\mu}\}.$$

□

Note that for cuts Γ_1 given by analytic arcs and for analytic boundaries Γ_2 and boundary data g and f_2, we can improve the above error estimate to the form

$$\|\tilde{\mu}_k - \tilde{\mu}_k^{(n_k)}\|_{m,\alpha} \leqslant C_k e^{-q_k n_k}, \quad k = 1,2,$$

for some constants $q_k > 0$ (see [15]). In addition, following [15, Section 13.4], the error analysis can also be carried out in a Sobolev space setting.

To compute the normal derivative on the cut, we use the jump relation

$$\frac{\partial u^{\pm}}{\partial \upsilon}(x) = \mp\frac{1}{2}\varphi_1(x) + \int_{\Gamma_1} \varphi_1(y)\frac{\partial\Phi(x,y)}{\partial\upsilon(x)}\,ds(y) + \int_{\Gamma_2} \varphi_2(y)\frac{\partial^2\Phi(x,y)}{\partial\upsilon(x)\partial\nu(y)}\,ds(y),$$

where $x \in \Gamma_1 \setminus \{x_{-1}^*, x_1^*\}$ and $\pm$ refer to the two sides of the cut Γ_1. The parametrization and quadrature rules (3.10) lead to the following approximation:

$$\frac{\partial u^{\pm}}{\partial \upsilon}(\tilde{x}_1(s_i)) \approx \mp\frac{\tilde{\mu}_{1i}}{2|\sin s_i||\tilde{x}_1(s_i)|} + \frac{1}{2n_2}\sum_{j=0}^{2n_2-1}\tilde{\mu}_{2j}L_2(s_i,t_j) +$$

$$\frac{1}{2n_1}\sum_{j=1}^{n_1-1}\tilde{\mu}_{1j}L_1(s_i,s_j)+\frac{1}{4n_1}\tilde{\mu}_{10}L_1(s_i,0)+\frac{1}{4n_1}\tilde{\mu}_{1n_1}L_1(s_i,\pi), \tag{3.15}$$

where $i = 1, \dots, n_1 - 1$,

$$L_1(s,\sigma) := \begin{cases} \dfrac{(\tilde{x}_1(\sigma)-\tilde{x}_1(s))\cdot\upsilon(\tilde{x}_1(s))}{|\tilde{x}_1(s)-\tilde{x}_1(\sigma)|^2} & \text{for} \quad s \neq \sigma, \\ \dfrac{\tilde{x}_1''(s)\cdot\upsilon(\tilde{x}_1(s))}{2|\tilde{x}_1'(s)|^2} & \text{for} \quad s = \sigma \end{cases}$$

and

$$L_2(s,\tau) := \frac{[(x_2(\tau)-\tilde{x}_1(s))\cdot\upsilon(\tilde{x}_1(s))][(\tilde{x}_1(s)-x_2(\tau))\cdot x_2'(\tau)^\perp]}{|\tilde{x}_1(s)-x_2(\tau)|^4}+\frac{\upsilon(\tilde{x}_1(s)\cdot x_2'(\tau)^\perp}{|\tilde{x}_1(s)-x_2(\tau)|^2}.$$

To generate Cauchy data in our numerical experiments, we have to calculate the trace of the solution on Γ_2. From the jump relations of the potentials the following representation holds:

$$u(x) = -\frac{1}{2}\varphi_2(x)+\int\limits_{\Gamma_1}\varphi_1(y)\Phi(x,y)\,ds(y)+\int\limits_{\Gamma_2}\varphi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\,ds(y), \quad x\in\Gamma_2. \tag{3.16}$$

Now, the approximate values for the solution u on Γ_2 can easily be obtained by the quadratures (3.10).

3.1.4. The case of a two-side Dirichlet boundary condition on the cut. Now we return to the more general case with the boundary conditions on the cut

$$u^\pm = g^\pm \quad \text{on} \quad \Gamma_1.$$

According to the results of [7], we seek the solution of the corresponding mixed Dirichlet — Neumann boundary value problem in the form

$$u(x) = \int\limits_{\Gamma_1}\varphi_1(y)\Phi(x,y)\,ds(y)+\int\limits_{\Gamma_1}[g](y)\frac{\partial\Phi(x,y)}{\partial\upsilon(y)}ds(y)+\int\limits_{\Gamma_2}\varphi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\,ds(y), \quad x\in D\setminus\Gamma_1,$$

where we denote $[g] := g^+ - g^-$.
Then the unknown densities satisfy the system of integral equations

$$\begin{cases} \displaystyle\int\limits_{\Gamma_1}\varphi_1(y)\Phi(x,y)\,ds(y)+\int\limits_{\Gamma_2}\varphi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\,ds(y) = \\ \displaystyle\frac{1}{2}(g^+(x)+g^-(x))-\int\limits_{\Gamma_1}[g](y)\frac{\partial\Phi(x,y)}{\partial\upsilon(y)}\,ds(y), & x\in\Gamma_1, \\ \displaystyle\int\limits_{\Gamma_1}\varphi_1(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y)+\frac{\partial}{\partial\nu(x)}\int\limits_{\Gamma_2}\varphi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\,ds(y) = \\ \displaystyle f_2(x)-\int\limits_{\Gamma_1}[g](y)\frac{\partial^2\Phi(x,y)}{\partial\nu(x)\partial\upsilon(y)}\,ds(y), & x\in\Gamma_2. \end{cases} \tag{3.17}$$

Thus, we have obtained integral equations with singularities analogously to the above considered case. Note that if $g^+ = g^-$, then system (3.17) transforms to (3.4).

Remark 3.2. The numerical solution of (3.17) can be realized again by the quadrature method. In this case, to obtain the convergence order proved in Theorem 3.4, we have to take into account the integrals contained in the term $[g]$ on the right-hand side of system (3.17). Interestingly, in the alternating method based on the periodic properties of the boundary functions in the corresponding mixed problems, we can use for their numerical calculation the quadrature rules (3.10).

3.2. Boundary value problem with a Neumann condition on the cut. We employ a similar potential based approach to solve the mixed Neumann — Dirichlet boundary value problem. Again, we shall work with integral equations of the first kind and firstly we present the solution method for the case of boundary functions $h^+ = h^- = h$. For our transformation in this case we need to change the assumption about the smoothness of the boundaries.

3.2.1. Reduction to boundary integral equations. We wish to find a function $u \in C^2(D \setminus \Gamma_1) \bigcap C(\bar{D} \setminus \Gamma_1)$ which satisfies the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad D \setminus \Gamma_1, \tag{3.18}$$

the Neumann boundary value condition

$$\frac{\partial u^\pm}{\partial \upsilon} = h \quad \text{on} \quad \Gamma_1 \setminus \{x^*_{-1}, x^*_1\} \tag{3.19}$$

and the Dirichlet boundary value condition

$$u = f_1 \quad \text{on} \quad \Gamma_2. \tag{3.20}$$

The gradient of the solution is assumed to satisfy (2.7). Analogously to the mixed problem studied in Section 3.1, we have the uniqueness.

Theorem 3.5. *The direct mixed Neumann — Dirichlet boundary value problem* (3.18)–(3.20) *has at most one solution.*

To construct a solution to (3.18)–(3.20), we again introduce u as a combination of a double- and a single-layer potentials:

$$u(x) = \int_{\Gamma_1} \varphi_1(y) \frac{\partial \Phi(x,y)}{\partial \upsilon(y)}\, ds(y) + \int_{\Gamma_2} \varphi_2(y) \Phi(x,y)\, ds(y), \quad x \in D.$$

To incorporate the known asymptotic at the endpoints of the cut Γ_1, the unknown density φ_1 is assumed to be of the form

$$\varphi_1(x) = \tilde{\varphi}_1(x) \sqrt{|x - x^*_{-1}||x - x^*_1|}, \quad x \in \Gamma_1, \quad \tilde{\varphi}_1 \in C^1(\Gamma_1).$$

By using the properties (jump relations) of the single- and double-layer potentials the mixed problem (3.18)–(3.20) can be reduced to the following system of integral equations of the first kind:

$$\begin{cases} \dfrac{\partial}{\partial \upsilon(x)} \displaystyle\int_{\Gamma_1} \varphi_1(y) \frac{\partial \Phi(x,y)}{\partial \upsilon(y)}\, ds(y) + \int_{\Gamma_2} \varphi_2(y) \frac{\partial \Phi(x,y)}{\partial \upsilon(x)}\, ds(y) = h(x), & x \in \Gamma_1, \\ \displaystyle\int_{\Gamma_1} \varphi_1(y) \frac{\partial \Phi(x,y)}{\partial \upsilon(y)}\, ds(y) + \int_{\Gamma_2} \varphi_2(y) \Phi(x,y)\, ds(y) = f_1(x), & x \in \Gamma_2. \end{cases} \tag{3.21}$$

Therefore, we again get a system of integral equations of the first kind containing both the logarithmic singularity and the hypersingularity in the kernels.

3.2.2. Numerical solution of the boundary integral equations. We assume that the cut Γ_1 and the closed boundary part Γ_2 have the parametric representation (3.5). Using the parametrization in (3.21) and employing the cosine- substitution in combination with some further transformations (for details, see [5, 21]), we obtain the following system of integral equations:

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\left\{\tilde{\mu}_1'(\sigma)\cot\frac{\sigma-s}{2}+\tilde{\mu}_1(\sigma)\tilde{K}_{11}(s,\sigma)\right\}d\sigma+\frac{1}{2\pi}\int_0^{2\pi}\tilde{\mu}_2(\tau)\tilde{K}_{12}(s,\tau)\,d\tau=\tilde{h}(s), \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\tilde{\mu}_1(\sigma)\tilde{K}_{21}(t,\sigma)d\sigma+\frac{1}{2\pi}\int_0^{2\pi}\tilde{\mu}_2(\tau)\left\{\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)+\tilde{K}_{22}(t,\tau)\right\}d\tau=\tilde{f}_1(t), \end{cases} \tag{3.22}$$

where $s,t\in[0,2\pi]$. Here, we introduced the functions $\tilde{\mu}_1(s):=\varphi_1(\tilde{x}_1(s))\mathrm{sign}(\pi-s)$, $\tilde{\mu}_2(t):=\varphi_2(x_2(t))|x_2'(t)|$, $\tilde{h}(s):=2h(\tilde{x}_1(s))|\tilde{x}_1(s)|\sin s$, $\tilde{f}_1(t):=-2f_1(x_2(t))$ and the kernels

$$\tilde{K}_{11}(s,\sigma):=\begin{cases} \left\{\dfrac{2[\tilde{x}_1'(s)\cdot(\tilde{x}_1(\sigma)-\tilde{x}_1(s))][\tilde{x}_1'(\sigma)\cdot(\tilde{x}_1(\sigma)-\tilde{x}_1(s))]}{|\tilde{x}_1(s)-\tilde{x}_1(\sigma)|^4}-\right. \\ \left.\dfrac{\tilde{x}_1'(s)\cdot\tilde{x}_1'(\sigma)}{|\tilde{x}_1(s)-\tilde{x}_1(\sigma)|^2}-\dfrac{1}{(\cos s-\cos\sigma)^2}\right\}\sin s\sin\sigma, & \text{for}\quad s\neq\sigma, \\ \left\{\dfrac{1}{6}\dfrac{\tilde{x}_1'(s)\cdot\tilde{x}_1'''(s)}{|\tilde{x}_1'(s)|^2}+\dfrac{1}{4}\dfrac{\tilde{x}_1''^2(s)}{|\tilde{x}_1'(s)|^2}-\dfrac{1}{2}\dfrac{(\tilde{x}_1'(t)\cdot\tilde{x}_1''(s))^2}{|\tilde{x}_1'(s)|^4}\right\}\sin^2 s, & \text{for}\quad s=\sigma, \end{cases}$$

$$\tilde{K}_{12}(s,\tau):=2\frac{(x_2(\tau)-\tilde{x}_1(s))\cdot\tilde{x}_1(s)^{\perp}}{|\tilde{x}_1(s)-x_2(\tau)|^2}\sin s,\quad \tilde{K}_{21}(t,\sigma):=\frac{(\tilde{x}_1(\sigma)-x_2(t))\cdot\upsilon(\tilde{x}_1(\sigma))}{|x_2(t)-\tilde{x}_1(\sigma)|^2}\sin\sigma,$$

$$\tilde{K}_{22}(t,\tau):=\begin{cases} \ln\dfrac{|x_2(t)-x_2(\tau)|^2}{\frac{4}{e}\sin^2\frac{t-\tau}{2}}, & \text{for}\quad t\neq\tau, \\ \ln|x_2'(t)|+1, & \text{for}\quad t=\tau. \end{cases}$$

One can check that $\tilde{\mu}_1(s)=-\tilde{\mu}_1(2\pi-s)$ for $s\in[0,\pi]$ and that the density $\tilde{\mu}_1$ is an odd function. Moreover, the kernel $\tilde{K}_{11}$ is also odd with respect to both variables and the kernels $\tilde{K}_{12}$ and $\tilde{K}_{21}$ are odd in one of the variables. Denote by $C^{m,\alpha}_{odd}[0,2\pi]$ the subspaces of the odd functions from $C^{m,\alpha}[0,2\pi]$. By arguments as in the proof of Theorem 3.2, we can prove the well-posedness of the integral equations system (3.22).

Theorem 3.6. *For $m\in\mathbb{N}$, $m\leqslant p$, where p describes the smoothness of Γ_1 and Γ_2, $\tilde{h}\in C^{m-1,\alpha}_{odd}[0,2\pi]$, $\tilde{f}_1\in C^{m,\alpha}[0,2\pi]$, the integral equation system* (3.22) *has exactly one solution $\tilde{\mu}_1\in C^{m,\alpha}_{odd}[0,2\pi]$, $\tilde{\mu}_2\in C^{m-1,\alpha}[0,2\pi]$.*

From this follows the existence result for the mixed problem (3.18)–(3.20).

Theorem 3.7. *For each $h\in C^{0,\alpha}(\Gamma_1)$ and $f_1\in C^{1,\alpha}(\Gamma_2)$, the Neumann — Dirichlet boundary value problem* (3.18)–(3.20) *has a unique solution which depends continuously on the boundary data.*

Near the endpoints of the cut Γ_1 a similar estimate holds as in Remark 3.1.

After using quadratures (3.10) for the corresponding integrals in (3.22) and by appropriate collocation, we obtain the $(n_1 + 2n_2 - 1) \times (n_1 + 2n_2 - 1)$ linear system

$$\begin{cases} \sum_{j=1}^{n_1-1} \tilde{\mu}_{1j} A_{ij}^{11} + \sum_{j=0}^{2n_2-1} \tilde{\mu}_{2j} A_{ij}^{12} = \tilde{h}_i, \quad i = 1, \dots, n_1 - 1, \\ \sum_{j=1}^{n_1-1} \tilde{\mu}_{1j} A_{ij}^{21} + \sum_{j=0}^{2n_2-1} \tilde{\mu}_{2j} A_{ij}^{22} = \tilde{f}_{1i}, \quad i = 0, \dots, 2n_2 - 1, \end{cases} \tag{3.23}$$

where $\tilde{\mu}_{1k} \approx \tilde{\mu}_1(s_k)$, $\tilde{h}_k = \tilde{h}(s_k)$, $k = 1, \dots, n_1 - 1$, $\tilde{\mu}_{2k} \approx \tilde{\mu}_2(t_k)$, $\tilde{f}_{1k} = \tilde{f}_1(t_k)$, $k = 0, \dots, 2n_2 - 1$, and the matrix coefficients have the form

$$A_{ij}^{11} = T_{|i-j|} - T_{2n_1-i-j} + \frac{1}{n_1} \tilde{K}_{11}(s_i, s_j), \quad i, j = 1, \dots, n_1 - 1,$$

$$A_{ij}^{12} = \frac{1}{2n_2} \tilde{K}_{12}(s_i, t_j), \quad i = 1, \dots, n_1 - 1, \quad j = 0, \dots, 2n_2 - 1,$$

$$A_{ij}^{21} = \frac{1}{n_1} \tilde{K}_{21}(t_i, s_j), \quad i = 0, \dots, 2n_2 - 1, \quad j = 1, \dots, n_1 - 1,$$

$$A_{ij}^{22} = R_{|i-j|} + \frac{1}{2n_2} \tilde{K}_{22}(t_i, t_j), \quad i, j = 0, \dots, 2n_2 - 1.$$

The convergence analysis and error estimate can be proved analogously to Theorem 3.4.

Theorem 3.8. *For $\tilde{f}_1 \in C^{\ell,\beta}[0, 2\pi]$, $\tilde{h} \in C_{odd}^{\ell-1,\beta}[0, 2\pi]$ and sufficiently large n_1 and n_2 the system of approximate equations* (3.23) *has a unique solution $\tilde{\mu}_1^{(n_1)} \in \mathcal{T}_{n_1}$ and $\tilde{\mu}_2^{(n_2)} \in \mathcal{T}_{n_2}$. For the exact solution $\tilde{\mu}_1$ and $\tilde{\mu}_2$ of* (3.22) *we have the error estimates*

$$\|\tilde{\mu}_k - \tilde{\mu}_k^{(n_k)}\|_{m,\alpha} \leqslant C_k \frac{\ln n_k}{n_k^{\ell-m+\beta-\alpha}} \|\tilde{\mu}_k\|_{\ell,\beta}, \quad k = 1, 2,$$

for $0 \leqslant m \leqslant \ell$, $0 < \alpha \leqslant \beta < 1$ and some constants C_1 and C_2 depending only on α, β, m, ℓ.

The function values of the solution restricted to the cut Γ_1 can be calculated as

$$u^{\pm}(x) = \pm \frac{1}{2} \varphi_1(x) + \int_{\Gamma_1} \varphi_1(y) \frac{\partial \Phi(x, y)}{\partial \upsilon(y)} \, ds(y) + \int_{\Gamma_2} \varphi_2(y) \Phi(x, y) \, ds(y), \quad x \in \Gamma_1.$$

After using parametrization (3.5) we can apply the trapezoidal quadrature rule from (3.10).

3.2.3. The case of two-side Neumann boundary condition on the cut. Now we consider the case of boundary conditions on two sides of the cut

$$\frac{\partial u^{\pm}}{\partial \upsilon} = h^{\pm} \quad \text{on } \Gamma_1.$$

Based on the results of [23] we construct the solution of the corresponding mixed Neumann — Dirichlet boundary value problem in the form

$$u(x) = \int_{\Gamma_1} \varphi_1(y) \frac{\partial \Phi(x, y)}{\partial \upsilon(y)} \, ds(y) - \int_{\Gamma_1} [h](y) \Phi(x, y) \, ds(y) + \int_{\Gamma_2} \varphi_2(y) \Phi(x, y) \, ds(y), \quad x \in D \setminus \Gamma_1.$$

Then the unknown densities solve the system of integral equations of the first kind

$$\begin{cases} \dfrac{\partial}{\partial \upsilon(x)} \displaystyle\int_{\Gamma_1} \varphi_1(y) \frac{\partial \Phi(x,y)}{\partial \upsilon(y)}\, ds(y) + \int_{\Gamma_2} \varphi_2(y) \frac{\partial \Phi(x,y)}{\partial \upsilon(x)}\, ds(y) = \\ \dfrac{1}{2}(h^+(x) + h^-(x)) + \displaystyle\int_{\Gamma_1} [h](y) \frac{\partial \Phi(x,y)}{\partial \upsilon(x)}\, ds(y), \qquad x \in \Gamma_1, \\ \displaystyle\int_{\Gamma_1} \varphi_1(y) \frac{\partial \Phi(x,y)}{\partial \upsilon(y)}\, ds(y) + \int_{\Gamma_2} \varphi_2(y) \Phi(x,y)\, ds(y) = \\ f_1(x) + \displaystyle\int_{\Gamma_1} [h](y) \Phi(x,y)\, ds(y), \qquad x \in \Gamma_2. \end{cases} \tag{3.24}$$

Thus, the system obtained is only different from the one considered above (see (3.21)) in terms of the right-hand side. Our Remark 3.2 concerning the calculation of integrals on the right-hand side of (3.17) remains valid in this case too.

4. Numerical experiments

Here we present the numerical results for three different examples. In the first example, we investigate the accuracy of the proposed numerical scheme presented in Section 3 for solving different mixed boundary value problems that occur in the alternating method. In the two remaining examples, we present the numerical investigations of the alternating method proposed in Section 2 for solving the Cauchy problem (1.1), (1.2).

Example 4.1 (Tests for the mixed problems). We consider two domains bounded by an ellipse and a rounded rectangle, respectively, containing a line segment $[-1, 1]$ on the axis Ox_1 constituting the cut Γ_1 (see Fig. 4.1). The boundary Γ_2 has the parametric representation

$$\Gamma_2^{(e)} := \{x_2(t) = (3\cos t, 2\sin t),\ 0 \leqslant t \leqslant 2\pi\}$$

respectively

$$\Gamma_2^{(r)} := \{x_2(t) = r(t)(\cos t, \sin t),\ 0 \leqslant t \leqslant 2\pi\},$$

where

$$r(t) = \left(\left(\frac{1}{2}\cos t \right)^{10} + \left(\frac{2}{3}\sin t \right)^{10} \right)^{-0.1}.$$

For our experiment we use the holomorphic square root function $F(z) = z\sqrt{1 - z^{-2}}$, $z \in \mathbb{C} \setminus (-1, 1)$. Thus for the Dirichlet — Neumann mixed problem we have the following boundary data:

$$g^{\pm}(x) = 0, \quad x \in \Gamma_1 \quad \text{and} \quad f_2(x) = \operatorname{Re}\{F(x_1 + ix_2)\}, \quad x = (x_1, x_2) \in \Gamma_2,$$

and for the Neumann — Dirichlet mixed problem the boundary functions are given in the form

$$h^{\pm}(x) = 0, \quad x \in \Gamma_1 \quad \text{and} \quad f_1(x) = \operatorname{Im}\{F(x_1 + ix_2)\}, \quad x \in \Gamma_2.$$

The numerical results are presented in the table below. Here, we show the errors for the solutions of the mixed problems in $D \setminus \Gamma_1$ (see Fig. 4.1). For the calculation of the norm $\| \cdot \|_{C(\tilde{D})}$ we search for the maximum of the absolute errors at 900 points uniformly distributed in $\tilde{D}$ in the case (a) and at 1500 points in the case (b). The domain $\tilde{D}$ is marked in Fig. 4.1 by gray color. The discretization parameters are chosen as $n = n_1 = n_2$.

From these results the theoretical error estimates, especially the exponential convergence, for our numerical method proved in the previous section are confirmed.

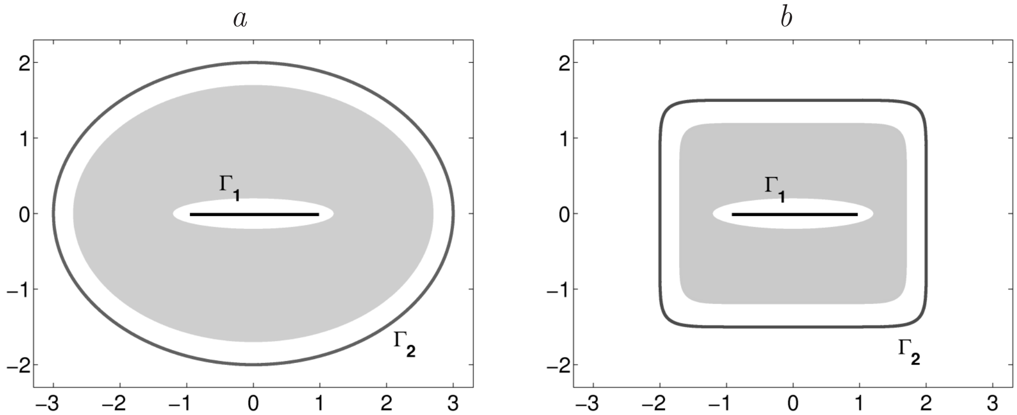


Fig 4.1. Domains with cuts: cut in the ellipse domain (a), cut in the rectangle domain (b)

$\|u - u_{ex}\|_{C(\tilde{D})}$-errors for the mixed problems

n	Neumann — Dirichlet problem		Dirichlet — Neumann problem	
	Ellipse domain	Rectangle domain	Ellipse domain	Rectangle domain
16	0.587853×10^{-2}	0.651126×10^{-2}	0.583429×10^{-1}	0.118655×10^{0}
32	0.105372×10^{-3}	0.164031×10^{-3}	0.128893×10^{-2}	0.407673×10^{-2}
64	0.703893×10^{-7}	0.103009×10^{-6}	0.642910×10^{-6}	0.566353×10^{-5}
128	0.657252×10^{-13}	0.120792×10^{-12}	0.143363×10^{-12}	0.130922×10^{-10}

Example 4.2 (Cauchy problem with a line cut). Now we use the proposed alternating potential approach for the Cauchy problem in the case where the planar domain is bounded by the rounded rectangle $\Gamma_2^{(r)}$ and contains the line cut Γ_1 (see Fig. 4.1,b). The Cauchy data are found by solving the direct Dirichlet — Neumann mixed problem with boundary functions $g^{\pm}(x_1, x_2) = x_1^2$ on Γ_1 and $f_2 = 1$ on $\Gamma_2^{(r)}$ and f_1 is calculated by approach (3.16). Figure 4.2 compares the exact boundary function g (here and throughout the examples, the dashed line is the analytical solution) with the reconstruction in the cases of exact data and 5% data perturbation, respectively. Figure 4.3 shows the result of the reconstruction of the normal derivative $\frac{\partial u^+}{\partial \nu}$ on the boundary Γ_1.

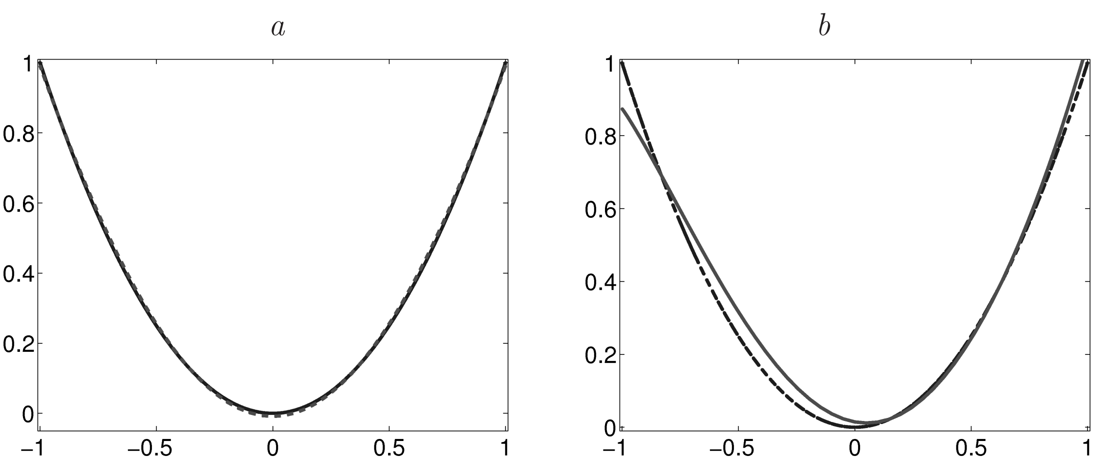
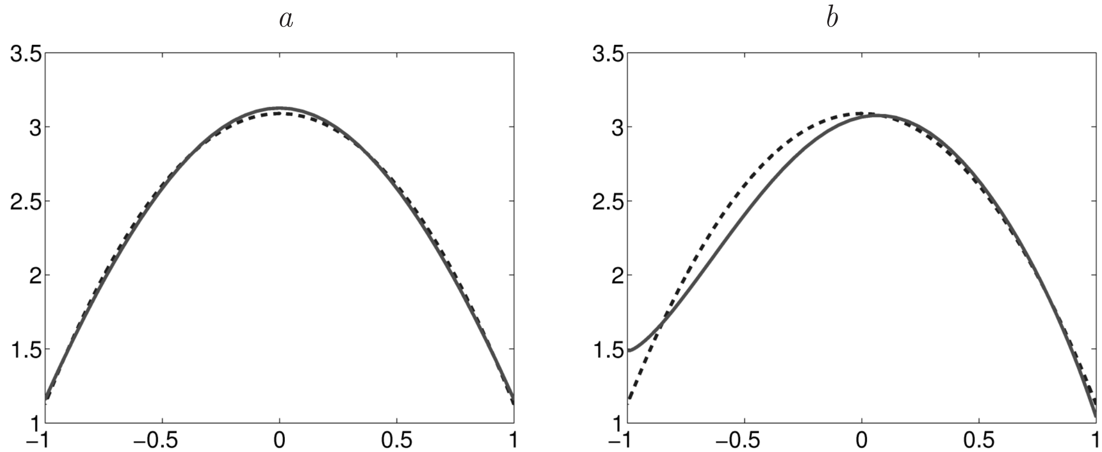
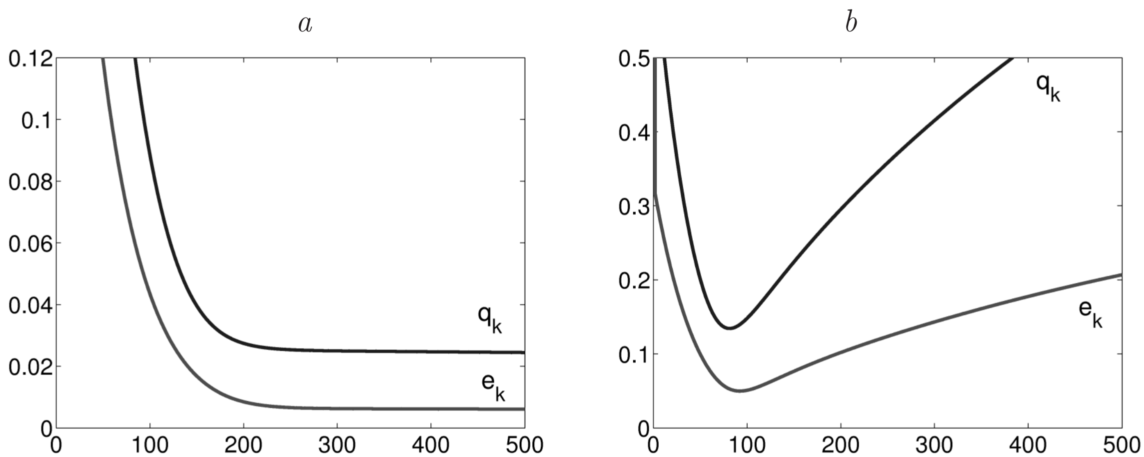


Fig 4.2. Reconstruction of the normal derivative $\frac{\partial u^+}{\partial v}$ on the line cut: exact data, $k^* = 500$ (a); 5% noise, $k^* = 92$ (b)



F i g 4.3. Reconstruction of the normal derivative $\frac{\partial u^+}{\partial \upsilon}$ on the line cut: exact data, $k^* = 500$ (a); 5% noise, $k^* = 81$ (b)

The L^2-errors $e_k := \|u^+_{2k} - g^+\|_{L^2(\Gamma_1)}$ and $q_k := \|\frac{\partial u^+_{2k+1}}{\partial \upsilon} - \frac{\partial u^+}{\partial \upsilon}\|_{L^2(\Gamma_1)}$ are given in Fig. 4.4. Here, we used the following discretization parameters: $n_1 = n_2 = 64$ and the initial guess $h^{\pm}_0 = 0$ on Γ_1. In the case of noisy data, we point out that the result corresponded to the minimum of the L_2-error.



F i g 4.4. L^2-error e_k and q_k for the line cut: exact data (a); 5% noise (b)

Example 4.3 (Cauchy problem with a parabolic cut). In this example, the cut Γ_1 has a more complicated geometry and is given as a parabola

$$\Gamma_1^{(p)} := \{(t, 0.3(t^2 - 0.5)), -1 \leqslant t \leqslant 1\},$$

and the boundary D is bounded by $\Gamma_2^{(e)}$. The boundary data are chosen as $g^{\pm}(x_1, x_2) = \exp(-x_1 - 4x_2)$ and $f_2(x_1, x_2) = 1$, and the Cauchy data are generated as described in the previous example.

The results of the reconstructions for the boundary function and the normal derivative are presented in Fig. 4.5 and in Fig. 4.6. The behavior of the L^2-error on each iteration step is illustrated in Fig. 4.7.

As can be seen from Figure 4.7, in order to still obtain a stable approximation for the solution u on Γ_1 in the case of noisy data, the iterations have to be terminated appropriately, otherwise, due to the ill-posedness of the Cauchy problem (1.1), (1.2), the error starts to magnify. As in the previous example, we use the following discretization parameters: $n_1 = n_2 = 64$ and the initial guess is $h^{\pm}_0 = 0$ on Γ_1. Moreover, the noise level is 5%.

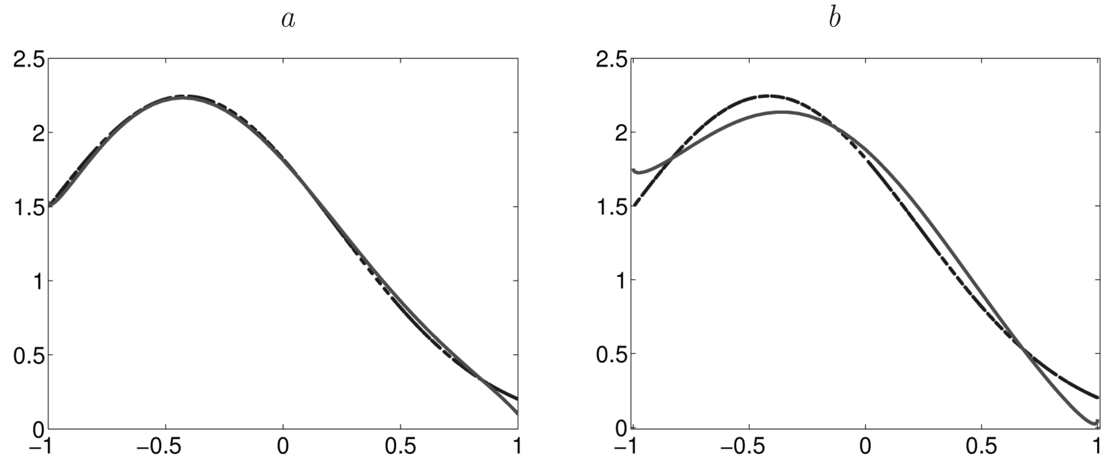


Fig 4.5. Reconstruction of the boundary function u^+ on the parabolic cut: exact data, $k^* = 3000$ (a); 5% noise, $k^* = 145$ (b)

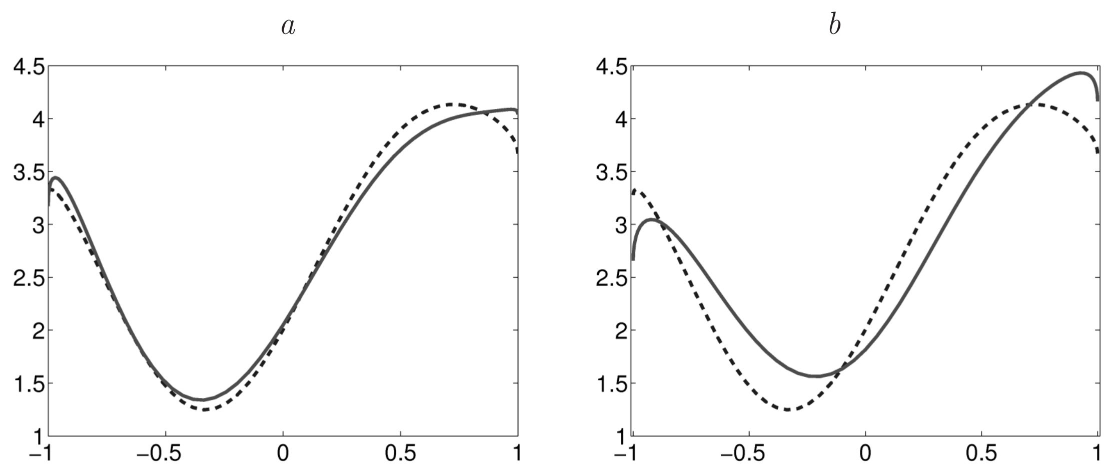


Fig 4.6. Reconstruction of the normal derivative $\frac{\partial u^+}{\partial \upsilon}$ on the parabolic cut: exact data, $k^* = 3000$ (a); 5% noise, $k^* = 140$ (b)

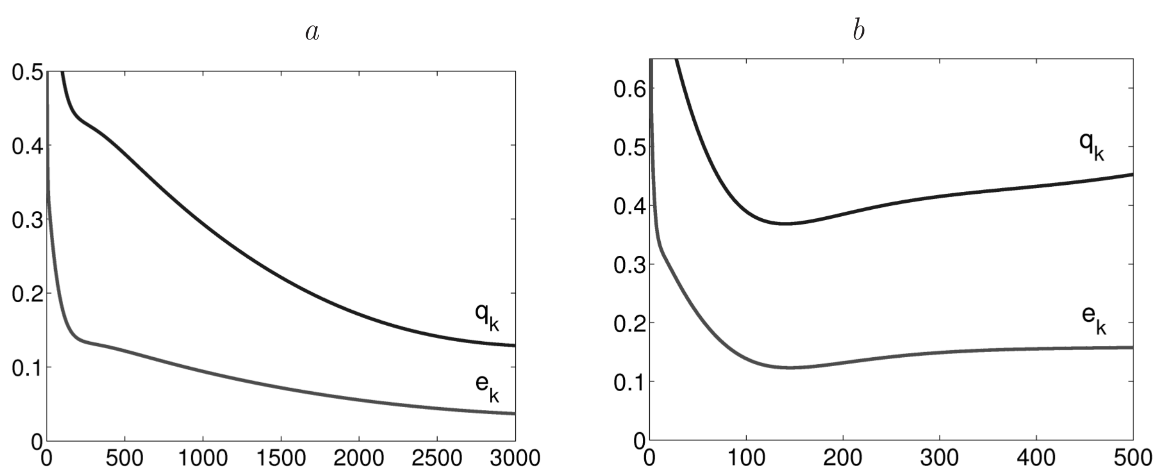


Fig 4.7. L^2-error e_k and q_k for the parabolic cut: exact data (a); 5% noise (b)

Remark 4.1. In Example 4.3, the proposed procedure was relatively slow and in the case of noisy data the reconstructions were not accurate enough. However, there is a possibility of accelerating the alternating procedure both to improve the convergence rate and accuracy (see, for example, [8]). Investigations of such a procedure are deferred to future work.

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