 CMES, vol.62, no.1, pp.57-75, 2010

On the numerical solution of a Cauchy problem in an elastostatic half-plane with a bounded inclusion

Roman Chapko [1], B. Tomas Johansson [2] and Oleh Sobeyko[1]

Abstract: We propose an iterative procedure for the inverse problem of determining the displacement vector on the boundary of a bounded planar inclusion given the displacement and stress fields on an infinite (planar) line-segment. At each iteration step mixed boundary value problems in an elastostatic half-plane containing the bounded inclusion are solved. For efficient numerical implementation of the procedure these mixed problems are reduced to integral equations over the bounded inclusion. Well-posedness and numerical solution of these boundary integral equations are presented, and a proof of convergence of the procedure for the inverse problem to the original solution is given. Numerical investigations are presented both for the direct and inverse problems, and these results show in particular that the displacement vector on the boundary of the inclusion can be found in an accurate and stable way with small computational cost.

Keywords: Alternating method, Boundary integral equations, Cauchy problem, Elastostatics, Green's function, Quadrature method, Trigonometric interpolation.

1 Introduction

In practical applications, to approximate the elastic plane stress field in a large domain, a widely used model is to consider the problem in an unbounded homogeneous elastic region such as a half-plane. Now, unbounded planar stress and strain field determination in unbounded domains are challenging since there can be infinite displacements subject to static load, which is not the case in three-dimensions. Due to this, displacement-based methods, such as the Finite Element Method, require additional non-trivial modifications to be applied in the planar case. Also, typically the domain is needed to be truncated to a finite one, often involving an artificial boundary being introduced. However, the stress and strain fields in planar

[1] Faculty of Applied Mathematics and Informatics, Ivan Franko National University of Lviv, 79000 Lviv, Ukraine

[2] School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK

unbounded domains are well-behaved, and are of practical interest to compute.

We shall propose a numerical procedure in elastostatics for the inverse problem of calculating the displacement on the boundary of a bounded planar inclusion given the values of the displacement and stress vectors on an infinite-line segment. To find this vector in a stable way, we extend the alternating method proposed for bounded domains in Kozlov and Maz'ya (1989), where at each iteration step mixed problems are needed to be solved. We point out that for stationary heat conduction governed by the Laplace equation the alternating method was successfully extended and implemented for unbounded regions in Chapko and Johansson (2008). For more on inverse problems in elasticity in bounded domains, see, for example, Kubo (1988); Maniatty, Zabaras and Stelson (1989); Marin, Hào and Lesnic (2002); Marin and Lesnic (2005) and references therein. For some recent results on the reconstruction of the solution from Cauchy data for the Laplace equation, see, for example, Ling and Takeuchi (2008); Mera, Elliott, Ingham, and Lesnic (2000).

To formulate the problem that we shall consider, let (x,y) denote an element in $\mathbb{R}^2$. Furthermore, let $D_1 \subset \mathbb{R}^2$ be the upper half-plane $y > 0$ with boundary

$$\Gamma_1 = \{\gamma_1(t) = (t,0) : t \in \mathbb{R}\}. \tag{1}$$

We assume that the half-plane D_1 is filled with an isotropic and homogeneous elastic medium with Lamé constants μ and λ satisfying $\lambda > -\mu$ and $\mu > 0$.

Let then D_2 be a simply connected bounded domain in $\mathbb{R}^2$ with C^2-smooth boundary Γ_2, such that $\bar{D}_2 \subset D_1$ (see Fig.1). Moreover, it is assumed that the boundary Γ_2 has the parametric representation

$$\Gamma_2 = \{\gamma_2(t) = (\gamma_{21}(t), \gamma_{22}(t)) : t \in [0,2\pi]\}, \tag{2}$$

where $\gamma_2 : [0,2\pi] \to \mathbb{R}^2$ is injective and two times continuously differentiable. By θ we denote the unit tangent vector to Γ_2 given by $\theta(t) = |\gamma_2'(t)|^{-1}\gamma_2'(t)$ and by $\upsilon = Q\theta$ the unit normal vector, where Q denotes the unitary matrix

$$Q := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, we denote by D the unbounded solution domain where $D := D_1 \setminus \bar{D}_2$, and let ν be the outward unit normal vector to Γ_1, see further Fig.1.

Given the displacement f_1 and stress f_2 on the boundary part Γ_1 we consider the inverse problem of reconstructing the displacement u on Γ_2, i.e. to construct a solution u such that

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u = 0 & \text{in } D, \\ u = f_1 & \text{on } \Gamma_1, \\ Tu = f_2 & \text{on } \Gamma_1, \end{cases} \tag{3}$$

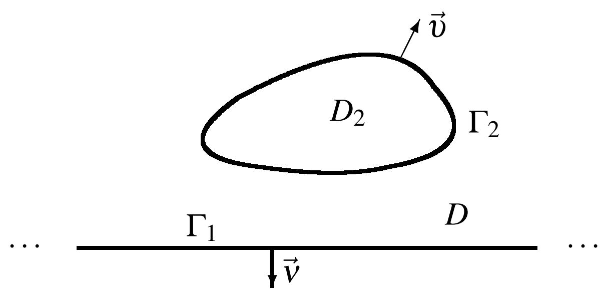

Figure 1: The upper half-plane with a bounded inclusion D_2.

where the equations in D in (3) are the Navier equations and T is the traction operator given by

$$Tu := \lambda\,\mathrm{div}(u)\,\nu + 2\mu\,(\nu \cdot \mathrm{grad})\,u + \mu\,\mathrm{div}(Qu)\,Q\nu. \tag{4}$$

This problem may be considered as a sub-problem of reconstructing the functions u and Tu (with a given normal vector) at any point of the domain D and particularly on the interior boundary Γ_2.

For regularity, we require that u is bounded and $u \in C^2(D) \cap C^1(\bar{D})$. Throughout the paper the function spaces such as $C^2(D)$ and $C(\Gamma_2)$ have to be understood as vector-valued, i.e. $u : D \to \mathbb{R}^2$ and $\psi : \Gamma_2 \to \mathbb{R}^2$.

Note that there is at most one solution to (3) due to unique continuation properties of elliptic equations and we shall assume that data are chosen such that there exists a solution, see further Yeih, Koya and Mura (1993).

At each iteration step of the method we propose for solving the above inverse problem we solve mixed boundary value problems for the elastostatic (Navier) equations. We need the solution of the Neumann-Dirichlet boundary value problem in which the vector function u satisfies the mixed boundary value problem

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u = 0 & \text{in } D, \\ u = f_1 & \text{on } \Gamma_1, \\ Tu = g_2 & \text{on } \Gamma_2. \end{cases} \tag{5}$$

Also, we have to construct the solution of the Dirichlet-Neumann boundary problem

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u = 0 & \text{in } D, \\ Tu = f_2 & \text{on } \Gamma_1, \\ u = g_1 & \text{on } \Gamma_2. \end{cases} \tag{6}$$

The procedure starts by solving (5) with an initial guess of g_2. Then g_1 is constructed from the solution of (5), and (6) is solved with this g_1 as data. We then again solve (5) for a certain value of g_2 constructed from the previous iteration, and the method continues in this way by updating g_2 and g_1 and solving (5) and (6), respectively, to obtain a sequence of approximations to (3), see further Section 2.

To find the solutions to (5) and (6) we propose an integral equation technique based on Green's formula in combination with certain fundamental solutions. The main advantage with our approach is that each mixed problem is reduced to a boundary integral equation over the boundary of the bounded inclusion, making it computationally efficient. We note that using results from complex variable theory, such as conformal mappings, it is possible in certain cases to obtain analytical formulas for the solution to planar elastostatic problems, see, for example, Sun and Peng (2003). However, the method we shall propose can easily be generalized to three-dimensions.

The outline of the paper is the following. In Section 2, we propose the numerical procedure for the inverse problem and outline the convergence of it. In Section 3, we show how to reduce the mixed problems (5) and (6) to boundary integral equations on the bounded boundary Γ_2 only by employing the fundamental solution for the elastostatic half-plane. In Section 4, we use the parameterisations of the different boundary parts and describe how to numerically implement the method for the direct problems taking the singularity of each of the kernels into account. Well-posedness of the boundary integral equations is also included, see Theorems 4.1 and 4.2. Finally, in Section 5, numerical experiments are presented.

2 An alternating procedure for the Cauchy problem (3)

Following Kozlov and Maz'ya (1989) we propose the following procedure to find the displacement vector u in (3):

- The first approximation u_0 to the solution u of (3) is obtained by solving (5) with $g_2 = g_0$, where g_0 is an arbitrary initial guess of the stress on the boundary Γ_2.
- Once u_{2k} has been constructed, we find u_{2k+1} by solving problem (6) with $g_1 = u_{2k}|_{\Gamma_2}$.
- Then the element u_{2k+2} is obtained by solving problem (5) with $g_2 = Tu_{2k+1}|_{\Gamma_2}$.

In the case of exact data the procedure then continues by iterating in the last two steps.

To give a stopping rule for the case of noisy data, let w_1 be the element obtained from the second approximation in the proposed alternating procedure, with initial guess $g_0 = 0$, and define the element $F(f_1, f_2)$ by

$$F(f_1, f_2) = Tw_1|_{\Gamma_2}. \tag{7}$$

Then, for noisy data f_1^δ and f_2^δ, where $\delta > 0$, and

$$\|F(f_1^\delta, f_2^\delta) - F(f_1, f_2)\| \leq \delta, \tag{8}$$

the discrepancy principle can be employed as a stopping rule. This implies in particular that if $k = k(\delta)$ is the smallest integer with

$$\|Tu_{2k+1}^\delta - Tu_{2k-1}^\delta\| \leq b\delta \tag{9}$$

for given $b > 1$, then $u_{k(\delta)}^\delta$ converges to the exact solution of (3) when $\delta \to 0$.
To see this let u_0 be the solution to (5), for given functions $g_2 = g$ and $f_1 = 0$. Then let u_1 be the solution to (6) with $f_2 = 0$ and $g_1 = u_0$ on Γ_2. The operator B is defined by

$$Bg = Tu_1|_{\Gamma_2}. \tag{10}$$

This is a well-defined linear operator. The Cauchy problem (3) is equivalent with the fixed point equation

$$Bg + F(f_1, f_2) = g. \tag{11}$$

Thus, for the convergence of alternating procedure one has to investigate the properties of the operator B. From Kozlov and Maz'ya (1989) and Chapko and Johansson (2008), it can be shown that B (using appropriate function spaces) is self-adjoint, non-negative, non-expansive, and the number one is not an eigenvalue. This imply convergence of the procedure. Moreover, according to Chapt. 3, Sect. 3 inVainikko and Veretennikov (1986), the discrepancy principle can be employed as a stopping rule in case of noisy data.

3 Reduction of the mixed boundary value problems to integral equations

As is well-known, see Chen and Zhou (1992), the fundamental solution to the Navier equation in (5) is given by

$$\Phi(x,y) := \frac{c_1}{4\pi}\,\Psi(x,y)I + \frac{c_2}{4\pi}\,J(x-y),$$

where

$$c_1 := \frac{\lambda + 3\mu}{\mu(\lambda + 2\mu)}, \qquad c_2 := \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}$$

and

$$\Psi(x,y) := \ln \frac{1}{|x-y|}, \quad x \neq y.$$

Here, I is the 2×2 identity matrix, and the matrix J is defined by $J(w) := ww^\top/|w|^2$ in terms of a dyadic product of $w \in \mathbb{R}^2 \setminus \{0\}$ and its transpose $w^\top$.

Since we are working in a semi-infinite solution domain and have an integral representation for the solution, the application of the Green's function technique is preferable in this case. The Green's function (or Green's tensor) for the Navier equation in (5) in the upper half-plane D_1 with the Dirichlet condition of (5) imposed on its boundary Γ_1, can be represented in the following way (see Sheremet (1984))

$$G(x,y) = \Phi(x,y) + U(x,y), \quad x,y \in D_1,$$

where the matrix U is composed by the elements

$$u_{11}(x,y) := \frac{1}{4\pi}\left[-c_1 - c_2(x_1 - y_1)\frac{\partial}{\partial y_1} + 2c_3 x_2 y_2 \frac{\partial^2}{\partial y_1^2}\right]\Psi(x,y^*),$$

$$u_{12}(x,y) := \frac{1}{4\pi}\left[-c_2(x_2 - y_2) + 2c_3 x_2 y_2 \frac{\partial}{\partial y_2}\right]\frac{\partial \Psi(x,y^*)}{\partial y_1},$$

$$u_{21}(x,y) := \frac{1}{4\pi}\left[-c_2(x_2 - y_2) - 2c_3 x_2 y_2 \frac{\partial}{\partial y_2}\right]\frac{\partial \Psi(x,y^*)}{\partial y_1},$$

$$u_{22}(x,y) := \frac{1}{4\pi}\left[-c_1 + c_2(x_2 + y_2)\frac{\partial}{\partial y_2} - 2c_3 x_2 y_2 \frac{\partial^2}{\partial y_2^2}\right]\Psi(x,y^*)$$

with $y^* := (y_1, -y_2)$ and $c_3 := \dfrac{(\lambda+\mu)^2}{\mu(\lambda+2\mu)(\lambda+3\mu)}$.

We shall use the indirect boundary integral equation method for the mixed boundary value problem (5). Taking into account that we are working in a semi-infinite region, we consider the following modification of the elastic single-layer potential

$$U(x) = \int_{\Gamma_2} G(x,y)\psi(y)\,ds(y),$$

with an unknown vector density ψ on the boundary Γ_2. Note here that the possibility to use the Green's function technique in the case of elasticity is discussed in

Kupradze (1965) in general and in Arens (1999) for the elastodynamic half-plane. Based on the definition of the Green function G we see that the above potential U has the properties of the classical elastic single-layer potential. Also, we note that $U(x) = 0$ for $x \in \Gamma_1$. Let w be some function, which satisfies the elastostatic equation in the semi-infinite domain D_1 and the boundary value condition $w = f_1$ on Γ_1. Now, we can seek the solution of the boundary value problem (5) in the form

$$u = U + w \quad \text{in } D.$$

It is straightforward to see that the function u satisfies the elastostatic equation and the Dirichlet boundary condition on the boundary Γ_1. In general the function w can be represented in the form

$$w(x) = -\int_{\Gamma_1} [T_y G(y,x)]^\top f_1(y)\, ds(y).$$

We assume the following asymptotic behaviour for the boundary function f_1

$$f_1(x) = O(|x|^{1-\varepsilon}), \quad \varepsilon > 0, \quad |x| \to \infty. \tag{12}$$

Summarizing we represent the solution of (5) as

$$u(x) = \int_{\Gamma_2} G(x,y)\psi(y)\, ds(y) - \int_{\Gamma_1} [T_y G(y,x)]^\top f_1(y)\, ds(y), \quad x \in D, \tag{13}$$

with an unknown vector density ψ on Γ_2 and T defined by (4).

Using the well-known properties of the restriction of the single-layer elasticity potential to the boundary, we reduce the boundary value problem (5) to the following integral equation of the second kind

$$-\frac{1}{2}\psi(x) + \int_{\Gamma_2} T_x G(x,y)\psi(y)\, ds(y) = g_2(x) + \int_{\Gamma_1} T_x [T_y G(y,x)]^\top f_1(y)\, ds(y), \quad x \in \Gamma_2. \tag{14}$$

The kernel of the equation (14) has a strong singularity and the corresponding integral is interpreted as a Cauchy principal value integral.

Also in the case of the Neumann boundary condition of (6) the indirect boundary integral equation method with the Green's function technique is used. The corresponding Green's function again has the form

$$N(x,y) = \Phi(x,y) + V(x,y), \quad x,y \in D_1,$$

where the regular matrix V is composed by the elements

$$v_{11}(x,y) := \frac{c_2}{4\pi}\left[-\frac{4x_2y_2(x_1-y_1)^2}{|x-y^*|^4}+\frac{c_4(x_1-y_1)^2+2x_2y_2}{|x-y^*|^2}-c_5\Psi(x,y^*)\right],$$

$$v_{12}(x,y) := \frac{c_2}{4\pi}\left[\frac{4x_2y_2(x_1-y_1)(x_2+y_2)}{|x-y^*|^4}+\frac{c_4(x_1-y_1)(x_2-y_2)}{|x-y^*|^2}+c_6\arctan\frac{y_1-x_1}{x_2+y_2}\right],$$

$$v_{21}(x,y) := \frac{c_2}{4\pi}\left[\frac{4x_2y_2(y_1-x_1)(x_2+y_2)}{|x-y^*|^4}+\frac{c_4(x_1-y_1)(x_2-y_2)}{|x-y^*|^2}-c_6\arctan\frac{y_1-x_1}{x_2+y_2}\right],$$

$$v_{22}(x,y) := \frac{c_2}{4\pi}\left[\frac{4x_2y_2(x_2+y_2)^2}{|x-y^*|^4}+\frac{c_4(x_2+y_2)^2-2x_2y_2}{|x-y^*|^2}-c_5\Psi(x,y^*)\right]$$

with constants $c_4 := (\lambda+3\mu)/(\lambda+\mu)$, $c_5 := -(\lambda^2+4\lambda\mu+5\mu^2)/(\lambda+\mu)^2$, $c_6 = 2\mu(\lambda+2\mu)/(\lambda+\mu^2)$.

Thus, the solution of the boundary value problem (6) can be written in the integral form

$$u(x) = \int_{\Gamma_2} N(x,y)\psi(y)ds(y) + \int_{\Gamma_1} N(x,y)f_2(y)ds(y) + \alpha, \quad x \in D, \tag{15}$$

with an unknown density ψ on Γ_2 and a constant vector α. In order to satisfy the boundedness condition at infinity, the side condition

$$\int\limits_{\Gamma_2} \psi(y)\,ds(y) = 0$$

is imposed. Also we suppose that the following conditions for the boundary function f_2 are satisfied

$$f_2(x) = O(|x|^{-1-\varepsilon}),\ \varepsilon > 0,\ |x| \to \infty, \quad \int\limits_{\Gamma_1} f_2(y)\,ds(y) = 0. \tag{16}$$

Considering well-known properties of a single-layer potential on the boundary the problem (6) can be reduced to a system of integral equations of the first kind

$$\begin{cases} \displaystyle\int_{\Gamma_2} N(x,y)\psi(y)ds(y) + \alpha = g_1(x) - \int_{\Gamma_1} N(x,y)f_2(y)ds(y), \quad x \in \Gamma_2, \\ \displaystyle\int_{\Gamma_2} \psi(y)ds(y) = 0. \end{cases} \tag{17}$$

We point out that the kernel of the left-hand side integral in the first equation contains a logarithmic singularity.

4 Numerical solution of the integral equations

4.1 Numerical solution of (14)

Using the parameterizations (1) and (2) for the boundaries of the solution domain D, we can transform (14) into the parametric form

$$-\frac{\varphi(t)}{|\gamma_2'(t)|}+\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{c_7}{|\gamma_2'(t)|}\cot\frac{t-\tau}{2}Q+K_1(t,\tau)\right]\varphi(\tau)\,d\tau=w(t),\ t\in[0,2\pi], \quad (18)$$

where $\varphi(t):=\psi(\gamma_2(t))|\gamma_2'(t)|$, $c_7:=\mu/(\lambda+2\mu)$ and

$$w(t):=2g_2(\gamma_2(t))+\frac{1}{2\pi}\int_{-\infty}^{\infty}K_2(t,\tau)f_1(\gamma_1(\tau))\,d\tau. \quad (19)$$

The kernel K_1 has the form:

$$K_1(t,\tau):=K_{11}(t,\tau)-K_{12}(t,\tau),$$

where

$$K_{11}(t,\tau):=\frac{2}{|\gamma_2'(t)|}\left\{c_7M_1(t,\tau)Q-M_2(t,\tau)[c_7I+c_8\widetilde{J}(t,\tau)]\right\},$$

and

$$K_{12}(t,\tau):=2\pi T_xU(x,y)\,\Big|_{x=\gamma_2(t),y=\gamma_2(\tau)}.$$

Here, we introduced the constant $c_8:=2(\lambda+\mu)/(\lambda+2\mu)$ and the matrices

$$M_1(t,\tau):=\begin{cases}\dfrac{1}{2}\cot\dfrac{\tau-t}{2}+\dfrac{(\gamma_2(t)-\gamma_2(\tau))\gamma_2'(t)}{|\gamma_2(t)-\gamma_2(\tau)|^2}, & \text{for} \quad t\neq\tau,\\[2ex] \dfrac{\gamma_2''(t)\gamma_2'(t)}{2|\gamma_2'(t)|^2}, & \text{for} \quad t=\tau,\end{cases}$$

$$M_2(t,\tau):=\begin{cases}\dfrac{(\gamma_2(t)-\gamma_2(\tau))Q\gamma_2'(t)}{|\gamma_2(t)-\gamma_2(\tau)|^2}, & \text{for} \quad t\neq\tau,\\[2ex] -\dfrac{\gamma_2''(t)Q\gamma_2'(t)}{2|\gamma_2'(t)|^2}, & \text{for} \quad t=\tau,\end{cases}$$

and

$$\widetilde{J}(t,\tau)=\begin{cases}J(\gamma_2(t)-\gamma_2(\tau)), & \text{for} \quad t\neq\tau,\\[2ex] \dfrac{\gamma_2'(t)\gamma_2'(t)^\top}{|\gamma_2'(t)|^2}, & \text{for} \quad t=\tau.\end{cases}$$

The kernel K_2 is given as

$$K_2(t,\tau) := 2\pi T_x[T_y G(y,x)]^\top \big|_{x=\gamma_2(t), y=\gamma_1(\tau)}.$$

Note that the kernels K_{12} and K_2 are continuous and their analytical expressions can be obtained by direct calculations by hand, however, the use of some system of computer algebra is recommended.

For $m \in \mathbb{N} \cup \{0\}$ and $0 < \alpha < 1$, by $C^{m,\alpha}[0,2\pi]$ we denote the space of m-times uniformly Hölder with exponent α continuously differentiable and 2π-periodic vector functions equipped with the usual Hölder norm. For the well-posedness of the integral equation (18) we have the following result:

Theorem 4.1 *For $m \in \mathbb{N} \cup \{0\}$, $\Gamma_2 \in C^{m+2}$, $g_2 \in C^{m,\alpha}[0,2\pi]$ and $f_1 \in C(\Gamma_1)$ satisfying the condition* (12)*, the integral equation* (18) *has exactly one solution $\varphi \in C^{m,\alpha}[0,2\pi]$.*

For the full discretization of the integral equation (18), which has a strong singularity, we apply a quadrature method together with the quadrature rule Chapko (2004); Kress (1999) based on trigonometric interpolation. For this purpose, we choose an equidistant mesh by setting

$$t_i := i\pi/M, \quad i = 0,\dots,2M-1, M \in \mathbb{N}$$

and use the quadrature rules

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{1}{2M}\sum_{j=0}^{2M-1} f(t_j) \tag{20}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\cot\frac{t-\tau}{2}\,d\tau \approx \sum_{j=0}^{2M-1} \tilde{T}_j(t) f(t_j)$$

with weight functions

$$\tilde{T}_k(t) = -\frac{1}{M}\sum_{m=1}^{M-1} \sin m(t-t_k) - \frac{1}{2M}\sin M(t-t_k).$$

For the numerical calculation of the integrals in (19), we apply the so-called sinc-quadrature rule

$$\int_{-\infty}^{\infty} f(\tau)\,d\tau \approx h_\infty \sum_{i=-M_1}^{M_1} f(ih_\infty), \quad M_1 \in \mathbb{N}, \quad h_\infty = \frac{c}{\sqrt{M_1}}, c > 0. \tag{21}$$

Thus, after the application of the Nyström method to the integral equations (18) and the quadrature rule (21) for the computation of the integral in the right-hand side of (19), we obtain the following system of linear equations

$$-\frac{\tilde{\varphi}(t_k)}{|\gamma_2'(t_k)|}+\sum_{j=0}^{2M-1}\left[\frac{c_7}{|\gamma_2'(t_k)|}\tilde{T}_j(t_k)Q+\frac{1}{2M}K_1(t_k,t_j)\right]\tilde{\varphi}(t_j)=\tilde{w}(t_k), \quad k=0,\dots,2M-1,$$

with the right-hand side

$$\tilde{w}(t_k):=2g_2(\gamma_2(t_k))+\frac{h_\infty}{2\pi}\sum_{i=-M_1}^{M_1}K_2(t_k,ih_\infty)f_1(\gamma_1(ih_\infty)).$$

A convergence and error analysis for the Nyström method can be found, for example, in Kress (1999). This analysis exhibits the dependence of the convergence rate on the smoothness of the boundary curve Γ_2 of the inclusion and the boundary function g_2, i.e. the proposed method belongs to the class of algorithms without "saturation effect". The numerical solution $u_M=(u_{1,M},u_{2,M})$ of the problem (5) can be obtained by discretization of the representation (13) via the quadratures (20) and (21).

4.2 Numerical solution of (17)

We can also numerically solve (17) with the above technique and we therefore only shortly describe the numerical solution of the system (17). After using the parametrizations (1) and (2) we arrive to the parametrized system of integral equations which could be transformed as follows:

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\left[-\frac{c_1}{4}\ln\left\{\frac{4}{e}\sin^2\frac{t-\tau}{2}\right\}I+H_1(t,\tau)\right]\varphi(\tau)d\tau+\alpha=\omega(t), \quad t\in[0,2\pi], \\ \displaystyle\int_0^{2\pi}\psi(\tau)d\tau=0, \end{cases} \tag{22}$$

where $\varphi(t):=\psi(\gamma_2(t))|\gamma_2'(t)|$ and the right hand side is given as

$$\omega(t):=g_1(\gamma_2(t))-\frac{1}{2\pi}\int_{-\infty}^{\infty}H_2(t,\tau)f_2(\gamma_1(\tau))d\tau. \tag{23}$$

The kernels have the form

$$H_1(t,\tau) := H_{11}(t,\tau) + H_{12}(t,\tau),$$

where

$$H_{11}(t,\tau) := \begin{cases} \dfrac{c_1}{4} \ln\left\{\dfrac{4}{e}\sin^2\dfrac{t-\tau}{2}\right\} I + \dfrac{c_2}{2}\tilde{J}(t,\tau) & \text{for} \quad t \neq \tau, \\ \dfrac{c_1}{4} \ln\left\{\dfrac{1}{e|\gamma_2'(t)|^2}\right\} + \dfrac{c_2}{2}\tilde{J}(t,t) & \text{for} \quad t = \tau, \end{cases}$$

and

$$H_{12}(t,\tau) := 2\pi V(\gamma_2(t),\gamma_2(\tau)), \qquad H_2(t,\tau) := 2\pi N(\gamma_2(t),\gamma_1(\tau)).$$

The following classical result about well-posedness holds Kress (1999); McLean (2000).

Theorem 4.2 *For $m \in \mathbb{N} \cup \{0\}$, $\Gamma_2 \in C^{m+2}$, $g_1 \in C^{m+1,\beta}[0,2\pi]$ and $f_2 \in C(\Gamma_1)$ satisfying the conditions* (16)*, the integral equation* (22) *has exactly one solution $\varphi \in C^{m,\beta}[0,2\pi]$ and $\alpha \in \mathbb{R}^2$.*

For the full discretization of (22) the quadrature method based on the trigonometric interpolation is applied again Chapko and Kress (1993); Kress (1999). We use the quadrature (20) for the integrals without singularities and in the case of a logarithmic singularity the following quadrature rule Kress (1999)

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left\{\frac{4}{e}\sin^2\frac{t-\tau}{2}\right\} f(\tau)d\tau \approx \sum_{j=0}^{2M-1} R_j(t) f(t_j) \tag{24}$$

with known weight functions

$$R_j(t) = -\frac{1}{2M}\left\{1 + 2\sum_{k=1}^{M-1}\frac{1}{k}\cos k(t-t_j) + \frac{1}{M}\cos(t-t_j)\right\}.$$

For the integrals in the right-hand side in (23) the sinc-quadrature (21) is applied. The numerical solution of the mixed problem (6) can be obtained by discretization of the representation (15) via the quadratures (20), (21) and (24).

5 Numerical experiments

Ex.1 First, we shall investigate the proposed numerical method for the direct mixed Neumann-Dirichlet problem (5). The chosen boundary curves are illustrated in Fig.2.

Here, Γ_1 and the boundary of the inclusion, i.e. Γ_2, are described by the parametrizations

$$\Gamma_1 = \big\{(x_1,x_2)| \quad x_1(t) = t, x_2(t) = 0, \quad -\infty < t < \infty\big\},$$

$$\Gamma_2 = \big\{(x_1,x_2)|x_1(t) = r(t)\cos t, x_2(t) = r(t)\sin t + 2, r(t) = \sqrt{\cos^2 t + 0.25\sin^2 t},$$

$$t \in [0,2\pi]\big\}.$$

The Lamé constants in (5) are $\mu = 1$ and $\lambda = 1$, and the boundary functions are given by

$$g_2(x) = (x_1 + x_2, x_1 - x_2) \quad \text{on} \quad \Gamma_2, \qquad f_1(x) = (x_1 e^{-x_1^2}, x_1 e^{-x_1^2}) \quad \text{on} \quad \Gamma_1.$$

To produce numerical results we choose three points in the solution domain D and these points have coordinates $(0,0.4)$, $(1.4,1)$, and $(0,3)$, respectively, and are marked out in Fig.2.

In Table 1 we demonstrate the convergence of our proposed boundary integral method from Sections 3 and 4, for the mixed Neumann-Dirichlet boundary value

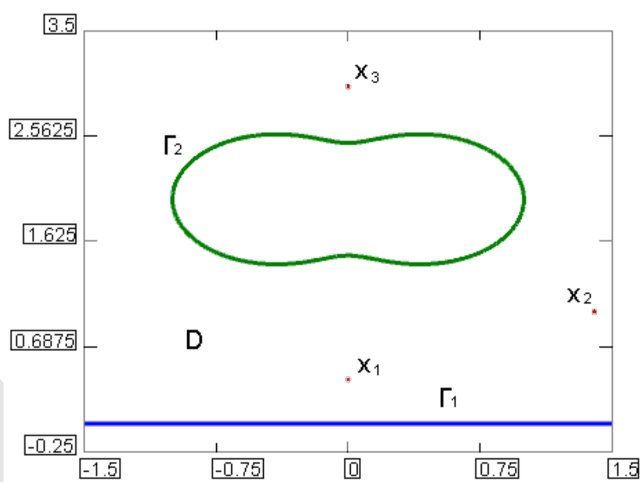

Figure 2: Half-plane with a bounded inclusion.

Table 1: Numerical results for the displacement solution $U_M = (U_{1,M,U_{2,M}})$ in the Neumann-Dirichlet problem (5).

x	M	$u_{1,M}$	$u_{2,M}$
(0,0.4)	16	-0.33684693676588	0.48560794330456
	32	-0.33684812513376	0.48560925550574
	64	-0.33684812524779	0.48560925562875
	128	-0.33684812524779	0.48560925562875
(1.4,1)	16	-1.27722557867857	0.30228160310729
	32	-1.27722902221229	0.30228260448122
	64	-1.27722902253654	0.30228260457376
	128	-1.27722902253654	0.30228260457376
(0,3)	16	-2.86917184535060	1.89717003519494
	32	-2.86918172557303	1.89718275876519
	64	-2.86918172650555	1.89718276002206
	128	-2.86918172650555	1.89718276002206

problem (5), with respect to the number of collocation points M for the three different points (in the solution domain D) given above and fixed $M_1 = 1000$. The exponential convergence with respect to the number M is clearly exhibited in all cases.

Ex.2 Here, we investigate the numerical method proposed for the direct mixed Dirichlet-Neumann boundary value problem (6). The boundary curves are as in the previous example, see Fig.2, and the boundary data functions are given by

$$g_1(x) = (x_1 + x_2, x_1 - x_2) \quad \text{on} \quad \Gamma_2, \qquad f_2(x) = \left(x_1 e^{-x_1^2}, x_1 e^{-x_1^2}\right) \quad \text{on} \quad \Gamma_1.$$

Moreover, the same three points as above are chosen in the solution domain D. In Table 2 we demonstrate the convergence of our method from Sections 3 and 4 for the mixed Dirichlet-Neumann boundary value problem (6) with respect to the number of collocation points M for fixed $M_1 = 1000$. Again, we clearly see the exponential convergence with respect to the number M.

Ex.3 In this numerical example we finally investigate convergence of the alternating algorithm for the Cauchy problem (3). We choose the exact solution of the problem

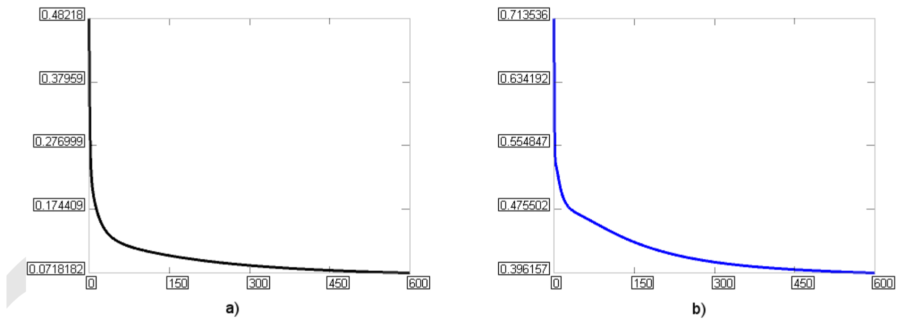

Figure 3: L_2-errors after 600 alternating iterations for the displacements (a)) and for the tractions (b)) in case of the exact Cauchy data.

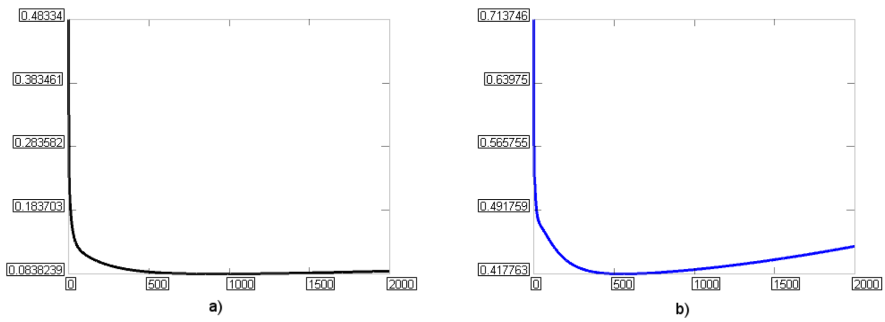

Figure 4: L_2-errors after 2000 alternating iterations for the displacements (a)) and for the tractions (b)) in case of a 10% input data perturbation.

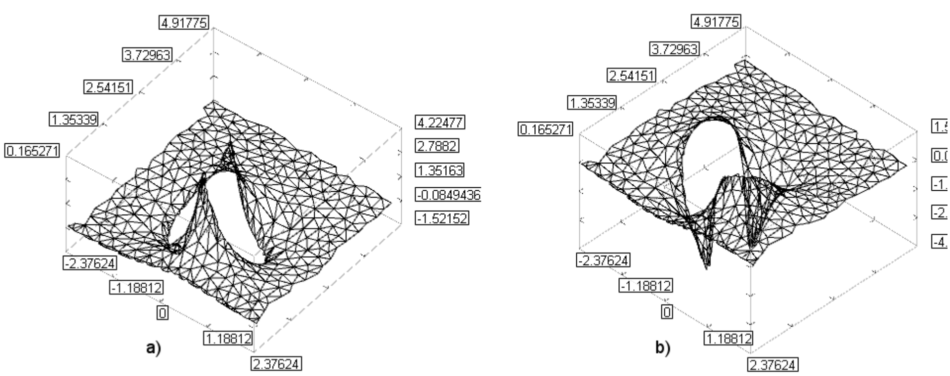

Figure 5: Exact displacement field in D. a) represents the displacement field component u_1 and b) represents the displacement field component u_2.

Table 2: Numerical results for the displacement solution $U_M = (U_{1,M,U_{2,M}})$ in the Dirichlet-Neumann problem (6)

x	M	$u_{1,M}$	$u_{2,M}$
(0,0.4)	16	1.64758725469458	-1.19947152991989
	32	1.64758725187306	-1.19947153235628
	64	1.64758725187314	-1.19947153235656
	128	1.64758725187314	-1.19947153235656
(1.4,1)	16	1.85977712177613	-1.35488948702158
	32	1.85977712931313	-1.35488950906986
	64	1.85977712931324	-1.35488950906975
	128	1.85977712931324	-1.35488950906975
(0,3)	16	2.61131218205450	-2.55788373094839
	32	2.61131222241267	-2.55788359505479
	64	2.61131222241262	-2.55788359505479
	128	2.61131222241262	-2.55788359505479

given by:

$$u(x) := \begin{pmatrix} \dfrac{-x_1^2 - 2x_1(x_2-2) + (x_2-2)^2}{(x_1^2 + (x_2-2)^2)^2} \\ \\ \dfrac{x_1^2 - 2x_1(x_2-2) - (x_2-2)^2}{(x_1^2 + (x_2-2)^2)^2} \end{pmatrix}. \quad (25)$$

The Lamé parameters are set to $\lambda = \mu = 1$. The discretization parameters for the direct problems are $M = 32$ and $M_1 = 500$. The boundary curves are the same as in the previous numerical examples.

In this test example, the exact input Cauchy data are generated from the exact solution (25). The initial guess for the alternating algorithm is chosen as a perturbed exact solution with high rate of perturbation. The perturbed Cauchy data were obtained from the exact input data by adding at each point some arbitrary value which is not greater than the predefined level of noise δ. For visual comparison purposes we demonstrate the exact components of the displacements field in (a part of) the unbounded domain D in Fig. $5^1 1$. The behaviour of the L_2-errors between the exact and computed displacements and stress (traction) field reconstructions, respectively, on the inner boundary Γ_2 are shown in Fig. 3 and Fig. 4 . The results of the reconstruction of the displacements on the inner boundary Γ_2 and inside of the domain in case of the exact input data and in case of the perturbed input data are

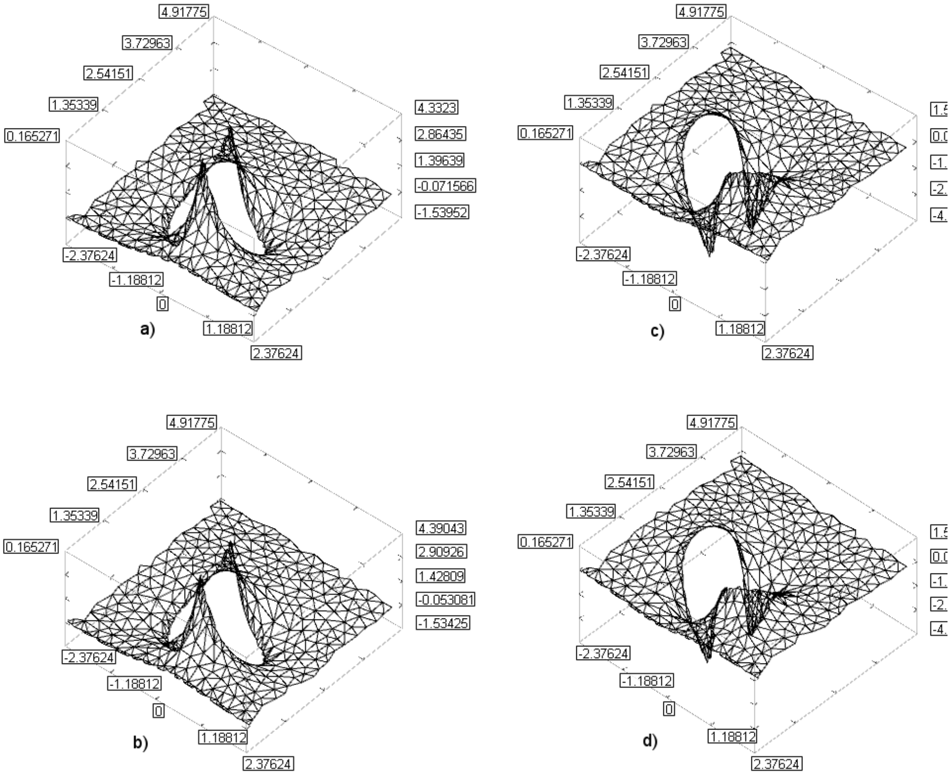

Figure 6: Reconstruction of the displacement field in D. a) and c) are the displacement field components u_1 and u_2 respectively after 1000 iterations in case of the exact input data; b) and d) are the corresponding results obtained after 600 iterations (minimal errors) in case of a 10% perturbation of the input data.

shown in Fig. 6[2]. As expected, the accuracy for the stress field reconstructions is less than the corresponding accuracy of the displacement field. However, it is still a reasonable accurate reconstruction also for the stress field. Furthermore, in the case of noisy data, due to the ill-posedness of the Cauchy problem (3), the alternating iterative procedure has to be terminated since the error in the data will start to magnify after a certain number of iterations. A stopping rule, the discrepancy principle, can be employed to terminate the iterations as explained in Section 2.

[2] We point out that the triangular mesh was used only for purpose of data visualisation and it has no relation to the process of solving of the inverse problem.

References

Arens, T. (1999): A new integral equation formulation for the scattering of plane elastic waves by diffraction gratings. *J. Integral Equations Appl.*, vol. 3 no. 11, pp. 275–297.

Chapko, R., Kress, R. (1993): On a quadrature method for a logarithmic integral equation of the first kind. *Contributions in numerical mathematics*. Ed.: Agarwal, R., World Sci. Ser. Appl. Anal., 2, World Sci. Publ., River Edge, NJ, Singapore, pp. 127–140.

Chapko, R. (2004): On the numerical solution of a boundary value problem in the plane elasticity for a double-connected domain. *Math. Comput. Simulation*, vol. 66, no. 4-5, pp. 425–438.

Chapko, R., Johansson, B. T. (2008): An alternating boundary integral based method for a Cauchy problem for Laplace equation in semi-infinite domains. *Inverse Probl. Imaging*, vol. 3 no. 2, pp. 317–333.

Chen, G., Zhou, J. (1992): Boundary Element Methods. Academic Press, San Diego.

Kozlov, V. A., Maz'ya, V. G. (1989): On iterative procedures for solving ill-posed boundary value problems that preserve differential equations, *Algebra i Analiz*, vol. 5 no. 1, pp. 144–170. English transl.: (1990): *Leningrad Math. J.*, vol. 5 no. 1, pp. 1207–1228.

Kress, R. (1999): *Linear Integral Equations*. 2nd. ed. Springer-Verlag, Heidelberg.

Kubo, S. (1988): Inverse problems related to the mechanics and fracture of solids and structures. *JSME Int. J.*, no. 2 vol. 31, pp. 157–166.

Kupradze, V. D. (1965): *Potential methods in the theory of elasticity*. Israel Program for Scientific Translations, Jerusalem.

Ling, L., Takeuchi, T. (2008): Boundary control for inverse Cauchy problems of the Laplace equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 29 no. 1, pp. 45–54.

Maniatty, A., Zabaras, N., Stelson, K. (1989): Finite element analysis of some elasticity problems.*J. Engng Mech. Div. ASCE*, vol. 6 no. 115, pp. 1302–1316.

McLean, W. (2000): *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press.

Marin, L., Hào, D. N., Lesnic, D. (2002) Conjugate gradient-boundary element method for the Cauchy problem in elasticity. *Quart. J. Mech. Appl. Math.*, vol. 2 no. 55, pp. 227–247.

Marin, L., Lesnic, D. (2005): Boundary element-Landweber method for the Cauchy problem in linear elasticity. *IMA J. Appl. Math.*, vol. 2 no. 70, pp. 1–18.

Mera, N. S., Elliott, L., Ingham, D. B., Lesnic, D. (2000): An iterative boundary element method for the solution of a Cauchy steady state heat conduction problem. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1 no. 3, pp. 101–106.

Sheremet, V. D. (1984): A statical equilibrium of an elastic half plane charged by concentrated force. *Prikladnaya Mekhanika (Applied Mechanics)*, Kyiv vol. 20, pp. 80–88 (in Russian).

Sun, Y. F., Peng, Y. Z. (2003): Analytic solutions for the problems of an inclusion of arbitrary shape embedded in a half-plane. *Appl. Math. Comput.*, vol 1 no. 140, pp. 105–113.

Vainikko, G. M., Veretennikov, A. Y. (1986): *Iteration Procedures in Ill-Posed Problems*. Nauka Publ., Moscow, (in Russian).

Yeih, W. C., Koya, T., Mura, T. (1993): An inverse problem in elasticity with partially overprescribed boundary conditions, I. Theoretical approach. *Trans. ASME J. Appl. Mech.*, vol. 3 no. 60, pp. 595–600.