THE DISCOVERY OF NEPTUNE -
A CRITICAL EXAMINATION OF THE THEORY OF LEVERRIER.

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SUMMARY

This thesis describes an investigation of the problems and queries surrounding the theory of Urbain Jean Joseph LeVerrier for the discovery of the planet Neptune.

Each step of the analytic treatment for each of his three solutions has been systematically examined and a comparison of his results with those of the present author is provided.

The effect of a variation in the number of perturbatory forces and equations of condition is taken into consideration, and orbital elements determined using all the available observations, culminating with a comparison with the final solution as derived by LeVerrier.

For the first time, following LeVerrier's procedure, solutions are given for different values of the ratio between the semi-major axes of Uranus and Neptune, comparing the orbital elements obtained with those of the true values available following its discovery.

The analysis concludes with a derivation of solutions at other epochs, and seeks to determine to what extent LeVerrier's solution was correct only at the time of discovery of Neptune.

Neptune
Orbit
Perturbations
Uranus

CONTENTS

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1. INTRODUCTION

One of the most remarkable episodes in the history of astronomy occurred on the evening of March 13th, 1781, when Sir William Herschel, whilst pursuing an ambitious project of examining every section of the northern skies, discovered near the foot of Gemini, a strangely blue object of unusual size. Suspecting that it might be a comet, he carefully noted its position. Further observations on the following evening confirmed its motion and the discovery was promptly conveyed to Greenwich and Oxford.

For a comet, the new object was certainly peculiar. It was so starlike that even experienced observers had difficulty in finding it. Nevil Maskeline, the Astronomer Royal, suspected that the object was a planet and not a comet, an idea quickly confirmed by the Russian astronomer, Andres Johann Lexell, who happened to be visiting England at the time. Lexell's computations showed that the new object moved in a nearly circular orbit, always remaining more distant than Saturn which was the most distant planet then known. The new object was Uranus, the seventh planet belonging to the solar system.

Joseph Jerome Lafrancais de Laland was among the first to prove by calculation of its orbit that Herschel's object was a planet. Because of the smallness of the eccentricity of the then known planets, it was first assumed that the eccentricity of the new planet Uranus was also small, and accordingly the early investigations by Lexell (1781), Laland

(1782) and Laplace (1783) were based on the assumption of a circular orbit. As more observational material became available Laplace (1784), Boscovitch (1785), Fixlmillner (1789) and others made several attempts to determine simple elliptic elements, whilst Oriani (1789), Delambre (1789) and Laplace (1802) took into account the corrections to be made due to the perturbations of the planet by Jupiter and Saturn. As a result of the steadily improving values found for the elements of the orbit, it soon became possible to determine with reasonable accuracy the position of Uranus prior to its discovery.

As Uranus was a little fainter than stars just visible to the naked eye it occurred to the German astronomer Johann Elert Bode that possibly Uranus had been mistaken for a star in the past, in which event its position would be recorded in the contemporary catalogues. A search of the older catalogues proved more successful than had been anticipated, for Uranus had been observed on no fewer than 19 different occasions, the observations most remote in time being those made in 1690 by Flamsteed, the first Astronomer Royal. The observations made between 1690 and 1771 were referred to as 'ancient' observations to distinguish them from the possibly somewhat more reliable observations made subsequent to 1781 and which later were termed the 'modern' observations. The ancient observations had the advantage of covering rather more than one complete revolution of Uranus in its orbit, while the modern observations covered rather less than half a revolution, but were regarded as being more accurate than the ancient ones.

However, even with this wider range of observations, all efforts to produce a satisfactory ephemeris proved only partially successful. For, although the theories appeared to be satisfied by the modern and, where applicable, the ancient observations, the disagreement between observations and theory became significant within a few years of publishing tables of the planet.

In 1821 the French astronomer, Alexis Bouvard, published his tables for the three exterior major planets, Jupiter, Saturn and Uranus, having forty years of modern observations available for the latter. Bouvard had faced a major problem. No orbit could be found that would satisfy both ancient and modern observations. If the two series of observations were utilised, the discrepancies between observation and theory were excessively large, ranging from $-40''.2$ to $+32''.0$ sexagesimal. On the other hand, if the orbit was derived from the modern observations alone the largest discrepancy was reduced to $+10''.5$. Using the ancient observations alone, the largest discrepancy amounted to $73''.8$. As a consequence, Bouvard rejected the ancient observations as unreliable and based his tables solely upon the modern observations, writing in the introduction: "I leave to the future the task of determining whether the difficulty arises from the inaccuracy of the ancient observations or whether it depends upon some strange and unknown cause which may have been acting on the planet."

Laplace spoke of "some extraneous and unknown influence which has acted upon the planet."

Within a few years of the publication of Bouvard's tables, Uranus was seen to deviate more and more from the predicted positions, the discrepancy in longitude increasing alarmingly.

The opinion became widespread among astronomers that there must be an unknown body disturbing the motion of Uranus. In the late eighteen thirties Bessel persuaded one of his students, F. W. Flemming, to attempt a computation of this unknown body from the deviations in the orbit of Uranus. Unfortunately Flemming died just after the work had started.

In 1841 the discrepancy in geocentric longitude reached 70", and it became evident that these discrepancies were almost certainly due to the presence of an unknown planet.

Such was the position in 1841 when John Couch Adams, an undergraduate at St. John's College, Cambridge, penned his celebrated memorandum of July 3rd in which he recorded his determination, as soon as he had taken his degree, to investigate the irregularities in the motion of Uranus on the hypothesis that these arose from the perturbation of an unknown planet. Adams began his investigations, producing between 1843 and 1846 six solutions of ever-increasing accuracy.

In the meantime, the problem was undertaken by Urbain Jean Joseph LeVerrier (1811-1877), quite independently of and unknown to Adams. In the summer of 1845, LeVerrier began a careful study of the nature of the irregularities of the motion of Uranus and the cause of the unexpected inequalities, seeking to discover the direction and magnitude of the disturbing body. There appeared to be three possibilities. Either

(i) the irregularities were caused by a large satellite accompanying Uranus,

or (ii) a comet had suddenly disturbed the motion of Uranus,

or (iii) the irregularities were attributable to the presence of a hitherto unknown planet.

The first possibility was ruled out as the corresponding oscillations produced in the motion of Uranus would be of short period whereas precisely the opposite results from the observations.

As for the cometary theory, LeVerrier was well satisfied with the movement of Uranus between 1781 and 1820 without any recourse to any extraordinary force. However, from about 1826 onward, Uranus began to deviate so much from its predicted position that such a theory became untenable.

There remained the third hypothesis, viz., that of a body continuously acting upon Uranus changing its movement very gradually. From what was known about the solar system at that time, such a body could only be a hitherto unknown planet.

Thus motivated, LeVerrier inquired: "Is it possible that the inequalities of Uranus could be due to the action of a planet situated in the ecliptic at an average distance double that of Uranus? If so, where is this planet actually situated? What is its mass? What are the elements of the orbit which it traverses?"

On 1st June, 1846, LeVerrier presented his first results to the Academy of Sciences, and on 23rd September of that year Neptune was discovered. When details of the orbit of the new planet finally became available it was found that the elements of both Adams and LeVerrier were almost totally erroneous, as demonstrated in the following table:

Orbital Elements	LeVerrier	Adams	Neptune
Semi-major Axis (A.U.)	36.15	37.25	30.07
Eccentricity	0.1076	0.1206	0.0086
Longitude of Perihelion	284° 45'	299° 11'	44°
Mass of sun/Mass of Neptune	9300	6666	19300
True Longitude (at time of discovery)	326° 0'	329° 27'	326° 57'

For this reason it has always seemed desirable that an investigation be conducted into the problems and queries surrounding the theoretical procedures involved in the theories of both Adams and LeVerrier to see to what extent their solutions were graced by good fortune. Whereas the work of Adams has been thoroughly investigated, (Brookes, 1970), that of LeVerrier has not.

The present research is a systematic analysis of LeVerrier's theory and seeks to determine to what extent LeVerrier's solution was correct only at the time of discovery of Neptune. This work is divided into six main sections:

The first section is an examination of the validity of the various stages of LeVerrier's first solution, including re-calculations of the elements of Neptune using the same data and making the same assumptions as LeVerrier. The second and third sections represent analyses of LeVerrier's second and final solutions, respectively. Section Four includes a determination of the elements of Neptune for different values of $\alpha = \frac{a}{a'} = 0.49, 0.51$ up to $\alpha = 0.60$, to determine to what extent LeVerrier's statement $\alpha < 0.5475$ is true. Section Five contains the derivation of orbital elements determinable at times other than 1845. The final section provides a comparison of the results of LeVerrier, Adams, Brookes and the present author in order to examine the final discrepancies in the orbit of Neptune.

2. LEVERRIER'S FIRST SOLUTION

The main aim of this solution is an initial determination of the mass m' of the disturbing planet. Let a denote the semi-major axis of the orbit of Uranus, e its eccentricity, $\tilde{\omega}$ and ε the longitudes of perihelion and of the epoch respectively. Let a', e', $\tilde{\omega}'$, ε' denote the corresponding orbital elements for Neptune, its mass m' being related to the ten-thousandth part of the sun's mass which was taken as unity. We write

$$\frac{a}{a'} = \alpha \;, \quad \text{and} \; \frac{n'}{n} = \alpha^{3/2} = \nu \;, \tag{2.1}$$

n,n' denoting the mean motion of Uranus and Neptune respectively.

If v_c represents the calculated longitude of Uranus at a given epoch t as determined from Bouvard's tables, and $n+\Delta n$, $\varepsilon+\Delta\varepsilon$, $e+\Delta e$ and $\tilde{\omega}+\Delta\tilde{\omega}$ represent the corrected values of the orbital elements, then the corresponding error in Bouvard's longitude is

$$\Delta v = \frac{\partial v}{\partial n}\cdot\Delta n + \frac{\partial v}{\partial \varepsilon}.\Delta\varepsilon + \frac{\partial v}{\partial e}.\Delta e + \frac{\partial v}{\partial \tilde{\omega}}.\Delta\tilde{\omega} \;,$$

the coefficients $\frac{\partial v}{\partial n}$, $\frac{\partial v}{\partial \varepsilon}$, $\frac{\partial v}{\partial e}$ and $\frac{\partial v}{\partial \tilde{\omega}}$ being well-known functions of t, n, ε, e and $\tilde{\omega}$. Thus, if v_o denotes the observed longitude for the epoch t, then

$$v_o - v_c = \mathfrak{L} + \frac{\partial v}{\partial n}.\Delta n + \frac{\partial v}{\partial \varepsilon}.\Delta\varepsilon + \frac{\partial v}{\partial e}.\Delta e + \frac{\partial v}{\partial \tilde{\omega}}.\Delta\tilde{\omega} \;, \tag{2.2}$$

$\mathfrak{L}$ denoting the perturbation in v due to the effect of

Neptune.
The origin of time was taken at midnight between 31st December, 1799 and 1st January, 1800.

The perturbations in heliocentric longitude introduced by LeVerrier in his first solution took the form:

$$\begin{aligned}
\delta v = {} & m'P^{(1)} \sin[(n'-n)t + \varepsilon'-\varepsilon] \\
& + m'P^{(2)} \sin[2(n'-n)t + 2(\varepsilon'-\varepsilon)] \\
& + m'P^{(3)} \sin[3(n'-n)t + 3(\varepsilon'-\varepsilon)] \\
& + \\
& m'N^{(1)} \sin(n't + \varepsilon'-\tilde{\omega}) \\
& + m'N^{(2)} \sin[(2n'-n)t + (2\varepsilon'-\varepsilon)-\tilde{\omega}] \\
& + m'N^{(3)} \sin[(3n'-2n)t + (3\varepsilon'-2\varepsilon)-\tilde{\omega}] \\
& + \\
& m'e'M^{(1)} \sin(n't + \varepsilon'-\tilde{\omega}') \\
& + m'e'M^{(2)} \sin[(2n'-n)t + (2\varepsilon'-\varepsilon)-\tilde{\omega}'] \\
& + m'e'M^{(3)} \sin[(3n'-2n)t + (3\varepsilon'-2\varepsilon)-\tilde{\omega}']
\end{aligned} \qquad (2.3)$$

the coefficients $P^{(i)}, N^{(i)}$ and $M^{(i)}$ ($\forall i = 1,2,3$) being functions of Laplace coefficients and their derivatives. For example,

$$P^{(1)} = \frac{\alpha}{(1-\nu)^2}\left\{1 - \frac{4}{\nu(2-\nu)}\right\}(\alpha - b_{\frac{1}{2}}^{(1)}) - \frac{2\alpha}{\nu(1-\nu)(2-\nu)} \times \left(\alpha - \alpha\,\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}\right)$$

$b^{(1)}_{\frac{1}{2}}$ being the Laplace coefficient associated with $P^{(1)}$, with similar expressions for the other terms. The coefficients $P^{(i)}$ and $M^{(i)}$ are independent of the eccentricity e, while $N^{(i)}$ is dependent on e.

For higher orders of eccentricity, the only perturbations of any significance are those of the second order, which depend on the argument (3n'-n), and become considerable because of the smallness of this argument. The corresponding perturbations of the mean motion contain the square of this argument as divisor, the relevant equation giving rise to this term being of the form

$$\frac{d^2\ell}{dt^2} = -\ 3an^2m'\frac{dR}{d\varepsilon} \tag{2.4}$$

wherein,

$$\begin{aligned} R = &+ \frac{e^2}{a'}\ B\cos(3\ell'-\ell-2\tilde{\omega}) \\ &+ \frac{ee'}{a'}\ C\cos(3\ell'-\ell-\tilde{\omega}'-\tilde{\omega}) \\ &+ \frac{e'^2}{a'}\ D\cos(3\ell'-\ell-2\tilde{\omega}')\ , \end{aligned} \tag{2.5}$$

neglecting terms dependent upon the inclination, B,C and D being functions of Laplace coefficients.

Forming $\frac{dR}{d\varepsilon}$, substituting in (2.4) and integrating with respect to the time t, we obtain the following perturbations in the mean longitude ℓ:

$$\begin{aligned} \delta\ell = &\ \frac{3B\alpha n^2}{10000''(3n'-n)^2}\ m'e^2\sin(3\ell'-\ell-2\tilde{\omega}) \\ &+ \frac{3C\alpha n^2}{10000''(3n'-n)^2}\ m'ee'\sin(3\ell'-\ell-\tilde{\omega}-\tilde{\omega}') \end{aligned}$$

$$+ \frac{3D\alpha n^2}{10000''(3n'-n)^2} m'e'^2 \sin(3\ell'-\ell-2\tilde{\omega}'). \qquad (2.6)$$

For a first approximation, however, LeVerrier retained only those terms up to the first-order in e, delaying the inclusion of the second-order terms until more accurate solutions were required.

With $\alpha = 0.5$, as given by Bode's Law, the values of $P^{(i)}$, $N^{(i)}$ and $M^{(i)}$ in equation (2.3) are:

i	$P^{(i)}$	$N^{(i)}$	$M^{(i)}$
1	18".491 (18".5)	1".895 (1".90)	43" (43")
2	29.480 (29.5)	17.002 (17.00)	121 (122)
3	2.899 (2.9)	24.411 (24.40)	930 (930)
4	0.641	0.850	33
5	0.184	0.154	7
6	0.065	0.067	2

Table 2.1 Numerical values of $P^{(i)}, N^{(i)}, M^{(i)}$ ($\forall i=1,2,\dots,6$).

the values in parentheses being those of LeVerrier. The magnitude of the ignored terms (for i=4,5,6) are included for further comparison.

With these numerical values available, and writing

$$\left.\begin{array}{ll} h = e\sin\tilde{\omega}, & h' = e'\sin\tilde{\omega}' \\ \ell = e\cos\tilde{\omega}, & \ell' = e'\cos\tilde{\omega}' \end{array}\right\} \qquad (2.7)$$

eight expressions of the form (2.3) were produced to give

$$A^{(i)} m' + H^{(i)} m'h' + L^{(i)} m'\ell' = P^{(i)} \qquad (2.8)$$

($\forall i=1,2,...,8$) representing the perturbations in heliocentric longitude at intervals of $\tau=14$ years, the epoch being 1747.7.

Eight equations of condition were then formed, between the corrections of the elliptic elements and the heliocentric tabular errors, for every 14 years from 1747.7 to 1845.7 neglecting terms of degree greater than the equation of centre, and omitting the terms in $e\delta\varepsilon$ and $e\delta n$, which would produce only small errors. The general form of the equation thus formed, is

$$\delta\varepsilon + (2j-1)\tau.\delta n + 2\sin[\zeta+(j-1)n\tau].\delta e - 2\cos[\zeta+(j-1)n\tau] \times e\delta\tilde{\omega} + \nu_j + \mathfrak{L}^{(j)} = 0 \qquad (2.9)$$

$\forall j=1,2,...,8$

ν_j being the excess of the calculated heliocentric longitude, given in Table 2.2, and ζ the mean anomaly at the epoch.

Eliminating $\delta\varepsilon$ by calculating the first-differences $\delta\nu_i$, and δn by calculating the second-differences $\delta^2\nu_i$ we have six equations in δe, $e\delta\tilde{\omega}$, $\delta^2\nu_i$ and $\delta^2\mathfrak{L}^{(i)}$ ($\forall i=1,2,...,6$), each of the form:

$$-8\sin^2\left(\frac{n\tau}{2}\right)\sin(\zeta+in\tau)\,\delta e + 8\sin^2\left(\frac{n\tau}{2}\right)\cos(\zeta+in\tau)\,e\delta\tilde{\omega} + \delta^2\nu_i + \delta^2\mathfrak{L}^{(i)} = 0 \qquad (2.10)$$

Epochs	Excess of Calculated Heliocentric Longitude ν_i	$\delta\nu_i$ First Differences	$\delta^2\nu_i$ Second Differences	$\delta^2\mathfrak{P}^{(i)}$
1747.7	+34".8	-	-	-
1761.7	+24.7	-10".1	-	-
1775.7	-3.7	-28.4	-18".3	+18".3
1789.7	-28.6	-24.9	+3.5	-3.5
1803.7	-33.6	-5.0	+19.9	-19.9
1817.7	-32.3	+1.3	+6.3	-6.3
1831.7	+3.4	+35.7	+34.4	-34.4
1845.7	+110.5	+107.1	+71.4	-71.4

Table 2.2 Numerical values of excess calculated heliocentric longitude for different epochs and the values of $\delta\nu_i$ $\delta^2\nu_i$ and $\delta^2\mathfrak{P}^{(i)}$ $(\forall i=1,2,\ldots,8)$

Substituting for $n\tau=60°$ into these equations and combining appropriately, gives

$$\left.\begin{aligned}
\delta^2\nu_1 + \delta^2\nu_4 + \delta^2\mathfrak{P}^{(1)} + \delta^2\mathfrak{P}^{(4)} &= 0\\
\delta^2\nu_2 + \delta^2\nu_5 + \delta^2\mathfrak{P}^{(2)} \quad \delta^2\mathfrak{P}^{(5)} &= 0\\
\delta^2\nu_3 + \delta^2\nu_6 + \delta^2\mathfrak{P}^{(3)} \quad \delta^2\mathfrak{P}^{(6)} &= 0
\end{aligned}\right\} \qquad (2.11)$$

Adopting, for example, a value $\varepsilon'=270°$(centesimal) equations (2.11) reduce to

$$\left.\begin{aligned}
24''.5m' + 24''m'h' - 156m'\ell' - 12''.0 &= 0\\
-12.7m' + 147m'h' - 85m'\ell' + 37.9 &= 0\\
-58.0m' + 160m'h' + 65m'\ell' + 91.3 &= 0
\end{aligned}\right\} \qquad (2.12)$$

from which we find

$$m' = 2.02\ (2.11).$$

3. LEVERRIER'S SECOND SOLUTION

It has been shown in the first solution that LeVerrier restricted himself to eight equations of condition, at intervals of fourteen years, between the epochs 1747.7 and 1845.7. Choosing epochs at intervals of seven years instead of fourteen gives rise to fifteen equations instead of eight, to which LeVerrier added three further equations corresponding to the epochs 1690.98, 1712.25 and 1715.23 of Flamsteed, giving eighteen equations of condition in all. The equations are:

1690.98: $0.977\delta\varepsilon - 106.5\delta n - 1.912\delta e + 0.487e\delta\tilde{\varpi} - 63.1 + (1) = 0$

1712.25: $1.097\delta\varepsilon - 96.2\delta n - 0.388\delta e - 2.086e\delta\tilde{\varpi} - 59.9 + (2) = 0$

1715.23: $1.099\delta\varepsilon - 93.2\delta n + 0.111\delta e - 2.121e\delta\tilde{\varpi} - 64.6 + P = 0$

1747.7 : $0.928\delta\varepsilon - 48.5\delta n + 1.127\delta e + 1.542e\delta\tilde{\varpi} + 34.8 + (3) = 0$

1754.7 : $0.912\delta\varepsilon - 41.3\delta n + 0.254\delta e + 1.874e\delta\tilde{\varpi} + 32.8 + (4) = 0$

1761.7 : $0.917\delta\varepsilon - 35.1\delta n - 0.669\delta e + 1.775e\delta\tilde{\varpi} + 24.7 + (5) = 0$

1768.7 : $0.941\delta\varepsilon - 29.5\delta n - 1.461\delta e + 1.257e\delta\tilde{\varpi} + 10.0 + (6) = 0$

1775.7 : $0.982\delta\varepsilon - 23.9\delta n - 1.939\delta e + 0.393e\delta\tilde{\varpi} - 3.7 + Q = 0$

1782.7 : $1.031\delta\varepsilon - 17.8\delta n - 1.932\delta e - 0.653e\delta\tilde{\varpi} - 17.4 + (7) = 0$

1789.7 : $1.074\delta\varepsilon - 11.1\delta n - 1.355\delta e - 1.595e\delta\tilde{\varpi} - 28.6 + (8) = 0$

1796.7 : $1.098\delta\varepsilon - 3.6\delta n - 0.320\delta e - 2.097e\delta\tilde{\varpi} - 29.8 + (9) = 0$

1803.7 : $1.091\delta\varepsilon + 4.0\delta n + 0.830\delta e - 1.943e\delta\tilde{\varpi} - 33.6 + (10) = 0$

1810.7 : $1.056\delta\varepsilon + 11.3\delta n + 1.686\delta e - 1.202e\delta\tilde{\varpi} - 35.3 + R = 0$

1817.7 : $1.008\delta\varepsilon + 17.8\delta n + 2.004\delta e - 0.171e\delta\tilde{\varpi} - 32.3 + (11) = 0$

1824.7 : $0.962\delta\varepsilon + 23.8\delta n + 1.773\delta e + 0.821e\delta\tilde{\varpi} - 24.5 + (12) = 0$

1831.7 : $0.928\delta\varepsilon + 29.4\delta n + 1.129\delta e + 1.541e\delta\tilde{\varpi} + 3.4 + (13) = 0$

1838.7 : $0.912\delta\varepsilon + 35.3\delta n + 0.255\delta e + 1.874e\delta\tilde{\varpi} + 50.0 + (14) = 0$

1845.7 : $0.917\delta\varepsilon + 41.9\delta n - 0.666\delta e + 1.775e\delta\tilde{\varpi} + 110.5 + S = 0$

The perturbations in the equations above, at the epochs 1715.23, 1775.7, 1810.7 and 1845.7 are represented by P,Q,R and S respectively, the others being numbered from (1) to (14). The four former equations were then solved for $\delta\varepsilon, \delta n, \delta e$ and $e\delta\tilde{\omega}$ in terms of P,Q,R and S which were then inserted in the remaining fourteen equations. The last twelve of these equations were then divided into three groups of four equations which were then combined into three final equations corresponding to the mean epochs in 1758, 1793 and 1828.

viz.,

$$
\begin{aligned}
1758&: 7.711P-8.094Q-6.114R+10.499S-[(3)+(4)+(5)+(6)]+805''.5 = 0\\
1793&: 3.897P-8.311Q-5.737R+6.154S+[(7)+(8)+(9)+(10)]+552''.1 = 0 \qquad (3.1)\\
1828&: 3.312P-5.516Q-0.752R+6.959S-[(11)+(12)+(13)+(14)]+605''.4 = 0
\end{aligned}
$$

Calculation of the coefficients did not yield any significant difference between them and those determined by LeVerrier. It was then assumed that the solution of equations (3.1) should satisfy all the equations of condition between the epochs 1775.7 and 1845.7 reasonably accurately. However, this was not necessarily so for those equations of condition corresponding to the epochs between 1715.23 and 1775.7. The equation for the epoch 1712.25 was discarded as it was thought to be too close to the epoch 1715.23 to be of any value.

The most appropriate value of ε' was to be decided upon by testing its value in the epochs 1747.7 and 1690.98. Having thus made the decision, it was necessary to proceed by evaluating A,H and L (the latter deduced from H), for each of the epochs chosen, as given by equations (2.3) and (2.8) of the previous section.

LeVerrier, then, rearranged equations (3.1) under the form:

$$\left.\begin{aligned} am'+bm'h'+cm'\ell' &= d\,, \\ a'm'+b'm'h'+c'm'\ell' &= d'\,, \\ a''m'+b''m'h'+c''m'\ell' &= d''\,, \end{aligned}\right\} \qquad (3.2)$$

where

$$d = -805''.5-7.711P'+8.094Q'\,,$$
$$d' = -552''.1-3.897P'+8.311Q'\,,$$
$$d'' = -605''.4-3.312P'+5.516Q'\,,$$

P' and Q' being the errors of the observations in the epochs 1715.23 and 1775.7, while the errors for the epochs 1810 and 1845 were neglected, as their observations were determined to a great degree of accuracy. With A,H and L known, the coefficients a,b,c,a',b',c',a'',b'' and c'' become functions of ε' only. For example:

$$a=+307''.0\sin\varepsilon'+312''.2\sin 2\varepsilon'+54''.7\sin 3\varepsilon'$$
$$+437''.5\cos\varepsilon'-358''.9\cos 2\varepsilon'-139''.2\cos 3\varepsilon',$$

$$b=-448''\sin\varepsilon'+1032''\sin 2\varepsilon'-6379''\sin 3\varepsilon'$$
$$-653''\cos\varepsilon'-1319''\cos 2\varepsilon'+1575''\cos 3\varepsilon',$$

$$c = +653'' \sin \varepsilon' + 1319'' \sin 2\varepsilon' - 1575'' \sin 3\varepsilon'$$
$$-448'' \cos \varepsilon' + 1032'' \cos 2\varepsilon' - 6379'' \cos 3\varepsilon' .$$

Similar expressions can be written for the remaining terms. Comparison of the coefficients determined by LeVerrier and those determined by the author, indicate overall differences of less than 3%.

The right-hand sides of the above expressions for the coefficients can be rearranged in the form:

$$\sum_{i=1}^{3} (b_i \sin i\varepsilon' + c_i \cos i\varepsilon') \tag{3.3}$$

which may be combined in pairs to give expressions of the form:

$$\sum_{i=1}^{3} a_i \sin(\alpha_i + i\varepsilon'). \tag{3.4}$$

Using equation (3.4), the value of the mass m' was determined by a transcendental equation which contains only the unknown ε'.

viz.,

$$m' = \frac{d(b'c''-c'b'')+d'(cb''-bc'')+d''(bc'-cb')}{a(b'c''-c'b'')+a'(cb''-bc'')+a''(bc'-cb')} = \frac{N}{D}$$

i.e.

$$Dm' - N = 0 , \tag{3.5}$$

where D and N are functions of ε' only and are considerably affected by the observational errors. Most of the coefficients appearing in the quantities D and N are very small compared to those appearing in the original equations, so that the slightest change in the data leads to considerable changes in m'.

In order to obtain an idea of the value of ε', LeVerrier first transformed expression (3.4) by substituting the value

$$\tan \frac{\varepsilon'}{2} = x ,$$

to give alternate expressions for the coefficients a,b,..., b",c", for example:

$$\begin{aligned}(1+x^2)^3 a = (c_1+c_2+c_3) + (2b_1+4b_2+6b_3)x + (c_1-5c_2-15c_3)x^2 \\ -(4b_1 \quad -20b_3)x^3-(c_1+5c_2-15c_3)x^4 \\ +(2b_1-4b_2+6b_3)x^5-(c_1-c_2+c_3)x^6.\end{aligned}$$

Similar expressions can be obtained for the remaining coefficients. Considering the previous calculations, the values of the differences (bc'-cb'),(cb"-bc") and (b'c"-c'b") are dependent on x^{12}, while each one of them contains only the trigonometric quantities ε' and $2\varepsilon'$. Transformation was made for each of the differences, such that it contained x^4 only, which simplifies the subsequent calculations. Appropriate expressions were determined for N and D in terms of polynomials in x of the fourth and tenth degrees respectively, concluding that $N < 0$ for all values of x.

The expression obtained for D was:

$$\left(\frac{1+x^2}{10}\right)^5 D = -(1+0.086767\,x)(x+1.12748)(x+0.011491)$$

$$\times(x-.123707)F^{(6)} \qquad (3.6)$$

where

$$F^{(6)} = 5602+11596x+29193x^2+25857x^3$$
$$+22430x^4+14560x^5+3447x^6.$$

$F^{(6)}$ does not indicate any real factor and excludes all the possibilities of a new positive factor.

From equation (3.6), bearing in mind that D must be negative in order that $m' > 0$, LeVerrier deduced that ε' must satisfy at least one of the inequalities:

$$96°40' < \varepsilon' < 189°55'$$
$$263°08' < \varepsilon' < 358°41'.$$

However, the expression given in equation (3.6) does not appear to represent D. The correct form is found to be:

$$\left(\frac{1+x^2}{10}\right)^5 D = -(1+0.064626x)(x+0.6921056)(x+0.01579194)$$
$$\times(x-1.0382822)F^{(6)} ,$$

using an algorithm developed by the present author (see Appendix),

where

$$F^{(6)} = 9700+16094x+45970x^2+27706x^3$$
$$+31753x^4+16416x^5+3547x^6.$$

The corresponding ranges of values of ε' are found to be

$$92°09' < \varepsilon' < 187°24'$$
$$290°38' < \varepsilon' < 358°11' \qquad (3.7)$$

The differences between these values and those of LeVerrier are attributable to the sensitivity of the numerical procedure. The limiting values of ε' produced by this technique should not be regarded as particularly significant. This can be demonstrated by reference to LeVerrier's subsequent treatment, in which he, presumably, tested the above theory by inserting different values of ε', from 0° to 351° at intervals of 9°, in the equations of condition.

Although LeVerrier concluded that ε' did indeed lie within his estimated ranges, the present research indicates some differences. For example, when ε' lies between 252° and 261°, we find that values of m', $m'h'$, $m'\ell'$ as determined from equation (3.2), are

ε'	m'	$m'h'$	$m'\ell'$
225°	+101.818(-ve...)	+7.929(-ve...)	+10.790(-ve...)
234°	+43.268(-ve...)	+4.575(-ve...)	+3.201(-ve...)
243°	+38.123(-ve...)	+4.298(-ve...)	+1.236(-ve...)
252°	+37.231(-ve...)	+3.934(-ve...)	-0.176(-ve...)
261°	+30.761(-ve...)	+2.268(-ve...)	-0.747(-ve...)

Table 3.1

indicating positive values of m'. Further confirmation appears later when we find that LeVerrier selects $\varepsilon'=252°$ which lies outside his chosen range. Additional values of m', m'h' and m'ℓ' are given in Table 3.2.

The errors corresponding to the epochs 1690 and 1747 were then calculated in order to test whether the previous limitations (3.7) would permit the prediction of the position of Uranus at the above-mentioned epochs. These errors were written in the following form:

$$\left.\begin{array}{l} 1690{:}\ a_0 m'+b_0 m'h'+c_0 m'\ell'-182''.6-1.913P'+0.904Q' \ , \\ 1747{:}\ a_1 m'+b_1 m'h'+c_1 m'\ell'-263.3-2.745P'+3.351Q' \ , \end{array}\right\} \quad (3.8)$$

the coefficients $a_0, b_0, c_0, a_1, b_1, c_1$ having forms similar to those of equation (3.2).

The numerical values of these coefficients, corresponding to different values of ε' between 0° and 351°, were calculated by neglecting the errors of observations P' and Q'. These coefficients, together with the values of m', m'h' and m'ℓ' form the theoretical errors in the above-mentioned epochs for 1690 and 1747. From an examination of these errors LeVerrier was forced to the conclusion that ε' must lie outside the limits estimated previously, in spite of his earlier conviction regarding the range of values of ε'. Corresponding expressions were derived for the sum of the errors, of the theory, of the longitudes of the four epochs 1747,1754, 1761 and 1768, in addition to those of 1690 and 1747.

ε'	m'		m'h'		m'ℓ'	
90°	-	-				
99°	+ 18.644	(+18.242)	-1.484	(-1.451)	- 1.254	(-1.233)
108°	+ 4.255	(+ 4.314)	-0.613	(-0.614)	- 0.376	(-0.380)
117	+ 2.753	(+ 2.764)	-0.588	(-0.584)	- 0.131	(-0.133)
126	+ 2.345	(+ 2.377)	-0.568	(-0.570)	+ 0.090	(+0.086)
135	+ 2.343	(+ 2.308)	-0.496	(-0.488)	+ 0.299	(+0.293)
144	+ 2.652	(+ 2.587)	-0.368	(-0.358)	+ 0.480	(+0.466)
153	+ 3.385	(+ 3.310)	-0.200	(-0.196)	+ 0.624	(+0.608)
162	+ 4.966	(+ 4.700)	-0.021	(-0.016)	+ 0.747	(+0.708)
171	+ 8.927	(+ 8.352)	+0.122	(+0.115)	+ 0.961	(+0.895)
180	+ 24.845	(+19.790)	+0.107	(+0.101)	+ 2.006	(+1.590)
189°	-	+				
216	-	-				
225°	+101.818	-	+7.929		+10.790	
234	+ 43.268	-	+4.575		+ 3.201	
243	+ 37.231	-	+4.298		+ 1.236	
252	+ 38.123	-	+3.934		- 0.176	
261	+ 30.761	-	+2.268		- 0.747	
270	+ 18.505	(+40.077)	+0.816	(+1.790)	- 0.302	(-0.940)
279	+ 11.272	(+14.411)	+0.410	(+0.495)	+ 0.128	(+0.099)
288	+ 7.931	(+ 8.606)	+0.474	(+0.495)	+ 0.267	(+0.276)
297	+ 6.430	(+ 6.381)	+0.641	(+0.637)	+ 0.171	(+0.167)
306	+ 5.833	(+ 5.780)	+0.746	(+0.743)	- 0.085	(-0.086)
315	+ 5.772	(+ 5.564)	+0.703	(+0.677)	- 0.409	(-0.402)
324	+ 6.152	(+ 5.742)	+0.491	(+0.452)	- 0.691	(-0.659)
333	+ 7.127	(+ 6.780)	+0.165	(+0.149)	- 0.834	(-0.805)
342	+ 9.487	(+ 9.005)	-0.154	(-0.160)	- 0.803	(-0.771)
351	+ 18.296	(+16.478)	-0.278	(-0.644)	- 0.708	(-0.282)
360°	-	-	-			

Table 3.2: Numerical values of m',m'h',m'ℓ' according to different values of ε'.

From an analysis of the above errors, and assuming that m' cannot be greater than 4 units without introducing to the longitude of Saturn inequalities which do not exist, LeVerrier deduced that for the three epochs considered, the errors continuously decrease as ε' increases. Furthermore, that the errors become infinitely small when ε' lies between 243° and 252°.

For example, with $m'=0.8$, $P'=-15''$ and $Q'=-10''$, only the following errors remain:

In	1758:	-5"(-6") ,
	1690:	-13"(-13"),
	1747:	0"(-2") ,

the figures in parentheses being those of LeVerrier.

From this brief discussion of the errors, LeVerrier reached the following conclusion:

> "There is in the ecliptic only one region in which the disturbing planet can be localized, so as to realize the movement of Uranus; that the mean longitude of this planet should be, on the first of January 1800, between 243° and 252°."

To assign the region of the ecliptic in which the unknown planet must be placed, assuming that the individual errors of the previous eighteen epochs, from 1690 to 1845, represent effectively the observations of the disturbing planet (whose longitude of epoch ε' lies in the range between 234° and 270°), let

$$\varepsilon' = 252° + \zeta , \qquad (3.9)$$

ζ being a suitably small number.

Introducing appropriate values for $\mathcal{L}$, expressions for the perturbations, the coefficients of the equations of condition, the values of the orbital elements and the true longitude of the disturbing planet, can be developed, in particular for

$$\varepsilon' = 234°,\ 243°,\ 252°,\ 261° \text{ and } 270°.$$

Numerical values of A,H and L were derived using the formulae of the previous section, for each value of ε' mentioned above, for all the eighteen equations of condition, (see Tables 3.3, 3.4 and 3.5).

The second and third equations of (3.2) were resolved with respect to $m'h'$ and $m'\ell'$ for all values of $\varepsilon'=234°$ up to 270° at intervals of 9°, to give:

<u>$\varepsilon'=234°$</u>:

$$m'h' = -0.28425 + 0.11231m' - 0.000961P' + 0.000364Q'$$

$$m'\ell' = -0.186626 + 0.07850m' - 0.000113P' - 0.002546Q'$$

<u>$\varepsilon'=243°$</u>:

$$m'h' = -0.33241 + 0.12436m' - 0.000824P' - 0.000697Q'$$

$$m'\ell' = -0.04115 + 0.03430m' + 0.000549P' - 0.002525Q'$$

<u>$\varepsilon'=252°$</u>:

$$m'h' = -0.31379 + 0.11142m' - 0.000505P' - 0.001679Q'$$

$$m'\ell' = +0.11215 - 0.00755m' + 0.000897P' - 0.002083Q'$$

<u>$\varepsilon'=261°$</u>:

$$m'h' = -0.22903 + 0.081175m' - 0.000047P' - 0.002428Q'$$

$$m'\ell' = +0.24555 - 0.03227m' + 0.001087P' - 0.001276Q'$$

ε'					
EPOCHS	234°	243°	252°	261°	270°
1690.98	+37".2	+27".7	+13".0	-40".9	-20".6
1712.25	+25.1	+43.7	+55.3	+57.8	+50.7
1715.23	+12.5	+34.0	+50.0	+57.9	+56.1
1747.7	-39.6	-48.7	-51.4	-47.3	-37.2
1754.7	-10.2	-25.5	-37.1	-43.3	-43.4
1761.7	+22.2	+ 6.0	-10.1	-23.6	-32.7
1768.7	+48.3	+36.8	+21.7	+ 5.5	- 9.3
1775.7	+60.0	+57.7	+48.9	+35.5	+20.0
1782.7	+53.0	+62.1	+63.5	+57.7	+46.6
1789.7	+29.3	+48.0	+60.3	+65.1	+62.5
1796.7	- 3.0	+20.3	+40.5	+55.1	+62.6
1803.7	-31.9	-10.9	+11.4	+31.8	+47.4
1810.7	-47.2	-34.1	-15.9	+ 6.3	+23.3
1817.7	-45.8	-42.4	-32.2	-17.1	+ 0.0
1824.7	-32.7	-36.8	-34.4	-26.3	-14.2
1831.7	-17.9	-25.0	-27.3	-24.6	-17.5
1838.7	-10.1	-16.5	-19.8	-19.2	-14.7
1845.7	-12.1	-16.4	-18.8	-18.1	-13.7

Table 3.3. The numerical values of **A**

		ε'			
EPOCHS	234°	243°	252°	261°	270°
1690.98	- 282"	+ 182"	+ 608"	+ 907"	+1015"
1712.25	-1005	- 968	- 729	- 338	+ 123
1715.23	- 954	-1009	- 855	- 522	- 80
1747.7	+ 679	+ 409	+ 28	- 378	- 719
1754.7	+ 767	+ 684	+ 430	+ 60	- 340
1761.7	+ 639	+ 763	+ 698	+ 458	+ 99
1768.7	+ 327	+ 626	+ 767	+ 719	+ 494
1775.7	- 95	+ 307	+ 621	+ 778	+ 746
1782.7	- 524	- 117	+ 295	+ 623	+ 796
1789.7	- 855	- 544	- 132	+ 289	+ 630
1796.7	-1006	- 870	- 559	- 142	+ 288
1803.7	- 937	-1013	- 879	- 568	- 147
1810.7	- 659	- 934	-1015	- 885	- 574
1817.7	- 232	- 649	- 929	-1014	- 888
1824.7	+ 246	- 218	- 638	- 922	-1012
1831.7	+ 670	+ 259	- 205	- 626	- 915
1838.7	+ 944	+ 678	+ 271	- 192	- 616
1845.7	+1008	+ 944	+ 683	+ 280	- 182

Table 3.4. The numerical values of H

			ε'		
EPOCHS	234°	243°	252°	261°	270°
1690.98	+ 968"	+ 992"	+ 807"	+ 452"	+ 1"
1712.25	- 147	+ 311	+ 706	+ 955	+1003
1715.23	- 337	+ 116	+ 550	+ 871	+1010
1747.7	- 418	- 712	- 848	- 791	- 550
1754.7	+ 14	- 367	- 667	- 815	- 772
1761.7	+ 437	+ 67	- 316	- 624	- 785
1768.7	+ 748	+ 487	+ 119	- 267	- 585
1775.7	+ 872	+ 790	+ 536	+ 169	- 222
1782.7	+ 776	+ 902	+ 830	+ 581	+ 215
1789.7	+ 482	+ 794	+ 931	+ 867	+ 622
1796.7	+ 59	+ 490	+ 810	+ 956	+ 899
1803.7	- 393	+ 59	+ 495	+ 823	+ 976
1810.7	- 770	- 398	+ 56	+ 497	+ 831
1817.7	- 982	- 775	- 404	+ 50	+ 494
1824.7	- 978	- 984	- 781	- 413	+ 41
1831.7	- 759	- 976	- 988	- 790	- 425
1838.7	- 377	- 755	- 976	- 992	- 799
1845.7	+ 78	- 373	- 752	- 976	- 997

Table 3.5. The numerical values of L

$\underline{\varepsilon'=270°}$:

$$m'h' = -0.08900 + 0.04890m' + 0.000481P' - 0.002811Q'$$

$$m'\ell' = +0.33213 - 0.03427m' + 0.001064P' - 0.000207Q'$$

Examination of the errors, Δ, that these different solutions produce in the theory for each of the eighteen equations of condition from 1690 to 1845, led LeVerrier to the conclusion that

$$1 \leq m' \leq \frac{3}{2} \tag{3.10}$$

i.e. the mass of the new planet must be greater than that of Uranus. The corresponding values of Δ determined by the present author, are given in Tables 3.6 to 3.10.

The epoch	The error Δ
1690.98	$- 3''.06 - 56''.94m' - 0''.56P' - 2''.11Q'$
1712.25	$+ 23.39 - 9.12m' - 0.94P' + 0.00Q'$
1715.23	$+ 0.00 + 0.00m' - 1.00P' + 0.00Q'$
1747.7	$- 27.93 - 7.78m' - 0.73P' - 1.52Q'$
1754.7	$- 39.04 - 0.55m' - 0.55P' - 1.71Q'$
1761.7	$- 36.70 + 5.26m' - 0.34P' - 1.73Q'$
1768.7	$- 21.60 + 4.92m' - 0.14P' - 1.48Q'$
1775.7	$+ 0.00 + 0.00m' + 0.00P' - 1.000Q'$
1782.7	$+ 10.37 - 3.98m' + 0.05P' - 0.41Q'$
1789.7	$+ 3.62 - 2.77m' + 0.02P' + 0.05Q'$
1796.7	$- 4.77 + 2.47m' - 0.03P' + 0.22Q'$
1803.7	$- 8.01 + 3.56m' - 0.04P' + 0.15Q'$
1810.7	$+ 0.00 + 0.00m' + 0.00P' + 0.00Q'$
1817.7	$+ 10.76 - 4.20m' + 0.03P' - 0.05Q'$
1824.7	$+ 5.02 - 2.67m' + 0.02P' + 0.01Q'$
1831.7	$- 5.23 + 2.51m' - 0.02P' + 0.03Q'$
1838.7	$- 10.51 + 4.55m' - 0.03P' + 0.03Q'$
1845.7	$+ 0.00 + 0.00m' + 0.00P' + 0.00Q'$

Table 3.6 : For $\varepsilon'=234°$

The epoch	The error Δ
1690.98	$-7''.76 - 52''.73m' - 0''.54P' - 2''.24Q'$
1712.25	$+22.66 - 10.91m' - 0.93P' - 0.02Q'$
1715.23	$+0.00 + 0.00m' - 1.00P' + 0.00Q'$
1747.7	$-24.53 - 5.03m' - 0.74P' - 1.44Q'$
1754.7	$-36.16 - 0.74m' - 0.56P' - 1.64Q'$
1761.7	$-34.74 + 3.92m' - 0.35P' - 1.68Q'$
1768.7	$-20.80 + 4.01m' - 0.15P' - 1.46Q'$
1775.7	$+0.00 + 0.00m' + 0.00P' - 1.00Q'$
1782.7	$+10.11 - 3.61m' + 0.05P' - 0.42Q'$
1789.7	$+3.47 - 2.69m' + 0.02P' + 0.05Q'$
1796.7	$-4.56 + 2.18m' - 0.03P' + 0.23Q'$
1803.7	$-8.48 + 3.66m' - 0.04P' + 0.15Q'$
1810.7	$+0.00 + 0.00m' + 0.00P' + 0.00Q'$
1817.7	$+10.85 - 4.68m' + 0.03P' - 0.05Q'$
1824.7	$+4.92 - 3.26m' + 0.02P' + 0.01Q'$
1831.7	$-5.10 + 2.53m' - 0.02P' + 0.03Q'$
1838.7	$-10.47 + 5.57m' - 0.03P' + 0.03Q'$
1845.7	$+0.00 + 0.00m' + 0.00P' + 0.00Q'$

Table 3.7 : For $\varepsilon'=243°$

The epoch	The error Δ
1690.98	- 8".72 - 44".90m' - 0".51P' - 2".38Q'
1712.25	+ 22.50 - 11.90m' - 0.93P' - 0.04Q'
1715.23	+ 0.00 + 0.00m' - 1.00P' + 0.00Q'
1747.7	- 23.52 - 1.80m' - 0.76P' - 1.35Q'
1754.7	- 35.18 - 0.91m' - 0.58P' - 1.57Q'
1761.7	- 34.06 + 2.18m' - 0.36P' - 1.63Q'
1768.7	- 20.56 + 2.74m' - 0.15P' - 1.44Q'
1775.7	+ 0.00 + 0.00m' + 0.00P' - 1.00Q'
1782.7	+ 10.04 - 2.92m' + 0.05P' - 0.43Q'
1789.7	+ 3.39 - 2.36m' + 0.02P' + 0.05Q'
1796.7	- 4.49 + 1.70m' - 0.03P' + 0.23Q'
1803.7	- 8.81 + 3.30m' - 0.04P' + 0.15Q'
1810.7	+ 0.00 + 0.00m' + 0.00P' + 0.00Q'
1817.7	+ 10.91 - 4.66m' + 0.03P' - 0.05Q'
1824.7	+ 4.89 - 3.54m' + 0.02P' - 0.02Q'
1831.7	- 5.03 + 2.30m' - 0.02P' + 0.04Q'
1838.7	- 10.54 + 6.07m' - 0.03P' + 0.03Q'
1845.7	+ 0.00 + 0.00m' + 0.00P' + 0.00Q'

Table 3.8 : For ε'=252°

The epoch	The error Δ
1690.98	$-\ 5''.44 - 34''.06m' - 0''.46P' - 2''.51Q'$
1712.25	$+\ 23.02 - 12.08m' - 0.92P' - 0.06Q'$
1715.23	$+\ 0.00 + 0.00m' - 1.00P' + 0.00Q'$
1747.7	$-\ 25.23 + 2.18m' - 0.79P' - 1.27Q'$
1754.7	$-\ 36.46 - 0.67m' - 0.60P' - 1.50Q'$
1761.7	$-\ 34.89 - 0.43m' - 0.38P' - 1.58Q'$
1768.7	$-\ 21.01 + 1.29m' - 0.16P' - 1.42Q'$
1775.7	$+\ 0.00 + 0.00m' + 0.00P' - 1.00Q'$
1782.7	$+\ 10.20 - 1.98m' + 0.05P' - 0.43Q'$
1789.7	$+\ 3.39 - 1.81m' + 0.02P' + 0.04Q'$
1796.7	$-\ 4.59 + 1.10m' - 0.04P' + 0.24Q'$
1803.7	$-\ 8.94 + 2.55m' - 0.04P' + 0.16Q'$
1810.7	$+\ 0.00 + 0.00m' + 0.00P' + 0.00Q'$
1817.7	$+\ 10.82 - 4.13m' + 0.03P' - 0.05Q'$
1824.7	$+\ 4.98 - 3.50m' + 0.02P' - 0.02Q'$
1831.7	$-\ 5.04 + 1.85m' - 0.02P' + 0.04Q'$
1838.7	$-\ 10.65 + 5.93m' - 0.04P' + 0.04Q'$
1845.7	$+\ 0.00 + 0.00m' + 0.00P' + 0.00Q'$

Table 3.9 : For $\varepsilon'=261°$

The epoch	The error Δ
1690.98	+ 2.″68 - 20.″42m' - 0.″40P' - 2.″64Q'
1712.25	+ 24.34 - 11.46m' - 0.91P' - 0.08Q'
1715.23	+ 0.00 + 0.00m' - 1.00P' + 0.00Q'
1747.7	- 30.09 + 7.17m' - 0.83P' - 1.18Q'
1754.7	- 40.50 - 0.51m' - 0.63P' - 1.42Q'
1761.7	- 37.60 - 0.86m' - 0.40P' - 1.53Q'
1768.7	- 22.31 - 0.08m' - 0.17P' - 1.39Q'
1775.7	+ 0.00 + 0.00m' + 0.00P' - 1.00Q'
1782.7	+ 10.67 - 0.93m' + 0.06P' - 0.44Q'
1789.7	+ 3.53 - 1.14m' + 0.02P' + 0.04Q'
1796.7	- 4.86 + 0.44m' - 0.04P' + 0.24Q'
1803.7	- 8.95 + 1.53m' - 0.04P' + 0.16Q'
1810.7	+ 0.00 + 0.00m' + 0.00P' + 0.00Q'
1817.7	+ 10.63 - 3.19m' + 0.03P' - 0.06Q'
1824.7	+ 5.18 - 3.10m' + 0.02P' - 0.02Q'
1831.7	- 5.15 + 1.24m' - 0.02P' + 0.04Q'
1838.7	- 10.84 + 5.20m' - 0.04P' + 0.04Q'
1845.7	+ 0.00 + 0.00m' + 0.00P' + 0.00Q'

Table 3.10 : For ε'=270°

Finally, LeVerrier carefully examined the errors of the theory for a specific value of ε', viz.,252°, and arbitrarily assumed that m'=1, P'=-15" and Q'=-10". The corresponding values of the errors were found to decrease as follows:

EPOCH	Δ	EPOCH	Δ	EPOCH	Δ
1690	-22"(-23")	1768	-1"(-2")	1810	0"(0")
1712	+25(+24)	1775	+10(+10)	1817	+5(+6)
1715	+15(+15)	1782	+11(+10)	1824	+1(+1)
1747	0(-1)	1789	+1(0)	1831	-2(-3)
1754	-12(-12)	1796	-5(-5)	1838	-4(-4)
1761	-10(-10)	1803	-6(-7)	1845	0(0)

Table 3.11

Among these errors, those of 1754 (-12") and 1782 (+11") are the most serious, since the position of Uranus is very well determined at these epochs. If, on the other hand, one examines the influence of the error R' and S' corresponding to 1810 and 1845, it is found that values of R' and S' equal to -3" are quite realistic, which have the effect of reducing the error of 1754 and 1782 to -2" and +8", respectively.

From these results, LeVerrier arrived at the conclusion that the observations of Uranus could be represented by means of the disturbing action of

Neptune, whose mean longitude of epoch ε' should lie in the vicinity of 252° corresponding to the 1st January 1800. For 1st January 1847, the corresponding heliocentric longitude v as deduced by LeVerrier, took the form:

$$v = 314\overset{\circ}{.}5 + 12.25\zeta + \frac{1}{m'}(20\overset{\circ}{.}82 - 10.97\zeta - 1.14\zeta^2) \; .... \qquad (3.11)$$

Relating the true heliocentric longitude to the limits in which m' and ζ must be contained, it follows that the true heliocentric longitude at that epoch is ~ 325°, a value not far removed from LeVerrier's final solution.

4. THE ELEMENTS OF THE DISTURBING PLANET - LEVERRIER'S FINAL SOLUTION

In the first solution, LeVerrier used eight equations of condition separated by intervals of fourteeen years covering ancient and modern observations over the period 1747.7 to 1845.7. The primary objective of the analysis was the determination of the approximate mass m' of the disturbing planet. For this reason he neglected the secular inequalities and the second order terms and assumed, in accordance with Bode's Law, that α, the ratio between the semi-major axes of Uranus and the new planet, was 0.50. Subsequently, LeVerrier obtained his first approximation to the mass of the disturbing planet, viz., $m' \sim 2.11 \times 10^{-4}$ the mass of the sun.

In the second solution, LeVerrier obtained fifteen equations of condition, covering the observations for the same period above but at intervals of seven years, together with three additional equations covering the earlier observations of Flamsteed. The eighteen equations of condition were used primarily to obtain an approximate value of the longitude of the epoch ε', which in turn was utilised to determine the mass of the new planet to a better approximation than the first solution. In this second solution he again neglected secular inequalities and second order terms and assumed that $\alpha = 0.50$. The equations of condition, in both solutions, were derived from considerations of heliocentric errors in the Tables of Uranus (i.e. from 1781 to 1845). Modern observations were chosen at opposition, in order to facilitate the formation of the equations of condition. Unfortunately, the ancient observations were not made at opposition, thus leading to inaccuracies within the theory:

(a) inaccuracies in the heliocentric longitude of Uranus, and

(b) inaccuracies in the radius vector of Uranus.

The latter had been neglected since it was assumed that the effect was minimal due to the inherent inaccuracy in the ancient observations.

The second solution yielded:

(i) $243° \leq \varepsilon' \leq 270°$,

(ii) $1 \leq m' \leq \frac{3}{2}$,

(iii) a value for the true longitude v' of the new planet in the region of 325°.

Although the two solutions confirmed the existence of a disturbing planet, and to some extent determined the true longitude, the planet eluded physical observation. It was thus essential that the assumptions and simplifications made be examined carefully.

The assertion of the existence of a disturbing planet was no longer in question, but the requirement now becomes one of pin-pointing its position more precisely, requiring accurate determination of the longitude of the epoch ε', the mass of the new planet m' and its true longitude v', from which we can determine all other elements of the orbit.

The equations of condition, between the geocentric errors of the tables and the corrections of the elements of the orbit of Uranus, were therefore completed by adding to their first members, the perturbations of the geocentric longitude G of Uranus, due to the action of the new planet. These geocentric perturbations, δG, were deduced from the perturbations of the heliocentric longitude, δv, and the perturbations of the radius vector, δr, by means of the following equations:

$$\tan\,(G-v-N) = \frac{-R\sin\,(♁-v)}{r-R\cos\,(♁-v)} \qquad (4.1)$$

$$\begin{aligned} \delta G &= P.\delta AR - Q.\delta D \\ \delta b &= -R.\delta AR + S.\delta D \end{aligned} \qquad (4.2)$$

where

N denotes the Nutation in longitude,

v, the heliocentric longitude of Uranus,

♁, the Earth's longitude,

R, the radius vector from Earth to Sun,

r, the radius vector of Uranus,

δb, the error of the observed latitude of Uranus,

δAR, the error of the observed right ascension,

δD, the error in declination,

and P,Q,R and S are values that could be obtained from the published tables (from the observations by Greenwich Observatory, 1836).

By eliminating δD, the following equation is obtained:

$$\delta G = \left(P - \frac{Q.R}{S}\right) \delta AR - \frac{Q}{S} \delta b \qquad (4.3)$$

Furthermore, higher precision was introduced into this final solution, by allowing for slight variations in $\alpha = \frac{a}{a'}$, periodic variations of the second order terms, and secular variations in the calculations of the longitude and radius vector.

Thirty-three equations of condition were then solved instead of the eight and eighteen equations in the first and second solutions respectively.

As mentioned previously, as a first approximation in the first and second solutions, α was assumed to be 0.50, giving rise to a value of the mean longitude of the epoch ε' equal to $252°(243° \leq \varepsilon \leq 270°)$. To achieve maximum precision, therefore, LeVerrier introduced the expressions

$$\alpha = 0.51 + 0.02\gamma, \qquad (4.4)$$

$$\varepsilon' = 252° + 18°\mathfrak{C}, \qquad (4.5)$$

wherein

$$-1 \leq \gamma \leq +1 ,$$

and

$$-1 \leq \mathfrak{C} \leq +1 ,$$

thereby widening the range to

$$0.49 \leq \alpha \leq 0.53 ,$$

and

$$234° \leq \varepsilon' \leq 270° .$$

The new equations, containing γ and $\mathcal{L}$, become too complicated for treatment by Taylor Series if γ and $\mathcal{L}$ are allowed to vary continuously. Thus, it is convenient to choose fixed values of γ and $\mathcal{L}$ equal to -1, 0 and +1. Equations were developed for six different combinations of γ and $\mathcal{L}$, viz.,

$\mathcal{L}$:	0	-1	0	+1	-1	0
γ :	-1	0	0	0	+1	+1

Further consideration of the exact solution of the algebraic expressions which satisfy the problem for values of α and ε', indicated that the absolute value of γ should be close to 1 while the value of $\mathcal{L}$ should lie between 0 and -1.

By this means the six cases are reduced (temporarily) to the following three:

$\mathcal{L}$:	0	0	0
γ :	-1	0	+1

The principal values of Laplace coefficients and their derivatives, as derived by the present author for the three cases of γ, did not indicate any significant differences from those given by LeVerrier.

The periodic perturbations of the heliocentric longitude v and the radius vector r, and hence the secular inequalities, to be considered, are of the zero and first orders. Higher order perturbations gave insignificant differences.

4.1. Perturbations in Longitude

Omitting higher powers of the eccentricities e and e', the terms retained by LeVerrier were now of the form

$$\delta v = m'\sum_{i=1}^{3} P^{(i)}\sin i\,(n't-nt+\varepsilon'-\varepsilon)$$

$$+m'\sum_{i=1}^{3} N^{(i)}\sin[i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\tilde{\omega}]$$

$$+m'e'\sum_{i=1}^{3} M^{(i)}\sin[i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\tilde{\omega}']$$

$$+m'f_1 nte\cos(nt+\varepsilon-\tilde{\omega})$$

$$+m'f_2 nte'\cos(nt+\varepsilon-\tilde{\omega}), \qquad (4.6)$$

$P^{(i)}$, $N^{(i)}$ and $M^{(i)}$ ($\forall i=1,2,3$) being as before, with

$$f_1 = -\alpha^2\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} - \frac{1}{2}\alpha^3\frac{d^2b_{\frac{1}{2}}^{(0)}}{d\alpha^2}, \qquad (4.7)$$

and

$$f_2 = -\frac{\alpha}{2}\,2b_{\frac{1}{2}}^{(1)} - 2\alpha\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} - \alpha^2\frac{d^2b_{\frac{1}{2}}^{(1)}}{d\alpha^2}. \qquad (4.8)$$

For convenience, the terms in e' (within the third summation), were regrouped in the form

$$m'\ell'\sum_{i=1}^{3} M^{(i)}\sin[i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon]$$

$$+ m'h'\sum_{i=1}^{3} M^{(i)}\cos[i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon], \quad (4.9)$$

ℓ' and h' having the same forms as before.

At this point the arguments themselves were re-labelled for future reference, the new notation being

$$\left.\begin{aligned} p_i &= i(n't-nt+\varepsilon'-\varepsilon), \\ n_i &= p_i+nt+\varepsilon \ , \\ p_{i+3} &= n_i-\tilde{\omega} \ , \end{aligned}\right\} \forall i=1,2,3 \qquad (4.10)$$

For comparison with the results of LeVerrier, the arguments of perturbations p_i and $n_i (\forall i=1,2,\ldots,6)$, are expressed in decimal degrees whilst the coefficients, $P^{(i)}$, $N^{(i)}$ and $M^{(i)}$, are in sexagesimal seconds.

From earlier calculations of perturbations of Jupiter and Saturn on Uranus, the following values were available:

$\theta \;=\; 72°59'\ 21''$

$\varepsilon = 192^g.98 = 173°.502$,

$e = 0.0466794$,

$n = 4^g.76103 = 4°.284927$,

$\phi = 0°46' \, 28''.4$,

$\tilde{\omega} = 186^g.12 = 167°.508$,

$a = 19.182729$ A.U. ,

enabling the numerical values of the periodic perturbations for each of the three different values of γ, to be determined, assuming, of course, that $\varepsilon'=252°(=280^g)$, corresponding to $\mathcal{L}=0$.
(See Tables 4.1, 4.2 and 4.3). Values for i = 4,5,6 have been included for reference only.

α	i	$P^{(i)}$	$N^{(i)}$	$M^{(i)}$
0.49	1	16".91 (16".9)	1".74 (1".7)	40" (40")
	2	24.97 (25.0)	14.26 (14.2)	100 (101)
	3	2.50 (2.5)	43.90 (43.8)	1706 (1706)
	4	0.55	0.68	27
	5	0.15	0.13	6
	6	0.05	0.05	2
0.51	1	20".22 (20".2)	2".07 (2".1)	46" (46")
	2	35.00 (35.0)	20.50 (20.5)	148 (148)
	3	3.36 (3.4)	18.62 (18.6)	696 (696)
	4	0.75	1.07	41
	5	0.22	0.19	8
	6	0.08	0.08	2
0.53	1	24".21 (24".2)	2".46 (2".5)	52" (52")
	2	50.48 (50.5)	31.13 (31.1)	230 (230)
	3	4.53 (4.5)	14.97 (14.9)	539 (539)
	4	1.03	1.73	64
	5	0.31	0.28	12
	6	0.12	0.13	3

Table 4.1 : Numerical values of $P^{(i)}$, $N^{(i)}$ and $M^{(i)}$.

α	i	p_i	n_i
0.49	1	$87\overset{g}{.}22 - 3\overset{g}{.}128t$	$280\overset{g}{.}00 + 1\overset{g}{.}633t$
	2	$374.44 - 6.256t$	$367.22 - 1.495t$
	3	$61.66 - 9.384t$	$54.44 - 4.623t$
	4	$93.88 + 1.633t$	
	5	$381.10 - 1.495t$	
	6	$68.32 - 4.623t$	
0.51	1	$87\overset{g}{.}22 - 3\overset{g}{.}027t$	$280\overset{g}{.}00 + 1\overset{g}{.}734t$
	2	$374.44 - 6.054t$	$367.22 - 1.293t$
	3	$61.66 - 9.081t$	$54.44 - 4.320t$
	4	$93.88 + 1.734t$	
	5	$381.10 - 1.293t$	
	6	$68.32 - 4.320t$	
0.53	1	$87\overset{g}{.}22 - 2\overset{g}{.}924t$	$280\overset{g}{.}22 + 1\overset{g}{.}837t$
	2	$374.44 - 5.848t$	$367.44 - 1.087t$
	3	$61.66 - 8.772t$	$54.44 - 4.011t$
	4	$93.88 + 1.837t$	
	5	$381.10 - 1.087t$	
	6	$68.32 - 4.011t$	

Table 4.2 : Arguments of perturbations.

	$\alpha = 0.49$	$\alpha = 0.51$	$\alpha = 0.53$
f_1:	$-0\overset{''}{.}02133(-0.0210)$	$-0\overset{''}{.}02529(-0\overset{''}{.}0257)$	$-0\overset{''}{.}2997(-0.0304)$
f_2:	$+0.27075(+0.270)$	$+0.33321(+0.334)$	$+0.40895(+0.408)$

Table 4.3 : Numerical comparison of the coefficients of δv.

4.2. Perturbations in the radius vector

For completeness, we compare also the perturbations in the radius vector, included by LeVerrier in his research.

Beginning with the fundamental equation

$$\frac{d^2}{dt^2}(rdr) + \mu\,\frac{r\delta r}{r^3} = 2\int d\dot{R} + r\left(\frac{dR}{dr}\right)$$

where R is the disturbing function and $\mu = n^2a^3$, it is found (for example, Somerville, 1931) that to first order in e,e',

$$\delta r = \frac{m'}{6}\alpha^2\,\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha} - \frac{m'}{2}\sum_{i=1}^{3} C^{(i)}\cos i\,[(n'-n)t+\varepsilon'-\varepsilon]$$

$$+m'ef_3\cos(nt+\varepsilon-\tilde{\omega})+m'e'f_4\cos(nt+\varepsilon-\tilde{\omega}')$$

$$+m'e\sum_{i=1}^{3} D^{(i)}\cos[i\{n't-nt+\varepsilon'-\varepsilon\}+nt+\varepsilon-\tilde{\omega}]$$

$$+m'e'\sum_{i=1}^{3} E^{(i)}\cos[i\{n't-nt+\varepsilon'-\varepsilon\}+nt+\varepsilon-\tilde{\omega}'], \quad \ldots \quad (4.11)$$

where

$$f_3 = -\frac{\alpha}{2}\left[\frac{7}{3}\alpha\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}+\alpha^2\frac{d^2b_{\frac{1}{2}}^{(0)}}{d\alpha^2}\right],$$

$$f_4 = +\frac{\alpha}{2}\left[\frac{3}{2}(\alpha-b_{\frac{1}{2}})-\frac{3}{2}\alpha\left(1-\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha}\right)+\alpha^2\frac{d^2b_{\frac{1}{2}}^{(1)}}{d\alpha^2}\right],$$

$$C^{(i)} = \frac{\alpha}{i^2(\nu-1)^2-1}\left[\frac{2}{1-\nu}b_{\frac{1}{2}}^{(i)}+\alpha\frac{db_{\frac{1}{2}}^{(i)}}{d\alpha}\right],$$

$$D^{(i)} = \frac{\alpha}{1-\{i(\nu-1)+1\}^2}\left[\frac{(i-1)(2i-1)}{i(\nu-1)+1}b_{\frac{1}{2}}^{(i-1)}\right.$$

$$\left.-\frac{i^2(\nu-1)+1}{i(\nu-1)+1}\alpha\frac{db_{\frac{1}{2}}^{(i-1)}}{d\alpha}-\frac{1}{2}\alpha^2\frac{d^2b_{\frac{1}{2}}^{(i)}}{d\alpha^2}\right],$$

and

$$E^{(i)} = \frac{1}{1-\{i(\nu-1)+1\}^2}\left[\frac{3\alpha}{1-\nu}b_{\frac{1}{2}}^{(i)}-\{i^2(\nu-1)(i(\nu-1)-1)-3\}C^{(i)}\right.$$

$$\left.+\frac{\alpha}{2}\cdot\alpha^2\frac{d^2b_{\frac{1}{2}}^{(i)}}{d\alpha^2}\right],$$

the arguments being rearranged as in the case of δv.

The terms retained by LeVerrier were those in $C^{(2)}$, $D^{(3)}$, $E^{(3)}$, f_3 and f_4, giving rise to the numerical values found in table 4.4

α	ef_3	f_4	$C^{(2)}$	$D^{(3)}$	$E^{(3)}$
0.49	-0″.0232	+0″.0922	-1″.43(-2.6)	+160″.(112″)	-4″.1(-3″.0)
0.51	-0.0275	+0.1147	-1.96(-4.3)	+ 63(76)	-1.6(-1.9)
0.53	-0.0325	+0.1423	-2.77(-6.1)	+ 46(59)	-1.2(-1.4)

<u>Table 4.4</u> : Numerical comparison of the coefficients of $\delta r/100$

The values found clearly differ from those of LeVerrier; perhaps some factor has been overlooked. The second order terms (i.e. in e^2) do account for the differences. Although the reason for the discrepancies is not clear we observe that some of the coefficients are directly proportional to those of LeVerrier, indicating, perhaps, the omission of some factor, presumably dependent upon α. For example,

for $\alpha = 0.49$

$$\left.\begin{matrix} D^{(3)} = 160'' \\ E^{(3)} = 4''.1 \end{matrix}\right\} \times 0.7 \simeq \left\{\begin{matrix} 112''\ (112'') \\ 2''.9\ (3''.0) \end{matrix}\right.$$

for $\alpha = 0.51$

$$\left.\begin{matrix} D^{(3)} = 63'' \\ E^{(3)} = 1''.6 \end{matrix}\right\} \times 1.2 \simeq \left\{\begin{matrix} 76''\ (76'') \\ 1''.9\ (1''.9) \end{matrix}\right.$$

for $\alpha = 0.53$

$$\left.\begin{matrix} D^{(3)} = 46'' \\ E^{(3)} = 1''.2 \end{matrix}\right\} \times 1.2 \simeq \left\{\begin{matrix} 55''\ (59'') \\ 1''.4\ (1''.4) \end{matrix}\right.$$

4.3 Second-Order Terms

Having considered the variation in α and the secular variations in the true longitude and the radius vector, it is now necessary to consider the periodic variation in the second-order terms. As indicated by the first solution, that part of the second order perturbations which is worthy of consideration, is that which depends on the small argument (3n'-n). It is, therefore, necessary to consider the perturbations in the mean longitude ℓ (not to be confused with $\ell = e\cos\tilde{\omega}$ introduced earlier) and thereafter to obtain the true longitude v.

Thus,

$$\delta\ell = \frac{3B\alpha n^2}{10000''(3n'-n)^2} m'e^2 \sin[(3n't-nt)+(3\varepsilon'-\varepsilon)-2\tilde{\omega}]$$

$$+ \frac{3C\alpha n^2}{10000''(3n'-n)^2} m'ee' \sin[(3n't-nt)+(3\varepsilon'-\varepsilon)-(\tilde{\omega}+\tilde{\omega}')]$$

$$+ \frac{3D\alpha n^2}{10000''(3n'-n)^2} m'e'^2 \sin[(3n't-nt)+(3\varepsilon'-\varepsilon)-2\tilde{\omega}'] \tag{4.12}$$

i.e.

$$\begin{aligned}\delta\ell = {} & B'm' \sin[(3n't-nt)+(3\varepsilon'-\varepsilon)-2\tilde{\omega}] \\ & +C'm'e' \sin[(3n't-nt)+(3\varepsilon'-\varepsilon)-(\tilde{\omega}+\tilde{\omega}')] \\ & +D'm'e'^2 \sin[(3n't-nt)+(3\varepsilon'-\varepsilon)-2\tilde{\omega}'],\end{aligned}$$

containing terms in m', $m'e'$ amd $m'e'^2$.

The expressions for the coefficients are:

$$B' = \frac{3\alpha e^2}{8(10^4)''(3\nu-1)^2}\left[21b_{\frac{1}{2}}^{(3)} + 10\alpha\frac{db_{\frac{1}{2}}^{(3)}}{d\alpha} + \alpha^2\frac{d^2b_{\frac{1}{2}}^{(3)}}{d\alpha^2}\right],$$

$$C' = -\frac{3\alpha e}{4(10^4)''(3\nu-1)^2}\left[20b_{\frac{1}{2}}^{(2)} + 10\alpha\frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} + \alpha^2\frac{d^2b_{\frac{1}{2}}^{(2)}}{d\alpha^2}\right],$$

$$D' = +\frac{3\alpha}{8(10^4)''(3\nu-1)^2}\left[17b_{\frac{1}{2}}^{(1)} + 10\alpha\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} + \alpha^2\frac{d^2b_{\frac{1}{2}}^{(1)}}{d\alpha^2}\right]. \tag{4.13}$$

LeVerrier considered the term in m' to be sufficiently small to be neglected, thus offering a further simplification of the solution without inducing appreciable inaccuracies in the mean motion, relative to the period of observation at his disposal. An indication of the magnitude of B' for the three values of γ, is given below:

γ :	-1	0	+1
α :	0.49	0.51	0.53
B':	0.005024	0.000590	0.000243

Consideration of the term in m'e' showed that it is the most appreciable of the three terms. Furthermore, its inclusion would not complicate the form of the equations. Since this term is dependent on m'h' and m'ℓ', it is therefore necessary to consider terms in t^2 which are obtained by the development of this part of the inequality.

The values of C' corresponding to the three values of γ and α are given below:

γ :	-1	0	+1
α :	0.49	0.51	0.53
C':	0.390424	0.044047	0.017483

The term in $m'e'^2$, containing a higher power of the eccentricity of the disturbed planet, would considerably complicate the solution if retained. Partly for this reason and partly because it could be confounded with the mean motion relative to the period of the observations under consideration, LeVerrier chose to neglect this term, even though the values of D' are appreciable, as illustrated below:

γ :	-1	0	+1
α :	0.49	0.51	0.53
D':	1.4045	0.166198	0.069115

Adopting this procedure, the equat ions to be examined would remain in linear form with respect to m', m'h' and m'ℓ'. Thus the perturbations are completed by adding to the heliocentric longitude, appropriate second-order terms containing t^2 for each of m'h' and m'ℓ', viz.,

γ	℘	δν
-1	0	$-0''.00526t^2m'h' + 0''.00751t^2m'\ell'$
0	-1	$-0.01056t^2m'h' + 0.00018t^2m'\ell'$
0	0	$-0.00606t^2m'h' + 0.00865t^2m'\ell'$
0	+1	$+0.00344t^2m'h' + 0.00999t^2m'\ell'$
+1	-1	$-0.01214t^2m'h' + 0.00020t^2m'\ell'$
+1	0	$-0.00696t^2m'h' + 0.00993t^2m'\ell'$

The coefficients within the linear equations in m', m'h' and m'ℓ', were then determined for the six combinations of γ and ℘, referred to previously.

4.4. Equations of Condition

The complete expression for the perturbations of the geocentric longitude, G, necessitated the determination of the coefficients of the terms in m', m'h' and m'ℓ' for 114 mean epochs, involving 279 observations. The problem remaining, therefore, was to reduce the number of the equations to a reasonable number whilst at the same time ensuring that an adequate solution could be obtained with a higher degree of accuracy than that provided by the eighteen equations solved previously. The ancient observations were considered accordingly and re-grouped to form seven mean observations, viz.,

(i) one observation by Flamsteed in 1690,

(ii) four observations by Flamsteed in 1712 and 1715, the latter made at opposition,

(iii) two observations by Lemonnier in 1750,

(iv) two very accurate observations made at opposition by Mayer and Bradley in 1753 and 1756 respectively,

(v) one observation by Lemonnier in 1764,

(vi) eight observations made at opposition by Lemonnier in 1768 and 1769,

and (vii) one observation by Lemonnier in 1771,

yielding seven equations of condition.

The remaining 260 modern observations were divided into groups of ten (approximately) yielding a further twenty-six equations of condition.

The equations of condition thus derived are given in Table 4.5, the perturbations in each equation being designated by the number [j], (j=1,2,...,33), of that equation. The symbols [1], [2],...,[33], were evaluated for each of the six combinations of γ and δ.

EPOCHS	No. of Observations	Equations of Condition
1690	1	$1.019\delta\varepsilon - 111.09\delta n - 1.986\delta e + 0.538e\delta\tilde{\omega} - 65''.9 + [1] = 0$
1712-1715	4	$1.152\delta\varepsilon - 98.45\delta n - 0.033\delta e - 2.213e\delta\tilde{\omega} - 66.5 + [2] = 0$
1750	2	$0.924\delta\varepsilon - 45.42\delta n + 0.793\delta e + 1.737e\delta\tilde{\omega} + 35.3 + [3] = 0$
1753-1756	2	$0.933\delta\varepsilon - 41.62\delta n + 0.194\delta e + 1.910e\delta\tilde{\omega} + 33.4 + [4] = 0$
1764	1	$0.913\delta\varepsilon - 32.83\delta n - 0.909\delta e + 1.652e\delta\tilde{\omega} + 21.2 + [5] = 0$
1768-1769	8	$0.948\delta\varepsilon - 29.36\delta n - 1.475\delta e + 1.264e\delta\tilde{\omega} + 8.2 + [6] = 0$
1771	1	$0.996\delta\varepsilon - 27.93\delta n - 1.788\delta e + 0.953e\delta\tilde{\omega} + 11.9 + [7] = 0$
1781-1782	10	$1.052\delta\varepsilon - 18.58\delta n - 1.988\delta e - 0.622e\delta\tilde{\omega} - 17.2 + [8] = 0$
1783-1784	9	$1.056\delta\varepsilon - 16.68\delta n - 1.881\delta e - 0.891e\delta\tilde{\omega} - 20.6 + [9] = 0$
1785-1788	10	$1.089\delta\varepsilon - 14.36\delta n - 1.696\delta e - 1.270e\delta\tilde{\omega} - 23.9 + [10] = 0$
1789-1790	11	$1.094\delta\varepsilon - 11.02\delta n - 1.337\delta e - 1.653e\delta\tilde{\omega} - 29.1 + [11] = 0$

Table 4.5

EPOCHS	No. of Observations	Equations of Condition
1791-1792	10	$1.111\delta\varepsilon - 9.01\delta n - 1.071\delta e - 1.866e\delta\tilde{\omega} - 27''.9 + [12] = 0$
1793-1794	9	$1.118\delta\varepsilon - 6.61\delta n - 0.728\delta e - 2.035e\delta\tilde{\omega} - 30.9 + [13] = 0$
1795-1797	9	$1.147\delta\varepsilon - 4.22\delta n - 0.384\delta e - 2.178e\delta\tilde{\omega} - 31.9 + [14] = 0$
1797-1801	9	$1.155\delta\varepsilon + 0.09\delta n + 0.266\delta e - 2.205e\delta\tilde{\omega} - 32.2 + [15] = 0$
1802-1804	11	$1.145\delta\varepsilon + 3.25\delta n + 0.739\delta e - 2.085e\delta\tilde{\omega} - 34.2 + [16] = 0$
1804-1806	10	$1.146\delta\varepsilon + 6.29\delta n + 1.147\delta e - 1.904e\delta\tilde{\omega} - 35.4 + [17] = 0$
1807-1808	12	$1.124\delta\varepsilon + 8.73\delta n + 1.456\delta e - 1.635e\delta\tilde{\omega} - 35.1 + [18] = 0$
1809-1810	10	$1.116\delta\varepsilon + 10.95\delta n + 1.689\delta e - 1.382e\delta\tilde{\omega} - 36.7 + [19] = 0$
1811-1813	11	$1.071\delta\varepsilon + 13.17\delta n + 1.870\delta e - 0.968e\delta\tilde{\omega} - 37.2 + [20] = 0$
1813-1815	11	$1.080\delta\varepsilon + 15.49\delta n + 2.011\delta e - 0.704e\delta\tilde{\omega} - 37.4 + [21] = 0$
1816-1817	9	$1.052\delta\varepsilon + 17.54\delta n + 2.068\delta e - 0.322e\delta\tilde{\omega} - 33.9 + [22] = 0$

Table 4.5 (continued)

EPOCHS	No. of Observations	Equations of Condition
1818-1820	12	$1.051\delta\varepsilon + 20.46\delta n + 2.101\delta e + 0.094e\delta\tilde{\omega} - 32''.5 + [23] = 0$
1821-1823	9	$1.028\delta\varepsilon + 23.13\delta n + 1.998\delta e + 0.551e\delta\tilde{\omega} - 27.4 + [24] = 0$
1824-1827	11	$1.006\delta\varepsilon + 26.06\delta n + 1.773\delta e + 1.009e\delta\tilde{\omega} - 25.2 + [25] = 0$
1828-1830	11	$0.966\delta\varepsilon + 28.91\delta n + 1.375\delta e + 1.424e\delta\tilde{\omega} - 7.5 + [26] = 0$
1835-1835	9	$0.962\delta\varepsilon + 34.22\delta n + 0.677\delta e + 1.871e\delta\tilde{\omega} + 27.9 + [27] = 0$
1835-1836	10	$0.930\delta\varepsilon + 33.88\delta n + 0.595\delta e + 1.829e\delta\tilde{\omega} + 30.6 + [28] = 0$
1837-1838	11	$0.939\delta\varepsilon + 35.88\delta n + 0.345\delta e + 1.912e\delta\tilde{\omega} + 45.7 + [29] = 0$
1839-1840	10	$0.946\delta\varepsilon + 38.08\delta n + 0.065\delta e + 1.957e\delta\tilde{\omega} + 63.1 + [30] = 0$
1841-1842	9	$0.941\delta\varepsilon + 39.66\delta n - 0.195\delta e + 1.940e\delta\tilde{\omega} + 77.0 + [31] = 0$
1842-1844	9	$0.928\delta\varepsilon + 40.29\delta n - 0.353\delta e + 1.887e\delta\tilde{\omega} + 86.4 + [32] = 0$
1844-1845	8	$0.951\delta\varepsilon + 42.82\delta n - 0.601\delta e + 1.873e\delta\tilde{\omega} + 105.7 + [33] = 0$

Table 4.5 (continued)

As mentioned above, a set of 33 equations of condition was obtained for each of the six combinations of γ and $\mathscr{L}$, which consisted of 260 new observations and 19 ancient observations.

In order to solve the thirty-three equations, for each combination of γ and $\mathscr{L}$, the new observations were grouped into six groups and the ancient observations were also divided among these six groups. Thus, six mean equations were obtained in terms of the six unknowns, namely: $\delta\varepsilon$, δn, δe, $e\delta\tilde{\omega}$, $m'h'$ and $m'\ell'$.

By elimination, the six equations were reduced in terms of m', for each combination of γ and $\mathscr{L}$. An example of the values for $\gamma = -1$ and $\mathscr{L} = 0$ is shown below:

$$\delta\varepsilon = -15''.664 - 1''.097m' ,$$

$$\delta n = -0.5250 + 0.0750m' ,$$

$$\delta e = -82.369 + 63.088m' ,$$

$$e\delta\tilde{\omega} = +69.190 + 3.766m' ,$$

$$m'h' = -0.15632 + 0.053734m' ,$$

$$m'\ell' = +0.016494 + 0.011559m' ,$$

The values of $\delta\varepsilon$, δn, δe, $e\delta\tilde{\omega}$, $m'h'$ and $m'\ell'$ thus found (in terms of m'), were then substituted into the 33 equations giving rise to equations of the form:

$$a_{i,1}+a_{i,2}\gamma+a_{i,3}\delta+a_{i,4}\gamma^2+a_{i,5}\delta^2+a_{i,6}\gamma\delta$$

$$+(b_{i,1}+b_{i,2}\gamma+b_{i,3}\delta+b_{i,4}\gamma^2+b_{i,5}\delta^2+b_{i,6}\gamma\delta)m' = 0 ,$$

for $i=1,2,\ldots,33$, $a_{i,j}$ and $b_{i,j}$ $(j=1,2,\ldots,6)$ being constants.

In order to obtain an accurate solution LeVerrier weighted the thirty-three equations by assuming that the relative accuracy of the last twenty-six equations, which represent the modern observations, be unity, while the relative accuracy of the first equation of 1690.98, was one quarter. The remaining six equations representing the rest of the ancient observations, were assumed to have relative accuracies of one half.

The algebraic development of these thirty-three new expressions resulted in an equation of the fourth degree in m', γ and δ. LeVerrier neglected all terms of the third and fourth degrees and formed the sum $\sum$ of the squares giving

$$\begin{aligned}\sum= &+644.40-337.27\gamma+34.33\delta+99.42\gamma^2+42.50\delta^2+32.50\gamma\delta\\ &-m'[600.20-17.29\gamma-121.88\delta-41.97\gamma^2-186.26\delta^2-108.98\gamma\delta]\\ &+m'^2[184.63+70.18\gamma-54.90\delta+12.30\gamma^2-48.90\delta^2-11.58\gamma\delta]\end{aligned} \qquad (4.14)$$

Three equations were then formed corresponding to

$$\frac{\partial\Sigma}{\partial\gamma} = 0, \quad \frac{\partial\Sigma}{\partial C} = 0 \quad \text{and} \quad \frac{\partial\Sigma}{\partial m'} = 0$$

viz.,

$$(198.84+83.94m'+24.60m'^2)\gamma+(32.50+108.98m'-11.58m'^2)C$$
$$-337.27+17.29m'+70.18m'^2 = 0 \qquad (4.15)$$

$$(32.50+108.98m'-11.58m'^2)\gamma+(85.00+372.52m'-97.80m'^2)C$$
$$+34.33+121.88m'-54.90m'^2 = 0 \qquad (4.16)$$

$$(24.60m'+41.97)\gamma^2+(-23.16m'+108.98)\gamma C+(-97.80m'+186.26)C^2$$
$$+(140.36m'+17.29)\gamma+(-109.80m'+121.88)C$$
$$+369.26m'-600.20 = 0 \qquad (4.17)$$

Although equations (4.15) and (4.16) are linear equations of γ and C, in terms of m', equation (4.17) can not be solved easily, and consequently a trial and error method is used. Equation (4.17) is equated to N (say). Values of m' are chosen for which values of γ, C and N are calculated as in the following table:

m'	γ	$\mathcal{G}$	N
0.9	+1.19502	-0.71657	-20.691
1.0	+1.09949	-0.67870	- 8.624
1.1	+1.00273	-0.63945	+ 3.204

Table 4.6

Inspection of this table indicates that equation (4.17) has its minimum when

$$1.0 < m' < 1.1$$

Further evaluation yields the mass of the disturbing planet m' and hence γ and $\mathcal{G}$.
Thus, we find

$$m' = 1.081068\ (1.072714)\ ,$$

$$\gamma = +1.021105(+1.02925)\ ,$$

and

$$\mathcal{G} = -0.646983(-0.65030)\ .$$

from which it follows that the ratio of the mass of the new planet to that of the sun is

$$\frac{1}{9250} \quad \left(\frac{1}{9322} \right) .$$

This means that the new planet will have a considerable mass of the order of two and a half times that of Uranus. In order of size, it will be the third planet next to Jupiter and Saturn. Hence, it will have an appreciable effect on the orbits of other planets.

For the new value of γ obtained,

$$\alpha = 0.530422\ (0.530585)$$

the corresponding mean semi-major axis of the orbit of the new planet being

$$a' = 36\overset{AU}{.}1650\ (36\overset{AU}{.}1539).$$

It follows that the duration T' of its sidereal revolution, expressed in Julian years, is

$$T' = 217\overset{y}{.}484\ (217\overset{y}{.}387).$$

The remaining elements follow accordingly. Thus the longitude of the epoch 1st January 1800 is given by

$$\varepsilon' = 240°21'15''\ (240°17'41'')$$

Adding the sidereal motion n't in 47 years and the motion of the equinox in the same time, the mean longitude L' on 1st January 1847 becomes

$L' = 318°56'27''$ $(318°47'4'')$.

The corresponding numerical values of h' and ℓ' are found to be

$$h' = -0.103134 \ (-0.10437) \ ,$$
$$\ell' = +0.027284 \ (+0.02621) \ ,$$

from which we can determine the eccentricity e', i.e.

$$e' = 0.1066804 \ (0.10761) \ .$$

Also, the longitude of perihelion on 1st January 1847 is

$$284°48'55'' \ (284°45'\ 8'').$$

The mean anomaly at the same epoch is

$$34°07'32'' \ (34°1'56'')$$

and the equation of centre,

$$7°24'33'' \ (7°44'44'').$$

We have, therefore, for the true heliocentric longitude v' of the new planet on 1st January 1847:

$$v' = 326°21'00'' \ (326°31'48'').$$

These results were presented to the Academie de Science on 31st August 1846 together with a personal reflection:

> "This true longitude differs little from 325°, the value which resulted from my first research work. The present determination is founded on numerous and accurate data. It places the new heavenly body about 5° to the east of the star δ of Capricorne.
>
> The opposition of the planet took place on the 19th August last. We are therefore presently at an epoch very favourable for its discovery."

LeVerrier assumed that the density of the new planet was equal to that of Uranus, since the densities of planets were thought to diminish as the distance of the planet from the sun increased.

By supposing, also, that the reflecting power of the surface of the new planet was similar to that of the surface of Uranus, LeVerrier deduced that its specific brightness should be about one third that of the specific brightness possessed by Uranus when at its mean distance from the sun.

The following table provides a comparison of the new theory with the modern observations,

Dates of observations	Excess obs.-cal. positions	Dates of observations	Excess obs.-cal. positions
1781 - 1782	+ 2."3	1813 - 1815	- 0."9
1783 - 1784	+ 0.1	1816 - 1817	+ 0.4
1785 - 1788	- 1.2	1818 - 1820	+ 0.4
1789 - 1790	- 3.4	1821 - 1823	+ 0.9
1791 - 1792	+ 0.3	1824 - 1827	- 5.4
1793 - 1794	- 0.5	1828 - 1830	- 2.2
1795 - 1797	- 1.0	1835	- 0.8
1797 - 1801	+ 0.9	1835 - 1836	+ 2.3
1802 - 1804	+ 0.8	1837 - 1838	+ 2.5
1804 - 1806	+ 0.8	1839 - 1840	+ 2.2
1807 - 1808	+ 2.1	1841 - 1842	- 0.2
1808 - 1810	+ 0.8	1842 - 1844	- 0.4
1811 - 1813	- 0.5	1844 - 1845	- 0.3

Table 4.7

and demonstrates how well the modern observations are represented.

The comparison with ancient observations, however, does not indicate the same level of success, as the following table shows:

Epoch	Excess obs.-cal. positions
1690	+ 19".9
1712 - 1715	- 5.5
1750	+ 7.4
1753 - 1756	+ 4.0
1764	+ 4.9
1768 - 1769	- 3.7

Table 4.8

5. THE EFFECT OF OTHER INEQUALITIES ON THE DEGREE OF ACCURACY OF THE ORBITAL ELEMENTS.

In order to implement a search for an accurate solution in which the error between theory and observation is reduced to its least value, it is necessary first to examine the relative magnitudes of each of the first and second-order perturbations. It should be recalled also that in the first approximation, LeVerrier neglected second-order terms and the secular inequalities, and similarly for his second solution, discussing rather the effect of the number of equations of condition on the accuracy of the determination of the orbital elements of the unknown planet.. LeVerrier had started with eight equations of condition, increased them to eighteen in his second approximation and continued thus until he had thirty-three equations covering the ancient and modern observations from 1690 up to 1845. Only when examining the latter group, as discussed in Section (4), did he consider those perturbations dependent on a higher power of eccentricity, neglecting some in order to facilitate the resolution of the equations of condition.

Consider, for example, the first-order perturbations, i.e. the perturbations, $P^{(i)}$, independent of the eccentricities e and e', the perturbations, $N^{(i)}$, dependent on the first power of the eccentricity, e, of Uranus and the inequalities, $M^{(i)}$, which are proportional to the first power of the eccentricity, e', of the unknown planet, for $i=1,2,...,6$ corresponding to $\alpha=0.49,0.50,...0.60$ respectively. These perturbations may be calculated by using formulae similar to those of Section 2, giving rise to the numerical values shown in Table 5.1. Corresponding second-order terms, B', C' and D', which depend on the small argument (3n'-n) may be calculated from equations (4.13) of Section 4, their numerical values being shown in Table 5.2. Tables 5.3 and 5.4 give respectively the numerical values of the coefficients of the error in heliocentric longitude, δv, as derived from equations (4.7) and (4.8), and the error in the radius vector, δr, as derived from equation (4.11).

Reference to Table 5.1 shows that for values of α outside the range considered by LeVerrier, terms corresponding to $i > 3$ can no longer be safely neglected. On the other hand the second-order terms appear to be insignificant.

However, since there is no way of knowing which terms would have been included by LeVerrier, had he chosen a value of α in excess of 0.53, any subsequent investigation using a range of values of $\alpha=0.49,0.50,...,0.60$ must necessarily include some perturbations corresponding to values of $i > 3$.

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60
$P^{(1)}$	16".91	18".49	20".22	22".12	24".21	26".49	29".01	31".78	34".84	38".21	46".94	46".08
$P^{(2)}$	24.97	29.48	35.00	41.86	50.48	61.50	75.88	95.16	121.87	160.65	220.74	323.83
$P^{(3)}$	2.50	2.90	3.36	3.90	4.53	5.28	6.16	7.20	8.43	9.91	11.69	13.83
$P^{(4)}$	0.55	0.64	0.75	0.88	1.03	1.21	1.42	1.66	1.96	2.30	2.71	3.21
$P^{(5)}$	0.15	0.18	0.22	0.26	0.31	0.37	0.44	0.52	0.62	0.73	0.87	1.04
$P^{(6)}$	0.05	0.06	0.08	0.10	0.12	0.14	0.17	0.21	0.25	0.31	0.37	0.45

Table 5.1

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60
$N^{(1)}$	1.74	1.90	2.06	2.25	2.45	2.68	2.92	3.19	3.49	3.82	4.18	4.59
$N^{(2)}$	14.26	17.00	20.47	25.03	31.08	39.37	51.07	68.33	95.18	140.12	223.59	405.45
$N^{(3)}$	43.90	24.41	18.60	16.10	14.94	14.51	14.54	14.94	15.66	16.71	18.11	19.91
$N^{(4)}$	0.68	0.85	1.07	1.35	1.73	2.24	2.93	3.91	5.33	7.49	11.00	17.26
$N^{(5)}$	0.13	0.15	0.19	0.23	0.28	0.34	0.41	0.50	0.61	0.74	0.91	1.12
$N^{(6)}$	0.05	0.07	0.08	0.10	0.13	0.16	0.21	0.26	0.33	0.41	0.52	0.67

Table 5.1 (continued)

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60
$M^{(1)}$	40"	43"	46"	49"	52"	56"	60"	64"	69"	74"	79"	85"
$M^{(2)}$	100	121	148	183	230	294	385	519	727	1077	1725	3136
$M^{(3)}$	1706	930	696	591	539	513	506	510	526	552	588	636
$M^{(4)}$	27	33	41	51	64	82	105	138	185	255	369	569
$M^{(5)}$	6	7	8	10	12	15	18	22	28	35	43	55
$M^{(6)}$	2	2	2	3	3	4	5	6	7	8	10	12

Table 5.1. (continued)

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60
B' $\times 10^{-2}$	0.502	0.126	0.059	0.035	0.024	0.018	0.014	0.012	0.010	0.009	0.008	0.007
C' $\times 10^{-2}$	39.04	9.578	4.405	2.596	1.748	1.280	0.993	0.803	0.670	0.574	0.501	0.445
D' $\times 10^{-2}$	140.5	35.29	16.62	10.03	6.911	5.179	4.109	3.399	2.901	2.539	2.266	2.057

Table 5.2 Second-Order Terms

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60
f_1 $\times 10^{-2}$	2.130	2.320	2.530	2.750	2.997	3.261	3.549	3.862	4.203	4.576	4.983	5.428
f_2 $\times 10^{-2}$	27.07	30.05	33.32	36.93	40.90	45.27	50.09	55.40	61.27	67.74	74.90	82.81

<u>Table 5.3</u> Coefficients of δv

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60
ef_3	-0.0213	-0.0253	-0.0275	-0.0299	-0.0325	-0.0354	-0.0385	-0.0418	-0.0496	-0.0495	-0.0538	-0.0586
f_4	0.0922	0.1029	0.1147	0.1278	0.1423	0.1584	0.1762	0.1960	0.2180	0.2425	0.2697	0.3000
$C^{(2)}$	-1.49	-1.74	-2.04	-2.42	-2.89	-3.48	-4.24	-5.26	-3.32	-4.32	-5.86	-8.48
$D^{(3)}$	+160	+96	+63	+58	+46	+44	+42	+42	+42	+43	+44	+46
$E^{(3)}$	-4.1	-2.2	-1.6	-1.4	-1.2	-1.1	-1.05	-1.03	-1.02	-1.03	-1.05	-1.09

Table 5.4. Numerical Values of the Coefficients of $\frac{\delta r}{100}$

5.1 Retention of Four Principal Inequalities

Throughout this section, the origin of time used is that adopted by LeVerrier in his research, viz. 1st January 1800.

Following LeVerrier's procedure outlined in Sections 3 and 4, but excluding second-order terms, the principal terms occurring in A, H and L are those corresponding to $i=1,2,3,4$. The perturbation in heliocentric longitude is therefore

$$\delta v = Am' + Hm'h' + Lm'\ell' ,$$

where

$$A = \sum_{i=1}^{4} P^{(i)} \sin i \{(n'-n)t+\varepsilon'-\varepsilon\}$$

$$+ \sum_{i=1}^{4} N^{(i)} \sin[i\{(n'-n)t+\varepsilon'-\varepsilon\}-(nt+\varepsilon+\tilde{\omega})] , \qquad (5.1)$$

$$H = -\sum_{i=1}^{4} M^{(i)} \cos[i\{(n'-n)t+\varepsilon'-\varepsilon\}-(nt+\varepsilon)] , \qquad (5.2)$$

and

$$L = +\sum_{i=1}^{4} M^{(i)} \sin[i\{(n'-n)t+\varepsilon'-\varepsilon\}-(nt+\varepsilon)] . \qquad (5.3)$$

Following the procedure of Section 3, the final equations for the correction of the elliptic elements were then derived with respect to $\delta\varepsilon$, δn, δe and $e\delta\tilde{\omega}$. The perturbations of the heliocentric longitude and the errors of the observations were then added to each equation. The 279 equations of condition covering both ancient and modern observations were then reduced to 114 by regrouping, as before.

The numerical expressions of the above perturbations, A, H and L, were formed for the different epochs using

$$\alpha = 0.51 + 0.02\gamma \quad , \quad -1 \leq \gamma \leq +1 ,$$

and

$$\varepsilon' = 252^{\circ} + 18C \quad , \quad -1 \leq C \leq +1 ,$$

as before.

The resulting equations were solved for γ and C, as in Section 4, and the values of the orbital elements found to be:

a'	$36^{AU}.04(36^{AU}.1539)$	ε'	$239^{\circ}\,41'\,7''\,(240^{\circ}\,17'\,41'')$
m'	0.98575(1.072714)	v'	$328^{\circ}\,57'\,00''(326^{\circ}\,32'\,00'')$
e'	0.11915(0.10761)	$\tilde{\omega}$	$277^{\circ}\,4'\,29''\,(284^{\circ}\,45'\,8'')$

Table 5.5

The previous work was then repeated with the addition of the second-order terms C' given in Section

4, i.e. by adding

$$\frac{3C\alpha e}{(10^4)''(3\nu-1)^2} m'e'\sin(3\ell'-\ell-\tilde{\omega}-\tilde{\omega}') ,$$

where

$$C = \frac{1}{4}\left[20b_{\frac{1}{2}}^{(2)} + 10\alpha \frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} + \alpha^2 \frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\alpha^2}\right] ,$$

and

$$\ell = nt + \varepsilon \quad , \ell' = n't + \varepsilon',$$

and the terms containing t^2 which are coefficients of $m'h'$ and $m'\ell'$.

Solution yielded:

	AU AU		
a'	36.1803(36.1539)	ε'	240° 26' 13" (240° 17' 41")
m'	1.092606(1.072714)	v'	326° 6' (326° 32')
e'	0.1054347(0.10761)	$\tilde{\omega}'$	286° 27' 25" (284° 45' 8")

Table 5.6

clearly indicating the sensitivity of the solutions to the inclusion of second-order terms, a conclusion also arrived at by Brookes (1970) in his analysis of the work of Adams.

5.2 Variations of the Semi-Major Axis

To examine the effect of values of α outside the range $0.51 + 0.02\gamma$ it was thought more appropriate to follow the method of LeVerrier's second solution. Consequently, solutions were sought corresponding to specific values of α, viz. $\alpha = 0.49, 0.50, \ldots, 0.60$ whilst still retaining the four principal inequalities referred to above, and in addition, the second-order term.

The results obtained are shown in Table 5.7 (using the whole 279 equations of condition) and Table 5.8 (using the reduced set of 114).

5.3 Retention of Six Principal Inequalities

As a final analysis of the effect of the erroneous value of a' and as a preliminary to a subsequent investigation into the significance of the timing of LeVerrier's research, it was thought desirable to include two further perturbations, all three second-order terms, and to utilise all 279 equations of condition.

Two separate techniques were used. With i=1,2,...,6 and writing

$$\alpha = 0.53 + \gamma \quad , \quad -1 \leq \gamma \leq +1$$

$$\varepsilon' = 234^\circ + 18\beta \quad , \quad -1 \leq \beta \leq +1$$

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
e' $\times 10^{-2}$	20.51	18.43	15.46	13.87	10.14	24.63	105.2	96.35	72.13	41.42	20.61	10.34	0.86
m'	3.041	2.599	2.118	1.546	1.142	0.941	0.825	0.722	0.679	0.526	0.504	0.465	0.259
v'	330°.5	325°.0	323°.5	325°.0	330°.3	334°.8	337°.9	343°.4	346°.3	354°.5	359°.1	2°.9	327°.4
$\tilde{\omega}'$	35°.1	11°.6	345°.7	318°.0	290°.6	273°.1	82°.5	73°.6	70°.1	58°.8	57°.3	54°.9	44°

Table 5.7

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
e' $\times 10^{-2}$	21.35	19.65	16.72	14.38	10.45	28.17	107.7	98.76	75.32	45.67	21.92	11.24	0.86
m'	3.073	2.654	2.146	1.613	1.163	0.969	0.872	0.745	0.694	0.581	0.532	0.500	0.259
v'	$331^\circ.2$	$325^\circ.4$	$323^\circ.5$	$325^\circ.5$	$330^\circ.7$	$335^\circ.6$	$339^\circ.8$	$344^\circ.7$	$347^\circ.2$	$359^\circ.8$	$2^\circ.3$	$7^\circ.2$	$327^\circ.4$
$\tilde{\omega}'$	$36^\circ.2$	$14^\circ.9$	$347^\circ.0$	$321^\circ.6$	$292^\circ.3$	$275^\circ.6$	$86^\circ.8$	$75^\circ.5$	$71^\circ.3$	$62^\circ.6$	$59^\circ.2$	$57^\circ.0$	44°

Table 5.8

the method of Section 5.1 gave rise to the results of Table 5.9.

a'	36.0903(36.1539) AU	v'	$327^\circ 40'(326^\circ 32')$
m'	1.024363(1.072714)	$\tilde{\omega}'$	$279^\circ 52' 3''(284^\circ 45' 8'')$
e'	0.11358(0.10761)	ε'	$239^\circ 57' 13''(240^\circ 17' 41'')$

Table 5.9

It is interesting to note that the inclusion of the additional terms, whilst not producing significant changes in the orbital elements, gives rise to a value of v' which is nearer to the actual value than that deduced by LeVerrier.

The corresponding solutions for specific values of α, following the procedure referred to in Section 5.2, are given in Table 5.10.

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.25	33.65	33.07	32.51	31.97	30.07
$e' \times 10^{-2}$	10.50	6.14	6.13	7.57	10.44	28.03	102.4	99.87	87.65	40.32	14.64	2.44	0.86
ε'	273°.64	259°.00	250°.19	244°.61	240°.51	236°.93	234°.72	232°.14	239°.77	251°.43	254°.52	256°.45	
$\tilde{\omega}'$	35°.56	14°.36	346°.12	321°.03	286°.67	60°.63	84°.25	115°.24	296°.42	327°.15	142°.64	129°.65	44°
$m' \times 10^{-4}$	3.07	2.66	2.14	1.62	1.10	0.56	0.44	0.32	0.28	0.25	0.24	0.23	0.26
v'	325°.6	329°.9	330°.0	328°.5	325°.9	350°.1	338°.2	330°.5	328°.3	326°.8	325°.0	328°.4	327°.4

Table 5.10. Numerical Values of Orbital Elements for Different Values of α for Neptune.

6. SOLUTIONS FOR OTHER EPOCHS

If the investigations had been instituted in any year prior to 1846, then the number of observations and hence the number of equations of condition that could have been formed would naturally have been reduced, the period over which the modern observations extended now being less than sixty-four years. As a result, it is no longer possible to retain an equal number of equations of condition for the new epochs, corresponding to the times

$$1800.604 + t \quad , \; \forall t = 10, 15, 20, \ldots, 40.$$

The number of equations of condition used, at each new epoch, will be sufficient to enable the calculations to be made in retrospect for any year between 1800 and 1845.

In Table 6.1, the number of available observations and their corresponding number of equations of condition are given for the years 1840, 1835, 1830, 1825, 1820, 1815 and 1810.

As mentioned earlier in Section 4, the equations of condition are those between the geocentric errors of the Tables and the corrections of the elements of the orbit of Uranus. They were therefore completed by adding to their first members the perturbations of the geocentric

Year of calculation	Observations used	No. of available observations	No. of equations of condition according to LeVerrier's technique	Actual No. of equations of condition used
1840	1690 : 1 : 1840	253	31	106
1835	1690 : 1 : 1835	222	29	96
1830	1690 : 1 : 1830	211	27	93
1825	1690 : 1 : 1825	200	25	86
1820	1690 : 1 : 1820	180	23	81
1815	1690 : 1 : 1815	159	21	75
1810	1690 : 1 : 1810	137	19	67

Table 6.1

longitude, G, of Uranus, which are due to the action of the disturbing planet. These were deduced from equations (4.1), (4.2) and (4.3).

The coefficients of the perturbations $P^{(i)}$, $N^{(i)}$ and $M^{(i)}$, i=1,2,...,6 are used for each solution corresponding to each one of the new epochs.

The second-order terms C' in the mean longitude and that containing t^2 for each of m'h' and m'ℓ' are also used, without, however, the addition of any other terms. This is felt to be more in accordance with LeVerrier's actual method.

The aim of this section is the recalculation of the orbital elements following the method of solution given in Section 4.

After the formation of a set of equations of condition for a specific epoch, the solution of the equations at this epoch was given, i.e. the seven unknown values $\delta\varepsilon$, δn, δe, $e\delta\tilde{\omega}$, m'h', m'ℓ' and m'.

The coefficients A, H and L of m', m'h' and m'ℓ' respectively were recalculated under the following assumptions, adopted by LeVerrier in his final solution.

(i) $\alpha = 0.53 + \gamma$, $\varepsilon' = 252^{\circ} + \beta$

(ii) $\alpha = 0.51 + \gamma$, $\varepsilon' = 234^\circ + \ell$

(iii) $\alpha = 0.53 + \gamma$, $\varepsilon' = 234^\circ + \ell$.

Consider, for example, the first possibility relating to the year 1840. Assuming that LeVerrier started his investigations into the problem in 1840, the possible number of equations of condition was 253, covering both ancient and modern observations. This number was reduced for convenience to 106 equations of condition by a method of regrouping in order to have a reasonable number of equations, which were then weighted approximately.

The resolution of the 106 equations of condition for each of the cases (i), (ii) and (iii) mentioned above, yielded values of the orbital elements and are given in Tables 6.2, 6.3 and 6.4, compared with those of true values of Neptune.

Solutions corresponding to 1835, 1830,..., 1810 were similarly obtained, using, of course, a reduced number of equations of condition in each case.

	1840	1835	1830	1825	1820	1815	1810	Neptune
ε'	$240^\circ.6$	$236^\circ.1$						
$\tilde{\omega}'$	$288^\circ.7$	$53^\circ.2$						44°
v'	$315^\circ.5$	$5^\circ.1$						*
a'	36 .2	35 .4						30.07
m'	1.1197	0.4373	-ve	-ve	-ve	-ve	-ve	0.2590
e'	0.10269	0.41255						0.0086

Table 6.2

	1840	1835	1830	1825	1820	1815	1810	Neptune
ε'	$241^\circ.7$	$238^\circ.7$	$234^\circ.7$	$233^\circ.9$	$234^\circ.5$	$234^\circ.4$	$235^\circ.4$	
$\tilde{\omega}'$	$299^\circ.6$	$84^\circ.3$	$42^\circ.6$	$38^\circ.6$	$41^\circ.7$	$41^\circ.3$	$47^\circ.2$	44°
v'	$313^\circ.26$	$311^\circ.24$	$67^\circ.1$	$150^\circ.4$	$78^\circ.8$	$88^\circ.6$	$28^\circ.1$	*
a'	36.40	35.86	35.20	35.13	35.18	35.17	35.28	30.07
m'	1.2582	0.8448	0.2301	0.1415	0.2114	0.1980	0.3272	0.2590
e'	0.09184	0.14769	0.99078	1.78777	1.10199	1.19505	0.62366	0.0086

Table 6.3

	1840	1835	1830	1825	1820	1815	1810	Neptune
ε'	$239^\circ.8$	$239^\circ.1$	$236^\circ.1$	$234^\circ.4$	$235^\circ.4$	$236^\circ.4$	$237^\circ.5$	
$\tilde{\omega}'$	$279^\circ.2$	$88^\circ.7$	$52^\circ.9$	$41^\circ.2$	$47^\circ.1$	$41^\circ.2$	$45^\circ.7$	44°
v'	$318^\circ.25$	$307^\circ.5$	$6^\circ.0$	$88^\circ.0$	$28^\circ.9$	$87^\circ.8$	$37^\circ.3$	*
a'	36.07	35.92	35.38	35.17	35.28	35.17	35.25	30.07
m'	1.0091	0.8930	0.4315	0.1988	0.3246	0.1991	0.2974	0.2590
e'	0.11569	0.13621	0.42078	1.189128	0.63044	1.18692	0.70975	0.0086

Table 6.4

* See Tables 6.5, 6.6, ..., 6.11

Following the procedures of Sections 5.2 and 5.3 solutions were also obtained for specific values of α, viz. α=0.49, 0.50, ..., 0.60 as before, using the reduced number of equations of condition where appropriate. A comparison of the orbital elements so determined, with those of Neptune, are given in Tables 6.5, 6.6, ..., 6.11, the value of 44° for the longitude of perihelion, $\tilde{\omega}'$, being an adequate approximation, for the purpose of comparison, of the values quoted in modern tables. The unit of mass is taken as 10^4 times that of the sun as adopted by LeVerrier.

Whether or not the predictions derived in other years and using different values of α would have led to the discovery of the new planet, is best illustrated in Tables 6.12 and 6.13. The procedure adopted is that given by Brookes (1970), in designating by $\alpha_i = 0.48 + 0.01i$ $(i=1,2,\ldots,12)$, the twelve values chosen and indicating by a positive sign if the planet lies within $\pm 5^\circ$ or $\pm 10^\circ$ of the predicted position, and by a negative sign if not.

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
$e' \times 10^{-2}$	14.54	13.88	12.54	11.33	10.42	10.06	112.97	91.79	70.89	40.65	20.48	10.26	0.86
m'	1.496	1.322	1.262	1.205	1.150	0.903	0.812	0.721	0.679	0.625	0.599	0.425	0.259
v'	$309^\circ.8$	$313^\circ.6$	$317^\circ.1$	$319^\circ.8$	$322^\circ.4$	$318^\circ.7$	$315^\circ.2$	$306^\circ.9$	$308^\circ.2$	$308^\circ.9$	$310^\circ.4$	$311^\circ.9$	$312^\circ.3$
$\tilde{\omega}'$	$36^\circ.2$	$14^\circ.9$	$8^\circ.7$	$2^\circ.4$	$358^\circ.7$	$354^\circ.2$	$349^\circ.6$	$46^\circ.5$	$39^\circ.4$	$354^\circ.1$	$356^\circ.2$	$358^\circ.0$	44°
ε'	$270^\circ.6$	$258^\circ.9$	$256^\circ.4$	$254^\circ.5$	$252^\circ.1$	$248^\circ.2$	$247^\circ.6$	$243^\circ.5$	$239^\circ.7$	$236^\circ.2$	$234^\circ.5$	$236^\circ.6$	

Table 6.5. Solution for the Epoch 1840

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
$e' \times 10^{-2}$	18.80	17.20	16.42	16.17	15.84	13.64	101.24	86.35	64.21	21.34	10.57	2.15	0.86
m'	1.777	1.587	1.398	1.274	1.256	1.102	0.928	0.787	0.677	0.592	0.514	0.446	0.259
v'	300°.9	303°.6	306°.8	309°.1	310°.2	298°.9	280°.8	317°.1	312°.5	309°.4	305°.2	302°.3	301°.8
$\tilde{\omega}'$	16°.8	14°.9	10°.2	8°.6	5°.4	2°.7	355°.6	351°.6	348°.2	356°.7	22°.7	298°.7	44°
ε'	268°.7	256°.9	250°.2	244°.6	240°.5	236°.9	239°.7	241°.4	242°.4	245°.7	251°.8	254°.2	

Table 6.6. Solution for the Epoch 1835

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
$e' \times 10^{-2}$	68.42	49.67	36.06	26.18	19.00	13.80	110.2	87.12	63.24	45.91	29.67	15.81	0.86
m'	0.472	0.343	0.249	0.181	0.276	0.255	0.436	0.715	0.615	0.449	0.316	0.143	0.259
v'	$356^\circ.2$	$350^\circ.4$	$332^\circ.7$	$331^\circ.2$	$328^\circ.4$	$318^\circ.9$	$314^\circ.6$	$310^\circ.7$	$305^\circ.2$	$300^\circ.1$	$298^\circ.7$	$300^\circ.4$	$290^\circ.5$
$\tilde{\omega}'$	$242^\circ.1$	$236^\circ.1$	$234^\circ.6$	$232^\circ.1$	$209^\circ.8$	$212^\circ.7$	$1^\circ.0$	$4^\circ.6$	$8^\circ.7$	$357^\circ.1$	$160^\circ.8$	$147^\circ.2$	44°
ε'	$256^\circ.4$	$252^\circ.6$	$250^\circ.1$	$248^\circ.1$	$243^\circ.7$	$240^\circ.8$	$251^\circ.4$	$252^\circ.3$	$247^\circ.5$	$240^\circ.1$	$238^\circ.7$	$236^\circ.6$	

Table 6.7. Solution for the Epoch 1830

α	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
e' $\times 10^{-2}$	19.46	17.24	16.82	16.36	15.42	13.43	112.66	92.11	74.23	43.57	12.46	10.66	0.86
m'	1.125	1.013	0.988	0.923	0.897	0.826	0.746	0.611	0.510	0.417	0.365	0.253	0.259
v'	280°.4	283°.7	288°.1	292°.4	286°.9	281°.3	290°.3	285°.5	278°.4	291°.2	283°.5	284°.1	279°.6
$\tilde{\omega}$'	335°.7	168°.7	334°.2	332°.1	156°.8	290°.4	270°.3	264°.1	201°.5	179°.6	194°.8	197°.1	44°
ε'	250°.1	248°.3	245°.6	241°.4	240°.4	238°.7	240°.9	236°.4	235°.1	239°.2	244°.7	252°.7	

Table 6.8. Solution for the Epoch 1825

α	0.59	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
e' $\times 10^{-2}$	18.64	17.03	16.45	15.13	14.22	12.35	109.21	87.65	65.35	42.15	28.76	7.68	0.86
m'	1.892	1.713	1.654	1.588	1.399	1.206	1.108	0.988	0.765	0.588	0.446	0.309	0.259
v'	$280^\circ.2$	$272^\circ.3$	$267^\circ.2$	$263^\circ.5$	$258^\circ.6$	$225^\circ.6$	$253^\circ.4$	$252^\circ.7$	$260^\circ.1$	$256^\circ.4$	$272^\circ.1$	$270^\circ.6$	$268^\circ.9$
$\tilde{\omega}'$	$325^\circ.7$	$323^\circ.6$	$320^\circ.9$	$317^\circ.6$	$316^\circ.7$	$315^\circ.4$	$310^\circ.9$	$354^\circ.0$	$356^\circ.1$	$10^\circ.6$	$61^\circ.2$	$34^\circ.2$	44°
ε'	$268^\circ.2$	$265^\circ.3$	$270^\circ.1$	$272^\circ.4$	$281^\circ.6$	$285^\circ.4$	$290^\circ.6$	$288^\circ.4$	$276^\circ.8$	$266^\circ.5$	$259^\circ.4$	$256^\circ.1$	

Table 6.9. Solution for the Epoch 1820

	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
e' $\times 10^{-2}$	6.12	7.62	8.21	9.12	10.96	11.24	123.57	98.79	79.85	42.31	12.36	8.56	0.86
m'	2.235	2.062	1.965	1.868	1.679	1.432	1.216	1.143	0.998	0.736	0.543	0.326	0.259
ν'	246°.7	245°.7	248°.1	252°.7	258°.4	260°.2	265°.7	270°.4	268°.2	275°.1	273°.5	270°.1	259°.8
$\tilde{\omega}$	160°.2	198°.7	348°.1	351°.6	355°.7	359°.4	2°.3	45°.7	47°.6	38°.7	178°.6	150°.2	44°
ε'	272°.1	269°.4	267°.8	265°.4	270°.3	269°.1	268°.7	267°.4	260°.6	255°.4	252°.6	250°.3	

Table 6.10. Solution for the Epoch 1815

	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	Neptune
a'	39.15	38.36	37.61	36.89	36.19	35.52	34.88	34.26	33.65	33.07	32.51	31.97	30.07
$e' \times 10^{-2}$	12.13	11.56	10.23	9.76	8.67	7.32	98.76	87.65	32.15	12.14	9.76	6.43	0.86
m'	3.245	3.013	2.688	2.423	2.056	1.988	1.568	1.326	1.246	0.876	0.643	0.523	0.259
v'	$234^{\circ}.7$	$236^{\circ}.2$	$246^{\circ}.6$	$249^{\circ}.8$	$259^{\circ}.1$	$257^{\circ}.6$	$256^{\circ}.8$	$235^{\circ}.2$	$266^{\circ}.7$	$262^{\circ}.1$	$254^{\circ}.2$	$252^{\circ}.8$	$247^{\circ}.3$
$\tilde{\omega}'$	$218^{\circ}.3$	$212^{\circ}.6$	$210^{\circ}.7$	$190^{\circ}.4$	$232^{\circ}.6$	$242^{\circ}.1$	$256^{\circ}.1$	$258^{\circ}.0$	$19^{\circ}.7$	$34^{\circ}.2$	$43^{\circ}.7$	105°	44°
ε'	$259^{\circ}.8$	$258^{\circ}.3$	$256^{\circ}.7$	$254^{\circ}.1$	$252^{\circ}.0$	$248^{\circ}.1$	$246^{\circ}.6$	$243^{\circ}.4$	$240^{\circ}.8$	$238^{\circ}.7$	$236^{\circ}.7$	$234^{\circ}.8$	

<u>Table 6.11.</u> Solution for the Epoch 1810

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
1846	+	+	+	+	+	-	-	?	?	?	+	+
1840	+	+	+	-	-	-	?	-	?	?	?	+
1835	+	+	+	-	-	+	-	-	-	-	+	+
1830	-	-	-	-	-	-	-	-	-	-	-	-
1825	+	+	-	-	-	+	-	-	?	-	+	+
1820	-	+	+	-	-	-	-	-	-	-	?	+
1815	-	-	-	-	+	+	-	-	-	-	-	-
1810	-	-	+	+	-	-	-	-	-	-	-	-

Table 6.12 (within $\pm 5^{\circ}$)

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
1846	+	+	+	+	+	-	-	?	?	?	+	+
1840	+	+	+	+	-	+	?	?	?	?	?	+
1835	+	+	+	+	+	+	-	-	-	?	+	+
1830	-	-	-	-	-	-	-	-	-	?	?	+
1825	+	+	+	-	+	+	-	?	?	-	+	+
1820	-	+	+	+	-	-	-	-	?	-	?	+
1815	-	-	-	+	+	+	?	-	?	-	-	-
1810	-	-	+	+	-	-	+	-	-	-	+	+

Table 6.13 (within $\pm 10^{\circ}$)

The ? indicates that although the predicted position is within the specified range, the corresponding value of the orbital eccentricity is unacceptably high, viz. $e' \geq 0.2$.

From Tables 6.12 and 6.13 it would seem, therefore, that the new planet would have been predicted with reasonable accuracy chiefly in the years 1835-1845, but essentially using erroneous values of α, viz. $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

It is significant, however, that the greatest success (within $\pm 10^\circ$) is achieved using the value of $\alpha = \alpha_{12}$ (corresponding to $a' = 32$ A.U.), i.e. the value of a' closest to the actual value $a' = 30$ A.U.

7. CONCLUSION

The main aim of LeVerrier's planetary theory was to determine the position of the disturbing planet, Neptune, from an analysis of the orbital motion of Uranus. In this respect he produced three solutions of increasing accuracy, classified as the First, the Second and the Third solution respectively, the goals and results of which are summarised below.

The First Solution aimed at determining the mass of Neptune on the assumption that the ratio, α, between the semi-major axes of Uranus and Neptune equals $\frac{1}{2}$, in keeping with Bode's Law. For this solution, eight equations of condition were solved corresponding to the years

$$1747.7 + 14(j-1) \quad , j = 1,2,\ldots,8,$$

covering both ancient and modern observations, but utilising only those perturbatory forces $P^{(i)}$, $N^{(i)}$, $M^{(i)}$ (i=1,2,3), of order zero and unity, in e and e'.

According to this solution, the mass of the disturbing planet was of order $\frac{1}{4700}$ times that of the Sun.

The Second Solution, aimed at determining the orbital elements of the disturbing planet and refining the value for the mass, previously determined. For this solution three further equations of condition were added corresponding to the epochs 1690, 1712 and 1715 respectively. The remaining equations of condition were rearranged into fifteen alternative equations, at intervals of seven years, corresponding to the epochs

$$1747.7 + 7(j-1) \quad , \; j=1,2,\ldots,15.$$

On solution, and following close examination of the errors between theory and observation for the epochs 1690, 1747, 1758, 1793 and 1828, LeVerrier came to the conclusion that the mass of the disturbing planet should lie between $\frac{1}{10,000}$ th and $\frac{1}{6700}$ th that of the Sun, and deduced that the true heliocentric longitude on 1st January 1847 should be approximately 325°.

The Third and final solution, called upon the experience gained in the First and Second Solutions, and attempted to improve the accuracy of the values for the mass and the orbital elements of the disturbing planet. Using a range of values of α from 0.49 to 0.53, LeVerrier formed thirty-three equations of condition covering a period of 155 years, and for the first time included certain second-order terms. The solution, as we have seen, yielded a value for the mass equal to approximately $\frac{1}{9300}$ times that of the Sun and a predicted true heliocentric longitude for 1847.0, less than one degree from its actual position.

Although some differences were found in the re-determination of LeVerrier's solutions, generally these could be attributed to the differences in computational techniques available then, and now. In a few instances, however, this is perhaps not so. For example, in the Second Solution, LeVerrier concluded that the mass m' was negative within the range

$$198^\circ \leq \varepsilon' \leq 270^\circ$$

whereas in fact, the re-calculation indicates positive values.

Present day computing facilities encouraged the author to attempt to improve LeVerrier's theory by taking into account the neglected perturbations of the first- and second-order. Moreover, the full 279 equations of condition could be retained instead of the thirty-three equations of the Third Solution. In addition, it was now feasible to seek solutions for each of a range of values of α, viz. $\alpha = 0.49, 0.50, \ldots, 0.60$.

It is intriguing to find that the addition of second-order terms as well as the six principal inequalities $P^{(i)}$, $N^{(i)}$ and $M^{(i)}$ $(i=1,2,\ldots,6)$, led to a reduction in the error of the actual position from $52'$ to $16'$, in spite of the adoption of an erroneous value of α.

Following the procedure of Brookes (1970) outlined in his investigation of the theory of Adams (1847) solutions have been obtained for different years before 1845, backwards in time to the year 1810 at intervals of 5 years, utilising only those observations that would have been available at that time.

No firm conclusions can be drawn from the result obtained due to the frequent occurrence of unacceptably high values of eccentricity. Nevertheless, there is perhaps some justification for stating that LeVerrier's technique would have led to a successful prediction in the years 1835 to 1845 for some values of α, notably $\alpha = 0.49$ to 0.52. Beyond these values, only one gave a sensible solution, viz. $\alpha = 0.60$.

Comparison of the predicted positions corresponding to the theory of both Adams (Brookes) and LeVerrier (Baghdady) is possible now, for the first time.

The two analyses have several epochs and values of α in common, and permit a direct comparison for values of $\alpha = 0.49, 0.50, 0.52, 0.53, 0.56, 0.57, 0.60$ (approximately), for the years 1846, 1840, 1835, 1830 and 1825. The results are presented in Tables 7.1 and 7.2, the letters A and L signifying successful predictions according to the theory of Adams and LeVerrier respectively. As in Section 6, predictions yielding values of $e' \geq 0.2$ are regarded as unsuccessful; and consequently are not included in these tables.

α	0.49	0.50	0.52	0.53	0.56	0.57	0.60
1846	AL	AL	AL	-L	--	--	-L
1840	AL	AL	AL	-L	--	-L	AL
1835	AL	AL	-L	--	A-	A-	AL
1830	--	--	--	--	--	--	--
1825	AL	-L	A-	--	--	-L	-L

Table 7.1 (within ±5°)

α	0.49	0.50	0.52	0.53	0.56	0.57	0.60
1846	AL	AL	AL	AL	A-	A-	AL
1840	AL	AL	AL	AL	A-	AL	AL
1835	AL	AL	AL	AL	A-	A-	AL
1830	--	--	--	--	--	--	-L
1825	AL	-L	A-	-L	A-	-L	AL

Table 7.2 (within ±10°)

It would seem, therefore, that LeVerrier's technique proves the more successful within the range $\pm 5^\circ$, whilst that of Adams proves more successful within the range $\pm 10^\circ$. Both techniques are particularly unsuccessful when applied in 1830 and only partially successful in 1825.

Of the two theories, that of Adams came closest to the correct value of eccentricity, viz. $e' = 0.0088$ (for $\alpha = 0.60$ in 1846) compared to the actual value of 0.0086. On the other hand, e' was always ≥ 0.2 for the year 1830. Values of e' according to the theory of LeVerrier were consistently out by a factor of at least ten.

The fortuitous timing of the predictions in 1846 is thus self-evident, especially if one recalls the usage of erroneous values of α.

However, the validity of both theories seems established, since the best solutions (in terms of a' and e'), are those corresponding to the highest value of α, viz. $\alpha = 0.60$ ($a' = 32$ A.U.), which is not far removed from reality, viz. $a' = 30$ A.U.

APPENDIX

Roots of a real polynomial by QD-algorithm with displacement.

The progressive QD-algorithm is a fast device for the calculation of all roots of a polynomial P(x) with real coefficients, when there are no approximations to the roots available.

The roots of P(x) = 0 are determined by solving for the poles of Q(x)/P(x) where Q(X) is some polynomial of smaller degree than P(x). In the following, n is used to denote the degree of P(x). Using the derivative $P'(x)$ for Q(x) has the advantage that the poles are simple, even in the case of multiple roots of P(x).

The start of the QD-algorithm requires the continued S-fraction:

$$\frac{c}{x-q_1} \; \frac{}{1-e_1} \; \frac{}{x-q_2} \; \frac{}{1-} \; \cdots \; \frac{q_n}{1} \quad \text{of } P'(x)/P(x)$$

This calculation is performed using the Euclidean-algorithm. By means of normalisation such that the highest coefficient is 1, $Q_0(x)$ is obtained from $P(x)$, and $P_1(x)$ from $P'(x)$. Then e_k and q_k are calculated from the iteration scheme:

$$a_k Q_k(x) = x.P_k(x) - Q_{k-1}(x) \quad ; \quad k=1,2,\ldots n \qquad (1)$$

$$e_k P_{k+1}(x) = Q_k(x) - P_k(x) \quad ; \quad k=1,2,\ldots n-1 \qquad (2)$$

using the fact that all elements $Q_i(x)$ and $P_i(x)$ are normalised by definition.

The following example is an illustration of this iteration scheme:

$P(x) = 1 + 3x + 3x^2 + x^3$, $n=3$

$Q_0(x) = x^3 + 3x^2 + 3x + 1$

$P_1(x) = x^2 + 2x + 1$, $q_1 Q_1(x) = -x^2-2x-1$, $q_1 = -1$

$Q_1(x) = x^2 + 2x + 1$, $e_1 P_2(x) = 0$, $e_1 = 0$

where $Q_1(x)$ is a common divisor of $P'(x)/P(x)$, and $q_1 = -1$ is the value of the real root which factors out first.

The above iteration scheme breaks down if some intermediate value of q_i, with $0 < i < n$ is equal or approximately equal to zero. In this case an error code should be used (in the program it is set to 4), which indicates that no S-fraction exists for $P'(x)/P(x)$.

If Q_0 and P_1 have a common divisor of degree j, this common divisor is obtained as Q_{n-j}, and necessarily $e_{n-j}P_{n-j+1} = 0$. Due to round-off errors, all coefficients of $e_{n-j}P_{n-j+1}$ cannot be expected to vanish exactly; therefore, allowance is made for coefficients of small absolute value.

If the highest coefficient of $e_{n-j}P_{n-j+1}$ is small in absolute value, but some lower coefficient is not, the error code is set to 4, and the subroutine is abandoned.

The coefficients e_k, q_k obtained by the Euclidean-algorithm, form the first row of the QD-array indicated by superscript $_0$.

e_0	q_1	e_1	q_2	e_2	q_m	e_m
0	$q_1^{(0)}$	$e_1^{(0)}$	$q_2^{(0)}$	$e_2^{(0)}$	$q_m^{(0)}$	$e_m^{(0)}$
0	$q_1^{(1)}$	$e_1^{(1)}$	$q_2^{(1)}$	$e_2^{(1)}$	$q_m^{(1)}$	$e_m^{(1)}$

....

Normally m is equal to n, but it may be smaller if a common factor exists.

The following rows are obtained by means of the relationships:

$$q_i^{(v+1)} = q_i^{(v)} + e_i^{(v)} - e_{i-1}^{(v+1)} \quad , \quad i=1,2,\ldots m \qquad (3)$$

$$e_i^{(v+1)} = \frac{e_i^{(v)} \cdot q_{i+1}^{(v)}}{q_i^{(v+1)}}, \quad i=1,2,\ldots m-1 \tag{4}$$

together with $e_0^{(v)} = e_m^{(v)} = 0$.

Let x_i denote the roots of P(x) ordered in decreasing absolute value. If P(x) has only roots of different absolute value, there is convergence of the q_k-column to the value x_k. Complex roots are indicated by oscillation of certain q-values.

If x_k and x_{k+1} are equal absolute values, then the roots of

$$x^2 - (q_k^{(v+1)} + q_{k+1}^{(v)})x + q_k^{(v)} q_{k+1}^{(v)} = 0 \tag{5}$$

converge to x_k and x_{k+1}. The same reasoning applies if the relation $|x_k|=|x_{k+1}|$ holds only approximately.

The QD-algorithm in the form (3),(4) suffers from the fact that convergence is rather slow. By means of displacement of the origin, a form is obtained which is asymptotically of quadratic convergency.

The strategy is as follows. Start with displacement $t = 0$.

I. As soon as convergence is indicated to a real root, then $e_{m-1}^{(v)}$ is sufficiently small (internal test value 0.01). Perform a displacement by the amount $q_m^{(v)}$.

Starting with $q_k^{(v)}$, $e_k^{(v)}$, use instead of (3) the iteration scheme:

$$t = t + q_m^{(v)} \qquad (6)$$

$$q_k^{(v+1)} = q_k^{(v)} + e_k^{(v)} - e_{k-1}^{(v+1)} - q_m^{(v)} \quad , k=1,2,...m \qquad (7)$$

$$e_k^{(v+1)} = q_{k+1}^{(v)} \cdot e_k^{(v)}/q_k^{(v+1)} \quad , k=1,2,...m-1 \qquad (8)$$

Values $e_0^{(v+1)}$ and $e_m^{(v)}$ are thereby set equal to zero.

II. If convergence is indicated to a root pair - that is, $e_{m-2}^{(v)}$ is less than $e_{m-1}^{(v)}$ and sufficiently small (internal test value 0.01) - calculate the discriminant of the quadratic equation (5):

$$D = P^*P - q_m^{(v)} * q_{m-1}$$

with

$$P = 0.5\ (q_{m-1}^{(v)} + e_{m-1}^{(v)} + q_m^{(v)})$$

If D is positive, provide for a real displacement of amount

P - √D or P + √D, whichever has the smaller absolute value, and proceed according to (7), (8). If D is negative, three complex displacements are applied in sequence: the first by amount $P+i\sqrt{|D|}$, the second by $-2i\sqrt{|D|}$, and the third by $V+i\sqrt{|D|}$, resulting in a total real displacement of amount P. Instead of (6), (7), (8) the following iteration scheme is then obtained. Starting with:

$$t = t + P \tag{9}$$

$$\overset{*}{q}_1 = \overset{(v)}{q}_1 + \overset{(v)}{e}_1 - P \tag{10}$$

$$P_1 = -D/\overset{*}{q}{}_1^{2} \tag{11}$$

$$\overset{*}{e}_1 = \overset{(v)}{e}_1 \overset{(v)}{q}_2 \, / \, [\overset{*}{q}_1(1+P_1)] \tag{12}$$

$$\overset{**}{q}_1 = \overset{*}{q}_1 + \overset{*}{e}_1 \tag{13}$$

calculate for k=2,3,...m:

$$\overset{*}{q}_k = \overset{(v)}{q}_k + \overset{(v)}{e}_k - \overset{*}{e}_{k-1} - P \tag{14}$$

$$\overset{**}{e}_{k-1} = \overset{*}{e}_{k-1} \overset{*}{q}_k / \overset{**}{q}_{k-1} \tag{15}$$

$$P_k = P_{k-1}(\overset{**}{q}_{k-1}/\overset{*}{q}_k)^2 \tag{16}$$

$$q_{k-1}^{(v+3)} = q_{k-1}^{**} + e_{k-1}^{**} - e_{k-2}^{(v+3)} \quad (17)$$

$$e_k^{*} = q_{k+1}^{(v)} e_k^{(v)}/[q_k^{*}(1+P_k)] \quad (18)$$

$$q_k^{**} = q_k^{*} + e_k^{*} - e_{k-1}^{**} \quad (19)$$

$$e_{k-1}^{(v+3)} = (e_{k-1}^{**} q_k^{**}/q_{k-1}^{(v+3)})(1+P_k) \quad (20)$$

Finally set:

$$q_m^{(v+3)} = q_m^{**} - e_{m-1}^{(v+3)} \quad (21)$$

values $e_0^{(v+3)}$, $e_m^{(v)}$ and $q_{m+1}^{(v)}$ are thereby set equal to zero.

III. If none of the values $e_{m-1}^{(v)}$ and $e_{m-2}^{(v)}$ is sufficiently small, the relationships (3),(4) are used with no displacement at all.

Regarding termination of the iterative scheme given by I, II, III, there are two possibilities:

1. If $e^{(v)}_{m-1}$ is negligible (internal test value is 10^{-6} in single precision and 10^{-16} in double precision), a real root is factored out.

2. If $e^{(v)}_{m-2}$ is negligible (with the same internal test values), a pair of roots is factored out.

A maximum of ten times the number of coefficients using I, II or III is allowed. At every iteration step for one and the same root or root pair, the internal test value for convergence and the internal test value for acceptance of a displacement are increased by 10%.

In case of convergence:

1. For a real root ...
   (a) real part of root = $t + q_m$
   (b) complex part of root = 0

2. For a real root pair (characterised by $D > 0$)
   (a) real part of first root = $t + P + \sqrt{D}$
   (b) complex part of first root = 0
   (c) real part of second root = $t + P - \sqrt{D}$
   (d) complex part of second root = 0

3. For a complex root pair (characterised by $D < 0$)
   (a) real part of first root = $t + P$
   (b) complex part of first root = $\sqrt{-D}$

(c) real part of second root = t + P

(d) complex part of second root = $-\sqrt{-D}$

As soon as a root or root pair has been factored out, m is reduced by 1 or 2 respectively and the whole procedure I, II, III is repeated with original values of internal test values, until m = 0. This means, all roots have been calculated, or m = 1, when the last real root is factored immediately. If P(x), P(x) have a common divisor, the whole process is repeated for this common divisor. Thus, the complete factorisation of the original polynomial P(x) is obtained.

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