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A Continuous Extension that preserves Concavity, Monotonicity and Lipschitz Continuity

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Abstract

The following is proven here: let $W : X \times C \longrightarrow \mathbb{R}$, where X is convex, be a continuous and bounded function such that for each $y \in C$, the function $W(\cdot, y) : X \longrightarrow \mathbb{R}$ is concave (resp. strongly concave; resp. Lipschitzian with constant M; resp. monotone; resp. strictly monotone) and let $Y \supseteq C$. If C is compact, then there exists a continuous extension of W, $U : X \times Y \longrightarrow \left[\inf_{X \times C} W, \sup_{X \times C} W\right]$, such that for each $y \in Y$, the function $U(\cdot, y) : X \longrightarrow \mathbb{R}$ is concave (resp. strongly concave; resp. Lipschitzian with constant M_y; resp. monotone; resp. strictly monotone).

There are several classical extension results in real, abstract and convex analysis. Urysohn's lemma (see Royden (1988), 8.3.7) says that, given a normal space Y,[1] if $A, B \subseteq Y$ are closed, then there exists $U : Y \longrightarrow [0,1]$, continuous, such that for each $y \in A$, $U(y) = 0$ and for each $y \in B$, $U(y) = 1$.

Tietze's extension theorem states that any continuous and bounded function defined from a closed subset of a metric space into the real line has a continuous extension to the whole of the space, with the same bounds as the original function. Formally:

Theorem 1 *Let $(Y, \|\cdot\|)$ be a metric space, and suppose that $A \subseteq Y$ is closed and $W : A \longrightarrow \mathbb{R}$ is continuous and bounded. There exists $U : Y \longrightarrow \mathbb{R}$,*

[1] That is, Y is assumed to be endowed with a topology $\Im$ such that

$$(\forall y, y' \in Y : y \neq y')\,(\exists O, O' \in \Im) : y \in O \backslash O' \wedge y' \in O' \backslash O$$

and

$$(\forall C, C' \subseteq Y : C \cap C' = \varnothing \wedge Y \backslash C \in \Im \wedge Y \backslash C' \in \Im)$$
$$(\exists O, O' \in \Im) : O \cap O' = \varnothing \wedge C \subseteq O \wedge C' \subseteq O'$$

continuous, such that

$$\begin{aligned}(\forall y \in A) \quad &: \quad U(y) = W(y) \\ \inf_{y \in Y} U(y) \quad &= \quad \inf_{y' \in A} W(y') \\ \sup_{y \in Y} U(y) \quad &= \quad \sup_{y' \in A} W(y')\end{aligned}$$

Proof. This is theorem 3.2.13 in Bridges (1998), pp. 144-145. ■

Regarding Lipschitz continuity,[2] Kirszbaum's theorem (see Federer (1969), 2.10.43) states that if $(Y, \|\cdot\|)$ is a metric space, $A \subseteq Y$ and $W : A \longrightarrow \mathbb{R}$ is Lipschitzian with constant $M \in \mathbb{R}$, then there exists an extension of W to the whole of Y, say $U : Y \longrightarrow \mathbb{R}$, which is also Lipschitzian with constant M.

In convex analysis, the classical extension result studies conditions under which a convex and bounded real valued function defined on the interior of a set can be extended to a continuous and convex function defined on the whole of the set.[3]

A problem similar to the one covered by Tietze's result is studied here. Suppose that $X \subseteq \mathbb{R}^J$ and $Y \subseteq \mathbb{R}^K$, where $J, K < \infty$. Suppose that X is convex and that $C \subseteq Y$. Let $W : X \times C \longrightarrow \mathbb{R}$ be continuous, bounded and such that for each $y \in C$, $W(\cdot, y) : X \longrightarrow \mathbb{R}$ is concave. I study conditions on C under which one can ensure that there exists $U : X \times Y \longrightarrow \mathbb{R}$ such that:

1. U is continuous;
2. $(\forall (x, y) \in X \times Y)$:
$$\inf_{(x', y') \in X \times C} W(x', y') \leqslant U(x, y) \leqslant \sup_{(x'', y'') \in X \times C} W(x'', y'')$$
3. $(\forall y \in Y) : U(\cdot, y) : X \longrightarrow \mathbb{R}$ is concave;
4. $(\forall (x, y) \in X \times C) : U(x, y) = W(x, y)$.

By adapting the proof of Tietze's extension theorem given by Bridges (1998), I find that compactness of C suffices. Moreover, the compactness assumption allows me to claim that if for each $y \in C$, $W(\cdot, y)$ is strongly concave

[2] If $(X, \|\cdot\|)$ is a metric space, a function $W : X \longrightarrow \mathbb{R}$ is said to be Lipschitz continuous, or Lipschitzian, if for some $M \in \mathbb{R}$, it is true that

$$(\forall x, x' \in X) : |W(x) - W(x')| \leqslant M \|x - x'\|$$

In this case, M is said to be a Lipschitz constant for W.

[3] If $C \subseteq \mathbb{R}^J$ is a locally simplicial, convex set and $W : ri(C) \longrightarrow \mathbb{R}$ is a convex function and such that $\forall D \subseteq ri(C)$, D bounded, $W[D]$ is bounded above, then there exists a unique continuous extension of W to C (see Rockafellar (1970), 10.3). A set C is said to be locally simplicial if $\forall x \in C$, there exists a finite collection of simplices, say $\{S_m\}_{m=1}^M$, such that $\forall m \in \{1, ..., M\}$, $S_m \subseteq C$ and there exists $U \subseteq \mathbb{R}^J$, open, such that $x \in U$ and

$$U \cap \left(\bigcup_{m=1}^{M} S_m \right) = U \cap C$$

(resp. Lipschitzian with constant M – independent of y; resp monotone[4]; resp. strictly monotone[5]), then U can further be found that satisfies that for each $y \in Y$, $U(\cdot, y)$ is strongly concave (resp. Lipschitzian with constant M_y; resp. monotone; resp. strictly monotone).

To the best of my knowledge, this result is new. There is, however some related literature. Stadje (1987) shows that if $A \subseteq (a,b)$ has full Lebesgue measure with respect to (a,b) (that is $M((a,b) \setminus A) = 0$) and $W : A \longrightarrow \mathbb{R}$ is measurable[6] and mid-convex,[7] then there exists a convex extension of W to (a,b). Neither continuity nor Lipschitz continuity are studied by Stadje.

Matoušková (2000) shows that if Y is a compact Hausdorff space,[8] $A \subseteq Y$ is closed, and $W : A \longrightarrow \mathbb{R}$ is continuous and Lipschitzian, then there exists a continuous extension of W, $U : Y \longrightarrow \mathbb{R}$ which is Lipschitzian, with the same constant as W, and has the same sup norm.[9] No concavity or monotonicity properties are studied by Matoušková.

On the other hand, Howe (1986) gives necessary and sufficient conditions under which for a finite collection $\left\{W^l\right\}_{l=1}^{L}$ of continuous and concave functions, $W^l : \mathbb{R}_+^J \longrightarrow \mathbb{R}_+$, there exist $M \in \mathbb{N} \cup \{0\}$ and $U : \mathbb{R}_+^J \times \mathbb{R}_+^M \longrightarrow \mathbb{R}_+$ such that:

1. U is continuous;
2. U is concave;
3. $(\forall l \in \{1, ..., L\}) \left(\exists x^l \in \mathbb{R}_+^M\right) : U\left(\cdot, x^l\right) = W^l(\cdot)$

Because this last problem deals with a setting very similar to the one studied here, the differences deserve to be pointed out. The first and more obvious one is that no smoothness or monotonicity properties are dealt with by Howe. The second one, which is fundamental, is that I am not assuming that C is finite, so that the finiteness assumption of Howe's does not fit in my setting. Besides, I take as given the set Y, and, therefore, cannot use its dimension as a variable. Moreover, I do not require concavity of the function U, but only of its cross

[4] $f : X \longrightarrow \mathbb{R}$ is monotone if for every $x, x' \in X$ such that $x >> x'$, it is true that $f(x) > f(x')$.

[5] $f : X \longrightarrow \mathbb{R}$ is strictly monotone if for every $x, x' \in X$ such that $x > x'$, it is true that $f(x) > f(x')$.

[6] That is to say that $\forall \alpha \in \mathbb{R}$, the set

$$f^{-1}[(\alpha, \infty)] = \{x \in (a,b) | f(x) > \alpha\}$$

is Lebesgue measurable.

[7] That is

$$\left(\forall x, x' \in A : \frac{1}{2}(x + x') \in A\right) : W\left(\frac{1}{2}(x + x')\right) \leqslant \frac{1}{2}(W(x) + W(x'))$$

[8] A topological space $(Y, \Im)$ is Hausdorff, or T_2, if

$$(\forall y, y' \in Y : y, y') (\exists O, O' \in \Im) : y \in O \wedge y' \in O' \wedge O \cap O' = \varnothing$$

[9] The target set need not be $\mathbb{R}$, but any metric space with lower semicontinuous metric.

sections ($U(\cdot, y)$ for each $y \in Y$), and Y need not even be a convex set, nor do I study the necessity of my assumptions.

In what follows, given a set $Y \subseteq \mathbb{R}^K$, I define the point-to-set distance function

$$dis : Y \times \mathcal{P}'(Y) \longrightarrow \mathbb{R}_+; dis(y, C) = \inf_{\widehat{y} \in C} \|y - \widehat{y}\|$$

where $\mathcal{P}(Y)$ represents the power set of set Y, $\mathcal{P}'(Y) = \mathcal{P}(Y) \setminus \{\varnothing\}$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^K$. For simplicity of notation, I use $\|\cdot\|$ for the Euclidean norm without being specific about the dimensionality of the space being considered. It is understood that I am using the norm corresponding to the dimensionality of the vector in question. Similarly, $B_d(z)$ denotes the open ball of radius d around z in the Euclidean space of dimension equal to the one of z, which is not explicitly noted. Given $Z \subseteq \mathbb{R}^K$, I denote by Z^0 its interior and by $\overline{Z}$ its closure, both on the Euclidean topology.

The main result obtained here is the following:

Theorem 2 *Let $X \subseteq \mathbb{R}^J$ and $Y \subseteq \mathbb{R}^K$, where $J, K \in \mathbb{N}$, be nonempty. Suppose that X is convex and $C \subseteq Y$ is compact. Suppose that $W : X \times C \longrightarrow \mathbb{R}$ is continuous and bounded, and that for each $y \in C$, $W(\cdot, y)$ is concave. Then, there exists $U : X \times Y \longrightarrow \mathbb{R}$, continuous, such that*

1. *For each $(x, y) \in X \times C$, $U(x, y) = W(x, y)$*
2. *For each $y \in Y$, $U(\cdot, y)$ is concave*
3. $$\begin{aligned} \inf_{(x,y) \in X \times Y} U(x, y) &= \inf_{(x', y') \in X \times C} W(x', y') \\ \sup_{(x,y) \in X \times Y} U(x, y) &= \sup_{(x', y') \in X \times C} W(x', y') \end{aligned}$$

Proof. If W is constant, the result is trivial. Else, let

$$l : \left[\inf_{(x,y) \in X \times C} W(x, y), \sup_{(x,y) \in X \times C} W(x, y) \right] \longrightarrow [1, 2]$$

be the affine increasing bijection. Both l and l^{-1} are concave, continuous and strictly increasing. Define $f : X \times C \longrightarrow [1, 2]$ by

$$f(x, y) = (l \circ W)(x, y)$$

By construction, f is continuous,

$$\begin{aligned} \inf_{(x,y) \in X \times C} f(x, y) &= 1 \\ \sup_{(x,y) \in X \times C} f(x, y) &= 2 \end{aligned}$$

and for each $y \in C$, $f(\cdot, y)$ is concave.

Now, since C is closed, it follows[10] that

$$(\forall y \in Y \backslash C) : dis(y, C) > 0$$

Define the function $F : X \times Y \longrightarrow \mathbb{R}$ by

$$F(x, y) = \begin{cases} f(x, y) & \text{if} \quad y \in C \\ \frac{\inf_{\widehat{y} \in C} f(x, \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} & \text{otherwise} \end{cases}$$

It is obvious that

$$(\forall (x, y) \in X \times C) : F(x, y) = f(x, y)$$

Moreover, F has the following properties:

Property 1: $(\forall (x, y) \in X \times Y) : F(x, y) \in [1, 2]$
Proof of property 1: By construction,

$$(\forall (x, y) \in X \times C) : F(x, y) = f(x, y) \in [1, 2]$$

Fix $(x, y) \in X \times (Y \backslash C)$. Clearly,

$$\left(\forall \widehat{\widehat{y}} \in C\right) : f\left(x, \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\| \leqslant 2 \left\|y - \widehat{\widehat{y}}\right\|$$

from where

$$(\forall \widehat{y} \in C) : \inf_{\widehat{\widehat{y}} \in C} f\left(x, \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\| \leqslant f(x, \widehat{y}) \|y - \widehat{y}\| \leqslant 2 \left\|y - \widehat{\widehat{y}}\right\|$$

so that

$$(\forall \widehat{y} \in C) : F(x, y) \leqslant \frac{2 \|y - \widehat{y}\|}{dis(y, C)}$$

However, since $\|\cdot\|$ is continuous and C is compact, it follows from Weierstrass' theorem that

$$Arg \min_{\widehat{\widehat{y}} \in C} \left\|y - \widehat{\widehat{y}}\right\| \neq \varnothing$$

so that

$$(\exists \widehat{y} \in C) : \|y - \widehat{y}\| = dis(y, C)$$

and, therefore, that

$$F(x, y) \leqslant 2$$

Moreover,

$$(\forall \widehat{y} \in C) : 1 \leqslant \frac{\|y - \widehat{y}\|}{dis(y, C)} \leqslant \frac{f(x, \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)}$$

[10] See for example Moore (1999b), proposition 7.54, page 33.

from where

$$F(x,y) \geqslant 1$$

Property 2: For each $y \in Y$, $F(\cdot, y)$ is concave.
Proof of property 2: This is obvious for $y \in C$.
Fix $y \in Y \backslash C$. Let $x, x' \in X$ and $\lambda \in [0.1]$. By definition,

$$F(\lambda x + (1-\lambda) x', y) = \frac{\inf_{\widehat{y} \in C} f(\lambda x + (1-\lambda) x', \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)}$$

By compactness of C and continuity of f and $\|\cdot\|$,

$$(\exists \widehat{y} \in C) : F(\lambda x + (1-\lambda) x', y) = \frac{f(\lambda x + (1-\lambda) x', \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)}$$

Fix one such $\widehat{y} \in C$. Since $f(\cdot, \widehat{y})$ is concave, $\|y - \widehat{y}\| > 0$ and $dis(y, C) > 0$.

$$\begin{aligned}
F(\lambda x + (1-\lambda) x', y) &\geqslant \frac{(\lambda f(x, \widehat{y}) + (1-\lambda) f(x', \widehat{y})) \|y - \widehat{y}\|}{dis(y, C)} \\
&= \lambda \frac{f(x, \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} + (1-\lambda) \frac{f(x', \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} \\
&\geqslant \lambda \frac{\inf_{\widehat{\widehat{y}} \in C} f\left(x, \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} + (1-\lambda) \frac{\inf_{\widehat{\widehat{y}} \in C} f\left(x', \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} \\
&= \lambda F(x, \widehat{y}) + (1-\lambda) F(x', \widehat{y})
\end{aligned}$$

Property 3: F is continuous.
Proof of property 3: It follows by construction that F is continuous at all $(x, y) \in X \times C^0$.
Consider now $(x, y) \in X \times (Y \backslash C)$. Define the function $h : X \times (Y \backslash C) \times C \longrightarrow \mathbb{R}$ by

$$h(x, y, \widehat{y}) = f(x, \widehat{y}) \|y - \widehat{y}\|$$

and the correspondence $\Gamma : X \times (Y \backslash C) \rightrightarrows C$, by

$$\Gamma(x, y) = C$$

h is continuous and Γ is compact-valued and both upper- and lower-hemicontinuous, so that, by the theorem of the maximum[11], the function $\widehat{h} : X \times (Y \backslash C) \longrightarrow \mathbb{R}$, defined by

$$\widehat{h}(x, y) = \min_{\widehat{y} \in \Gamma(x,y)} h(x, y, \widehat{y})$$

is continuous. But, by construction,

$$(\forall (x, y) \in X \times (Y \backslash C)) : \widehat{h}(x, y) = \inf_{\widehat{y} \in C} f(x, \widehat{y}) \|y - \widehat{y}\|$$

[11] See, for example, Stokey and Lucas (1989), theorem 3.6, page 62.

Moreover, the function $\rho : X \times (Y \backslash C) \longrightarrow \mathbb{R}_{++}$, defined by

$$\rho(x, y) = dis(y, C)$$

is continuous[12], from where it follows that F is continuous at each $(x, y) \in X \times (Y \backslash C)$.

Finally, I prove that F is continuous at each $(x, y) \in X \times \left(C \cap \overline{(Y \backslash C)}\right)$.[13] Fix $(x, y) \in X \times \left(C \cap \overline{(Y \backslash C)}\right)$ and $\varepsilon \in (0, 1)$. I prove this result in a series of claims:

Claim 1: $(\exists r \in \mathbb{R}_{++})(\forall (\widetilde{x}, \widetilde{y}) \in (B_r(x) \times B_r(y)) \cap (X \times C)) :$

$$|f(x, y) - f(\widetilde{x}, \widetilde{y})| < \varepsilon$$

Proof of claim 1: Since $(x, y) \in X \times C$ and f is continuous, there exists $\overline{r} \in \mathbb{R}_{++}$ such that

$$(\forall (\widetilde{x}, \widetilde{y}) \in B_{\overline{r}}(x, y) \cap (X \times C)) : |f(x, y) - f(\widetilde{x}, \widetilde{y})| < \varepsilon$$

Fix one such $\overline{r}$, and define $r = \frac{\overline{r}}{2} \in \mathbb{R}_{++}$. All I have to show is that $B_r(x) \times B_r(y) \subseteq B_{\overline{r}}(x, y)$. Let $(\widetilde{x}, \widetilde{y}) \in (B_r(x) \times B_r(y))$. Obviously, $(\widetilde{x}, \widetilde{y}) = (x, y) + (0, \widetilde{y} - y) + (\widetilde{x} - x, 0)$, from where

$$\begin{aligned} \|(\widetilde{x}, \widetilde{y}) - (x, y)\| &= \|(0, \widetilde{y} - y) + (\widetilde{x} - x, 0)\| \\ &\leqslant \|(0, \widetilde{y} - y)\| + \|(\widetilde{x} - x, 0)\| \\ &< 2r \\ &= \overline{r} \end{aligned}$$

Note: For the following claims, I take r as given in claim 1.

Claim 2: $\left(\forall \widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)\right) : dis(\widetilde{y}, C) = dis(\widetilde{y}, B_r(y) \cap C)$

[12] See, for example, Moore (1999b), proposition 7.53, page 32.

[13] It is obviuos that

$$C^0 \cup (Y \backslash C) \cup \left(C \cap \overline{(Y \backslash C)}\right) \subseteq Y$$

To see that

$$Y \subseteq C^0 \cup (Y \backslash C) \cup \left(C \cap \overline{(Y \backslash C)}\right)$$

let $y \in Y$. Suppose that $y \notin Y \backslash C$. If $y \notin C^0$, then

$$(\forall \varepsilon \in \mathbb{R}_{++}) : B_\varepsilon(y) \cap (Y \backslash C) \neq \emptyset$$

which implies that $y \in \overline{(Y \backslash C)}$

Proof of claim 2: Fix $\widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)$. For each $\widehat{y} \in C \backslash B_r(y)$,

$$\begin{aligned} \|\widetilde{y} - \widehat{y}\| &\geqslant \|y - \widehat{y}\| - \|y - \widetilde{y}\| \\ &> \frac{3r}{4} \\ &> 2\|\widetilde{y} - y\| \\ &\geqslant 2 \inf_{\widehat{y} \in B_r(y) \cap C} \left\|\widetilde{y} - \widehat{y}\right\| \\ &= 2dis\left(\widetilde{y}, B_r(y) \cap C\right) \\ &\geqslant dis\left(\widetilde{y}, B_r(y) \cap C\right) \end{aligned}$$

which establishes the result.

Claim 3: $(\forall \widetilde{x} \in B_r(x) \cap X)\left(\forall \widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)\right)$:

$$\inf_{\widehat{y} \in C} f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\| = \inf_{\widehat{y} \in B_r(y) \cap C} f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\|$$

Proof of claim 3: Fix $\widetilde{x} \in B_r(x) \cap X$ and $\widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)$. For each $\widehat{y} \in C \backslash B_r(y)$, since $f(\widetilde{x}, \widehat{y}) \geqslant 1$ and $f(\widetilde{x}, y) \leqslant 2$, it follows from the set of inequalities in the proof of claim 2 that

$$\begin{aligned} f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\| &> \frac{3r}{4} \\ &\geqslant f(\widetilde{x}, y) \|\widetilde{y} - y\| \\ &\geqslant \inf_{\widehat{y} \in B_r(y) \cap C} f\left(\widetilde{x}, \widehat{y}\right) \left\|\widetilde{y} - \widehat{y}\right\| \end{aligned}$$

which establishes the result.

Claim 4: $(\forall \widetilde{x} \in B_r(x) \cap X)\left(\forall \widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)\right)$:

$$F(x, y) - \varepsilon \leqslant F(\widetilde{x}, \widetilde{y}) \leqslant F(x, y) + \varepsilon$$

Proof of claim 4: Fix $\widetilde{x} \in B_r(x) \cap X$ and $\widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)$. For each $\widehat{y} \in B_r(y) \cap C$, since

$$(\widetilde{x}, \widehat{y}) \in (B_r(x) \times B_r(y)) \cap (X \times C)$$

it follows from claim 1 that

$$f(x, y) - \varepsilon < f(\widetilde{x}, \widehat{y}) < f(x, y) - \varepsilon$$

so that, since $f(x, y) - \varepsilon > 0$ and $0 < dis(\widetilde{y}, C) \leqslant \|\widetilde{y} - \widehat{y}\|$, one has that

$$\begin{aligned} (f(x, y) - \varepsilon) dis(\widetilde{y}, C) &\leqslant (f(x, y) - \varepsilon) \|\widetilde{y} - \widehat{y}\| \\ &\leqslant f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\| \\ &\leqslant (f(x, y) + \varepsilon) \|\widetilde{y} - \widehat{y}\| \end{aligned}$$

and, therefore,

$$\begin{aligned}(f(x,y)-\varepsilon)\,dis\,(\widetilde{y},C) &\leqslant \inf_{\widehat{y}\in B_r(y)\cap C} f(\widetilde{x},\widehat{y})\,\|\widetilde{y}-\widehat{y}\| \\ &\leqslant (f(x,y)+\varepsilon)\inf_{\widehat{y}\in B_r(y)\cap C}\|\widetilde{y}-\widehat{y}\| \\ &= (f(x,y)+\varepsilon)\,dis\,(\widetilde{y},B_r(y)\cap C)\end{aligned}$$

The result then follows from claims 3 and 2, which imply that

$$(f(x,y)-\varepsilon)\,dis\,(\widetilde{y},C)\leqslant \inf_{\widehat{y}\in C} f(\widetilde{x},\widehat{y})\,\|\widetilde{y}-\widehat{y}\|\leqslant (f(x,y)+\varepsilon)\,dis\,(\widetilde{y},C)$$

and, therefore, since $dis\,(\widetilde{y},C)>0$, that

$$f(x,y)-\varepsilon\leqslant F(\widetilde{x},\widetilde{y})\leqslant f(x,y)+\varepsilon$$

Hence, to establish continuity at (x,y), define $\delta=\frac{r}{4}\in\mathbb{R}_{++}$. It follows from claims 1 and 4 that

$$(\forall(\widetilde{x},\widetilde{y})\in B_\delta(x,y)\cap(X\times Y)):|F(x,y)-F(\widetilde{x},\widetilde{y})|\leqslant\varepsilon$$

which suffices to prove property 3.

Now, define $U:X\times Y\longrightarrow\mathbb{R}$ by

$$U(x,y)=\left(l^{-1}\circ F\right)(x,y)$$

which is well defined given property 1. Given property 3, since l^{-1} is continuous, so is U. Also, if $(x,y)\in X\times C$, by construction,

$$\begin{aligned}U(x,y) &= l^{-1}(F(x,y)) \\ &= l^{-1}(f(x,y)) \\ &= l^{-1}(l(W(x,y))) \\ &= W(x,y)\end{aligned}$$

Moreover, since l^{-1} is increasing and concave, property 2 implies that for each $y\in Y$, $U(\cdot,y)$ is concave . Finally, since

$$l^{-1}:[1,2]\longrightarrow\left[\inf_{(x,y)\in X\times C}W(x,y),\sup_{(x,y)\in X\times C}W(x,y)\right]$$

it is obvious that

$$\inf_{(x,y)\in X\times Y}U(x,y)\geqslant\inf_{(x,y)\in X\times C}W(x,y)$$

whereas, by definition,

$$\left(\forall r>\inf_{(x,y)\in X\times C}W(x,y)\right)(\exists(\widehat{x},\widehat{y})\in X\times C):W(\widehat{x},\widehat{y})<r$$

which implies that

$$\left(\forall r > \inf_{(x,y)\in X\times C} W(x,y)\right)(\exists(\widehat{x},\widehat{y})\in X\times Y) : U(\widehat{x},\widehat{y}) < r$$

and therefore

$$\inf_{(x,y)\in X\times Y} U(x,y) \leqslant \inf_{(x,y)\in X\times C} W(x,y)$$

A similar reasoning establishes that

$$\sup_{(x,y)\in X\times Y} U(x,y) = \sup_{(x,y)\in X\times C} W(x,y)$$

■

Although the compactness of C was used in the proof of properties 1 and 2, this is by no means necessary. Property 1 can be established assuming only closedness as in Bridges (1998) pages 144-145, whereas concavity of $F(\cdot,y)$ for $y\in Y\backslash C$ could be argued as follows:

Fix $y\in Y\backslash C$. Consider the following family of functions:

$$\mathcal{F}_y = \{\, g : X \longrightarrow \mathbb{R} |\, (\exists\widehat{y}\in C) : g(\cdot) = f(\cdot,\widehat{y})\,\|y-\widehat{y}\|\}$$

$\mathcal{F}_y$ is a family of concave and bounded below functions, since for each $\widehat{y}\in C$, $f(\cdot,\widehat{y})$ is concave and bounded below and $\|y-\widehat{y}\|>0$. Define the function $\widehat{g}_y : X\longrightarrow\mathbb{R}$, by

$$\widehat{g}_y(x) = \inf_{g\in\mathcal{F}_y} g(x)$$

It follows from theorem 5.72 in Moore (1999a), pages 313 and 314, that $\widehat{g}_y$ is concave. By construction,

$$(\forall x\in X) : \widehat{g}_y(x) = \inf_{\widehat{y}\in C} f(x,\widehat{y})\,\|y-\widehat{y}\|$$

which implies, since $dis(y,C)>0$, that $F(\cdot,y)$ is concave.

Whether property 3 can be argued with only closedness of C is unknown to me. However, the presence of the compactness assumption allows the following:

Corollary 1 *Let $X\subseteq\mathbb{R}^J$ and $Y\subseteq\mathbb{R}^K$, where $J,K\in\mathbb{N}$, be nonempty. Suppose that X is convex and $C\subseteq Y$ is compact. Suppose that $W : X\times C\longrightarrow\mathbb{R}$ is continuous and bounded, and that for each $y\in C$, $W(\cdot,y)$ is strongly concave (resp. Lipschitzian with constant M; resp. monotone; resp. strictly monotone). Then, there exists $U : X\times Y\longrightarrow\mathbb{R}$, continuous, such that*

1. *For each $(x,y)\in X\times C$, $U(x,y) = W(x,y)$*
2. *For each $y\in Y$, $U(\cdot,y)$ is strongly concave (resp. Lipschitzian with constant M_y; resp. monotone; resp. strictly monotone.)*

3.

$$\begin{aligned}\inf_{(x,y)\in X\times Y} U(x,y) &= \inf_{(x,y)\in X\times C} W(x,y)\\ \sup_{(x,y)\in X\times Y} U(x,y) &= \sup_{(x,y)\in X\times C} W(x,y)\end{aligned}$$

Proof. If X is a singleton, the result follows trivially from theorem 2. Else, recall all the definitions given in the proof theorem 2.

For strong concavity, it suffices to show that for each $y \in C$, $f(\cdot, y)$ is strongly concave, that for each $y \in Y\backslash C$, $F(\cdot, y)$ is strongly concave and that l^{-1} is strictly increasing.

By strong concavity,

$$\sup_{(x,y)\in X\times C} W(x,y) > \inf_{(x,y)\in X\times C} W(x,y)$$

from where both l and l^{-1} are strictly increasing. Notice, then, that for each $y \in C$, $f(\cdot, y)$ is strongly concave: fix $y \in C$. Let $x, x' \in X$, $x \neq x'$ and $\lambda \in (0,1)$. By strong concavity of $W(\cdot, y)$ and concavity and strict monotonicity of l,

$$\begin{aligned}W(\lambda x + (1-\lambda)x', y) &> \lambda W(x,y) + (1-\lambda) W(x', y)\\ l(W(\lambda x + (1-\lambda)x', y)) &> l(\lambda W(x,y) + (1-\lambda) W(x', y))\\ &\geqslant \lambda l(W(x,y)) + (1-\lambda) l(W(x', y))\end{aligned}$$

Finally, fix $y \in Y\backslash C$. Let $x, x' \in X$, $x \neq x'$ and $\lambda \in (0,1)$. By definition,

$$F(\lambda x + (1-\lambda)x', y) = \frac{\inf_{\widehat{y}\in C} f(\lambda x + (1-\lambda)x', \widehat{y})\, \|y - \widehat{y}\|}{dis(y, C)}$$

By compactness of C and continuity of f and $\|\cdot\|$,

$$(\exists \widehat{y} \in C) : F(\lambda x + (1-\lambda)x', y) = \frac{f(\lambda x + (1-\lambda)x', \widehat{y})\, \|y - \widehat{y}\|}{dis(y, C)}$$

Fix one such $\widehat{y} \in C$. Since $f(\cdot, \widehat{y})$ is strongly concave, $\|y - \widehat{y}\| > 0$ and $dis(y, C) > 0$.

$$\begin{aligned}F(\lambda x + (1-\lambda)x', y) &> \frac{(\lambda f(x, \widehat{y}) + (1-\lambda) f(x', \widehat{y}))\, \|y - \widehat{y}\|}{dis(y, C)}\\ &= \lambda \frac{f(x, \widehat{y})\, \|y - \widehat{y}\|}{dis(y, C)} + (1-\lambda) \frac{f(x', \widehat{y})\, \|y - \widehat{y}\|}{dis(y, C)}\\ &\geqslant \lambda \frac{\inf_{\widehat{\widehat{y}}\in C} f\left(x, \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} + (1-\lambda) \frac{\inf_{\widehat{\widehat{y}}\in C} f\left(x', \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)}\\ &= \lambda F(x, \widehat{y}) + (1-\lambda) F(x', \widehat{y})\end{aligned}$$

I now show that if for each $y \in C$, $W(\cdot, y)$ is Lipschitzian with constant M (independent of y), then for each $y \in Y$, $U(\cdot, y)$ is Lipschitzian with some constant M_y.

If W is constant, the result is trivial. Hence, I assume that the affine bijection l has slope $a > 0$.

It follows by construction that for each $y \in C$, $U(\cdot, y)$ is Lipschitzian with constant $M_y = M$. Since for each $y \in C$, $W(\cdot, y)$ is Lipschitzian with constant M, one has that $f(\cdot, y)$ is Lipschitzian with constant aM. Fix $y \in Y \backslash C$ and $x, x' \in X$. By definition of F, compactness of C and continuity of f and $\|\cdot\|$, as before, there exist $\widehat{y}, \widehat{\widehat{y}} \in C$ such that

$$\begin{aligned} F(x, y) &= \frac{f(x, \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} \\ F(x', y) &= \frac{f\left(x', \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} \end{aligned}$$

Fix such $\widehat{y}, \widehat{\widehat{y}} \in C$. By definition

$$\begin{aligned} \frac{f(x, \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} &\leqslant \frac{f\left(x, \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} \\ \frac{f\left(x', \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} &\leqslant \frac{f(x', \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} \end{aligned}$$

whereas, since both $f(\cdot, \widehat{y})$ and $f\left(\cdot, \widehat{\widehat{y}}\right)$ are Lipschitzian with constant M/a,

$$\begin{aligned} |f(x, \widehat{y}) - f(x', \widehat{y})| &\leqslant aM \|x - x'\| \\ \left|f\left(x, \widehat{\widehat{y}}\right) - f\left(x', \widehat{\widehat{y}}\right)\right| &\leqslant aM \|x - x'\| \end{aligned}$$

and, therefore,

$$\begin{aligned} \left|\frac{f(x, \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} - \frac{f(x', \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)}\right| &\leqslant aM \frac{\|x - x'\| \|y - \widehat{y}\|}{dis(y, C)} \\ \left|\frac{f\left(x, \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} - \frac{f\left(x', \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)}\right| &\leqslant aM \frac{\|x - x'\| \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} \end{aligned}$$

Define

$$M_y = \frac{M}{dis(y, C)} \max_{y' \in C} \|y - y'\|$$

which exists and satisfies $M_y > 0$, because C is compact and $\|\cdot\|$ is continuous.

Clearly,

$$\begin{aligned}\left|\frac{f\left(x,\widehat{y}\right)\left\|y-\widehat{y}\right\|}{dis\left(y,C\right)}-\frac{f\left(x',\widehat{y}\right)\left\|y-\widehat{y}\right\|}{dis\left(y,C\right)}\right| &\leqslant aM_y\left\|x-x'\right\| \\ \left|\frac{f\left(x,\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis\left(y,C\right)}-\frac{f\left(x',\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis\left(y,C\right)}\right| &\leqslant aM_y\left\|x-x'\right\|\end{aligned}$$

Now, if

$$\frac{f\left(x,\widehat{y}\right)\left\|y-\widehat{y}\right\|}{dis\left(y,C\right)}\leqslant\frac{f\left(x',\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis\left(y,C\right)}$$

it follows that

$$F\left(x,y\right)\leqslant F\left(x',y\right)\leqslant\frac{f\left(x',\widehat{y}\right)\left\|y-\widehat{y}\right\|}{dis\left(y,C\right)}$$

from where

$$\left|F\left(x,y\right)-F\left(x',y\right)\right|\leqslant M_y\left\|x-x'\right\|$$

If, on the other hand,

$$\frac{f\left(x',\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis\left(y,C\right)}<\frac{f\left(x,\widehat{y}\right)\left\|y-\widehat{y}\right\|}{dis\left(y,C\right)}$$

then

$$F\left(x',y\right)<F\left(x,y\right)\leqslant\frac{f\left(x,\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis\left(y,C\right)}$$

from where, again

$$\left|F\left(x,y\right)-F\left(x',y\right)\right|\leqslant aM_y\left\|x-x'\right\|$$

Hence, it follows that $F\left(\cdot,y\right)$ is Lipschitzian with constant aM_y, and, therefore, that $U\left(\cdot,y\right)=\left(l^{-1}\circ F\right)\left(\cdot,y\right)$ is Lipschitzian with constant M_y.

Finally, I show that if for each $y\in C$, $W\left(\cdot,y\right)$ is monotone (resp. strictly monotone), then for each $y\in Y$, $U\left(\cdot,y\right)$ is monotone (resp. strictly monotone).

If there do not exist $x,x'\in X$ such that $x\gg x'$ (resp. $x>x'$) the result is trivial. Else, fix $x,x'\in X$, $x\gg x'$ (resp. $x>x'$) and $y\in Y$. Since for each $\widehat{y}\in C$, $W\left(x,\widehat{y}\right)>W\left(x',\widehat{y}\right)$, it follows that both l and $l^{-1}$are strictly increasing and, hence, that for each $\widehat{y}\in C$, $f\left(\cdot,\widehat{y}\right)$ is monotone (resp. strictly monotone).

Then, if $y\in C$, the result is trivial and I now assume that $y\in Y\backslash C$. By compactness of C and continuity of f, there exists $\widehat{y}\in C$ such that

$$F\left(x,y\right)=\frac{f\left(x,\widehat{y}\right)\left\|y-\widehat{y}\right\|}{dis\left(y,C\right)}$$

Fix one such $\widehat{y} \in C$. Since $f(\cdot, \widehat{y})$ is monotone (resp. strongly monotone), $\|y - \widehat{y}\| > 0$ and $dis(y, C) > 0$, one has that

$$\begin{aligned} F(x,y) &= \frac{f(x,\widehat{y})\|y-\widehat{y}\|}{dis(y,C)} \\ &> \frac{f(x',\widehat{y})\|y-\widehat{y}\|}{dis(y,C)} \\ &\geqslant \inf_{\widehat{\widehat{y}}\in C} \frac{f\left(x',\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis(y,C)} \\ &= F(x',y) \end{aligned}$$

showing that $F(\cdot, y)$ is monotone (resp. strictly monotone). Since l^{-1} is strictly increasing, it follows that $U(\cdot, y)$ is monotone (resp. strictly monotone). ■

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