M. A. M. EL NADI

THE QUANTUM THEORY OF THE ELECTROMAGNETIC FIELD

The present work is concerned with de Broglie's quantum theory of light.

It is assumed that the photon is described by a hermitian wave function ψ_{ik} with 16 components. Using this wave function, it is shown that the 32 of de Broglie's equations are reduced to one set of 16 equations in the form:

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = \alpha_1\frac{\partial\psi}{\partial x} + \alpha_2\frac{\partial\psi}{\partial y} + \alpha_3\frac{\partial\psi}{\partial z} + \frac{i}{\hbar}\mu_0 c\,\alpha_4\psi \qquad (1)$$

where the α's are Dirac matrices and ψ is a matrix with 16 components.

The electromagnetic quantities associated with the photon are described by means of the Dirac matrices operating on ψ in a specified way. It is shown also that these electromagnetic quantities satisfy Maxwell's equations as a result of the equation (1).

The interaction between an electron and a photon is developed and the matrix elements for the radiation transitions are calculated.

It is further shown that the above wave equation can be considered as the superposition of two similar wave equations, one for the positive energy photons and the other for the negative energy photons. To each of these states there corresponds electromagnetic quantities defined by the above method. It is the superposition of these fields which gives rise to the reality of the electromagnetic field found in experience.

The wave mechanics of the positive energy photon is discussed and the method of second quantization is applied to its wave function, from which we deduce the commutation relations for the complex fields.

London University.

THESIS

FOR THE DEGREE OF PH.D.

BY

MOHAMED ABDEL MAKSOUD EL NADI

B.Sc., M.Sc.

(FOUAD The 1st University, Cairo, EGYPT)

JUNE 1948.

THE QUANTUM THEORY

OF

THE ELECTRO-MAGNETIC FIELD.

CONTENTS

CONTENTS (Contd.)

PART II:

THE NATURE OF LIGHT

The development of the theories of the nature of light shows a remarkable alternation between the corpuscular conception on the one hand and the wave conception on the other. Newton is usually considered as the founder of the corpuscular theory (of light) and Huygens as the founder of the wave theory. Newton regarded the light rays as the paths of definite light corpuscles which are emitted in all directions from the glowing body and they move with uniform motion in a straight line, i.e. just like ordinary material particles in the absence of external fields. Reflection and refraction of light rays at the surface between two different homogeneous bodies e.g. air and water, are explained, in the corpuscular theory, by the action of definite forces active in the surface of the transition layer.(1)

When light rays travel from air into water, Newton's corpuscular theory tells us that the velocity of the light corpuscles in water will be greater than that in air due to the surface forces, while the opposite case is given by the wave theory. Foucault(2) actually measured the velocity of light in water and found that it is smaller than that in air. This experiment showed the incorrectness of Newton's theory and led towards the acceptance of the wave theory. At the beginning of the twentieth century the corpuscular theory was revived in another form at the hands of Einstein, as a result of his special theory of relativity together with

Plancks' quantum theory.

This revival of the corpuscular theory was introduced for the explanation of the photo-electric effect. Einstein assumed the light to be composed of particles, called photons, moving with constant velocity C, and each having an energy hv and momentum $\frac{hv}{C}$. On that assumption the photo-electric effect was successfully explained and the relation between the energy of the ejected photo-electrons (T) and the frequency of the incident light is expressed by:

$$T = hv - Eo \quad (Eo = \text{constant}) \quad (1)$$

which is the equation for the conservation of energy, for the collision between the incident photon and the ejected electron.

Now according to the theory of relativity the energy and momentum of a particle of rest mass μ_0 and velocity V, is given by:

$$E = \frac{\mu_0 c^2}{\sqrt{1-\beta^2}} , \quad \vec{p} = \frac{\mu_0 \vec{v}}{\sqrt{1-\beta^2}} \quad \left(\beta = \frac{v}{c}\right) \quad (2)$$

$$\text{i.e.} \quad \vec{p} = \frac{E}{c^2} \vec{v} \quad (3)$$

As the photon moves with velocity C and energy E = hv, then (3) becomes:

$$p = \frac{h\nu}{c} \quad (4)$$

If $\mu_0 = 0$, then (3) is valid; but from (2) we must regard μ_0 as tending to zero as $v \to c$. Thus if the photon is supposed to have a velocity C, its rest mass μ_0 must be zero.

Another support for Einstein's theory came from the discovery of Compton effect in 1922 and its successful explanation on the basis of Einstein's corpuscular theory.

A monochromatic beam of x-rays passing through a Wilson chamber is scattered by free electrons in the gas molecules. An electron is then ejected in a certain direction, forming a track in the chamber. The x-ray is scattered in another direction which can also be observed from the electrons ejected by the scattered wave. It has been found that the directions of the ejected electron and the scattered x-ray bear exactly the same relation as in an elastic collision of two particles according to classical mechanics (conservation of energy and momentum).

On the other hand, a beam of x-rays shows the well known interference phenomena which are proofs of the wave nature of x-rays.

It is this dual nature of light which inspired De Broglie in 1924 to assume the same dual nature for particles and this has led to the developments of wave mechanics.

The hypothesis of photons enables us to find the correct value for the radiation pressure. Maxwell's electromagnetic theory has lead to the following conclusion, which was discovered by Poynting and proved experimentally by Lebedew, Nichols and Hull:[3]

"If a beam of light with energy density W falls on a plane absorbing screen making an angle of incidence θ, it exerts on this screen a pressure equal to $W\cos^2\theta$. If the screen is perfectly reflective, the radiation pressure is again equal to $W\cos^2\theta$, where W now means the total density of energy (incident and reflected) near the mirror".

Let us try to explain this experiment on the basis of Newton's corpuscular theory. Let a beam of incident particles with masses μ and velocity v falls at an angle θ on a perfectly reflecting surface, the density of energy in the light beam being: $N \cdot \frac{mv^2}{2}$, where N is the number of particles per unit volume. In unit time, a unit surface element will receive $N v \cos\theta$ corpuscles, each carrying a normal momentum $= m v \cos\theta$. Thus the pressure suffered by the surface will be equal: $N m v^2 \cos^2\theta$, i.e., $2W\cos^2\theta$. This is twice as big as was found experimentally. Let us now apply relativistic mechanics to our problem. The number of photons falling per unit time on unit surface of the screen is $Nc\cos\theta$ and the energy density in the beam is NE, the normal momentum received per unit surface area per unit time is $\frac{E}{c}\cos\theta \cdot Nc\cos\theta = NE\cos^2\theta = W\cos^2\theta$, which is exactly the result obtained by the electromagnetic theory and confirmed by experience.

But up to the present, establishing a quantum mechanics for the photon has proved to be a very difficult task. Many trials have been made by authorities in the subject of quantum mechanics but most of these are not considered satisfactory (See Heisenberg-Pauli, Frenkel, Fock, Landau - Peierl, Oppenheimer, C.G.Darwin, De Broglie and Jordan-Kronig).(4)

It is probably agreed, as Darwin said in 1932,(5) that among all the recent developments of the quantum theory, one of the least satisfactory is the theory of radiation.

There are ~~two~~ three main problems connected with the theory of the photon, about which opinions and ideas differ quite considerably.

1. The idea that the photon has a rest mass.
2. The fact that to the photon are associated real fields and not imaginary ones.
3. The fact that photons could be created and annihilated.

THE REST MASS OF THE PHOTON

After the establishment of quantum mechanics which explained quite satisfactorily almost all the dual properties of electrons, it occurred to De Broglie to try to establish a similar quantüm mechanics for the photon, on attributing to it a very small rest mass.[6] The idea was very fruitful and was carried out by many physicists in different directions. G.I. Pokrowski[7] suggested that the rest mass of light quanta might be of electrostatic origin. On that assumption he used the data of Campbell and Trumpler in the 1922 solar eclipse in the neighbourhood of the sun's magnetic pole to deduce $\frac{e}{m_0}$ which he found to be equal to that of protons. The charge of the photon was shown to be positive.

W. Anderson[8] assumed that a light quantum consists of an electric dipole of which the positive and negative charges equal those of the proton and electron respectively. The electrostatic energy of the dipole gives it its "restenergy" which is normally much smaller than the total energy hν. On collision, however, the dipole distorts and its kinetic energy is transformed into potential energy. The accompanying increase of electric force between the poles of the dipole is then adequate to produce photo-electric effect.

De Broglie proceeded to develop a complete quantum theory for the photon, assuming it to have a small rest mass. At first he assumed it to be a single particle, but meeting considerable difficulties,[9] he then assumed it to be a complex particle formed by two elementary ones. Assuming the two elementary particles to be a neutrino and an anti-neutrino, Jordan[4] has carried for many years a formulation for the neutrino theory of light which still is unsatisfactory.[10]

Thus apparently, out of all these theories which were based on assuming a rest mass for the photon, De Broglie's theory was the most satisfactory. It was carried out much further than all the others.

Now, we shall discuss the implications of assuming that the photon possesses a small rest mass μ_0.
The relativistic mechanics give us for the energy and momentum of a particle with rest mass μ_0 and velocity $\vec{v}$;

$$E = \frac{\mu_0 c^2}{\sqrt{1-\beta^2}} \quad ; \quad \vec{p} = \frac{\mu_0 \vec{v}}{\sqrt{1-\beta^2}} \quad , \quad \beta = \frac{v}{c} \tag{2}$$

from which we deduce

$$\vec{p} = \frac{E}{c^2} \vec{v} \tag{3}$$

As it is mentioned before, if $\mu_0 = 0$, v will be equal to c from (2) and (3) becomes:

$$\vec{p} = \frac{E}{c} \tag{4}$$

Thus assuming $\mu_0 = 0$ we have the advantage of always attributing to light the velocity C.

Considering, on the other hand, that the photon has a small rest mass μ_0, this rest mass will vary with velocity according to the relativity theory. Then to the smaller velocities correspond smaller energies and thus smaller frequencies. Thus there exists a dispersion for the vacuum, the red light will propagate with smaller velocity in vacuum than the violet light. But assuming μ_0 to be very small it could be shown that this vacuum - dispersion is negligible in all the domain of measurable frequencies.

De Broglie has calculated the order of magnitude of the rest mass of the photon which agrees with the measurable physical facts and it came out to be:[11]

$$\mu_0 \leqslant 10^{-44} \text{ grams.} \tag{5}$$

which is very small with respect to the rest mass of the electron: $\mu_0 \sim 0.9 \times 10^{-27}$ grams. (6)

Thus the photon could be supposed to behave like a particle with rest mass of the order $\sim 10^{-44}$ gram, moving with velocity very near to C, i.e. varies from C-ε[12] to C according to the different types of radiations.

COMPLEX FIELDS

In the electromagnetic theory of light, the fields are considered to be quantities essentially real since they are, in principle, measurable quantities. Frenkel[13] wrote sometime ago: It is sometimes argued that for light-vibrations we can measure the average value of the energy only, and not the field strengths as a function of the time. But, on the other hand, for electromagnetic oscillations of moderate frequency we can certainly measure the field strength itself and follow its variation with time (with the help of an oscillograph for example). It is true that the quantity directly measured is not the field strength itself but the force i.e. the product of E by the electrical charge of the test particle upon which it is supposed to act. But this charge can be considered for electrons, for example, as a known quantity, and on the other hand, the force acting on it cannot be transformed into an expression quadratic in the resulting field strength and corresponding to the electromagnetic energy of the resulting field. It is therefore erroneous to think that the measurement of electric force can be reduced to the measurement of electromagnetic energy.

On the other hand, when the problems of the interaction between radiation and matter were investigated recently, it was shown that for representing the interaction, complex expressions for the electromagnetic fields should be used.(14)

CREATION AND ANNIHILATION OF PHOTONS.

In the recent developments of the theory of interaction between radiation and matter, it was assumed that photons could be created and annihilated. Thus the photon should have a structure, characteristic for it, and very much different from that of the other elementary particles.

It might be mentioned that, in the quantum mechanical explanation of the Compton effect, the primary photon is supposed to be annihilated (absorbed) and the secondary is created. This was the basis of the Klein-Nishina formula which proved to be in excellent agreement with experience.

On the other hand, Compton and Darwin (15) were able to derive formulae for the scattering which are in good agreement with experience, on treating the problem as a simple collision between two elastic bodies (the photon and the electron).

Darwin in his trial to form a theory for the photon, neglected the idea of creation and annihilation, and considered the photon to be of a structure similar to that of other elementary particles.

DE BROGLIE'S THEORY OF THE PHOTON.

Introduction:

Dirac's theory of the electron offers a good basis for constructing a quantum theory for the photon since:-

A. It is relativistically invariant and thus is valid for particles moving with very high velocity.

B. It introduces the spin which is supposed to have certain analogies with polarization.

To build a quantum theory for the photon, much importance must be directed to the definition of the electromagnetic quantities and to Maxwell's equation which they satisfy. In most current theories (e.g. Darwin, Frenkel, Proca and Kemmer) (16) the field quantities are identified with the components of the wave function of the photon. To define this relation in his theory, De Broglie noticed that:

a. The electromagnetic field of the photon describes the way in which this particle acts on matter and thus it should be represented by the density of the matrix element for the transition from one state to another.

b. As the electromagnetic field associated with the photon is always unique and well determined, then the final state of the transition must be known in advance.

Now the photo-electric effect and the absorption phenomena may seem to show that the photons are annihilated when they come into contact with matter. This suggests that the structure of the photon should have a peculiarity which does not exist in the cases of other elementary particles. Such structure is now easy to visualize according to Dirac's theory of the electron which introduces the conception of complementary particles.

According to Dirac's theory, there corresponds to each particle of a given proper mass and electric charge, a complementary particle, which forms together with the original particle an entity capable of being annihilated.

Assuming the photon to be formed of such an entity, the phenomenon of annihilation could be easily explained. Such a structure has another advantage concerning statistics. Asumming each of the elementary particles to possess a spin momentum $\frac{h}{4\pi}$ i.e. $\frac{\hbar}{2}$, our entity will have a spin momentum of value $\hbar$ or 0 according to the direction of the spin of each constituent particle. Now, if we assume that elementary particles of spin $\frac{\hbar}{2}$ follow the Fermi-Dirac statistics, it is shown that complex particles formed by an even number of elementary particles follow the Bose-Einstein statistics. Hence, the assumption that the photon is composed of two demi photons explains why photons obey the Bose-Einstein statistics, which was difficult to understand earlier.

Admitting now the complex structure of the photon, let us go back to the definition of the electromagnetic quantities as the densities of matrix elements corresponding to transitions from one state to another, with the final state well defined. The first idea was to define the electromagnetic field quantity associated with the field operator F_{op} as: $\psi'^{*} \cdot F_{op} \psi$

where ψ and ψ' are the proper functions of one demi-photon and its complementary. But such expressions, in addition to being unsymmetrical in the position of the two wave functions with respect to the operator, do not satisfy the superposition principle which is responsible for the interpretation of the interference phenomena.(17)

This principle implies that: if the wave function ψ of the photon is formed by a superposition of plane monochromatic waves, then the corresponding electromagnetic field is obtained by the superposition of electromagnetic fields which correspond to each monochromatic component of ψ considered separately. As an example, let us consider the case when the function ψ is formed by the superposition of two plane monochromatic waves ψ_1 and ψ_2. To these waves correspond the complementary wave functions ϕ_1 and ϕ_2; and the two corresponding fields are:

$$\sum_k \phi_1 \cdot F_{op} \psi_1 \quad \text{and} \quad \sum_k \phi_2 \cdot F \psi_2 \tag{7}$$

Then it is clear that:

$$\sum_k (\phi_1 + \phi_2)_k \cdot F_{op} (\psi_1 + \psi_2)_k \neq \sum_k \phi_1 F_{op} \psi_1 + \sum_k \phi_2 F_{op} \psi_2 \tag{8}$$

where the term on the left side of the inequality is the field associated with the superposed wave.

In order to satify the superposition principle, the field quantities must be expressed linearly in terms of the components of the wave function. This happens only if the final state is represented by a constant function which is always the same, in which case, they will be expressed linearly in terms of the components of the wave function representing the initial state. This constant and unique wave function will be taken to represent the annihilation state ϕ° and then the electromagnetic field quantity $\phi^{\circ *} \cdot F_{op} \phi$ represents the density of the matrix element corresponding to the transition from the initial state of the photon to the annihilation state ϕ°. In such a case the inequality (8) becomes:

$$\sum_k \phi_k^{\circ *} \cdot F_{op} (\psi_1 + \psi_2)_k = \sum_k \phi_k^{\circ *} \cdot F_{op} (\psi_1)_k + \sum_k \phi_k^{\circ *} \cdot F_{op} (\psi_2)_k \tag{9}$$

which satisfies the superposition principle. We shall show later that the annihilation state could be represented by a constant wave function, as is required.

DERIVATION OF THE WAVE EQUATION:

If we assume that ψ_i is the wave function of one demi-photon and ϕ_k the wave function of the complementary particle, then according to De Broglie[(18)] the Dirac equations for thw two demi-photons are:

$$\frac{1}{c}\frac{\partial \psi_i}{\partial t} = \alpha_1 \frac{\partial \psi_i}{\partial x} + \alpha_2 \frac{\partial \psi_i}{\partial y} + \alpha_3 \frac{\partial \psi_i}{\partial z} + \varkappa \frac{\mu_0 c}{2} \alpha_4 \psi_i$$

and (10)

$$\frac{1}{c}\frac{\partial \phi_k}{\partial t} = \alpha_1 \frac{\partial \phi_k}{\partial x} - \alpha_2 \frac{\partial \phi_k}{\partial y} + \alpha_3 \frac{\partial \phi_k}{\partial z} - \varkappa \frac{\mu_0 c}{2} \alpha_4 \phi_k \qquad (\varkappa = \tfrac{i}{\hbar})$$

where i and k run from 1 to 4, $\frac{\mu_0}{2}$ is the mass of each demi-photon while W, $\vec{p}$ and μ_0 will be the energy, momentum and mass of the photon as a whole. As the two demi-photons will have the same energy and momentum, we have:(19)

$$\phi_k \frac{\partial \psi_i}{\partial q} = \frac{\partial \phi_k}{\partial q}\psi_i = \frac{1}{2}\frac{\partial}{\partial q}(\phi_k \psi_i) = \frac{1}{2}\frac{\partial \phi_{ik}}{\partial q} \qquad (11)$$

the variable q stands for x, y or z and ϕ_{ik} $(= \psi_i \phi_k)$ is a single component of the 16 components of the wave function ϕ_{ik}, which will be the wave function of the photon and could be used for the description of the motion of the ensemble of the two demi-photons as one entity.

Let us now introduce the eight matrices A_r and B_r with $r = 1,2,3,4$, each having 16 columns and rows:

$$(A_r)_{ik,lm} = (\alpha_r)_{il}\,\delta_{km} \quad ; (B_r)_{ik,lm} = -(-1)^r (\alpha_r)_{km}\,\delta_{il} \qquad (12)$$

All the indices run from 1 to 4. These matrices (12) satisfy the relations:

$$A_r A_s + A_s A_r = 2\,\delta_{rs}\cdot 1 \quad ; B_r B_s + B_s B_r = 2\,\delta_{rs}\cdot 1$$

$$A_r B_s - B_s A_r = 0 \qquad (13)$$

Multiplying the two equations (10) by ϕ_k and ψ_i respectively, we get, taking (12) and (13) into consideration:

$$\frac{1}{c}\frac{\partial \phi_{ik}}{\partial t} = \left(\frac{\partial}{\partial x}A_1 + \frac{\partial}{\partial y}A_2 + \frac{\partial}{\partial z}A_3 + \varkappa \mu_0 c A_4\right)\phi_{ik}$$

$$\frac{1}{c}\frac{\partial \phi_{ik}}{\partial t} = \left(\frac{\partial}{\partial x}B_1 + \frac{\partial}{\partial y}B_2 + \frac{\partial}{\partial z}B_3 + \varkappa \mu_0 c B_4\right)\phi_{ik} \qquad (14)$$

with the notation:

$$A_1 \phi_{ik} = \sum_{l,m=1}^{4} (A_1)_{ik,lm} \phi_{lm}, \text{ etc} \tag{15}$$

(14) is a system of 32 partial differential equations, which must be satisfied by the 16 components of the wave function ϕ_{ik}.

By addition and subtraction we get from equations (14):

$$\frac{1}{c}\frac{\partial \phi_{ik}}{\partial t} = \left[\frac{\partial}{\partial x}\frac{A_1+B_1}{2} + \frac{\partial}{\partial y}\frac{A_2+B_2}{2} + \frac{\partial}{\partial z}\frac{A_3+B_3}{2} + \varkappa\mu_0 c\,\frac{A_4+B_4}{2}\right]\phi_{ik} \tag{16a}$$

$$0 = \left[\frac{\partial}{\partial x}\frac{A_1-B_1}{2} + \frac{\partial}{\partial y}\frac{A_2-B_2}{2} + \frac{\partial}{\partial z}\frac{A_3-B_3}{2} + \varkappa\mu_0 c\,\frac{A_4-B_4}{2}\right]\phi_{ik} \tag{16b}$$

The equations in the form (16) are divided into two parts similar to Maxwell's equations which are divided into two groups, one containing $\frac{\partial}{\partial t}$ and the other not containing the time derivative (div E = div H = 0).

THE FUNCTION OF ANNIHILATION:(20)

The equations (16a) admit solutions independent of x, y, z, t. These solutions must satisfy the equations:

$$(A_4 + B_4)\,\phi_{ik} = 0 \tag{17}$$

As the matrix α_4 is chosen in this work to be diagonal, equation (17) can be satisfied by the solution:

$$\phi^0_{ik} = \delta_{ik} \tag{18}$$

This solution satisfies equation (16a) but not (16b), because of the last term containing the rest mass. This is not a very satisfactory feature of the theory, but it leads to linear expressions for the electromagnetic quantities satisfying the superposition principle.

DEFINITION OF THE ELECTROMAGNETIC QUANTITIES:

These quantities are defined as the density expressions

of the matrix elements corresponding to transitions from the actual states of the photons to the state of annihilation. To define these operators let us denote by $\vec{A}$ and $\vec{B}$ those matrix-vectors with components: A_1, A_2, A_3 and: B_1, B_2, B_3 which were defined by equations (12).
The electromagnetic potential will be identified with the operators:

$$(\vec{A})_{op} = -K\,\frac{\vec{A}+\vec{B}}{2}\ , \qquad (V)_{op} = K\,\frac{1+1}{2} = K\cdot 1 \tag{19}$$

K is a numerical constant which is put equal to $\frac{\hbar}{2\sqrt{\mu_0}}$ (20)
Then the operators corresponding to the electromagnetic fields will be defined in the usual way:

$$(\vec{H})_{op} = \operatorname{curl}\,(\vec{A})_{op} = -K\,\operatorname{curl}\,\frac{\vec{A}+\vec{B}}{2} \tag{21}$$

$$(\vec{E})_{op} = -\operatorname{grad}\,(V)_{op} - \frac{1}{c}\frac{\partial}{\partial t}(\vec{A})_{op} = K\left[-\operatorname{grad} + \frac{1}{c}\frac{\partial}{\partial t}\,\frac{\vec{A}+\vec{B}}{2}\right] \tag{22}$$

With the help of the equations of the photon, we can get rid of the differential terms in the definitions (21, 22) and get the following operators:

$$\vec{E} = K\varkappa\mu_0 c\,\frac{i\vec{A}A_4 + i\vec{B}B_4}{2}\ , \qquad \vec{H} = -K\varkappa\mu_0 c\,\frac{\vec{S}^{(a)} + \vec{S}^{(b)}}{2} \tag{23}$$

where $\vec{S}^{(a)}$ and $\vec{S}^{(b)}$ are respectively the matrix vectors with the components:

$$iA_2A_3,\ iA_3A_1,\ iA_1A_2 \quad \text{and} \quad iB_2B_3,\ iB_3B_1,\ iB_1B_2 \tag{24}$$

Now, we know that the Dirac α-operators form 16 independent product combinations. Similarly the A's and B's of de Broglie have each 16 independent products, from which we can form the following 32 combinations, 16 of which are symmetric and the others are antisymmetric

$$\frac{1+1}{2}\,,\ \frac{A_1 \pm B_1}{2}\,,\ \frac{A_2 \pm B_2}{2}\,,\ \frac{A_3 \pm B_3}{2}\,,\ \frac{iA_1A_4 \pm iB_1B_4}{2}\,,$$
$$\frac{iA_2A_4 \pm iB_2B_4}{2}\,,\ \frac{iA_3A_4 \pm iB_3B_4}{2}\,,\ \frac{iA_2A_3A_4 \pm iB_2B_3B_4}{2} \qquad (25)$$
$$\frac{iA_3A_1A_4 \pm iB_3B_1B_4}{2}\,,\ \frac{iA_1A_2A_4 \pm iB_1B_2B_4}{2}.$$

and

$$\frac{A_4 \pm B_4}{2}$$
$$\frac{iA_2A_3 \pm iB_2B_3}{2}\,,\ \frac{iA_3A_1 \pm iB_3B_1}{2}\,,\ \frac{iA_1A_2 \pm iB_1B_2}{2}\,,$$
$$\frac{iA_1A_2A_3 \pm iB_1B_2B_3}{2}\,,\ \frac{A_1A_2A_3A_4 \pm B_1B_2B_3B_4}{2} \qquad (26)$$

On taking the ten combinations with negative sign from (25) we find that the quantities $\phi^{0*} F_{op} \phi$ formed from them vanish. Thus we are left with the other 10 symmetrical terms with the positive sign, whose density expressions are the above-defined electromagnetic quantities. On the other hand taking the six combinations with positive sign from (26), we find that their density expressions vanish while those with negative signd survive. The density expressions of these latter operators are associated with non-Maxwellian electromagnetic operators in the following way:

$$I_1 = \sum_{ik} \delta_{ik} \frac{A_4 - B_4}{2} \phi_{ik}$$
$$\sigma_x = -\sum_{ik} \delta_{ik} \frac{iA_2A_3 - iB_2B_3}{2} \phi_{ik}\,, \ldots\ldots, \ldots\,, \sigma_4 = \sum_{ik} \delta_{ik} \frac{iA_1A_2A_3 - iB_1B_2B_3}{2} \phi_{ik}$$
$$I_2 = \sum_{ik} \delta_{ik} \frac{A_1A_2A_3A_4 - B_1B_2B_3B_4}{2} \phi_{ik} \qquad (27)$$

THE EQUATIONS OF ELECTROMAGNETISM:

From the wave equations of the photon, 32 in number, satisfied by the 16 components of ϕ_{ik}, we can deduce the following 32 electromagnetic equations, divided into two groups:

1. $$-\frac{1}{c}\frac{\partial \vec{H}}{\partial t} = \mathrm{curl}\,\vec{E} \quad , \quad \mathrm{div}\,\vec{H} = 0 \qquad (28)$$

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \mathrm{curl}\,\vec{H} - \varkappa^2\mu_0^2c^2\vec{A} \quad ; \quad \mathrm{div}\,\vec{E} = \varkappa^2\mu_0^2c^2V$$

$$\vec{H} = \mathrm{curl}\,\vec{A} \quad , \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \overrightarrow{\mathrm{grad}}\,V$$

$$\frac{1}{c}\frac{\partial V}{\partial t} + \mathrm{div}\,\vec{A} = 0$$

together with an identity (16 equations)

These are exactly the Maxwellian equations in the case $\mu_0 \to 0$

2.
$$\left.\begin{array}{ll} & \mu_0 I_1 = 0 \; , \\ \frac{1}{c}\frac{\partial I_1}{\partial t} = 0 & , \; \overrightarrow{\mathrm{grad}}\,I_1 = 0 \end{array}\right\} (a)$$

$$\left.\begin{array}{ll} -\frac{1}{c}\frac{\partial I_2}{\partial t} = \varkappa\mu_0 c\sigma_4 & , \; \overrightarrow{\mathrm{grad}}\,I_2 = \varkappa\mu_0 c\vec{\sigma} \\ \mathrm{curl}\,\vec{\sigma} = 0 & , \; \frac{1}{c}\frac{\partial \vec{\sigma}}{\partial t} + \overrightarrow{\mathrm{grad}}\,\sigma_4 = 0 \\ \frac{1}{c}\frac{\partial \sigma_4}{\partial t} + \mathrm{div}\,\vec{\sigma} = \varkappa\mu_0 c I_2 & \end{array}\right\} (b) \qquad (29)$$

(16 equations)

These are the non-Maxwellian equations. As it is clear from (28) and (29) these equations are not all independent. From these equations also it can be deduced that each of the 16 electromagnetic quantities satisfies the second order equation:

$$\Box F = \varkappa^2\mu_0^2c^2F = -\frac{\mu_0^2c^2}{\hbar^2}F \qquad (30)$$

F denoting a typical quantity.

THE ELECTROMAGNETIC QUANTITIES EXPRESSED EXPLICITY IN TERMS OF WAVE FUNCTIONS:

We shall adopt the following representation for the Dirac -matrices throughout all our work:

$$\alpha_1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \alpha_2 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{vmatrix}, \quad \alpha_3 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \alpha_4 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Calculating the matrix densities of the operators associated with the electromagnetic quantities we get:

$$\begin{aligned}
A_x &= -K\left(\phi_{41} + \phi_{32} + \phi_{23} + \phi_{14}\right) \\
A_y &= -iK\left(\phi_{41} - \phi_{32} + \phi_{23} - \phi_{14}\right) \\
A_z &= -K\left(\phi_{31} - \phi_{42} + \phi_{13} - \phi_{24}\right) \\
V &= K\left(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44}\right) \\
E_x &= -i\frac{\mu_0 c}{\hbar}K\left(-\phi_{41} - \phi_{32} + \phi_{23} + \phi_{14}\right) \\
E_y &= \frac{\mu_0 c}{\hbar}K\left(-\phi_{41} + \phi_{32} + \phi_{23} - \phi_{14}\right) \\
E_z &= -i\frac{\mu_0 c}{\hbar}K\left(-\phi_{31} + \phi_{42} + \phi_{13} - \phi_{24}\right) \\
H_x &= \frac{\mu_0 c}{\hbar}K\left(\phi_{21} + \phi_{12} - \phi_{34} - \phi_{43}\right) \\
H_y &= -i\frac{\mu_0 c}{\hbar}K\left(-\phi_{21} + \phi_{12} - \phi_{34} + \phi_{43}\right) \\
H_z &= \frac{\mu_0 c}{\hbar}K\left(\phi_{11} - \phi_{22} - \phi_{33} + \phi_{44}\right) \\
I_1 &= -i\left(-\phi_{11} - \phi_{22} + \phi_{33} + \phi_{44}\right) \\
\sigma_x &= -i\left(\phi_{12} + \phi_{21} + \phi_{34} + \phi_{43}\right) \\
\sigma_y &= \left(\phi_{12} - \phi_{21} + \phi_{34} - \phi_{43}\right) \\
\sigma_z &= -i\left(\phi_{11} - \phi_{22} + \phi_{33} - \phi_{44}\right) \\
\sigma_4 &= -i\left(\phi_{13} + \phi_{24} + \phi_{31} + \phi_{42}\right) \\
I_2 &= i\left(\phi_{13} - \phi_{31} + \phi_{24} - \phi_{42}\right)
\end{aligned} \tag{31}$$

?

This name is never twice spelled in the same way!

The reverse relations were deduced by Geteniau and shall be called after him henceforth:

$$
\begin{aligned}
\phi_{11} &= \tfrac{1}{4}\left(\frac{\hbar}{\kappa\mu_0 c}H_z + \frac{1}{\kappa}V - I_1 + i\sigma_z\right)\\
\phi_{22} &= \tfrac{1}{4}\left(-\frac{\hbar}{\kappa\mu_0 c}H_z + \frac{1}{\kappa}V - I_1 - i\sigma_z\right)\\
\phi_{33} &= \tfrac{1}{4}\left(-\frac{\hbar}{\kappa\mu_0 c}H_z + \frac{1}{\kappa}V + I_1 + i\sigma_z\right)\\
\phi_{44} &= \tfrac{1}{4}\left(\frac{\hbar}{\kappa\mu_0 c}H_z + \frac{1}{\kappa}V + I_1 - i\sigma_z\right)\\
\phi_{13} &= \tfrac{1}{4}\left(\frac{i\hbar}{\mu_0 c\kappa}E_z - \frac{1}{\kappa}A_z - iI_2 + i\sigma_4\right)\\
\phi_{31} &= \tfrac{1}{4}\left(\frac{-i\hbar}{\mu_0 c\kappa}E_z - \frac{1}{\kappa}A_z + iI_2 + i\sigma_4\right)\\
\phi_{24} &= \tfrac{1}{4}\left(\frac{-i\hbar}{\mu_0 c\kappa}E_z + \frac{1}{\kappa}A_z - iI_2 + i\sigma_4\right)\\
\phi_{42} &= \tfrac{1}{4}\left(\frac{i\hbar}{\mu_0 c\kappa}E_z + \frac{1}{\kappa}A_z + iI_2 + i\sigma_4\right)\\
\phi_{12} &= \tfrac{1}{4}\left[\frac{\hbar}{\mu_0 c\kappa}(H_x + iH_y) + i(\sigma_x - i\sigma_y)\right]\\
\phi_{21} &= \tfrac{1}{4}\left[\frac{\hbar}{\mu_0 c\kappa}(H_x - iH_y) + i(\sigma_x + i\sigma_y)\right]\\
\phi_{14} &= \tfrac{1}{4}\left[\frac{i\hbar}{\mu_0 c\kappa}(E_x + iE_y) - \frac{1}{\kappa}(A_x + iA_y)\right]\\
\phi_{41} &= \tfrac{1}{4}\left[\frac{-i\hbar}{\mu_0 c\kappa}(E_x - iE_y) - \frac{1}{\kappa}(A_x - iA_y)\right]\\
\phi_{32} &= \tfrac{1}{4}\left[\frac{-i\hbar}{\mu_0 c\kappa}(E_x + iE_y) - \frac{1}{\kappa}(A_x + iA_y)\right]\\
\phi_{23} &= \tfrac{1}{4}\left[\frac{i\hbar}{\mu_0 c\kappa}(E_x - iE_y) - \frac{1}{\kappa}(A_x - iA_y)\right]\\
\phi_{34} &= \tfrac{1}{4}\left[\frac{-\hbar}{\mu_0 c\kappa}(H_x + iH_y) + i(\sigma_x - i\sigma_y)\right]\\
\phi_{43} &= \tfrac{1}{4}\left[\frac{-\hbar}{\mu_0 c\kappa}(H_x - iH_y) + i(\sigma_x + i\sigma_y)\right]
\end{aligned}
\qquad (32)
$$

It might be mentioned that the indeterminacy of the potentials occurring in the electromagnetic theory of Maxwell i.e. gauge invariance, does not exist now because of the appearance of the mass term in the new electromagnetic equations.

PLANE WAVE SOLUTION.

Let us associate with the motion of the photon the plane wave:

$$\phi_{ij} = a_{ij}\, e^{i(kct - \vec{k}\cdot\vec{r})} \qquad (i,j = 1,2,3,4) \tag{33}$$

where the a_{ij} are constants. As the wave function ϕ_{ij} satisfies equations (30) we get: $k^2 = |\vec{k}|^2 + \frac{\mu_0^2 c^2}{\hbar^2}$ (34)

where $$|\vec{k}|^2 = k_x^2 + k_y^2 + k_z^2 \tag{35}$$

Taking the direction of propagation $\vec{k}$ as the z-axis, putting:

$$\Delta = k + \frac{\mu_0 c}{\hbar} \qquad \text{and} \qquad P = e^{i(kct - \vec{k}\cdot\vec{r})} \tag{36}$$

we find:

$$\begin{aligned}
&\phi_{11} = \phi_{33} = \frac{|\vec{k}|}{\Delta} c_3 P \quad ; \quad \phi_{22} = \phi_{44} = \frac{|\vec{k}|}{\Delta} c_4 P \\
&\phi_{31} = -c_3 P \ , \ \phi_{13} = -\frac{|\vec{k}|^2}{\Delta^2} c_3 P \ ; \ \phi_{42} = c_4 P \ ; \ \phi_{24} = \frac{|\vec{k}|^2}{\Delta^2} c_4 P \\
&\phi_{41} = -c_2 P \ , \ \phi_{23} = \frac{|\vec{k}|^2}{\Delta^2} c_2 P \ ; \ \phi_{43} = -\phi_{21} = \frac{|\vec{k}|}{\Delta} c_2 P \\
&\phi_{32} = c_1 P \ ; \ \phi_{14} = -\frac{|\vec{k}|^2}{\Delta^2} c_1 P \ ; \ \phi_{34} = -\phi_{12} = \frac{|\vec{k}|}{\Delta} c_1 P
\end{aligned} \tag{37}$$

where the four c_i's are arbitary constants, subjected to the normalization condition.

Let us now calculate the Maxwellian electromagnetic quantities which are associated with the plane wave (37), these are:

$$A_x = A_1 = (c_2 - c_1) \frac{\sqrt{\mu_0 c^2}}{\Delta} P$$
$$A_y = A_2 = (c_2 + c_1) i \frac{\sqrt{\mu_0 c^2}}{\Delta} P$$
$$A_z = A_3 = (c_3 + c_4) \frac{\hbar}{\sqrt{\mu_0}} \cdot \frac{k}{\Delta} P$$
$$V = A_4 = (c_3 + c_4) \frac{\hbar}{\sqrt{\mu_0}} \frac{|\vec{k}|}{\Delta} P$$
$$E_x = -ik(c_2 - c_1) \frac{\sqrt{\mu_0 c^2}}{\Delta} P$$
$$E_y = k(c_2 + c_1) \frac{\sqrt{\mu_0 c^2}}{\Delta} P$$
$$E_z = -i \frac{\mu_0^{3/2} c^2}{k\Delta} 2\pi (c_3 + c_4) P.$$
$$H_x = -|\vec{k}| (c_2 + c_1) \frac{\sqrt{\mu_0 c^2}}{\Delta} P.$$
$$H_y = -i|\vec{k}| (c_2 - c_1) \frac{\sqrt{\mu_0 c^2}}{\Delta} P.$$
$$H_z = 0 \tag{38}$$

As we see the magnetic field is always transverse, as required by the equation div $\vec{H} = 0$. On the other hand $E_z \neq 0$, i.e. the electric field is not transverse which corresponds to the fact that in this theory div E $\neq 0$, but becomes transverse when $\mu_0 \to 0$. It may be mentioned here that these longitudinal waves play a fundamental role in the theory of the interaction between two charged particles, so that their occurrence in the theory of the photon is not a disturbing feature.

GENERAL FORMALISM OF THE THEORY OF THE PHOTON.

1. As the wave function of the photon ϕ_{ik} is quadratic in the Dirac wave function ψ_i, we may expect that the density ~~equations (   ) the field quantities associated~~

expressions:

$$\rho = \sum_{ik} \phi_{ik}^{\wedge} \phi_{ik} \quad ; \quad \vec{f} = -c \sum_{ik} \phi_{ik}^{\wedge} \frac{\vec{A}+\vec{B}}{2} \phi_{ik} \tag{39}$$

do not form a four-vector. They really form the fourth components of a tensor of the second rank. On the other hand the following combinations form a four vector:

$$\rho = \sum_{ik} \phi_{ik}^{*} \frac{A_4+B_4}{2} \phi_{ik} \quad , \quad \vec{f} = -c \sum_{ik} \phi_{ik}^{*} \frac{\vec{A}B_4+\vec{B}A_4}{2} \phi_{ik} \tag{40}$$

It can be shown that the expressions (40), satisfy the continuity equation as a result of the wave equations (14):

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{f} = 0 \tag{41}$$

Thus the normalization conditions becomes:

$$\int_D \rho \, d\tau = \int_D \sum_{ik} \phi_{ik}^{\wedge} \frac{A_4+B_4}{2} \phi_{ik} \, d\tau = 1 \tag{42}$$

The explicit expression for ρ is:

$$\rho = |\phi_{31}|^2 + |\phi_{32}|^2 + |\phi_{41}|^2 + |\phi_{42}|^2 - |\phi_{13}|^2 - |\phi_{14}|^2 - |\phi_{23}|^2 - |\phi_{24}|^2 . \tag{43}$$

which for a plane wave is equal:

$$\rho = \sum_{i=1}^{4} |C_i|^2 \frac{8\pi k \mu_0 c}{h \Delta^2} \tag{44}$$

Now, the definition (40) for the probability density is not positive definite. For a negative energy wave, which is a solution of the wave equation of the photon, the expression (44) is negative as it contains k. De Broglie suggested that normalization should be performed by:[22]

$$\int_D \rho W \, d\tau = h\nu \tag{45}$$

It should be mentioned that the current opinion among physicists is that the probability density has no meaning for the photon. As Pauli wrote some years ago, after discussing Landau-Peierl's work on the theory of the photon: "There exists no density-current four-vector for the light quanta, which satisfies the continuity equation and has positive definite density. Only one of the two requirements might be fulfilled formally, either the vector character of the density-current as regards Lorentz transformations or the positive definite character of the density. This stands in striking contradiction to the description of material particles in Dirac's theory where both requirements are fulfilled. From the non-existence of a density for the light quanta, it results that the position of a light quantum is not observable".

In quantum mechanics, the positive definite ρ plays the two roles:

1. The quantity $\rho(xyzt)$ represents at each instant the probability that an observation could locate the particle in the volume element at that instant.

2. The integral $\int_{D} \rho d\tau$ represents the total number of present particles.

According to the above arguments, the property (1) cannot be applied to the photon while the second one can still be used.

2. If F is an operator in the theory of the photon, it can always be written in the form: $F = F^{(1)} + F^{(2)}$ (46)

where $F^{(1)}$ and $F^{(2)}$ are linear and hermitian operators, which act on the x,y,z variables in the same way; but $F^{(1)}$ acts only on the first index of the function ϕ_{ik} while $F^{(2)}$ acts on the second index. As an example consider the Hamiltonian operator:

$$H_{op} = \frac{c}{\varkappa}\left(\frac{\partial}{\partial x}\frac{A_1+B_1}{2} + \frac{\partial}{\partial y}\frac{A_2+B_2}{2} + \frac{\partial}{\partial z}\frac{A_3+B_3}{2} + \varkappa\mu_0 c\frac{A_4+B_4}{2}\right) \tag{47}$$

as deduced from equations (14). It is the sum of the two operators:

$$H^1 = \frac{c}{2\varkappa}\left(\frac{\partial}{\partial x}A_1 + \frac{\partial}{\partial y}A_2 + \frac{\partial}{\partial z}A_3 + \varkappa\mu_0 c A_4\right) \tag{48}$$

$$H^2 = \frac{c}{2\varkappa}\left(\frac{\partial}{\partial x}B_1 + \frac{\partial}{\partial y}B_2 + \frac{\partial}{\partial z}B_3 + \varkappa\mu_0 c B_4\right) \tag{49}$$

The proper values of the operators $F^{(1)}$ and $F^{(2)}$ will be given by:

$$F^{(1)}\phi_m^{(1)} = f_m^{(1)}\phi_m^{(1)} \quad ; \quad F^{(2)}\phi_n^{(2)} = f_n^{(2)}\phi_n^{(2)} \tag{50}$$

The proper values of $f_m^{(1)}$ and $f_n^{(2)}$ are not independent. If $F^{(1)}$ and $F^{(2)}$ depend on the variables x,y,z, $f_m^{(2)}$ must be equal to $f_m^{(1)}$.

If $F^{(1)}$ and $F^{(2)}$ do not depend on x,y,z i.e. if they act only on the indices then $f_m^{(2)}$ must be equal to $\pm f_m^{(1)}$.

3. The proper functions are normalized by the formula:

$$\int_D (\phi_{ik})^*_m \frac{A_4+B_4}{2}(\phi_{ik})_m \, d\tau = 1 \tag{51}$$

where D is the entire field space. We say that the proper functions $(\phi_{ik})_m$ are normalized by $\frac{A_4+B_4}{2}$. It can also be shown that the proper functions are also orthogonal through $\frac{A_4+B_4}{2}$, i.e.,

$$\int (\phi^*_{ik})_{m'} \frac{A_4+B_4}{2}(\phi_{ik})_m \, d\tau = \delta_{mm'} \tag{52}$$

For the plane wave $\Phi(\vec{k})$ it is clear that:

$$\int_D \Phi^*(\vec{k}) \frac{A_4 - B_4}{2} \Phi(k)\, d\tau = 0 \tag{53}$$

and as the $(\phi_{ik})_m$ can always be developed in terms of the $\Phi(\vec{k})$, we deduce that:

$$\int_D \phi^*_{m'} \frac{A_4 - B_4}{2} \phi_m\, d\tau = 0 \tag{54}$$

from which we deduce that:

$$\int_D \phi^*_m A_4 \phi_m\, d\tau = \int_D \phi^*_m B_4 \phi_m\, d\tau = \int_D \phi^*_m \frac{A_4 + B_4}{2} \phi_m\, d\tau \tag{55}$$

where ϕ_m is the abbreviation for $(\phi_{ik})_m$.

4. If the state of the photon is represented by a certain wave function ϕ, and if this wave function can be developed in terms of the proper values ϕ_m of the operator F in the form:

$$\phi = \sum_m c_m \phi_m \tag{56}$$

then the probability of finding the value f_m for the observable quantity F in the state represented by ϕ will be $|c_m|^2$.

Now:

$$\int \phi^* \frac{A_4 + B_4}{2} \phi\, d\tau = \sum_{mm'} c_m c_{m'} \int_D \phi^*_{m'} \frac{A_4 + B_4}{2} \phi_m\, d\tau = \sum |c_m|^2 = 1 \tag{51}$$

which shows that the theory of the total probability is satisfied. Then the average value of the quantity F will be:

$$\bar{F} = \sum_m f_m |c_m|^2 \tag{58}$$

which was shown to be equivalent to:

$$\bar{F} = \int \sum_{ik=1}^{4} \phi^*_{ik} (F^1 B_4 + F^2 A_4) \phi_{ik}\, d\tau \tag{59}$$

The definition (59) for the average, or the expectation value of the physical quantity F leads to the following definition of the matrix elements corresponding to the operator F:

$$F_{mn} = \int_D \sum \phi_m^* (F^1 \beta_4 + F^2 A_4) \phi_n \, d\tau \qquad (60)$$

Let us assume that the ϕ_n are the system of plane waves $\Phi(\vec{k})$ which are proper functions for the operator H. We then have:

$$F_{\vec{k}'\vec{k}} = \int_D \Phi^*(\vec{k}')[F^1 B_4 + F^2 A_4] \Phi(\vec{k}) d\tau \qquad (61)$$

Assuming that $F^{(1)}$ and $F^{(2)}$ are independent of time, then the condition that F will be a constant of motion is:

$$\frac{\partial F_{\vec{k}'\vec{k}}}{\partial t} = \int_D \Big[\frac{\partial \Phi^*(\vec{k}')}{\partial t} (F^{(1)} B_4 + F^{(2)} A_4) \Phi(\vec{k}) + \Phi^*(\vec{k}') (F^1 B_4 + F^2 A_4) \frac{\partial \Phi(\vec{k})}{\partial t} \Big] d\tau = 0 \qquad (62)$$

for every $\vec{k}$ and $\vec{k}'$. Inserting the definitions (48), (49) in (14) we get:

$$\frac{\partial \phi}{\partial t} = 2\varkappa H^{(1)} \phi = 2\varkappa H^{(2)} \phi \qquad (63)$$

Equation (62) becomes:

$$\frac{\partial F_{\vec{k}'k}}{\partial t} = 2\varkappa \int \Big[-H^{(1)*} \Phi^*(\vec{k}') F^1 B_4 \Phi(\vec{k}) + \Phi^*(\vec{k}') F^1 B_4 H^1 \Phi(\vec{k}) - H^{(2)*} \Phi^*(\vec{k}') F^2 A_4 \Phi(\vec{k}) + \Phi^*(\vec{k}') F^{(2)} A_4 H^{(2)} \Phi(k) \Big] = 0$$

are

fined!

As $H^{(1)}$ and $H^{(2)}$ are Hermitian operators, we get:

$$\frac{\partial F_{\vec{k}'\vec{k}}}{\partial t} = 2\varkappa \int_D \Phi^*(\vec{k}') \{ [F^1, H^1] B_4 + [F^{(2)}, H^{(2)}] A_4 \} \Phi(k) d\tau = 0 \qquad (64)$$

If F is a constant of the motion, the right side of (64) vanishes for every $\vec{k}$ and $\vec{k}'$, i.e. all the matrix elements which correspond to: $[F^1, H^1] + [F^2, H^2]$ vanish.

$$\therefore \; [F^1, H^1] + [F^2, H^2] \equiv 0 \qquad (65)$$

This is the condition for F to be a constant of the motion.

De Broglie assumed that the operators $F^{(1)}$ and $F^{(2)}$ also satisfy the relation:

$$[F^{(1)}, H^{(1)}] + [F^{(2)}, H^{(2)}] = 0 \qquad (66)$$

Adding (65) and (66) we get the condition:

$$[F,H] = 0 \qquad (67)$$

which is to be satisfied by the operator F in order to be a constant of the motion.

An example in which this applies is the case of conservation of total angular momentum of the photon.

THE SPIN OF THE PHOTON:

Let us denote by $\vec{M}$ the orbital momentum vector, defined by: $(M_x)_{op} = i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$,

$$(M_y)_{op} = i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \quad (M_z)_{op} = i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \qquad (68)$$

The Hamiltonian operator for the photon is given by equation (47):

$$H_{op} = \frac{c}{x}\left(\frac{\partial}{\partial x}\frac{A_1+B_1}{2} + \frac{\partial}{\partial y}\frac{A_2+B_2}{2} + \frac{\partial}{\partial z}\frac{A_3+B_3}{2} + x\mu_0 c\frac{A_4+B_4}{2}\right) \qquad (47)$$

Applying the condition (67) to the components of the orbital momentum operator, we find that theyare not constants of the motion. They could be so if we add to the vector $\vec{M}$ another vector $\vec{S}$, defined by: $(S_x)_{op} = \hbar\frac{iA_2A_3 + iB_2B_3}{2}$,

$$(S_y)_{op} = \hbar\frac{iA_3A_1 + iB_3B_1}{2}, \quad (S_z)_{op} = \hbar\frac{iA_1A_2 + iB_1B_2}{2} \qquad (69)$$

in which case we find that:

$$[M_x+S_x, H] = [M_y+S_y, H] = [M_z+S_z, H] \equiv 0 \qquad (70)$$

The operators (69), which are called the spin operators of the photon, have the eigenvalues $\pm\hbar$ and 0.

It is clear that these proper values of the spin of the photon denote the sum of the spin momenta of the two constituent particles.

PROPOSED MODIFICATIONS

1. Now, we know the reality property of the electromagnetic quantities is fundamental in the theory of light, as it represents a physically measurable quantity. The peculiar structure of the expressions for the electromagnetic quantities, as given by equation (31), shows that if we assume the function ϕ_{ik} to be a hermitian matrix, then the Maxwellian quantities in question will be real. Actually all the quantities in (31) will be real in this case with the exception of σ_x, σ_y, σ_z and σ_4 which are purely imaginary.

Thus we are going to assume that the state of the photon is described by a hermitian matrix ψ_{ik}, differing from De Broglie's ϕ_{ik}. We shall see now that the hermiticity of ψ_{ik} has the further advantage that it reduces De Broglie's 32 equations of the photons into a system of 16 equations only, put in the form of the simple Dirac equation, but using the matrix ψ_{ik} for the photon instead of the one column matrix function for the electron.

Now, De Broglie's equations of the photons (14) can be written with the help of equations (12) and (15) in the form:

$$\frac{1}{c}\frac{\partial \phi_{ik}}{\partial t} = (\alpha_1)_{il}\frac{\partial \phi_{lk}}{\partial x} + (\alpha_2)_{il}\frac{\partial \phi_{lk}}{\partial y} + (\alpha_3)_{il}\frac{\partial \phi_{lk}}{\partial z} + \chi\mu_0 c(\alpha_4)_{il}\phi_{lk}:$$

and

$$\frac{1}{c}\frac{\partial \phi_{ik}}{\partial t} = \frac{\partial \phi_{il}}{\partial x}(\alpha_1)_{lk} + \frac{\partial \phi_{il}}{\partial y}(\alpha_2)_{lk} + \frac{\partial \phi_{il}}{\partial z}(\alpha_3)_{lk} - \chi\mu_0 c\,\phi_{il}(\alpha_4)_{lk}.$$

making use of the fact that α_2 is antisymmetric.

These can be written simply in the form:

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = \alpha_1\frac{\partial \phi}{\partial x} + \alpha_2\frac{\partial \phi}{\partial y} + \alpha_3\frac{\partial \phi}{\partial z} + \varkappa\mu_0 c\alpha_4\phi \tag{71}$$

and

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\alpha_1 + \frac{\partial \phi}{\partial y}\alpha_2 + \frac{\partial \phi}{\partial z}\alpha_3 - \varkappa\mu_0 c\phi\alpha_4 \tag{72}$$

Taking the adjoints of equation (71) and (72), remembering that the α's are hermitian we get

$$\frac{1}{c}\frac{\partial \phi^+}{\partial t} = \frac{\partial \phi^+}{\partial x}\alpha_1 + \frac{\partial \phi^+}{\partial y}\alpha_2 + \frac{\partial \phi^+}{\partial z}\alpha_3 - \varkappa\mu_0 c\phi^+\alpha_4 \tag{73}$$

and

$$\frac{1}{c}\frac{\partial \phi^+}{\partial t} = \alpha_1\frac{\partial \phi^+}{\partial x} + \alpha_2\frac{\partial \phi^+}{\partial y} + \alpha_3\frac{\partial \phi^+}{\partial z} + \varkappa\mu_0 c\alpha_4\phi^+ \tag{74}$$

Now, adding equations (71) to (74) and (72) to (73) respectively we get the 2 equations:

$$\frac{1}{c}\frac{\partial}{\partial t}(\phi+\phi^+) = \alpha_1\frac{\partial}{\partial x}(\phi+\phi^+) + \alpha_2\frac{\partial}{\partial y}(\phi+\phi^+) + \cdots + \varkappa\mu_0 c\alpha_4(\phi+\phi^+) \tag{75}$$

and

$$\frac{1}{c}\frac{\partial}{\partial t}(\phi+\phi^+) = \frac{\partial}{\partial x}(\phi+\phi^+)\alpha_1 + \frac{\partial}{\partial y}(\phi+\phi^+)\alpha_2 + \cdots - \varkappa\mu_0 c(\phi+\phi^+)\alpha_4 \tag{76}$$

Now the function $\phi+\phi^+ = \psi$ is hermitian, since $\psi = \psi^+$ and this equation (76) becomes the adjoint of equation (75), which can be written as:

$$\frac{1}{c}\frac{\partial \psi}{\partial t} = \alpha_1\frac{\partial \psi}{\partial x} + \alpha_2\frac{\partial \psi}{\partial y} + \alpha_3\frac{\partial \psi}{\partial z} + \chi\mu_0 c\alpha_4\psi \tag{77}$$

Now equation (77) will be the wave equation of the photon, the state of which will be described by the hermitian matrix function ψ_{ik} with 16 components. In this way the 32 equations of De Broglie (14) are reduced to only a system of 16 equations (77) for the determination of the 16 components ψ_{ik}.

2. The electromagnetic quantities will be defined by using Dirac's matrices and the state function ψ_{ik}. (It is to be noted that the function ψ_{ik} is of the dimensions of "density expressions" in Dirac's theory as it is a quadratic

quantity in Dirac's wave functions). Thus if we assume the operator F to be associated with an electromagnetic field quantity, then, we shall define the value of the observable electromagnetic quantity F as:

$$F = \sum_n (F\psi)_{nn} = \sum_{n,m} F_{nm}\psi_{mn} \equiv F : \psi \tag{78}$$

Thus we assume that the electromagnetic quantities are associated with the stationary states of the photon ψ^{ℓ}, and obtain the expression for the value of any field quantity by the double scalar multiplication (D.S.M.) of the corresponding operator by the state eigenfunction ψ^{ℓ}_{ik} as shown in (78).

We mention some of the properties of the double scalar multiplication process, which we use in our calculations:

$$A : B = \sum A_{mn} B_{nm} = \sum B_{nm} A_{mn} = B : A \tag{79}$$

$$A : BC = \sum_{m,n,p} A_{mn} B_{np} C_{pm} = AB : C = CA : B = BC : A \tag{80}$$

$$(A+B) : C = \sum (A+B)_{mn} C_{nm} = A : C + B : C \tag{81}$$

ELECTROMAGNETIC OPERATORS:

We introduce the four-vector matrix operator $(\alpha_1, \alpha_2, \alpha_3, 1)$ to represent the four-vector electromagnetic potential (A_x, A_y, A_z, V), i.e. $\vec{A}_{op} = -K\vec{\alpha}$, $V_{op} = K.1$ (82)

where K is a constant. In analogy with the classical theory we define the operators associated with the electric and magnetic fields as

$$\vec{H}_{op} = \text{curl}\, \vec{A}_{op} = -K\, \text{curl}\, \vec{\alpha} \tag{83}$$

$$\vec{E} = -\overrightarrow{\text{grad}}\, V_{op} - \frac{1}{c}\frac{\partial \vec{A}_{op}}{\partial t} = K\left[-\text{grad} + \frac{1}{c}\frac{\partial}{\partial t}\vec{\alpha}\right] \tag{84}$$

Let us consider the wave equation of the photon and its adjoint:

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = \alpha_1\frac{\partial\psi}{\partial x} + \alpha_2\frac{\partial\psi}{\partial y} + \alpha_3\frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_4\psi \tag{85}$$

and

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial x}\alpha_1 + \frac{\partial\psi}{\partial y}\alpha_2 + \frac{\partial\psi}{\partial z}\alpha_3 - \varkappa\mu_0 c\,\psi\alpha_4 \tag{86}$$

Multiplying (85) from left by α_1, by the double scalar multiplication (D.S.M.), we get, using equation (80):

$$\frac{1}{c}\alpha_1 : \frac{\partial\psi}{\partial t} - 1 : \frac{\partial\psi}{\partial x} = \alpha_1\alpha_2 : \frac{\partial\psi}{\partial y} + \alpha_1\alpha_3 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1\alpha_4 : \psi$$

$$= \frac{\partial\psi}{\partial y} : \alpha_1\alpha_2 + \frac{\partial\psi}{\partial z} : \alpha_1\alpha_3 + \varkappa\mu_0 c\,\psi : \alpha_1\alpha_4 \tag{87}$$

or

$$\frac{1}{c}\alpha_1 : \frac{\partial\psi}{\partial t} - 1 : \frac{\partial\psi}{\partial x} = -\frac{\partial\psi}{\partial y} : \alpha_2\alpha_1 - \frac{\partial\psi}{\partial z} : \alpha_3\alpha_1 + \varkappa\mu_0 c\,\psi : \alpha_1\alpha_4 .$$

Similarly, multiplying (86) from right by α_1, by D.S.M. we get:

$$\frac{1}{c}\alpha_1 : \frac{\partial\psi}{\partial t} - 1 : \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} : \alpha_2\alpha_1 + \frac{\partial\psi}{\partial z} : \alpha_3\alpha_1 + \varkappa\mu_0 c\,\psi : \alpha_1\alpha_4 \tag{88}$$

Adding (87) and (88) we get: $\frac{1}{c}\alpha_1 : \frac{\partial\psi}{\partial t} - 1 : \frac{\partial\psi}{\partial x} = \varkappa\mu_0 c\,\psi : \alpha_1\alpha_4$

i.e. $$\left[\frac{1}{c}\alpha_1\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right] : \psi = \varkappa\mu_0 c\,\alpha_1\alpha_4 : \psi \tag{89}$$

From eqution (84) and (89) we deduce that:

$$(E_x)_{op} = K\left[-\frac{\partial}{\partial x} + \frac{1}{c}\alpha_1\frac{\partial}{\partial t}\right] = K\varkappa\mu_0 c\,\alpha_1\alpha_4 \tag{90}$$

Similarly, by multiplying equation (85) and (86) by α_2 from left and right, and once more doing the same with α_3 we get the following operators:

$$(E_y)_{op} = K\varkappa\mu_0 c\,\alpha_2\alpha_4 \tag{91}$$

and

$$(E_z)_{op} = K\varkappa\mu_0 c\,\alpha_3\alpha_4 \tag{92}$$

a result which has to be effected, since the density of electric moment operators in Dirac's theory: $\vec{L} = \frac{e\hbar}{2m_0 c}\left[i\alpha_1\alpha_4, i\alpha_2\alpha_4, i\alpha_3\alpha_4\right]$ has the dimensions of an electric field intensity.

To deduce the operators which have to be associated with the magnetic field intensity, we have:

$$H_x = [\operatorname{curl} \vec{A}]_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -K\frac{\partial \alpha_3}{\partial y} + K\frac{\partial \alpha_2}{\partial z}$$
$$= K\left(\alpha_2 \frac{\partial}{\partial z} - \alpha_3 \frac{\partial}{\partial y}\right) \qquad (93)$$

Similarly $H_y = K\left(\alpha_3 \frac{\partial}{\partial x} - \alpha_1 \frac{\partial}{\partial z}\right)$; $H_z = K\left(\alpha_1 \frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right)$ (94)

Multiplying Dirac's equation (85) by $\alpha_2\alpha_3$ from left by D.S.M. we get:

$$\frac{1}{c}\alpha_2\alpha_3 \vdots \frac{\partial \psi}{\partial t} = \alpha_2\alpha_3\alpha_1 \vdots \frac{\partial \psi}{\partial x} - \alpha_3 \vdots \frac{\partial \psi}{\partial y} + \alpha_2 \vdots \frac{\partial \psi}{\partial z} + \varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_4 \vdots \psi \qquad (95)$$

and multiplying the adjoint equation (86) by $\alpha_2\alpha_3$ from right by D.S.M. we get:

$$\frac{1}{c}\alpha_2\alpha_3 \vdots \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x} \vdots \alpha_1\alpha_2\alpha_3 + \frac{\partial \psi}{\partial y} \vdots \alpha_3 - \frac{\partial \psi}{\partial z} \vdots \alpha_2 - \varkappa\mu_0 c\,\psi \vdots \alpha_4\alpha_2\alpha_3 \qquad (96)$$
$$= \alpha_2\alpha_3\alpha_1 \vdots \frac{\partial \psi}{\partial x} + \alpha_3 \vdots \frac{\partial \psi}{\partial y} - \alpha_2 \vdots \frac{\partial \psi}{\partial z} - \varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_4 \vdots \psi$$

subtracting (96) from (95) we get:

$$0 = -2\alpha_3 \vdots \frac{\partial \psi}{\partial y} + 2\alpha_2 \vdots \frac{\partial \psi}{\partial z} + 2\varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_4 \vdots \psi$$
$$\therefore H_x = K\left(\alpha_2 \frac{\partial}{\partial z} - \alpha_3 \frac{\partial}{\partial y}\right) = -K\varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_4. \qquad (97)$$

Similarly we deduce:

$$H_y = K\left(\alpha_3 \frac{\partial}{\partial x} - \alpha_1 \frac{\partial}{\partial z}\right) = K\varkappa\mu_0 c\,\alpha_1\alpha_3\alpha_4 \qquad (98)$$

and

$$H_z = K\left(\alpha_1 \frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right) = -K\varkappa\mu_0 c\,\alpha_1\alpha_2\alpha_4.$$

The operators (97) and (98) are associated in Dirac's theory of the electron, with the magnetic moment of the electron and their density expressions have the physical dimensions of the magnetic field.

The operators (82), (90), (91), (92), (97) and (98) represent 10 of the 16 matrix operators introduced in Dirac's theory. The other 6 operators, which define non-Maxwellian quantities are introduced:

$$\Gamma_1 = \alpha_4 \quad , \quad \Gamma_2 = \alpha_1\alpha_2\alpha_3\alpha_4 \qquad (99)$$
$$\sigma_x = -i\alpha_2\alpha_3 \quad ; \quad \sigma_y = -i\alpha_3\alpha_1 \quad ; \quad \sigma_z = -i\alpha_1\alpha_2 \quad ; \quad \sigma_4 = i\alpha_1\alpha_2\alpha_3$$

In order to determine the values of the observables associated with the above 16 operators, when the photon is in its stationary state ψ_{ik}, we have to apply these 16 operators, by D.S.M. on the matrix function ψ_{ik}. In doing that we will get for these observables the values mentioned in equation (31). Thus we see that the Maxwellian quantities have real values as should be expected.

We can now deduce the equations (28), (29a) and (29b) for the Maxwellian quantities which go over into Maxwell's electromagnetic equation when $\mu_0 \to 0$

DEDUCTION OF THE EQUATIONS OF ELECTROMAGNETISM:

The wave equation of the photon and its adjoint are written as:

$$\frac{1}{c}\frac{\partial \psi}{\partial t} = \alpha_1 \frac{\partial \psi}{\partial x} + \alpha_2 \frac{\partial \psi}{\partial y} + \alpha_3 \frac{\partial \psi}{\partial z} + \varkappa \mu_0 c\, \alpha_4 \psi \tag{100}$$

$$\frac{1}{c}\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x}\alpha_1 + \frac{\partial \psi}{\partial y}\alpha_2 + \frac{\partial \psi}{\partial z}\alpha_3 - \varkappa \mu_0 c\, \psi \alpha_4 \tag{101}$$

1. Now, let us multiply equation (100) from left and (101) from right by D.S.M. by the operator $\alpha_2\alpha_3\alpha_4$, we get, using the rules of the D.S.M. the following two equations:

$$\alpha_2\alpha_3\alpha_4 : \frac{\partial \psi}{c\,\partial t} = -\alpha_1\alpha_2\alpha_3\alpha_4 : \frac{\partial \psi}{\partial x} + \alpha_3\alpha_4 : \frac{\partial \psi}{\partial y} - \alpha_2\alpha_4 : \frac{\partial \psi}{\partial z} + \varkappa\mu_0 c\,\alpha_2\alpha_3 : \psi \tag{102}$$

$$\alpha_2\alpha_3\alpha_4 : \frac{\partial \psi}{c\,\partial t} = \alpha_1\alpha_2\alpha_3\alpha_4 : \frac{\partial \psi}{\partial x} + \alpha_3\alpha_4 : \frac{\partial \psi}{\partial y} - \alpha_2\alpha_4 : \frac{\partial \psi}{\partial z} - \varkappa\mu_0 c\,\alpha_2\alpha_3 : \psi \tag{103}$$

Adding (102) and (103) we get:

$$\alpha_2\alpha_3\alpha_4 : \frac{\partial \psi}{c\,\partial t} = \alpha_3\alpha_4 : \frac{\partial \psi}{\partial y} - \alpha_2\alpha_4 : \frac{\partial \psi}{\partial z} \tag{104}$$

Subtracting (102) from (103) we get:

$$0 = \alpha_1\alpha_2\alpha_3\alpha_4 : \frac{\partial\psi}{\partial x} - \varkappa\mu_0 c\, \alpha_2\alpha_3 : \psi \tag{105}$$

By the definitions of the electromagnetic quantities in terms of the operators (104) and (105) can be written in the form:

$$-\frac{1}{c}\frac{\partial H_x}{\partial t} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = (\mathrm{curl}\,\vec{E})_x \tag{106}$$

and

$$\frac{\partial I_2}{\partial x} = -\frac{\mu_0 c}{\hbar}\sigma_x \tag{107}$$

Similarly by multiplying (100) and (101) from left and right respectively by D.S.M. by $\alpha_3\alpha_1\alpha_4$ and $\alpha_1\alpha_2\alpha_4$ and then adding and subtracting in both cases, we get a couple of two equations which with (106) and (107) can be written in the form:

$$-\frac{1}{c}\frac{\partial \vec{H}}{\partial t} = \mathrm{curl}\,\vec{E} \tag{108}$$

and

$$\overrightarrow{\mathrm{grad}}\; I_2 = -\frac{\mu_0 c}{\hbar}\vec{\sigma} \tag{109}$$

2. Multiplying (100) from left by D.S.M. by $\alpha_2\alpha_3\alpha_4\alpha_1$, we get:

$$\alpha_2\alpha_3\alpha_4\alpha_1 : \frac{\partial\psi}{c\partial t} = \alpha_2\alpha_3\alpha_4 : \frac{\partial\psi}{\partial x} - \alpha_3\alpha_4\alpha_1 : \frac{\partial\psi}{\partial y} + \alpha_2\alpha_4\alpha_1\frac{\partial\psi}{\partial z} - \varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_1 : \psi \tag{110}$$

and multiplying (101) from right by D.S.M. by $\alpha_1\alpha_2\alpha_3\alpha_4$, we get:

$$-\alpha_2\alpha_3\alpha_4\alpha_1 : \frac{\partial\psi}{c\partial t} = \alpha_2\alpha_3\alpha_4 : \frac{\partial\psi}{\partial x} - \alpha_3\alpha_4\alpha_1 : \frac{\partial\psi}{\partial y} + \alpha_2\alpha_4\alpha_1 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_1 : \psi \tag{111}$$

Adding (110) and (111) we get:

$$0 = \alpha_2\alpha_3\alpha_4 : \frac{\partial\psi}{\partial x} + \alpha_3\alpha_1\alpha_4 : \frac{\partial\psi}{\partial y} + \alpha_1\alpha_2\alpha_4 : \frac{\partial\psi}{\partial z} \tag{112}$$

Subtracting (111) from (110) we get:

$$\alpha_1\alpha_2\alpha_3\alpha_4 : \frac{\partial\psi}{c\partial t} = \varkappa\mu_0 c\,\alpha_1\alpha_2\alpha_3 : \psi \tag{113}$$

(112) and (113) are the operator-equations of the electromagnetic equations:

$$\operatorname{div} \vec{H} = 0 \tag{114}$$

and

$$\frac{1}{c}\frac{\partial \Gamma_2}{\partial t} = \frac{\mu_0 c}{\hbar}\sigma_4 \tag{115}$$

3. Multiplying equation (100) by $\alpha_1\alpha_4$ by D.S.M. from left we get:

$$\frac{1}{c}\alpha_1\alpha_4 : \frac{\partial \psi}{\partial t} = -\alpha_4 : \frac{\partial \psi}{\partial x} - \alpha_1\alpha_2\alpha_4 : \frac{\partial \psi}{\partial y} + \alpha_3\alpha_1\alpha_4 : \frac{\partial \psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1 : \psi \tag{116}$$

Multiplying (101) by $\alpha_1\alpha_4$ from left we get:

$$\frac{1}{c}\alpha_1\alpha_4 : \frac{\partial \psi}{\partial t} = \alpha_4 : \frac{\partial \psi}{\partial x} - \alpha_1\alpha_2\alpha_4 : \frac{\partial \psi}{\partial y} + \alpha_3\alpha_1\alpha_4 : \frac{\partial \psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1 : \psi \tag{117}$$

Adding (116) and (117) we get:

$$\frac{1}{c}\alpha_1\alpha_4 : \frac{\partial \psi}{\partial t} = -\alpha_1\alpha_2\alpha_4 : \frac{\partial \psi}{\partial y} + \alpha_3\alpha_1\alpha_4 : \frac{\partial \psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1 : \psi \tag{118}$$

Subtracting we deduce:

$$0 = \alpha_4 : \frac{\partial \psi}{\partial x} \tag{119}$$

(118) and (119) express the electromagnetic equations:

$$\frac{1}{c}\frac{\partial}{\partial t}E_x = (\operatorname{Curl} H)_x - \varkappa^2\mu_0^2 c^2 A_x$$

and

$$\frac{\partial \Gamma_1}{\partial x} = 0 .$$

Similarly we can deduce the complementary of the above two equations which thus take the form:

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \operatorname{curl} \vec{H} - \varkappa^2\mu_0^2 c^2 \vec{A} \tag{120}$$

$$\vec{\operatorname{grad}}\, \Gamma_1 = 0 \tag{121}$$

4. Multiplying equation (100) from left and (101) from right by α_4 by D.S.M. we get:

$$\frac{1}{c}\alpha_4 : \frac{\partial \psi}{\partial t} = -\alpha_1\alpha_4 : \frac{\partial \psi}{\partial x} - \alpha_2\alpha_4 : \frac{\partial \psi}{\partial y} - \alpha_3\alpha_4 : \frac{\partial \psi}{\partial z} + \varkappa\mu_0 c : \psi \tag{122}$$

and

$$\frac{1}{c}\alpha_4 : \frac{\partial \psi}{\partial t} = \alpha_1\alpha_4 : \frac{\partial \psi}{\partial x} + \alpha_2\alpha_4 : \frac{\partial \psi}{\partial y} + \alpha_3\alpha_4 : \frac{\partial \psi}{\partial z} - \varkappa\mu_0 c : \psi \tag{123}$$

Subtracting (122) from (123) we get:

$$0 = \alpha_1\alpha_4 : \frac{\partial\psi}{\partial x} + \alpha_2\alpha_4 : \frac{\partial\psi}{\partial y} + \alpha_3\alpha_4 : \frac{\partial\psi}{\partial z} - \varkappa\mu_0 c \, 1 : \psi \tag{124}$$

Adding (122) and (123) we get:

$$\frac{1}{c}\alpha_4 : \frac{\partial\psi}{\partial t} = 0 \tag{125}$$

Equation (124) and (125) express the two electromagnetic equations:

$$div\,\vec{E} = \varkappa^2\mu_0^2 c^2 V \tag{126}$$

$$\frac{1}{c}\frac{\partial I_1}{\partial t} = 0 \tag{127}$$

5. Multiplying (100) by $\alpha_2\alpha_3$ from left and (101) by $\alpha_2\alpha_3$ from right by D.S.M. we get:

$$\alpha_2\alpha_3 : \frac{\partial\psi}{c\partial t} = \alpha_1\alpha_2\alpha_3 : \frac{\partial\psi}{\partial x} - \alpha_3 : \frac{\partial\psi}{\partial y} + \alpha_2 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_4 : \psi \tag{128}$$

$$\alpha_2\alpha_3 : \frac{\partial\psi}{c\partial t} = \alpha_1\alpha_2\alpha_3 : \frac{\partial\psi}{\partial x} + \alpha_3 : \frac{\partial\psi}{\partial y} - \alpha_2 : \frac{\partial\psi}{\partial z} - \varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_4 : \psi \tag{129}$$

Subtracting we get: $0 = \alpha_3 : \frac{\partial\psi}{\partial y} - \alpha_2 : \frac{\partial\psi}{\partial z} - \varkappa\mu_0 c\,\alpha_2\alpha_3\alpha_4 : \psi$

and adding we get: $\alpha_2\alpha_3 : \frac{\partial\psi}{c\partial t} = \alpha_1\alpha_2\alpha_3 : \frac{\partial\psi}{\partial x}$

which can be written explicitly as the x-components of the following two equations:

$$\vec{H} = curl\,\vec{A} \tag{130}$$

$$\frac{1}{c}\frac{\partial\vec{\sigma}}{\partial t} + \overrightarrow{grad}\,\sigma_4 = 0 \tag{131}$$

6. Multiplying (100) by α_1, from left and (101) from right by D.S.M. we get:

$$\frac{1}{c}\alpha_1 : \frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial x} + \alpha_1\alpha_2 : \frac{\partial\psi}{\partial y} + \alpha_1\alpha_3 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1\alpha_4 : \psi \tag{132}$$

$$\frac{1}{c}\alpha_1 : \frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial x} - \alpha_1\alpha_2 : \frac{\partial\psi}{\partial y} - \alpha_1\alpha_3 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1\alpha_4 : \psi \tag{133}$$

Adding and subtracting we get:

$$\frac{1}{c}\alpha_1 : \frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial x} + \varkappa\mu_0 c\,\alpha_1\alpha_4 : \psi$$

and

$$0 = \alpha_1\alpha_2 : \frac{\partial\psi}{\partial y} + \alpha_1\alpha_3 : \frac{\partial\psi}{\partial z}$$

which are the operator equations of the x-component of the electromagnetic equations:

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \overrightarrow{grad}\,V \tag{134}$$

$$\text{Curl}\ \vec{\sigma} = 0 \tag{135}$$

7. Multiplying (100) and (101) by the unit matrix by D.S.M. we get:

$$\frac{1}{c} : \frac{\partial\psi}{\partial t} = \alpha_1 : \frac{\partial\psi}{\partial x} + \alpha_2 : \frac{\partial\psi}{\partial y} + \alpha_3 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_4 : \psi \tag{136}$$

$$\frac{1}{c} : \frac{\partial\psi}{\partial t} = \alpha_1 : \frac{\partial\psi}{\partial x} + \alpha_2 : \frac{\partial\psi}{\partial y} + \alpha_3 : \frac{\partial\psi}{\partial z} - \varkappa\mu_0 c\,\alpha_4 : \psi \tag{137}$$

Adding we get:

$$\frac{1}{c} : \frac{\partial\psi}{\partial t} = \alpha_1 : \frac{\partial\psi}{\partial x} + \alpha_2 : \frac{\partial\psi}{\partial y} + \alpha_3 : \frac{\partial\psi}{\partial z}$$

and subtracting we get:

$$\alpha_4 : \psi = 0$$

which are equivalent to:

$$\frac{1}{c}\frac{\partial V}{\partial t} + \operatorname{div}\vec{A} = 0 \tag{138}$$

$$\Gamma_1 = 0 \tag{139}$$

8. Multiplying (100) from left and (101) from right by D.S.M. by $\alpha_1\alpha_2\alpha_3$ we get:

$$\frac{1}{c}\alpha_1\alpha_2\alpha_3 : \frac{\partial\psi}{\partial t} = \alpha_2\alpha_3 : \frac{\partial\psi}{\partial x} - \alpha_1\alpha_3 : \frac{\partial\psi}{\partial y} + \alpha_1\alpha_2 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1\alpha_2\alpha_3\alpha_4 : \psi \tag{140}$$

$$\frac{1}{c}\alpha_1\alpha_2\alpha_3 : \frac{\partial\psi}{\partial t} = \alpha_2\alpha_3 : \frac{\partial\psi}{\partial x} - \alpha_1\alpha_3 : \frac{\partial\psi}{\partial y} + \alpha_1\alpha_2 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1\alpha_2\alpha_3\alpha_4 : \psi \tag{141}$$

Adding we get:

$$\frac{1}{c}\alpha_1\alpha_2\alpha_3 : \frac{\partial\psi}{\partial t} = \alpha_2\alpha_3 : \frac{\partial\psi}{\partial x} - \alpha_1\alpha_3 : \frac{\partial\psi}{\partial y} + \alpha_1\alpha_2 : \frac{\partial\psi}{\partial z} + \varkappa\mu_0 c\,\alpha_1\alpha_2\alpha_3\alpha_4 : \psi \tag{142}$$

Subtracting we get:

$$0 = 0$$

(142) can be written as:

$$i\alpha_1\alpha_2\alpha_3 : \frac{\partial \psi}{c\,\partial t} = i\alpha_2\alpha_3 : \frac{\partial \psi}{\partial x} + i\alpha_3\alpha_1 : \frac{\partial \psi}{\partial y} + i\alpha_1\alpha_2 : \frac{\partial \psi}{\partial z} + i\chi\mu_0 c\,\alpha_1\alpha_2\alpha_3\alpha_4 : \psi$$

or

$$\frac{1}{c}\frac{\partial \sigma_4}{\partial t} + \operatorname{div} \vec{\sigma} = -\frac{\mu_0 c}{\hbar} I_2 \qquad (143)$$

Thus the tensorial equations of the photon can be written in the form:

$$\left.\begin{array}{l} -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} = \operatorname{curl} \vec{E} \;;\; \operatorname{div} \vec{H} = 0 \\ \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \operatorname{curl} \vec{H} - \chi^2\mu_0^2 c^2 \vec{A} \;;\; \operatorname{div} \vec{E} = \chi^2\mu_0^2 c^2 V \\ \vec{H} = \operatorname{curl} \vec{A} \;,\; \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \operatorname{grad} V \\ \frac{1}{c}\frac{\partial V}{\partial t} + \operatorname{div} \vec{A} = 0 \end{array}\right\} (A)$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \operatorname{grad} I_2 = -\frac{\mu_0 c}{\hbar}\vec{\sigma} \;;\; \frac{1}{c}\frac{\partial I_2}{\partial t} = \frac{\mu_0 c}{\hbar}\sigma_4 \\ \frac{1}{c}\frac{\partial \vec{\sigma}}{\partial t} + \operatorname{grad} \sigma_4 = 0 \;,\; \operatorname{curl} \vec{\sigma} = 0 \\ \frac{1}{c}\frac{\partial \sigma_4}{\partial t} + \operatorname{div} \vec{\sigma} = -\frac{\mu_0 c}{\hbar} I_2 \end{array}\right\} (B_1) \\ \left.\begin{array}{l} I_1 = 0 \;,\; \frac{1}{c}\frac{\partial I_1}{\partial t} = 0 \;, \\ \overrightarrow{\operatorname{grad}}\, I_1 = 0 \end{array}\right\} (B_2) \end{array}\right\} (B)$$

From the above equations we can deduce for any of the electromagnetic quantities, the equation:

$$\Box^2 F = -\frac{\mu_0^2 c^2}{\hbar^2} F \qquad (144)$$

where F is any of the above 16 quantities. Equation (144) is clear, as each of the electromagnetic quantities is formed by linear combinations of the ψ_{ik}'s which obey the wave equations.

PLANE-WAVE SOLUTION:

The energy of material particles of all kinds is described by means of exponentials,(23) and we would like to regard the energy of the photon in the same way. It is then natural to regard the wave function of the photon -being Hermitian- as the superposition of two plane waves, one with positive energy and momentum, and the other with negative. Thus let us put:

$$\psi_{ik} = a_{ik}\, e^{i(kct - \vec{k}\cdot\vec{r})} + a^{\times}_{ki}\, e^{-i(kct - \vec{k}\cdot\vec{r})} \qquad (145)$$

$$= \phi_{ik} + \phi^{\times}_{ki} = \phi_{ik} + \phi^{+}_{ik} \qquad (146)$$

where ϕ_{ik} and $\phi^{\times}_{ki}$ are given by:

$$\phi_{ik} = a_{ik}\, e^{i(kct - \vec{k}\cdot\vec{r})}, \quad \phi^{\times}_{ki} = \phi^{+}_{ik} = a^{\times}_{ki}\, e^{-i(kct - \vec{k}\cdot\vec{r})} \qquad (147)$$

In the above equations the constants a_{ik} are complex. Now let us substitute in the wave equation of the photon for ψ by equation (146), in which case the resulting equation will be the superposition of the following two equations:

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = \alpha_1 \frac{\partial \phi}{\partial x} + \alpha_2 \frac{\partial \phi}{\partial y} + \alpha_3 \frac{\partial \phi}{\partial z} + \varkappa \mu_0 c\, \alpha_4 \phi \qquad (148)$$

and

$$\frac{1}{c}\frac{\partial \phi^{+}}{\partial t} = \alpha_1 \frac{\partial \phi^{+}}{\partial x} + \alpha_2 \frac{\partial \phi^{+}}{\partial y} + \alpha_3 \frac{\partial \phi^{+}}{\partial z} + \varkappa \mu_0 c\, \alpha_4 \phi^{+} \qquad (149)$$

Consider the first equation (148) and put for ϕ:

$$\phi_{ik} = a_{ik}\, e^{i(kct - \vec{k}\cdot\vec{r})} \qquad (150)$$

We get as the condition for the existence of such a solution (150) is:

$$k^2 = |\vec{k}|^2 + \frac{\mu_0^2 c^2}{\hbar^2} \quad ; \quad (|\vec{k}|^2 = k_x^2 + k_y^2 + k_z^2) \tag{151}$$

Let us consider the case when the wave travels along the z-axis, introducing the notation:

$$\Delta = k + \frac{\mu_0 c}{\hbar} \qquad P = e^{i(kct - \vec{k}\cdot\vec{r})} \tag{152}$$

then we shall get for the values of the constants a_{ik}:

$$\begin{aligned}
&a_{11} = a_{33} = \frac{|\vec{k}|}{\Delta} c_3 \quad ; \quad a_{22} = a_{44} = \frac{|\vec{k}|}{\Delta} c_4 \ ; \\
&a_{31} = -c_3 \quad ; \quad a_{13} = -\frac{|\vec{k}|^2}{\Delta^2} c_3 \ ; \ a_{42} = c_4 \ ; \ a_{24} = \frac{|\vec{k}|^2}{\Delta^2} \ ; \\
&a_{41} = -c_2 \quad ; \quad a_{23} = \frac{|\vec{k}|^2}{\Delta^2} c_2 \ ; \ a_{43} = -a_{21} = \frac{|\vec{k}|}{\Delta} c_2 \ ; \\
&a_{32} = c_1 \quad ; \quad a_{14} = -\frac{|\vec{k}|^2}{\Delta^2} c_1 \ ; \ a_{34} = -a_{12} = \frac{|\vec{k}|}{\Delta} c_1 \ .
\end{aligned} \tag{153}$$

where C_1, C_2, C_3, and C_4 are four arbitrary complex constants.

It can be shown that equation (149) for ϕ^+ is also satisfied by the same values for the coefficients (153).

Using the equations (31) in order to determine the magnitude of the electromagnetic quantities associated with the plane wave of the photon we get:

$$
\begin{aligned}
A_x &= \frac{2\mu_0 c}{\hbar}\,\frac{K}{\Delta}\left[(c_2-c_1)P + (c_2^* - c_1^*)\bar{P}\right]\\
A_y &= \frac{2\mu_0 c}{\hbar}\,\frac{iK}{\Delta}\left[(c_2+c_1)P - (c_2^* + c_1^*)\bar{P}\right]\\
A_z &= 2K\,\frac{k}{\Delta}\left[(c_3+c_4)P + (c_3^* + c_4^*)\bar{P}\right]\\
V &= 2K\,\frac{|\vec{k}|}{\Delta}\left[(c_3+c_4)P + (c_3^* + c_4^*)\bar{P}\right]\\
E_x &= -\frac{2\mu_0 c}{\hbar}\,\frac{k}{\Delta}\,iK\left[(c_2-c_1)P - (c_2^* - c_1^*)\bar{P}\right]\\
E_y &= \frac{2\mu_0 c}{\hbar}\,\frac{k}{\Delta}\,K\left[(c_2+c_1)P + (c_2^* + c_1^*)\bar{P}\right]\\
E_z &= -\frac{4\pi\mu_0^2 c^2}{k\hbar}\,\frac{iK}{\Delta}\left[(c_3+c_4)P - (c_3^* + c_4^*)\bar{P}\right]\\
H_x &= -\frac{2\mu_0 c}{\hbar}\,K\,\frac{|\vec{k}|}{\Delta}\left[(c_2+c_1)P + (c_2^* + c_1^*)\bar{P}\right]\\
H_y &= -\frac{2\mu_0 c}{\hbar}\,iK\,\frac{|\vec{k}|}{\Delta}\left[(c_2-c_1)P - (c_2^* - c_1^*)\bar{P}\right]\\
H_z &= 0
\end{aligned}
\tag{154}
$$

We notice that the magnetic field is transverse while the electric field is nearly so, as the longitudinal component is proportional to μ_0^2, which is extremely small.

We also notice that the non-Maxwellian quantities depend on $(C_3 - C_4)$ while the longitudinal Maxwellian quantities, such as A_z and E_z, depend on $(C_3 + C_4)$.

THE GENERALIZED MAXWELL'S TENSOR.

In the Maxwellian electromagnetic theory, a four-dimensional symmetrical tensor M_{ij} ($i, j = 1, 2, 3, 4$) of the second order exists which might be called the energy-momentum tensor of the photon. The space-time part of this tensor (M_{i4}) represents the Poynting flux while the pure time part (M_{44}) represents the energy density of the field. The pure space part is identical with the Maxwell stress tensor. This tensor satisfies the following conservation formula in the case of the pure radiation field:

$$\sum_j \frac{\partial M_{ij}}{\partial x_j} = 0 \qquad (155)$$

If we now consider a volume V in which a pure radiation field exists with no electric charges inside, and if on the boundary: $M_{ij} = 0$ (156), then it can be proved that:

$$\int M_{4j}\, d\tau = j^{th} \text{ component of a four vector,}^{(24)} \qquad (157)$$

where $d\tau$ is the volume element in the ordinary 3 dimensional space.

De Broglie defined the tensor M_{ij} in the following way

(a) $M_{kl} = M_{lk} = \mu_0 c^2 \cdot \phi^x \frac{A_k B_l + A_l B_k}{2} \phi \quad (k,l = 1, 2, 3)$

(b) $M_{k4} = M_{4k} = -\mu_0 c^2 \cdot \phi^x \frac{A_k + B_k}{2} \phi \quad (k = 1, 2, 3) \qquad (158)$

(c) $M_{44} = \mu_0 c^2 \cdot \phi^x \cdot \phi$

Using Gehenian's equations, (158 b) and (158 c) can be expressed as:

$$M_{k4} = [\vec{E}^*, \vec{H}]_k + [\vec{E}, \vec{H}^*]_k - \frac{\mu_0^2 c^2}{\hbar^2}(VA_k^* + V^*A_k) \quad (k=1,2,3)$$
$$M_{44} = |\vec{E}|^2 + |\vec{H}|^2 + \frac{\mu_0^2 c^2}{\hbar^2}(|\vec{A}|^2 + |V|^2) \tag{159}$$

In equations (158) we notice that ϕ is the wave function of the photon representing the initial state and ϕ^* is its complex conjugate, i.e. the annihilation state is not included in this expression. Moreover, we also notice that the expression (158c) for the energy density M_{44} is constant and not an oscillating quantity like the square of a sine function in the usual Maxwellian theory.

In accordance with the notation of this paper let us define the Maxwell tensor $\vec{M}$ by the formulae:

$$M_{ij} = M_{ji} = \frac{2K^2\mu_0^2c^2}{\hbar^2}(\alpha_i \psi \,:\, \alpha_j \psi) \tag{160}$$
$$M_{i4} = M_{4i} = -\frac{2K^2\mu_0^2c^2}{\hbar^2}(\alpha_i \psi \,:\, \psi) \tag{161}$$
$$M_{44} = \frac{2K^2\mu_0^2c^2}{\hbar^2}(\psi \,:\, \psi) \tag{162}$$

If we now introduce the matrices β_i with the following properties:

$$\beta_i = -\alpha_i \quad \text{for } i = 1, 2, 3.$$
$$\beta_4 = 1 \quad \text{(unit matrix)} \tag{163}$$

Equations (160), (161) and (162) can be put in the form:

$$M_{ij} = M_{ji} = \frac{2K^2\mu_0^2c^2}{\hbar^2}(\beta_i \psi \,:\, \beta_j \psi) \quad (i, j = 1, 2, 3, 4) \tag{164}$$

which shows clearly the symmetrical properties of the tensor.

To see the analogy between the tensor (164) and the classical Maxwell tensor, let us calculate the following components M_{i4}. We find, using the inverse equations of Geheniau for the values of ψ_{ik} in terms of the electromagnetic field quantities,

$$M_{14} = M_{xt} = [\vec{E},\vec{H}]_x + \frac{\mu_0^2 c^2}{\hbar^2}(VA_x) + \frac{K^2\mu_0^2 c^2}{\hbar^2}(\sigma_x \sigma_4) \tag{165}$$

$$M_{24} = M_{yt} = [\vec{E},\vec{H}]_y + \frac{\mu_0^2 c^2}{\hbar^2}(VA_Y) + \frac{K^2\mu_0^2 c^2}{\hbar^2}(\sigma_y \sigma_4) \tag{166}$$

$$M_{34} = M_{zt} = [\vec{E},\vec{H}]_z + \frac{\mu_0^2 c^2}{\hbar^2}(VA_z) + \frac{K^2\mu_0^2 c^2}{\hbar^2}(\sigma_z \sigma_4) \tag{167}$$

and:

$$M_{44} = |\vec{E}|^2 + |\vec{H}|^2 + \frac{\mu_0^2 c^2}{\hbar^2}(|\vec{A}|^2 + V^2) + \frac{K^2\mu_0^2 c^2}{\hbar^2}(\vec{\sigma}^2 + \sigma_4^2 + I_1^2 + I_2^2) \tag{168}$$

where $\vec{\sigma}$ is the vector $(\sigma_1, \sigma_2, \sigma_3)$

In passing to the limit when $\mu_0 \to 0$, or neglecting the quantities of the second order containing μ_0^2 we get the classical Maxwellian quantities. We may notice here that the expression for the energy density M_{44} as given by equation (168) is an oscillating quantity, since E and H are real quantities. This is exactly as in the Maxwellian theory.

Now, let us consider the wave equation of the photon and its adjoint:

$$\frac{1}{c}\frac{\partial \psi}{\partial t} = \alpha_1 \frac{\partial \psi}{\partial x} + \alpha_2 \frac{\partial \psi}{\partial y} + \alpha_3 \frac{\partial \psi}{\partial z} + \varkappa \mu_0 c \alpha_4 \psi. \tag{169}$$

$$\frac{1}{c}\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x}\alpha_1 + \frac{\partial \psi}{\partial y}\alpha_2 + \frac{\partial \psi}{\partial z}\alpha_3 - \varkappa \mu_0 c \psi \alpha_4. \tag{170}$$

Multiplying (169) from the left by $\alpha_i \cdot \psi$, by double scalar multiplication, we get,

$$\frac{\alpha_i\cdot\psi}{c} : \frac{\partial\psi}{\partial t} = \alpha_i\psi : \alpha_1\frac{\partial\psi}{\partial x} + \cdots + \cdots + \varkappa\mu_0 c\,\alpha_i\cdot\psi : \alpha_4\psi \cdot \qquad (171)$$

Similarly multiplying (170) by ψ from the right and by α_i from left we get:

$$\left(\frac{\partial}{\partial t}\frac{\alpha_i\psi}{c}\right) : \psi = \left(\frac{\partial}{\partial x}\alpha_i\psi\right) : \alpha_1\psi + \cdots + \cdots - \varkappa\mu_0 c\,\alpha_i\cdot\psi : \alpha_4\psi \,. \qquad (172)$$

Adding (171) and (172) we get:

$$\frac{\partial}{\partial x}(\alpha_i\cdot\psi : \alpha_1\psi) + \frac{\partial}{\partial y}(\alpha_i\cdot\psi : \alpha_2\psi) + \frac{\partial}{\partial z}(\alpha_i\cdot\psi : \alpha_3\psi) + \frac{\partial}{c\partial t}(-\psi : \alpha_i\cdot\psi) = 0 \qquad (173)$$

which can be written in the form:

$$\sum_{j=1}^{4} \frac{\partial}{\partial x_j} M_{ij} = 0 \qquad , i = 1, 2, 3, 4. \qquad (174)$$

(174) corresponds to the classical formulas of the conservation of electromagnetic energy and momentum in Maxwell's theory.

<u>NORMALIZATION OF THE PLANE WAVES:</u> (25)

In the classical theory it was shown that, if we consider a region of space in which the field is different from zero, and assume that the field vanishes outside it, i.e. if

$$\sum_k \frac{\partial M_{ik}}{\partial x_k} = 0 \qquad (175)$$

and $M_{ik} = 0$ on the boundary

then $\int M_{4k}\, d\tau = k^{th}$ component of a 4-vector which is the energy momentum tensor. For a plane monochromatic wave by equating the total energy in the field to the energy of the photon,

$$\int M_{44}\, d\tau = 2h\nu \tag{176}$$

i.e. $$\mu_0 c^2 \int \psi^{\times}_{ik} \psi_{ik}\, d\tau = 2h\nu \tag{177}$$

Calculating the value of $\sum_{ik} \psi^{\times}_{ik} \psi_{ik}$ for a plane monochromatic wave, we get a term that oscillates with a frequency 2ν about an average value exactly similar to the expression for the real energy density in the classical theory. This oscillating term vanishes after integration and we get:

$$\mu_0 c^2 \frac{8k^2}{\Delta^2} \mathcal{V} \sum_{i=1}^{4} C_i^2 = 2h\nu = 2\hbar c k \tag{178}$$

where $\mathcal{V}$ is the volume in which the radiation is contained, and which is supposed to be very large with respect to the wave length of the light waves.

$$\therefore \sum_{i=1}^{4} C_i^2 = \frac{2\hbar \Delta^2}{8\mu_0 c k \mathcal{V}} \tag{179}$$

We can specify this normalization better by considering separately the two transverse waves polarized along Ox and Oy and the longitudinal wave, as was done by de Broglie in the following way:

1. <u>A transverse wave with the electric vibration along Ox:</u>

To represent this wave let the transverse components be E_x and H_y, all other components vanishing. Reference to equations (154) page (41) shows that the representation can be made by placing:

$$C_1 = -C_2 \quad , \quad C_3 = C_4 = 0$$

The electromagnetic potential is:

$$A_x = -\frac{4\mu_0 c}{\hbar} \cdot \frac{K}{\Delta}(c_1 P + c_1^* P^-)$$

$$A_y = A_z = V = 0 \qquad (180)$$

Assuming now the value $\frac{\hbar}{\sqrt{4\mu_0}}$ for K, which will be deduced later, we get:

$$A_x = -\frac{c\sqrt{4\mu_0}}{\Delta}(c_1 P + c_1^* P^-)$$

where

$$P = e^{i(kct - \vec{k}\cdot\vec{r})} \quad \text{and} \quad P^- = e^{-i(kct - \vec{k}\cdot\vec{r})} \qquad (181)$$

To normalize this equation we use the relations:

$$\sum_{i=1}^{4} c_i^2 = 2c_1^2 = \frac{2\hbar\Delta^2}{8\mu_0 c k\nu}$$

$$\therefore |c_1|^2 = \frac{2\hbar\Delta^2}{16\mu_0 c k\nu} \qquad (182)$$

Thus $$A_x = \sqrt{\frac{\hbar c}{2k\nu}}(P + P^-) \qquad (183)$$

if c_1 is placed equal to $-\sqrt{\frac{2\hbar}{16\mu_0 c k\nu}} \cdot \Delta$. (184)

2. <u>A transverse wave with the electric vibration along Oy :</u>

In this case we put $c_3 = c_4 = 0$, $c_1 = c_2$ obtaining

$$A_y = \frac{2ic}{\Delta}\sqrt{4\mu_0}(c_1 P - c_1^* P^-) \qquad (185)$$

$$A_x = A_z = V = 0$$

Taking for $|c_1|$ the value (184), equation (185) becomes:

$$A_y = -i\sqrt{\frac{\hbar c}{2k\nu}}(P + P^-)$$

$$A_x = A_z = V = 0 \qquad (186)$$

The above normalized equation (186) can be written, neglecting a phase factor:

$$A_y = \sqrt{\frac{\hbar c}{2k\nu}}(P + P^-)$$

$$A_x = A_z = V = 0 \qquad (187)$$

From equations (183) and (187), introducing the unit vector $\vec{a}$ which defines the direction of polarization of the vector potential, we get for the normalized value of the potential:

$$\vec{A} = \vec{a}\sqrt{\frac{\hbar c}{2k\nu}}\,(P + P^{-}) \qquad (188)$$

3. Longitudinal wave:

Let us consider the case in which the electrical intensity is along the direction of propagation oz, and in which there are no other components. The formulae (154) on page (41) show that the representation of this case can be made by placing:

$$C_1 = C_2 = 0 \quad , \quad c_3 = c_4 .$$

Thus we get:

$$A_x = A_y = 0 \quad , \quad A_z = \frac{4k}{\Delta}\frac{\hbar}{\sqrt{4\mu_0}}(c_3 P + c_3^* P^{-})$$

$$V = \frac{4|\vec{k}|}{\Delta}\frac{\hbar}{\sqrt{4\mu_0}}(c_3 P + c_3^* P^{-})$$

From equation (179) above we deduce:

$$|c_3|^2 = \frac{2\Delta^2\hbar}{16\mu_0 c k\nu}$$

$$\therefore |c_3| = \sqrt{\frac{2\hbar}{16\mu_0 c k\nu}}\cdot\Delta$$

$$\therefore |A_z| = \frac{\hbar}{\mu_0}\sqrt{\frac{\hbar k}{2c\nu}}\,(P + P^{-})$$

$$V = \frac{\hbar|\vec{k}|}{\mu_0}\sqrt{\frac{\hbar}{2ck\nu}}\,(P + P^{-}) \qquad (189)$$

with $\quad A_x = A_y = 0$

The Interaction of an electron with radiation:

Dirac's equation for a free electron runs as follows:

$$\frac{\hbar}{i}\frac{1}{c}\frac{\partial \psi_k}{\partial t} = \frac{\hbar}{i}\left[\frac{\partial}{\partial x}\alpha_1 + \frac{\partial}{\partial y}\alpha_2 + \frac{\partial}{\partial z}\alpha_3\right]\psi_k + m_0 c\,\alpha_4\psi_k \qquad (190)$$

$$k = 1, 2, 3, 4.$$

Introducing the operators:

$$(p_i)_{op} = -\frac{\hbar}{i}\frac{\partial}{\partial q_i} \quad (i=1,2,3) \qquad , \qquad W_{op} = \frac{\hbar}{i}\frac{\partial}{\partial T} \tag{191}$$

Equation (190) becomes:

$$\frac{1}{c} W_{op} \psi_k + \vec{p}_{op}\cdot\vec{\alpha}\,\psi_k = m_0 c^2 \alpha_4 \psi_k \tag{192}$$

If the Dirac electron is under the influence of an electromagnetic field defined by the potentials ($\vec{A}$, V), then the wave equation is derived by replacing W_{op} and $\vec{p}_{op}$ in (192) by the operators:

$$W_{op} \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial T} + eV \qquad , \qquad \vec{p}_{i\,op} \longrightarrow -\frac{\partial}{\partial q_i} + \frac{e}{c}\vec{A} \tag{193}$$

and thus the wave equation becomes:

$$\left(\frac{\hbar}{i}\frac{1}{c}\frac{\partial}{\partial T} + \frac{e}{c}V\right)\psi_k = \sum_{j=1}^{3}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - \frac{e}{c}A_j\right)\alpha_j\psi_k + m_0 c\,\alpha_4\psi_k \tag{194}$$

which can be written as:

$$\frac{\hbar}{i}\frac{\partial \psi_k}{\partial T} = H\psi_k \tag{195}$$

where the Dirac Hamiltonian H is given by:

$$H = \frac{c}{\varkappa}\left(\alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z}\right) + m_0 c^2\alpha_4 - e\left[V + (\vec{\alpha}\cdot\vec{A})\right] \tag{196}$$

and where $\qquad \varkappa = \frac{i}{\hbar}$

In this equation the terms independent of e form the Dirac Hamiltonian in the absence of the field. The terms

$$H' = -e\left[V + (\vec{\alpha}\cdot\vec{A})\right] \tag{197}$$

are the interaction terms between the electron and the radiation field as they contain terms belonging to both sides.

If we like to consider transverse radiations only, we can choose the abitrary constants as in (188) in which case the interaction terms become: $H' = -e(\vec{\alpha}\cdot\vec{A})$ (198)

as V and A_z vanish and $\vec{A}$ is given by equation (188)

Now, according to the perturbation theory, the interaction potential H' between the electron and the photon (radiation field) will cause a transition of the electron from one energy state A to another one B with the emission or absorption of one photon.

In the expression of the perturbing potential (198) we have to insert the expression (188) for the vector potential, when dealing with transverse waves. This expression being real is expressed in terms of the exponentials: $e^{i(kct-\vec{k}\cdot\vec{r})} + e^{-i(kct-\vec{k}\cdot\vec{r})}$

The correspondence principle,(26) when applied to the quantum theory of radiation, as was done by Klein, shows that in the real expression of the electromagnetic quantities concerned with the interaction phenomena, the part with $e^{-2\pi i\nu t}$ corresponds to the emission of photons while the part with $e^{2\pi i\nu t}$ corresponds to the inverse process of absorption. Thus from the expression $\vec{A} = \vec{a}\sqrt{\frac{\hbar c}{2k V}}\left(e^{+i(kct-\vec{k}\cdot\vec{r})} + e^{-i(kct-\vec{k}\cdot\vec{r})}\right)$ we get for the transition of the electron from the state A to the state B, with the emission of one photon the interaction term:

$$H'_{em} = -e\left[\vec{\alpha}\cdot\vec{a}\sqrt{\frac{\hbar c}{2kV}}\, e^{-i(kct-\vec{k}\cdot\vec{r})}\right]$$

$$= -e\sqrt{\frac{\hbar c}{2kV}}\left[\vec{\alpha}\cdot\vec{a}_l\, e^{-i(kct-\vec{k}\cdot\vec{r})}\right] \qquad (199)$$

while we get for the absorption transition:

$$H'_{ab} = -e\sqrt{\frac{\hbar c}{2kV}}\left[\vec{\alpha}\cdot\vec{a}_l\, e^{i(k_l ct - \vec{k}_l\cdot\vec{r})}\right] \tag{200}$$

Let us calculate the matrix element corresponding to the emission transition:

The electron in the state A has the wave function:

$$\phi_A(xyzt) = u_A(xyz)\, e^{iK_A ct} \tag{201}$$

and in the state B has the wave function:

$$\phi_B(xyzt) = u_B(xyz)\, e^{iK_B ct} \tag{202}$$

where u_A and u_B are the amplitudes of the proper functions in the corresponding stationary states. Thus the matrix element of H' corresponding to the transition from the state A to B with the emission of a photon can be written in the form:

$$H'_{A0,Bl} = -e\sqrt{\frac{\hbar c}{2kV}}\int \phi_B^*(\vec{r})\left[\vec{\alpha}\cdot\vec{a}_l\, e^{-i(k_l ct - \vec{k}\cdot\vec{r})}\right]\phi_A(\vec{r})\, d\vec{r} \tag{203}$$

It should be mentioned here that the electromagnetic field at a point of space acts on the electric charge (electron) which exists at the same point of space. Thus $\vec{r}$ is the same in both parts of the integrand of $H'_{A0,Bl}$. Substituting for ϕ_A and ϕ_B in (203), we get:

$$H'_{A0,Bl} = -e\sqrt{\frac{\hbar c}{2kV}}\int\left[u_B^*(\vec{r})\cdot(\vec{a}_l\cdot\vec{\alpha})\, u_A(\vec{r})\right]\times e^{i(\vec{k}_l\cdot\vec{r})}\, e^{i(-k_l - K_B + K_A)ct}\, d\vec{r} \tag{204}$$

The expression (204) is independent of the choice of axes as it is a scalar product.

Now for either emission or absorption phenomena we have a continuous spectrum for either the initial or the final state. In accordance with the quantum theory the conservation of energy is ensured during transitions

when $$k_l = K_A - K_B \tag{205}$$

i.e. $$\nu_l = \frac{1}{h}(W_A - W_B) \tag{206}$$

which is Bohr's frequency condition.

Thus we get:

$$\begin{aligned} H_{A0,Bl} &= -e\sqrt{\frac{\hbar c}{2k V}} \int \left[u_B^*(\vec{r}) \cdot (\vec{a}_l \cdot \vec{\alpha}) u_A(\vec{r}) \right] e^{i(\vec{k}_l \cdot \vec{r})} d\vec{r} \\ &= -e\sqrt{\frac{\hbar c}{2k V}} \int u_B^*(\vec{r}) \alpha_l e^{i(\vec{k}_l \cdot \vec{r})} u_A(\vec{r}) d\vec{r} \end{aligned} \tag{207}$$

where α_l represents the component of the matrix vector $\vec{\alpha}$ in the direction of polarization of the light quantum. The matrix element of H' corresponding to the absorption transition can be similarly deduced:

$$H'_{Al,B0} = -e\sqrt{\frac{\hbar c}{2k V}} \int u_B^*(\vec{r}) \alpha_l e^{-i(\vec{k}_l \cdot \vec{r})} u_A(\vec{r}) d\vec{r} \tag{208}$$

Equations (207) and (208) are similar to those deduced in the quantum theory of radiation (Heitler) and can be applied to different processes.

Part ii.

CHAPTER I

From the previous work we noticed that the assumption that the wave function of the photon ψ_{ik} is hermitian has given rise to the reality property of the electromagnetic field quantities, and has simplified the form of the wave equation of the photon.

Moreover, it is clear from this work (pages 28 and 29), that the wave function ψ_{ik} is the superposition of two wave functions ϕ_{ik} and ϕ^{+}_{ik}, where the first is de Broglie's wave function.

1. The plane wave associated with the photon is:

$$\Psi_{ik} = a_{ik}\, e^{i(kct - \vec{k}\cdot\vec{r})} + a^{+}_{ik}\, e^{-i(kct - \vec{k}\cdot\vec{r})} \qquad (1)$$

which is the superposition of the following two plane waves:

$$\Phi_{ik} = a_{ik}\, e^{i(kct - \vec{k}\cdot\vec{r})}, \quad \Phi^{+}_{ik} = a^{+}_{ik}\, e^{-i(kct - \vec{k}\cdot\vec{r})} \qquad (2)$$

where Φ_{ik} represents the motion of a particle with positive energy and momentum and Φ^{+}_{ik} represents the motion of another with both negative. But in both cases the corpuscular velocity of the particles associated with these two waves (i.e. the group velocity) is the same and is given by:[27]

$$v = \frac{\partial W}{\partial p} \qquad (3)$$

2. Considering now the general case of a wave function formed by the superposition of plane monochromatic waves, then the 16 components of the wave functions are expressed by Fourier integrals in the form:

$$\psi_{ik}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int a_{ik}(\vec{k})\,e^{i(kct-\vec{k}.\vec{r})}\,d\vec{k}+\frac{1}{(2\pi)^{3/2}}\int a^{+}_{ik}(\vec{k})\,e^{-i(kct-\vec{k}.\vec{r})}\,d\vec{k} \qquad (4)$$

$$\phi_{ik}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int a_{ik}(\vec{k})\,e^{i(kct-\vec{k}.\vec{r})}\,d\vec{k}. \qquad (5)$$

$$\phi^{+}_{ik}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int a^{+}_{ik}(\vec{k})\,e^{-i(kct-\vec{k}.\vec{r})}\,d\vec{k} \qquad (6)$$

Thus the two components ϕ_{ik} and ϕ^{+}_{ik} of the wave function of the photon are expressed in terms of positive and negative energy and momentum exponentials respectively.

3. From (1) and (4) we notice that negative energy expressions take part in the present development of the theory of the photon. In de Broglie's theory, the photon is regarded as a complex particle with positive energy and momentum. Thus in that case, the plane wave associated with the photon is:

$$\phi_{ik} = a_{ik}\,e^{i(kct-\vec{k}.\vec{r})} \qquad (7)$$

On the other hand in the present work the plane wave associated with the photon, being hermitian, is expressed by equation (1), which is the superposition of the two plane waves with different energies. Thus we may think of the wave function of the photon ψ_{ik} as superposition of two states represented by the wave function ϕ_{ik} with positive energy and momentum and ϕ^+_{ik} with both negative, i.e:

$$\psi_{ik} = \phi_{ik} + \phi^+_{ik} \tag{8}$$

The wave equation of the photon and its adjoint can be written in the form:

$$\frac{1}{c}\frac{\partial}{\partial t}(\phi+\phi^+) = \alpha_1\frac{\partial}{\partial x}(\phi+\phi^+)+\alpha_2\frac{\partial}{\partial y}(\phi+\phi^+)+\alpha_3\frac{\partial}{\partial z}(\phi+\phi^+)+\varkappa\mu_0 c\alpha_4(\phi+\phi^+) \tag{9}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\phi+\phi^+) = \frac{\partial}{\partial x}(\phi+\phi^+)\alpha_1+\frac{\partial}{\partial y}(\phi+\phi^+)\alpha_2+\frac{\partial}{\partial z}(\phi+\phi^+)\alpha_3-\varkappa\mu_0 c(\phi+\phi^+)\alpha_4 \tag{9^1}$$

which can be considered as the superposition of the two systems of wave equations:

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \alpha_1\frac{\partial\phi}{\partial x}+\alpha_2\frac{\partial\phi}{\partial y}+\alpha_3\frac{\partial\phi}{\partial z}+\varkappa\mu_0 c\alpha_4\phi\ , \tag{10}$$

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial x}\alpha_1+\frac{\partial\phi}{\partial y}\alpha_2+\frac{\partial\phi}{\partial z}\alpha_3-\varkappa\mu_0 c\phi\alpha_4 \tag{10^1}$$

and

$$\frac{1}{c}\frac{\partial\phi^+}{\partial t} = \alpha_1\frac{\partial\phi^+}{\partial x}+\alpha_2\frac{\partial\phi^+}{\partial y}+\alpha_3\frac{\partial\phi^+}{\partial z}+\varkappa\mu_0 c\alpha_4\phi^+ \tag{11}$$

$$\frac{1}{c}\frac{\partial\phi^+}{\partial t} = \frac{\partial\phi^+}{\partial x}\alpha_1+\frac{\partial\phi^+}{\partial y}\alpha_2+\frac{\partial\phi^+}{\partial z}\alpha_3-\varkappa\mu_0 c\phi^+\alpha_4 \tag{11^1}$$

The first set (10, 10^1) can be considered as describing the positive energy state of the photon, while the second describes its negative energy state.

The consistency of the set (11, 11^1) with (10, 10^1) is clear, as (11^1) and (11) are simply the hermitian conjugate of (10) and (10^1).

These two sets were considered before, in the derivation of the wave equation of the photon from de Broglie's set of equations which is equivalent to the set ($10, 10^1$).

That the electromagnetic field quantities are the superposition of the two states of opposite energies is clear from their reality property. The dissocation into these two states in the form of exponentials was done by Darwin who wrote in 1932:[28]

"It would seem natural to regard the wave function of the photon, being real, as the superposition of two waves, one with positive energy and momentum, and the other with both negative. In the case of the electron, Dirac has excluded the negative energy electrons with the help of the exclusion principle. We cannot apply the same idea here because of the different statistics of the photons. It might be thought that we are increasing the troubles of the quantum theory by introducing a new case of negative energy, but it is usually found that the best hope of resolving a deep difficulty is to extend its applications as widely as possible."

Darwin simply excluded the negative energy part and considered only the positive one. The same process has been adopted by other workers who developed the theory of radiation interaction, the principle of second quantization and the commutation relations of the electromagnetic field quantities.

Negative energy states play a fundamental rôle in the relativistic quantum theory. In the applications of Dirac's wave equation to the theory of the Compton effect, use has been made of the negative energy states as intermediate states. The results of the theory (Klein-Nishina formula) were found to be in very good agreement (29) with experiment up to energies of at least 10 m c^2. If we had only taken intermediate states with positive energy, we should have obtained a formula for the Compton scattering deviating largely from the Klein-Nishina formula, and the agreement with the experiments would have been destroyed. The negative energy states thus form a necessary part of the present theory of radiation phenomena. Then we expect that, as the present quantum theory is successful in the low energy domain, the negative energy states must have some physical meaning.

4. One advantage of rejecting one of the energy states exponentials (the negative energy part) and considering the other (the positive) is that the field quantities will be comparable to the complex wave function of the electron in the quantum theory. In this case expressions, quadratic in the field quantities, will be constant in time and a quantum mechanics for the photon, similar to that of the electron can be developed.

Let us consider, as an example, the density expression for the average value of the operator F. This is defined to be:

$$\psi^*_{ik} \cdot F \psi_{ik} \equiv \psi^*_{ik} \cdot (F\psi)_{ik} \tag{12}$$

As ψ is assumed to be hermitian, (12) becomes:

$$\psi : F\psi \equiv \psi_{ki} \cdot (F\psi)_{ik} \tag{13}$$

Substituting for ψ its value for the plane wave, we find that the resulting expression contains an oscillating term. This is also evident from the expressions of the components of the Maxwell electromagnetic tensor, defined by equations (168) on page (44).

This indicates that Hermitian wave functions can describe field quantities but are not suitable for the definition of corpuscular quantities.

5. Another reason for this unsuitability is the following:

In the ordinary quantum mechanics the operator: $i\hbar \frac{\partial}{\partial x}$ is associated with the x - component of the momentum as is clear from the case of the plane wave:

$$\phi_i = a_i e^{i(kct - \vec{k}\cdot\vec{r})} \tag{13}$$

where $k = E/\hbar$ and $\vec{k} = \vec{p}/\hbar$ (14)

since we have:

$$\left(i\hbar \frac{\partial}{\partial x}\right)\phi_i = \left(i\hbar \frac{\partial}{\partial x}\right) a_i e^{i(kct - \vec{k}\cdot\vec{r})}$$

$$= p_x \cdot a_i e^{i(kct - \vec{k}\cdot\vec{r})} = p_x \cdot \phi_i \tag{15}$$

Thus p_x is the proper value of the momentum operator $i\hbar\frac{\partial}{\partial x}$ belonging to the plane wave (13).

Let us now consider the hermitian plane wave, associated with our photon:

$$\psi_{ik} = a_{ik}\, e^{i(kct - \vec{k}\cdot\vec{x})} + a_{ik}^{+}\, e^{-i(kct - \vec{k}\cdot\vec{x})}$$

$$\therefore \left(i\hbar\frac{\partial}{\partial x}\right)\psi_{ik} = \left(i\hbar\frac{\partial}{\partial x}\right)(\phi_{ik} + \phi_{ik}^{+}) = (p_x\phi_{ik} - p_x\phi_{ik}^{+}) \qquad (16)$$

$$= p_x(\phi_{ik} - \phi_{ik}^{+}) \neq p_x\psi_{ik} \qquad (17)$$

Thus we see that the operator $i\hbar\frac{\partial}{\partial x}$ cannot be associated with the momentum as usual, for that representation of ψ . By adopting Darwin's idea of separating the two superposed parts, we can still retain the property of the momentum operator, exactly as in the case of the electron.

That has been done by most quantum physicists and the definition of the momentum operator is retained. As an example we refer to Pauli's article on the quantization of the radiation field.

6. It might be mentioned here that in 1930, V.Fock[(4)] suggested putting Maxwell's equations:

$$-\operatorname{curl} H + \frac{1}{c}\frac{\partial E}{\partial t} = 0$$

$$\operatorname{curl} E + \frac{1}{c}\frac{\partial H}{\partial t} = 0$$

in the form:

$$\mathcal{H}F + \frac{\hbar}{i}\frac{\partial F}{\partial t} = 0$$

where $\mathcal{H}$ is the operator:

$$\mathcal{H} \equiv \frac{\hbar c}{i} \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}$$

and

$$F \equiv \begin{pmatrix} E \\ H \end{pmatrix}$$

in which case the operator $\frac{\hbar}{i}\frac{\partial}{\partial x}$ is supposed to have no meaning. But this theory was not extended any further.

Almost all other theories are developed on the same lines as those for the electron.

Part ii.

CHAPTER II

1. Let us now consider the following two sets of equations which were dealt with before:

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial \phi}{\partial t} &= \alpha_1 \frac{\partial \phi}{\partial x} + \alpha_2 \frac{\partial \phi}{\partial y} + \alpha_3 \frac{\partial \phi}{\partial z} + \chi \mu_0 c \alpha_4 \phi \\ \frac{1}{c}\frac{\partial \phi}{\partial t} &= \frac{\partial \phi}{\partial x}\alpha_1 + \frac{\partial \phi}{\partial y}\alpha_2 + \frac{\partial \phi}{\partial z}\alpha_3 - \chi \mu_0 c \phi \alpha_4 \end{aligned}\right\} \tag{18}$$

and

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial \phi^\dagger}{\partial t} &= \alpha_1 \frac{\partial \phi^\dagger}{\partial x} + \alpha_2 \frac{\partial \phi^\dagger}{\partial y} + \alpha_3 \frac{\partial \phi^\dagger}{\partial z} + \chi \mu_0 c \alpha_4 \phi^\dagger \\ \frac{1}{c}\frac{\partial \phi^\dagger}{\partial t} &= \frac{\partial \phi^\dagger}{\partial x}\alpha_1 + \frac{\partial \phi^\dagger}{\partial y}\alpha_2 + \frac{\partial \phi^\dagger}{\partial z}\alpha_3 - \chi \mu_0 c \phi^\dagger \alpha_4 \end{aligned}\right\} \tag{19}$$

We shall consider the equations (18) as describing the positive energy photon and the equations (19) as describing the negative energy one.

We shall associate with each of these two states, electromagnetic quantities defined by the double scalar multiplication of the Dirac α-operators and the respective wave functions ϕ and $\phi^\dagger$, as:

$$(\vec{A}\,,\, V) \equiv (\vec{\alpha} : \phi\,,\, 1 : \phi) \tag{20}$$

$$(\vec{A}^\dagger, V^\dagger) \equiv (\vec{\alpha} : \phi^\dagger\,,\, 1 : \phi^\dagger) \tag{21}$$

and similarly with the other operators which were associated with the electromagnetic quantities in the first part of this work. It is easy to see that, by the superposition of the corresponding quantities from the systems (20) and (21), we get our original expressions which gave rise to the reality property of the Maxwellian quantities. For instance we see that:

$$\vec{A} + \vec{A}^{+} = \vec{\alpha} : \phi + \vec{\alpha} : \phi^{+} = \vec{\alpha} : (\phi + \phi^{+}) = \vec{\alpha} : \psi \qquad (22)$$

From the definitions (20), (21) and the two sets of equations (18), (19) we can deduce the Maxwell's equations for the two types of fields belonging to the states ϕ and ϕ^{+}.

This may justify the use of complex quantities for the quantum theory of radiation and the assumption that Maxwell's equations are satisfied by these complex field quantities. It gives a physical picture to that method which was derived from the correspondence principle, and supported by more studies of the nature of the radiation phenomena.

2. We have shown that while the hermitian wave function ψ was suitable for the field description of the photon , the constituent wave functions ϕ and ϕ^{+} are convenient for the corpuscular description. Using the wave functions ϕ and ϕ^{+} the momentum operator $i\hbar \frac{\partial}{\partial x_n}$ can still retain its meaning. Hence the spin operators can be deduced, in a similar way to that used for the electron in Dirac's theory. The spin of the photon is of fundamental importance for the explanation of the properties of polarization.

3. We shall now proceed to deduce the Hamiltonian operators for both the photons ϕ and ϕ^{+}. It is clear from equations (18) and (19) that the Hamiltonian operator

will be the same for the two states, as the two sets are exactly similar, apart from the wave functions. The equations (18) for ϕ , give, by adding them together:

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{\alpha_1 0 + 0\alpha_1}{2}\frac{\partial \phi}{\partial x} + \frac{\alpha_2 0 + 0\alpha_2}{2}\frac{\partial \phi}{\partial y} + \frac{\alpha_3 0 + 0\alpha_3}{2}\frac{\partial \phi}{\partial z} + \chi\mu_0 c(\alpha_4 0 - 0\alpha_4)\phi \tag{23}$$

where for convenience, the following notation is used:

$$\left(\frac{\alpha_1 0 + 0\alpha_1}{2}\right)F = \frac{\alpha_1 F + F\alpha_1}{2} \tag{24}$$

Equation (23) can be written as:

$$\frac{1}{\chi}\frac{\partial \phi}{\partial t} = \frac{c}{\chi}\mathcal{H}\phi \tag{25}$$

where the Hamiltonian operator $\mathcal{H}$ is given by:

$$\mathcal{H} = \frac{\partial}{\partial x}\frac{\alpha_1 0 + 0\alpha_1}{2} + \frac{\partial}{\partial y}\frac{\alpha_2 0 + 0\alpha_2}{2} + \frac{\partial}{\partial z}\frac{\alpha_3 0 + 0\alpha_3}{2} + \chi\mu_0 c(\alpha_4 0 - 0\alpha_4) \tag{26}$$

(26) is clearly also the Hamiltonian deduced from the set of equations (19) for the state ϕ^+. This can be seen by adding the two equations (19) from which we get:

$$\frac{1}{c}\frac{\partial \phi^+}{\partial t} = \frac{\alpha_1 0 + 0\alpha_1}{2}\frac{\partial \phi^+}{\partial x} + \cdots + \cdots + \chi\mu_0 c(\alpha_4 0 - 0\alpha_4)\phi^+ \tag{27}$$

Thus we see that the operator $\mathcal{H}$ represents the two states ϕ and ϕ^+. These are the two energy states of the photon whose Hamiltonian is (26).

4. In working out with the Hamiltonian (25) we meet two kinds of operations, the first one given by the operator $\alpha_1 0$, where the operator stands on the left of the operand as in the usual Dirac's theory. The other kind of operation

is given by the operator $O\alpha_1$, which, when acting on the function ϕ , acts on it from right, i.e. $(O\alpha_1)\phi = \phi\alpha_1$ This type of operator has different properties from the previous type; e.g.:

$$O\alpha_i \times (O\alpha_j) = O\alpha_j \alpha_i$$
$$O\alpha_i \cdot \alpha_j \times (O\alpha_j) = O\alpha_j \cdot \alpha_i \cdot \alpha_j = -\alpha_i \tag{28}$$

As regards the mixed operations, we have:

$$\alpha_i \cdot \alpha_j \cdot O \times O\alpha_k = O\alpha_k \times \alpha_i \cdot \alpha_j \cdot O = \alpha_i \cdot \alpha_j \cdot O\alpha_k \tag{29}$$

i.e. the mixed operators commute.

Actually the first type of operators corresponds to de Broglie's matrix operators A and the second to the matrices B, and their commutation relations which correspond to those above mentioned, are expressed by equations (13) page (13).

5. From the form of the wave equation (23) it might be suggested that the field operators should be defined using operators similar to $\frac{\alpha_i \cdot O + O\alpha_i}{2}$, etc instead of the simple ones. Because of the properties of the double scalar operations, we find that the two definitions are the same, as can be seen from the following:

$$\tfrac{1}{2}(\alpha_i \cdot O + O\alpha_i) : \phi = \tfrac{1}{2}(\alpha_i : \phi + \phi : \alpha_i)$$
$$= \alpha_i : \phi \tag{30}$$

THE SPIN OF THE PHOTON

6. Proceeding along the same lines as in Dirac's theory, we can show that constants of the motion can be defined as those quantities which commute with the Hamiltonian of the system (25). It can be proved that the components of the orbital momentum:

$$(M_x)_{op} = (y p_z - z p_y) = i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \; ; \; \cdots \; , \; \cdots \qquad (31)$$

of the photon do not commute with H_{op}.

We can obtain a constant of the motion for the photon if we add to the orbital momentum vector $\vec{M}$, the vector $\vec{S}$ with components:

$$(S_x)_{op} = i\hbar \frac{\alpha_2 \alpha_3 O + O \alpha_3 \alpha_2}{2} \; ; \; (S_y)_{op} = i\hbar \frac{\alpha_3 \alpha_1 O + O \alpha_1 \alpha_3}{2}$$
$$(S_z)_{op} = i\hbar \frac{\alpha_1 \alpha_2 O + O \alpha_2 \alpha_1}{2} \qquad (32)$$

since then we find that:

$$[M_x + S_x, H_{op}] = [M_y + S_y, H_{op}] = [M_z + S_z, H_{op}] = 0$$

as can be easily shown taking into account equations (28) and (29). Hop is the operator given by equation (25). The operators (32) are those associated with the components of the spin angular momentum and have the proper values ± 1 and zero.

It is known that if S_x, S_y and S_z are the spin operators for a particle of spin 1, -1 and zero, then we must have the following relations:[31]

$$S_i^3 = \hbar^2 S_i$$
$$S_x S_y - S_y S_x = i\hbar S_z \qquad (33)$$

Using the properties (28) and (29) it can be easily shown that the operators (32) satisfy equations (33).

As an illustration we shall show the following:

$$S_x^2 = -\frac{\hbar^2}{4}(\alpha_2\alpha_3 O + O\alpha_3\alpha_2)(\alpha_2\alpha_3 O + O\alpha_3\alpha_2)$$
$$= -\frac{\hbar^2}{4}(\alpha_2\alpha_3\alpha_2\alpha_3 O + 2\alpha_2\alpha_3 O\alpha_3\alpha_2 + O\alpha_3\alpha_2\alpha_3\alpha_2)$$
$$= -\frac{\hbar^2}{4}(-1 + 2\alpha_2\alpha_3 O\alpha_3\alpha_2 - 1)$$
$$= -\frac{\hbar^2}{2}(\alpha_2\alpha_3 O\alpha_3\alpha_2 - 1)$$
$$S_x^3 = -\frac{\hbar^2}{2}\frac{i\hbar}{2}(\alpha_2\alpha_3 O + O\alpha_3\alpha_2)(\alpha_2\alpha_3 O\alpha_3\alpha_2 - 1)$$
$$= -\hbar^2\frac{\hbar i}{4}(-O\alpha_3\alpha_2 - \alpha_2\alpha_3 O - \alpha_2\alpha_3 O - O\alpha_3\alpha_2)$$
$$= -\hbar^2\frac{\hbar i}{2}(-\alpha_2\alpha_3 O - O\alpha_3\alpha_2)$$
$$= \hbar^2 \cdot \frac{\hbar i}{2}(\alpha_2\alpha_3 O + O\alpha_3\alpha_2) = \hbar^2 \cdot S_x \qquad (34)$$

Similarly the other equations can be verified.

5- Let us now examine, in particular, the operator S_z ; defined by equation (32); we have:

$$S_z\phi_{ik} = \frac{\hbar}{2}\left[(i\alpha_1\alpha_2\phi)_{ik} + (\phi i\alpha_2\alpha_1)_{ik}\right]$$
$$= \hbar\begin{vmatrix} 0 & \phi_{12} & 0 & \phi_{14} \\ -\phi_{21} & 0 & -\phi_{23} & 0 \\ 0 & \phi_{32} & 0 & \phi_{34} \\ -\phi_{41} & 0 & -\phi_{43} & 0 \end{vmatrix} \qquad (35)$$

From which we see that the components ϕ_{11} , ϕ_{22} ϕ_{33} ϕ_{44} , ϕ_{13} , ϕ_{24} ϕ_{31} , ϕ_{42} correspond to the value zero of the spin ; the four components ϕ_{21} , ϕ_{23} , ϕ_{41} ϕ_{43} , to the value $-\hbar$ and the components ϕ_{12} ϕ_{14} ϕ_{32} ϕ_{34} , to the value $+\hbar$.

The explanation of the polarization phenomena is based on the spin properties of the components of the wave function of the photon (ϕ), and is developed by de Broglie. The explanation is completely satisfactory and we assume that it applies here to both the states ϕ and ϕ^{+}. For that explanation we refer to de Broglie's book "La Lumiere, Herman 1940, pages 168-172".

CHAPTER III

QUADRATIC QUANTITIES AND THE NORMALIZATION OF THE FUNCTION ϕ :

1. We shall now consider the state ϕ , knowing that similar treatments can be applied to the wave function ϕ^{+} . We know that the quantity $\sum_{i,k=1}^{4} \phi_{ik}^{*} \cdot \phi_{ik}$ does not transform like the fourth component of a vector but rather like the pure time component of a tensor of the second rank.

Now, to the state ϕ , we can ascribe two energy-momentum tensors similar to those introduced by de Broglie, the first is the Maxwell electromagnetic field tensor while the second is the corpuscular tensor introduced in Dirac's theory. The Maxwell field tensor is defined by:

$$
\begin{aligned}
M_{ij} &= M_{ji} = \mu_0 c^2 \alpha_i \cdot \phi^{+} : \alpha_j \phi \\
M_{i4} &= M_{4i} = -\mu_0 c^2 \alpha_i \cdot \phi^{+} : \phi \\
M_{44} &= \mu_0 c^2 \quad \phi^{+} : \phi
\end{aligned}
\qquad (36)
$$

As on the first part page (46), we shall put:

$$\mu_0 c^2 \int \phi^{+} \vdots \phi \, d\tau = \mu_0 c^2 \int \phi_{ik}^{*} \cdot \phi_{ik} \, d\tau = h\nu = h \frac{kc}{2\pi} = \hbar c k \qquad (37)$$

i.e.

$$\frac{\mu_0 c}{\hbar} \int \phi^{*} \cdot \phi \, d\tau = k$$

or

$$\frac{\mu_0 c}{\hbar} \int \phi^{*} \cdot \frac{1}{k} \phi \, d\tau = 1 \qquad (38)$$

(32)

Now denoting by W the operator $\frac{\cancel{\partial}\sqrt{\chi^2+\vec{k}^{2}}}{\cancel{i \cdot c\, \partial t}}$, we

$\left(\chi = \frac{\mu c}{\hbar}\right)$

notice:

$$W e^{i(kct - \vec{k}\cdot\vec{r})} = |k|\, e^{i(kct - \vec{k}\cdot\vec{r})}$$

$$\frac{1}{W} e^{i(kct - \vec{k}\cdot\vec{r})} = \frac{1}{|k|} e^{i(kct - \vec{k}\cdot\vec{r})} \tag{39}$$

From (39) equation (38) becomes:

$$\frac{\mu_0 c}{\hbar}\int \phi^{\times} \frac{1}{W} \phi \, d\tau = 1 \tag{40}$$

This will be our normalizing equation for the wave function ϕ. The operator W and the similar operator $\sqrt{-\nabla}$ were introduced in the quantum theory of the electromagnetic field as was suggested by Frenkel,[33] Pauli-Heisenberg,[4] Landau - Peierls.[4] The reason for the use of the operator W here while $\sqrt{-\nabla}$ was used by Peierls, is that in de Broglie's theory of the photon the quantities k and $|\vec{k}|$ are different from each other because of the existence of a rest mass for the photon. They are related to each other by the equation:

$$k^2 = |\vec{k}|^2 + \frac{\mu_0^2 c^2}{\hbar^2} \tag{41}$$

It is clear that the expression:

$$\rho = \frac{\mu_0 c}{\hbar} \phi^{\times} \frac{1}{W} \phi \tag{42}$$

is the fourth component of a four-vector, since from equation (36) the quantity $\mu_0 c^2 \phi^{\times} \cdot \phi$ is the pure time component of a tensor of the second rank and W is the fourth component of a vector.

The operator $\frac{1}{W}$ must be taken into account, when defining the density expressions of the corpuscular quantities. As an example the momentum of the photon in the state ϕ, will be represented by:

$$P_i = \frac{\mu_0 c}{\hbar} \int \phi^* i\hbar \frac{\partial}{\partial x_i} \frac{1}{W} \phi \, d\tau$$

i.e.

$$P_i = i\mu_0 c \int \phi^* \frac{\partial}{\partial x_i} \frac{1}{W} \phi \, d\tau \tag{43}$$

which can be compared with the corresponding expression given by Pauli:(34)

$$P_i = \frac{2}{c} \int \vec{F}^*(x) \frac{1}{i} \frac{\partial}{\partial x_i} \frac{1}{\sqrt{-\Delta}} \vec{F}(x) \, dx \tag{44}$$

2. Now, in Dirac's theory of the electron, an energy-momentum tensor is defined for the electron in the following way:(35)

$$T_{ij} = T_{ji} = \frac{\hbar c}{4i} \left[\psi^* \alpha_i \frac{\partial \psi}{\partial x_j} - \frac{\partial \psi^*}{\partial x_j} \alpha_i \psi + \psi^* \alpha_j \frac{\partial \psi}{\partial x_i} - \frac{\partial \psi^*}{\partial x_i} \alpha_j \psi \right]$$

$$T_{i4} = T_{4i} = -\frac{\hbar c}{4i} \left[\psi^* \frac{\partial \psi}{\partial x_i} - \frac{\partial \psi^*}{\partial x_i} \psi + \psi^* \alpha_i \frac{\partial \psi}{\partial x_4} - \frac{\partial \psi^*}{\partial x_4} \alpha_i \psi \right]$$

$$T_{44} = \frac{\hbar c}{2i} \left[\psi^* \frac{\partial \psi}{\partial x_4} - \frac{\partial \psi^*}{\partial x_4} \psi \right] \tag{45}$$

T_{44} represents the energy density associated with the wave ψ, the three T_{i4} components represent the density of the energy flux while the T_{ij} ($i, j = 1, 2, 3$) represent the density of flux of the three components of momentum along the three axes.

In the absence of external fields the components (45) satisfy the conservation equation:

$$\sum_{j=1}^{4} \frac{\partial T_{ij}}{\partial x_j} = 0 \qquad (i = 1, 2, 3, 4)$$

The tensor (45) is based on the definition of the energy momentum operators:

$$p_x = i\hbar \frac{\partial}{\partial x}\,, \quad p_y = i\hbar \frac{\partial}{\partial y}\,, \quad p_z = i\hbar \frac{\partial}{\partial z}\,, \quad p_4 = -i\hbar \frac{\partial}{c\partial t} \tag{46}$$

and

$$u_x = -c\alpha_1\,, \quad u_y = -c\alpha_2\,, \quad u_z = -c\alpha_3\,, \quad u_4 = c \cdot 1 \tag{47}$$

The components of the tensor (45) are formed by taking the density expressions of the product of the operators (46) and (47) e.g. $\psi^{*}(\alpha_i \frac{\partial}{\partial x_j})\psi$, neglecting the constants. The term $\psi^{*}(\alpha_i \frac{\partial}{\partial x_j})\psi$ is added to make the expression symmetrical and the complex conjugate of these two terms are added to make the expréssion real.

This tensor is defined in de Broglie's theory as:

$$\begin{aligned}
T_{ij} = T_{ji} &= \frac{\hbar c}{4i}\Big[\phi^{*}\frac{A_iB_4+B_iA_4}{2}\frac{\partial\phi}{\partial x_j} - \frac{\partial\phi^{*}}{\partial x_j}\frac{A_iB_4+B_iA_4}{2}\phi \qquad (i,j=1,2,3)\\
&\qquad + \phi^{*}\frac{A_jB_4+B_jA_4}{2}\frac{\partial\phi}{\partial x_i} - \frac{\partial\phi^{*}}{\partial x_i}\frac{A_jB_4+B_jA_4}{2}\phi\Big]\\
T_{i4} = T_{4i} &= -\frac{\hbar c}{4i}\Big[\phi^{*}\frac{A_4+B_4}{2}\frac{\partial\phi}{\partial x_i} - \frac{\partial\phi^{*}}{\partial x_i}\frac{A_4+B_4}{2}\phi \qquad (i=1,2,3)\\
&\qquad + \phi^{*}\frac{A_iB_4+B_iA_4}{2}\frac{\partial\phi}{\partial x_4} - \frac{\partial\phi^{*}}{\partial x_4}\frac{A_iB_4+B_iA_4}{2}\phi\Big]\\
T_{44} &= \frac{\hbar c}{2i}\Big[\phi^{*}\frac{A_4+B_4}{2}\frac{\partial\phi}{\partial x_4} - \frac{\partial\phi^{*}}{\partial x_4}\frac{A_4+B_4}{2}\phi\Big]
\end{aligned} \tag{48}$$

The operator $\frac{A_4+B_4}{2}$ is added because of the normalization process.

In our case the normalization operator is $\frac{\mu_0 c}{\hbar W}$, and thus the corpuscular energy momentum tensor will be defined as:

$$T_{ij} = T_{ji} = \frac{\hbar c}{4i} \cdot \frac{\mu_0 c}{\hbar}\left[\phi^* \frac{\alpha_i}{W}\frac{\partial \phi}{\partial x_j} - \frac{\partial \phi^*}{\partial x_j}\frac{\alpha_i}{W}\phi + \phi^* \frac{\alpha_j}{W}\frac{\partial \phi}{\partial x_i} - \frac{\partial \phi^*}{\partial x_i}\frac{\alpha_j}{W}\phi\right] \tag{49}$$

$$T_{i4} = T_{4i} = -\frac{\hbar c}{4i} \cdot \frac{\mu_0 c}{\hbar}\left[\phi^* \frac{1}{W}\frac{\partial \phi}{\partial x_i} - \frac{\partial \phi^*}{\partial x_i}\frac{1}{W}\phi + \phi^* \frac{\alpha_i}{W}\frac{\partial \phi}{\partial x_4} - \frac{\partial \phi^*}{\partial x_4}\frac{\alpha_i}{W}\phi\right]$$

$$T_{44} = \frac{\hbar c}{4i} \cdot \frac{\mu_0 c}{\hbar}\left[\phi^* \frac{1}{W}\frac{\partial \phi}{\partial x_4} - \frac{\partial \phi^*}{\partial x_4}\frac{1}{W}\phi\right]$$

As we see, the tensor is the same as that of the electron apart from the factor $\frac{\mu_0 c}{\hbar}\frac{1}{W}$ which is introduced for the purpose of normalization. From the wave equation

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = \alpha_1 \frac{\partial \phi}{\partial x} + \alpha_2 \frac{\partial \phi}{\partial y} + \alpha_3 \frac{\partial \phi}{\partial z} + \varkappa \mu_0 c\, \alpha_4 \phi \tag{50}$$

and its adjoint

$$\frac{1}{c}\frac{\partial \phi^+}{\partial t} = \frac{\partial \phi^+}{\partial x}\alpha_1 + \frac{\partial \phi^+}{\partial y}\alpha_2 + \frac{\partial \phi^+}{\partial z}\alpha_3 - \varkappa \mu_0 c\, \phi^+ \alpha_4 \tag{51}$$

we can deduce the conservation formulae for the components of the tensor (49).

The two quantities M_{44} and T_{44} are the expressions for the energy density in the two tensors, introduced above. It is important to show now that they are equivalent. Let us test this for the case of the plane monochromatic wave:

$$\phi_{ik} = a_{ik}\, e^{i(kct - \vec{k}\cdot\vec{r})} \tag{52}$$

we find that

$$M_{44} = \mu_0 c^2 a^+_{ik} : a_{ik} \tag{53}$$

and

$$T_{44} = \frac{\mu_0 c^2}{2i}\left[\phi^\times ik \frac{1}{k}\phi + ik\phi^\times \frac{1}{k}\phi\right]$$

$$= \mu_0 c^2 \phi^\times \cdot \phi = \mu_0 c^2 \phi^+ : \phi = \mu_0 c^2 a^+_{ik} : a_{ik} \quad (54)$$

From (53) and (54) we see the equivalence of the value of the energy density for the plane wave (52).

Does this hold generally?

As an example to show the formalism of the present theory and to justify the use of the equation:

$$\left[H, F \right] = 0 \tag{55}$$

for the constant of motion F , let us consider the two equations:

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \alpha_1\frac{\partial\phi}{\partial x} + \alpha_2\frac{\partial\phi}{\partial y} + \alpha_3\frac{\partial\phi}{\partial z} + \varkappa\mu_0 c\,\alpha_4\phi \tag{56}$$

and

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial x}\alpha_1 + \frac{\partial\phi}{\partial y}\alpha_2 + \frac{\partial\phi}{\partial z}\alpha_3 - \varkappa\mu_0 c\,\phi\,\alpha_4 \tag{57}$$

which can be written as:

$$\frac{\partial\phi}{\partial t} = \frac{i}{\hbar} H_1\phi = \varkappa H_1\phi = \varkappa H_2\phi = \varkappa\frac{H_1+H_2}{2}\phi$$
$$= \varkappa H\phi. \quad \left(\varkappa = \frac{i}{\hbar}\right) \tag{58}$$

where $H_1 \equiv \frac{c}{\varkappa}\left(\alpha_1\frac{\partial 0}{\partial x} + \alpha_2\frac{\partial 0}{\partial y} + \alpha_3\frac{\partial 0}{\partial z} + \varkappa\mu_0 c\,\alpha_4 0\right)$ (59)

and $H_2 \equiv \frac{c}{\varkappa}\left(\frac{\partial 0}{\partial x}\alpha_1 + \frac{\partial 0}{\partial y}\alpha_2 + \frac{\partial 0}{\partial z}\alpha_3 - \varkappa\mu_0 c\;0\,\alpha_4\right)$ (60)

and $H \equiv \frac{c}{\varkappa}\left[\frac{\alpha_1 0 + 0\alpha_1}{2}\frac{\partial}{\partial x} + \cdots + \cdots + \varkappa\mu_0 c\,(\alpha_4 0 - 0\alpha_4)\right]$ (61)

The operator F may for example be the momentum or energy operator, or composed of the operators α'_ρ , which may be put in the way in which the α-operators are introduced in equation (24). Let us assume that this operator F is independent of time, and consider our wave functions to be a system of plane waves $\Phi(\vec{k})$.

$$\therefore F_{\vec{k}'\vec{k}} = \int_D \Phi^*(\vec{k}')\frac{F}{W}\Phi(\vec{k})\,d\tau$$

$$\therefore \frac{\partial F_{\vec{k}'\vec{k}}}{\partial t} = \int_D\left[\frac{\partial\Phi^*(\vec{k}')}{\partial t}\frac{F}{W}\Phi(k) + \Phi^*(\vec{k}')\frac{F}{W}\frac{\partial\Phi(k)}{\partial t}\right]d\tau = 0 \tag{62}$$

Now:

$$\frac{\partial \Phi(\vec{k})}{\partial t} = \varkappa H \Phi(\vec{k}) \tag{63}$$

and

$$\frac{\partial \Phi^{\times}(\vec{k'})}{\partial t} = -\varkappa \left[H \Phi(\vec{k'}) \right]^{\times} \tag{64}$$

$\therefore$ Equation (62) becomes:

$$\frac{\partial F}{\partial t}_{k'k} = \int \left\{ -\varkappa [H \phi^{\times}(\vec{k'})]^{\times} \frac{F}{W} \phi(\vec{k}) + \phi^{\times}(\vec{k'}) \frac{F}{W} \varkappa H \phi(\vec{k}) \right\} d\tau = 0 \tag{65}$$

Since the operators H_1, H_2 and H are hermitian, we have from the definition of a hermitian operator:(30)

$$\int \phi^{\times} H \left(\frac{F}{W} \phi \right) d\tau = \int (H\phi)^{\times} \left(\frac{F}{W} \phi \right) d\tau \tag{66'}$$

$\therefore$ we have from equation (65):

$$\int \left\{ -\varkappa \Phi^{\times}(\vec{k'}) H \frac{F}{W} \Phi(\vec{k}) + \Phi^{\times}(\vec{k'}) \frac{F}{W} \varkappa H \phi(\vec{k}) \right\} d\tau = 0$$

i.e.

$$\int \left\{ \Phi^{\times}(\vec{k'}) [FH - HF] \frac{1}{W} \Phi(\vec{k}) \, d\tau = 0 \right. \tag{67}$$

If F is a constant of the motion, equation (67) is zero for every wave k and k'. This means that all the matrix elements belonging to the operator $[FH - HF] \equiv [F, H]$ are zero whatever wave functions $\Phi(\vec{k})$ and $\Phi(\vec{k'})$ are used. Thus the operator itself: $(FH - HF)$ must be zero.

$$\therefore \quad [F, H] = 0 \tag{68}$$

which is used for the derivation of the operators associated with the spin momentum.

Thus we see that the use of the operator $\frac{1}{W}$ for the purpose of normalization has simplified the formalism, more simply than the use of the operators A_4 and B_4 on p.(26).

CHAPTER IV

THE SECOND QUANTIZATION IN THE THEORY OF THE PHOTON: (37)

1. For electrons the theory of second quantization expresses their corpuscular character, i.e., the fact that we have always complete numbers of them. In order to prove the same fact for the photon we have to subject either one of the wave functions ψ_{ik} $(= \psi_{ik}^{\dagger})$ or ϕ_{ik} to second quantization. As both methods are equivalent we shall quantize the wave function ϕ_{ik}, from which we shall deduce the integral property for the photons associated with it. We shall proceed to develop the wave function ϕ_{ik} in terms of plane waves and then assume non-commutation relations between the coefficients and their conjugates, i.e. transform these coefficients into operators.

Let us consider a plane monochromatic wave whose components are expressed in terms of the four constants C_i. The normalization equation (40) can be expressed in the form:

$$\int \rho \, d\tau = 1 \qquad (69)$$

where ρ is given by:

$$\begin{aligned} \rho &= \frac{\mu_0 c}{\hbar} \phi^{*} \frac{1}{W} \phi \\ &= \frac{\mu_0 c}{\hbar} \frac{1}{k} \phi^{*} \phi \end{aligned} \qquad (70)$$

Since for the plane wave, we have:

$$\begin{aligned} \phi_{ik}^{*} \phi_{ik} &= a_{ik}^{*} a_{ik} = \frac{4k^2}{\Delta^2} \sum_{i=1}^{4} C_i^2 \\ \therefore \quad \rho &= \frac{\mu_0 c}{\hbar} \cdot \frac{4k}{\Delta^2} \sum_{i=1}^{4} |C_i|^2 . \end{aligned} \qquad (71)$$

where $\Delta = k + \frac{\mu_0 c}{\hbar}$

If we put:

$$b_i = \sqrt{\frac{4\mu_0 c k}{\hbar \Delta^2}} \cdot c_i \tag{72}$$

we get:

$$\rho = \sum_{i=1}^{4} b_i^{\times} b_i \tag{73}$$

These quantities, b_i ..., should be subjected to the commutation conditions. Since the photons obey the Bose-Einstein statistics, we must have:

$$\left[b_i^{\times}, b_j \right] = -\delta_{ij} \tag{74}$$

Other quantities commute with each other, i.e.

$$\left[b_i, b_j \right] = \left[b_i^{\times}, b_j^{\times} \right] = 0 \tag{75}$$

Equations (74) express the second quantization of the plane monochromatic wave:

$$\phi_{ik} = a_{ik} e^{i(kct - \vec{k}\cdot\vec{r})} \tag{76}$$

Consider the general case of the wave $\Phi(\vec{r}, t)$ formed by the superposition of plane monochromatic waves; the 16 components Φ_{ik} of Φ are expressed by Fourier integrals of the form:

$$\Phi_{ik}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int a_{ik}(\vec{k})\, e^{i(kct - \vec{k}\cdot\vec{r})} d\vec{k} \tag{77}$$

the $a_{ik}(\vec{k})$ are expressed linearly in terms of the four constants $c_i(\vec{k})$ and then we have the normalization formula:

$$N = \int \sum_{i=1}^{4} |b_i(\vec{k})|^2 d\vec{k} \tag{78}$$

the $b_i(\vec{k})$ are related to the $C_i(\vec{k})$ in the same way as above. It was shown in the second quantization of Dirac's theory that the two quantities $\frac{d\vec{k}}{k}$ and $k\delta(\vec{k}-\vec{k}')$ are relativistic invariants. From the above equation we notice that $b_i(\vec{k})\sqrt{k}$ is an invariant and thus we can put our commutation relations in the form:

$$\left[b_i^*(\vec{k}'), b_k(\vec{k})\right] = -\delta_{ik}\,\delta(\vec{k}-\vec{k}') \tag{79}$$

Thus:

$$\left[C_i^*(\vec{k}'), C_j(\vec{k})\right] = -\frac{\hbar\Delta^2}{4\mu_0 ck}\delta_{ik}\,\delta(\vec{k}-\vec{k}')$$

$$= -R\,\delta_{ik}\,\delta(\vec{k}-\vec{k}') \tag{80}$$

where R denotes the corresponding constant. Also we have:

c_j^*

$$\left[C_i(\vec{k}), C_j(\vec{k}')\right] = \left[C_i^*(\vec{k}'), C_j(\vec{k})\right] = 0 \tag{81}$$

2. Now, let us assume that the plane wave $\phi_{ik} = a_{ik}\,e^{i(kct-\vec{k}\cdot\vec{r})}$ represents N photons of positive energy, thus from our normalization formula:

$$\int \rho\, d\tau = N = \frac{\mu_0 c}{\hbar}\cdot\frac{4k}{\Delta^2}\sum_{i=1}^{4}|c_i|^2\int d\tau$$

$$= \sum_{i=1}^{4} b_i^* b_i \int d\tau = V\sum_{i=1}^{4} b_i^* b_i\,. \tag{82}$$

Denoting by n the number of particles per unit volume in the beam represented by the plane wave we get:

$$n = \frac{N}{V} = \sum_{i=1}^{4} b_i^* b_i \tag{83}$$

If we now define:

$$p_i = \sqrt{\frac{\hbar\nu}{2}}\,(b_i + b_i^*)\,, \qquad q_i = i\sqrt{\frac{\hbar}{2\nu}}\,(b_i - b_i^*) \tag{84}$$

then we get:

$$p_i q_j = i\frac{\hbar}{2}[b_i b_j - b_i b_j^{\times} + b_i^{\times} b_j - b_i^{\times} b_j^{\times}]$$

$$q_j p_i = i\frac{\hbar}{2}[b_j b_i - b_j^{\times} b_i + b_j b_i^{\times} - b_j^{\times} b_i^{\times}]$$

$$\therefore p_i q_j - q_j p_i = i\frac{\hbar}{2}\{[b_i, b_j] - [b_i, b_j^{\times}] + [b_i^{\times}, b_j] - [b_i^{\times}, b_j^{\times}]\}$$

$$= i\frac{\hbar}{2}\{0 - \delta_{ij} - \delta_{ij}\} = -i\hbar\,\delta_{ij} \qquad (85)$$

Also:

$$p_i^2 = \frac{\hbar\nu}{2}[b_i b_i + b_i b_i^{\times} + b_i^{\times} b_i + b_i^{\times} b_i^{\times}]$$

$$\nu^2 q_i^2 = -\frac{\hbar\nu}{2}[b_i b_i - b_i b_i^{\times} - b_i^{\times} b_i + b_i^{\times} b_i^{\times}]$$

$$\therefore \quad p_i^2 + \nu^2 q_i^2 = \hbar\nu(b_i^{\times} b_i + b_i b_i^{\times})$$

$$= \hbar\nu(2b_i^{\times} b_i + 1)$$ using the commutation relations.

(86)

Using equation (83), we get

$$p_i^2 + \nu^2 q_i^2 = \hbar\nu(2n+1)$$

$$\therefore \frac{1}{2}(p_i^2 + \nu^2 q_i^2) = (n + \frac{1}{2})\hbar\nu \qquad (87)$$

Has this any significance?

which shows that the quantities p_i and q_i, defined by (84) are the coordinate and momenta of a simple harmonic oscillator with unit mass and orbital frequency ν.

3. The commutation relations (80) and (81) will enable us to deduce the commutation relations for the electromagnetic field quantities. Let us consider a plane monochromatic wave (with positive energy) with the propagation vector k along the $o'z'$ axis of the coordinate system $x'y'z'$. Denoting by the symbol P the exponential $e^{i(kct - \vec{k}\cdot\vec{r})}$, we can deduce from

equations (p.41) the field quantities associated with the plane wave:

$$\phi_{ik} = a_{ik}\, e^{i(kct - \vec{k}\cdot\vec{r})} = a_{ik} P \tag{88}$$

as follows:

$$
\begin{aligned}
A_1 &= A_x = \frac{\sqrt{\mu_0 c^2}}{\Delta}(c_2 - c_1) P \\
A_2 &= A_y = i\frac{\sqrt{\mu_0 c^2}}{\Delta}(c_2 + c_1) P \\
A_3 &= A_z = 2\frac{k}{\Delta}\frac{\hbar}{\sqrt{\mu_0}}(c_3 + c_4) P \\
A_4 &= V = 2\frac{|\vec{k}|}{\Delta}\frac{\hbar}{\sqrt{\mu_0}}(c_3 + c_4) P
\end{aligned}
\tag{89}
$$

From these expressions we deduce the following commutation relations between the electromagnetic quantities associated with the above mentioned plane wave:

This needs some specification

$$[A^*_{i'}(\vec{k}'), A_{j'}(\vec{k})] = -\frac{\hbar c}{2k}\delta_{i'j'}\,\delta(\vec{k} - \vec{k}') \tag{90}$$

$$[A^*_{i'}(\vec{k}'), A_{j'}(\vec{k})] = 0, \quad i' = 1,2;\ j' = 3,4 \text{ or vice versa}$$

$$
\left.
\begin{aligned}
[A^*_{3'}(\vec{k}'), A_{3'}(\vec{k})] &= -\frac{\hbar^3}{2\mu_0^2 c}k\,\delta(\vec{k} - \vec{k}') \\
[A^*_{3'}(\vec{k}'), A'_4(\vec{k})] &= -\frac{\hbar^3}{2\mu_0^2 c}|\vec{k}|\,\delta(\vec{k} - \vec{k}') \\
[A^*_{4'}(\vec{k}'), A_{4'}(\vec{k})] &= -\frac{\hbar^3}{2\mu_0^2 c}\frac{|\vec{k}|^2}{k}\,\delta(\vec{k} - \vec{k}')
\end{aligned}
\right\}
\tag{91}
$$

The commutation relations (90) just as (91) are only valid in the coordinate system (x'y'z') where oz' coincides with the direction of the propagation vector $\vec{k}$. This is a special case and we must translate these equations into any system of rectangular coordinates (xyz).

Performing these transformations we get the following commutation relations which are valid in any coordinate system:

$$\left.\begin{aligned} \left[A_i^*(\vec{k}'), A_j(\vec{k})\right] &= -\frac{\hbar c}{2k}\left(\delta_{ik} + \frac{\hbar^2}{\mu_0^2 c^2} k_i k_j\right)\delta(\vec{k}-\vec{k}') \\ \left[A_i^*(\vec{k}'), A_4(\vec{k})\right] &= -\frac{\hbar^3}{2\mu_0^2 c} k_i\, \delta(\vec{k}-\vec{k}') \\ \left[A_4^*(\vec{k}'), A_4(\vec{k})\right] &= -\frac{\hbar^3}{2\mu_0^2 c}\frac{|\vec{k}|^2}{k}\,\delta(\vec{k}-\vec{k}') \end{aligned}\right\} \quad (92)$$

with $i, j = 1, 2, 3$.

These are the fundamental relations of commutation between the components of the four-vector potential in the quantum theory of the photon.

Using the fact that the two expressions:(39)

$$\frac{d\vec{k}}{k} \quad \text{and} \quad k'\,\delta(\vec{k}-\vec{k}')$$

are relativistically invariant, we can deduce the relativistic invariance of the commutation relations (92)

Now, starting from the commutation relations (92), it is easy to deduce the commutation relations between components of the potential and of the field or between two field components. As an example, let us consider the commutation relations between the components of $\vec{A}(\vec{k})$ and those of $\vec{E}(\vec{k})$.

For a plane wave, we have:

$$E_j = -\frac{1}{c}\frac{\partial A_j}{\partial t} - \frac{\partial V}{\partial x_j} = -ik A_j + ik_j V \qquad (93)$$

Multiplying the first equation of (92) by $-ik$ and the second by ik_j and adding we get:

$$[A_i^*(\vec{k}'), E_j(\vec{k})] = -\frac{\hbar c}{2i}\delta_{ij}\,\delta(\vec{k}-\vec{k}') \qquad (94)$$

The commutation relations (92) and (94) are expressed between the "spectral" components of the field, i.e., between the coefficients of the Fourier development. We can easily deduce the commutation relations between the local (spatial) components of the electromagnetic fields at each point.

Consider the Fourier developments of the electromagnetic quantities in the form:

$$A_i(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int A_i(\vec{k})\, e^{i(kct-\vec{k}\cdot\vec{r})}\, d\vec{k} \qquad (95)$$

and similar expressions for the other quantities, we find that:

$$\left[A_i^{*}(\vec{r}\,'), A_4(\vec{r})\right] = \frac{1}{8\pi^3}\int d\vec{k}'\int d\vec{k}\left[A_i^{*}(\vec{k}'), A_4(\vec{k})\right] e^{i[(k-k')ct-(\vec{k}\cdot\vec{r}-\vec{k}'\cdot\vec{r}\,')]}$$

$$= -\frac{\hbar^3}{2\mu_0^2 c}\cdot\frac{1}{8\pi^3}\int d\vec{k}\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, k_i$$

$$= \frac{\hbar^3}{2i\mu_0^2 c}\cdot\frac{1}{8\pi^3}\frac{\partial}{\partial x_i}\int e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, d\vec{k}$$

$$= \frac{\hbar^3}{2i\mu_0^2 c}\frac{\partial}{\partial x_i}\,\delta(\vec{r}-\vec{r}\,') \tag{96}$$

Similarly:

$$\left[A_i^{*}(\vec{r}\,'); E_j(\vec{r})\right] = \frac{1}{8\pi^3}\int d\vec{k}'\int d\vec{k}\left[A_i^{*}(\vec{k}'), E_j(\vec{k})\right] e^{i[(k-k')ct-(\vec{k}\cdot\vec{r}-\vec{k}'\cdot\vec{r}\,')]}$$

$$= -\frac{\hbar c}{2i}\,\delta_{ij}\,\frac{1}{8\pi^3}\int d\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}$$

$$= -\frac{\hbar c}{2i}\,\delta_{ij}\,\delta(\vec{r}-\vec{r}\,') \tag{97}$$

Similarly for all the other commutation relations. Now, let us apply to the relation (97) the operation $\sum_{j=1}^{3}\frac{\partial}{\partial x_j}$, we get:

$$\left[A_i^{*}(\vec{r}\,'), \operatorname{div}\vec{E}(\vec{r})\right] = -\frac{\hbar c}{2i}\frac{\partial}{\partial x_i}\,\delta(\vec{r}-\vec{r}\,') \tag{98}$$

In the quantum theory of the electromagnetic field of Heisenberg and Pauli, a relation similar to (98) was deduced. According to Maxwell's theory, we have $\operatorname{div}\vec{E}(\vec{r})=0$ in the absence of charges, from which equation (98) leads to a contradiction, as it becomes:

$$[A_i^\times(\vec{r}'),\ 0] = -\frac{\hbar c}{2i}\frac{\partial}{\partial x_i}\delta(\vec{r}-\vec{r}') \tag{99}$$

Heisenberg and Pauli tried to solve this difficulty by introducing, in the classical Maxwellian equations, a very small term, in which case Maxwell's equations become the limiting form of the equations containing the supplementary term. This is exactly what we have found in de Broglie's theory of the photon, which adds to Maxwell's equations a small term proportional to μ_0^2 i.e. $\sim (10^{-44})^2$. At the limit when $\mu_0 \to 0$ we get Maxwell's equations.

In de Broglie's theory of the photen (i.e. with rest mass) we have:

$$\operatorname{div}\vec{E} = \varkappa^2\mu_0^2c^2V = -\frac{\mu_0^2c^2}{\hbar^2}A_4 \tag{100}$$

Thus equation (98) becomes:

$$[A_i^\times(\vec{r}'), A_4(\vec{r})] = \frac{\hbar^3}{2\mu_0^2 c i}\cdot\frac{\partial}{\partial x_i}\cdot\delta(\vec{r}-\vec{r}') \tag{101}$$

which is the same as equation (96) found above. This is a remarkable point in favour of the present theory of attributing a small rest mass to the photon.

Finally let us deduce the commutation relations between the components of $\vec{E}$ and $\vec{H}$. We have:

$$H_i(\vec{r}) = \frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} \tag{102}$$

Thus from the relations (97) we deduce that:

$$[H_i^\wedge(\vec{r}'), E_j(\vec{r})] = \frac{\hbar c}{2i}\left(\frac{\partial}{\partial x_l}\delta_{kj} - \frac{\partial}{\partial x_k}\delta_{lj}\right)\delta(\vec{r}-\vec{r}') \tag{103}$$

where i, j, k denote permutations of the numbers 1,2,3.

THE SECOND QUANTIZATION AND THE RADIATION INTERACTION.

In the theory of radiation interaction developed in Part 1, we have considered the interaction between a single electron and one photon. As photons obey both Einstein statistics, we can have more than one photon in one state and thus this must be taken into consideration in the theory of interaction. Thus we have to refer to the theory of second quantization. This can be done, as in the earlier theories, by studying the evolution with respect to time of the distribution function $R(n_0, n_1, n_2, \cdots, t)$ of the photons among the different possible states (n_0 being the number of photons in the state of annihilation, n_ℓ in the state ℓ.)

The results deduced by this method are exactly similar to those deduced by the above mentioned process, if we take into consideration the rule:

"That each transition probability must be multiplied by the product of the number of photons in the initial state before the transition and of the number of photons in the final state after the transition."

Let us consider the absorption transition:

$$(\ell, m) \longrightarrow (0, m'),$$

in which the photon in the state ℓ is absorbed with the state of the electron changing from m to m^1.

The transition probability, given by equation (208) on page (52), has to be multiplied by $n_\ell(n_0+1) = n_\ell \cdot n_0$ as the number of photons in the annihilation state is enormously great.

For the emission transition $(0, m') \rightarrow (\ell, m)$, we have to multiply our expression for the transition probability given by equation (207) on page (52) by $n_0(n_\ell+1)$

Thus we get the well-known formula:

$$\frac{\text{Emission probability}}{\text{Absorption probability}} = \frac{n_\ell + 1}{n_\ell} \qquad (104)$$

The above considerations must be taken into account in the calculations of the probability transitions of the different radiation processes.

Discussions and Summary.

The present work is based essentially on the particular form of de Broglie's equations of the photon:

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial \phi_{ik}}{\partial t} &= \left(\frac{\partial}{\partial x}A_1 + \frac{\partial}{\partial y}A_2 + \frac{\partial}{\partial z}A_3 + \varkappa\mu_0 c A_4\right)\phi_{ik} \\ \frac{1}{c}\frac{\partial \phi_{ik}}{\partial t} &= \left(\frac{\partial}{\partial x}B_1 + \frac{\partial}{\partial y}B_2 + \frac{\partial}{\partial z}B_3 + \varkappa\mu_0 c B_4\right)\phi_{ik} \end{aligned}\right\} \quad (1)$$

$$(i, k = 1, 2, 3, 4)$$

which can be written as:

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = \alpha_1\frac{\partial \phi}{\partial x} + \alpha_2\frac{\partial \phi}{\partial y} + \alpha_3\frac{\partial \phi}{\partial z} + \varkappa\mu_0 c\,\alpha_4\phi. \quad (2)$$

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\alpha_1 + \frac{\partial \phi}{\partial y}\alpha_2 + \frac{\partial \phi}{\partial z}\alpha_3 - \varkappa\mu_0 c\,\phi\alpha_4. \quad (3)$$

Taking their adjoint we get:

$$\frac{1}{c}\frac{\partial \phi^\dagger}{\partial t} = \frac{\partial \phi^\dagger}{\partial x}\alpha_1 + \frac{\partial \phi^\dagger}{\partial y}\alpha_2 + \frac{\partial \phi^\dagger}{\partial z}\alpha_3 - \varkappa\mu_0 c\,\phi^\dagger\alpha_4 \quad (4)$$

$$\frac{1}{c}\frac{\partial \phi^\dagger}{\partial t} = \alpha_1\frac{\partial \phi^\dagger}{\partial x} + \alpha_2\frac{\partial \phi^\dagger}{\partial y} + \alpha_3\frac{\partial \phi^\dagger}{\partial z} + \varkappa\mu_0 c\,\alpha_4\phi \quad (5)$$

Of these two systems of equations (2,3) and (4,5) we notice the following points:

1. Equation (2) is the same as (5) apart from the fact that ϕ is replaced by $\phi^\dagger$. Also (3) is the same as (4). This shows that if the system (2,3) describes a particle with positive energy and momentum, the system (4,5) will describe another with both negative. Each of these systems has the same Hamiltonian:

$$\mathcal{H} = \frac{c}{\varkappa}\left[\frac{\alpha_1 0 + 0\alpha_1}{2}\frac{\partial}{\partial x} + \cdots + \cdots + \varkappa\mu_0 c\,(\alpha_4 0 - 0\alpha_4)\right] \quad (6)$$

2. Adding (2) and (5), (3) and (4) we get two

equations of which the second is the hermitian conjugate of the first which runs as follows:

$$\frac{1}{c}\frac{\partial \psi}{\partial t} = \alpha_1 \frac{\partial \psi}{\partial x} + \alpha_2 \frac{\partial \psi}{\partial y} + \alpha_3 \frac{\partial \psi}{\partial z} + \varkappa \mu_0 c \alpha_4 \psi \tag{7}$$

$$\text{where} \quad \psi = \phi + \phi^{\dagger} \tag{8}$$

As we noticed above the definitions of the electromagnetic quantities by means of (8) gave rise to real expressions of the field quantities, which are really the superposition of two quantities, the importance of which has been separately realized in the theory of radiation.

Although de Broglie's description of the electromagnetic quantities as densities of matrix elements for the annihilation phenomena seems attractive, we have preferred to consider the fields as related directly to the wave function. Although mathematically both methods are the same, the latter interpretation has the advantage of avoiding difficulties concerned with the function of annihilation and moreover it brings the theory nearer to the current meson theories, where the field quantities (vector and others) are considered as wave functions.

The definition of the electromagnetic stress tensor has enabled us to introduce the operator $\frac{1}{W}$, similar to those operators suggested by Pauli, Frenkel, Landau-Peierls, for the definition of the normalization formula. It has also simplified the formalism of the theory.

The requirements which should be satisfied by an expression for the probability density are:(40)

1. It should transform like the fourth component of a vector.

How nice!

2. It should, together with the expression for the probability current density, satisfy a continuity equator.

3. It should be positive definite.

These were the conditions for the existence of a probability density, which were imposed during the developments of the quantum theory.

Gordon's expression for the probability density:

$$\frac{1}{4\pi i}\left(\frac{\partial \psi^{*}}{\partial x_0}\psi - \psi^{*}\frac{\partial \psi}{\partial x_0}\right) \qquad (9)$$

satisfies the conditions (1) and (2) but not (3). Its imployment would result in having negative probability.(41) Gordon's theory itself, moreover, allows of negative values for the energy p_0 as well as positive values.

Dirac discovered the first order wave equations of the electron, in which $\sum_i |\psi_i(x)|^2$, summed for the components of ψ_i, turns out to be the time component of a four-vector and further the divergence of this form-vector vanishes. Thus it is satisfactory to use this expression as the probability per unit volume of the particle being at any place at any time. One does not now have any negative probabilities in the theory. However, the negative energies remain.

In all recent theories of the photon, expressions for the probability density are not positive definite. As an example de Broglie's expression for the probability density for a plane wave:

$$\rho = \sum_{ik} \phi^*_{ik} \frac{A_4 + B_4}{2} \phi_{ik} = \sum_{i=1}^{4} \frac{4k\mu_0 c}{\hbar \Delta^2} |C_i|^2 \qquad (10)$$

which depends on the sign of the energy k. Similarly also in the present work, the value of the same expression is:

$$\rho = \frac{\mu_0 c}{\hbar} \sum_{ik} \phi^*_{ik} \frac{1}{W} \phi_{ik} = \frac{\mu_0 c}{\hbar} \frac{1}{k} \frac{4k^2}{\Delta^2} \sum_{i=1}^{4} C_i^2 = \frac{\mu_0 c}{\hbar} \cdot \frac{4k}{\Delta^2} \sum_{i=1}^{4} C_i^2 \qquad (11)$$

which is exactly the same as de Broglie's.

In his new theory of quantum electrodynamics, Dirac said:[42]

"There are always states of negative energy as well as those of positive energy. For particles whose spin is an integral number of quanta, the negative energy states occur with negative probability and the positive energy ones with a positive probability , while for particles whose spin is a half-odd integral number of quanta, all states occur with a positive probability."

"Negative energies and probabilities should not be considered as nonsense. They are well defined concepts mathematically. Thus negative energies and probabilities should be considered simply as things which do not appear in experimental results."

He concluded by writing that:

"It appears that, whether one is dealing with particles of integral spin or half-odd spin, one is led to a similar conclusion, namely, that the mathematical methods at present in use in quantum mechanics are capable of direct interpretation only in terms of a hypothetical world differing very markedly from the actual one. These mathematical methods can be made into a physical theory by the assumption that results about collision processes (including radiation interactions) are the same for the hypothetical world as the actual one. One thus gets back to Heisenberg's view about physical theory - that all it does is to provide a consistent means of calculating experimental results".

APPENDIX I.

We have noticed in Part II of this thesis that the positive energy state ϕ of the photon is described by the wave equations:

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \alpha_1\frac{\partial\phi}{\partial x} + \alpha_2\frac{\partial\phi}{\partial y} + \alpha_3\frac{\partial\phi}{\partial z} + \varkappa\mu_0 c\,\alpha_4\phi. \qquad (1)$$

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial x}\alpha_1 + \frac{\partial\phi}{\partial y}\alpha_2 + \frac{\partial\phi}{\partial z}\alpha_3 - \varkappa\mu_0 c\,\phi\alpha_4. \qquad (2)$$

which is in reality the same as de Broglie's wave equations:

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = A_1\frac{\partial\phi}{\partial x} + A_2\frac{\partial\phi}{\partial y} + A_3\frac{\partial\phi}{\partial z} + \varkappa\mu_0 c A_4\phi. \qquad (3)$$

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = B_1\frac{\partial\phi}{\partial x} + B_2\frac{\partial\phi}{\partial y} + B_3\frac{\partial\phi}{\partial z} + \varkappa\mu_0 c B_4\phi. \qquad (4)$$

As was mentioned before, De Broglie noticed that the expressions:

$$\rho = \sum_{ik}\phi_{ik}^{*}\phi_{ik}, \quad \rho\vec{u} = -c\sum_{ik}\phi_{ik}^{*}\cdot\frac{\vec{A}+\vec{B}}{2}\phi_{ik} \qquad (5)$$

do not form four vectors as regards Lorentz transformations, and thus cannot be used to represent the probability current-density vector. De Broglie suggested that the vector:

$$\rho = \sum_{ik}\phi_{ik}^{*}\frac{A_4+B_4}{2}\phi_{ik}, \quad \rho\vec{u} = -c\sum_{ik}\phi_{ik}^{*}\frac{B_4\vec{A}+A_4\vec{B}}{2}\phi_{ik} \qquad (6)$$

can be used as the probability current density vector in the theory of the photon.

For our work we preferred to use the operator W (32) defined by:

$$W = \sqrt{\chi^2 - \Delta} \tag{7}$$

$$\frac{1}{W^n} e^{i\vec{k}\vec{x}} = \frac{e^{i\vec{k}\vec{x}}}{k_0^n} \quad \text{with} \quad k_0 = +\sqrt{\chi^2 + \vec{k}^2} \tag{8}$$

where $$\chi = \frac{\mu_0 c}{\hbar} \tag{9}$$

when $\mu_0 = 0$, the operator W goes over into Landau-Peierls' operator $\sqrt{-\Delta}$.

The use of the operator W for the definition of the probability density has simplified the formalism of the theory.

We shall now show that an operator similar to (6) can be introduced in our work, which can be used to make the above theory run parallel to de Broglie's work. It is clear that both definitions lead to the same results as can be seen from equations (44) and (179) on p.(22) and page (46).

Starting with de Broglie's expressions we notice that they can be put in the form:

$$\begin{aligned}
\rho &= \sum_{ik} \phi_{ik}^{*} \frac{A_4 + B_4}{2} \phi_{ik} = \frac{1}{2} \phi_{ik}^{*} \left[(A_4)_{ik,lm} \phi_{lm} + (B_4)_{ik,lm} \phi_{lm} \right] \\
&= \sum \frac{1}{2} \phi_{ik}^{*} \left[(\alpha_4)_{il} \delta_{km} \phi_{lm} - (\alpha_4)_{km} \delta_{il} \phi_{lm} \right] = \frac{1}{2} \phi_{ik}^{*} \left[(\alpha_4)_{il} \phi_{lk} - (\alpha_4)_{km} \phi_{im} \right] \\
&= \frac{1}{2} \phi_{ik}^{*} \left[(\alpha_4 \phi)_{ik} - (\phi \alpha_4)_{ik} \right] = \phi_{ik}^{*} \frac{\alpha_4 O - O \alpha_4}{2} \phi_{ik} \\
&= \phi_{ki}^{*} \cdot \frac{\alpha_4 O - O \alpha_4}{2} \phi_{ik} = \phi^{\dagger} : \frac{\alpha_4 O - O \alpha_4}{2} \phi
\end{aligned} \tag{10}$$

$$f_1 = -c\sum_{ik}\phi^*_{ik}\frac{B_4A_1+A_4B_1}{2}\phi_{ik}$$
$$= -\frac{c}{2}\phi^*_{ik}\left[(B_4A_1)_{ik,lm}\phi_{lm} + (A_4B_1)_{ik,np}\phi_{np}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[(B_4)_{ik,rs}(A_1)_{rs,lm}\phi_{lm} + (A_4)_{ik,rs}(B_1)_{rs,np}\phi_{np}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[-(\alpha_4)_{ks}\delta_{ir}(\alpha_1)_{rl}\delta_{sm}\phi_{lm} + (\alpha_4)_{ir}\delta_{ks}(\alpha_1)_{sp}\delta_{rn}\phi_{np}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[-(\alpha_4)_{ks}(\alpha_1)_{il}\phi_{ls} + (\alpha_4)_{ir}(\alpha_1)_{kp}\phi_{rp}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[-(\alpha_1\phi\alpha_4)_{ik} + (\alpha_4\phi\alpha_1)_{ik}\right] = -c\phi^*_{ik}\frac{\alpha_4\circ\alpha_1-\alpha_1\circ\alpha_4}{2}\phi_{ik} \quad (11)$$

$$f_2 = -c\sum\phi^*_{ik}\frac{B_4A_2+A_4B_2}{2}\phi_{ik}$$
$$= -\frac{c}{2}\phi^*_{ik}\left[(B_4)_{ik,pr}(A_2)_{pr,lm}\phi_{lm} + (A_4)_{ik,pr}(B_2)_{pr,lm}\phi_{lm}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[-(\alpha_4)_{kr}\delta_{ip}(\alpha_2)_{pl}\delta_{rm}\phi_{lm} - (\alpha_4)_{ip}\delta_{kn}(\alpha_2)_{nm}\delta_{pl}\phi_{lm}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[-(\alpha_4)_{kr}(\alpha_2)_{il}\phi_{lr} - (\alpha_4)_{ip}(\alpha_2)_{km}\phi_{pm}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[(\alpha_4\phi\alpha_2)_{ik} - (\alpha_2\phi\alpha_4)_{ik}\right] = -c\sum\phi^*_{ik}\frac{\alpha_4\circ\alpha_2-\alpha_2\circ\alpha_4}{2}\phi_{ik} \quad (12)$$

$$f_3 = -\frac{c}{2}\phi^*_{ik}\left[B_4A_3+A_4B_3\right]\phi_{ik}$$
$$= -\frac{c}{2}\phi^*_{ik}\left[(B_4)_{ik,lm}(A_3)_{lm,ns}\phi_{ns} + (A_4)_{ik,lm}(B_3)_{lm,ns}\phi_{ns}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[-(\alpha_4)_{km}\delta_{il}(\alpha_3)_{ln}\delta_{ms}\phi_{ns} + (\alpha_4)_{il}\delta_{km}(\alpha_3)_{ms}\delta_{ln}\phi_{ns}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[-(\alpha_4)_{km}(\alpha_3)_{in}\phi_{nm} + (\alpha_4)_{il}(\alpha_3)_{ks}\phi_{ls}\right]$$
$$= -\frac{c}{2}\phi^*_{ik}\left[-(\alpha_3\phi\alpha_4)_{ik} + (\alpha_4\phi\alpha_3)_{ik}\right] = -c\phi^*_{ik}\frac{\alpha_4\circ\alpha_3-\alpha_3\circ\alpha_4}{2}\phi_{ik} \quad (13)$$

Using the fact that $\alpha_1, \alpha_3, \alpha_4$ are symmetrical and α_2 antisymmetrical operators.

Thus we can define, in the new notation, the probability current density operators, as:

$$(\iota, \vec{u}) \equiv \left(\frac{\alpha_4 0 - 0 \alpha_4}{2} , -c \frac{\alpha_4 0 \vec{\alpha} - \vec{\alpha} 0 \alpha_4}{2} \right) \qquad (14)$$

and the density expressions can be put as:

$$\phi^{\dagger} : \iota \phi \quad , \quad \phi^{\dagger} : u \phi \qquad (15)$$

To show that the expressions (15) satisfy the continuity relations, let us write down the wave equations and their adjoints:

$$\left.\begin{array}{l}(a)\ \frac{1}{c}\frac{\partial \phi}{\partial t} = \alpha_1 \frac{\partial \phi}{\partial x} + \alpha_2 \frac{\partial \phi}{\partial y} + \alpha_3 \frac{\partial \phi}{\partial z} + \varkappa \mu_0 c \alpha_4 \phi \\ (b)\ \frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\alpha_1 + \frac{\partial \phi}{\partial y}\alpha_2 + \frac{\partial \phi}{\partial z}\alpha_3 - \varkappa \mu_0 c \phi \alpha_4\end{array}\right\} \qquad (16)$$

$$\left.\begin{array}{l}(a)\ \frac{1}{c}\frac{\partial \phi^{\dagger}}{\partial t} = \alpha_1 \frac{\partial \phi^{\dagger}}{\partial x} + \alpha_2 \frac{\partial \phi^{\dagger}}{\partial y} + \alpha_3 \frac{\partial \phi^{\dagger}}{\partial z} + \varkappa \mu_0 c \alpha_4 \phi^{\dagger} \\ (b)\ \frac{1}{c}\frac{\partial \phi^{\dagger}}{\partial t} = \frac{\partial \phi^{\dagger}}{\partial x}\alpha_1 + \frac{\partial \phi^{\dagger}}{\partial y}\alpha_2 + \frac{\partial \phi^{\dagger}}{\partial z}\alpha_3 - \varkappa \mu_0 c \phi^{\dagger} \alpha_4\end{array}\right\} \qquad (17)$$

Multiplying (16a) by $\phi^{\dagger}:$ from left and by α_4 from right we get:

$$\phi^{\dagger} : \frac{\partial \phi}{\partial t}\frac{\alpha_4}{c} = \phi^{\dagger} : \alpha_1 \frac{\partial \phi}{\partial x}\alpha_4 + \phi^{\dagger} : \alpha_2 \frac{\partial \phi}{\partial y}\alpha_4 + \phi^{\dagger} : \alpha_3 \frac{\partial \phi}{\partial z}\alpha_4 + \varkappa \mu_0 c \phi^{\dagger} \alpha_4 : \phi \alpha_4 \qquad (18)$$

Multiplying (16b) by α_4 from left and $:\phi^{\dagger}$ from right, we get:

$$\frac{\alpha_4}{c}\frac{\partial \phi}{\partial t} : \phi^{\dagger} = \alpha_4 \frac{\partial \phi}{\partial x}\alpha_1 : \phi^{\dagger} + \alpha_4 \frac{\partial \phi}{\partial y}\alpha_2 : \phi^{\dagger} + \alpha_4 \frac{\partial \phi}{\partial z}\alpha_3 : \phi^{\dagger} - \varkappa \mu_0 c \alpha_4 \phi \alpha_4 : \phi^{\dagger} \qquad (19)$$

Subtracting (18) and (19) we get:

$$\frac{1}{c}\phi^{\dagger}\frac{\alpha_4 0 - 0 \alpha_4}{2}\frac{\partial \phi}{\partial t} = \phi^{\dagger}\frac{\alpha_4 0 \alpha_1 - \alpha_1 0 \alpha_4}{2}\frac{\partial \phi}{\partial x} + \cdots + \cdots - \varkappa \mu_0 c \alpha_4 \phi : \alpha_4 \phi^{\dagger} \qquad (20)$$

Multiplying (17a) from right by $:\alpha_4 \phi$ we get:

$$\frac{1}{c}\frac{\partial \phi^{\dagger}}{\partial t} : \alpha_4 \phi = \frac{\partial \phi^{\dagger}}{\partial x} : \alpha_4 \phi \alpha_1 + \frac{\partial \phi^{\dagger}}{\partial y} : \alpha_4 \phi \alpha_2 + \frac{\partial \phi^{\dagger}}{\partial z} : \alpha_4 \phi \alpha_3 + \varkappa \mu_0 c \alpha_4 \phi^{\dagger} : \alpha_4 \phi \qquad (21)$$

Multiplying (17b) from right by : $\phi\alpha_4$ we get:

$$\frac{1}{c}\frac{\partial\phi^{+}}{\partial t}:\phi\alpha_4 = \frac{\partial\phi^{+}}{\partial x}:\alpha_1\phi\alpha_4 + \frac{\partial\phi^{+}}{\partial y}:\alpha_2\phi\alpha_4 + \frac{\partial\phi^{+}}{\partial z}:\alpha_3\phi\alpha_4 - \varkappa\mu_0 c\alpha_4\phi^{+}:\alpha_4\phi \quad (22)$$

Subtracting (22) from (21) we get:

$$\frac{1}{c}\frac{\partial\phi^{+}}{\partial t}\frac{\alpha_4 0 - 0\alpha_4}{2}\phi = \frac{\partial\phi^{+}}{\partial x}\frac{\alpha_4 0\alpha_1 - \alpha_1 0\alpha_4}{2}\phi + \frac{\partial\phi^{+}}{\partial y}\frac{\alpha_4 0\alpha_2 - \alpha_2 0\alpha_4}{2}\phi$$
$$+ \frac{\partial\phi^{+}}{\partial z}\frac{\alpha_4 0\alpha_3 - \alpha_3 0\alpha_4}{2}\phi + \varkappa\mu_0 c\,\alpha_4\phi^{+}:\alpha_4\phi. \quad (23)$$

Adding (20) and (23) we get:

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\phi^{+}:\frac{\alpha_4 0 - 0\alpha_4}{2}\phi\right) = \frac{\partial}{\partial x}\left(\phi^{+}:\frac{\alpha_4 0\alpha_1 - \alpha_1 0\alpha_4}{2}\phi\right)$$
$$+\frac{\partial}{\partial y}\left(\phi^{+}:\frac{\alpha_4 0\alpha_2 - \alpha_2 0\alpha_4}{2}\phi\right) + \frac{\partial}{\partial z}\left(\phi^{+}:\frac{\alpha_4 0\alpha_3 - \alpha_3 0\alpha_4}{2}\phi\right)$$

which is the required conservation formula.

The similarity of the new operators ($0\vec{\alpha}$) to the $\vec{B}$ introduced by de Broglie is confirmed by the form of the probability-current density (14).

Similarly we can go on the same lines building the formalism of the theory.

As an illustration we shall construct the corresponding energy-momentum tensor in the following way:

$$T_{ij} = T_{ji} = \frac{1}{4}\frac{\hbar c}{i}\Big[\phi^{\times}\frac{\alpha_4 0\alpha_i - \alpha_i 0\alpha_4}{2}\frac{\partial\phi}{\partial x_j} - \frac{\partial\phi^{\times}}{\partial x_j}\frac{\alpha_4 0\alpha_i - \alpha_i 0\alpha_4}{2}\phi$$
$$+\phi^{\times}\frac{\alpha_4 0\alpha_j - \alpha_j 0\alpha_4}{2}\frac{\partial\phi}{\partial x_i} - \frac{\partial\phi^{\times}}{\partial x_i}\frac{\alpha_4 0\alpha_j - \alpha_j 0\alpha_4}{2}\phi\Big]$$
$$T_{i4} = T_{4i} = -\frac{1}{4}\frac{\hbar c}{i}\Big[\phi^{\times}\frac{\alpha_4 0 - 0\alpha_4}{2}\frac{\partial\phi}{\partial x_i} - \frac{\partial\phi^{\times}}{\partial x_i}\frac{\alpha_4 0 - 0\alpha_4}{2}\phi \tag{24}$$
$$+\phi^{\times}\frac{\alpha_4 0\alpha_i - \alpha_i 0\alpha_4}{2}\frac{\partial\phi}{\partial x_4} - \frac{\partial\phi^{\times}}{\partial x_4}\frac{\alpha_4 0\alpha_i - \alpha_i 0\alpha_4}{2}\phi\Big]$$
$$T_{44} = \frac{1}{2}\frac{\hbar c}{i}\Big[\phi^{\times}\frac{\alpha_4 0 - 0\alpha_4}{2}\frac{\partial\phi}{\partial x_4} - \frac{\partial\phi^{\times}}{\partial x_4}\cdot\frac{\alpha_4 0 - 0\alpha_4}{2}\phi\Big]$$
with $x_4 = ct$ and $i = 1, 2, 3$.

These expressions can be compared with those of de Broglie given on p. (71).

The equivalence of this value of T_{44} to the one given before by equation (49) on p. (72) is clear and can be easily checked for the plane wave:

$$\phi_{ik} = a_{ik}\, e^{i(kx_4 - \vec{k}\cdot\vec{r})}$$

In this case, from the above definition of T_{44} we get:

$$T_{44} = \hbar c k\left(\phi^{\times}\frac{\alpha_4 0 - 0\alpha_4}{2}\phi\right) = \hbar c k\big[|\phi_{31}|^2 + |\phi_{32}|^2 + |\phi_{41}|^2 + |\phi_{42}|^2$$
$$-|\phi_{13}|^2 - |\phi_{14}|^2 - |\phi_{23}|^2 - |\phi_{24}|^2\big] = \mu_0 c^2\,\frac{4k^2}{\Delta^2}\sum_{i=1}^{4}|C_i|^2 \tag{25}$$

The value of T_{44} as given by equation (54) on p. (73) is:

$$T_{44} = \mu_0 c^2\, a^{\times}_{ik} a_{ik} = \mu_0 c^2\cdot\frac{4k^2}{\Delta^2}\sum_{i=1}^{4}|C_i|^2 \tag{26}$$

which is the same as (25)

II.

A striking similarity exists between the work developed here in the second part for the photon and the work published a few years ago by Kemmer (16) on the particle aspect of the meson theory, using the well-known Kemmer β_μ matrices.

A - Kemmer has defined two energy momentum tensors for the meson $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ exactly similar to the tensors $T_{\mu\nu}$ and $M_{\mu\nu}$ introduced above. μ and ν run from 1 to 4. These tensors are defined as:

$$T_{\mu\nu} = \frac{c}{2}\left[\psi^{+}\beta_\nu \frac{h}{i}\frac{\partial}{\partial x_\mu}\psi - \left(\frac{h}{i}\frac{\partial}{\partial x_\mu}\psi^{+}\right)\beta_\nu\psi\right] \qquad (1)$$

the second term being added in order to make $T_{\mu\nu}$ real. This tensor (1) is not symmetrical, but it can be made so, as in the usual case for the electron given by Tetrode.

The other energy-momentum tensor for the meson runs as:

$$\Theta_{\mu\nu} = \frac{-mc^2}{i}\left[\psi^{+}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu) - \delta_{\mu\nu}\psi^{+}\psi\right] \qquad (2)$$

The tensor (2) is symmetrical and it gives for the energy density:

$$\Theta_{44} = -mc^2\psi^{*}\psi \qquad (3)$$

which is the same as the expression for the Maxwellian energy tensor M_{44} in the theory of the photon.

B - As a result of the existence of the matrix (2), we can get expressions for the expectation values of energy and momentum, which are different from the usual ones, similar to those introduced for the electron. As an example, the momentum which is defined by the formula

$$\overline{p}_\mu = \frac{1}{i} \int \psi^+ \beta_4 \frac{\hbar}{i} \frac{\partial}{\partial x_\mu} \psi \, d\tau \qquad (4)$$

can be given by the formula:

$$\overline{p}_\mu = \frac{1}{ic} \int (17)_{\mu 4} \, d\tau \qquad (5)$$

Kemmer remarked that from the two equations (4), (5) we see that it is not necessary to abandon the connection between the operators $\frac{\hbar}{i} \frac{\partial}{\partial x_\mu}$ and the momenta, as seemed to be the case in previous presentations of the meson theory.

In the theory of the photon the operator $\frac{\hbar}{i} \frac{\partial}{\partial x_\mu}$ could not represent momentum when the photon is represented by the Hermitian wave function $\psi_{i,k}$. From (5) we get for the energy density the expression:

$$(17)_{44} = -mc^2 \psi^\times \psi . \qquad (6)$$

III.

A- Using the Lorentz-Heaviside units, the energy density in Maxwell's theory is given by:

But these units were not used previously!

$$D = \frac{1}{2}(\vec{E}^2 + \vec{H}^2) \tag{1}$$

where $\vec{E}$ and $\vec{H}$ are real quantities.

Now assuming $\vec{E}$ and $\vec{H}$ to be complex, the expression (1) becomes:

$$D = |\vec{E}|^2 + |\vec{H}|^2 \tag{2}$$

(2) is the corresponding expression which is constant with time.

Substituting for $\vec{E}$, $\vec{H}$ and the other corresponding quantities in the expression for the energy density from Gétériau's equations we get:

$$D = \frac{4K^2\mu_0^2c^2}{\hbar^2}\phi^{\times}\cdot\phi \tag{3}$$

where K is the constant of proportionality between the field quantities and the components ϕ_{ik}

Comparing the value (3) with the corresponding value given by the component M_{44} of the Maxwellian tensor:

$$M_{44} = \mu_0 c^2 \phi^{\times}\cdot\phi \tag{4}$$

?

[see equation (36) on page], we deduce:

$$K = \frac{\hbar}{2\sqrt{\mu_0}} \tag{5}$$

which was assumed above.

B- For the case, when the hermitian function is used, we have:

$$M_{44} = \mu_0 c^2 \psi^{\dagger} : \psi = 2\mu_0 c^2 \phi^{\times} \cdot \phi \qquad (6)$$

neglecting the two oscillating terms

(6) shows us that M_{44} for the real fields is equal to the energy density for both the states ϕ and $\phi^{\dagger}$. Both states have the same value of the energy, as:

$\phi^{\times} \cdot \phi = \frac{4k^2}{\Delta^2} \sum_{i=1}^{4} C_i^2$ (for a plane wave); which is quadratic in k, and thus positive definite.

Thus we must put:

$$\mu_0 c^2 \int \psi^{\dagger} : \psi \, d\tau = 2\mu_0 c^2 \int \phi^{\times} \cdot \phi \, d\tau = 2h\nu \qquad (7)$$

from eqn. (162) page 43 with $K = \frac{\hbar}{2\sqrt{\mu_0}}$ follows

$$M_{44} = \frac{\mu_0 c^2}{2} (\psi : \psi)$$

Thus the above consideration is not correct.

IV.

We would like to mention here explicitly the way in which the operators $\alpha_i \cdot 0$, $0\alpha_j$, $\alpha_i 0 \alpha_j$ act on the wave function ϕ_{ij}. This is clear from the way by which they are defined:

$$(\alpha_i \cdot 0)\,\phi_{lm} = (\alpha_i \cdot \phi)_{lm} \quad ; \quad (0\,\alpha_j)\,\phi_{lm} = (\phi\,\alpha_j)_{lm}$$

$$(\alpha_i \cdot 0\, \alpha_j)\,\phi_{lm} = (\alpha_i \cdot \phi\, \alpha_j)_{lm}$$

Similarly:

$$\sum_{l,m} \phi^{*}_{lm} (\alpha_i \cdot 0\, \alpha_j) \cdot \phi_{lm} = \sum_{l,m} \phi^{*}_{lm} (\alpha_i \cdot \phi\, \alpha_j)_{lm}$$

V.

By means of the 16 linearly independent Dirac α-operators we were able to define 16 electromagnetic quantities:

$$(\vec{A}, V ; \vec{E}, \vec{H}) \quad , \quad (I_2, \sigma_x, \sigma_y, \sigma_z) \; ; \; (I_1) \tag{1}$$

satisfying the equations:

$$\left\{\begin{array}{l} -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} = \operatorname{curl}\vec{E} \; ; \; \operatorname{div}\vec{H} = 0 \; ; \; \vec{E} = -\operatorname{grad}V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \; ; \; \vec{H} = \operatorname{curl}\vec{A} \; , \\ \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \operatorname{curl}\vec{H} + \frac{\mu_0^2 c^2}{k^2}\vec{A} \; ; \; \operatorname{div}\vec{E} = -\frac{\mu_0^2 c^2}{k^2} V \; ; \; \frac{1}{c}\frac{\partial V}{\partial t} + \operatorname{div}\vec{A} = 0 \end{array}\right\} \tag{2}$$

$$\left\{\begin{array}{l} \frac{1}{c}\frac{\partial I_2}{\partial t} = -\varkappa\mu_0 c\,\alpha_4 \; ; \; \vec{\operatorname{grad}}\, I_2 = \varkappa\mu_0 c\,\vec{\sigma} \; , \; \frac{1}{c}\frac{\partial \sigma_4}{\partial t} + \operatorname{div}\vec{\sigma} = \varkappa\mu_0 c\, I_2 \\ \operatorname{curl}\vec{\sigma} = 0 \; ; \; \frac{1}{c}\frac{\partial \vec{\sigma}}{\partial t} + \vec{\operatorname{grad}}\,\sigma_4 = 0 \end{array}\right\} \tag{3}$$

$$\mu_0 I_1 = 0 \quad , \quad \frac{1}{c}\frac{\partial I_1}{\partial t} = 0 \quad , \quad \vec{\operatorname{grad}}\, I_1 = 0 \tag{4}$$

The group of equations (4) gives the equations of evolution of the invariant quantity I_1. They are of little interest, especially for a particle with non-vanishing rest mass, since then I_1, will vanish, and the other equations of the same group are identically satisfied.

The group (2) are of the Maxwellian type. The only difference between (2) and Maxwell's equations is that the equations for $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ and $\operatorname{div}\vec{E}$ contain a term proportional to μ_0^2. The form of these Maxwellian equations shows that the quantities A_x, A_y, A_z and V form the four components of a four vector "the potential four vectors". Similarly $\vec{E}$ and $\vec{H}$ are transformed with respect to Lorentz transformations as the six distinct components of an antisymmetric tensor of the second rank. For the change of the systems of reference without relative motion, $\vec{E}$ on one hand and $\vec{H}$ on the other are transformed as the components of two space-vectors.

The group of the non-Maxwellian equations (3), defined in terms of the five quantities σ_1, σ_2, σ_3, σ_4 and I_2, is not known in the classical theory of electromagnetism. The quantity I_2 is a completely antisymmetric tensor of the fourth rank, i.e. a pseudo invariant. The quantities σ_1, σ_2, σ_3 & σ_4 form the four components of a pseudo space-time four vector, i.e. a completely antisymmetric tensor of the third rank.

Now, recent developments of the meson theory show us that the Maxwellian equations (2) describe a particle of spin 1, while the non-Maxwellian pseudo scalar equations (3) describe a particle with the total spin zero. As is seen from (3) the quantities σ_1, σ_2, σ_3 and σ_4 are all derived from the pseudo scalar quantity I_2.

Now, these non-Maxwellian quantities (or photons) were not known classically. Considerations of these fields in the meson theory and the important role which they play suggest that, in the future, these photons may help us to understand something about the Poincaré tension force which keeps the electron in a stable state and prevents it from explosion because of the electrostatic repulsion.

ACKNOWLEDGEMENT.

The writer wishes to express his great indebtedness to Professor H. T. FLINT for suggesting the subject of this thesis, for his continuous guidance, interest and helpful advice during the progress of the work and also for the help given to him in writing this thesis.

REFERENCES

I. References on de Broglie's Theory:

A. Louis de Broglie, Nouvelles Recherches sur La Lumière, Hermann & Cie., Paris, 1936.

B. Louis de Broglie, Une Nouvelle Théorie de La Lumière, Hermann & Cie., two volumes, Vol. 1, 1940; Vol. II, 1942.

C. Louis de Broglie, Théorie Générale des Particules A Spin, Gauthier-Villars, Paris 1943.

D. Louis de Broglie, De La Mécanique Ondulatoire à la Théorie du Noyau, Tome II, Hermann, 1945.

E. Jules Gépériau, Contribution à la Théorie de la Lumière de L. de Broglie, Hermann, Paris, 1939.

II. General References

(1) Frenkel J., Wave Mechanics, elementary theory, pp. 1 - 6. Oxford, 1936.

(2) de Broglie, Nouvelle Théorie de la Lumière, vol 1, p. 35.

(3) de Broglie, Nouvelle Théorie de la Lumière, vol 1, pp. 44-45

(4) Heisenberg - Pauli, Zeits fur Phys. 56 (1929), p. 1; 59 (1930) p. 169
Frenkel, Wave Mechanics (Oxford), Elementary Theory, p. 187 (1936)
Fock: Mechanics of Photons, Comptes Rendus, 190, pp. 1399-1401, 1930.
Landau - Peierls, Zeits. fur Phys. vol 62, p 188, 1930
Oppenheimer, Phys. Ref., vol. 38, p 725, 1931.
Darwin, C.G., Proc. Roy. Soc., A, vol. 136, p. 36, 1932
Jordan P., Zeits fur Phys. 93, p 464, 1935; 98, p. 709, 1936, 99, p. 109, 1936.
Kronig R. de L., Arn. de H. Poincarée, vol. 6, p. 213, 1936.

(5) Darwin, loc. cit. in ref. no. (4), p. 36.

(6) de Broglie L., Comptes Rendus 195, pp. 536, 577, 862 (1932);
" " 197, p. 1377 (1933);
" " 198, p. 135 (1934);
" " 199, pp.445, 1165, (1934).

(7) Pokrowski, Zeits. fur Phys. 57, pp. 566 - 569, 1929.

(8) Anderson, W. Zeits. fur Phys. 58, pp. 841 - 857, 1929.

(9) de Broglie, La Lumière, vol. I, 1940, Chap. VI, p. 132.

(10) Pryce, Proc. Roy. Soc., A, vol. 165 A, pp. 247 - 271, 1938.

(11) de Broglie, La Lumière, vol I, 1940, p. 39.

(12) is an infinitesimal quantity

(13) Frenkel J., Wave Mechanics, Elementary Theory (1936), p. 189.

(14) de Broglie, La Lumière, 1940, p. 128.

de Broglie, "La Matière et le Rayonnement", Hermann, Paris (1938), pp. 34 - 39.

Pauli W., Handb. der Physik, 2nd edition, XXIV, 1, pp.201 - 210.

(15) Darwin, C.G., Proc. Roy. Soc., A, vol. 136, p. 37 (1932), and
" " " " vol. 124, p. 375 (1929)

(16) Kemmer N., " " " " vol. 173, p. 91 (1939)

(17) de Broglie, La Lumière (1940) 1, pp. 144 - 145

(18) de Broglie, " " " ", pp. 115 and 148

(19) de Broglie, loc. cit., p. 147

(20) de Broglie, Nouvelles Récherches sur la Lumière (1936), pp. 10 - 12 and pp. 41 - 42.

(21) de Broglie, La Lumière, 1, 1940, p. 178.

(22) de Broglie, loc. cit., p. 175

(23) Darwin, Proc. Roy. Soc., A', vol. 136, p. 41 (1932).

(24) Heitler W., The Quantum Theory of Radiation (Oxford 1944); pp. 17 - 19.

(25) de Broglie, La Lumière, II, 1942, p. 67 et seq.

(26) " " " " " " pp. 46 - 54.
Heitler W., loc. cit. pp. 94 - 96

(27) de Broglie, La Lumière, I, 1940, p. 114.

(28) Darwin, C.G., Proc. Roy. Soc., A, vol. 136, p. 41 (1932)

(29) Heitler, loc. cit. pp. 186 - 187.

(30) Rojanksy, Quantum Mechanics, Prentice-Hall, New York, 1942, pp. 253 - 5.

For a Hermitian operator H, we have :

$$\int_{-\infty}^{+\infty} \phi^{x} H \psi \, d\tau = \int_{-\infty}^{+\infty} (H\phi)^{x} \psi \, d\tau$$

from which we derive equation (66 p 75) by putting

$$\psi = \left(\frac{E}{W}\phi\right)$$

(31) Heitler W., On the particle equation of the meson, Proc. of the Roy. Irish Acad. vol. XLIX, Sect. A, pp. 1 - 4 equations (5a) and (b).

(32) Le Couteur, Proc. Camb. Phil. Soc., vol. 44, 2, 1948. pp. 229 - 241, section 2, equation (15)

(33) Frenkel, Wave Mechanics, Elementary Theory, pp. 188 - 189 equation (1226)

(34) Pauli, W., Handb. der Phys., 2nd edition, XXIV, 1, p. 255, equation (152).

(4)

(35) de Broglie, La Lumière, I, 1940, p. 105

Tetrode

(36) de Broglie, loc. cit. p. 187

(37) " " " " pp. 254 - 263

(38) " " La Lumière, II, (1942), pp. 85 - 86

" " La Lumière, I, (1940), pp. 222 - 24

(39) " " " " " " pp. 234 - 235

(40) Pauli, W., Handb. der Physik, 2nd edition, XXIV, i, pp. 216 - 218

(41) Dirac, Proc. Roy. Soc. A., 180, pp. 6 - 7

(42) Dirac, loc. cit. p. 8.