"CONTRIBUTIONS TO THE THEORY OF GENERALISED HYPERGEOMETRIC SERIES"

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Abstract of the thesis on "Contributions to the theory of Generalised Hypergeometric Series ".

This thesis deals with various aspects of development in the field of both ordinary and basic hypergeometric functions. It comprises six chapters. The first chapter gives a brief survey of some of the recent developments in this field including the work done in the present thesis. The second chapter gives a systematic study of the transformations connected with the partial sums of generalised hypergeometric series, both ordinary and basic. The main theorem proved in the chapter gives the most general relation of its type. The third chapter is concerned with the development of the transformation theory of bilateral cognate trigonometrical series and generalises all the known results in that field. The fourth chapter gives the integral analogues of some of the transformations of basic series analogous to those for the ordinary series in Chapter VI of Bailey's Cambridge Tract. The fifth chapter deals with a systematic classification and study of two and three term relations between special kinds of well-poised series of the type ${}_8\Phi_7$ and gives a new method, by integrals, of deducing these transformations easily. The last and the sixth chapter gives a number of identities involving basic analogues of Appell's hypergeometric functions of two variables and some associated functions.

CONTENTS

PREFACE

The present thesis on " Contributions to the theory of Generalised Hypergeometric Series " embodies the researches carried on by me since October 1951 to May 1953 as a Research student for the degree of Ph.D. at Bedford College (University of London), London. The thesis contains six chapters on various aspects of the subject and the following five papers, under publication, form a shortened version of it

(i) "The partial sums of series of hypergeometric type"

(Accepted for publication in the Proc. Camb. Phil. Soc.)

(ii) " General transformations of bilateral cognate trigonometrical series of ordinary hypergeometric type "

(Under publication in the Canadian Journal of Math.).

(iii) " On integral analogues of certain transformations of well-poised basic hypergeometric series "

(Accepted for publication in the Quart. J. of Math. (Oxford)).

(iv) " Some transformations of well-poised basic hypergeometric series of the type ${}_8\Phi_7$ "

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(v) " Some basic hypergeometric identities "

(Under publication in the Quart. J. of Math. (Oxford)).

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R. P. Agarwal
1.5.53

CHAPTER I

SOME RECENT DEVELOPMENTS IN THE THEORY OF HYPERGEOMETRIC FUNCTIONS

(1.1) In this chapter I give a brief historical survey of the researches carried out in the field of hypergeometric functions since the publication of the Cambridge tract on "Generalised Hypergeometric Series" by W.N.Bailey in 1935.No attempt has been made to give a complete chronological account of all the developments in this field,but only the important aspects of the subject and allied topics closely connected with the present thesis have been dealt with.

During the last two decades relatively fewer contributions have been made to the theory of ordinary hypergeometric series than to basic hypergeometric series which have proved to have applications in the theory of numbers and combinatory analysis.A study of these researches reveal that the subject has,broadly speaking,developed from three main points of view.The first and the most important being the transformation theory,the second is the application of the basic hypergeometric functions to transformations and identities occurring in combinatory analysis,including the famous Rogers-Ramanujan identities. Lastly,the theory has developed from the study of hypergeometric differential equations and the use of differential operators with applications to expansions and identities involving hypergeometric functions of two variables and other miscellaneous functions.

We shall deal with the subject under these three wide classifications systematically. In many places the chronological sequence of researches has been ignored in favour of a connected presentation of the subject. The following notation will be used throughout this chapter and the present thesis.

Let

$$(a)_n = \Gamma(a+n)/\,\Gamma(a) = a(a+1)(a+2)\ldots.(a+n-1);$$

$$(a)_0 = 1\ ;\quad (a)_{-n} = (-)^n/(1-a)_n\quad ;$$

$$(1)\quad {}_rF_s\left[\begin{matrix}a_1,\ldots.,a_r;\\ b_1,\ldots.,b_s,\end{matrix}\,x\right] = \sum_{n=0}^{\infty}\frac{(a^1)_n\ldots..(a_r)_n}{(b_1)_n\ldots..(b_s)_n n!}x^n\ .$$

The series ${}_rF_s$ converges for $|x|<1$ when $r=s+1$ and for all x when $r<s+1$ and $r=s$.

When $r=s+1$, the series 1.1(1) is called "Saalschützian" when $\Sigma b - \Sigma a = 1$, and well-poised when

$$1+a_1 = b_1+a_2 = \ldots\ldots\ldots.=b_s+a_{s+1}\ .$$

When x=1 in 1.1(1) as usual it shall be omitted from the symbol.

In the case of basic hypergeometric series, to avoid suffixes, we will write

$(a;n) = (1-a)(1-aq)..........(1-aq^{n-1})$; $(a;0) = 1$; $|q|<1$,

$(a;-n) = 1/(1-a/q)(1-a/q^2).....(1-a/q^n)$

$= (-)^n q^{\frac{1}{2}n(n+1)} a^{-n}/(q/a;n)$;

$$(2) \quad {}_r\Phi_s\begin{bmatrix} a_1,.....,a_r; \\ b_1,.....,b_s, \end{bmatrix} x = \sum_{n=0}^{\infty} \frac{(a_1;n)......(a_r;n)}{(b_1;n)......(b_s;n)} \frac{x^n}{(1;n)} .$$

When r=s+1, the series ${}_r\Phi_s$ converges for $|x|<1$ and for all x when $r<s+1$. The series* 1.1(2) is called "Saalschützian" when $b_1b_2....b_s = qa_1....a_{s+1}$, and well-poised when

$$a_1q = b_1a_2 = ...= b_sa_{s+1} .$$

(1.2) W.N.Bailey (5),in 1936,defined the bilateral ordinary and basic hypergeometric series which helped in generalising most of the known transformations of hypergeometric series.The first explicit definition and study of an ordinary bilateral series is found in a paper "On Vandermonde's theorem and some more general expansions" by John Dougall (1) in 1907.An ordinary bilateral hypergeometric series of order p is defined as

$$(1) \quad {}_pH_p\begin{bmatrix} a_1,.....,a_p; \\ b_1,.....,b_p; \end{bmatrix} x = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n.......(a_p)_n}{(b_1)_n.......(b_p)_n} x^n ,$$

which can be easily written as the sum of two ordinary

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* In the case when $r = s+1$.

hypergeometric series of the type $_{p+1}F_p$. In the particular case when x=1 the series $_pH_p(1)$ converges for $Rl(\Sigma b-\Sigma a -1)>0$. Similarly, when the argument is -1, the series $_pH_p(-1)$ is convergent when $Rl(\Sigma b-\Sigma a) > 0$.

When $b_p=1$, the series $_pH_p$ evidently reduces to a series of the type $_pF_{p-1}$.

A consideration of similar results for basic hypergeometric series is suggested by Jacobi's classical formula

$$\sum_{-\infty}^{\infty}(-)^n a^n q^{n^2} = \prod_{n=1}^{\infty}\left[(1-q^{2n})(1-aq^{2n-1})(1-q^{2n-1}/a)\right] .$$

So, we define a basic bilateral series as

$$(2) \qquad {}_r\Psi_r\left[\begin{matrix} a_1,\ldots\ldots,a_r; \\ b_1,\ldots\ldots,b_r \end{matrix} x\right] = \sum_{-\infty}^{\infty}\frac{(a_1;n)\ldots\ldots(a_r;n)}{(b_1;n)\ldots\ldots(b_r;n)}\, x^n .$$

This series converges in the annular region $|b_1\ldots\ldots b_r/a_1\ldots\ldots a_r x|<1, \quad |x|<1$.

For $b_r=q$, the series 1.2(2) reduces to one of the type $_r\Phi_{r-1}$.

For series of the type 1.2(1) Bailey proved in his paper that

$$(3) \qquad {}_6H_6\left[\begin{matrix} 1+\frac{1}{2}a, & b, & c, & d, & e, & f; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f; \end{matrix} -1\right]$$

=

$$= \frac{\Gamma(1-b)\,\Gamma(1-c)\,\Gamma(1-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+2a-b-c-d)}{\Gamma(1+a)\,\Gamma(1-a)\,\Gamma(1+a-b-d)\,\Gamma(1+a-c-d)\,\Gamma(1+a-e-f)\,\Gamma(1+a-b-c)}$$

$$\times\ {}_3H_3\left[\begin{matrix}1+2a-b-c-d,\ e\ ,\ f\ ;\\ 1+a-b,1+a-c,1+a-d,\end{matrix}\right] ,$$

where one of the parameters $(1+a-b),(1+a-c),(1+a-d)$ is a positive integer, so that both the series terminate below. This gives a relation between a well-poised ${}_6H_6(-1)$ and a general ${}_3H_3$.

The case when the series do not terminate was explicitly given later by M. Jackson (4) in 1952 who used it to derive a number of other transformations of bilateral series. In another paper (3) she has generalised the theorems of ~~Watson~~ Thomae and Whipple which give two and three term relations between ${}_3F_2$, by deducing transformations of series of the type ${}_3H_3$.

Bailey (5) also deduced two and three term relations between well-poised basic series of the type ${}_8\Phi_7$. The three term relations were the analogues of some of Whipple's transformations between three well-poised ${}_7F_6$'s given earlier in 1935. An interesting transformation was

$$(4)\qquad \chi(a;b,c,d,e,f) = \prod_{n=1}^{\infty}\frac{(1-aq^n/cf)(1-aq^n/df)(1-aq^n/ef)}{(1-q^n/c)(1-q^n/d)(1-q^n/e)}\times$$

$$\prod_{n=1}^{\infty}\frac{(1-bq^{n-1})(1-aq^n/b)(1-bq^{n-1}/a)}{(1-bfq^{n-1}/a)(1-fq^n/b)(1-bq^{n-1}/f)}\times$$

$$\times\,\chi(f^2/a;bf/a,cf/a,df/a,ef/a,f)\ +$$

+ a similar expression with f and b interchanged,

where

$$\chi(a;b,c,d,e,f)$$

$$= \prod_{n=1}^{\infty} \frac{(1-aq^n/b)(1-aq^n/c)(1-aq^n/d)(1-aq^n/e)(1-aq^n/f)(1-a^2q^{n+1}/bcdef)}{(1-aq^n)}$$

$$\times {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & f; \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, \end{matrix} a^2q^2/bcdef\right].$$

If f=q in 1.2(4) the second series on the right reduces to a multiple of a well-poised, summable ${}_6\Phi_5$ and we get

$$(5) \qquad {}_6\Psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e; \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, \end{matrix} a^2q/bcde\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-q^n/a)(1-q^n)(1-aq^n/bc)(1-aq^n/bd)(1-aq^n/be)}{(1-q^n/b)(1-q^n/c)(1-q^n/d)(1-q^n/e)(1-aq^n/b)(1-aq^n/c)} \times \right.$$

$$\left. \times \frac{(1-aq^n/cd)(1-aq^n/ce)(1-aq^n/de)}{(1-aq^n/d)(1-aq^n/e)(1-a^2q^n/bcde)}\right].$$

The formula 1.2(5) has proved to have applications in the theory of elliptic functions, combinatory analysis and the deduction of Rogers-Ramanujan identities. We shall discuss these in the next section.

But so far a systematic study of these three term relations between ${}_8\Phi_7$'s was lacking and only a few scattered results were to be found in the litreature. In Chapter V of the present thesis these two and three term relations are systematically studied and classified. Some new three term relations are given and a very simple method of deducing all these relations by means of basic integrals of the Barnes type given by Watson (2) is evolved. This method by integrals is more fully exploited earlier in Chapter IV, where it is used to deduce integrals representing well-poised basic hypergeometric functions and the integral equivalent of Jackson's analogue of Dougall's theorem (T. 8.3(1)), thus filling a long existing significant gap in the theory of basic series. The results investigated are the basic analogues of some of the results given in Chapter VI of Bailey's tract.

Later in 1947 Bailey (6) gave a two term relation between two well-poised ${}_8\Phi_7$'s similar to one proved by him earlier for two ${}_7F_6$'s (T.6.6(3)). This was the relation giving the form Jackson's analogue of Dougall's theorem assumes when we remove the restriction that one of the numerator parameters must be of the form q^{-N}. He used this result to deduce a relation between four non-terminating ${}_{10}\Phi_9$'s similar to that obtained by him earlier for four ${}_9F_8$'s. The proof of these analogues was suggested by the method of obtaining transformations of integrals of Barnes type by which the corresponding transformations for the ordinary series were discovered, although the details in the case of basic transformations were more complicated.

Till recently, the relations between four $_9F_8$'s or $_{10}\Phi_9$'s were the most general known transformations of their kind.

In 1951, however, D.B.Sears (1-5) in a series of five papers published in the Proceedings of the London Mathematical Society developed a novel method by which he obtained some most remarkable and general transformations of well-poised and general hypergeometric functions, both ordinary and basic. Sears (3) gave a transformation connecting (M+N+1) general ordinary hypergeometric series of order (M+N+1). His proof, which is based on Hill's (1) identity for terminating series, uses double induction in M and N, and from this result he deduced several general transformations of well-poised ordinary hypergeometric series. He also gave there-in a systematic study of the transformation theory of cognate trigonometrical series of the hypergeometric type, introduced into analysis by Whipple (3) in 1936. The transformations of these trigonometrical series include as particular cases the general and well-poised transformations of ordinary hypergeometric series.

In a later paper (5) he extends the methods of this paper to deduce a general transformation between (M+N+1) series of the type $_{M+N+1}\Phi_{M+N}$, which he used to deduce transformations of well-poised basic series of any order.

These transformations between general and well-poised hypergeometric series both in the case of ordinary and basic functions have been very recently generalised by L.J.Slater (3 & 4). She has deduced from Sears' known transformations relations between general and well-poised bilateral series. Thus, her results include Sears' results as particular cases. Slater not only deduced these general bilateral transformations but also has given very elegant direct proofs of Sears' and her own bilateral theorems. She has made use of certain basic integrals different from those used earlier by Watson (2), to prove the basic transformations and has used contour integrals of Barnes type to prove the transformations of ordinary hypergeometric series. I give below an example of the type of basic contour integrals given by Slater. She (4) has considered the integral

$$(6) \qquad \frac{1}{2\pi i}\int_{-i\pi/t}^{i\pi/t} \prod\left[\begin{matrix}(1-A-s),(1+a_1-A+s),1+\tfrac{1}{2}a_1+s,1-\tfrac{1}{2}a_1-s,\\ \qquad 1+\tfrac{1}{2}a_1+s+\pi i/t,1-\tfrac{1}{2}a_1-s+\pi i/t;\\ (a+s),(a-a_1-s)\ ;\end{matrix}\right] q^{sds},$$

where there are (M+1) of the 'a' parameters, and (M-1) of the A parameters and

$$\prod\left[\begin{matrix}c_1,\ldots\ldots,c_p;\\ d_1,\ldots\ldots,d_m;\end{matrix}\right] = \prod_{n=0}^{\infty}\frac{(1-q^{c_1+n})\ldots\ldots(1-q^{c_p+n})}{(1-q^{d_1+n})\ldots\ldots(1-q^{d_m+n})},$$

and the left hand symbol has been further abbreviated as $\prod\left[\begin{smallmatrix}(c);\\(d);\end{smallmatrix}\right]$,where the number of parameters of each kind is stated.

The line of integration is the imaginary axis taken from $-\pi/t$ to π/t, where $q=e^{-t}$, t positive.

The integral 1.2(6) represents one of the transformations of well-poised basic series due to Sears (5;7.2).Similar integrals have been used by her to deduce other transformations as well. Slater thus generalised all the ordinary and basic transformations given by Sears. In Chapter III of this thesis,I give ,however,still more generalised transformations of ordinary hypergeometric series.Using Sears results on general transformations of cognate trigonometrical series I deduce and give direct proofs as well,of the general and well-poised transformations of bilateral cognate trigonometrical series.They thus generalise all the general transformations given by Sears for ordinary and cognate trigonometrical series,and also the bilateral transformations for ordinary series given by Slater (3).

In another paper,Sears (4) has given a very s systematic and complete theory of two,three and four term relations between series of the type ${}_3\Phi_2$ which give the analogues of Thomae's and Whipple's relations for the series

$_3F_2$ .All these relations have been recently generalised by M.Jackson (5) by deducing transformations between bilateral series of the type $_3\Psi_3$.

In Chapter II of the thesis transformations of a entirely different nature from the above have been deduced and their consequences studied.The study concerns the transformations of partial sums of series of the hypergeometric type.The subject had previously caught the attention of a number of mathematicians such as Ramanujan,Watson (1), Whipple (1),Darling (1),Hodgkinson (1),and Bailey (4) in the case of ordinary hypergeometric series. The present interest in the subject arose from a theorem due to Bailey (4) given some years back which is generalised in this chapter and the most general relation of that type has beem given.Similar transformations for basic series have also been studied there in. A recent paper by Hermann Von Schelling* suggests that the transformations studied in this chapter may be applied to certain statistical problems, but this aspect, being out of our present scope,has not been studied.

Iconclude this section with the remark that with Sears' work on the transformation theory of hypergeometric

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* Hermann Von Schelling:- "A second formula for the partial sum of hypergeometric series having unity as the fourth argument." Annals of Math.Statistics,2(1950)458-60. The result proved is a particular case of a known formula due to Bailey.

series and its consequent generalisations by Slater, M.Jackson and others,the theory can reasonably be said to be a almost complete one.

(1.3) We pass on now to the consideration of some basic hypergeometric identities connected with combinatory analysis and identities of the Rogers-Ramanujan type. Bailey (7) in 1947 considerably simplified L.J.Rogers'* work of expressing special series as infinite products.Most of Rogers work,which was very involved,was inspired by a thorough study of theta-series and the properties of elliptic functions.Bailey,however,deduced some of the identities given by Rogers as limiting cases of more general basic hypergeometric identities.One of the theorems that he gave was that ,if

$$B_n = \sum_{r=0}^{n} \frac{(x^{-n};r)(kx^n;r)}{(\alpha x^{n+1};r)(\alpha x^{1-n}/k;r)} A_r ,$$

where A_r is independant of n , then

$$(1) \qquad \sum_{n=0}^{\infty} \frac{(k;n)(k\alpha;n)(a_1;n)(a_2;n)}{(x;n)(x\alpha;n)(b_1;n)(b_2;n)} t^n B_n =$$

* 1.L.J.Rogers:- "Second Memoir on the expansions of certain infinite products." Proc.London Math. Soc.(1),25(1894), 318-43.

2. ....:- "Third Memoir on the expansions of certain infinite products." Proc.London Math.Soc.(1),26(1895), 15-32.

3. ....:- "On two theorems of combinatory analysis and some allied identities." Proc.London Math.Soc.(2),16 (1917),315-36.

$$= \sum_{r=0}^{\infty} \frac{(k;2r)(a_1;r)(a_2;r)}{(\alpha x;2r)(b_1;r)(b_2;r)} (kt/\alpha x)^r A_r \times$$

$$\times \Phi\left[\begin{matrix} kx^{2r}, k/\alpha, a_1x^r, a_2x^r \, ; \\ \alpha x^{2r+1}, b_1x^r, b_2x^r \, , \end{matrix} \; t\right] ,$$

where there may be any number of numbers a and b.

With the help of this and other transformations he not only deduced many of Rogers' identities but also deduced general transformations of basic series. From 1.3(1) he obtained a transformation giving a Saalchützian nearly-poised ${}_5\Phi_4$ in terms of a well-poised ${}_{12}\Phi_{11}$. This was the first transformation given for a nearly-poised basic series. Of course, later on Bailey (8) and Sears (2) gave some more transformations for nearly-poised series. In another paper Bailey (9) developed a new method which led to a remarkable simplification in the ideas underlying the methods of finding transformations of basic hypergeometric series and also identities of the Rogers-Ramanujan type. The simple fundamental result proved was that if

(2) $$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n ,$$

assuming, of course that the series converge and that certain other conditions are satisfied. By giving suitable values to the numbers u_n, v_n, α_r, and δ_r he got a number of interesting new identities besides some of the identities given earlier by Rogers. It may be mentioned that F.J.Dyson (1) had previously given new proofs of three of Rogers' formulae.

In a recent paper L.J.Slater(1) has given some very general transformation theorems by which she not only obtained all the known identities of Rogers but obtained numerous others, some of them filling significant gaps in Rogers' work. In fact, her work simplifies considerably the very complicated work of Rogers. The transformation 1.3(2) is fundamental in her work as well, although she also proved a number of other very general theorems of this type. She used a particular case of 1.3(2) in the form, that if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;n-r)(x;n+r)}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(q;r-n)(x;r+n)}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \quad .$$

The summation formula 1.2(5) has, in general, been used to evaluate the sum β_n. The most interesting of the new identities of the Rogers-Ramanujan type deduced by Slater are those involving products of the types

$$\prod_{n=1}^{\infty}(1-q^{32n}) \quad , \quad \prod_{n=1}^{\infty}(1-q^{36n}) \quad , \quad \prod_{n=1}^{\infty}(1-q^{42n}) \quad ,$$

$$\prod_{n=1}^{\infty}(1-q^{48n}) \quad , \quad \prod_{n=1}^{\infty}(1-q^{54n}) \quad \text{and} \quad \prod_{n=1}^{\infty}(1-q^{64n}).$$

(1.4) Lastly,coming to some of the work done in various other fields of hypergeometric series we find that a number of interesting papers have been written on infinite expansions of Appell's double hypergeometric series and their basic analogues.These expansions were first studied by J.L.Burchnall and T.W.Chaundy (1) in 1940.It was shown about twenty years back by Bailey that Appell's double hypergeometric function of the fourth type reduces to a product of two ordinary hypergeometric functions when its parameters are connected by a single relation. Burchnall and Chaundy,however,showed that in the case of an unrestricted Appell's function of the fourth type that product is represented as the first term of an infinite series of such products.It was from this enquiry that they were further led to a number of other infinite expansions for other types of double hypergeometric functions as well. The expansions have been obtained by the help of certain symbolic differential operators of which the following is the fundamental one

$$\nabla(h) \equiv \frac{\Gamma(h)\,\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\Gamma(\delta'+h)} ,$$

where $\delta , \delta' \equiv x\frac{\partial}{\partial x} , y\frac{\partial}{\partial y}$ respectively. In a later paper* they studied the confluent forms of the expansions given earlier. F.H.Jackson (1) following Burchnall and Chaundy, defined the basic analogues of Appell's functions and deduced a number of transformations which give the basic analogues of Burchnall-Chaundy expansions for double hypergeometric functions and confluent hypergeometric functions in two variables. In the case of basic expansions certain other types of functions called "abnormal" functions by Jackson occur. The confluent forms of these "abnormal" functions are closely connected with the basic analogues of Bessel functions, Laguerre functions and Whittaker functions given many years back by Jackson.

In a later communication Chaundy (1) gave some more very general expansions of Appell's double hypergeometric functions and functions of higher order in two variables. In Chapter VI of the thesis the basic generalisations of some of these expansions have been deduced and their confluent forms have been studied. Some of these expansions contain "abnormal" functions as well.

Chaundy (2) in another paper in 1943 used an argument of a rather different character to define what he calls an "extended" hypergeometric function. It is known

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* Burchnall and Chaundy (2)

that a hypergeometric series $_rF_s(a_1,....,a_r;b_1,....,b_s;)$ terminates if one of the numerator parameters 'a' is a negative integer and further that under certain circumstances it can be summed up to a 'gamma product' of the type $(A_1)_n(A_2)_n...../(B_1)_n(B_2)_n....$. Calling such a series a "reducible" hypergeometric series it is easily seen that if F_n is a "reducible" hypergeometric series,then the infinite series $\sum \gamma_n F_n x^n/n!$ is merely a hypergeometric series of the same order,where γ_n is any 'gamma product' of the type mentioned above.This led him to study infinite series of the above type when F_n is no longer reducible. It is such a series that he calls "extended" hypergeometric series.He has studied the differential equations of a number of such "extended" functions of different types.

The study of differential equations associated with the generalised hypergeometric series and Appell's double hypergeometric functions has been made from time to time by various other authors,as well .

I conclude this chapter by briefly mentioning a somewhat different approach to the subject of hypergeometric series found in some recent papers by W.Hahn (1 & 2). With the help of the basic hypergeometric series he has defined the basic analogues of the sine,cosine,exponential and Bessel functions.These definitions are similar to those given by F.H.Jackson some fifty years back. Hahn has made a study of the q-difference equations, the recurrence relations and other analytic properties of these functions. With the help of his basic exponential function he has given a basic analogue of the well-known Laplace integral transform.

Chapter II

THE PARTIAL SUMS OF SERIES OF HYPERGEOMETRIC TYPE

(2.1) <u>Introduction.</u> This chapter deals with a study of the partial sums of the series of the coefficients of the ordinary and basic hypergeometric series. A number of scattered results were given about twenty years back expressing the sum of N terms of a hypergeometric series with unit argument, in terms of an infinite series of the generalised hypergeometric type. The subject had previously been studied by Hill and Whipple as early as 1907, but new interest was aroused when Ramanujan gave the following theorem

(1) $$1/n + (1/2)^2 1/(n+1) + (1.3/2.4)^2 1/(n+2) + \ldots\ldots$$

$$= \{\Gamma(n)/\Gamma(n+\tfrac{1}{2})\}^2 \ \{1 + (1/2)^2 + (1.3/2.4)^2 + \ldots \text{ to n terms}\}.$$

This was proved by Watson (1), Darling (1), and later generalised by Whipple (1), Hodgkinson (1) and Bailey (4). Watson obtained the result as a particular case of a limiting case of a known formula due to Thomae (T.3.8(1). Whipple generalised Ramanujan's result and proved that

(2) $$ {}_2F_1(a,b;c) \text{ to n terms} = \frac{\Gamma(1+a-c)\Gamma(1+b-c)}{\Gamma(1-c)\Gamma(1+a+b-c)} \times\left\{1 - \frac{(a)_n(b)_n}{n!\,(c-1)_n}\ {}_3F_2\left[\begin{matrix}1-a, 1-b, n;\\ 2-c, n+1\end{matrix}\right]\right\},$$

provided $a+b-c > -1$.

Whipple's result covers the most interesting case for which the complete hypergeometric series is not convergent, but still there remains the problem of finding the corresponding sum when $a+b-c \leqslant -1$. This case was covered by Bailey (3) who proved that for all values of the parameters the sum of the first n terms of ${}_2F_1(a,b;c)$ is

$$\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(n)\,\Gamma(a+b+n)}\ {}_3F_2\left[\begin{matrix}a,b,c+n-1;\\ c,a+b+n\end{matrix}\right].$$

Later, Bailey proved a more general and particularly simple result, namely that

$$(3)\qquad \frac{\Gamma(x+m)\Gamma(y+m)}{\Gamma(m)\Gamma(x+y+m)}\ {}_3F_2\left[\begin{matrix}x,y,v+m-1;\\ v,x+y+m\end{matrix}\right] \quad \text{to n terms}$$

$$= \frac{\Gamma(x+n)\Gamma(y+n)}{\Gamma(n)\Gamma(x+y+n)}\ {}_3F_2\left[\begin{matrix}x,y,v+n-1;\\ v,x+y+n\end{matrix}\right] \quad \text{to m terms.}$$

When $x=y=\frac{1}{2}$, $v=1$, this result reduces to the theorem that

$$\left\{\Gamma(m+\tfrac{1}{2})/\Gamma(m)\right\}^2 \left\{1/m + (1/2)^2 1/(m+1) + (1.3/2.4)^2 1/(m+2)+\ldots \ldots\text{to n terms}\right\}$$

$$= \left\{\Gamma(n+\tfrac{1}{2})/\Gamma(n)\right\}^2 \left\{1/n + (1/2)^2 1/(n+1) + (1.3/2.4)^2 1/(n+2)+\ldots \ldots\text{to m terms}\right\},$$

which gives Ramanujan's theorem when m tends to infinity.

In this chapter I have generalised all the above known results by deriving a relation of the type 2.1(3) between two ${}_7F_6$'s with unit argument and have deduced a number of other transformations connecting partial sums of hypergeometric series. The main theorem has been proved in 2.2. Two proofs of this theorem have been given. The first is the one from which the theorem was first derived. The second one is as simple a proof as one can expect and involves very little algebra. It is interesting to find that all the results on the partial sums of ordinary hypergeometric series have corresponding analogues for basic series as well. They have also been studied in this chapter.

(2.2) The main theorem.

We shall now prove that

(1)

$$\frac{\Gamma(1+X)\Gamma(1+X-D-E)\Gamma(1+E+M)\Gamma(1+D+M)}{\Gamma(1+X-D)\Gamma(1+X-E)\Gamma(1+D+E+M)\Gamma(1+M)} \times$$

$$\times\, {}_7F_6\left[\begin{matrix} X, 1+\tfrac{1}{2}X, 1-C+X, & C+M, & A-D-E, & D, & E; \\ \tfrac{1}{2}X, & C, & 1+X-C-M, & 1+D+E+M, & 1+X-D, & 1+X-E \end{matrix}\right]$$

to (N+1) terms

$$= \frac{\Gamma(1+X')\Gamma(1+X'-D-E)\Gamma(1+E+N)\Gamma(1+D+N)}{\Gamma(1+X'-D)\Gamma(1+X'-E)\Gamma(1+D+E+N)\Gamma(1+N)} \times$$

$$\times\, {}_7F_6\left[\begin{matrix} X', 1+\tfrac{1}{2}X', 1-C+X', & C+N, & A-D-E, & D, & E; \\ \tfrac{1}{2}X', & C, & 1+X'-C-N, & 1+D+E+N, & 1+X'-D, & 1+X'-E \end{matrix}\right]$$

to (M+1) terms,

where $X=A+M$ and $X'=A+N$.

<u>First Proof.</u> Let us consider the general relation connecting two terminating well-poised ${}_9F_8$'s (T.4.3(7)), namely that

$$(2)\qquad {}_9F_8\left[\begin{matrix}a, 1+\tfrac{1}{2}a, & b, & c, & d, & e, & f, & g, & h\ ; \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g, & 1+a-h\end{matrix}\right]$$

$$=\frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-h)\Gamma(1+a-e-g-h)\Gamma(1+a-e-f-h)\,\Gamma(1+a-f-g-h)\,\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-e-f)\Gamma(1+a-e-g)\Gamma(1+a-e-h)\Gamma(1+a-f-g)\Gamma(1+a-f-h)\,\Gamma(1+a-g-h)\Gamma(1+a-e-f-g-h)}$$

$$\times\,{}_9F_8\left[\begin{matrix}k, 1+\tfrac{1}{2}k, k+b-a, k+c-a, k+d-a, & e, & f, & g, & h\ ; \\ \tfrac{1}{2}k, 1+a-b, 1+a-c, 1+a-d, & 1+k-e, & 1+k-f, & 1+k-g, & 1+k-h\end{matrix}\right],$$

where $k=1+2a-b-c-d,$

$$2+3a=b+c+d+e+f+g+h,$$

and h is a negative integer say, $-N$.

Putting $f=1+k+N$ in the above relation we get

$${}_7F_6\left[\begin{matrix}k, 1+\tfrac{1}{2}k, k+b-a, k+c-a, k+d-a, & e, & g\ ; \\ \tfrac{1}{2}k, 1+a-b, 1+a-c, 1+a-d, & 1+k-e, & 1+k-g\end{matrix}\right] \text{ to } (N+1) \text{ terms}$$

$$=\frac{\Gamma(1+a)\Gamma(a-e-k-N)\Gamma(1+a-e-g)\Gamma(1+a-e+N)\Gamma(a-g-k-N)\Gamma(a-k)\,\Gamma(1+a-g+N)\Gamma(a-e-g-k)}{\Gamma(1+a-e)\Gamma(a-k-N)\Gamma(1+a-g)\Gamma(1+a+N)\Gamma(a-g-k)\Gamma(1+a-e-g+N)\,\Gamma(a-e-k)\Gamma(a-e-k-g-N)}$$

$$\times\,{}_9F_8\left[\begin{matrix}a, 1+\tfrac{1}{2}a, & b, & c, & d, & e, & 1+k+N, & g, & -N\ ; \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & a-k-N, & 1+a-g, & 1+a+N\end{matrix}\right]$$

where $k=1+2a-b-c-d$ and $a=g+e$.

Now let c tend to -M (M,a positive integer) and replace a by g+e, then we get

$$ {}_7F_6\left[\begin{matrix} k, 1+\frac{1}{2}k, k+b-g-e, k-M-g-e, k+d-g-e, \quad e \;, \quad g \;; \\ \frac{1}{2}k, 1+g+e-b, 1+g+e+M, 1+g+e-d, 1+k-e, 1+k-g \end{matrix}\right] \text{ to } (N+1) \text{ terms} $$

$$ = \frac{\Gamma(1+g+e)\Gamma(g+e-k)\Gamma(-k)\Gamma(1+g+N)\Gamma(1+e+N)\Gamma(g-k-N)\Gamma(e-k-N)}{\Gamma(1+g)\Gamma(1+e)\Gamma(e-k)\Gamma(g-k)\Gamma(1+g+e+N)\Gamma(1+N)\Gamma(g+e-k-N)\Gamma(-k-N)} $$

$$ \times {}_9F_8\left[\begin{matrix} g+e, 1+\frac{1}{2}(g+e), \quad b \;, \quad -M \;, \quad d \;, e \;, 1+k+N \;, g \;, -N; \\ \frac{1}{2}(g+e), 1+g+e-b, 1+g+e+M, 1+g+e-d, 1+g, g+e-k-N, 1+e, 1+g+e+N \end{matrix}\right] $$

where $\qquad k=1+2g+2e-b-d+M.$

In the above relation the right hand ${}_9F_8$ is symmetrical in M and N and hence after some transposition we have

$$ \frac{\Gamma(e-k)\Gamma(g-k)\Gamma(1+e+M)\Gamma(1+g+M)}{\Gamma(g+e-k)\Gamma(-k)\Gamma(1+g+e+M)\Gamma(1+M)} \times $$

$$ \times {}_7F_6\left[\begin{matrix} k, 1+\frac{1}{2}k, k+b-g-e, k-g-e-M, k+d-g-e, \quad e \;, \quad g \;; \\ \frac{1}{2}k, 1+g+e-b, 1+g+e+M, 1+g+e-d, 1+k-e, 1+k-g \end{matrix}\right] \text{ to } (N+1) \text{ terms} $$

= the same expression with M and N interchanged,

where $\qquad k=1+2g+2e-b-d+M.$

Changing the notation by putting

$$ 1+2g+2e-b-d = A $$

$$ 1-b+g+e = C $$

$$e = D$$

$$g = E$$

$$A+M = X$$

$$A+N = X',$$

and simplifying we get the result in the required form.

Second Proof of 2.2(1).

Let us take $N > M$. Then the terms containing powers of C on the left hand side of 2.2(1) are of the type

$$\frac{(1+X-C)_r\ (C+M)_r}{(C)_r\ (1+A-C)_r} \quad .$$

If we now multiply both sides of 2.2(1) by $(C)_M(1+A-C)_M$, the highest power of C on both sides becomes C^{2M}. Hence, we have got two polynomials in C of degree 2M.

So, if we can prove that the polynomials are equal for (2M+1) values of C, we shall have proved the result.

To prove this let $C = -N, -N-1, \ldots\ldots, -N-M, 2+A+N, \ldots\ldots, M+1+A+N$ be the (2M+1) values. The partial series in 2.2(1) become complete hypergeometric series which can be summed by Dougall's theorem. We can thus easily verify the result in these cases, and so the theorem is proved.

(2.3) Particular cases.

In this section we shall deal with some of the particular cases of the main theorem and obtain the known results mentioned in 2.1.

(i) Let A tend to infinity in 2.2(1), then we get

$$\frac{\Gamma(1+E+M)\Gamma(1+D+M)}{\Gamma(1+D+E+M)\Gamma(1+M)}\ {}_3F_2\begin{bmatrix}C+M,D,E;\\C,1+D+E+M\end{bmatrix} \text{ to } (N+1) \text{ terms}$$

$$= \frac{\Gamma(1+E+N)\Gamma(1+D+N)}{\Gamma(1+D+E+N)\Gamma(1+N)}\ {}_3F_2\begin{bmatrix}C+N,D,E;\\C,1+D+E+N\end{bmatrix} \text{ to } (M+1) \text{ terms},$$

a result due to Bailey given in 2.1(3).

(ii) If we let M tend to infinity in (i) above we get

$${}_2F_1(D,E;C) \text{ to } (N+1) \text{ terms} = \frac{\Gamma(1+E+N)\Gamma(1+D+N)}{\Gamma(1+D+E+N)\Gamma(1+N)}\ {}_3F_2\begin{bmatrix}C+N,D,E;\\C,1+D+E+N\end{bmatrix},$$

another result due to Bailey (3).

(iii) The ${}_3F_2$ in the above case (ii) can be transformed in various ways. In particular using the relation between three series of the type* $F_p(0;4,5)$, $F_p(4;0,1)$ and $F_p(1;0,4)$, we get the result 2.1(2) due to Whipple. Also if we use the relation

$$F_p(0;4,5) = F_p(0;1,4)$$

in (ii) we get a result due to Hodgkinson (1), namely

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* These are well-known relations due to Thomae. For the notation see T.p.17.

$_2F_1(D,E;C)$ to $(N+1)$ terms

$$= \frac{\Gamma(1+D+N)\Gamma(1+E+N)}{\Gamma(1+N)\Gamma(1+D+E-C)\Gamma(1+C+N)}\ {}_3F_2\left[\begin{matrix}C-D, C-E, C+N;\\ C, 1+C+N\end{matrix}\right],$$

valid for $Rl(1+D+E-C) > 0$.

(iv) If we let M tend to infinity in 2.2(1), we get

$${}_3F_2\left[\begin{matrix}A-D-E, D, E;\\ C, 1+A-C\end{matrix}\right] \text{ to } (N+1) \text{ terms}$$

$$= \frac{\Gamma(1+X')\Gamma(1+X'-D-E)\Gamma(1+E+N)\Gamma(1+D+N)}{\Gamma(1+X'-D)\Gamma(1+X'-E)\Gamma(1+D+E+N)\Gamma(1+N)} \times$$

$$\times\ {}_7F_6\left[\begin{matrix}X', 1+\tfrac{1}{2}X', 1-C+X', & C+N, & A-D-E, & D, & E;\\ \tfrac{1}{2}X', \quad C & , 1+X'-C-N, & 1+D+E+N, & 1+X'-D, & 1+X'-E\end{matrix}\right],$$

where $X'=A+N$. This gives the sum of a Saalschützian $_3F_2$ to $(N+1)$ terms in terms of an infinite well-poised $_7F_6$, a result given by Bailey (3).The well-poised $_7F_6$ in this can be transformed into two Saalchützian $_4F_3$'s (T.7.5(3)) to give the following result by Darling (2)

$${}_3F_2\left[\begin{matrix}A-D-E, D, E;\\ C, 1+A-C\end{matrix}\right] \text{ to } (N+1) \text{ terms}$$

$$= \frac{\Gamma(1+A-D-E+N)\Gamma(1+E+N)\Gamma(1+D+N)}{\Gamma(1+N)\Gamma(C+N)\Gamma(1+A+N-C)} \times$$

$$\left\{ \frac{\Gamma(C)\Gamma(2C-1-A)}{(1+A-C+N)\Gamma(C-D)\Gamma(C-E)\Gamma(C-A+D+E)} \times \right.$$

$$\times {}_4F_3 \left[\begin{matrix} 1+D+E-C, 1+A-C-D, 1+A-C-E, 1+A-C+N; \\ 2+A+N-C, 1+A-C, 2+A-2C \end{matrix} \right]$$

$$+ \frac{\Gamma(1+A-C)\Gamma(1+A-2C)}{(C+N)\Gamma(1+A-C-D)\Gamma(1+A-C-E)\Gamma(1+D+E-C)} \times$$

$$\left. \times {}_4F_3 \left[\begin{matrix} C-A+D+E, C-D, C-E, C+N; \\ 1+C+N, 2C-A, C \end{matrix} \right] \right\}.$$

(v) If we put E=1+A-C in 2.2(1) we get a relation of the same type between two ${}_5F_4$'s, namely

$$\frac{\Gamma(1+X)\Gamma(C-D+M)\Gamma(1+D+M)\Gamma(2+A-C+M)}{\Gamma(1+X-D)\Gamma(C+M)\Gamma(1+M)\Gamma(2+A+D-C+M)} \times$$

$$\times {}_5F_4 \left[\begin{matrix} X, 1+\tfrac{1}{2}X, 1-C+X, & C-D-1 & , & D & ; \\ & \tfrac{1}{2}X, & C & , 2+A+D-C+M, 1+X-D \end{matrix} \right] \text{ to } (N+1) \text{ terms}$$

= a similar expression with M and N interchanged, where X=A+M.

(vi) Lastly, if we put h=-N, $f=\frac{1}{2}(1+a)$ and let g tend to (1+a+N) in 2.2(2) we get the sum of a well-poised ${}_6F_5$ to (N+1) terms in terms of a terminating well-poised ${}_9F_8$, namely

$$ {}_6F_5\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b\ , & c\ , & d\ , & e\ ; \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, 1+a-e \end{matrix}\right] \text{ to } (N+1) \text{ terms} $$

$$ = \frac{(1+a)_N(3/2+\tfrac{1}{2}a)_N(1+e)_N(\tfrac{1}{2}+\tfrac{1}{2}a-e)_N}{(1+a-e)_N(\tfrac{1}{2}+\tfrac{1}{2}a)_N(3/2+\tfrac{1}{2}a+e)_N\ N!} \times $$

$$ {}_9F_8\left[\begin{matrix} k, 1+\tfrac{1}{2}k, k+b-a, k+c-a, k+d-a, \ e\ , \tfrac{1}{2}(1+a)\ , 1+a+N, \ -N\ ; \\ \tfrac{1}{2}k, 1+a-b, 1+a-c, 1+a-d, 1+k-e, k+\tfrac{1}{2}(1-a), k-a-N, 1+k+N \end{matrix}\right], $$

where $k=1+2a-b-c-d$ and $b+c+d+e=\tfrac{1}{2}(1+3a)$.

(2.4) Next, let us consider a transformation due to Whipple (T.4.3(4)), namely

(1) $$ {}_7F_6\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b\ , & c\ , & d\ , & e\ , & -m\ ; \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, 1+a-e, 1+a+m \end{matrix}\right] $$

$$ = \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}\ {}_4F_3\left[\begin{matrix} 1+a-b-c, d, \ e\ , \ -m\ ; \\ 1+a-b, 1+a-c, d+e-a-m \end{matrix}\right], $$

which transforms a terminating well-poised ${}_7F_6$ into a Saalchützian ${}_4F_3$.

Put **b**=1+a+m and we get

$$ {}_5F_4\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & c\ , & d\ , & e\ ; \\ & \tfrac{1}{2}a, & 1+a-c, & 1+a-d, 1+a-e \end{matrix}\right] \text{ to } (m+1) \text{ terms} $$

$$ = \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}\ {}_3F_2\left[\begin{matrix} -c-m, d\ , \ e\ ; \\ 1+a-c, d+e-a-m \end{matrix}\right] \text{ to } (m+1) \text{ terms}. $$

Transforming the Saalchützian ${}_3F_2$ on the right by means of the result of Bailey derived in 2.3(iv) ,we get the sum of a~~n~~ ~~inf~~ well-poised ${}_5F_4$ upto (m+1) terms in terms of an infinite well-poised ${}_7F_6$,in fact

(2) $${}_5F_4\left[\begin{matrix} a, 1+\frac{1}{2}a, & c & , & d & , & e & ; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix}\right] \text{ to } (m+1) \text{ terms}$$

$$= \frac{(1+a)_m(1+a-d-e)_m\Gamma(1+d+e-c)\Gamma(1-c)\Gamma(1+d+m)\Gamma(1+e+m)}{(1+a-d)_m(1+a-e)_m\Gamma(1+d+e+m)\Gamma(e-c+1)\Gamma(d-c+1)\Gamma(m+1)} \times$$

$${}_7F_6\left[\begin{matrix} k, 1+\frac{1}{2}k, d+e-a, 1+a-c+m, & -c-m & , & d & , & e & ; \\ \frac{1}{2}k, 1+a-c, d+e-a-m, d+e+m+1, & e-c+1 & , & d-c+1 & & & \end{matrix}\right],$$

where $k=d+e-c$.

If now we let $e=\frac{1}{2}(1+a)$ in 2.4(2) we obtain

(3) $${}_4F_3\left[\begin{matrix} a, 1+\frac{1}{2}a, & c & , & d & ; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d & \end{matrix}\right] \text{ to } (m+1) \text{ terms}$$

$$= \frac{\Gamma(1+a+m)\Gamma(1+d+m)\Gamma(1+a-d)\Gamma(1-c)\Gamma(\frac{1}{2}+\frac{1}{2}a)}{\Gamma(1+a-d+m)\Gamma(1+m)\Gamma(1+a)\Gamma(d-c+1)\Gamma(\frac{1}{2}+\frac{1}{2}a+m)} \times$$

$$\times \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a-d+m)\Gamma(3/2+\frac{1}{2}a+m)\Gamma(3/2+\frac{1}{2}a+d-c)}{\Gamma(3/2+\frac{1}{2}a+d+m)\Gamma(\frac{1}{2}+\frac{1}{2}a-d)\Gamma(3/2+\frac{1}{2}a-c)} \times$$

$${}_7F_6\left[\begin{matrix} k, 1+\frac{1}{2}k, \frac{1}{2}(1-a)+d, & 1+a-c+m & , & -c-m & , & d & , & \frac{1}{2}+\frac{1}{2}a \;; \\ \frac{1}{2}k, 1+a-c, \frac{1}{2}(1-a)+d-m, & \frac{1}{2}(3+a)+d+m, & & \frac{1}{2}(3+a)-c & , & d-c+1 & & \end{matrix}\right],$$

where $k=\frac{1}{2}(1+a)+d-c$, which gives the sum of a well-poised

${}_4F_3$ to (m+1) terms in terms of a non-terminating well-poised ${}_7F_6$.

However, if we let e tend to (1+a+m) and then let d tend to infinity in 2.4(1), we get

$$(4)\qquad {}_4F_3\left[\begin{matrix}a, 1+\tfrac{1}{2}a, & b\ , & c\ ; \\ \tfrac{1}{2}a, 1+a-b, 1+a-c & & \end{matrix}\ -1\right] \text{ to } (m+1) \text{ terms}$$

$$= \frac{(-1)^m\ (1+a)_m}{m!}\ {}_3F_2\left[\begin{matrix}1+a-b-c, 1+a+m, -m; \\ 1+a-b\ ,\ 1+a-c\end{matrix}\right],$$

a formula due to Bailey (2) which gives the sum to (m+1) terms of a well-poised ${}_4F_3(-1)$ with special kind of second parameter, in terms of a terminating ${}_3F_2$.

If we now let c tend to infinity in 2.4(4) we get the sum of a well-poised ${}_3F_2$ to (m+1) terms in terms of a terminating ${}_2F_1$ with unit argument. Summing the ${}_2F_1$ by Gauss's theorem we get

$${}_3F_2\left[\begin{matrix}a, 1+\tfrac{1}{2}a, & b\ ; \\ \tfrac{1}{2}a, 1+a-b & \end{matrix}\right] \text{ to } (m+1) \text{ terms } = \frac{(1+a)_m(1+b)_m}{m!\ (1+a-b)_m}.$$

An interesting case is obtained if we take a=c=d=e=½ , in 2.4(2). We get

$$\left[1 + 5(1/2)^4 + 9(1.3/2.4)^4 + \ldots\ldots\ldots \text{to } (m+1) \text{ terms}\right]$$

$$= \left\{\Gamma(m+3/2)/\Gamma(m+1)\right\}^4 \left\{1/(m+1)(m+\tfrac{1}{2}) + 5/(m+2)(m-\tfrac{1}{2})\ (\tfrac{1}{2})^4 + 9/(m+3)(m-3/2)\ (1.3/2.4)^4 + \ldots\right\},$$

a result similar to 2.1(1) given by Ramanujan.

(2.5) <u>Basic Series</u>. Passing on to the consideration of the partial sums of series of basic hypergeometric type, we shall first give some elementary results and then discuss the analogues of the theorems already derived for ordinary hypergeometric series.

Let us start with Watson's formula (T.8.5(2)) for transforming a well-poised terminating ${}_8\Phi_7$ into a Saalchützian ${}_4\Phi_3$, viz.

(1) $${}_8\Phi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N};\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{N+1}\end{matrix}\; a^2q^{N+2}/bcde\right]$$

$$= \prod_{n=1}^{N}\frac{(1-aq^n)(1-aq^n/de)}{(1-aq^n/d)(1-aq^n/e)}\; {}_4\Phi_3\left[\begin{matrix}aq/bc, d, e, q^{-N};\\ deq^{-N}/a, aq/b, aq/c\end{matrix}\; q\right].$$

If in 2.5(1) we take $bcde=a^2q^{N+1}$, we get the analogue of Dougall's theorem, viz.

(2) $${}_8\Phi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N};\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{N+1}\end{matrix}\; q\right]$$

$$= \frac{(aq;N)(aq/cd;N)(aq/bd;N)(aq/bc;N)}{(aq/b;N)(aq/c;N)(aq/d;N)(aq/bcd;N)},$$

provided $bcde=a^2q^{N+1}$.

Let e tend to aq^{N+1} in 2.5(2) then we get

(3) $${}_6\Phi_5\left[\begin{matrix}a, q\sqrt{a}, & -q\sqrt{a}, & b\;, & c\;, & d\;; \\ & \sqrt{a}, & -\sqrt{a}\;, aq/b, & aq/c, & aq/d\end{matrix}\; q\right] \text{ to } (N+1) \text{ terms}$$

$$= \frac{(aq;N)(bq;N)(cq;N)(dq;N)}{(aq/b;N)(aq/c;N)(aq/d;N)(q;N)},$$

provided bcd=a, which gives the sum of a well-poised ${}_6\Phi_5$ upto (N+1) terms.

Substituting for d from bcd=a in 2.5(3) and letting a tend to zero we get

(4) $${}_2\Phi_1\left[\begin{matrix}b\;, & c\;; \\ bcq & \end{matrix}\; q\right] \text{ to } (N+1) \text{ terms } = \frac{(bq;N)(cq;N)}{(bcq;N)(q;N)},$$

which gives the basic analogue of 2.3(ii) in particular case when C=D+E+1.

In 2.5(1) let b tend to aq^{N+1}. This gives

(5) $${}_6\Phi_5\left[\begin{matrix}a, q\sqrt{a}, & -q\sqrt{a}, & c\;, & d\;, & e\;; \\ & \sqrt{a}, & -\sqrt{a}\;, aq/c, & aq/d, & aq/e\end{matrix}\; aq/cde\right] \text{ to } (N+1) \text{ terms}$$

$$= \prod_{n=1}^{N} \frac{(1-aq^n)(1-aq^n/de)}{(1-aq^n/d)(1-aq^n/e)}\; {}_3\Phi_2\left[\begin{matrix}q^{-N}/c, & d\;, & e\;; \\ deq^{-N}/a, & aq/c & \end{matrix}\; q\right] \text{ to } (N+1) \text{ terms},$$

which is the basic analogue of the sum of (N+1) terms of a well-poised ${}_5F_4$ in terms of the sum of (N+1) terms of a ${}_3F_2$ discussed in 2.4.

If we let c=aq/d in 2.5(5) we have

$${}_4\Phi_3\left[\begin{matrix}a, q\sqrt{a}\ -q\sqrt{a},\ e\ ; \\ \sqrt{a},\ -\sqrt{a}, aq/e\end{matrix}\ 1/e\right] \text{ to } (N+1) \text{ terms}$$

$$= \prod_{n=1}^{N} \frac{(1-aq^n)(1-aq^n/de)}{(1-aq^n/d)(1-aq^n/e)}\ {}_2\Phi_1\left[\begin{matrix}dq^{-N-1}/a,\ e\ ; \\ deq^{-N}/a\end{matrix}\ q\right] \text{ to } (N+1) \text{ terms.}$$

Summing the ${}_2\Phi_1$ on the right by the result 2.5(4) we get

$${}_4\Phi_3\left[\begin{matrix}a, q\sqrt{a},\ -q\sqrt{a},\ e\ ; \\ \sqrt{a},\ -\sqrt{a}, aq/e\end{matrix}\ 1/e\right] \text{ to } (N+1) \text{ terms}$$

$$= \prod_{n=1}^{N} \frac{(1-aq^n)(1-eq^n)}{(1-q^n)(e-aq^n)} .$$

If however, we let e tend to aq^{N+1} in 2.5(1) we get the sum of a ${}_6\Phi_5$ upto (N+1) terms in terms of a terminating ${}_4\Phi_3$.

(2.6) In this section we will now derive the basic analogue of the main theorem 2.2(1). Consider now one of the most general transformations for basic series due to Bailey (T.8.5(1)),namely

$${}_{10}\Phi_9\left[\begin{matrix}a, q\sqrt{a},\ -q\sqrt{a},\ b\ ,\ c\ ,\ d\ ,\ e\ ,\ f\ , akq^{N+1}/ef,\ q^{-N}\ ; \\ \sqrt{a},\ -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, ef/kq^N\ , aq^{N+1}\end{matrix}\ q\right]$$

(1) $$= \prod_{n=1}^{N} \frac{(1-aq^n)(e-kq^n)(f-kq^n)(ef-aq^n)}{(1-kq^n)(e-aq^n)(f-aq^n)(ef-kq^n)} \times$$

$$\times {}_{10}\Phi_9\left[\begin{matrix} k, q\sqrt{k}, & -q\sqrt{k}, kb/a, kc/a, kd/a, & e, & f, akq^{N+1}/ef, & q^{-N}; \\ \sqrt{k}, & -\sqrt{k}, aq/b, aq/c, aq/d, kq/e, kq/f, fe/aq^N & & , kq^{N+1} & \end{matrix} q\right],$$

where $k=a^2q/bcd$.

Putting $f=a/e$ we get

(2)

$${}_8\Phi_7\left[\begin{matrix} k, q\sqrt{k}, & -q\sqrt{k}, kb/a, kc/a, kd/a, & e, & a/e; \\ \sqrt{k}, & -\sqrt{k}, aq/b, aq/c, aq/d, kq/e, keq/a & & \end{matrix} q\right] \text{ to } (N+1) \text{ terms}$$

$$= \prod_{n=1}^{N} \frac{(1-kq^n)(e-aq^n)(1-eq^n)(a-kq^n)}{(1-aq^n)(e-kq^n)(1-q^n)(a-keq^n)} \times$$

$$\times {}_{10}\Phi_9\left[\begin{matrix} a, q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, a/e, & kq^{N+1}, & q^{-N}; \\ \sqrt{a}, & -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, & eq, aq^{-N}/k, aq^{N+1} \end{matrix} q\right],$$

where $k=a^2q/bcd$.

Now put $c=q^{-M}$ (M, a positive integer) in 2.6(2) and we see that the ${}_{10}\Phi_9$ on the right becomes symmetrical in M and N and hence

$$\prod_{n=1}^{N} \frac{(1-kq^n/e)(1-aq^n)(1-keq^n/a)(1-q^n)}{(1-kq^n)(1-aq^n/e)(1-eq^n)(1-kq^n/a)} \times$$

$${}_8\Phi_7\left[\begin{matrix} k, q\sqrt{k}, & -q\sqrt{k}, kb/a, aq/bd, kd/a, & e, & a/e; \\ \sqrt{k}, & -\sqrt{k}, aq/b, aq^{M+1}, aq/d, kq/e, keq/a & & \end{matrix} q\right] \text{ to } (N+1) \text{ terms}$$

= the same expression with M and N interchanged,

where $k=a^2q^{M+1}/bd$.

This on some simplification of the products gives

$$\frac{(a^2q^2/bd;M)(aq^2/bd;M)(aq/e;M)(eq;M)}{(a^2q^2/bde;M)(aeq^2/bd;M)(aq;M)(q;M)} \times$$

$${}_8\Phi_7\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, kb/a, aq/bd, kd/a, e, a/e ; \\ \sqrt{k}, -\sqrt{k}, aq/b, aq^{M+1}, aq/d, kq/e, keq/a \end{matrix} q\right] \text{ to } (N+1) \text{ terms}$$

= the same expression with M and N interchanged,

where $k=a^2q^{M+1}/bd$.

Again putting

$$k = X$$

$$e = D$$

$$a/e = E$$

$$a^2q/bd = A$$

$$aq/b = C,$$

we get the final result in the form

$$(3) \qquad \frac{(Aq;M)(Aq/DE;M)(Eq;M)(Dq;M)}{(Aq/D;M)(Aq/E;M)(DEq;M)(q;M)} \times$$

$${}_8\Phi_7\left[\begin{matrix} X, q\sqrt{X}, -q\sqrt{X}, qX/C, A/DE, Cq^M, D, E ; \\ \sqrt{X}, -\sqrt{X}, C, DEq^{M+1}, Aq/C, qX/D, qX/E \end{matrix} q\right] \text{ to } (N+1) \text{ terms}$$

= the same expression with M and N interchanged,

where $X=Aq^M$.

This gives the exact basic analogue for the main theorem proved in 2.2. As in 2.2 I give here also a very simple alternative proof for the above theorem.

Second Proof of 2.6(3). Let us suppose as before that $N > M$. Then the terms containing powers of C on the left hand side of 2.6(3) are of the form

$$\frac{(qX/C;r)(Cq^{M};r)}{(C;r)(Aq/C;r)} .$$

Now multiplying both sides by $(C;M)(Aq/C;M)\ C^{M}$ we find that the highest power of C on either sides of 2.6(3) is 2M. Thus, the relation 2.6(3) can be treated as a relation between two polynomials in C of degree 2M. So, if we can show that these polynomials are equal for (2M+1) different values of C we shall have proved the result.

To do so let C have the (2M+1) values $q^{-N}, q^{-N-1}, \ldots\ldots, q^{-N-M}, Aq^{2+N}, Aq^{3+N}, \ldots\ldots, Aq^{M+N+1}$. It is easily seen that for each of these (2M+1) values of C the series become complete hypergeometric series which are summable by the analogue of Dougall's theorem and the verification is immediate.

Hence, our theorem follows.

(2.7) Particular cases of 2.6(3).

(i) Letting M tend to infinity in 2.6(3) we get

$${}_3\Phi_2\left[\begin{matrix}A/DE,\ D\ ,\ E\ ; \\ C\ ,\ Aq/C\end{matrix}\ q\right] \text{ to } (N+1) \text{ terms}$$

$=$

$$= \prod_{n=N}^{\infty} \frac{(1-Aq^{n+1}/D)(1-Aq^{n+1}/E)(1-DEq^{n+1})(1-q^{n+1})}{(1-Aq^{n+1})(1-Aq^{n+1}/DE)(1-Eq^{n+1})(1-Dq^{n+1})} \times$$

$$\times {}_8\Phi_7\left[\begin{matrix} X', q\sqrt{X'}, -q\sqrt{X'}, qX'/C, A/DE, Cq^N, D, E; \\ \sqrt{X'}, -\sqrt{X'}, C, DEq^{N+1}, Aq/C, qX'/D, qX'/E \end{matrix} q\right],$$

where $X' = Aq^N$.

This gives the basic analogue of Bailey's result 2.3(iv), for the partial sum of a ${}_3F_2$ in terms of an infinite ${}_7F_6$. If we transform the ${}_8\Phi_7$ on the right into two Saalchützian ${}_4\Phi_3$ by means of the following known transformation due to Bailey (5;4.5)

$${}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f; \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix} a^2q^2/bcdef\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-aq^n/cd)(1-aq^n/cf)(1-aq^n/ce)(1-bq^{n-1})(1-aq^n)}{(1-aq^n/c)(1-aq^n/d)(1-aq^n/e)(1-aq^n/f)(1-bq^{n-1}/c)} \times \frac{(1-a^2q^{n+1}/bdef)}{(1-a^2q^{n+1}/bcdef)}\right] \times$$

$$\times {}_4\Phi_3\left[\begin{matrix} aq/bd, aq/be, aq/bf, c; \\ a^2q^2/bdef, aq/b, cq/b \end{matrix} q\right]$$

+ a similar expression with b and c interchanged,

we get

$$ {}_3\Phi_2\left[\begin{matrix} A/DE\ ,\ D\ ,\ E\ ; \\ C\ ,\ Aq/C \end{matrix}\ q\right] \quad \text{to } (N+1) \text{ terms} $$

$$ = \prod_{n=N}^{\infty} \frac{(1-Aq^{n+1}/C)(1-Cq^{n})(1-q^{n+1})}{(1-Aq^{n+1}/DE)(1-Eq^{n+1})(1-Dq^{n+1})} \times $$

$$ \times\left[\ 1/(1-Cq^{N}) \prod_{n=1}^{\infty} \frac{(1-DEq^{n}/C)(1-Aq^{n}/CD)(1-Aq^{n}/CE)}{(1-Aq^{n}/C)(1-Aq^{n}/C^{2})(1-q^{n})} \times \right. $$

$$ \times\ {}_4\Phi_3\left[\begin{matrix} CDE/A, C/D, C/E, Cq^{N}\ ; \\ Cq^{N+1}, C, C^{2}/A \end{matrix}\ q\right] + $$

$$ + 1/(1-Aq^{N+1}/C) \prod_{n=1}^{\infty} \frac{(1-CDEq^{n-1}/A)(1-Cq^{n-1}/D)(1-Cq^{n-1}/E)}{(1-Cq^{n-1})(1-C^{2}q^{n-2}/A)(1-q^{n})} \times $$

$$ \left. \times\ {}_4\Phi_3\left[\begin{matrix} DEq/C, Aq/CD, Aq/CE, Aq^{N+1}/C; \\ Aq^{N+2}/C, Aq/C, Aq^{2}/C^{2} \end{matrix}\ q\right]\ \right], $$

which is the basic analogue of Darling's result given in 2.3(IV).

(ii) Putting $E= Aq/C$ in 2.6(3) we get

$$ \frac{(Aq;M)(C/D;M)(Aq^{2}/C;M)(Dq;M)}{(Aq/D;M)(C;M)(ADq^{2}/C;M)(q;M)} \times $$

$$ \times\ {}_6\Phi_5\left[\begin{matrix} X, q\sqrt{X}, -q\sqrt{X}, qX/C, & C/Dq\ , & D\ ; \\ \sqrt{X}, -\sqrt{X}, C\ , & ADq^{M+2}/C, & qX/D \end{matrix}\ q\right] \quad \text{to } (N+1) \text{ terms} $$

= the same expression with M and N interchanged, where $X=Aq^{M}$. This is the analogue of the result 2.3(v).

(iii) Let A tend to infinity in 2.6(3) and we get

$$\frac{(Eq;M)(Dq;M)}{(DEq;M)(q;M)} \; {}_3\Phi_2\left[\begin{matrix} Cq^M , D , E ; \\ C , DEq^{M+1} \end{matrix} q\right] \text{ to } (N+1) \text{ terms}$$

= the same expression with M and N interchanged.

This is the exact analogue of Bailey's result 2.3(i), from which was deduced the theorem due to Ramanujan stated in 2.1(1).

(iv) If in (iii) above we make M tend to infinity we get

$${}_2\Phi_1\left[\begin{matrix} D , E ; \\ C \end{matrix} q\right] \text{ to } (N+1) \text{ terms}$$

$$= \prod_{n=N}^{\infty} \frac{(1-DEq^{n+1})(1-q^{n+1})}{(1-Eq^{n+1})(1-Dq^{n+1})} \; {}_3\Phi_2\left[\begin{matrix} Cq^N , D , E ; \\ C , DEq^{N+1} \end{matrix} q\right] ,$$

which is the basic analogue of the result given in 2.3(ii).

(v) Next, putting C=DEq in (iv), we get the sum of the Saalchützian ${}_2\Phi_1$ upto (N+1) terms given in 2.5(4).

(vi) Putting $e^2=kq$ in 2.6(2), we get

$${}_7\Phi_6\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, a/\sqrt{kq} ; \\ \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, (kq)^{3/2}/a \end{matrix} q\right] \text{ to } (N+1) \text{ terms}$$

=

$$= \prod_{n=1}^{N} \frac{(1-kq^n)(1-aq^{n-\frac{1}{2}}/\sqrt{k})(1-\sqrt{k}\, q^{n+\frac{1}{2}})(1-kq^n/a)}{(1-aq^n)(1-\sqrt{k}\, q^{n-\frac{1}{2}})(1-q^n)(1-k\sqrt{k}\, q^{n+\frac{1}{2}}/a)} \times$$

$$\times {}_{10}\Phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, (kq)^{\frac{1}{2}}, a/(kq)^{\frac{1}{2}}, kq^{N+1}, q^{-N}; \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq^{\frac{1}{2}}/k^{\frac{1}{2}}, k^{\frac{1}{2}}q^{3/2}, aq^{-N}/k, aq^{N+1} \end{matrix} q\right],$$

which gives the sum to (N+1) terms of a well-poised ${}_7\Phi_6$ in terms of a terminating well-poised ${}_{10}\Phi_9$.

(vii) Again, taking $e = aq/d$ in 2.6(2) we get

$${}_6\Phi_5\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, d/q; \\ \sqrt{k}, -\sqrt{k}, aq/b, aq/c, kq^2/d \end{matrix} q\right] \text{ to (N+1) terms}$$

$$= \prod_{n=1}^{N} \frac{(1-kq^n)(1-dq^{n-1})(1-aq^{n+1}/d)(1-kq^n/a)}{(1-aq^n)(1-kdq^{n-1}/a)(1-kq^{n+1}/d)(1-q^n)} \times$$

$$\times {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d/q, kq^{N+1}, q^{-N}; \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^2/d, aq^{-N}/k, aq^{N+1} \end{matrix} q\right],$$

where $k = a^2q/bcd$. This gives the partial sum of a well-poised ${}_6\Phi_5$ in terms of a terminating ${}_8\Phi_7$.

Finally, it may be mentioned, for completeness, that from relations connecting four well-poised ${}_{10}\Phi_9$'s given by Bailey (6;7.2 & 8.1), by putting $c = q^{-N}$, and then letting b tend to aq^{N+1} , we obtain two formulae each of which gives the sum of (N+1) terms of the series

$$ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, & -q\sqrt{a}, & d, & e, & f, & g, & h; \\ \sqrt{a}, & -\sqrt{a}, & aq/d, & aq/e, & aq/f, & aq/g, & aq/h \end{matrix} q\right] $$

where $a^2q = defgh$, in terms of two infinite well-poised series. The formulae are too complicated to be of any interest and hence have not been discussed.

It may also be remarked that it does not seem possible to give a more general result of the type 2.2(1) or 2.6(3) than these. Also, one can apply the method to finding the partial sums of ordinary and basic bilateral series of the hypergeometric type given by Bailey (5), but it has not been found possible to obtain results of the type 2.2(1) or 2.6(3) in the case of these series. We can either find results by terminating a bilateral series on one side only or on both sides. The former type of transformations are equivalent to the results on partial sums of the unilateral series discussed in this chapter and the only importance of the second type of results appears to be in their mere existance, and hence they have not been discussed here.

CHAPTER III

GENERAL TRANSFORMATIONS OF BILATERAL COGNATE TRIGONOMETRICAL SERIES OF ORDINARY HYPERGEOMETRIC TYPE

(3.1) Introduction. This chapter is concerned with the development of the transformation theory of bilateral trigonometric series of the hypergeometric type. The trigonometric series arise as limiting cases of hypergeometric series on their circle of convergence. These series were first studied by Whipple (3) in 1936 who considered the transformations connecting well-poised hypergeometric series as particular cases of relations between cognate trigonometrical series. He used the integrals of the Barnes's type to deduce such transformations. Later, very recently Sears (3) gave a systematic theory of general and well-poised transformations of trigonometrical series of any order, which include Whipple's result as particular cases. In this chapter I have used Sears' (3) results on the transformations of trigonometrical series to deduce general transformations connecting bilteral trigonometrical series. These transformations for particular values of θ yield the known results of Bailey (5), Slater (3) and Sears (3).

Let u_n denote the (n+1)th term of the hypergeometric series

$$F(a_1,\ldots,a_{M+1};b_1,\ldots\ldots,b_M;)\ .$$

Then the series

$$\sum_0^\infty(-)^n u_n \sin(K+2n)\theta \quad , \quad \sum_0^\infty u_n \sin(K+2n)\theta$$

$$\sum_0^\infty(-)^n u_n \cos(K+2n)\theta \quad , \quad \sum_0^\infty u_n \cos(K+2n)\theta$$

will be denoted by the symbols

$$_{M+1}S_M\begin{bmatrix}a_1,\ldots\ldots,a_{M+1};\theta\\ b_1,\ldots\ldots,b_M\ \ ;K\end{bmatrix} \quad , \quad _{M+1}S'_M\begin{bmatrix}a_1,\ldots\ldots,a_{M+1};\theta\\ b_1,\ldots\ldots,b_M\ \ ;K\end{bmatrix}$$

$$_{M+1}C_M\begin{bmatrix}a_1,\ldots\ldots,a_{M+1};\theta\\ b_1,\ldots\ldots,b_M\ \ ;K\end{bmatrix} \quad , \quad _{M+1}C'_M\begin{bmatrix}a_1,\ldots\ldots,a_{M+1};\theta\\ b_1,\ldots\ldots,b_M\ \ ;K\end{bmatrix},$$

respectively.When $K=a_1$, the first numerator parameter, it will be omitted from each symbol,and following Whipple,series of the type

$$_{M+1}S_M\begin{bmatrix}a_1,a_2,\ldots\ldots\ldots,a_{M+1};\theta\\ 1+a_1-a_2,\ldots\ldots,1+a_1-a_{M+1}\end{bmatrix}$$

in which $K=a_1$ necessarily,will be called well-poised. We will sometimes use the abbreviated notation $_{M+1}S_M(a_1)$ for this series.We will also make use of the following notation for the well-poised trigonometric series

$$ {}_{M+1}S_M(a_r) \equiv {}_{M+1}S_M \left[\begin{matrix} 2a_r-a_1, a_r, a_r+a_2-a_1, \ldots\ldots, a_r+a_{M+1}-a_1; \theta \\ 1+a_r-a_1, \ldots\ldots\ldots\ldots\ldots\ldots, 1+a_r-a_{M+1} \end{matrix} \right] $$

for $r > 1$; where the numerator parameter $(a_r+a_r-a_1)$ is the first to occur and the denominator parameter $(1+a_r-a_r)$ is omitted. Similar notations will be used for the series C,S',C'.

Now let us define the four bilateral trigonometrical series as follows:

$$ \begin{matrix} (1) \\ (2) \end{matrix} \qquad \begin{matrix} {}_MX_M \\ {}_MZ_M \end{matrix} \left[\begin{matrix} a_1, \ldots\ldots\ldots, a_M; \theta \\ b_1, \ldots\ldots\ldots, b_M; K \end{matrix} \right] $$

$$ = \sum_{n=-\infty}^{\infty} (-1)^n \frac{(a_1)_n \ldots\ldots (a_M)_n}{(b_1)_n \ldots\ldots (b_M)_n} \begin{matrix} \sin \\ \cos \end{matrix} \left[(K+2n)\theta \right] , $$

$$ \begin{matrix} (3) \\ (4) \end{matrix} \qquad \begin{matrix} {}_MX'_M \\ {}_MZ'_M \end{matrix} \left[\begin{matrix} a_1, \ldots\ldots\ldots, a_M; \theta \\ b_1, \ldots\ldots\ldots, b_M; K \end{matrix} \right] $$

$$ = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n \ldots\ldots (a_M)_n}{(b_1)_n \ldots\ldots (b_M)_n} \begin{matrix} \sin \\ \cos \end{matrix} \left[(K+2n)\theta \right] . $$

It is easily seen that

$$ \begin{matrix} (5) \\ (6) \end{matrix} \qquad \begin{matrix} {}_MX_M \\ {}_MZ_M \end{matrix} \left[\begin{matrix} a_1, \ldots\ldots\ldots, a_M; \theta \\ b_1, \ldots\ldots\ldots, b_M; K \end{matrix} \right] $$

$=$

$$= \begin{matrix} {}_{M+1}S_M \\ {}_{M+1}C_M \end{matrix} \begin{bmatrix} 1, a_1, \ldots\ldots, a_M; \theta \\ b_1, \ldots\ldots, b_M; K \end{bmatrix} \pm \frac{(1-b_1)\ldots\ldots(1-b_M)}{(1-a_1)\ldots\ldots(1-a_M)} \times$$

$$\times \begin{matrix} {}_{M+1}S_M \\ {}_{M+1}C_M \end{matrix} \begin{bmatrix} 1, 2-b_1, \ldots\ldots, 2-b_M; \theta \\ 2-a_1, \ldots\ldots, 2-a_M; 2-K \end{bmatrix} ,$$

where on the right hand side the positive or negative sign is to be taken according as the series is of the type ${}_M X_M$ or ${}_M Z_M$.

Similarly,

(7)
(8)

$$\begin{matrix} {}_M X'_M \\ {}_M Z'_M \end{matrix} \begin{bmatrix} a_1, \ldots\ldots\ldots, a_M; \theta \\ b_1, \ldots\ldots\ldots, b_M; K \end{bmatrix}$$

$$= \begin{matrix} {}_{M+1}S'_M \\ {}_{M+1}C'_M \end{matrix} \begin{bmatrix} 1, a_1, \ldots\ldots, a_M; \theta \\ b_1, \ldots\ldots, b_M; K \end{bmatrix} \mp \frac{(1-b_1)\ldots\ldots(1-b_M)}{(1-a_1)\ldots\ldots(1-a_M)} \times$$

$$\times \begin{matrix} {}_{M+1}S'_M \\ {}_{M+1}C'_M \end{matrix} \begin{bmatrix} 1, 2-b_1, \ldots\ldots, 2-b_M; \theta \\ 2-a_1, \ldots\ldots, 2-a_M; 2-K \end{bmatrix} ,$$

where on the right hand side the negative or positive sign is to be taken according as the series is of the type ${}_M X'_M$ or ${}_M Z'_M$.

The convergence factor $Rl(\sum_1^M b_r - \sum_1^M a_r - 1)$ will be, henceforth, denoted by "y".

The series 3.1(1) and 3.1(2) converge when either

$$y > 0 \quad , \quad -\pi \leq 2\theta \leq \pi$$

or

$$-1 < y \leq 0 \quad , \quad -\pi < 2\theta < \pi \quad .$$

The series 3.1(3) and 3.1(4) converge when either

$$y > 0 \quad , \quad 0 \leq \theta \leq \pi$$

or

$$-1 < y \leq 0 \quad , \quad 0 < \theta < \pi .$$

When $y > 0$ all the series converge uniformly and absolutely in the variable θ or in K, but when $-1 < y \leq 0$ the convergence is, in general, conditional. θ is restricted to be real.

(3.2) Notation. The following notation due to Sears will be used throughout this chapter.

Let

$$G(a_1, \ldots, a_M; b_1, \ldots, b_N) = \left\{\prod_1^M \Gamma(a_r)\right\} \left\{\prod_1^N \Gamma(b_r)\right\}^{-1} ,$$

$$A = \left\{\prod_1^M G(a_r; b_r)\right\} \left\{\prod_{M+1}^{M+N} G(1-b_r; 1-a_r)\right\} ,$$

$$A(a_1) = 1/\Gamma(b_1-a_1) \left\{\prod_2^M G(a_r-a_1; b_r-a_1)\right\} \left\{\prod_{M+1}^{M+N} G(1+a_1-b_r; 1+a_1-a_r)\right\},$$

$$A(b_{M+1}) = 1/\Gamma(b_{M+1}-a_{M+1}) \left\{\prod_1^M G(1+a_r-b_{M+1}; 1+b_r-b_{M+1})\right\} \times \left\{\prod_{M+2}^{M+N} G(b_{M+1}-b_r; b_{M+1}-a_r)\right\} ,$$

$$B = (A \operatorname{cosec} \pi a_{M+1}) / \Gamma(1-a_{M+N+1}) ,$$

$$B(a_1) = A(a_1) \; G(a_1; 1+a_1-a_{M+N+1}) \operatorname{cosec} \pi(a_{M+1}-a_1) ,$$

$$B(b_{M+1}) = - A(b_{M+1}) \; G(b_{M+1}-1; b_{M+1}-a_{M+N+1}) \operatorname{cosec} \pi(b_{M+1}-a_{M+1})$$

$$P = \Gamma(a_1) \left\{\prod_2^{M+1} G(a_r, a_r-a_1)\right\} \left\{\prod_{M+2}^{2M} G(1+a_1-a_r, 1-a_r)\right\}^{-1} ,$$

$$Q(a_2) = \left\{\prod_{1}^{M+1}{}' G(a_r-a_2, a_2+a_r-a_1)\right\} \left\{\prod_{M+2}^{2M} G(1+a_2-a_r, 1+a_1-a_2-a_r)\right\}^{-1},$$

$$R = P/\ G(1+a_1-a_{2M+1}, 1-a_{2M+1}) ,$$

$$T(a_2) = Q(a_2)/\ G(1+a_2-a_{2M+1}, 1+a_1-a_2-a_{2M+1}) ,$$

$$U = \Gamma(a_1)\left\{\prod_{2}^{M+2} G(a_r, a_r-a_1)\right\} \left\{\prod_{M+3}^{2M+1} G(1+a_1-a_r, 1-a_r)\right\}^{-1},$$

$$V(a_2) = \left\{\prod_{1}^{M+2}{}' G(a_r-a_2, a_r+a_2-a_1)\right\} \left\{\prod_{M+3}^{2M+1} G(1+a_2-a_r, 1+a_1-a_2-a_r)\right\}^{-1},$$

$$X = U\ G(1+a_1-a_{2M+1}, 1-a_{2M+1}) ,$$

$$Y(a_2) = V(a_2)\ G(1+a_2-a_{2M+1}, 1+a_1-a_2-a_{2M+1}) ,$$

" idem (a;b)" means that the preceding expression is to be repeated with b written in place of a and vice-versa. The accents in the product symbols denote the omission of the gamma functions with zero argument.

(3.3) <u>The general bilateral transformations.</u>

Sears (3;10.2-10.5) has proved the following four general transformations for general trigonometrical series

$$\begin{matrix}(1)\\(2)\end{matrix} \quad B \begin{matrix}S\\C\end{matrix}\left[\begin{matrix}a_1,\dots\dots,a_{M+N+1};\theta\\ b_1,\dots\dots,b_{M+N}\quad;K\end{matrix}\right]$$

$$= \mp B(a_1) \begin{matrix}S\\C\end{matrix}\left[\begin{matrix}a_1, 1+a_1-b_1,\dots\dots, 1+a_1-b_{M+N};\theta\\ 1+a_1-a_2,\dots\dots, 1+a_1-a_{M+N+1};K_1\end{matrix}\right] \mp$$

$$\mp \text{idem}(a_1; a_2,\dots\dots, a_{M+1}) +$$

$$+ B(b_{M+1}) \quad \begin{matrix} S \\ C \end{matrix} \begin{bmatrix} 1+a_1-b_{M+1}, \ldots\ldots\ldots\ldots, 1+a_{M+N+1}-b_{M+1}; \theta \\ 2-b_{M+1}, 1+b_1-b_{M+1}, \ldots, 1+b_{M+N}-b_{M+1} \; ; K_2 \end{bmatrix} +$$

$$+ \text{idem}(b_{M+1}; b_{M+2}, \ldots\ldots, b_{M+N}) \; .$$

where $K_1 = -K+2a_1(1-m\pi/\theta)$,

$K_2 = K+2-2b_{M+1}(1-m\pi/\theta)$.

These results are valid when either

$y > 0$ and $(2m-1)\pi \leqslant 2\theta \leqslant (2m+1)\pi$,

or $-1 < y \leqslant 0$ and $(2m-1)\pi < 2\theta < (2m+1)\pi$,

and on the right the upper or the lower sign is to be taken according as we consider the S- or the C-transformation.

The other two transformations are

$$\begin{matrix} (3) \\ (4) \end{matrix} \quad B \quad \begin{matrix} S' \\ C' \end{matrix} \begin{bmatrix} a_1, \ldots\ldots\ldots\ldots, a_{M+N+1}; \theta \\ b_1, \ldots\ldots\ldots\ldots, b_{M+N} \; ; K \end{bmatrix}$$

$$= \mp B(a_1) \quad \begin{matrix} S' \\ C' \end{matrix} \begin{bmatrix} a_1, 1+a_1-b_1, \ldots\ldots, 1+a_1-b_{M+N}; \; \theta \\ 1+a_1-a_2, \ldots\ldots\ldots, 1+a_1-a_{M+N+1}; K_3 \end{bmatrix} \mp$$

$$\mp \text{idem}(a_1; a_2, \ldots\ldots, a_{M+1}) -$$

$$-B(b_{M+1}) \quad \begin{matrix} S' \\ C' \end{matrix} \begin{bmatrix} 1+a_1-b_{M+1}, \ldots\ldots\ldots, 1+a_{M+N+1}-b_{M+1}; \theta \\ 2-b_{M+1}, 1+b_1-b_{M+1}, \ldots, 1+b_{M+N}-b_{M+1}; K_4 \end{bmatrix} -$$

$$- \text{idem}(b_{M+1}; b_{M+2}, \ldots\ldots, b_{M+N}) \quad ,$$

where

$$K_3 = -K+2a_1-(2m+1)\pi a_1/\theta \quad ,$$
$$K_4 = K+2(1-b_{M+1})+(2m+1)b_{M+1}\pi/\theta \quad ,$$

These results are valid when either

$$y > 0 \quad \text{and} \quad m\pi \leqslant \theta \leqslant (m+1)\pi \quad ,$$

or $$-1 < y \leqslant 0 \quad \text{and} \quad m\pi \leqslant \theta < (m+1)\pi \quad ,$$

where m is a positive integer, and on the right the upper or the lower sign is to be taken according as we consider the S'- or the C'-transformation.

Now, in the four transformations 3.3(1-4) let us take N=M and put $b_{M+1}=a_1, \ldots\ldots, b_{M+N}=a_M$. We find that all the series reduce to the series of the type ${}_{M+1}S_M$ or ${}_{M+1}C_M$ or ${}_{M+1}S'_M$ or ${}_{M+1}C'_M$ respectively, according to the transformation considered. Letting now a_{M+1} tend to unity the series corresponding to the parameter a_{M+1} on the right combines with the series on the left hand side in each case to give a bilateral series of the type ${}_MX_M$ or ${}_MZ_M$ or ${}_MX'_M$ or ${}_MZ'_M$, as the case may be. Also, on the right hand side the series corresponding to the parameters a_i and b_{M+i} (i=1,2....,M) , combine to give bilateral series in each case. Thus, each of the transformations 3.3(1-4) gives a transformation connecting (M+1) bilateral series. Finally, changing the notation by putting $a_{M+2}=c_1$, $a_{M+3}=c_2$, etc. and in general, $a_{2M+1}=c_M$, we can write the four transformations combined into two as follows

(5)
(6)

$$\prod_{1}^{M} G(a_r, 1-a_r; b_r, 1-c_r) \quad \begin{matrix} {}_M X_M \\ {}_M Z_M \end{matrix} \begin{bmatrix} c_1, \ldots\ldots, c_M; \theta \\ b_1, \ldots\ldots, b_M; K \end{bmatrix}$$

$$= \mp \Gamma(a_1)\, \Gamma(1-a_1) \prod_{1}^{M}{}' G(a_r-a_1, 1+a_1-a_r; b_r-a_1, 1+a_1-c_r) \times$$

$$\times \begin{matrix} {}_M X_M \\ {}_M Z_M \end{matrix} \begin{bmatrix} 1+a_1-b_1, \ldots\ldots\ldots, 1+a_1-b_M; \theta \\ 1+a_1-c_1, \ldots\ldots\ldots, 1+a_1-c_M; K_1 \end{bmatrix} \mp$$

$$\mp \ \mathrm{idem}(a_1; a_2, \ldots\ldots, a_M) \quad ,$$

valid when either $y > 0$ and $(2m-1)\pi \leqslant 2\theta \leqslant (2m+1)\pi$,

or $-1 < y \leqslant 0$ and $(2m-1)\pi < 2\theta < (2m+1)\pi$.

The other two transformations are

(7)
(8)

$$\prod_{1}^{M} G(a_r, 1-a_r; b_r, 1-c_r) \quad \begin{matrix} {}_M X'_M \\ {}_M Z'_M \end{matrix} \begin{bmatrix} c_1, \ldots\ldots\ldots, c_M; \theta \\ b_1, \ldots\ldots\ldots, b_M; K \end{bmatrix}$$

$$= \mp \Gamma(a_1)\, \Gamma(1-a_1) \prod_{1}^{M}{}' G(a_r-a_1, 1+a_1-a_r; b_r-a_1, 1+a_1-c_r) \times$$

$$\times \begin{matrix} {}_M X'_M \\ {}_M Z'_M \end{matrix} \begin{bmatrix} 1+a_1-b_1, \ldots\ldots\ldots\ldots, 1+a_1-b_M; \theta \\ 1+a_1-c_1, \ldots\ldots\ldots\ldots, 1+a_1-c_M; K_3 \end{bmatrix} \mp$$

$$\mp \ \mathrm{idem}(a_1; a_2, \ldots\ldots\ldots, a_M) \quad ,$$

valid when either $y > 0$ and $m\pi \leqslant \theta \leqslant (m+1)\pi$,

or $-1 < y \leqslant 0$ and $m\pi < \theta < (m+1)\pi$.

If in any of the above bilateral transformations say in ,3.3(5) we put $b_M=1$, the left hand series becomes a ${}_MS_{M-1}$ series. If further, we put $a_1=c_1, a_2=c_2, \ldots\ldots, a_R=c_R$, the first R ${}_MX_M$ series on the right also become ${}_MS_{M-1}$ series. Similarly, if we reverse the order of summation of the last (M-R) series by means of the relation

$${}_MX_M\begin{bmatrix}a_1,\ldots\ldots,a_M;\theta\\ b_1,\ldots\ldots,b_M;K\end{bmatrix}$$

$$= {}_MX_M\begin{bmatrix}1-b_1,\ldots\ldots,1-b_M;\theta\\ 1-a_1,\ldots\ldots,1-a_M;\ \pi/\theta\ -K\end{bmatrix}$$

and then put $1+a_{R+1}=b_R$, $1+a_{R+2}=b_{R+1}$,....., $1+a_M=b_{M-1}$, these bilateral series, as well, reduce to ${}_MS_{M-1}$ series. Carrying out all these three operations 3.3(5) reduces to a restatement of Sears result 3.3(1). Similarly, we can get Sears other original transformations also from these bilateral transformations.

Each of these four transformations 3.3(5-8) contains as a special case transformations connecting (M+1) series of the type ${}_MH_M$ with argument +1 or -1. Thus putting $\theta=0$ and m=0 in 3.3(6), we get

(9) $$\prod_1^M G(a_r,1-a_r;b_r,1-c_r)\ {}_MH_M\begin{bmatrix}c_1,\ldots\ldots,c_M; & \\ & -1\\ b_1,\ldots\ldots,b_M; & \end{bmatrix}$$

$$= \Gamma(a_1)\,\Gamma(1-a_1)\ \prod_1^{M\,\prime} G(a_r-a_1,1+a_1-a_r;b_r-a_1,1+a_1-c_r)\ \times$$

$$\times\ {}_MH_M\left[\begin{matrix}1+a_1-b_1,\ldots\ldots,1+a_1-b_M; \\ 1+a_1-c_1,\ldots\ldots,1+a_1-c_M;\end{matrix}\ -1\right] + \text{idem}(a_1;a_2,\ldots,a_M),$$

which gives a relation connecting (M+1) general series of the type ${}_MH_M(-1)$.

If, however, we take $b_M=1$, and $a_r=c_r$ $(r=1,2,\ldots M)$, in 3.3(9), we get

$$(10)\qquad \Gamma(c_M)\ \prod_1^{M-1} G(c_r;b_r)\ \ {}_MF_{M-1}\left[\begin{matrix}c_1,\ldots\ldots,c_M; \\ b_1,\ldots\ldots,b_{M-1}\end{matrix}\ -1\right]$$

$$= \Gamma(c_1)\ \Gamma(c_M-c_1)\ \prod_1^{M-1}{}' G(c_r-c_1;b_r-c_1)\ \times$$

$$\times\ {}_MF_{M-1}\left[\begin{matrix}c_1,1+c_1-b_1,\ldots\ldots,1+c_1-b_{M-1}; \\ 1+c_1-c_2,\ldots\ldots\ldots,1+c_1-c_M\ ;\end{matrix}\ -1\right] +$$

$$+\ \text{idem}(c_1;c_2,\ldots\ldots,c_M)\quad,$$

which gives a relation between (M+1) general series of the type ${}_MF_{M-1}(-1)$. It is a particular case of a result due to Sears (3;10.6 for N=0).

(3.4) <u>Well-poised trigonometrical transformations</u>.

Sears (3;11.1-11.8 & 11.10) has proved the following nine transformations for well-poised trigonometrical series, for $M \geqslant 1$

(1) $$P \sin\tfrac{1}{2}\pi a_1 \; {}_{2M}C_{2M-1}(a_1) +$$
$$+ Q(a_2) \sin\tfrac{1}{2}\pi(2a_2-a_1) \; {}_{2M}C_{2M-1}(a_2) +$$
$$+ \text{idem}(a_2;a_3,\ldots\ldots,a_{M+1}) = 0 \;,$$

(2) $$P \cos\tfrac{1}{2}\pi a_1 \; {}_{2M}S_{2M-1}(a_1) +$$
$$+ Q(a_2) \cos\tfrac{1}{2}\pi(2a_2-a_1) \; {}_{2M}S_{2M-1}(a_2) +$$
$$+ \text{idem}(a_2;a_3,\ldots\ldots,a_{M+1}) = 0 \;,$$

(3) $$P \; {}_{2M}S'_{2M-1}(a_1) + Q(a_2) \; {}_{2M}S'_{2M-1}(a_2) +$$
$$+ \text{idem}(a_2;a_3,\ldots\ldots,a_{M+1}) = 0 \;,$$

(4) $$X \sin\pi a_1 \; {}_{2M}C'_{2M-1}(a_1) +$$
$$+ Y(a_2) \sin\pi(2a_2-a_1) \; {}_{2M}C'_{2M-1}(a_2) +$$
$$+ \text{idem}(a_2;a_3,\ldots\ldots,a_{M+2}) = 0 \;,$$

(5) $$R \; {}_{2M+1}S_{2M}(a_1) + T(a_2) \; {}_{2M+1}S_{2M}(a_2) +$$
$$+ \text{idem}(a_2;a_3,\ldots\ldots,a_{M+1}) = 0 \;,$$

(6) $$U \sin\pi a_1 \; {}_{1+2M}C_{2M}(a_1) +$$
$$+ V(a_2) \sin\pi(2a_2-a_1) \; {}_{2M+1}C_{2M}(a_2) +$$
$$+ \text{idem}(a_2;a_3,\ldots\ldots,a_{M+2}) = 0 \;,$$

(7) $$U \cos\tfrac{1}{2}\pi a_1 \;{}_{2M+1}S'_{2M}(a_1) + V(a_2) \cos\tfrac{1}{2}\pi(2a_2-a_1)\;{}_{2M+1}S'_{2M}(a_2) + \text{idem}(a_2;a_3,\ldots\ldots,a_{M+2}) = 0 \quad ,$$

(8) $$U \sin\tfrac{1}{2}\pi a_1 \;{}_{2M+1}C'_{2M}(a_1) + V(a_2) \sin\tfrac{1}{2}\pi(2a_2-a_1)\;{}_{2M+1}C'_{2M}(a_2) + \text{idem}(a_2;a_3,\ldots\ldots,a_{M+2}) = 0 \quad ,$$

(9) $$U \;{}_{2M+1}S_{2M}(a_1) + V(a_2)\;{}_{2M+1}S_{2M}(a_2) + \text{idem}(a_2;a_3,\ldots\ldots,a_{M+2}) = 0 \quad ,$$

The formulae 3.4(1),3.4(2),3.4(5),3.4(6) are valid when

$$y > 0\ ,\ |2\theta| \leqslant \pi \ \text{ or } \ -1 < y \leqslant 0,\ \ |2\theta| < \pi\ .$$

Formulae 3.4(3),3.4(4),3.4(7) and 3.4(8) are valid when

$$y > 0\ ,\ 0 \leqslant \theta \leqslant \pi \ \text{ or } \ -1 < y \leqslant 0,\ 0 < \theta < \pi\ .$$

The formula 3.4(9) is valid when

$$y > 0,\ |2\theta| \leqslant 3\pi \ \text{ or } \ -1 < y \leqslant 0,\ |2\theta| < 3\pi$$

$(2\theta \neq \pm\pi)$.

(3.5) <u>Transformations of well-poised bilateral series.</u>

We will now use the above relations of Sears to deduce transformations of bilateral trigonometrical series. Let us first consider the transformations 3.4(1), 3.4(2) and 3.4(3). In each one of these suppose that M is odd, i.e. $M=2N+1$, and then let $a_{2N+2}=1$, $a_2=1+a_1-a_3$, $a_4=1+a_1-a_5$, etc. and, in general, $a_{2N}=1+a_1-a_{2N+1}$. Then the series in these relations reduce to one of the type ${}_{2N+1}C_{2N}$, ${}_{2N+1}S_{2N}$ and ${}_{2N+1}S'_{2N}$ respectively. Simplifying the coefficients of the series on the left and finally writing a for a_1, b_1 for a_{2N+3}, b_2 for a_{2N+4}, etc., in general, b_{2N} for a_{4N+2} and also a_1 for a_3, a_2 for a_5 etc. and in general, a_N for a_{2N+1}, we get on combining the series in pairs as in 3.3 the following three bilateral transformations

(1) $$P' \sin\tfrac{1}{2}\pi a \; {}_{2N}Z_{2N}(a)$$

$$= Q'(a_1) \sin\tfrac{1}{2}\pi(2a_1-a) \; {}_{2N}Z_{2N}(a_1) + \text{idem}(a_1;a_2,\dots,a_N),$$

(2) $$P' \cos\tfrac{1}{2}\pi a \; {}_{2N}X_{2N}(a)$$

$$= Q'(a_1) \cos\tfrac{1}{2}\pi(2a_1-a) \; {}_{2N}X_{2N}(a_1) + \text{idem}(a_1;a_2,\dots,a_N),$$

(3) $$P' \; {}_{2N}X'_{2N}(a)$$

$$= Q'(a_1) \; {}_{2N}X'_{2N}(a_1) + \text{idem}(a_1;a_2,\dots,a_N),$$

where

$$P' = \Gamma(a)\,\Gamma(1-a) \left\{\prod_1^N G(a_r, 1-a_r, 1+a-a_r, a_r-a)\right\} \times \left\{\prod_1^{2N} G(1+a-b_r, 1-b_r)\right\}^{-1},$$

$$Q'(a_1) = \Gamma(a_1)\,\Gamma(1-a_1)\,\Gamma(a_1-a)\,\Gamma(1+a-a_1)\,\times$$

$$\times\left\{\prod_{1}^{N\,\prime} G(1+a-a_1-a_r,\, a_1+a_r-a,\, a_r-a_1,\, 1+a_1-a_r)\right\}\times$$

$$\left\{\prod_{1}^{2N} G(1+a_1-b_r,\, 1+a-a_1-b_r)\right\}^{-1}$$

and

$${}_RZ_R(a) = {}_RZ_R\begin{bmatrix} b_1,\dots\dots\dots,b_R\ ;\theta \\ 1+a-b_1,\dots,1+a-b_R;a \end{bmatrix},$$

$${}_RZ_R(a_r) = {}_RZ_R\begin{bmatrix} a_r+b_1-a,\dots\dots,a_r+b_R-a;\theta \\ 1+a_r-b_1,\dots\dots,1+a_r-b_R;2a_r-a \end{bmatrix}$$

with similar notations for the series ${}_RX_R$, ${}_RX'_R$ and ${}_RZ'_R$. These notations will be frequently used in the following sections. The transformations are valid under the same conditions as for 3.4(1) ,3.4(2) and 3.4(3) respectively.

(3.6) Next, let us consider the transformations 3.4(4) and 3.4(5). Taking $M=2N$ (an even integer) in 3.4(4) and using the method of 3.5 we get on changing N to (N+1) in the final transformation

$$(1)\qquad P'\ \Gamma(1+a-a_{N+1})\,\Gamma(a_{N+1}-a)\,\Gamma(1-a_{N+1})\,\Gamma(a_{N+1})\,\times$$

$$\times\ \sin\pi a\ \ {}_{2N}Z'_{2N}(a)$$

$$= Q'(a_1)\ \Gamma(a_1-a+a_{N+1})\,\Gamma(1+a-a_1-a_{N+1})\,\Gamma(a_{N+1}-a_1)\,\times$$

$$\times\Gamma(1+a_1-a_{N+1})\ \sin\pi(2a_1-a)\ \ {}_{2N}Z'_{2N}(a_1)\ +$$

$$+\ \text{idem}(a_1;a_2,\dots\dots,a_{N+1})\ .$$

This gives a relation between (N+2) well-poised series of the type ${}_{2N}Z'_{2N}$, valid under the same conditions as for 3.4(4).

Similarly, if we take M=2N+1 in 3.4(5) and proceed as before we get the transformation

$$(2) \qquad P'/\Gamma(1+a-b_{2N+1})\,\Gamma(1-b_{2N+1}) \quad {}_{2N+1}X_{2N+1}(a)$$

$$= \quad Q'(a_1)/\Gamma(1+a_1-b_{2N+1})\,\Gamma(1+a-a_1-b_{2N+1}) \quad {}_{2N+1}X_{2N+1}(a_1)+$$

$$+ \text{ idem}(a_1;a_2,\dots\dots,a_N) \quad .$$

This gives a transformation connecting (N+1) series of the type ${}_{2N+1}X_{2N+1}$, valid under the same conditions as for 3.4(5).

(3.7) Finally, taking M=2N , in the transformations 3.4(6-9) and proceeding as in 3.5 we get the following four bilateral transformations

$$(1) \qquad P' \;\Gamma(1+a-b_{2N})\,\Gamma(1-b_{2N}) \;\sin\pi a \;\; {}_{2N-1}Z_{2N-1}(a)$$

$$= \quad Q'(a_1) \;\;\Gamma(1+a_1-b_{2N})\,\Gamma(1+a-a_1-b_{2N}) \times$$

$$\times \sin\pi(2a_1-a) \;\; {}_{2N-1}Z_{2N-1}(a_1) +$$

$$+ \text{ idem}(a_1;a_2,\dots\dots\dots,a_N) \quad ,$$

$$(2) \qquad P' \quad \Gamma(1+a-b_{2N})\,\Gamma(1-b_{2N}) \;\cos\tfrac{1}{2}\pi a \;\; {}_{2N-1}X'_{2N-1}(a)$$

$$= Q'(a_1)\ \Gamma(1+a_1-b_{2N})\ \Gamma(1+a-a_1-b_{2N})\ \cos\tfrac{1}{2}\pi(2a_1-a)\ \times$$

$$\times\ {}_{2N-1}X'_{2N-1}(a_1)\ +$$

$$+\ \text{idem}(a_1;a_2,\ldots\ldots,a_N)\ ,$$

(3) $$P'\ \Gamma(1+a-b_{2N})\ \Gamma(1-b_{2N})\ \sin\tfrac{1}{2}\pi a\ \ {}_{2N-1}Z'_{2N-1}(a)$$

$$= Q'(a_1)\ \Gamma(1+a_1-b_{2N})\ \Gamma(1+a-a_1-b_{2N})\ \sin\tfrac{1}{2}\pi(2a_1-a)\ \times$$

$$\times\ {}_{2N-1}Z'_{2N-1}(a_1)\ +$$

$$+\ \text{idem}(a_1;a_2,\ldots\ldots\ldots,a_N)\ ,$$

(4) $$P'\ \Gamma(1+a-b_{2N})\ \Gamma(1-b_{2N})\ \ {}_{2N-1}X_{2N-1}(a)$$

$$= Q'(a_1)\ \Gamma(1+a_1-b_{2N})\ \Gamma(1+a-a_1-b_{2N})\ \ {}_{2N-1}X_{2N-1}(a_1)\ +$$

$$+\ \text{idem}(a_1;a_2,\ldots\ldots\ldots\ldots,a_N)\ .$$

These are valid under the same conditions as are required for 3.4(6-9) respectively.

It may be noted that the formula 3.7(1),3.7(2) and 3.7(4) can also be obtained by putting $b_{2N}=\frac{1}{2}(1+a)$, in 3.5(1),3.5(3) and 3.5(2) respectively.

(3.8) In this section I shall discuss some of the interesting special cases of the well-poised bilateral transformations deduced in 3.5-3.7.

If we take $\theta=0$ in 3.5(1) or take $\theta=\pi/2$, in 3.5(3) we get the following result due to Slater (3;11)

$$(1) \qquad P' \sin\tfrac{1}{2}\pi a \; {}_{2N}H_{2N}\left[\begin{matrix} b_1, \ldots\ldots, b_{2N}; \\ 1+a-b_1, \ldots, 1+a-b_{2N}; \end{matrix} -1\right]$$

$$= Q'(a_1) \; \sin\tfrac{1}{2}\pi(2a_1-a) \; {}_{2N}H_{2N}\left[\begin{matrix} a_1+b_1-a, \ldots\ldots\ldots, a_1+b_{2N}-a; \\ 1+a_1-b_1, \ldots\ldots\ldots, 1+a_1-b_{2N}; \end{matrix} -1\right]$$

$$+ \text{idem}(a_1; a_2, \ldots\ldots\ldots, a_N) \quad .$$

Again, taking $\theta=0$ in 3.7(1) we get the following relation between (N+1) well-poised series of the type ${}_{2N-1}H_{2N-1}(-1)$ given by Slater (3;14)

$$(2) \qquad P' \quad \Gamma(1+a-b_{2N})\,\Gamma(1-b_{2N}) \; \sin\pi a \; \times$$

$$\times \; {}_{2N-1}H_{2N-1}\left[\begin{matrix} b_1, \ldots\ldots\ldots\ldots, b_{2N-1}; \\ 1+a-b_1, \ldots\ldots, 1+a-b_{2N-1}; \end{matrix} -1\right]$$

$$= \; Q'(a_1) \quad \Gamma(1+a_1-b_{2N})\,\Gamma(1+a-a_1-b_{2N}) \; \sin\pi(2a_1-a) \; \times$$

$$\times \; {}_{2N-1}H_{2N-1}\left[\begin{matrix} a_1+b_1-a, \ldots\ldots\ldots, a_1+b_{2N-1}-a; \\ 1+a_1-b_1, \ldots\ldots\ldots, 1+a_1-b_{2N-1}; \end{matrix} -1\right] +$$

$$+ \text{idem}(a_1; a_2, \ldots\ldots, a_N) \quad .$$

Similarly, if we take $\theta=0$ in 3.7(3) we get a relation between (N+1) well-poised series of the type ${}_{2N-1}H_{2N-1}(1)$ given by Slater (3;13). Next, if we take $\theta=0$ in 3.6(1) we get another result due to Slater (3;12) viz.,

$$(3)\quad P' \; \Gamma(1+a-a_{N+1})\,\Gamma(a_{N+1}-a)\,\Gamma(1-a_{N+1})\,\Gamma(a_{N+1}) \times$$

$$\times \sin\pi a \;\; {}_{2N}H_{2N}\begin{bmatrix} b_1,\ldots\ldots\ldots,b_{2N}; & 1 \\ 1+a-b_1,\ldots,1+a-b_{2N}; & \end{bmatrix}$$

$$= Q'(a_1)\;\Gamma(a_1-a+a_{N+1})\,\Gamma(1+a-a_1-a_{N+1})\,\Gamma(a_{N+1}-a_1)\times$$

$$\times\Gamma(1+a_1-a_{N+1})\;\sin\pi(2a_1-a)\;{}_{2N}H_{2N}\begin{bmatrix} a_1+b_1-a,\ldots\ldots,a_1+b_{2N}-a; & 1 \\ 1+a_1-b_1,\ldots\ldots,1+a_1-b_{2N}; & \end{bmatrix}+$$

$$+\ \text{idem}(a_1;a_2,\ldots\ldots\ldots,a_{N+1})\quad .$$

Thus, we have obtained all the well-poised results deduced by Slater as particular cases of the well-poised bilateral transformations of cognate trigonometrical series.

We can, also, by suitable substitutions of the type used in 3.3, rediscover from these bilateral transformations Sears's original transformations. From these then we get for suitable choice of θ all the well-poised relations of hypergeometric series of any order given by Sears (3;11.11-11.14).

If further, we take $m=0$, $M=2$ and $\theta=0$ in 3.3(8), we get a relation between three series of the type ${}_2H_2(1)$. Taking $a_1=c_1$, and $a_2=c_2$ in this, two of the series reduce to a summable ${}_2F_1(1)$ and simplification gives the sum of a general ${}_2H_2(1)$ given by Bailey (5;1.3).

As shown by Slater (3) we get other results of Bailey (5) also, as special cases of the foregoing well-poised bilateral transformations.

(3.9) Bilateral trigonometrical integrals.

We will now give direct proofs of the general and well-poised transformations of bilateral trigonometrical series deduced in the previous sections. We shall prove in details the transformations 3.3(5 & 6) in the case m=0 . Consider the integral of the Barnes' type

$$(1) \qquad I_R = \int_C G\begin{bmatrix} a_1+s,\ldots\ldots,a_M+s,1-a_1-s,\ldots\ldots,1-a_M-s,-s,1+s; \\ b_1+s,\ldots\ldots,b_M+s,1-c_1-s,\ldots\ldots,1-c_M-s \end{bmatrix} \times \exp(2is\theta)\ ds\ .$$

The contour of integration C is a large circle of radius R, with centre as origin, and R is so chosen that the circle does not pass through any of the poles of the integrand. The parameters a's , b's , and c's are supposed to be real for the sake of simplicity in the proof and such that none of the members of the two sequences

$$-1-n,-n-a_1,\ldots\ldots,-n-a_M;$$

and

$$n,1-a_1+n,\ldots\ldots\ldots,1-a_M+n\ ;$$

coincide. θ is restricted to be real. The extension to the case when a's, b's and the c's are complex is easy.

Now, for $Rl s > 0$ the integrand can be written as

$$\frac{\Gamma(c_1+s).....\Gamma(c_M+s)}{\Gamma(b_1+s).....\Gamma(b_M+s)} \times \frac{\pi \sin\pi(c_1+s)....\sin\pi(c_M+s)}{\sin\pi s \ \sin\pi(a_1+s)....\sin\pi(a_M+s)} \times$$
$$\times \exp(2is\theta) \quad .$$

Writing $s = R e^{i\phi}$, $-\pi/2 \leqslant \phi \leqslant \pi/2$ we find that the first fraction in the above product is $O(R^{\sum_1^M c_r - \sum_1^M b_r})$ and the second fraction is (i) bounded for R tending to infinity if $-\pi \leqslant 2\theta \leqslant \pi$ and is (ii) $O(\exp(-2\theta R \sin\phi - R\pi |\sin\phi|))$ when R tends to infinity.

From (i) it follows that the integral round the semi-circle on the right of the imaginary axis tends to zero if

$$\left(\sum_1^M b_r - \sum_1^M c_r \right) > 1 \quad \text{and} \quad -\pi \leqslant 2\theta \leqslant \pi \ .$$

Again from (ii) it follows with the help of Jordan's lemma that the integral round the same semi-circle tends to zero if

$$\left(\sum_1^M b_r - \sum_1^M c_r \right) > 0 \quad \text{and} \quad -\pi < 2\theta < \pi \quad ,$$

Similarly, when $Rl s < 0$, we can write the integrand as

$$\frac{\Gamma(1-b_1-s).....\Gamma(1-b_M-s)}{\Gamma(1-c_1-s).....\Gamma(1-c_M-s)} \times \frac{\pi \sin\pi(b_1+s).......\sin\pi(b_M+s)}{\sin\pi s \ \sin\pi(a_1+s)...\sin\pi(a_M+s)} \times$$
$$\times \exp(2is\theta) \quad ,$$

and similar remarks follow under the same conditions. Thus, we have shown that the integral I_R tends to zero as R tends to infinity under the above conditions.

Now, the value of the integral is equal to $2\pi i$ (the sum of the residues at the poles of the integrand), where R is infinitely large. The integrand has poles at the set of points given by

$$s = -1-n \ , \ -n-a_1 \ , \ldots\ldots , -n-a_M \ ,$$

and

$$s = n \ , \ 1-a_1+n \ , \ldots\ldots , 1-a_M+n \ .$$

The sum of the residues at these poles of the integrand is given by

$$(2)\ \frac{1}{2\pi i} I_R = \prod_1^M \frac{\sin\pi b_r}{\sin\pi a_r} \sum_{n=0}^{N_1} (-1)^n G\begin{bmatrix} 2+n-b_1,\ldots,2+n-b_M; \\ 2+n-c_1,\ldots,2+n-c_M \end{bmatrix} \times \exp(-2i\theta(n+1)) +$$

$$+ \prod_1^M \frac{\sin\pi c_r}{\sin\pi a_r} \sum_{n=0}^{N_2} (-1)^{n+1} G\begin{bmatrix} c_1+n,\ldots,c_M+n; \\ b_1+n,\ldots,b_M+n \end{bmatrix} \exp(2in\theta) -$$

$$- \Gamma(a_1)\,\Gamma(1-a_1) \prod_1^{M}{}' \frac{\sin\pi(a_1-b_r)}{\sin\pi(a_1-a_r)} \times$$

$$\times \sum_{n=0}^{N_{a_1}} (-1)^n G\begin{bmatrix} 1+a_1-b_1+n,\ldots\ldots,1+a_1-b_M+n; \\ 1+a_1-c_1+n,\ldots\ldots,1+a_1-c_M+n \end{bmatrix} \times \exp(-2i\theta(n+a_1)) -$$

$$- \mathrm{idem}(a_1;a_2,\ldots,a_M) -$$

$$- \Gamma(a_1)\,\Gamma(1-a_1) \prod_{1}^{M}{}' \frac{\sin\pi(a_1-c_r)}{\sin\pi(a_1-a_r)} \times$$

$$\times \sum_{n=0}^{N'_{a_1}} (-1)^{n+1}\, G \begin{bmatrix} 1+c_1-a_1+n, \ldots\ldots\ldots, 1+c_M-a_1+n; \\ 1+b_1-a_1+n, \ldots\ldots\ldots, 1+b_M-a_1+n \end{bmatrix} \times$$

$$\times \exp(2i\theta(1+n-a_1)) -$$

$$- \text{idem}(a_1; a_2, \ldots\ldots, a_M) \quad ;$$

where $N_1+1, N_2+1, N_{a_1}+1$, and $N'_{a_1}+1$ are the number of poles of $\Gamma(1+s)$, $\Gamma(-s)$, $\Gamma(a_1+s)$, $\Gamma(1-a_1-s)$ respectively which lie in the circle $|s| = R$, for any fixed R, and tend to infinity as R tends to infinity.

We first suppose that the conditions

$$\sum_{1}^{M} b_r - \sum_{1}^{M} c_r > 1 \quad , \quad -\pi \leq 2\theta \leq \pi \ ,$$

are satisfied. In case these conditions are satisfied the series of residues are uniformly and absolutely convergent for R tending to infinity and since I_R tends to zero as R tends to infinity we have from 3.9(2) ~~that~~ on multiplying throughout by $\exp(iK\theta)$ that

$$(3) \prod_{1}^{M} \frac{\sin\pi b_r}{\sin\pi a_r} \sum_{0}^{\infty} (-1)^n\, G \begin{bmatrix} 2-b_1+n, \ldots\ldots, 2-b_M+n; \\ 2-c_1+n, \ldots\ldots, 2-c_M+n \end{bmatrix} \times$$

$$\times \exp(i\theta(K-2n-2)) -$$

$$- \prod_{1}^{M} \frac{\sin\pi c_r}{\sin\pi a_r} \sum_{n=0}^{\infty} (-1)^n G \begin{bmatrix} c_1+n, \ldots\ldots, c_M+n; \\ b_1+n, \ldots\ldots, b_M+n \end{bmatrix} \exp(i\theta(K+2n)) -$$

$$- \Gamma(a_1)\,\Gamma(1-a_1) \prod_{1}^{M}{}' \frac{\sin\pi(a_1-b_r)}{\sin\pi(a_1-a_r)} \times$$

$$\times \sum_{n=0}^{\infty} (-1)^n G \begin{bmatrix} 1+a_1-b_1+n, \ldots\ldots, 1+a_1-b_M+n; \\ 1+a_1-c_1+n, \ldots\ldots, 1+a_1-c_M+n \end{bmatrix} \exp(i\theta(K-2n-2a_1))$$

$$- \text{idem}(a_1; a_2, \ldots, a_M) +$$

$$+ \Gamma(a_1)\,\Gamma(1-a_1) \prod_{1}^{M}{}' \frac{\sin\pi(a_1-c_r)}{\sin\pi(a_1-a_r)} \times$$

$$\times \sum_{n=0}^{\infty} (-1)^n G \begin{bmatrix} 1+c_1-a_1+n, \ldots\ldots, 1+c_M-a_1+n; \\ 1+b_1-a_1+n, \ldots\ldots, 1+b_M-a_1+n \end{bmatrix} \times$$

$$\times \exp(i\theta(K+2+2n-2a_1)) +$$

$$+ \text{idem}\ (a_1; a_2, \ldots, a_M)$$

$$= 0 \quad .$$

Now let us equate the real parts in the above equation 3.9(3) ※. We get on some reduction

$$\prod_{1}^{M} G(a_r, 1-a_r; b_r, 1-c_r) \left\{ {}_{M+1}C_M \begin{bmatrix} 1, c_1, \ldots, c_M; \theta \\ b_1, \ldots, b_M; K \end{bmatrix} - \right.$$

$$\left. - \frac{(1-b_1)\ldots(1-b_M)}{(1-c_1)\ldots(1-c_M)}\, {}_{M+1}C_M \begin{bmatrix} 1, 2-b_1, \ldots, 2-b_M; \theta \\ 2-c_1, \ldots, 2-c_M; 2-K \end{bmatrix} \right\} =$$

※ We could also change the sign of θ in 3.9(3) and add the resulting equation to 3.9(3), but since the parameters are supposed to be real we can easily equate real and imaginary parts.

$$= \Gamma(a_1)\,\Gamma(1-a_1) \prod_1^{M}{}' G(a_r-a_1, 1+a_1-a_r; b_r-a_1, 1+a_1-c_r) \times$$

$$\times \left\{ {}_{M+1}C_M \left[\begin{matrix} 1, 1+a_1-b_1, \ldots\ldots, 1+a_1-b_M; \theta \\ 1+a_1-c_1, \ldots\ldots, 1+a_1-c_M; 2a_1-K \end{matrix}\right] - \right.$$

$$\left. - \frac{(c_1-a_1)\ldots\ldots(c_M-a_1)}{(b_1-a_1)\ldots\ldots(b_M-a_1)}\, {}_{M+1}C_M \left[\begin{matrix} 1, 1+c_1-a_1, \ldots\ldots, 1+c_M-a_1; \theta \\ 1+b_1-a_1, \ldots\ldots, 1+b_M-a_1; 2+K-2a_1 \end{matrix}\right] \right\} +$$

$$+ \text{idem}(a_1; a_2, \ldots\ldots a_M) \quad .$$

This by 3.1(6) gives the required transformation 3.3(6) , under the conditions

$$\sum_1^M b_r - \sum_1^M c_r > 1 \quad , \quad -\pi \leqslant 2\theta \leqslant \pi \quad .$$

Similarly, the imaginary part of 3.9(3) gives the transformation 3.3(5), under the above conditions.

We next suppose that the conditions

$$\sum_1^M b_r - \sum_1^M c_r > 0 \quad , \quad -\pi < 2\theta < \pi \quad ;$$

are satisfied. In these circumstances the series of residues in 3.9(2) converge but not absolutely as R tends to infinity. Hence, we need a more delicate argument to justify the limit R tending to infinity. Let us write 3.9(2) in the abbreviated notation

$$I_R = \sum_0^{N_1} R_{n,1} + \sum_0^{N_2} R_{n,2} + \sum_{r=1}^{M} \sum_0^{N_{a_r}} r_{n,a_r} + \sum_{r=1}^{M} \sum_0^{N'_{a_r}} r'_{n,a_r} \quad .$$

Since we know that I_R tends to zero as R tends to infinity, we have

$$\underset{R\to\infty}{\text{Lt}} \left[\sum_0^{N_1} R_{n,1} + \sum_0^{N_2} R_{n,2} + \sum_{r=1}^{M} \sum_0^{N_{a_r}} r_{n,a_r} + \sum_{r=1}^{M} \sum_0^{N'_{a_r}} r'_{n,a_r} \right] = 0 \quad .$$

Now, (N_2-N_1) , $(N'_{a_r} \; -N_{a_r})$ $(r=1,2,\ldots M)$, all remain bounded as R tends to infinity; and so, since $R_{n,1}$, $R_{n,2}$, r_{n,a_r} , and r'_{n,a_r} tend to zero as n tends to infinity, we have

$$\underset{R\to\infty}{\mathrm{Lt}}\left[\sum_{n=0}^{N_1}(R_{n,1}+R_{n,2}) + \sum_{r=1}^{M}\left(\sum_{n=0}^{N_{a_r}}(r_{n,a_r}+r'_{n,a_r})\right)\right] = 0 .$$

But, since the limits

$$\sum_{n=0}^{\infty}(R_{n,1}+R_{n,2}) \quad \text{and} \quad \sum_{n=0}^{\infty}(r_{n,a_r}+r'_{n,a_r}) \; , r=1,\ldots M$$

exist, we have

$$\sum_{n=0}^{\infty}(R_{n,1}+R_{n,2}) + \sum_{r=1}^{M}\left[\sum_{n=0}^{\infty}(r_{n,a_r}+r'_{n,a_r})\right] = 0 ,$$

and we get the required transformations as in the previous case.

Thus, we have proved the transformations 3.3(5-6) under the conditions

$$\sum_1^M b_r - \sum_1^M c_r > 1 \quad , \qquad -\pi \leq 2\theta \leq \pi$$

or

$$\sum_1^M b_r - \sum_1^M c_r > 0 \quad , \qquad -\pi < 2\theta < \pi \quad .$$

In order to obtain the transformations 3.3(7) and 3.3(8) we have to consider the integral 3.9(1) with $\exp(is(\pi+2\theta))$, instead of $\exp(2is\theta)$ in the integrand and proceed in exactly the same manner.

(3.10) Well-poised bilateral trigonometrical integrals.

Let

$$P_N(s) = \Gamma(a+s)\,\Gamma(1-a-s)\,\Gamma(-s)\,\Gamma(1+s)\ \times$$

$$\times \prod_{1}^{N} G(a_r+s, 1-a_r-s, 1+a-a_r+s, a_r-a-s) \quad ,$$

and

$$Q_{2N}(s) = \left\{ \prod_{1}^{2N} G(1+a-b_r+s, 1-b_r-s) \right\}^{-1} \quad .$$

Then

(1) $$\frac{1}{2\pi i}\int_C P_N(s)\ Q_{2N}(s)\ \ \sin\pi(s+\tfrac{1}{2}a)\ \ \exp(2s+a)i\theta\ \ ds \quad ,$$

(2) $$\frac{1}{2\pi i}\int_C P_N(s)\ Q_{2N}(s)\ \ \cos\pi(s+\tfrac{1}{2}a)\ \ \exp(2s+a)i\theta\ \ ds \quad ,$$

(3) $$\frac{1}{2\pi i}\int_C P_N(s)\ Q_{2N}(s)\ \ \exp(2s+a)i\theta\ \ ds \quad ;$$

where C is the same contour as before ,give the transformations 3.5(1),3.5(2) and 3.5(3) respectively. The other transformations can be got by suitable modifications of the above integrals.

CHAPTER IV

ON INTEGRAL ANALOGUES OF CERTAIN TRANSFORMATIONS OF WELL-POISED BASIC HYPERGEOMETRIC SERIES

(4.1) Introduction. As long ago as 1909 Watson (2) used basic integrals of Barnes type to obtain transformations of general basic series. Recently Sears (5) used a method , not involving integrals , to obtain transformations of well-poised basic series of any order, and Slater (4) more recently has used special kinds of basic integrals (not of the Watson type) to prove these results. Sears also used his methods to obtain transformations of well-poised basic series corresponding to the transformations of the series of the ordinary hypergeometric type given in Chapter VI of Bailey's tract (T). Bailey's method makes use of contour integrals of the Barnes type. But so far the corresponding methods by integrals, for the basic analogues of these results (T. Chapter VI) , has not been given. This significant gap to some extent is filled by this chapter. The integrals used are similar to those used by Watson.

(4.2) Notation . The following notation will be used in this and the succeeding chapters.

Let

$$[a;n] = (q^a;n) = (1-q^a)(1-q^{a+1}).....(1-q^{a+n-1}), \quad |q|<1 ,$$

$$[a;-n] = (q^a;-n) = (-1)^n q^{\frac{1}{2}n(n+1)-an} / [1-a;n] ,$$

$$[(a)^*;n] = (-q^a;n) ,$$

(1) $${}_r\mathcal{F}_s\left[\begin{matrix}a_1,\ldots\ldots,a_r;\\ b_1,\ldots\ldots,b_s\end{matrix}\ x\right] \equiv {}_r\Phi_s\left[\begin{matrix}q^{a_1},\ldots\ldots,q^{a_r};\\ q^{b_1},\ldots\ldots,q^{b_s}\end{matrix}\ x\right] .$$

Whenever there is a parameter of the type $-q^{a_m}$ in ${}_r\Phi_s$,it will be represented by $(a_m)^*$ in ${}_r\mathcal{F}_s$. For example

$${}_3\mathcal{F}_2\left[\begin{matrix}a,\ (b)^*\ ,\ c;\\ d\ ,\ e\end{matrix}\ x\right] \equiv {}_3\Phi_2\left[\begin{matrix}q^a,\ -q^b,q^c;\\ q^d,\ q^e\end{matrix}\ x\right]$$

$$= \sum_{n=0}^{\infty} \frac{[a;n]\ [(b)^*;n]\ [c;n]}{[1;n]\ [d;n]\ [e;n]} x^n .$$

For a well-poised series of the type

(2) $${}_{s+1}\mathcal{F}_s\left[\begin{matrix}a,1+\frac{1}{2}a,(1+\frac{1}{2}a)^*\ ,b,c,\ldots\ldots;\\ \frac{1}{2}a,\ (\frac{1}{2}a)^*,1+a-b,1+a-c,\ldots\end{matrix}\ q^{2+2a-b-c-\ldots\ldots}\right]$$

we shall use the abbreviated notation

${}_{s+1}W_s$ $(a;b,c,\ldots\ldots)$ or simply $W(a;b,c,\ldots\ldots)$ when there is no need of specifying the number of parameters.

Also

(3) $$\delta\begin{bmatrix} a_1,\ldots\ldots,a_r; \\ b_1,\ldots\ldots,b_s; \end{bmatrix} = \prod_{n=0}^{\infty} \frac{(1-q^{a_1+n})\ldots\ldots(1-q^{a_r+n})}{(1-q^{b_1+n})\ldots\ldots(1-q^{b_s+n})},$$

Further, whenever there is a factor of the type $(1+q^{a+n})$ in the product on the right of 4.2(3), it will be, as before, represented as $(a)^*$ in the symbol on the left.

For the ordinary products we will use the notation

(4) $$\prod_{n=0}^{N}\begin{bmatrix} a_1,\ldots\ldots,a_r; \\ b_1,\ldots\ldots,b_s; \end{bmatrix} = \prod_{n=0}^{N} \frac{(1-a_1q^n)\ldots\ldots(1-a_rq^n)}{(1-b_1q^n)\ldots\ldots(1-b_sq^n)}.$$

When the limits in the product symbol on the left are left out we will mean that the limits are from $n=0$ to $n=\infty$.

We shall also put $\log q = -\omega = -(\omega_1 + i\,\omega_2)$ where ω, ω_1, ω_2 are definite quantities, ω_1 and ω_2 being real and since $|q| < 1$, $\omega_1 > 0$.

The contour of integration D over which the integrals have been taken throughout this chapter consists of an arc C' of a large circle the centre of which is at the origin; the arc lying to the right of a contour C, which is parallel to the line $Rl(\omega s) = 0$, with loops, if necessary, to ensure that the poles forming an increasing sequence lie to the right of the contour while the poles forming a decreasing sequence lie to the left of the contour. The contour C' is terminated by C.

We will use the following result of Littlewood† giving the asymptotic expansion of

$$S(x) = 1/\prod(1-q^{n+x}) \tag{5}$$

for large values of $|x|$, to deduce the convergence of the integrals.

He has proved that

(i) when $Rl(\omega x) > 0$ and large, $S(x)$ tends uniformly to unity as $|x|$ tends to infinity,

(ii) when $Rl(\omega x)$ is large and negative and $|x-x_0| > \epsilon$, where x_0 is a pole of $S(x)$ and ϵ is an assigned quantity which is not zero,

$$Rl \log S(x) = -\tfrac{1}{2\omega_1}\{Rl(\omega x)\}^2 - \tfrac{1}{2}Rl(\omega x) + J$$

where $|J|$ does not exceed a finite quantity depending on ϵ.

(4.3) <u>The basic analogue of Barnes Second Lemma</u>.

Watson†† (2) has proved the following basic analogue of Barnes first lemma (T.1.7)

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† Littlewood: Proc.London Math.Soc., (2), 5(1907), 395-98.

†† Watson (2) p.286, Lemma B : The notation has been changed by using an equivalent expression in infinite products instead of Jackson's basic gamma functions.

(1) $$\frac{1}{2\pi i}\int_C \wp\begin{bmatrix}1-\xi+s\ , & 1-\eta+s; \\ a_1+s\ , & a_2+s\ ;\end{bmatrix} \frac{\pi q^s\, ds}{\sin\pi(\xi-s)\,\sin\pi(\eta-s)}$$

$$= \frac{q^{\xi+\eta}}{\sin\pi(\xi-\eta)\,(q^{\xi}-q^{\eta})}\,\wp\begin{bmatrix}1, \xi-\eta\ , & \eta-\xi\ , & a_1+a_2+\xi+\eta\ ; \\ a_1+\xi\ , a_1+\eta\ , & a_2+\xi\ , & a_2+\eta\ ;\end{bmatrix} .$$

Let $\xi = n$ (a positive integer) and $\eta = c_1-a_1-a_2$ in 4.3(1) , then we get after some simplification

(2) $$\frac{1}{2\pi i}\int_C \wp\begin{bmatrix}1-n+s, 1-c_1+a_1+a_2+s; \\ a_1+s\ ,\ a_2+s\ ;\end{bmatrix} \frac{\pi q^s\, ds}{\sin\pi s\,\sin\pi(c_1-a_1-a_2-s)}$$

$$= \operatorname{cosec}\pi(a_1+a_2-c_1)\ \wp\begin{bmatrix}1+a_1+a_2-c_1, c_1-a_1-a_2, c_1, 1\ ; \\ a_1\ , a_2\ , c_1-a_1\ , c_1-a_2\ ;\end{bmatrix} \times$$

$$\times\ (-1)^n\, q^{-\frac{1}{2}n(n-1)+n(c_1-a_1-a_2)}\ \frac{[a_1;n]\,[a_2;n]}{[c_1;n]} .$$

Now, multiplying both sides of 4.3(2) by

$$q^{n(b_2-c)}\ [c;n]/[1;n]\,[b_2;n]$$

and summing for n from zero to infinity, we get on reduction and change of order of integration and summation, a process easily justifiable,

(3) $$\operatorname{cosec}\pi(a_1+a_2-c_1)\ \wp\begin{bmatrix}1+a_1+a_2-c_1, c_1-a_1-a_2, c_1, 1; \\ a_1\ , a_2\ , c_1-a_1\ , c_1-a_2\ ;\end{bmatrix} \times$$

$$\times {}_3\mathcal{F}_2\left[\begin{matrix} a_1, & a_2, & c; \\ c_1, & b_2 \end{matrix}\; q^{c_1+b_2-a_1-a_2-c}\right]$$

$$= \frac{1}{2\pi i}\int_C \wp\left[\begin{matrix} 1+s, 1-c_1+a_1+a_2+s; \\ a_1+s, \; a_2+s \; ; \end{matrix}\right] \frac{\pi q^s \, ds}{\sin\pi s \, \sin\pi(c_1-a_1-a_2-s)} \times$$

$$\times \left\{ {}_2\mathcal{F}_1\left[\begin{matrix} -s, c; \\ b_2 \end{matrix}\; q^{s+b_2-c}\right]\right\} .$$

Summing the ${}_2\mathcal{F}_1$ on the right by the basic analogue of Gauss's theorem (T.8.4(3)), we get

(4) $$\operatorname{cosec}\pi(a_1+a_2-c_1)\, \wp\left[\begin{matrix} 1+a_1+a_2-c_1, c_1-a_1-a_2, c_1, b_2, 1; \\ a_1, a_2, c_1-a_1, c_1-a_2, b_2-c \; ; \end{matrix}\right] \times$$

$$\times {}_3\mathcal{F}_2\left[\begin{matrix} a_1, a_2, c \; ; \\ c_1, b_2 \end{matrix}\; q^{c_1+b_2-a_1-a_2-c}\right]$$

$$= \frac{1}{2\pi i}\int_C \wp\left[\begin{matrix} 1+s, b_2+s, 1-c_1+a_1+a_2+s; \\ a_1+s, \; a_2+s, \; b_2-c+s \; ; \end{matrix}\right] \frac{\pi q^s \, ds}{\sin\pi s \, \sin\pi(c_1-a_1-a_2-s)} .$$

Now, take $c_1 = c$; the series on the left can be summed by the basic analogue of Gauss's theorem and we get , on putting

$$b_1 = 1-c+a_1+a_2 \quad , \quad a_3 = b_2-c$$

i.e. $$b_1+b_2 = 1+a_1+a_2+a_3$$

that

(5) $$\frac{1}{2\pi i}\int_C \phi\begin{bmatrix}1+s, b_1+s, b_2+s; \\ a_1+s, a_2+s, a_3+s;\end{bmatrix} \frac{\pi q^s \, ds}{\sin\pi s \, \sin\pi(1-b_1-s)}$$

$$= \operatorname{cosec}\pi(b_1-1) \; \phi\begin{bmatrix}1, b_1, 1-b_1, b_2-a_1, b_2-a_2, b_2-a_3; \\ a_1, a_2, a_3, 1+a_1-b_1, 1+a_2-b_1, 1+a_3-b_1;\end{bmatrix},$$

provided $b_1+b_2 = 1+a_1+a_2+a_3$.

Using the asymptotic expansion of 4.2(5) the integral is covergent when $Rl\left[s\log q - \log(\sin\pi s \sin\pi(1-b_1-s))\right] < 0$ for $|s|$ large.

This gives the basic analogue of Barnes Second Lemma.

(4.4) Integrals representing basic well-poised series.

It is easily verified by 4.3(5) that

$$\operatorname{cosec}\pi(b_1-1) \; \phi\begin{bmatrix}1, b_1+n, 1-b_1-n, k-a_1+n, k-a_2+n, k-a_3+n; \\ a_1+n, a_2+n, a_3+n, k-a_2-a_3, k-a_1-a_2, k-a_1-a_3;\end{bmatrix}$$

$$= \frac{1}{2\pi i}\int_C \phi\begin{bmatrix}k+s, 1-k+a_1+a_2+a_3+s, 1+s; \\ a_1+s \;,\; a_2+s \;,\; a_3+s \;;\end{bmatrix} \frac{\pi(-1)^n \, q^{s(n+1)-\frac{1}{2}n(n+1)} [-s;n] \, ds}{\sin\pi s \, \sin\pi(1-b_1-s) \; [k+s;n]}$$

where $b_1 = 1-k+a_1+a_2+a_3$.

Simplifying the left hand side it follows, by expansion and the interchange in order of integration and summation, that

$$(1)\quad \operatorname{cosec}\pi(b_1-1)\ \prod\left[\begin{matrix}1, b_1, 1-b_1, k-a_1, k-a_2, k-a_3;\\ a_1, a_2, a_3, k-a_1-a_2, k-a_1-a_3, k-a_2-a_3;\end{matrix}\right]\times$$

$$\times\ {}_{r+4}\mathcal{F}_{r+3}\left[\begin{matrix}a, a_1, a_2, a_3, \rho_1, \ldots\ldots, \rho_r\ ;\\ k-a_1, k-a_2, k-a_3, \sigma_1, \ldots, \sigma_r\end{matrix}\ q^{\eta-b_1}\right]$$

$$= \frac{1}{2\pi i}\int_C \prod\left[\begin{matrix}k+s, 1-k+a_1+a_2+a_3+s, 1+s;\\ a_1+s\ ,\ a_2+s\ ,\ a_3+s\ ;\end{matrix}\right]\frac{\pi\ q^s}{\sin\pi s\ \sin\pi(1-b_1-s)}\times$$

$$\times\ {}_{r+2}\mathcal{F}_{r+1}\left[\begin{matrix}a, \rho_1, \ldots\ldots, \rho_r, -s\ ;\\ \sigma_1, \ldots\ldots, \sigma_r, k+s\end{matrix}\ q^{\eta+s-1}\right]\ ds\ ,$$

where $b_1 = 1-k+a_1+a_2+a_3$ and η is an arbitrary parameter.

Thus, if we can sum the series on the right of 4.4(1) in terms of infinite products of the above type, we can find an integral representing the series on the left.

As an illustration take $r=4$, $k=1+a$, $\rho_1=1+\frac{1}{2}a$, $\rho_2=(1+\frac{1}{2}a)^*$, $\rho_3=b$, $\rho_4=c$, $\sigma_1=\frac{1}{2}a$, $\sigma_2=(\frac{1}{2}a)^*$, $\sigma_3=1+a-b$, $\sigma_4=1+a-c$ and $\eta=2+a-b-c$. On the right we get a ${}_6\mathcal{F}_5$ series which can be summed† and we thus obtain an integral representing a well-poised ${}_8\mathcal{F}_7$, namely

---

† $${}_6\mathcal{F}_5\left[\begin{matrix}a, 1+\frac{1}{2}a, (1+\frac{1}{2}a)^*,\ b\ ,\ c\ ,\ d\ ;\\ \frac{1}{2}a,\ (\frac{1}{2}a)^*, 1+a-b, 1+a-c, 1+a-d\end{matrix}\ q^{1+a-b-c-d}\right]$$

$$= \prod\left[\begin{matrix}1+a, 1+a-b-c, 1+a-b-d, 1+a-c-d;\\ 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d;\end{matrix}\right],$$

which is a particular case of T.8.5(3), when $1+a=d+e$.

(2) $$ {}_8W_7(a;b,c,a_1,a_2,a_3) $$

$$ = \sin\pi(a_1+a_2+a_3-a-1)\times $$

$$ \times \mathcal{P}\begin{bmatrix} 1+a,a_1,a_2,a_3,1+a-a_1-a_2,1+a-a_2-a_3,1+a-a_1-a_3,1+a-b-c; \\ 1,b_1,1-b_1,1+a-a_1,1+a-a_2,1+a-a_3,1+a-b,1+a-c; \end{bmatrix} \times $$

$$ \times \frac{1}{2\pi i}\int_C \mathcal{P}\begin{bmatrix} a_1+a_2+a_3-a+s,1+a-b+s,1+a-c+s,1+s; \\ a_1+s\ ,\ a_2+s\ ,\ a_3+s\ ,1+a-b-c+s\ ; \end{bmatrix} \frac{\pi q^s\ ds}{\sin\pi s\ \sin\pi(1-b_1-s)}, $$

where $b_1 = a_1+a_2+a_3-a$.

If, however, following Watson, we evaluate the integral on the right of 4.4(2) by considering the residues at the poles of $\operatorname{cosec}\pi s$ and $\operatorname{cosec}\pi(1-b_1-s)$ lying to the right of the contour C , we get the wellknown relation between a ${}_8\mathcal{F}_7$ and two Saalchützian ${}_4\mathcal{F}_3$'s †.

As another interesting illustration let us replace a by k, b by b+k-a , c by c+k-a ,and a_1 by a_1+k-a and take $k+a_1+b+c = 1+2a$, in 4.4(2). Then the right hand integral in 4.4(2) remains unchanged and we get a relation between two well-poised series of the type ${}_8\mathcal{F}_7$. Changing q^a to a ,and so on this gives a transformation due to Bailey (5;4.3) , namely

---

† After changing q^a to a, and so on, this relation becomes the relation between a well-poised ${}_8\Phi_7$ and two Saalchützian ${}_4\Phi_3$'s given in T.8.5(3) .

$$ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a_1, a_2, a_3, b, c; \\ \sqrt{a}, -\sqrt{a}, aq/a_1, aq/a_2, aq/a_3, aq/b, aq/c \end{matrix} qk/a_2a_3\right] $$

$$ = \prod\left[\begin{matrix} aq, aq/a_2a_3, kq/a_2, kq/a_3; \\ kq, kq/a_2a_3, aq/a_2, aq/a_3; \end{matrix}\right] \times $$

$$ \times {}_8\Phi_7\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, a_1k/a, a_2, a_3, bk/a, ck/a; \\ \sqrt{k}, -\sqrt{k}, aq/a_1, kq/a_2, kq/a_3, aq/b, aq/c \end{matrix} aq/a_2a_3\right], $$

for $\quad ka_1bc = a^2q$.

It gives an exact analogue of a relation between two well-poised ${}_7F_6$'s given in T.7.5(1). This relation between two well-poised ${}_8\Phi_7$'s and similar other relations will be considered in greater details in the next chapter.

(4.5) Integral analogue of basic-Dougall's theorem.

In this section we will deduce an integral equivalent for the Jackson's analogue of Dougall's theorem (T.8.3(1)).

In 4.4(2) replacing a_1, a_2, a_3, by d,e,f , respectively we get, on imposing the additional restriction on the parameters

$$ 1+2a = b+c+d+e+f $$

that

(1) $$w(a;b,c,d,e,f;q)$$

$$= \sin\pi(d+e+f-a-1)\ \Pi\begin{bmatrix}1+a-d-e, 1+a-e-f, 1+a-d-f; \\ 1,\ 1-\beta,\ b,\ c;\end{bmatrix} \times$$

$$\times \frac{1}{2\pi i}\int_C \Pi\begin{bmatrix}1+a-b+s, 1+a-c+s, 1+s; \\ d+s,\ e+s,\ f+s;\end{bmatrix} \frac{\pi q^s\, ds}{\sin\pi s\,\sin\pi(1-\beta-s)},$$

where $\beta = d+e+f-a$, and

$$w(a;b,c,d,e,f;q^{2+2a-b-c-d-e-f})$$

$$= \Pi\begin{bmatrix}1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f; \\ 1+a,\ b,\ c,\ d,\ e,\ f;\end{bmatrix} {}_8W_7(a;b,c,d,e,f).$$

By analogy we get that

(2) $$w(2b-a;b,b+c-a,b+d-a,b+e-a,b+f-a;q)$$

$$= -\sin\pi c\ \ \Pi\begin{bmatrix}1+a-d-e, 1+a-e-f, 1+a-d-f; \\ 1,\ 1-\beta,\ b,\ c;\end{bmatrix} \times$$

$$\times \frac{1}{2\pi i}\int_C \Pi\begin{bmatrix}1+b-a+s, 1+b-c+s, 1+s; \\ b+d-a+s, b+e-a+s, b+f-a+s;\end{bmatrix} \frac{\pi q^s\, ds}{\sin\pi s\,\sin\pi(c-s)},$$

provided that $1+2a = b+c+d+e+f$.

Putting $s+b-a = t$ in 4.5(2), so that the products in the integrand in 4.5(2) become the same as in 4.5(1) and adding to 4.5(1) we get, after slightly simplifying the integrand on the right, that

(3) $$\Big[w(a;b,c,d,e,f;q) - q^{b-a}\,w(2b-a;b,b+c-a,b+d-a,b+e-a,b+f-a;q)\Big]$$

$$= \rho\begin{bmatrix}1+a-d-e,1+a-e-f,1+a-d-f;\\ 1\ ,\ b\ ,\ c\ ,\ 1-\beta\ ;\end{bmatrix}\times$$

$$\times\frac{1}{2\pi i}\int_C \rho\begin{bmatrix}1+a-b+s,1+a-c+s,1+s;\\ d+s\ ,\ e+s\ ,\ f+s\ ;\end{bmatrix}\frac{\pi q^s\ \sin\pi(a-b)\ ds}{\sin\pi s\ \sin\pi(b-a-s)},$$

provided that $1+2a = b+c+d+e+f$.

Now, following Watson, it can be easily verified by means of the Calculus of Residues, by considering the residues at the poles of the integrand that lie to the right of C, i.e., the positive poles of $\operatorname{cosec}\pi s$ and $\operatorname{cosec}\pi(b-a-s)$, that the left hand expression in 4.5(3) is equal to the integral

(4) $$1/\prod_{n=0}^{\infty}(1-q^{n+1})\times$$

$$\times\frac{1}{2\pi i}\int_C\Bigg\{\rho\begin{bmatrix}1+s,\frac{1}{2}a+s,(\frac{1}{2}a+s)^*,1+a-b+s,1+a-c+s,1+a-d+s,1+a-e+s,1+a-f+s;\\ a+s,1+\frac{1}{2}a+s,(1+\frac{1}{2}a+s)^*,b+s,c+s,d+s,e+s,f+s\ ;\end{bmatrix}\times$$

$$\times\frac{\pi q^s\ \sin\pi(a-b)\ ds}{\sin\pi s\ \sin\pi(b-a-s)}\Bigg\}.$$

Equating 4.5(4) and the right hand integral of 4.5(3) we get on simplification

(5)

$$\frac{1}{2\pi i}\int_C \left\{ \mathcal{P}\begin{bmatrix} 1+s, \tfrac{1}{2}a+s, (\tfrac{1}{2}a+s)^*, 1+a-b+s, 1+a-c+s, 1+a-d+s, 1+a-e+s, 1+a-f+s; \\ a+s, 1+\tfrac{1}{2}a+s, (1+\tfrac{1}{2}a+s)^*, b+s, c+s, d+s, e+s, f+s; \end{bmatrix} \frac{\pi q^s \, ds}{\sin\pi s \sin\pi(b-a-s)} \right\}$$

$$= \mathcal{P}\begin{bmatrix} 1+a-d-e, 1+a-d-f, 1+a-e-f; \\ b, c, 1-\beta; \end{bmatrix} \times$$

$$\times \frac{1}{2\pi i}\int_C \mathcal{P}\begin{bmatrix} 1+a-b+s, 1+a-c+s, 1+s; \\ d+s, e+s, f+s; \end{bmatrix} \frac{\pi q^s \, ds}{\sin\pi s \sin\pi(b-a-s)},$$

provided that $1+2a = b+c+d+e+f$ and where $\beta = d+e+f-a$.

Now, the right hand integral of 4.5(5) can be evaluated by the help of 4.3(5) under the condition

$$1+2a = b+c+d+e+f,$$

and hence we obtain the result

(6)

$$\frac{1}{2\pi i}\int_C \left\{ \mathcal{P}\begin{bmatrix} 1+s, \tfrac{1}{2}a+s, (\tfrac{1}{2}a+s)^*, 1+a-b+s, 1+a-c+s, 1+a-d+s, 1+a-e+s, 1+a-f+s; \\ a+s, 1+\tfrac{1}{2}a+s, (1+\tfrac{1}{2}a+s)^*, b+s, c+s, d+s, e+s, f+s; \end{bmatrix} \times \frac{\pi q^s \, ds}{\sin\pi s \sin\pi(b-a-s)} \right\}$$

$$= \operatorname{cosec}\pi(a-b) \times$$

$$\times \mathcal{P}\begin{bmatrix} 1+a-d-e, 1+a-e-f, 1+a-d-f, 1+a-c-d, 1+a-c-e, 1+a-c-f, 1+a-b, b-a, 1; \\ b, c, d, e, f, b+c-a, d+b-a, e+b-a, f+b-a; \end{bmatrix},$$

provided $1+2a = b+c+d+e+f$.

This gives us the integral analogue of basic-Dougall's theorem. If, however, we evaluate the integral in 4.5(6) by considering the residues at the poles of $\operatorname{cosec}\pi s$ and $\operatorname{cosec}\pi(b-a-s)$ that lie to the right of C, we get a relation between two well-poised ${}_8\Phi_7$'s series given by Bailey (6;3.3) in 1947. This is the form which the basic analogue of Dougall's theorem assumes when we remove the restriction that one of the numerator parameters must be a negative integer. It can be easily shown with the help of the asymptotic expansion of 4.2(5) that the integral converges when $Rl\,[s \log q - \log(\sin\pi s \sin\pi(b-a-s))] < 0$ for large values of $|s|$.

It may be remarked that it is not possible to use 4.5(6) to deduce the integrals giving the transformation between four well-poised ${}_{10}\Phi_9$'s, as was done by Bailey to deduce the relation between four ${}_9F_8$'s from the integral analogue of Dougall's theorem (T. 6.8) .

CHAPTER V

SOME TRANSFORMATIONS OF WELL-POISED BASIC HYPERGEOMETRIC SERIES OF THE TYPE ${}_8\Phi_7$

(5.1) Introduction. This chapter is concerned with a systematic study of some transformations of well-poised series of the type ${}_8\Phi_7$,as was done by Whipple (2) in the case of a well-poised ${}_7F_6$. Transformations of the well-poised basic series ${}_8\Phi_7$ have been studied by Bailey (5),Sears (5) and Slater (3).Bailey (5) in 1936 gave two and three term relations connecting well-poised ${}_8\Phi_7$ series with special forms of second and third parameters in the numerator.Recently,Sears (2;10.2) has given one such transformation connecting three well-poised ${}_8\Phi_7$ of the above type , which as will be shown later in the chapter is really equivalent to one of the transformations of Bailey.Sears also gave general transformations connecting basic hypergeometric series of any order without the special forms of the parameters.These have been later studied and proved in other ways by Slater (4).These general transformations yield as particular cases the transformations connecting ${}_8\Phi_7$ series without special forms of the second and third numerator parameters.But a systematic study of these two and three term relations connecting well-poised ${}_8\Phi_7$ series has not yet been made, and the relations appear to be rather isolated ones.

In this chapter, following Whipple, I classify the well-poised ${}_8\Phi_7$ series in different groups and study systematically the relations existing between the 192 allied series that occur in the classification. The fundamental three term relation due to Bailey (5;5.1) has been employed to obtain the various relations, of which some are believed to be new.

In the last two sections of the chapter I have employed basic integrals of the Barnes type, discussed in the previous chapter, to give extremely simple and elegant proofs of the two term relations and the fundamental three term relation of Bailey.

The general notation employed is that of the previous chapter.

(5.2) In this section I give in a tabular form the numerator parameters of the well-poised ${}_8\Phi_7$'s of the type

$$(1)\quad \chi(a;b,c,d,e,f) \equiv \prod\left[\begin{matrix}aq/b,aq/c,aq/d,aq/e,aq/f,a^2q^2/bcdef;\\ aq;\end{matrix}\right]$$

$$\times\, {}_8\Phi_7\left[\begin{matrix}a,q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ f;\\ \sqrt{a},\ -\sqrt{a},aq/b,aq/c,aq/d,aq/e,aq/f\end{matrix}\ a^2q^2/bcdef\right],$$

that occur in different relations in the following sections.

Parameters of 192 allied χ-functions

$s = bcdef/a^2q$

Name of group	Numerator parameters	Convergence Indicator $\|I\|<1$	No. of permutations of each form
	a;b,c,d,e,f	q/s	1
$G_p(0)$	bc/s;b,c,aq/de,aq/ef,aq/df	qa/bc	10
	aq/bs;q/s,aq/bc,aq/bd,aq/be,aq/bf	b	5
	b^2/a;b,bc/a,bd/a,be/a,bf/a	q/s	1
$G_p(1)$	qb/as;q/s,q/c,q/d,q/e,q/f	b	1
($G_p(2)$	b^2c/sa;b,bc/a,aq/de,aq/ef,aq/df	q/c	4
etc are permut-	qb/sc;q/s,q/c,aq/cd,aq/ce,aq/cf	bc/a	4
ations)	qb/cd;q/c,q/d,be/a,bf/a,aq/cd	aq/ef	6
	q/a;q/b,q/c,q/d,q/e,q/f	s	1
$G_n(0)$	qs/bc;q/b,q/c,ef/a,df/a,de/a	bc/a	10
	bs/a;s,bc/a,bd/a,be/a,bf/a	q/b	5
	aq/b^2;q/b,aq/bc,aq/bd,aq/be,aq/bf	s	1
$G_n(1)$	as/b;s,c,d,e,f	q/b	1
($G_n(2)$	asq/b^2c;q/b,aq/bc,ef/a,df/a,de/a	c	4
etc are permut-	sc/b;s,c,cd/a,ce/a,cf/a	aq/bc	4
ations)	cd/b;c,d,aq/be,aq/bf,cd/a	ef/a	6

The group G_n has been obtained from the respective group G_p by dividing q by the numerator parameters in order. It is to be understood that, in the expansions of $G_p(0)$ and $G_n(0)$, permutations of the letters b to f are allowed. In the expansions of $G_p(1)$ and $G_n(1)$ permutations of the letters c to f are allowed and so on. The number of permutations of each form is given in the last column of the table.

(5.3) <u>Relations between two ${}_8\Phi_7$ series</u>.

The fundamental two term relations of ${}_8\Phi_7$ series were given by Bailey (5;4.3 & 4.4), namely

(1) $\chi(a;b,c,d,e,f) = \chi(a^2q/def;b,c,aq/de,aq/df,aq/ef)$,

and

(2) $\chi(a;b,c,d,e,f) =$

$$\chi(a^3q^2/b^2cdef;aq/bc,aq/bd,aq/be,aq/bf,a^2q^2/bcdef).$$

These two relations express the equality between the sixteen series $G_p(0)$.

In 5.3(1) and 5.3(2) if we change a to q/a, b to q/b and so on, we get the equality between the sixteen series $G_n(0)$.

Also replacing a by b^2/a , c by bc/a, d by bd/a, e by be/a and f by bf/a in 5.3(1) we get the equality between the first and the third series of the group $G_p(1)$. Similarly, by proper substitutions we can show that all the sixteen series of each of the groups $G_p(r)$ or $G_n(r)$, r=1,2,3,4,5 are equal among themselves.

(5.4) Three term relations between the allied series.

It can be easily shown that there are (apart from mere interchange of parameters) 110 different representations of a given ${}_8\Phi_7$ in terms of two other ${}_8\Phi_7$ series, the series being of the types given in table of 5.2. For, taking say, $G_p(0)$ to be the standard series $\chi(a;b,c,d,e,f)$ we have that there are

(i) 15 distinct representations of a $G_p(0)$ in terms of a $G_p(1)$ and a $G_p(2)$,

(ii) 25 .... $G_p(1)$ and a $G_n(2)$,

(iii) 15 .... $G_n(1)$ and a $G_n(2)$,

(iv) 25 .... $G_p(1)$ and a $G_n(1)$,

(v) 15 .... $G_p(1)$ and a $G_n(0)$,

(vi) 15 .... $G_n(1)$ and a $G_n(0)$.

Thus, with $\chi(a;b,c,d,e,f)$ as the standard $G_p(0)$ series there are six typical transformations between this and two other ${}_8\Phi_7$ series, which we proceed to deduce systematically. Two of these relations, namely, a relation between $G_p(0)$, $G_p(1)$ and $G_n(2)$ and the relation* between $G_p(0)$, $G_p(1)$ and $G_p(2)$ were given by Bailey (5;5.1 & 4.6).

The fundamental three term relation of Bailey between series of the type $G_p(0)$, $G_n(2)$ and $G_p(1)$ is

$$(1)\qquad \Pi(aq/def, def/a, bd/a, be/a, bf/a, q/c)\times \chi(a;b,c,d,e,f)$$

$$= \Pi(aq/b, b/a, aq/ef, aq/df, aq/de, a^2q^2/bcdef)\times \chi(ef/c;e,f,aq/bc,aq/cd,ef/a)$$

$$+ (b/a)\,\Pi(d,e,f,aq/bc,a^2q/bdef,bdef/a^2)\times \chi(b^2/a;b,bc/a,bd/a,be/a,bf/a)\,.$$

Interchanging b and d in 5.4(1), the first two series remain unchanged and we get a relation between three series of the type $G_p(0)$, $G_n(2)$ and $G_p(3)$. From this relation and 5.4(1) if we eliminate the series $G_n(2)$ we get

$$(2)\qquad \Pi(q/c,q/e,q/f,bd/a,b/d,qd/b)\times \chi(a;b,c,d,e,f)$$

$$= \Pi(b,aq/cd,aq/de,aq/df,aq/b,b/a)\times \chi(d^2/a;d,bd/a,cd/a,ed/a,fd/a)\ -$$

* The relation given by Bailey (5;4.6) is actually one between a $G_p(0)$, $G_p(1)$ and $G_p(5)$. But this equivalent to a relation between a $G_p(0)$, $G_p(1)$ and $G_p(2)$ by merely interchanging c and f.

$$- (b/d) \prod(d, aq/bc, aq/bf, aq/be, aq/d, d/a) \times \chi(b^2/a; b, bc/a, bd/a, be/a, bf/a) \quad .$$

This gives a relation between three series of the type $G_p(0)$, $G_p(3)$ and $G_p(1)$ *. It is equivalent to Bailey's result (5;4.6) which is obtained from this by merely interchanging d and f .

If in 5.4(1) we replace the $G_n(2)$ series by its equivalent series $\chi(bdef/aq; s, b, d, e, f)$, we get

$$(3) \quad \prod(aq/def, def/a, bd/a, be/a, bf/a, q/c) \times \chi(a; b, c, d, e, f)$$

$$= \prod(aq/b, b/a, aq/ef, aq/df, aq/de, a^2q^2/bcdef) \times \chi(bdef/aq; s, b, d, e, f) +$$

$$+ (b/a) \prod(d, e, f, aq/bc, a^2q/bdef, bdef/a^2) \times \chi(b^2/a; b, bc/a, bd/a, be/a, bf/a) \; .$$

In 5.4(3) interchange c and f .We then get a result given by Sears (2;10.2).Next,interchanging c and e in 5.4(3) and then eliminating $G_p(1)$ from the

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*In simplifying the coefficients of the series we have used the formula, for $|q| < 1$

$$(zb)^{-1} S(za, z/a, bc, b/c) = \text{idem}(a; b, c) \quad ,$$

where $S(x) = \prod(x, q/x)$ and $S(a, b, \ldots) = \prod(a, q/a, b, q/b, \ldots)$, and so on. This formula will be used very often in simplifying the coefficients of the series without further reference.

resulting transformation and 5.4(3) we get, after interchanging b and e in the final result, the relation

$$(4)\qquad c\ \prod(de/a, df/a, ef/a, b/c, qc/b, bcdef/a^2, a^2q/bcdef)\times \chi(a;b,c,d,e,f)$$

$$= \prod(aq/cf, aq/cd, aq/ce, b, a^2q^2/bcdef, a^2q/bdef, bdef/a^2)\times \chi(cdef/aq;s,c,d,e,f) -$$

$$- \text{idem}(c;b)\ .$$

This gives a relation between three series of the type $G_p(0)$, $G_n(1)$ and $G_n(2)$.

Now, eliminating $G_n(2)$ between 5.4(3) and 5.4(4) we get after some simplification

$$(5)\qquad \Big[S(def/a, bd/a, be/a, bf/a, cdef/a^2, c) + {}+c\ S(de/a, ef/a, df/a, bcdef/a^2, b/c, b/a)\Big]\times \chi(a;b,c,d,e,f)$$

$$= (b/a)\ \prod\Big[(aq/bc, aq/bf, aq/bd, aq/be, c, d, e, f, a^2q/bdef, bdef/a^2, a^2q/cdef, cdef/a^2)\Big]\times \chi(b^2/a;b, bc/a, bd/a, be/a, bf/a) +$$

$$+ \prod\Big[(aq/ce, aq/cd, aq/cf, aq/ed, aq/df, aq/b, b/a, b, a^2q/bdef, aq/ef, bdef/a^2, a^2q^2/bcdef)\Big]\times \chi(cdef/aq;s,c,d,e,f)\quad .$$

This gives a relation between three series of the type $G_p(0)$, $G_p(1)$ and $G_n(1)$.

If in 5.4(5) we put $b^2/a = A$, $bc/a = C$, $bd/a = D$, $be/a = E$, $bf/a = F$ and $b = B$, the $G_p(0)$ series becomes one of the type $G_p(1)$ and vice-versa. Also the $G_n(1)$ series is transformed into a $G_n(0)$ series and hence we get

$$(6)\qquad \Big[S(BDEF/A^2,D,E,F,CDEF/A^2,BC/A) + (BC/A)\, S(DE/A,EF/A,DF/A,BCDEF/A^2,A/C,A/B)\Big] \times \chi(B^2/A;B,BC/A,BD/A,BE/A,BF/A)$$

$$= (A/B)\prod\Big[(q/C,q/D,q/E,q/F,BC/A,BD/A,BE/A,BF/A,Aq/DEF,DEF/A,CDEF/A^2,A^2q/CDEF)\Big] \times \chi(A;B,C,D,E,F)$$

$$+\prod\Big[(Aq/CE,Aq/CD,Aq/CF,Aq/ED,Aq/EF,Aq/DF,Bq/A,A/B,B,Aq/DEF,DEF/A,A^2q^2/BCDEF)\Big] \times \chi(B^2CDEF/A^3q;BCDEF/A^2q,BC/A,BD/A,BE/A,BF/A) ,$$

which is a transformation connecting three series of the types $G_p(1)$, $G_p(0)$ and $G_n(0)$ respectively.

Finally, in 5.4(5) let us put $cdef/aq = A$, $bcdef/a^2q = B$ and replace c,d,e,f by the corresponding capital letters. Then the first $G_p(0)$ series is transformed into the series $\chi(CDEF/Aq;BCDEF/A^2q,C,D,E,F)$ which is of the type $G_n(1)$. The second $G_p(1)$ series becomes the series $\chi(B^2CDEF/A^3q;BCDEF/A^2q,BC/A,BD/A,BE/A,BF/A)$, which is of the type $G_n(0)$ and the last $G_n(1)$ series becomes the series $\chi(A;B,C,D,E,F)$, which is of the type $G_p(0)$. Thus, we get the transformation

(7) $$\left[S(Aq/C,BD/A,BE/A,BF/A,A^2q^2/CDEF,C) + C\, S(Aq/CF,Aq/CD,Aq/CE,Bq,BDEF/A^2q,B/A)\right] \times \chi(CDEF/Aq;BCDEF/A^2q,C,D,E,F)$$

$$= (B/A)\prod\left[(Aq/BC,Aq/BD,Aq/BE,Aq/BF,C,D,E,F,A^2q^2/CDEF, CDEF/A^2q,C/B,qB/C)\right] \times \chi(B^2CDEF/A^3q;BCDEF/A^2q,BC/A,BD/A,BE/A,BF/A) +$$

$$+ \prod\left[(DF/A,EF/A,DE/A,CD/A,CE/A,CF/A,Aq/B,B/A,BCDEF/A^2q, C/B,qB/C,q/B)\right] \times \chi(A;B,C,D,E,F) \quad ,$$

which gives a relation between three series of the type $G_n(1)$, $G_n(0)$ and $G_p(0)$.

This completes the transformation theory of expressing $\chi(a;b,c,d,e,f)$ in terms of two other ${}_8\Phi_7$ series of the type given in the table of 5.2. Similar remarks follow for relations between a given series of any other group and two other χ- series.

(5.5) <u>Integrals representing two term relations</u>.

In this section I will use integrals of the type discussed in Chapter IV to give direct proofs of the fundamental two term relations given in 5.3. It is proved in 4.4(2) that

(1) $${}_8W_7(a;b,c,d,e,f) = \sin\pi(d+e+f-a-1) \times$$

$$\times \wp\begin{bmatrix}1+a,d,e,f,1+a-d-e,1+a-e-f,1+a-d-f,1+a-b-c; \\ 1,d+e+f-a,1+a-d-e-f,1+a-b,1+a-c,1+a-d,1+a-e,1+a-f;\end{bmatrix} \times$$

$$\frac{1}{2\pi i}\int_C \wp\begin{bmatrix}d+e+f-a+s,1+a-b+s,1+a-c+s,1+s; \\ d+s\ ,\ e+s\ ,\ f+s\ ,\ 1+a-b-c+s\ ;\end{bmatrix} \frac{\pi\ q^s\ ds}{\sin\pi s\ \sin\pi(1+a-d-e-f-s)} .$$

Let

(2) $$\mathcal{J}(A,B,C,D;E,F)$$

$$= \frac{1}{2\pi i}\int_C \wp\begin{bmatrix}1+s,E+s,F+s,1+A+B+C+D-E-F+s; \\ A+s\ ,\ B+s\ ,\ C+s\ ,\ D+s\ ;\end{bmatrix} \frac{\pi\ q^s\ ds}{\sin\pi s\ \sin\pi(E+F-A-B-C-D-s)} .$$

Now, the left hand side in 5.5(1) is symmetrical in b,c,d,e and f. Hence, interchanging f and b in 5.5(1) and writing in the notation of 5.5(2) we get the transformation

(3) $$\mathcal{J}(d,e,f,1+a-b-c;1+a-b,1+a-c)$$

$$= \frac{\sin\pi(d+e+b-a)}{\sin\pi(d+e+f-a)} \wp\begin{bmatrix}b,1+a-b-d,1+a-b-e,1+a-c-f,d+e+f-a, \\ 1+a-d-e-f; \\ f,d+e+b-a,1+a-d-e-b,1+a-e-f,1+a-d-f, \\ 1+a-b-c;\end{bmatrix} \times$$

$$\mathcal{J}(d,e,b,1+a-c-f;1+a-f,1+a-c) .$$

Also interchanging d with c and e with b in 5.5(1) and writing in the above notation we get

$$(4)\qquad \mathcal{J}(d,e,f,1+a-b-c;1+a-b,1+a-c)$$

$$= \frac{\sin\pi(b+c+f-a)}{\sin\pi(d+e+f-a)}\,\rho\begin{bmatrix} b,c,1+a-b-f,1+a-c-f,d+e+f-a,1+a-d-e-f; \\ b+c+f-a,1+a-b-c-f,d,e,1+a-d-f,1+a-e-f; \end{bmatrix}\times$$

$$\mathcal{J}(c,b,f,1+a-d-e;1+a-e,1+a-d)\quad .$$

Putting d =A , e = B , f = C , 1+a-b-c = D, 1+a-b = E and 1+a-c = F i.e., c = E-D , b = F-D and a = E+F-D-1 we get from 5.5(3) and 5.5(4) that

$$(5)\qquad \mathcal{J}(A,B,C,D;E,F)$$

$$= \frac{\sin\pi(A+B-E)}{\sin\pi(A+B+C+D-E-F)}\,\rho\begin{bmatrix} F-D,E-A,F-C,E-B,1+A+B+C+D-E-F, \\ E+F-A-B-C-D; \\ 1+A+B-E,E-A-B,C,E+F-A-C-D,E+F-B-C-D,D; \end{bmatrix}$$

$$\times\mathcal{J}(A,B,F-D,F-C;E+F-D-C,F)\qquad ,$$

and

$$(6)\qquad \mathcal{J}(A,B,C,D;E,F)$$

$$= \frac{\sin\pi(C-D)}{\sin\pi(A+B+C+D-E-F)}\,\rho\begin{bmatrix} F-D,F-C,E-D,E-C,A+B+C+D-E-F+1, \\ E+F-A-B-C-D; \\ 1+C-D,D-C,A,B,E+F-B-C-D,E+F-A-C-D; \end{bmatrix}\times$$

$$\mathcal{J}(E-D,F-D,C,E+F-A-B-D;E+F-A-D,E+F-B-D)\quad .$$

In 5.5(6) if we put (F-A) for A , (F-B) for B, (F-D) for C , (F-C) for D , F for E and (2F+E-A-B-C-D) for F , the right hand integral remains unaltered and hence we get the transformation

(7) $$\mathcal{J}(A,B,C,D;E,F)$$

$$= \frac{\sin\pi(F-E)}{\sin\pi(A+B+C+D-E-F)} \times$$

$$\times \rho\begin{bmatrix}E-A,E-B,E-C,E-D,F-A,F-B,F-C,F-D,E+F-A-B-C-D,1+A+B+C+D-E-F;\\ 1+F-E,E-F,A,B,C,D,E+F-A-B-C,E+F-A-B-D,E+F-A-C-D,E+F-B-C-D;\end{bmatrix}$$

$$\times \mathcal{J}(F-A,F-B,F-C,F-D;F,2F+E-A-B-C-D) \quad .$$

The relations 5.5(5) and 5.5(7) give the integrals representing the two term relations 5.3(1) and 5.3(2) respectively.We can employ 5.5(5),5.5(6) and 5.5(7) to give us relations connecting four Saalchützian ${}_4\Phi_3$ also. In fact,evaluating the integrals on both sides of 5.5(6) by taking the residues at the poles of the integrand that lie to the right of the contour C,we get, after putting A for q^A and so on , the transformation

$$\prod\begin{bmatrix}E,F,G;\\ A,B,C,D;\end{bmatrix} {}_4\Phi_3\begin{bmatrix}A,B,C,D\ ;\\ E,F,G\end{bmatrix} q -$$

$$- (q/G)\prod\begin{bmatrix}q^2/G,qE/G,qF/G;\\ qA/G,qB/G,qC/G,qD/G;\end{bmatrix} {}_4\Phi_3\begin{bmatrix}qA/G,qB/G,qC/G,qD/G;\\ q^2/G,qE/G,qF/G\end{bmatrix} q =$$

$$= \prod\begin{bmatrix} EF/AD, EF/BD, F/C, E/C, G, q/G\ ; \\ D/C, A, B, C, qA/G, qB/G, qC/G\ ; \end{bmatrix} \times$$

$$\times\ {}_4\Phi_3\begin{bmatrix} C, E/D, F/D, qC/G\ ; \\ qC/D, EF/AD, EF/BD \end{bmatrix} q \Bigg] +$$

$$+ \text{idem}(C;D) \quad ,$$

with $EFG = qABCD$, a relation between four Saalchützian ${}_4\Phi_3$'s given by Sears (2;11.1) .

(5.6) <u>Integral representing the fundamental three term relation 5.4(1).</u>

Using the notation of 4.5 ,we can write 5.5(1) in the form

(1) $$w(a;b,c,d,e,f;x)$$

$$= \sin\pi(d+e+f-a-1)\ \wp\begin{bmatrix} 1+a-b-c, 1+a-d-e, 1+a-d-f, 1+a-e-f; \\ 1, b, c, d+e+f-a, 1+a-d-e-f\ ; \end{bmatrix} \times$$

$$\times \frac{1}{2\pi i}\int_C \wp\begin{bmatrix} d+e+f-a+s, 1+a-b+s, 1+a-c+s, 1+s; \\ d+s\ ,\ e+s\ ,\ f+s\ ,\ 1+a-b-c+s\ ; \end{bmatrix} \frac{\pi\ q^s\ ds}{\sin\pi s\ \sin\pi(1+a-d-e-f-s)}\ ,$$

where $x = q^{2+2a-b-c-d-e-f}$.

Hence,by analogy

(2) $$w(2b-a;b,b+c-a,b+d-a,b+e-a,b+f-a;x)$$

$$= \sin\pi(b+d+e+f-2a-1)\wp\begin{bmatrix} 1-c, 1+a-d-e, 1+a-e-f, 1+a-d-f; \\ 1, b, b+c-a, b+d+e+f-2a, 1+2a-b-d-e-f; \end{bmatrix} \times$$

$$\times \frac{1}{2\pi i}\int_C \wp\begin{bmatrix} b+d+e+f-2a+s, 1+b-a+s, 1+b-c+s, 1+s; \\ b+d-a+s, b+e-a+s, b+f-a+s, 1-c+s; \end{bmatrix} \frac{\pi q^s \, ds}{\sin\pi s \sin\pi(1+2a-b-d-e-f-s)}.$$

Putting s+b-a = t in 5.6(2), so that the products under the integral sign become the same as in 5.6(1), and combining with 5.6(1), we get after slight trigonometrical simplification that

$$(3) \quad \left\{ \wp\begin{bmatrix} 1, b, c, d+e+f-a, 1+a-d-e-f; \\ 1+a-b-c, 1+a-d-e, 1+a-e-f, 1+a-d-f; \end{bmatrix} \times w(a;b,c,d,e,f;x) - \right.$$

$$\left. -q^{b-a} \wp\begin{bmatrix} 1, b, b+c-a, b+d+e+f-2a, 1+2a-b-d-e-f; \\ 1-c, 1+a-d-e, 1+a-e-f, 1+a-d-f; \end{bmatrix} \times w(2b-a;b,b+c-a,b+d-a,b+e-a,b+f-a;x) \right\}$$

$$= \sin\pi(a-b) \frac{1}{2\pi i}\int_C \wp\begin{bmatrix} 1+s, d+e+f-a+s, 1+a-b+s, 1+a-c+s; \\ d+s \, , \, e+s \, , \, f+s \, , \, 1+a-b-c+s \, ; \end{bmatrix} \frac{\pi q^s \, ds}{\sin\pi s \sin\pi(b-a-s)}.$$

Now, compare the right hand integral in the above with the integral 5.5(2) after making in it the substitutions E=d+e+f-a , F=1+a-c , A=e , B=f , C=1+a-b-c and D=d . Then, using the result 5.5(1) and in the final result putting a for q^a , b for q^b and so on, we get the transformation 5.4(1).

It may be remarked that, as shown in 5.4 , the relation 5.4(1) can be used to find all other relations between three series of the type ${}_8\Phi_7$ and so we can prove all the relations connecting three ${}_8\Phi_7$ by means of simple transformations and manipulations of integrals of the type 5.6(1).

CHAPTER VI

SOME BASIC HYPERGEOMETRIC IDENTITIES

(6.1) Introduction & Notation. A number of basic hypergeometric identities have been given some years back by F.H.Jackson (1 & 2).These identities are the basic generalisations of some identities given earlier by Burchnall and Chaundy (1 & 2).In this chapter I give some more basic hypergeometric identities analogous to those given by Chaundy (1) for ordinary hypergeometric functions.Since basic hypergeometric functions are of some importance in the theory of numbers and combinatory analysis it is thought worthwhile studying the basic analogues of Chaundy's results (1).Besides the analogues of some of Chaundy's results a formal expansion giving a generating function for a basic polynomial sequence has been derived in 6.9 of the chapter.

We will use the following notation throughout this chapter,in addition to the notation already given in Chapter IV.

Let

$$[\delta;n] = (1-q^{\delta})(1-q^{\delta+1}).....(1-q^{\delta+n-1}), \quad |q| < 1,$$

with similar expression for $[\delta';n]$, where $\delta , \delta' = x\frac{\partial}{\partial x}$,

$y\frac{\partial}{\partial y}$ and $q^{\delta} = \exp(\delta \log q)$ and $q^{\delta'} = \exp(\delta' \log q)$.

The basic hypergeometric functions of two variables are defined as $(K \geq 0)$

$$\overset{(1)}{\Phi}(a;b,b';c;x,y;K) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;m+n]\ [b;m]\ [b';n]}{[1;m]\ [1;n]\ [c;m+n]}\, x^m y^n q^{Kn(n-1)},$$

$$\overset{(2)}{\Phi}(a;b,b';c,c';x,y;K) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;m+n]\ [b;m]\ [b';n]}{[1;m]\ [1;n]\ [c;m]\ [c';n]}\, x^m y^n q^{Kn(n-1)},$$

$$\overset{(3)}{\Phi}(a,a';b,b';c;x,y;K) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;m]\ [a';n]\ [b;m]\ [b';n]}{[1;m]\ [1;n]\ [c;m+n]}\, x^m y^n q^{Kn(n-1)},$$

$$\overset{(4)}{\Phi}(a,b;c,c';x,y;K) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;m+n]\ [b;m+n]}{[1;m]\ [1;n]\ [c;m]\ [c';n]}\, x^m y^n q^{Kn(n-1)},$$

The confluent hypergeometric functions used in the chapter are defined as follows

$$\Upsilon_1(a;b;c;x,y;K) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;m+n]\ [b;m]}{[1;m]\ [1;n]\ [c;m+n]}\, x^m y^n q^{Kn(n-1)},$$

$$\mathcal{E}_1(a,a';b;c;x,y;K) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;m]\ [a';n]\ [b;m]}{[1;m]\ [1;n]\ [c;m+n]}\, x^m y^n q^{Kn(n-1)},$$

$$\xi_2(a;b;c;x,y;K) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a;m]\;[b;m]}{[1;m]\;[1;n]\;[c;m+n]}\, x^m y^n q^{Kn(n-1)} .$$

Following Jackson, the functions in whose series definition there is a factor of the type $q^{Kn(n-1)}$ will be called "abnormal" functions. For $K=0$, we call the functions "normal".

(6.2) I give below some fundamental transformations and results which will be used in the chapter very frequently. They have been proved fully by Jackson (1 & 2).

(1) $$[-\delta;r]\,[-\delta';r]\left\{\frac{x^m y^n}{[1;m]\;[1;n]}\right\} = \frac{[-m;r]\;[-n;r]}{[1;m]\;[1;n]}\, x^m y^n$$

$$= \frac{q^{r(r-m-n-1)}}{[1;m-r]\,[1;n-r]}\, x^m y^n .$$

(2) $$[1-h-\delta-\delta';r]\; x^m y^n = (-1)^r\,[h+m+n-r;r]\; q^{-r(h+m+n)} \times q^{\frac{1}{2}r(r+1)} x^m y^n .$$

(3) $$1/[\delta+c;r]\;{}_2\mathcal{F}_1(a,b;c;x) = 1/[c;r]\;{}_2\mathcal{F}_1(a,b;c+r;x) .$$

(4) $$\nabla(h) = \wp\begin{bmatrix}\delta+h, & \delta'+h; \\ h, h+\delta+\delta'; & \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[-\delta;r]\;[-\delta';r]}{[1;r]\;[h;r]}\, q^{r(\delta+\delta'+h)} ,$$

and its inverse

$$(5)^{*} \quad \Delta(h) = \sum_{r=0}^{\infty} \frac{[-\delta;r]\,[-\delta';r]}{[1;r]\,[1-h-\delta-\delta';r]}\, q^{r}$$

$$(6) \quad = \sum_{r=0}^{\infty} \frac{(-1)^{r}\,[h;2r]\,[-\delta;r]\,[-\delta';r]}{[1;r]\,[h+r-1;r]\,[h+\delta;r]\,[h+\delta';r]}\, q^{r(\delta+\delta'+h)+(r)} \quad .$$

Also

$$(7) \quad \nabla(h)\,\Delta(k) = \sum_{r=0}^{\infty} \frac{[k;2r]\,[k-h;r]\,[-\delta;r]\,[-\delta';r]}{[1;r]\,[k+r-1;r]\,[k+\delta;r]\,[k+\delta';r]\,[h;r]}\, q^{r(\delta+\delta'+h)} \quad ,$$

$$(8) \quad = \sum_{r=0}^{\infty} \frac{[h-k;r]\,[-\delta;r]\,[-\delta';r]}{[1;r]\,[h;r]\,[1-k-\delta-\delta';r]}\, q^{r} \quad .$$

$$(9) \begin{cases} \Phi^{(1)}(a;b,b';c;x,y) = \nabla(a)\quad \Phi^{(3)}(a,a;b,b';c;x,y) \quad , \\ \Phi^{(1)}(a;b,b';c;x,y) = \nabla(a)\,\Delta(c)\,{}_2\mathcal{F}_1(a,b;c;x)\,{}_2\mathcal{F}_1(a,b';c;y) \quad , \\ \Phi^{(3)}(a,a';b,b';c;x,y) = \Delta(c)\,{}_2\mathcal{F}_1(a,b;c;x)\,{}_2\mathcal{F}_1(a',b';c;y) \quad . \end{cases}$$

(6.3) We begin with two simple identities giving the expansion of a ${}_2\mathcal{F}_1$ in terms of another ${}_2\mathcal{F}_1$, namely

$$(1) \quad {}_2\mathcal{F}_1(A,B;C;x) = \sum_{r=0}^{\infty} \frac{(-1)^{r}\,[a;r]\,[b;r]}{[1;r]\,[c;r]}\, {}_4\mathcal{F}_3\left[\begin{matrix} A,B,c,-r; \\ a,b,C \end{matrix}\; q\right] \times$$

$$\times\, x^{r}\, q^{(r)}\, {}_2\mathcal{F}_1(a+r,b+r;c+r;x) \quad ,$$

and

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* We will sometimes denote

"$q^{\frac{1}{2}ns(s-1)}$" by the abbreviated notation "$q^{n(s)}$" where n is any positive integer; e.g.,

$$q^{r(r-1)} \equiv q^{2(r)} \; .$$

(2) $$ {}_2\mathcal{F}_1(A,B;C;x) = \sum_{r=0}^{\infty} \frac{(-1)^r [a;r] [b;r]}{[1;r] [c+r-1;r]} {}_4\mathcal{F}_3\left[\begin{matrix} A,B,c+r-1,-r; \\ a,b,C \end{matrix} q\right] \times $$

$$ \times x^r q^{(r)} {}_2\mathcal{F}_1(a+r,b+r;c+2r;x) . $$

The proof of 6.3(1) and 6.3(2) depends upon the following two lemmas.

<u>Lemma I.</u>

(3) $$ \sum_{s=0}^{R} \frac{(-1)^s q^{(s)}}{[1;R-s] [1;s]} = \begin{cases} 0 & (R>0) , \\ 1 & (R=0) . \end{cases} $$

<u>Proof.</u> The left hand side of 6.3(3) can be written as

$$ 1/[1;R] \ {}_1\mathcal{F}_0(-R; q^R) . $$

This by the basic analogue of Vandermonde's theorem(T.a particular case of 8.4(3)) can be easily shown to give the required result.

<u>Lemma II.</u>

(4) $$ \sum_{s=0}^{R} \frac{(-1)^s q^{(s)}}{[1;s] [1;R-s] [c+2s;R-s] [c+s-1;s]} = \begin{cases} 0 & (R>0), \\ 1 & (R=0). \end{cases} $$

<u>Proof.</u> We have the left hand side of 6.3(4) equal to

$$\sum_{s=0}^{R} \frac{(-1)^s\ q^{(s)}(1-q^{c+2s-1})}{[1;s]\ [1;R-s]\ [c+s-1;R+1]}$$

$$= 1/(1-q^R) \sum_{s=0}^{R} \frac{(-1)^s\ q^{(s)} \left\{ q^s(1-q^{R-s})(1-q^{c+s-1})+(1-q^s)(1-q^{c+s+R-1}) \right\}}{[1;s]\ [1;R-s]\ [c+s-1;R+1]}$$

$$=1/(1-q^R) \left[\sum_{s=0}^{R-1} \frac{(-1)^s\ q^{(s)+s}}{[1;s]\,[1;R-s-1]\,[c+s;R]} - \sum_{s=1}^{R} \frac{(-1)^{s-1}\ q^{(s)}}{[1;s-1]\,[1;R-s]\,[c+s-1;R]} \right] .$$

This evidently vanishes identically for all $R > 0$.

This proves the lemma.

Now, to prove 6.3(1) we find that the coefficient of

(5) $$\frac{[A;n]\ [B;n]\ [c;n]}{[1;n]\ [a;n]\ [b;n]\ [C;n]}\, q^n$$

on the right hand side is

$$\sum_{r=0}^{\infty} \frac{(-1)^r\ [a;r]\ [b;r]\ [-r;n]}{[1;r]\ [c;r]} x^r\ q^{(r)}\,{}_2\mathcal{F}_1(a+r, b+r; c+r; x) \quad .$$

Granted absolute convergence we can rearrange this as

$$\sum_{R=0}^{\infty} \frac{[a;R]\ [b;R]}{[c;R]} x^R \sum_{r=n}^{R} \frac{(-1)^{r+n}\quad q^{(n)+(r)-rn}}{[1;R-r]\ [1;r-n]} \quad .$$

The inner sum is equal to q^{-n} by Lemma I ,for R=n and is zero,otherwise.

Hence the coefficient of 6.3(5) is simply $[a;n]\,[b;n]\;x^n q^{-n}/[c;n]$, which proves the identity 6.3(1).

Similarly,to prove 6.3(2) we collect the coefficient of

$$[A;n]\,[B;n]\;q^n\,/\,[a;n]\,[b;n]\,[C;n]\,[1;n]$$

on the right of 6.3(2) and proceeding as above get the result on applying Lemma II .

(6.4) If in 6.3(2) we suppose

$$A+B-C = a+b-c$$

and change a,b,A,B and x to c-a,c-b,C-A,C-B and xq^{A+B-C} ($= xq^{a+b-c}$) respectively,we can use the following well-known identity (T.8.4(2) after changing a to q^a and so on)

$$(1)\quad {}_2\mathfrak{F}_1(a,b;c;x) = \prod_{n=0}^{\infty}\frac{(1-xq^{a+b-c+n})}{(1-xq^n)}\;{}_2\mathfrak{F}_1(c-a,c-b;c;xq^{a+b-c}),$$

to replace the ${}_2\mathfrak{F}_1$ functions on both sides of 6.3(2) to get

$$(2)\quad {}_2\mathfrak{F}_1(A,B;C;x) = \sum_{r=0}^{\infty}\frac{[c-a;r]\,[c-b;r]}{[1;r]\,[c+r-1;r]}\,(-x)^r\,q^{r(a+b-c)+(r)}\times$$

$$\times\;{}_4\mathfrak{F}_3\left[\begin{matrix}C-A,C-B,c+r-1,-r;\\ c-a,c-b,C\end{matrix}\;q\right]{}_2\mathfrak{F}_1\left[\begin{matrix}a+r,b+r;\\ c+2r\end{matrix}\;x\right].$$

This gives an alternative form for 6.3(2).

Again putting A,B,C = a,b+h and c+h respectively in 6.3(2), the ${}_4\mathcal{F}_3$ on the right reduces to ${}_3\mathcal{F}_2$, which can be summed by the basic analogue of Saalchütz's theorem (T.8.4(1)) to give

$$(3)\ {}_2\mathcal{F}_1(a,b+h;c+h;x) = \sum_{r=0}^{\infty} \frac{(-1)^r [a;r][c-b;r][-h;r]}{[1;r][c+r-1;r][c+h;r]} x^r q^{r(b+h)} \times$$

$$\times\, q^{(r)}\, {}_2\mathcal{F}_1(a+r,b+r;c+2r;x) \quad .$$

(6.5) Basic hypergeometric functions of two variables.

Now, we shall prove the following four identities which involve the basic hypergeometric functions of two variables defined in 6.1 and functions of higher order, similar to those defined by Jackson (2;8). The following are the four expansions

$$(1)\quad \Phi^{(1)}(A;B,B';C;x,y;K) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^{r+s} \frac{[a;r+s][b;r][b';s]}{[1;r][1;s][c;r+s]} \times$$

$$\times\, x^r y^s\, q^{(s)+(r)}\, \Phi\begin{bmatrix} A,c:B,-r;B',-s; & \\ & q,q;K \\ a,C:b;b'; & \end{bmatrix} \times$$

$$\times\, \Phi^{(1)}(a+r+s;b+r,b'+s;c+r+s;x,y) \quad ,$$

$$(2)\quad \Phi^{(2)}(A;B,B';C,C';x,y;K) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{r+s} [a;r+s][b;r][b';s]}{[1;r][1;s][c;r][c';s]} \times$$

$$\times\, x^r y^s\, q^{(r)+(s)}\; \Phi\begin{bmatrix} A:B,c,-r;B',c',-s; & \\ & q,q;K \\ a:b,C;b',C'; & \end{bmatrix} \times$$

$$\times\, \overset{(2)}{\Phi}(a+r+s;b+r,b'+s;c+r,c'+s;x,y) \quad ,$$

(3) $\overset{(3)}{\Phi}(A,A';B,B';C;x,y;K) =$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^{r+s} \frac{[a;r][a';s][b;r][b';s]}{[1;r][1;s][c;r+s]}\; \Phi\begin{bmatrix} c:A,B,-r;A',B',-s; & \\ & q,q;K \\ C:a,b;a',b'; \quad . & \end{bmatrix} \times$$

$$\times\, x^r y^s\, q^{(r)+(s)}\; \overset{(3)}{\Phi}(a+r,a'+s;b+r,b'+s;c+r+s;x,y) \quad ,$$

(4) $\overset{(4)}{\Phi}(A,B;C,C';x,y;K) =$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{r+s}[a;r+s][b;r+s]}{[1;r][1;s][c;r][c';s]}\; \Phi\begin{bmatrix} A,B:c,-r;c',-s; & \\ & q,q;K \\ a,b:C;C'; & \end{bmatrix} \times$$

$$\times\, x^r y^s\, q^{(r)+(s)}\; \overset{(4)}{\Phi}(a+r+s,b+r+s;c+r,c'+s;x,y) \quad .$$

In 6.5(1) , $\Phi\begin{bmatrix} A,c:B,-r;B',-s; & \\ & q,q;K \\ a,C:b;b'; & \end{bmatrix}$ denotes

the double hypergeometric series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[A;m+n]\,[c;m+n]\,[B;m]\,[-r;m]\,[B';n]\,[-s;n]}{[1;m][1;n]\,[a;m+n]\,[C;m+n]\,[b;m]\,[b';n]} q^{m+n+Kn(n-1)} \quad .$$

So,in other 'coefficient' Φ's, the colons mark off the 'double' parameters (i.e. those occurring with the parameters m+n in the square brackets), the semi-colons the 'simple' parameters (occurring with the parameter m or n). Thus, the pattern of the above four formulae is quite clear.

The above identities are interesting in the sense that they express an "abnormal" function in a series of the "normal" function of the same type. For $K=\frac{1}{2}$, the left hand functions become similar to those studied by Jackson (1;9).

We shall prove the first of these formulae 6.5(1) and the rest follow by an exactly similar procedure.

Proof of 6.5(1).

The coefficient of

$$\frac{[A;m+n]\,[c;m+n]\,[B;m]\,[B';n]}{[a;m+n]\,[C;m+n]\,[b;m]\,[b';n]}\,q^{m+n+Kn(n-1)}$$

on the right hand side is

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{[a;r+s]\,[b;r]\,[b';s]}{[1;r]\,[1;s]\,[c;r+s]}\,x^r y^s\,q^{(r)+(s)}\,[-r;m][-s;n]\,\times$$

$$\times\,\Phi^{(1)}(a+r+s;b+r,b'+s;c+r+s;x,y)\quad .$$

This, granted absolute convergence, can be easily written in the form

$$\sum_{R=0}^{\infty}\sum_{S=0}^{\infty}\frac{[a;R+S][b;R][b';S]}{[c;R+S]}\,x^R y^S\sum_{r=m}^{R}\sum_{s=n}^{S}\frac{(-1)^{r+s+m+n}\;q^{(r)+(s)+(m)+(n)-sn-rm}}{[1;R-r]\,[1;S-s]\,[1;r-m]\,[1;s-n]}\,.$$

The value of the inner sum is found on simplification and a consequent double application of Lemma I. It is zero for $R>0$, $S>0$ and is q^{-m-n} when $R = m$ and $S = n$. Hence we get the required result as in 6.3(1).

(6.6) In this section we will consider some expansions involving the basic hypergeometric functions of two variables and generalised hypergeometric functions. They differ in nature and method of proof from the foregoing expansions. We shall prove the following identities

$$(1)\quad \prod_{n=0}^{\infty} \frac{(1-xq^n)(1-yq^n)}{(1-xq^{a+b-c+n})(1-yq^{a+b'-c+n})} \overset{(1)}{\Phi}(a;b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\,[c-b-b';r]\,[a;r]\,[c-a;r]}{[1;r]\,[c;2r]} q^{(r)+r(a+b+b'-c)} \times$$

$$\times\, x^r y^r \;\overset{(1)}{\Phi}(c-a+r;c-b+r,c-b'+r;c+2r;xq^{a+b-c},yq^{a+b'-c}),$$

$$(2)\quad \prod_{n=0}^{\infty} \frac{(1-xq^n)}{(1-xq^{a+b-c+n})} \overset{(3)}{\Phi}(a,a';b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} \frac{[c-a-b;r]\,[a';r]\,[b';r]}{[1;r]\,[c;2r]} q^{2(r)+r(a+b)} \; x^r y^r \;\times$$

$$\times\, \overset{(3)}{\Phi}(c-a+r,a'+r;c-b+r,b'+r;c+2r;xq^{a+b-c};y),$$

(3) $$\prod_{n=0}^{\infty} \frac{(1-xq^n)(1-yq^n)}{(1-xq^{a+b-c+n})(1-yq^{a+b'-c+n})} \Phi^{(1)}(a;b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} \frac{[c-a;r]\,[c-b;r]\,[c-b';r]}{[1;r]\,[c;2r]} x^r y^r\, q^{2(r)+r(a+b+b'-c)} \times$$

$$\times\, {}_3\mathcal{F}_2\left[\begin{matrix} c-b-b', -r, a; \\ c-b\ ,\ c-b' \end{matrix}\; q\right] \Phi^{(3)}(c-a+r, c-a+r; c-b+r, c-b'+r; c+2r; xq^{a+b-c}, yq^{a+b'-c}),$$

(4) $$\prod_{n=0}^{\infty} \frac{(1-xq^n)(1-yq^n)}{(1-xq^{a+b-c+n})(1-yq^{a+b'-c+n})} \Phi^{(3)}(a,a;b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{[a;r]\,[c-a;r]\,[c-b-b';r]}{[1;r]\,[c;2r]} q^{r(a+b+b'-c)+(r)} x^r y^r \times$$

$$\times\, {}_3\mathcal{F}_2\left[\begin{matrix} b, b', -r; \\ c-a, 1-r+b+b'-c \end{matrix}\; q\right] \Phi^{(1)}(c-a+r; c-b+r, c-b'+r; c+2r; xq^{a+b-c}, yq^{a+b'-c}),$$

(5) $$\prod_{n=0}^{\infty} \frac{(1-xq^n)}{(1-xq^{a+b-c+n})} \Phi^{(3)}(a,c-a;b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{[c-a;r]\,[a;r]\,[b';r]}{[1;r]\,[c;2r]} q^{rb+(r)} x^r y^r \times$$

$$\times\, \Phi^{(1)}(c-a+r; c-b+r, b'+r; c+2r; xq^{a+b-c}, y),$$

$$(6)\ \prod_{n=0}^{\infty} \frac{(1-xq^{n})}{(1-xq^{a+b-c+n})} \overset{(1)}{\Phi}(a;b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} \frac{[c-a;r]\,[a;r]\,[b';r]}{[1;r]\,[c;2r]} q^{ra+2(r)}\, x^{r}y^{r} \times$$

$$\times \overset{(3)}{\Phi}(c-a+r, a+r; c-b+r, b'+r; c+2r; xq^{a+b-c}, y) \quad ,$$

$$(7)\ \prod_{n=0}^{\infty} \frac{(1-xq^{n})(1-yq^{n})}{(1-xq^{a+b-c+n})(1-yq^{a'+b'-c+n})} \overset{(3)}{\Phi}(a,a';b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} \frac{[c-a;r]\,[c-b;r]\,[c-a'-b';r]}{[1;r]\,[c;2r]} q^{2(r)+r(a+a'+b+b'-c)}\, x^{r}y^{r} \times$$

$$\times {}_4\mathcal{F}_3\left[\begin{matrix} c-a-b, a', b', -r; \\ c-a, c-b, 1-r+a'+b'-c \end{matrix}\ q\right] \overset{(3)}{\Phi}(c-a+r, c-a'+r; c-b+r, c-b'+r;$$

$$c+2r; xq^{a+b-c}, yq^{a'+b'-c}).$$

<u>Proof of 6.6(1).</u>

Consider the following identity due to Jackson *,

(1;37)

$$(8)\qquad \overset{(1)}{\Phi}(a;b,b';c;x,y) = \sum_{r=0}^{\infty} \frac{[a;r][b;r][b';r][c-a;r]}{[1;r][c+r-1;r][c;2r]} q^{ra+2(r)} \times$$

$$\times\, x^{r}y^{r}\, {}_2\mathcal{F}_1(a+r, b+r; c+2r; x) \times$$

$$\times\, {}_2\mathcal{F}_1(a+r, b'+r; c+2r; y) \,.$$

---

* There is a misprint in Jackson's result. The solitary quadratic factor in the series on the right should be

$q^{ra+2(r)}$ and not $q^{rc+3(r)}$

Using 6.4(1) to replace the ${}_2\mathcal{F}_1$'s on the right we get

(9) $$\prod_{n=0}^{\infty} \frac{(1-xq^n)(1-yq^n)}{(1-xq^{a+b-c+n})(1-yq^{a+b'-c+n})} \Phi^{(1)}(a;b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} \frac{[a;r]\,[b;r]\,[b';r]\,[c-a;r]}{[1;r]\,[c+r-1;r]\,[c;2r]} q^{ra+2(r)} \quad x^r y^r \quad \times$$

$$\times {}_2\mathcal{F}_1\left[\begin{matrix} c-a+r, c-b+r; \\ c+2r \end{matrix} xq^{a+b-c}\right] {}_2\mathcal{F}_1\left[\begin{matrix} c-a+r, c-b'+r; \\ c+2r \end{matrix} yq^{a+b'-c}\right] .$$

The series on the right of 6.6(9) is symbolically equal to

(10)

$$\sum_{r=0}^{\infty} \frac{[c;2r]\,[a;r]\,[b;r]\,[b';r]\,[-\delta;r]\,[-\delta';r]}{[1;r]\,[c+r-1;r]\,[c-a;r]\,[c-b;r]\,[c-b';r]\,[c+\delta;r]\,[c+\delta';r]} q^{r(\delta+\delta'+2c-a-b-b')} \times$$

$$\times \; {}_2\mathcal{F}_1(c-a,c-b;c;xq^{a+b-c}) \quad {}_2\mathcal{F}_1(c-a,c-b';c;yq^{a+b'-c}) \quad ,$$

provided one of the numerator parameters δ or δ' is a positive integer. Using a formula due to Watson (T.8.5(2)) the operator in 6.6(10) can be expressed as a ${}_4\mathcal{F}_3$ to give

$${}_4\mathcal{F}_3\left[\begin{matrix} c-b-b', a, -\delta, -\delta'; \\ c-b, c-b', 1-\delta-\delta'+a-c \end{matrix} q\right] \quad \nabla(c-a) \;\; \Delta(c) \quad \times$$

$$\times \; {}_2\mathcal{F}_1(c-a,c-b;c;xq^{a+b-c}) \quad {}_2\mathcal{F}_1(c-a,c-b';c;yq^{a+b'-c})$$

$$= {}_4\mathcal{F}_3\left[\begin{matrix} c-b-b', a, -\delta, -\delta' ; \\ c-b, c-b', 1-\delta-\delta'+a-c \end{matrix} q\right] \overset{(1)}{\Phi}(c-a;c-b,c-b';c;xq^{a+b-c},yq^{a+b'-c})$$

by 6.2(9).

This on simplification and using the operator ${}_4\mathcal{F}_3$ gives the required result.

<u>Proof of 6.6(2).</u>

In order to prove 6.6(2) we use the following identity due to Jackson※ (1;35)

$$\overset{(3)}{\Phi}(a,a';b,b';c;x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r [a;r] [a';r] [b;r] [b';r]}{[1;r] [c;2r] [c+r-1;r]} \times$$

$$\times q^{rc+3(r)} x^r y^r {}_2\mathcal{F}_1(a+r,b+r;c+2r;x) \times$$

$$\times {}_2\mathcal{F}_1(a'+r,b'+r;c+2r;y) .$$

Now, using 6.4(1) to transform the first ${}_2\mathcal{F}_1$ on the right we get

$$(11) \prod_{n=0}^{\infty} \frac{(1-xq^n)}{(1-xq^{a+b-c+n})} \overset{(3)}{\Phi}(a,a';b,b';c;x,y)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [a;r] [a';r] [b;r] [b';r]}{[1;r] [c;2r] [c+r-1;r]} q^{rc+3(r)} x^r y^r \times$$

$$\times {}_2\mathcal{F}_1(c-a+r,c-b+r;c+2r;xq^{a+b-c}) {}_2\mathcal{F}_1(a'+r,b'+r;c+2r;y) .$$

※ There is a misprint in Jackson's result in the solitary factor $q^{(r)}$ on the right. It should be $q^{3(r)}$.

The right hand side of 6.6(11) can easily be shown to be equal to

(12)

$$\sum_{r=0}^{\infty} \frac{(-1)^r [c;2r][a;r][b;r][-\delta;r][-\delta';r]}{[1;r][c+r-1;r][c-a;r][c-b;r][c+\delta;r][c+\delta';r]} q^{r(\delta+\delta'+2c-a-b)+(r)} \times$$

$$\times {}_2\mathcal{F}_1(c-a,c-b;c;xq^{a+b-c}) \quad {}_2\mathcal{F}_1(a',b';c;y) \quad .$$

Using a limiting case of a result due to Watson (a limiting case of T.8.5(2)) we can write 6.6(12) in the form

$${}_3\mathcal{F}_2\left[\begin{matrix}c-a-b,-\delta,-\delta'; \\ c-a,c-b\end{matrix} q^{c+\delta+\delta'}\right] \Delta(c)\left\{{}_2\mathcal{F}_1\left[\begin{matrix}c-a,c-b; \\ c\end{matrix} xq^{a+b-c}\right] {}_2\mathcal{F}_1\left[\begin{matrix}a',b'; \\ c\end{matrix} y\right]\right\}$$

$$= {}_3\mathcal{F}_2\left[\begin{matrix}c-a-b,-\delta,-\delta'; \\ c-a,c-b\end{matrix} q^{c+\delta+\delta'}\right] \overset{(3)}{\Phi}(c-a,a';c-b,b';c;xq^{a+b-c},y) ,$$

on using the result of 6.2(9).

This on simplification gives the desired identity.

(6.7) To deduce 6.6(3) we use the following result due to Jackson (1;39)

$$(1) \quad \overset{(1)}{\Phi}(a;b,b';c;x,y) = \sum_{r=0}^{\infty} \frac{[a;r]\ [b;r]\ [b';r]}{[1;r]\ [c;2r]} x^r y^r q^{ra+2(r)} \times$$

$$\times \overset{(3)}{\Phi}(a+r,a+r;b+r,b'+r;c+2r;x,y)$$

on the right hand side of 6.6(1) to replace $\Phi^{(1)}$ by $\Phi^{(3)}$ and diagonal summation gives the required result. To obtain 6.6(4) we start with Jackson (1;40), viz.;

$$(2) \quad \Phi^{(3)}(a,a;b,b';c;x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r [a;r][b;r][b';r]}{[1;r][c;2r]} x^r y^r q^{ra+(r)} \times$$

$$\times \Phi^{(1)}(a+r;b+r,b'+r;c+2r;x,y)$$

and replace the $\Phi^{(1)}$ on the right of it by 6.6(1). Then diagonal summation and simplification yields 6.6(4). To prove 6.6(5) we use 6.6(2) with $a' = c-a$ and replace the $\Phi^{(3)}$ on the right by a $\Phi^{(1)}$ using 6.7(2). Consequent diagonal summation and an application of the basic analogue of Vandermonde's theorem (T.8.4(3), for $b=q^{-N}$) gives the result.

Next, to deduce 6.6(6) we begin with Jackson's result 6.7(1) and use 6.6(2) with $a'=a$ on the right hand side of 6.7(1) to replace its $\Phi^{(3)}$ by another $\Phi^{(3)}$. Summing diagonally and applying the basic analogue of Vandermonde's theorem we get the expansion.

Next, to deduce 6.6(7) we begin with 6.6(2) and with the similar expression for $\prod_{n=0}^{\infty}[(1-yq^n)/(1-yq^{a'+b'-c+n})]\times$ $\times\Phi^{(3)}$ get a double series in $\Phi^{(3)}$. Simplification and summation as before gives the required identity.

(6.8) In this section I shall state, without proof, some identities involving basic confluent hypergeometric functions. These identities are of special interest being closely connected with the basic Bessel, Laguerre and Whittaker's functions discussed by Jackson. They can either be obtained directly from the identities of Jackson (2) or as limiting cases of the results already discussed in 6.6 and 6.7, as is indicated in the section later. The expansions are as follows

$$(1)\prod_{n=0}^{\infty}\frac{(1-xq^{n})}{(1-xq^{a+b-c+n})(1+yq^{a-c+n})}\Upsilon_1(a;b;c;x,y;\tfrac{1}{2})$$

$$=\sum_{r=0}^{\infty}\frac{[a;r][c-a;r]}{[1;r][c;2r]}x^r y^r q^{r(a+b-c)+(r)}\Upsilon_1(c-a+r;c-b+r;c+2r;xq^{a+b-c},-yq^{a-c}),$$

$$(2)\prod_{n=0}^{\infty}\frac{(1-xq^{n})}{(1-xq^{a+b-c+n})}\mathcal{E}_1(a,a';b;c;x,y;K)$$

$$=\sum_{r=0}^{\infty}\frac{[c-a-b;r]\;[a';r]}{[1;r]\;[c;2r]}x^r y^r q^{r(a+b)+(K+1)r(r-1)}\times$$

$$\mathcal{E}_1(c-a+r,a'+r;c-b+r;c+2r;xq^{a+b-c},yq^{2Kr};K),$$

$$(3)\prod_{n=0}^{\infty}\frac{(1-xq^{n})}{(1-xq^{a+b-c+n})(1+yq^{a-c+n})}\Upsilon_1(a;b;c;x,y;\tfrac{1}{2})$$

=

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [c-a;r][c-b;r]}{[1;r]\ [c;2r]} x^r y^r\ q^{r(a+b-c)+2(r)}\ {}_2\mathcal{F}_1\left[\begin{matrix} a,-r; \\ c-b \end{matrix}\ q\right] \times$$

$$\times\ \mathcal{E}_1(c-a+r, c-a+r; c-b+r; c+2r; xq^{a+b-c}, -yq^{a-c}),$$

$$(4) \prod_{n=0}^{\infty} \frac{(1-xq^n)}{(1-xq^{a+b-c+n})(1+yq^{a-c+n})}\ \mathcal{E}_1(a,a;b;c;x,y;\tfrac{1}{2})$$

$$= \sum_{r=0}^{\infty} \frac{[a;r]\ [c-a;r]}{[1;r]\ [c;2r]} x^r y^r\ q^{r(a+b-c)+(r)}\ {}_2\mathcal{F}_1\left[\begin{matrix} b,-r; \\ c-a \end{matrix}\ q^{r+c-b}\right] \times$$

$$\times\ \Upsilon_1(c-a+r; c-b+r; c+2r; xq^{a+b-c}, -yq^{a-c}) \quad ,$$

$$(5) \prod_{n=0}^{\infty} \frac{(1-xq^n)}{(1-xq^{a+b-c+n})}\ \mathcal{E}_1(a, c-a; b; c; x, y; K)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [c-a;r]\ [a;r]}{[1;r]\ [c;2r]} x^r y^r\ q^{rb+(2K+1)(r)} \quad \times$$

$$\times\ \Upsilon_1(c-a+r; c-b+r; c+2r; xq^{a+b-c}, yq^{2Kr}; K) \quad ,$$

$$(6) \prod_{n=0}^{\infty} \frac{(1-xq^n)}{(1-xq^{a+b-c+n})}\ \Upsilon_1(a;b;c;x,y;K)$$

$$= \sum_{r=0}^{\infty} \frac{[c-a;r]\ [a;r]}{[1;r]\ [c;2r]}\ x^r y^r\ q^{ra+2(K+1)(r)} \quad \times$$

$$\mathcal{E}_1(c-a+r, a+r; c-b+r; c+2r; xq^{a+b-c}, yq^{2Kr}; K)$$

,

(7) $$\prod_{n=0}^{\infty} \frac{(1-xq^{n})}{(1-xq^{a+b-c+n})(1+yq^{a'-c+n})} \mathcal{E}_1(a,a';b;c;x,y;\tfrac{1}{2})$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [c-a;r]\,[c-b;r]}{[1;r]\,[c;2r]} x^r y^r q^{r(a+a'+b-c)+2(r)} \times$$

$$\times {}_3\mathcal{F}_2\left[\begin{matrix} c-a-b, a', -r; \\ c-a, c-b \end{matrix} \; q^{r+c-a'}\right] \mathcal{E}_1(c-a+r, c-a'+r; c-b+r; c+2r; xq^{a+b-c}, -yq^{a'-c}) ,$$

(8) $$\prod_{n=0}^{\infty} \frac{(1-xq^{n})}{(1-xq^{a+b-c+n})} \mathcal{E}_2(a;b;c;x,y;K')$$

$$= \sum_{r=0}^{\infty} \frac{[c-a-b;r]}{[1;r]\,[c;2r]} x^r y^r q^{ra+2(K'+1)(r)+rb} \times$$

$$\times \mathcal{E}_2(c-a+r; c-b+r; c+2r; xq^{a+b-c}, yq^{2K'r}; K') .$$

$(K' = K+\tfrac{1}{2})$

The identities 6.8(1),6.8(3),6.8(4) and 6.8(7) are the limiting cases of 6.6(1),6.6(3),6.6(4) and 6.6(7) respectively.The identities 6.8(2),6.8(5) and 6.8(6) are the limiting cases of 6.6(2),6.6(5) and 6.6(6) respectively when $K=\frac{1}{2}$.For other values of K they can be directly derived from known formulae due to Jackson (2). The identity 6.8(8) is a limiting case of 6.8(2).

(6.9) I conclude these expansions by giving a formal expansion which gives the generating function for a sequence of basic polynomials, in particular, of a general polynomial of the basic hypergeometric type.

Let $\Phi(t)$ be an analytic function which does not vanish for t=0 and let it be defined by an absolutely convergent series

$$\Phi(t) = \sum_{n=0}^{\infty} \frac{b_n\, t^n}{[1;n]} ,$$

and let

$$f(xt) = {}_0\mathcal{F}_K(-;\beta_1,\ldots\ldots,\beta_K;xtq^{\sigma}) , \sigma>0 .$$

Then, it can easily be shown by simple rearrangement of the series that

$$(1) \qquad \Phi(t)\, f(xt) = \sum_{n=0}^{\infty} y_n(x)\, t^n ,$$

where $y_n(x)$ is a polynomial of degree n given by

$$(2) \qquad y_n(x) = \sum_{s=0}^{n} \frac{b_{n-s}\, x^s\, q^{\sigma s}}{[1;s]\,[1;n-s][\beta_1;s]\cdots\cdots[\beta_K;s]} .$$

Hence, 6.9(1) can be taken to be the generating function of the polynomials 6.9(2).

A particular case of interest to us is obtained when $\Phi(t)$ itself is a basic hypergeometric function with

$$b_n = \frac{[a_1;n] \ldots\ldots\ldots\ldots [a_p;n]}{[c_1;n] \ldots\ldots\ldots\ldots [c_r;n]} \, q^{n\rho} \qquad .$$

We get from 6.9(2) that $y_n(x)$ is given by

$$y_n(x) = \frac{[a_1;n] \ldots\ldots [a_p;n]}{[1;n][c_1;n] \ldots [c_r;n]} \, q^{n\rho} \times$$

$$\times \sum_{s=0}^{n} \frac{(-1)^{s(r+p+1)} \; [-n;s] \; [1-n-c_1;s] \ldots\ldots [1-n-c_r;s]}{[1;s] \; [\beta_1;s] \ldots\ldots [\beta_k;s] \; [1-n-a_1;s] \ldots\ldots [1-n-a_p;s]} \, x^s \, q^T \quad ,$$

where $\quad T = \frac{1}{2}s(s+1)(p-r-1)+s(\sum_1^r c_r - \sum_1^p a_p + \sigma - \rho + 1 - np + nr + n)$.

In particular, if $r=p-1$ we get the generating function for

$$y_n(x) = \frac{[a_1;n] \ldots\ldots [a_p;n]}{[1;n] \; [c_1;n] \ldots\ldots [c_{p-1};n]} \, q^{n\rho} \times$$

$$\times \; {}_p\mathcal{F}_{k+p}\left[\begin{matrix} -n, 1-c_1-n, \ldots\ldots\ldots, 1-c_{p-1}-n; \\ \beta_1, \ldots\ldots, \beta_k, 1-a_1-n, \ldots\ldots, 1-a_p-n \end{matrix} \; x \, q^{\sum c_{p-1} - \sum a_p + \sigma - \rho + 1}\right].$$

(6.10) <u>Convergence conditions</u> . In order that the various rearrangements be justified in the formal proofs of the foregoing expansions deduced in 6.3-6.6, the conditions of absolute convergence of the expansions must be stated. We will take all the parameters in the hypergeometric functions to be real and positive; the variables x and y should be replaced by their absolute values $|x|$, $|y|$, but, for simplicity, we shall for the moment regard x,y themselves as positive, replacing them by $|x|$, $|y|$ in the final statement of the convergence conditions. The base q is also supposed to be real and positive such that $0 < q < 1$.

We need then know simple bounds for the basic hypergeometric functions with positive variables when their positive parameters diverge to infinity in certain ways. We establish these bounds by proving the following simple lemmas.

Lemma I.

$$\begin{array}{lll}
\text{(i)} & \Phi^{(1)}(a+1;b+1,b';c+1;x,y) & < A\ \Phi^{(1)}, \\
\text{(ii)} & \Phi^{(2)}(a+1;b+1,b';c+1,c';x,y) & < A\ \Phi^{(2)}, \\
\text{(iii)} & \Phi^{(3)}(a+1,a';b+1,b';c+1;x,y) & < A\ \Phi^{(3)}, \\
\text{(iv)} & \Phi^{(4)}(a+1,b+1;c+1,c';x,y) & < A\ \Phi^{(4)}, \\
\text{(v)} & {}_2\mathcal{F}_1(a+1,b+1;c+1;x) & < A\ {}_2\mathcal{F}_1\ ;
\end{array}$$

where $A = 1/(1-q^a)(1-q^b)$, and $\Phi^{(1)}$ stands for the function $\Phi^{(1)}(a;b,b';c;x,y)$, with similar abbreviated notations for other functions.

Proof. (i) Let $R_{m,n}$ be the ratio of the coefficients of $x^m y^n$ on both sides, then

$$R_{m,n} = \frac{(1-q^{c})(1-q^{a+m+n})(1-q^{b+m})}{(1-q^{c+m+n})}$$

$$< 1 \quad ,\text{since} \qquad (1-q^{c})/(1-q^{c+m+n}) \leqslant 1,$$

for all m and n and $0 < q < 1$. Hence the result follows.

The proofs of (ii-v) follow in exactly the same way.

<u>Corollary to Lemma I</u>. Repeated application of the results of Lemma I give

(i) $\Phi^{(1)}(a+r+s;b+r,b'+s;c+r+s;x,y) < \frac{1}{[a;r+s][b;r][b';s]} \Phi^{(1)}$,

(ii) $\Phi^{(2)}(a+r+s;b+r,b'+s;c+r,c'+s;x,y) < \frac{1}{[a;r+s][b;r][b';s]} \Phi^{(2)}$,

(iii) $\Phi^{(3)}(a+r,a'+s;b+r,b'+s;c+r+s;x,y) < \frac{1}{[a;r][a';s][b;r][b';s]} \Phi^{(3)}$,

(iv) $\Phi^{(4)}(a+r+s,b+r+s;c+r,c'+s;x,y) < \frac{1}{[a;r+s][b;r+s]} \Phi^{(4)}$,

(v) ${}_2\mathcal{F}_1(a+r,b+r;c+r;x) < \frac{1}{[a;r]\,[b;r]}\ {}_2\mathcal{F}_1$.

<u>Lemma II.</u>

(i) ${}_2\mathcal{F}_1(a+1,b;c+1;x) < \frac{(1-q^{c})}{(1-q^{A})}\ {}_2\mathcal{F}_1(a,b;c;x)$,

(ii) $\Phi^{(1)}(a+1;b+1,b'+1;c+2;x,y) < \dfrac{(1-q^{c})(1-q^{c+1})\ \Phi^{(1)}}{(1-q^{A})(1-q^{a})(1-q^{b})(1-q^{b'})}$,

where $c > a > A$.

Proof. (i) As before

$$R_m = \frac{(1-q^{A})(1-q^{a+m})}{(1-q^{a})(1-q^{c+m})} < 1 \text{ , if } c > a > A \text{ and } 0 < q < 1.$$

Thus, the lemma is proved.

(ii)

$$R_{m,n} = \frac{(1-q^{a+m+n})(1-q^{b+m})(1-q^{b'+n})(1-q^{A})}{(1-q^{c+m+n})(1-q^{c+m+n+1})}$$

$$< 1 \quad ,$$

under the stated conditions.

The inequality follows.

Corollary to Lemma II.*

(i) $_2\mathcal{F}_1(a+r,b+r;c+2r;x) < \dfrac{[c;2r]}{[A;r]\ [A';r]}\ {}_2\mathcal{F}_1$,

where $c > a > A$, $c > b > A'$.

---

* There is no loss of generality in taking $c > a$ or b since c diverges to infinity as $c+2r$ and a and b only as $a+r$, $b+r$. Hence c can be easily supposed to be greater than a and b.

(ii) $\overset{(1)}{\Phi}(a+r;b+r,b'+r;c+2r;x,y) < \dfrac{[c;2r]}{[A;r][a;r][b;r]~~[a';r]~~[b';r]}\overset{(1)}{\Phi}$,

where $c > a > A$.

Lemma III.

$$[c;m+n] \geqslant [c;m]\,[c;n]$$
$$< [c;m]\,[c^*;n] \quad ,$$

where $[c^*;n] = (1+q^c)(1+q^{c+1})\ldots\ldots(1+q^{c+n-1})$.

The proof is obvious.

Lemma IV. For $m \leqslant r$; r and m being positive integers

$$\left|[-r;m]\right| < q^{-\frac{1}{2}r(r+1)} \quad .$$

Proof.

$$\left|[-r;m]\right| = \left|(1-q^{-r})(1-q^{-r+1})\ldots\ldots(1-q^{-r+m-1})\right|$$

$$= \left|q^{-mr+\frac{1}{2}m(m-1)}(-1)^m(1-q^r)\ldots\ldots(1-q^{r-m+1})\right|$$

$$< q^{-mr+\frac{1}{2}m(m-1)} < q^{-\frac{1}{2}r(r+1)} \quad ,$$

for all $m \leqslant r$, $0 < q < 1$.

Corollary to Lemma IV.

(i) For large positive values of r and s and $0 < q < 1$

$$\Phi\left[\begin{matrix}c: A, B, -r; A', B', -s; \\ C: a, b; a', b' ;\end{matrix}\quad q,q\right] < K\,q^{-\frac{1}{2}r(r+1)-\frac{1}{2}s(s+1)},$$

where K is a constant independant of r and s.

(ii) Under the same conditions as for (i)

$$\Phi\begin{bmatrix} A, B: C, -r; C', -s ; \\ a , b: c ; c' ; \end{bmatrix} q,q \Big] < K\ q^{-\frac{1}{2}r(r+1)-\frac{1}{2}s(s+1)} .$$

(iii) $${}_4\mathcal{F}_3\begin{bmatrix} a,b,c,-r; \\ d,e,f \end{bmatrix} q \Big] < K\ q^{-\frac{1}{2}r(r+1)}$$, for large r and $0<q<1$.

(iv) $${}_3\mathcal{F}_2\begin{bmatrix} a,b,-r; \\ d,e \end{bmatrix} q \Big] < K\ q^{-\frac{1}{2}r(r+1)}$$, for large r and $0<q<1$.

(v) $${}_4\mathcal{F}_3\begin{bmatrix} a,b,c+r,-r; \\ d,e,f \end{bmatrix} q \Big] < (K/[c;r])\ q^{-\frac{1}{2}r(r+1)},$$ for large r

and $0 < q < 1$.

(vi) $${}_4\mathcal{F}_3\begin{bmatrix} a,b,c,-r; \\ d,e,f-r \end{bmatrix} q \Big] < Kq^{-rf},$$ for large r and $0 < q < 1$.

(vii) $${}_3\mathcal{F}_2\begin{bmatrix} a,b,-r; \\ d,e-r \end{bmatrix} q \Big] < Kq^{-re},$$ for large r and $0 < q < 1$.

<u>Proof of (i)</u>. The series on the left of (i) is less than

$$\left| \sum_{m=0}^{r} \sum_{n=0}^{s} \frac{[c;m+n]\ [A;m]\ [B;m]\ [-r;m]\ [A';n]\ [B';n]\ [-s;n]}{[1;m]\ [1;n]\ [a;m]\ [b;m]\ [a';n]\ [b';n]\ [C;m+n]}\, q^{m+n} \right|$$

$$< \sum_{m=0}^{r} \sum_{n=0}^{s} \frac{[c;m+n]\,[A;m]\,[B;m]\,[A';n]\,[B';n]\;\;|[-r;m]|\,|[-s;n]|}{[1;m]\,[1;n]\,[a;m]\,[b;m]\,[a';n]\,[b';n]\,[C;m+n]}\, q^{m+n}$$

$$< \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[c;m+n]\,[A;m]\,[B;m]\,[A';n]\,[B';n]}{[1;m]\,[1;n]\,[a;m]\,[b;m]\,[a';n]\,[b';n]\,[C;m+n]}\, q^{-\frac{1}{2}r(r+1)-\frac{1}{2}s(s+1)+m+n}$$

(by Lemma IV)

$< \;\; K\, q^{-\frac{1}{2}r(r+1)-\frac{1}{2}s(s+1)}$, since $0<q<1$ (by Lemma III).

The proof of (ii),(iii),(iv) and (v) follows similarly.In proving (v) we have only to note that

$$[c;r]\,[c+r;m] = [c;m+r] < [c;m] .$$

Next,to prove (vi) we note that for $m \leq r$

$$\frac{[-r;m]}{[f-r;m]} = q^{-mf}\frac{(1-q^{r-m+1})\ldots\ldots(1-q^{r})}{(1-q^{r-f-m+1}),,,,,(1-q^{r-f})}$$

$< C q^{-rf}$, for large r and $0<f<1$.

Hence the required result.

The proof of (vii) follows similarly.

Now,using the results of the lemmas and their corollaries we can state the following intervals of convergence for our expansions

$|x|<q$ in 6.3(1) ; $|x|<q$ in 6.3(2) ;

$|x|<q$, $|y|<q$ in 6.5(1,2,3,4) ;

$|x| < \min(1, q^{c-a-b})$, $|y| < \min(1, q^{c-a-b'})$ in 6.6(1), 6.6(3) and 6.6(4) ;

$|x| < \min(1, q^{c-a-b})$, $|y| < 1$ in 6.6(2), 6.6(5) and 6.6(6) ;

$|x| < \min(1, q^{c-a-b})$, $|y| < \min(1, q^{c-a'-b'})$ in 6.6(7).

The conditions of convergence for the expansions 6.5(1)-6.5(4) have been stated in the case K=0. For other values of K , the convergence is even sharper under the same conditions due to the presence of the solitary quadratic factors of the type $q^{Kn(n-1)}$. It may also be remarked that these conditions are quite stringent and probably could be relaxed by proving finer inequalities , but this would require much more detailed and complicated analysis than the possible importance of these expansions seems to justify. For similar reasons the results have not been extended to general values of q , and only the most interesting region $0 < q < 1$ has been considered. The expansions could, however, be shown true for complex q as well.

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