EXTENSION OF VALUATIONS IN SKEW FIELDS

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TO NOUHA AND GASSAN

ABSTRACT

The aim of this thesis is to study the extension of valuations in skew field extensions.

In Chapter I we look at the following problem.

Let K be a field and V a valuation ring of rank 1 in K. Let H be a crossed product division algebra over K. Then we study conditions under which there exists a matrix local ring R in H lying over V and generating H as K-space. We then find that R is a valuation ring in H lying over V iff R is local. Moreover if V is discrete of rank 1, then R is a maximal order in H.

In Chapter II we study directly conditions under which a valuation on the centre of a finite dimensional central division algebra can be extended to the whole algebra. In particular if $H = (E/K; \sigma, a)$ is a cyclic division algebra and v is a discrete rank 1 valuation on K, then the extension of v to H depends on v(a). We then carry on the study of the extension problem for the tensor product of algebras. In particular if $H \cong H_1 \underset{K}{\otimes} \cdots \underset{K}{\otimes} H_r$ and V a rank 1 valuation ring in K and if there exists a valuation ring W in H lying over V with $W \cap H_i = W_i (i = 1, \ldots, r)$, we study conditions under which $W \cong W_1 \underset{V}{\otimes} \cdots \underset{V}{\otimes} W_r$.

In Chapter III we look at infinite skew field extensions. We study valuations in skew function fields. The application will include among others, free algebras, universal associative envelopes of Lie algebras and generic crossed product. However our main concern in this chapter is the following question raised by P.M. Cohn.

Let K_1, K_2 be two skew fields with a common subfield K and let v_1, v_2 be real valued valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$.

Do v_1, v_2 have a common extension to $H = K_1 \underset{K}{\circ} K_2$ (the field coproduct of K_1 and K_2)?

We show that in general the answer is no. Nevertheless we find conditions under which v_1, v_2 have a common extension to H.

TABLES OF CONTENTS

Terminology and notations

Throughout this thesis, all rings occurring are associative, but not necessarily commutative. Every ring has a unit element, denoted by 1, which is preserved by homomorphisms and inherited by subrings. An integral domain R is said to be a right Ore domain if any two non-zero elements of R have a non zero-common right multiple. Left Ore rings are defined similarly. A non-zero ring in which every non-zero element has a two sided inverse will be called a skew field, and a commutative skew field will be called a field.

If $\sigma: A \rightarrow B$ is a map, then the image of an element $a \in A$ is denoted $\sigma(a)$ and sometimes a^{σ}.

Let R be a ring and S a subring of R, then Z(R) denotes the centre of R while $C_R(S)$ denotes the centralizer of S in R.

By J(R) we shall mean the Jacobson radical of R.

CHAPTER O

Preliminaries

In this chapter we collect some facts on rings and give the conventions we will follow throughout the work.

In Section 1 we define valuations on skew fields and we state Cohn-Krasner's theorem plus P.M. Cohn's theorems on finite dimensional central division algebras and total rings.

In Section 2 we define maximal orders and we state the main theorem needed for our work.

In Section 3 we define skew polynomial rings, while in Section 4 we define universal skew fields of fractions.

Section 5 will be devoted to the definitions of firs and the coproduct of fields over a subfield.

§1 Valuations on skew fields

Let K be a skew field and Γ a totally ordered additive group. A function v on K with values in $\Gamma \cup \{\infty\}$ is called a valuation on K if the following conditions are satisfied;

v.1 $v(a) = \infty$ if and only if $a = 0$ for all $a \in K$.

v.2 $v(a-b) \geq \min\{v(a), v(b)\}$ for all $a,b \in K$.

v.3 $v(ab) = v(a) + v(b)$ for all $a,b \in K$.

The image of $K^* = (K\backslash\{0\})$ is called the precise value group of v. If $U = \{x \in K;\ v(x) = 0\}$, then $\operatorname{im} v \cong K^*/U$. This of course follows from the fact that v is a group homomorphism of K^* onto $\operatorname{im} v$. If Γ is abelian then v is said to be abelian.

A subring V of K is said to be total if for every $a \in K^*$, $a \in V$ or $a^{-1} \in V$; it is invariant if $a^{-1}Va = V$ for all $a \in K^*$. By a valuation ring we understand a total invariant subring of K. It is easily verified that for any valuation v, the set

$$V = \{x \in K | v(x) \geq 0\}$$

is a valuation ring in K, and conversely, every valuation ring in K determines a valuation on K which is unique up to an isomorphism of the precise value group. V is said to be associated to v.

Remarks. Let v be a valuation on a skew field K, then

1) $v(a) \neq v(b)$ implies $v(a-b) = \min\{v(a), v(b)\}$

2) the valuation ring V is local with maximal ideal $\mathfrak{J} = \{x \in K | v(x) > 0\}$ $\mathfrak{J}$ is called the radical of v and $\bar{V} = V/\mathfrak{J}$ is called the residue class field of v

3) it is known that every valuation on K defines a topology on K; however K is not necessarily complete for this topology and its completion $\tilde{K}$ is called the completion of K relative to v.

The following theorem by P.M. Cohn generalizes theorem 9 of ([18]).

Theorem 0.1.1. Let D be a finite dimensional division algebra over its centre K and suppose that K has a real valued valuation v. Then the following conditions are equivalent, where $\tilde{K}$ denotes the completion of K relative to v.

(a) D is a topological skew field with a topology inducing the valuation topology on K, and D has a completion $\tilde{D}$ which is a division algebra.

(b) $D \otimes_K \tilde{K}$ is a division algebra.

(c) $F \otimes_K \tilde{K}$ is a field, for any commutative subfield F of D.

(d) v has a unique extension to every commutative subfield of D

(e) v can be extended to a valuation on D.

Proof. Cohn ([5] Theorem 1)

The following theorem is also due to P.M. Cohn.

Theorem 0.1.2. Let D be a finite dimensional central division algebra. Then any total subring of D inducing a real valued valuation on the centre of D is a valuation ring.

Proof. Cohn ([5] Theorem 3)

We now state Krasner-Cohn's theorem.

Theorem 0.1.3. Let $K \subseteq L$ be any skew field extension. Given any abelian valuation v on K with associated valuation ring V and radical $\mathfrak{M}$, there is an abelian extension ω of v to L iff ML^c is a proper ideal of VL^c, where L^c is the commutator group of L.

Proof. cf. ([8] Theorem 2.3)

N.B. This theorem is used indirectly in our work.

§2. Maximal orders

Let R be a noetherian commutative integral domain with a field of fractions K and let A be a central simple K-algebra; an R-module M is called an R-lattice if it is a finitely generated R-torsion free R-module. M is said to be a full R-lattice in A if M generates A as K-space.

An R-order in the K-algebra A is a subring Λ of A which is a full R-lattice in A.

A maximal R-order in A is an R-order which is not properly contained in any other R-order in A.

Throughout our work A will be assumed to be a skew field. A ring S is said to be matrix local if $S/J(S)$ is simple artinian i.e. $S/J(S) \cong M_n(L)$ where L is a skew field. n is called the capacity of S. In what follows R is assumed to be a discrete rank 1 valuation ring in K. Then we have

Theorem 0.2.1. Let Λ be an R-order in the skew field A, then Λ is a maximal order iff Λ is hereditary and matrix local. Moreover if Λ is a maximal order in A then Λ is a valuation ring iff its capacity is 1.

Proof. The first part of the theorem is ([17] Theorem 18.4) where the second part can be deduced from ([17] 18.7 and 18.8) and theorem 0.1.1.

In fact if the capacity of Λ is 1 then Λ is the unique maximal R-order.

§3. Skew polynomial rings

Let R be any ring. By a degree function we understand a function $d: R \to \mathbb{Z} \cup \{-\infty\}$ satisfying the following properties.

D.1. For $a \in R^* = R - \{0\}$, $d(a) \geqslant 0$, while $d(0) = -\infty$

D.2. $d(a-b) \leqslant \max\{d(a), d(b)\}$ for all $a,b \in R$

D.3. $d(ab) = d(a)+d(b)$ for all $a,b \in R$

D.3 implies $d(1) = 0$; and by D.1 and D.3, R^* is closed under multiplication; i.e. every ring with a degree function is necessarily an integral domain.

Given a ring R, let S be a ring containing R as subring, as well as an element x such that every element of the ring A generated by R and x is uniquely expressible in the form

$$f(x) = a_0 + xa_1 + \ldots + x^n a_n, \quad a_i \in R \tag{1}$$

Furthermore, we assume that $d(f) = \max\{i; a_i \neq 0\}$ is a degree function on A. This implies that R is an integral domain and moreover, for any $a \in R$, there exists a^α, a^δ in R such that

$$ax = xa^\alpha + a^\delta. \tag{2}$$

Firstly we note that a^α, a^δ are uniquely determined by a and $a = 0$ if and only if $a^\alpha = 0$. Secondly, by (1), we have $(a+b)x = x(a+b)^\alpha + (a+b)^\delta$, $ax+bx = xa^\alpha + a^\delta + xb^\alpha + b^\delta$. Therefore, $(a+b)^\alpha = a^\alpha + b^\alpha$, $(a+b)^\delta = a^\delta + b^\delta$ so α, δ are additive mappings of R. Similarly, by comparing $a(bx)$ and $(ab)x$ we obtain

$$(ab)^\alpha = a^\alpha b^\alpha, \quad (ab)^\delta = a^\delta b^\alpha + ab^\delta.$$

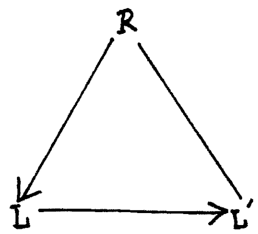
Putting $a = b = 1$, we find $1^\alpha = 1$, $1^\delta = 0$. Hence α is a monomorphism and δ is an α-derivation of R. The relation (2), with the uniqueness of (1), suffices to determine the multiplication in A in terms of R, α and δ. Thus, given R, α and δ, A is completely fixed. We shall write

$A = R[x;\alpha,\delta]$ and call A the skew polynomial ring in x over R determined by α and δ. When $\delta = 0$ we simply write $R[x;\alpha]$ instead of $R[x;\alpha,0]$.

Skew polynomial rings turn out to be useful in chapter III in providing examples and counter examples. We note that when R = K is a skew field, then $K[x;\alpha,\delta]$ is a right ore domain (cf. [3] pp.36) and thus has a field of fractions $K(x;\alpha,\delta)$, say which is called a skew function field.

4. Universal field of fractions

Given a ring R, by an R-ring we understand a ring L with a homomorphism $R \to L$. The R-rings (for fixed ring R) form a category in which the maps are ring homomorphisms $L \to L'$ such that the triangle



commutes.

By an epic R-field we shall mean an R-ring K which is a skew field, and such that K is the least skew field containing the image of R. If, moreover, the canonical mapping $R \to K$ is injective, we call K a field of fractions for R. Of course for some rings R there may be no epic R-fields at all. The only R-ring homomorphism possible between epic R-fields is an isomorphism. For any homomorphism between skew fields is injective, and in this case the image will be a skew field containing the image of R, hence we have a surjection, and therefore an isomorphism. This shows the need to consider more general maps.

Let us define a specialization between epic R-fields K,L as an R-ring homomorphism $f: K_0 \to L$ from an R-subring K_0 of K to L such that any element of K_0 not in the kernel of f has an inverse in K_0. The definition shows that K_0 is a local ring with maximal ideal Ker f, hence $K_0/\mathrm{Ker}\, f$ is a skew field and by the definition of L; $L \cong K_0/\mathrm{Ker}\, f$.

Thus any specialization of epic R-fields is surjective.

Two specializations from K to L are considered equal if they agree on a subring K_0 of K and the common restriction to K_0 is again a specialization. By ([3] pp.253) the epic R-fields and specializations again form a category $\mathcal{F}_R$ say.

An initial object in the category $\mathcal{F}_R$ is called a <u>universal epic R-field</u>. Explicitly a universal epic R-field is an epic R-field U such that for any epic R-field K, there is a unique specialization $U \to K$. Clearly a universal epic R-field, if it exists at all, is unique up to isomorphism. In general a ring R need not have a universal epic R-field (e.g. a commutative ring has a universal epic R-field if its nil radical is prime).

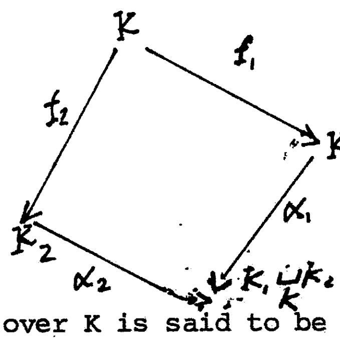
Suppose that R has a universal epic R-field U. Then R has a field of fractions iff $f: R \to U$ is injective. If f is injective then U is called the <u>universal skew field of fractions of R</u>.

§5. <u>Firs and free products</u>

A ring R is said to be a <u>right fir</u> if every right ideal is free of unique rank as right R-module.

<u>Left fir</u> is defined similarly.

A ring R is said to be a <u>fir</u> if it is right and left fir. We now consider a fixed ring K and K-rings K_1, K_2 then the <u>coproduct</u> of K_1, K_2 over K is their pushout



The coproduct of K_1, K_2 over K is said to be <u>faithful</u> if f_1, f_2 are injective. The coproduct is said to be <u>separating</u> if $K_1 \cap K_2 = K$ in $K_1 \underset{K}{\sqcup} K_2$.

The coproduct of K_1 and K_2 over K is called a free product over K if it is both faithful and separating. It is known that if K_1, K_2, K are skew fields then $K_1 \underset{K}{\sqcup} K_2$ is a free product and moreover $K_1 \underset{K}{\sqcup} K_2$ is a fir, hence it has a universal skew field of fractions (for proof of the above see ([4] and [3]. pp 106 and 283).

CHAPTER I

MATRIX LOCAL RINGS IN SKEW FIELDS

The purpose of this chapter is to study generalisations of the following well known result to non-discrete valuation rings of rank 1.

Let H be a finite dimensional central division K-algebra and let V be a discrete rank 1 valuation ring in K; then there exists a maximal V-order R in H. Moreover R is a valuation ring iff its capacity is 1.

In section 1) we define matrix local rings and we study their basic properties.

In section 2) we study the case of crossed product division algebras and we obtain the main theorem of this chapter. NAMELY: Let H = (E/K;f) be a crossed product division algebra and let V be a rank 1 valuation ring in K such that the following conditions are satisfied.

i) there exists a unique valuation ring W in E lying over V

ii) the inertia group of W is {1}

iii) Imf $\subseteq$ U(W) (the group of units of W).

Then there exists a matrix local ring in H generating H as E-space, lying over V and given by

$$R = \sum_{\sigma} Wu_{\sigma} \quad \text{where} \quad \sigma \in \text{Gal}(E/K) \quad \text{and} \quad u_{\sigma}u_{\tau} = f_{\sigma,\tau}u_{\sigma\tau} \text{ for all } \sigma,\tau \in \text{Gal}(E/K).$$

Moreover R is a valuation ring iff its capacity is 1. In section 3) we shall study the case im$f \not\subseteq$ U(W) and deduce that condition iii) of the above theorem cannot be omitted.

§1. Definition and basic properties of matrix local rings

Definition (1.1.1). A ring R is called matrix local ring if R/J(R) is simple artinian; i.e. R/J(R) $\simeq L_n$, where L is a skew field. n is called the capacity of R and will be denoted cap R.

We shall mainly consider matrix local rings which are contained in skew fields. So let H be a skew field and R a matrix local ring in H with cap R = n, let f_{ij} (i,j = 1,...,n) be a set of elements in R such that

(1) $f_{ij}f_{k\ell} \equiv \delta_{jk}f_{i\ell} (\text{mod } J(R))$, (2) $\sum f_{ii} \equiv 1 (\text{mod } J(R))$

We shall study the set

$$S_f = \{x \in R;\ xf_{ij} - f_{ij}\,x \in J(R)\}.$$

But first we have

Lemma (1.1.2). Let B be any ring, X a subset of B and O the centralizer of X(mod J(B)), then O is a subring of B and if a is a unit in B which lies in O then a is a unit in O.

Proof. O is a subring of B

1. O is an additive subgroup of (B,+) because J(B) is
2. O is multiplicatively closed. For

   $\alpha \in O \Rightarrow \alpha x - x\alpha \in J(B)$ for all $x \in X$

   $\beta \in O \Rightarrow \beta x - x\beta \in J(B)$ for all $x \in X$.

   Hence $\alpha\beta x - x\alpha\beta = \alpha(\beta x - x\beta) - (\alpha x - x\alpha)\beta \in J(B)$ for all $x \in X$,

   whence $\alpha\beta \in O$
3. $1 \in O$ because $x - x = 0 \in J(B)$ for all $x \in X$

   thus O is a subring of B.

For the second part we consider a unit a in B which lies in O; then there exists $b \in B$ such that

$$ab = ba = 1$$

Moreover $b(ax-xa)b = baxb - bxab = xb - bx \in J(B)$ for any $x \in X$, hence $b \in O$.

We can now have

Proposition (1.1.3). S_f is a subring of R which is independent of the choice of the f_{ij}'s.

Proof. S_f is a subring of R by Lemma (1.1.2) and S_f is independent of the f_{ij}'s because the $\overline{f_{ij}}$'s (mod J(R)) are matrix units of $\bar{R} = R/J(R)$.

This result allows us to denote S_f by S and we shall do so throughout this section.

Lemma (1.1.4). $J(R) \subseteq J(S)$.

Proof. 1. $J(R) \subseteq S$ by the definition of S

2. $J(R) \subseteq J(S)$ for let $a \in J(R)$, then for any $s \in S, x = 1+as$ is in S and is a unit in R, hence by (1.1.2) x is a unit in S, thus $a \in J(S)$ and $J(R) \subseteq J(S)$.

Proposition (1.1.5). $J(R) = J(S)$ and S is a local subring of R.

Proof. Consider $\theta: R \to R/J(R) \cong L_n$ where L is a skew field. Put $\bar{S} = \theta(S) = S/J(R)$; then $\bar{S}$ centralizes the matrix units in L_n hence $\phi: M_n(L) \to M_n(\bar{S})$ is an isomorphism which induces an isomorphism between L and $\bar{S}$; but this means that $\bar{S}$ is a skew field.

Hence the ideal J(R) in S is maximal (as left, right, two sided) ideal in S, so $J(R) = J(S)$.

J(S) is therefore the only maximal ideal (left, right, two sided) in S and S is a local subring of R.

Lemma (1.1.6). Let R be a ring contained in a skew field D which is generated by R and let O be a subring of R which contains a non-zero right ideal I of R then D is also generated by O.

Proof. Denote by $\langle I \rangle$, $\langle O \rangle$ the subfields of D generated by I, O respectively and let i be a non-zero element in I.

Consider $r \in R$ then $ir = j \in I$ hence $r = i^{-1}j \in \langle I \rangle$ thus $R \subseteq \langle I \rangle$ which implies $D = \langle I \rangle$ so $D \supseteq \langle O \rangle \supseteq \langle I \rangle = D$ and D is generated by O.

We can now describe the matrix local rings.

Proposition (1.1.7). Let R be a matrix local ring in a skew field H, then R contains a local subring S such that $\natural R \natural$ is generated by S and R is an O-algebra over a local subring of the centre of H.

Proof. We consider $\theta: R \to R/J(R) = \bar{R} \cong M_n(L)$ and we let e_{ij} $(i,j = 1,\ldots,n)$ be the set of matrix units of $\bar{R}$. We pick $f_{ij} \in \theta^{-1}(e_{ij})$ and we put

$$S = \{X \in R;\ Xf_{ij} - f_{ij}X \in J(R)\}.$$

Then by applying prop. (1.1.5) S is a local subring of R and applying lemma (1.1.6) yields the first part of the proposition.

For the second part we let K be the centre of H and we put $O = S \cap K$ then O is a local subring of K and R is an O-algebra. In fact $O = R \cap K$ because $R \cap K = S \cap K$.

Before proceeding to our next result in this section, we recall some definitions. By a global field we shall mean either an algebraic number field or else a field of rational functions in one indeterminate over a finite field. We observe that every valuation on a global field is discrete of rank 1.

A matrix local ring R will be called non-trivial if $R \neq O$ and R is not a skew field.

In the rest of this section, all matrix local rings are assumed non-trivial.

We first have

Lemma (1.1.8). Let H be a finite dimensional central division algebra over a global field K and let R be a matrix local ring in H with $O = R \cap K$. Then there exists a non-trivial valuation ring V in K such that $V \supseteq O$.

Proof. If O is not a field, then V is a maximal element for domination among local subrings of K containing O, see e.g. ([10] pp.65).

If K is an algebraic number field then O cannot be a field since otherwise (K:O) is finite. Hence (H:O) is finite, thus (R:O) is finite, whence R is a field being without zero-divisors so we have a contradiction because R is non-trivial.

If $K = F(X)$ where F is finite, then

O is a field $\Rightarrow$ { Either (K:O) is finite hence contradiction or else K is non-algebraic over O and by ([10] pp.63) $\exists$ a valuation ring V in K containing O }

and the lemma is proved.

Proposition (1.1.9). Let H be a finite dimensional central division algebra over a global field K, then any matrix local ring in H is contained as an additive group in a full lattice M over a valuation ring V in K.

Proof. By (1.1.8) $R = \sum_{\alpha} OC_{\alpha}$ where $O = R \cap K$ and $\{C_{\alpha}\}_{\alpha}$ is a generating set of R as O-module. By (1.1.8) $\exists$ in K, a discrete rank 1 valuation ring $V \supseteq O$.
there exists

Let us write

$$H = \sum_{i=1}^{n^2} Ku_i \quad \text{as K-space}$$

then

$$C_{\alpha} = \alpha_1 u_1 + \ldots + \alpha_{n^2} u_{n^2} \quad \text{where the } \alpha_i\text{'s} \in K.$$

If some of the α_i's does not belong to V then by a suitable change of basis we may assume $C_{\alpha} \in \sum V\mu_i$ where $\mu_i = \alpha_j u_i$ (α_j is such that $v(\alpha_j) = \min_{i=1,\ldots,n^2} v(\alpha_i)$ where v corresponds to V). By a successive change of basis we may assume W.L.O.G. that all the $C_{\alpha} \in M = \sum_{i=1}^{n^2} V\mu_i$.
Hence $R \subseteq M$. Now M is clearly a full V-lattice in H.

Example and remarks. Let $H = \left(\frac{-1,-1}{Q}\right)$ be the quaternion algebra and let ν_p ($P \neq 2$) be the p-adic valuation on Q with associated valuation ring Z_p.

Put $R = Z_p + iZ_p + jZ_p + ijZ_p$ where $i^2 = j^2 = -1$.

Let $J = PR$ and consider $\bar{R} = R/J$;

then $\bar{R} \cong F+F\bar{i}+F\bar{j}+F\bar{i}\bar{j} \cong (F(\bar{i})/F;\ \sigma,\ -1)$

where $\sigma: \bar{i} \to -\bar{i}$ and $F \cong Z_p/pZ_p \cong \mathbb{Z}/(p)$.

Since F is finite $\bar{R}$ splits over F, so $\bar{R} \cong M_2(F)$, hence J is maximal (as two-sided ideal), whence $J = J(R)$ because $J(R) \supseteq J$ by ([17] theorem 6.15), thus R is a matrix local ring. In fact 1) this shows that R is a P.I.D. (a principal ideal domain) while S is not since

$$S = Z_p + ipZ_p + pjZ_p + pijZ_p$$

and $$J(S) = J(R) = pZ_p + ip\,Z_p + jp\,Z_p + ijp\,Z_p.$$

2) If R is a matrix local ring with Cap $R \neq 1$, then R is not invariant since otherwise cap $R = 1$ and from the above example we see that S is not necessarily invariant. For assume that $P = 3$ and

let $x = 3 + 3i + 3j \in H$

and $y = 1 + 3i \in S$

then $XYX^{-1} = 1 + i + 2j - 2ij \notin S$.

§2. Matrix local rings in crossed product division algebras

Let H be a crossed product division algebra over the Galois extension E/K so that,

$H = (E/K;\ f)$ where f is a factor set from G to E*,

then $H = \sum_{\sigma \in G} Eu_\sigma$ where $G = \mathrm{Gal}(E/K)$

$$u_\sigma u_\tau = f_{\sigma,\tau} u_{\sigma\tau} \text{ for all } \sigma, \tau \in G$$

$$u_\sigma a = a^\sigma u_\sigma \text{ for all } a \in E \text{ and } \sigma \in G.$$

We note that the centre of H is K and $(H:K) = n^2$ where $n = \mathrm{ord}\ G$. Throughout this section we are given a rank 1 valuation v on K with associated valuation ring V and a residue class field $\bar{V} = V/m$ where m is the unique maximal ideal of V.

Let $\omega_1, \omega_2, \ldots, \omega_r$ $(r \leq n)$ be the distinct valuation on E which extend v, see e.g. ([10] theorem 2.12).

We pick one of them which we call ω with associated valuation ring W, maximal ideal $\mathcal{P}$, residue class field $\bar{W} = W/\mathcal{P}$ and group of units U(W).

We consider the set $D = \{\sigma \in G;\ \sigma W = W\}$ which is the decomposition group of W.

Each $\sigma \in D$ defines by passage to the residue class a $\bar{V}$-automorphism $\bar{\sigma}$ of $\bar{W}$ and we obtain a homomorphism $\varepsilon\colon D \to \mathrm{Aut}(\bar{W}/\bar{V})$ whose kernel is called the inertia group of W and will be denoted by T.

It is well known (see e.g. the above reference) that $\bar{W}/\bar{V}$ is normal and that $D/T \cong \mathrm{Aut}(\bar{W}/\bar{V})$.

Let K_D be the fixed field of D, i.e. the decomposition field

Let K_T be the fixed field of T, i.e. the inertia field of W.

$W_D = W \cap K_D$ with value group Γ_D and residue class field $\bar{W}_D$

$W_T = W \cap K_T$ with value group Γ_T and residue class field $\bar{W}_T$.

Then from the above reference we have

$\Gamma_D = \Gamma_T = \Delta$ where Δ is the value group of **v**.

$\bar{W}_D = \bar{V}$ and $\bar{W}_T$ is the separable closure of $\bar{V}$ in $\bar{W}$.

We now consider

$$A = \sum_{\sigma \in D} E u_\sigma$$

Proposition (1.2.1). A with the multiplication and addition induced from H is a subring of H which is a crossed product division algebra over the Galois extension E/K_D.

Proof. A with the induced multiplication and addition is clearly a subring of H.

Now the restriction of the factor set f to D yields a factor set from D to E*, hence A is crossed product over E/K_D; thus A is central simple as K_D-algebra whence A is a division subring of H because A has no zero-divisors.

Throughout this section we shall write

$$A = \sum_{\sigma \in D} E u_\sigma \cong (E/K_D;\ f_D) \text{ where } f_D = f/D\times D$$

We shall assume that there exists a factor set (from D to E*) equivalent to f_D and whose image is $\subseteq U(W)$. For simplicity we shall assume $\operatorname{im} f_D \subseteq U(W)$.

We consider the left W-module

$$R = \sum_{\sigma \in D} W u_\sigma$$

Then by the above assumption R with induced addition and multiplication forms a ring which generates A as E-space and such that $R \cap K = V$.

Much of the remaining is devoted to the study of this ring.

First we have the following lemma.

Lemma (1.2.2). Let B be any ring containing a local ring O such that B is finitely generated as left (respectively right) O-module then $J(B) \supseteq PB$ (respectively $J(B) \supseteq BP$) where P is the unique maximal ideal of O.

Proof. Direct application of Nakayama lemma.

We now go back to hypothesis and notations preceding Lemma 1.2.2. We put $J = \mathcal{P}R$ and $J' = \sum_{\sigma \in T} R(u_\sigma - 1)R$, then

Lemma (1.2.3). $M = J+J'$ is a two sided ideal of R.

Proof. J' is a two-sided ideal of R.

Now J is a right ideal of R.

Let $x = a_{\sigma_1} u_{\sigma_1} + \ldots + a_{\sigma_s} u_{\sigma_s}$ be a non-zero element in R,

Let a be a non-zero element of $\mathcal{P}$

$xa = a_{\sigma_1} a^{\sigma_1} u_{\sigma_1} + \ldots + a_{\sigma_s} a^{\sigma_s} u_{\sigma_s}$. $a^{\sigma_i} \in \mathcal{P}$ because $\sigma_i \in D$ $(i = 1,\ldots,s)$ and so every term belongs to J, it follows that $xa \in J$, hence J is two sided, whence M is two sided.

Proposition (1.2.4). M is either equal to R or is a maximal two-sided ideal of R.

Proof. If J' = R then M = R otherwise M is proper since J is proper and since if $1 = x+y$ then $y = 1-x$ is a unit because $x \in J(R)$.

We claim that M is maximal as two-sided ideal of R. For we let $\bar{R} = R/M$ and we prove that $\bar{R}$ is a simple ring.

If $\bar{R}$ is not simple then there is a proper two-sided ideal X in $\bar{R}$.

We write $\bar{R} = \sum_{\sigma} \overline{W}\bar{u}_{\sigma}$;

then there is a finite basis for $\bar{R}$ as left $\bar{W}$-space; if $\sigma \in T$ then $\bar{U}_{\sigma} = \bar{1} = \bar{U}_1$ hence the only $\sigma \in T$ which appears as a suffix for a basis is the identity.

If $\sigma \equiv \tau (\text{mod } T)$ then either σ or τ appears as a suffix since otherwise $\bar{u}_{\sigma} - \bar{\alpha}\bar{u}_{\tau} = 0$ where $\bar{\alpha} \in \bar{W}$ and $\bar{\alpha} = \overline{f_{\sigma,\tau^{-1}}}\ \overline{f_{\tau,\tau^{-1}}}^{-1}$ which is a contradiction.

We now consider a non-zero element of X,

$$\bar{x} = \bar{a}_{\sigma_1}\bar{u}_{\sigma_1} + \ldots + \bar{a}_{\sigma_t}\bar{u}_{\sigma_t} \quad \text{with } t \text{ minimal}$$

then $t > 1$ otherwise $\bar{x}$ is a unit in $\bar{R}$, since $\bar{a}_{\sigma_1}$ and $\bar{u}_{\sigma_1}$ are units; we can now choose $\bar{b} \in \bar{W}$ with $\overline{\sigma_1(b)} \neq \overline{\sigma_2(b)}$ because σ_1, σ_2 are not both in T and σ_1, σ_2 are not equivalent (mod T).

Now we put $\bar{y} = \bar{x} - \overline{\sigma_1(b)}^{-1}\bar{x}\bar{b}$ and $\bar{y}$ is clearly in X.

then
$$\begin{aligned}\bar{y} &= \bar{x} - \overline{\sigma_1(b)}^{-1}\bar{a}_{\sigma_1}\bar{u}_{\sigma_1}\bar{b} + \overline{\sigma_1(b)}^{-1}\bar{a}_{\sigma_2}\bar{u}_{\sigma_2}\bar{b} + \ldots + \overline{\sigma_1(b)}^{-1}\bar{a}_{\sigma t}\bar{u}_{\sigma t}\bar{b} \\ &= \bar{x} - \overline{\sigma_1(b)}^{-1}\bar{a}_{\sigma_1}\overline{\sigma_1(b)}\,\bar{u}_{\sigma_1} + \overline{\sigma_1(b)}^{-1}\bar{a}_{\sigma_2}\overline{\sigma_2(b)}\,\bar{u}_{\sigma_2} + \ldots + \overline{\sigma_1(b)}^{-1} \\ &\qquad \bar{u}_{\sigma_t}\overline{\sigma_t(b)}\,\bar{u}_{\sigma_t} \\ &= \bar{x} - \bar{a}_{\sigma_1}\bar{u}_{\sigma_1} + \overline{\sigma_1(b)}^{-1}\overline{\sigma_2(b)}\,\bar{a}_{\sigma_2}\bar{u}_{\sigma_2} + \ldots + \overline{\sigma_1(b)}^{-1}\overline{\sigma_t(b)}\bar{a}_{\sigma_t}\bar{u}_{\sigma_t}\end{aligned}$$

after simplification we see that $\bar{y}$ is a non-zero element which is shorter then t, hence a contradiction; whence $\bar{R}$ is simple and M is a maximal two-sided ideal in R.

We can now describe the ring R after keeping all the hypotheses and notations introduced before.

Corollary (1.2.5). Consider $A = \sum_{\sigma\in D} Eu_\sigma \cong (E/K_D; f_D)$ and assume that

1) $\operatorname{Im} f \subseteq U(W)$ 2) $T = \{1\}$.

Then $R = \sum_{\sigma\in D} Wu_\sigma$ is a matrix local ring generating A as E-space and such that $R \cap K = V$.

Moreover $\bar{R} = R/J(R)$ is a crossed product over $\bar{W}/\bar{V}$ and $\bar{R}$ splits over $\bar{V}$ iff f_D can be chosen so that $(f_{\sigma,\tau}-1) \in \mathcal{Y}$ for all $\sigma,\tau \in D$.

Proof. Since $T = \{1\}$, applying proposition (1.2.4) yields that $M = \mathcal{Y}R$ is maximal as two-sided ideal of R and applying lemma (1.2.2) yields that $J(R) \supseteq M$, hence $J(R) = M$ whence R is a matrix local ring because $\bar{R}$ is simple artinian (note that $\bar{R}$ is artinian because $\bar{R}$ is finite dimensional as $\bar{V}$-algebra). Now R clearly generates A as left E-space since $EW = E$; and $R \cap K = V$ since $R \cap K = W \cap K$.

We claim that $\bar{R}$ is a crossed product.

$\bar{R} = \sum_{\sigma\in D} \bar{W}\bar{u}_\sigma$ where the $\{\bar{u}_\sigma;\ \sigma \in D\}$ is a basis of $\bar{R}$ as left $\bar{W}$-space. Now $T = \{1\}$ yields that $\bar{W}/\bar{V}$ is a Galois extension with Galois group D after identifying σ in D with $\bar{\sigma}$ in $D/\{1\}$. So we can define $\bar{f}_D$: $D\times D \to \bar{W}^*$ by $\bar{f}_D(\sigma,\tau) = \overline{f_{D\sigma,\tau}}$ and $\bar{f}_D$ is easily seen to be a factor set from D to $\bar{W}^*$ hence $\bar{R}$ is a crossed product algebra over $\bar{W}/\bar{V}$. Now the last part is trivial since $\bar{R}$ splits iff $\bar{f}_D$ is trivial (cf.[19]), iff $(f_{D\sigma,\tau}-1) \in \mathcal{Y}$ and the corollary is proved.

Before proceeding to our main result we shall adapt some definitions but first we recall that if E/K is a field extension and V a valuation ring in K, then a valuation ring W in E is said to lie over V if $W \cap K = V$.

Let H be a crossed product division algebra over the Galois extension E/K and let V be a rank 1 valuation ring in K with W a valuation ring in E lying over V with a decomposition group D. Then $A = \sum_{\sigma\in D} Eu_\sigma$ is called the division subring of H associated to W. V is said to be extendable to A if there exists a matrix local ring R lying over V (i.e. $R \cap K = V$) and such that R generates A as left E-space.

We shall now state and prove our main theorem.

Theorem (1.2.6). Let H be a crossed product division algebra over the Galois extension E/K; so that $H = \sum_{\sigma \in G} Eu_\sigma = (E/K;\ f)$ where $G = Gal(E/K)$. Let V be a rank 1 valuation ring in K such that the following conditions are satisfied

i) There exists a unique valuation ring W in E lying over V

ii) The inertia group T of W is {1}

iii) $Im f \subseteq U(W)$ where U(W) is the group of units of W.

Then V is extendable to $R = \sum_{\sigma \in G} Wu_\sigma$ in H.

Moreover R is a valuation ring in H iff the capacity of R is 1.

Proof. Consider $H = \sum_{\sigma\ G} Eu_\sigma$ since W is the only valuation ring in E lying over V, the decomposition group of W is the whole of G. Hence Corollary (1.2.5) yields that $R = \sum_{\sigma \in G} Wu_\sigma$ is a matrix local ring which extends V to H and part 1 of the theorem is proved.

If R is a valuation ring then R is local, hence cap R = 1.

If cap R = 1, then R is a local ring.

We claim that R is a valuation ring.

We shall prove first that R is a total ring in H i.e. for every $x \in H$; either $x \in R$ or $x^{-1} \in R$. Let $h \in H\backslash R$ be a non-zero element, then

$$x = a_{\sigma_1} u_{\sigma_1} + \ldots + a_{\sigma_m} u_{\sigma_m} \text{ where some of the } a_{\sigma_i} \notin W (\sigma_i \in G).$$

If m = 1 then $x = a_{\sigma_1} u_{\sigma_1}$ where $a_{\sigma_1} \notin W$. Hence $x^{-1} = u_{\sigma_1}^{-1} a_{\sigma_1}^{-1} \in R$ because $a_{\sigma_1}^{-1} \in W$ and u_{σ_1} is a unit in R, so we assume W.L.O.G. that m > 1 and we let ω be the valuation on E which corresponds to W with ideal $\mathcal{P}$.

Let a_{σ_j} be such that $\omega(a_{\sigma_j}) = \min_{i=o,\ldots,m} \omega(a_{\sigma_i})$

Then $x = a_{\sigma_j}(a_{\sigma_1}/a_{\sigma_j} u_{\sigma_1} + \ldots + u_{\sigma_j} + \ldots + a_{\sigma_m}/a_{\sigma_j} u_{\sigma_m})$

where $\omega(a_{\sigma_i}/a_{\sigma_j}) \geq 0 \quad i = 1,\ldots,m.$

Now by Corollary (1.2.5) $J(R) = \mathcal{P}R$, hence the element

$y = a_{\sigma_1}/a_{\sigma_j} u_{\sigma_1} + \ldots + u_{\sigma_j} + \ldots + a_{\sigma_m}/a_{\sigma_j} u_{\sigma_m} \in J(R)$, this implies that y is a unit in R because R is local. Thus $x^{-1} = y^{-1}a_{\sigma_j}^{-1} \in R$ since $y^{-1} \in R$ and $a_{\sigma_j}^{-1} \in W \subset R$ so R is a total subring of H.

By theorem (0.1.2) every total subring of a finite dimensional central division algebra, inducing a rank 1 valuation ring in the centre is invariant. Hence R is a valuation ring and the theorem is proved.

As a corollary we have

Corollary (1.2.7). Let $H = \sum_{\sigma \in G} Eu_\sigma = (E/K; f)$ be a crossed product division algebra over the Galois extension E/K with G = Gal(E/K). Let V be a rank 1 valuation ring in K.

$W_1,\ldots,W_r$ the valuation rings in E lying over V.

$D_1,\ldots,D_r$ the decomposition groups of $W_1,\ldots,W_r$.

$T_1,\ldots,T_r$ the inertia groups of $W_1,\ldots,W_r$.

$A_1,\ldots,A_r$ the associated division subrings of H with

$f_1,\ldots,f_r$ their corresponding factor sets

and suppose that $\operatorname{im} f_i \subseteq U(W_i)$ $(i = 1,\ldots,r)$ where $U(W_i)$ is the group of units of W_i.

If $T_1 = \ldots = T_r = 1$, then

$$R_i = \sum_{\sigma^{(i)} \in D_i} W_i u_\sigma(i) \text{ is a matrix local ring extending V to A}$$

$(i=1,\ldots,r)$

Moreover R_i is a valuation ring in A iff Cap $R_i = 1$ $(i = 1,\ldots,r)$.

Proof. Direct application of corollary (1.2.5) and theorem (1.2.6).

Remarks and example: (i) If in theorem (1.2.6) V was discrete of rank 1, then conditions i), ii), iii) are redundant and H can be taken to be any finite dimensional central division algebra, since there is a maximal order R over V and R is a valuation ring iff Cap R = 1. The theorem can be considered as a generalization in the case of crossed product (note that R is matrix local).

(2) If v is the valuation which correspond to V then theorem (1.2.6) says that if the conditions i), ii) and iii) are satisfied then

v extends to a valuation on H precisely when cap R = 1.

(3) The condition on T cannot be omitted in general as the following example shows.

Example. Let $H = (\frac{-1,-1}{Q}) \cong (Q(i)/Q;\ \sigma,\ -1)$ be the quaternion algebra over the rationals where $i^2 = -1$ and $\sigma: i \to -i$.

Let ν_2 be the 2-adic valuation on Q with associated valuation ring $\mathbb{Z}_2$; then there is one valuation ring in Q(i) and only one lying over $\mathbb{Z}_2$, namely

$$W = \mathbb{Z}_2 + \mathbb{Z}_2[i] \text{ with } J(W) = 2W + (i-1)W \quad \text{and } \bar{W} \cong \mathbb{Z}/(2).$$

Now $-1 \in U(W)$ (the group of units of W), hence conditions i) and iii) of theorem (1.2.6) are satisfied.

However condition ii) is not satisfied since

$Gal(Q(i)/Q) = D = T = \{1,\sigma\}$ where D is the decomposition

group of W and T is the inertia group.

Now $R = W+Wj$ where $j^2 = -1$ is a matrix local ring in H generating H as Q(i)-space and lying over $\mathbb{Z}_2$, hence $\mathbb{Z}_2$ is extendable to R in H. Moreover cap R = 1 since $J(R) = J(W)R + (j-1)R + (ij-1)R$ and $\bar{R} = R/J(R) \cong \mathbb{Z}_2/2\mathbb{Z}_2 \cong \mathbb{Z}/(2)$.

However R is not a valuation ring in H since if it were then $\frac{1}{2}(1+i+j+ij) \in R$ which is not the case. This proves that condition ii) cannot be omitted. In fact we shall see later that R is contained in a valuation ring in H lying over $\mathbb{Z}_2$, though this valuation ring does not have the normal form exhibited in theorem (1.2.6). [Example(2.1.9).(1.1)]

(4) In section 3 we shall see that condition iii) can not be omitted.

§3. The cyclic case

Let H be a crossed product division algebra over a cyclic Galois extension E/K with cyclic group $G = \{1,\sigma,\ldots,\sigma^{n-1}\}$; so that $H = (E/K;\ \sigma,\ a)$ where $a \in K^*$.

i.e. $H = \sum_{i=0}^{n-1} Eu^j$ where multiplication is defined as follows.

(1) $u^i a = a^{\sigma^i} u^i$ for all $a \in E$ and $i \in \{0,\ldots,n-1\}$

(2) $u^i u^j = \begin{cases} u^{i+j} & \text{if } i+j < n \\ au^{i+j-n} & \text{if } i+j \geq n \end{cases}$

u^0 will be identified with 1 and H is called a cyclic algebra. Now let V be a rank 1 valuation ring with maximal ideal $\mathfrak{m}$ and residue classified $\bar{V} = V/\mathfrak{m}$.

Our aim in this section is to study conditions under which V is extendable to a matrix local ring in H and to prove that condition iii) of theorem (1.2.6) cannot be omitted. Let $W_1, W_2, \ldots, W_r$ $(r \leq n)$ be the distinct valuation rings of E lying over V.

We shall treat the case r = 1 first so assume that there is only one valuation ring W lying over V with maximal ideal $\mathcal{P}$ and residue class field $\bar{W} = W/\mathcal{P}$. H is always assumed non trivial i.e. $H \neq K$.

First we have the following lemma.

<u>Lemma (1.3.1)</u>. Let $H = (E/K;\ \sigma,\ a)$ by a cyclic algebra and let V be a valuation ring in K then $H \cong (E/K;\ \sigma,\ b)$ where $b \in V$.

<u>Proof</u>. Let v be the valuation on K which corresponds to V.

Since $a \in K^*$ we look at v(a).

If $v(a) \geq 0$ then we can take b = a,

if $v(a) < 0$ then $v(a^{-n+1}) = (-n+1)v(a) > 0$

because $n = [E:K]$ is > 1.

Now $a/a^{-n+1} = a^n \in N_{E/K}(E^*)$, hence if we put $b = a^{-n+1}$ then $(E/K;\ \sigma,\ a) \cong (E/K;\ \sigma,\ b)$ see for e.g. ([17] theorem 30.4).

Remark. b can always be chosen so that b is a non unit in V, if a is a unit it suffices to put $b = c^n a$ where $v(c) > 0$ and $c \neq 0$. Since the case of the imf $\subseteq U(W)$ (f the factor set and U(W) group of units) has been discussed in §2. We shall assume throughout this section that we are given

$$H = (E/K;\ \sigma,\ a) \text{ where } a \in m.$$

We consider $R = \sum_{i=0}^{n-1} Wu^i$. R with multiplication and addition induced from H is a subring of H generating H as left E-space and such that $R \cap K = V$. We shall study this ring.

Lemma (1.3.2). $I = \mathcal{P}R + Ru$ is a proper maximal two-sided ideal of R.

Proof. 1. $\mathcal{P}R$ is two-sided. (see the proof of Lemma (1.2.3)).

2. Ru is a proper right ideal since u is not a unit in R.

Now Ru is two-sided because if $x \neq 0$ element in R,

then $x = \alpha_0 + \alpha_1 u + \dots \alpha_s u^s$ where $0 \leq s \leq n-1$

and $ux = (\alpha_0^\sigma + \alpha_1^\sigma u + \dots + \alpha_s^\sigma u^s)u \in Ru$

because $\alpha_0^\sigma, \dots, \alpha_s^\sigma \in W$.

Now I is proper since if not then there exist $x \in \mathcal{P}R$ and $y \in Ru$ such that $1 = x+y$. But this implies that $y = 1-x$ is a unit in R because $x \in \mathcal{P}R \subseteq J(R)$. We now observe that the map $W \to R/I$ is surjective, hence it induces an isomorphism between $\bar{W}$ and $\bar{R} = R/I$, thus I is a maximal two-sided ideal in R.

Before we show that I is the Jacobson radical of R we shall need a criterion for an element in R to lie in J(R). But first we recall the following proposition.

Proposition (1.3.3). Let R be any ring and J(R) its Jacobson radical then J(R) contains every left (right) nilpotent ideal.

Proof. (cf. [17] proof of theorem 6.9).

We now state and prove the criterion.

Lemma (1.3.4). Let B be any ring and x an invariant element of B. Then

$$x^n \in J(B) \Rightarrow x \in J(B).$$

Proof. Consider $x \in B$ such that $xB = Bx$ and $x^n \in J(B)$. If $x \notin J(B)$ then $\bar{x}$ is a non-zero element in $\bar{B} \simeq B/J(B)$ and $\bar{x}$ generates a nilpotent ideal since

$$(\overline{Bx})^n = \overline{BxBx} \ldots \overline{Bx} = \overline{Bx^n} = \overline{Bx^n} = \bar{0}$$

By proposition (1.3.3) $\overline{Bx} \subset J(B/J(B)) = 0$, hence $\bar{x} = \bar{0}$ whence $x \in J(B)$ and the lemma is proved.

We are now ready to prove the main result of this section. We shall keep all the definitions and notations introduced in sections 1 and 2.

Proposition (1.3.5). Let H be a cyclic division algebra over the Galois extension E/K and let V be a rank 1 valuation ring in K. Assume that there exists a unique valuation ring W in E lying over V with $\mathcal{Y}$ as maximal ideal.

Write $H = \sum_{i=0}^{n-1} Eu^i$ where $u^n \in K^*$ and u can be chosen such that $u^n \in \mathcal{M}$ ($\mathcal{M}$ is the maximal ideal of V).

Then there exist infinitely many rings extending V to H and given by $R_c = \sum_{i=0}^{n-1} W(cu)^i$ where $c \in V$. However if V is non-discrete, none of the R_c's is a valuation ring of H although cap $R_c = 1$ for all $c \in V$.

Proof. Consider $R = \sum_{i=0}^{n-1} Wu^i$, then applying lemma (1.2.2) yields $J(R_1) \supseteq \mathcal{Y} R_1$ and applying lemma (1.3.4) yields $u \in J(R)$ since $u^n \in J(R_1)$. Now applying lemma (1.3.2) yields that $J(R_1) = \mathcal{Y} R_1 + uR_1$ since $\mathcal{Y} R_1 + uR_1$ is maximal two sided ideal which is contained in $J(R_1)$.

Now by (1.3.2) $\bar{R}_1 \simeq \bar{W} = W/\mathcal{Y}$ (i.e. R_1 is local) and R_1 extends V to H because R_1 generates H as left E-space and $R_1 \cap K = V$. If c is a unit in V, then $R_1 \simeq R_c$.

For c a non unit in V the proof is the same and the first part of the proposition is proved.

For the second part we notice first that cap $R_c = 1$ for all c. So we assume that V is non-discrete of rank 1 and we shall prove that R_1 is not a valuation ring. Let v the valuation on K which corresponds to V; then since v is non-discrete $\exists\, b \in \mathfrak{m}$ such that $v(b) < v(u^n)$. Now we consider $x = ub^{-1}$ then $x \notin R_1$ because $v(b^{-1}) < 0$. Now $x^{-1} = bu^{-1} = bu^{n-1}u^{-n} = u^{n-1}bu^{-n} = u^{n-1}d$ where $d = bu^{-n}$.

Now $v(d) = v(b)-v(u^n) < 0$ since $v(u^n) > v(b)$.

Hence $d \notin V$, whence x, x^{-1} do not belong to R_1; thus R_1 is not total and a fortiori R_1 is not a valuation ring.

For $c \neq 1$ we follow the same proof and the proposition is proved.

N.B.: if $v(c_1) \geq v(c_2)$ then $R_{c_1} \subseteq R_{c_2}$.

Before stating a corollary let us notice that if V is a valuation ring in K and $W_1,\ldots,W_r$ $(r \leq [E:K])$ are the distinct valuation rings lying over V. Then they have a common decomposition group, hence a common inertia group because E/K is cyclic. This implies that there is one and only one associated division subring A (as defined in section 1) and A has dimension $s = n/r$ as E-space where $n = [E:K]$. Then we have the corollary.

<u>Corollary (1.3.6)</u>. Let $H = \sum_{i=0}^{n-1} Eu^i$ be a cyclic division algebra over E/K and let V be a non-discrete rank 1 valuation ring V in K with ideal $\mathfrak{m}$ such that $u^n \in \mathfrak{m}$. Let $W_1,\ldots,W_r$ $(r \leq n)$ be the distinct valuation rings in E lying over V.

Consider the associated ring $A = \sum_{i=1}^{s-1} Eu^{ri}$, then

$$R^{(j)}_{(c)} = \sum_{i=1}^{s-1} W_j (cu)^{ri} \text{ for all } c \in V \text{ and } j = 1,\ldots,r$$

are local rings which extend V to A.

However none of the $R^{(j)}_{(c)}$'s is a valuation ring.

Proof. Consider $H = \sum_{i=0}^{n-1} Eu^i$ and let D be the decomposition group of any valuation ring in E lying over V, then order of D = s = n/r and $A = \sum_{i=0}^{s-1} Eu^{ri}$ is the common associated division ring, hence by applying proposition (1.3.5) we achieve the proof of the corollary.

N.B.: if $v(c_1) \geq v(c_2)$, then $R_{c_1}^{(j)} \subseteq R_{c_2}^{(j)}$ but $R_{c_1}^{(j)} \nsubseteq R_{c_2}^{(k)}$ for $j \neq k$.

Remarks and Example. 1) The second part of proposition (1.3.5) tells us that condition iii) of theorem (1.2.6) cannot be omitted for (the non-discrete case) since R_c is a local ring in H extending V and R_c is not a valuation ring.

2) The condition that V is non-discrete for the second part of (1.3.5) cannot be omitted as the following example shows.

Let $H = (\frac{-1,-3}{Q})$ be the quaternion algebra over the rationals then $H \cong (Q(i)/Q;\ \sigma,\ -3)$ where $\sigma: i \to -i$ and H is a division algebra since $-3 \notin N_{Q(i)/Q}(Q(i)^*)$. We consider the 3-adic valuation on Q with associated valuation ring $\mathbb{Z}_3$.

$R = \mathbb{Z}_3 + \mathbb{Z}_3[i] + \mathbb{Z}_3[u] + \mathbb{Z}_3[iu]$ is a valuation ring in H extending $\mathbb{Z}_3$ where $u^2 = -3$.

(For the proof see Chapter II §1, corollary (2.1.3)

3) Assume that in the hypothesis of (1.3.5) V is discrete, then from the N.B. which followed (1.3.5) we see that the order on the value group of V induces an order on R_c. Each R_c is clearly a V-order in H; we consider the maximal element for this order which exists because V is discrete, if this element is maximal among all V-orders in H then it is a valuation ring in H extending V since it has a capacity 1.

CHAPTER II

VALUATIONS IN FINITE DIMENSIONAL CENTRAL DIVISION ALGEBRAS

Our object is to consider central division algebras over a valuated field K and to investigate conditions under which the valuation v on K can be extended to the algebra.

In section 1) we consider cyclic division algebras and we assume v discrete rank 1. The main theorem will be the following.

Let $H = (E/K;\ \sigma,\ a)$ be a cyclic division algebra and v a normalized valuation on K with ramification index $e = 1$ in E.

Assume that v(a) is prime to deg H.

Then v extends to a valuation ϕ on H iff v is indecomposed in E. Moreover ϕ is unique.

As examples show the condition that v(a) is prime to deg H is not necessary; however we shall show that it is so when K is a global field.

In section 2) we shall introduce the notion of Azumaya valuation over V and carry on the study of the extension problem for the tensor product of algebras. In particular if V is henselian so that W exists and $H \cong H_1 \otimes H_2 \otimes \ldots \otimes H_r$ with $W_1, W_2, \ldots, W_r$ the valuation rings in H_i lying over V we study conditions under which $W \cong W_1 \otimes W_2 \otimes \ldots \otimes W_r$.

The application will be mainly to symmetric algebras and crossed product algebras with nilpotent Galois group.

In section 3) we study primary algebras while in section 4) we look at central extensions. In particular we shall give a counter example showing that v does not extend to central extensions in general.

§1. Extension of valuations in cyclic algebras.

Let $H = (E/K;\ \sigma,\ a)$ be a cyclic division algebra where σ is a generator of Gal(E/K); $\sigma^n = 1$ where $n = [E:K]$. Given a discrete rank 1 valuation v on K, we aim to study the extension of v to the whole of H.

We recall that every discrete rank 1 valautíon can be normalized i.e. its value group can be reduced to $\mathbb{Z}$ (the ring of integers). Recall also that $H = \sum_{i=0}^{n-1} Eu^i$, where u is such that $u^n \in K^*$ v is said to be <u>indecomposed</u> in E if there is only one valuation on E extending v. Our first result gives conditions for a valuation on E to extend to H.

<u>Theorem (2.1.1)</u>. Let $H = (E/K;\ \sigma,\ a) \simeq \sum_{i=0}^{n-1} Eu^i$ be a cyclic division algebra and let ω be a normalized discrete rank 1 valuation on E such that $\omega(a) = 1$, then ω extends to a valuation ϕ on H iff σ preserves ω.

Moreover ϕ is the unique valuation on H extending ω and is given by

$$(1) \qquad \phi\left(\sum_{i=0}^{n-1} a_i u^i\right) = \min_{i=0,\dots,n-1} \left\{\omega(a_i) + \frac{i}{n}\right\}$$

<u>Proof</u>: 1) The condition is necessary; if ϕ exists then ϕ satisfies (1) since the $\omega(a_i)$'s are integers and $\frac{i}{n} < 1$ $(i = 0,\dots,n-1)$, $\phi(u^n) = \phi(a) = 1$.

Now $ub = b^\sigma u$ for all $b \in E$, hence

$$\left.\begin{aligned} \phi(ub) &= \phi(u) + \phi(b) = \frac{1}{n} + \omega(b) \\ \phi(ub) &= \phi(b^\sigma) + \phi(u) = \omega(b^\sigma) + \frac{1}{n} \end{aligned}\right\} \Rightarrow \omega(b) = \omega(b^\sigma)$$

hence σ preserves the valuation.

2) The condition is sufficient.

Assume that σ preserves ω and consider

$$\phi : H \to \frac{1}{n}\mathbb{Z} \cup \{\infty\}$$

defined by (1), then ϕ is a well defined map because $1, u, \dots, u^{n-1}$ are linearly independent over E.

We claim that ϕ satisfies the axioms of a valuation on H. For,

v.1) $\phi(x) = \infty \Leftrightarrow x = 0$ for every $x \in H$ by definition of ϕ

v.2) $\phi(x-y) \geq \min(\phi(x), \phi(y))$ for all $x, y \in H\backslash\{0\}$.

Let $x = a_0 + a_1u + \dots + a_{n-1}u^{n-1}$

$y = b_0 + b_1u + \dots + b_{n-1}u^{n-1}$

and assume first that a_i, b_j are different from zero $(i,j = 0,\dots,n-1)$.

Then $x-y = (a_0-b_0) + (a_1-b_1)u + \dots + (a_{n-1}-b_{n-1})u^{n-1}$

$$\phi(x-y) = \min\{\omega(a_0-b_0), \omega(a_1-b_1) + \tfrac{1}{n}, \dots, \omega(a_{n-1}-b_{n-1}) + \tfrac{n-1}{n}\}$$

$$\geq \min\{\min(\omega(a_0),\omega(b_0)), \min(\omega(a_1),\omega(b_1)) + \tfrac{1}{n}, \dots, \min(\omega(a_{n-1}),\omega(b_{n-1})) + \tfrac{n-1}{n}\}$$

$$= \min\{\min(\omega(a_0),\omega(b_0)), \min(\omega(a_1)+\tfrac{1}{n}, \omega(b_1)+\tfrac{1}{n}), \dots, \min(\omega(a_{n-1}) + \tfrac{n-1}{n}, \omega(b_{n-1}) + \tfrac{n-1}{n})\}$$

$$= \min\{\omega(a_0), \omega(b_0), \omega(a_1) + \tfrac{1}{n}, \dots, \omega(b_{n-1}) + \tfrac{n-1}{n}\}$$

$$= \min\{\min(\omega(a_0),\dots,\omega(a_{n-1}) + \tfrac{n-1}{n}), \min(\omega(b_0),\dots,\omega(b_{n-1}) + \tfrac{n-1}{n})\}$$

$$= \min(\phi(x),\phi(y)).$$

If in the expressions x or y some of the coefficients are 0 the calculation is not affected and v.2) holds.

v.3) $\phi(xy) = \phi(x) + \phi(y)$ for all $x,y \in H$.

Let us first prove two remarks.

i) Let $x = a_0+a_1u + \dots + a_{n-1}u^{n-1}$ and assume that $a_i \neq 0$ $(i = 0,\dots,n-1)$ and that $\phi(x) = \omega(a_i) + \frac{i}{n}$

$$x = (a_0a^{-1}u^{n-i}+\dots+a_{i-1}a^{-1}u^{n-1}+a_i+a_{i+1}u+\dots+a_{n-1}u^{n-1-i})u^i$$

(because $u^{-i} = a^{-1}u^{n-i}$)

hence $x = (a_i+a_{i+1}u+\dots+a_0a^{-1}u^{n-i}+\dots+a_{i-1}a^{-1}u^{n-1})u^i = x'u^i$

where $\phi(x') = \omega(a_i)$ because $\omega(a_ja^{-1}) = \omega(a_j)-1 \geq \omega(a_i)$ for $j = 0,\dots,i-1$ and $\omega(a_k) \geq \omega(a_i)$ for $k = i,\dots,n-1$.

b) If $x = a_0 + a_1u + \dots + a_{n-1}u^{n-1}$ where $a_i \neq 0$ $(i = 0,\dots,n-1)$; then

$\phi(xu^j) = \phi(x) + j/n$ for any $j = 0,\dots,n-1$.

Note that these two remarks are not affected if some of the coefficients are 0.

After these two remarks we shall prove the following special case. We let $x = a_0+a_1u+\dots+a_{n-1}u^{n-1}$ where $a_0 \neq 0$ and $\phi(x) = \omega(a_0)$. And $y = b_0+b_1u+\dots+b_{n-1}u^{n-1}$ where $b_0 \neq 0$ and $\phi(y) = \omega(b_0)$.

We write $x = a_0 + \sum_{h\neq 0} a_h u^h$ and $y = b_0 + \sum_{k\neq 0} b_k u^k$,

then $xy = a_0b_0 + \sum_{r=0}^{n-1} c_r u^r = \sum_{r=0}^{n-1} \gamma_r u^r$.

If $r = 0$ then $\gamma_0 = a_0b_0+c_0$.

Now $c_r = \sum_{h=0}^{r} a_h b_{r-h}^{\sigma^h} + \sum_{h=r+1}^{n-1} a_h b_{r+n-h} a$,

hence $c_0 = \sum_{h=1}^{n-1} a_h b_{n-h}^{\sigma^h} a$,

whence $\gamma_0 = a_0b_0 + \sum_{h=1}^{n-1} a_h b_{n-h}^{\sigma^h} a$.

Now $\omega(\gamma_0) \geqslant \min(\omega(a_0b_0),\omega(c_0))$ and since $\omega(a_0b_0)$ is strictly less than the value of each term of the expression c_0, we have

$$\omega(\gamma_0) = \omega(a_0b_0) = \omega(a_0) + \omega(b_0).$$

We now observe that $\omega(\gamma_0) \leqslant \omega(\gamma_k)$ for $k = 1,\dots,n-1$.

Thus $\phi(xy) = \omega(\gamma_0) = \omega(a_0) + \omega(b_0) = \phi(x) + \phi(y)$.

Let us put $x = \sum_{n=0}^{n-1} a_h u^h$ and $y = \sum_{k=0}^{n-1} b_k u^k$ and assume that

$$\phi(x) = \omega(a_i) + i/n \quad \text{and} \quad \phi(y) = \omega(a_j) + j/n$$

By the remark a) $x = x'u^i$ where $x' = a_i + \dots$ and $\phi(x') = \omega(a_i)$.

$y = y'u^j$ where $y' = b_j + \dots$ and $\phi(y') = \omega(b_j)$.

Hence $xy = x'u^iy'u^j = x'zu^iu^j$ where $\phi(z) = \phi(y') = \omega(b_j)$.

We now look at the following two cases.

α) $i + j < n$.

Then $\phi(xy) = \phi(x'z) + \frac{i+j}{n} = \omega(a_i)+\omega(b_j) + \frac{i}{n}+\frac{j}{n} = \omega(a_i) + \frac{i}{n}+ \omega(b_j) + \frac{j}{n}$

$= \phi(x) + \phi(y)$.

β) $i + j \geq n$.

Then $xy = xz'au^p$ with $p = 0$ if $i+j = n$, otherwise $1 \leq p \leq n-1$.

If we multiply the coefficients of xz' by a and apply b) we see that

$$\phi(xy) = \omega(xz') + 1 + \frac{p}{n} = \omega(a_i)+ (b_j) + \frac{i+j}{n} = \omega(a_i) + \frac{i}{n}+ \omega(b_j) + \frac{j}{n}$$

$$= \phi(x) + \phi(y).$$

Hence v.3) is proved and ϕ is a valuation on H which clearly extends ω.

3) ϕ is unique.

Assume that there is another valuation ϕ' on H extending ω and let $x = a_k u^k + \ldots + a_s u^s$ an element of H, then

$$\phi'(u) = \frac{1}{n}, \text{ hence } \phi'(a_i u^i) \neq \phi'(a_j u^j) \text{ where } i,j = k,\ldots,s, \text{ whence}$$

$$\phi'(x) = \min\{\omega(a_k) + k/n_j,\ldots,\omega(a_s) + s/n\}, \text{ thus } \phi' = \phi$$

and the theorem is proved.

Before applying this theorem to the extension problem indicated in the introduction we shall need the following lemma.

<u>Lemma (2.1.2)</u>. Let $H = (E/K; \sigma, a)$ be a cyclic division algebra, v a normalized discrete rank 1 valuation on K such that $(v(a),n) = 1$ where $n = [E:K]$, then $\exists\, b \in K^*$ such that $H \cong (E/K; \sigma^r, b)$ where $v(b) = 1$ and $(r,n) = 1$.

<u>Proof</u>. We put $v(a) \equiv d (\text{mod } n)$ where $(d,n) = 1$.

If $d = 1$ then $v(a) = 1+mn$ for some $m \in \mathbb{Z}$. Let $c \in K$ such that $v(c) = -m$, put $b = ac^n$ then $v(b) = 1+mn-mn = 1$ and $b/a \in N_{E/K}(E^*)$ hence $H \cong (E/K; \sigma, b)$ where $v(b) = 1$; here $r = 1$ if $d \neq 1$ then $\exists\, n',d' \in \mathbb{Z}$ such that $n'n+d'd = 1$, hence $d'd = 1-n'n$ and $(n,d') = 1$, thus by ([17] pp.260) $H \cong (E/K; \sigma^{d'}, a^{d'})$ where $v(a^{d'}) = d'v(a) = d'd = 1-nn'$, hence by the first part of the proof there exists $b \in K^*$ such that

$H \cong (E/K;\ \sigma^{d'},\ b)$ where $v(b) = 1$. Here $r = d'$ and the lemma is proved.

Recall that if E/K is Galois and v is a valuation on K of which $\omega_1, \omega_2, \ldots, \omega_r$ are the distinct valuations on E extending v then they have a common ramification index e called the ramification index of v in E. Now we have the following.

Corollary (2.1.3). Let $H = (E/K;\ \sigma,\ a)$ be a cyclic division algebra, and let v be a normalized valuation on K with ramification index $e = 1$ in E.

Assume that $(v(a),n) = 1$ where $n = [E:K]$.

Then v extends to a valuation ϕ on H iff v is indecomposed in E.

Moreover ϕ is unique.

Proof. By Lemma (2.1.2) $H \cong (E/K;\ \sigma^r,\ b)$ where $v(b) = 1$.

Let ω be a valuation on E which extends v; since $e = 1$ ω is a normalized valuation on E with $\omega(b) = v(b) = 1$ we now observe that the condition that v is indecomposed in E is equivalent to σ^r preserves ω, hence applying theorem (2.1.1) yields the corollary.

Corollary (2.1.4). Let $H = (E/K;\ \sigma,\ a)$ be a cyclic algebra and let v be a normalized valuation on K such that v is indecomposed in E and $(v(a),n) = 1$ (where $n = [E:K]$) with ramification index $e = 1$.

Then H is a division algebra.

Proof. By lemma (2.1.2) $H = \sum_{i=0}^{h-1} Eu^i$ where $u^n \in K^*$ and $v(u^n) = 1$. Let ω be the unique valuation on E extending v, then ω satisfies the condition of theorem (2.1.1), hence the map $\phi\colon H \to \frac{1}{n}\mathbb{Z} \cup \{+\infty\}$ defined in the theorem satisfies $\phi(xy) = \phi(x)+\phi(y)$; whence H is an integral domain, thus H is a division algebra.

The following corollary gives the extension to subrings of H.

Corollary (2.1.5). Let $H = (E/K;\ \sigma,\ a)$ be a cyclic division algebra and let v be a normalized valuation on K with $\omega_1, \ldots, \omega_r$ the distinct valuations on E extending v with a common ramification index $e = 1$ and let $n = [E:K]$ and $s = n/r$. If $(v(a),n) = 1$, then v can be extended to r distinct valuations on the associated division subring of H.

Proof. If $(v(a),n) = 1$ then by the lemma $H \cong (E/K;\ \sigma^t,\ b)$ where $(t,n) = 1$ and $v(b) = 1$. Put $\sigma^t = \tau$ we write $H = \sum_{i=0}^{n-1} Eu^i$ where $u^n \in K^*$ and $v(u^n) = 1$. We consider the associated division subring $A = \sum_{i=0}^{s-1} Eu^{ir} \cong (E/K_D;\ \tau^r,\ b)$ where K_D is the decomposition field. Then $\omega_1, \omega_2, \ldots, \omega_r$ satisfy the conditions of theorem (2.1.1) on E, hence $\omega_1, \omega_2, \ldots, \omega_r$ extend to r-distinct valuations on A.

Remark (2.1.6). There is an alternative approach to corollary (2.1.3) based on the results of Chapter I, it is much longer than the above approach. However it has the advantage that it gives a precise description of the valuation ring in H lying over the valuation ring V associated to v. It consists in writing $H = \sum_{i=0}^{n-1} Eu^i$ with $v(u^n) = 1$ where v is the normalized valuation on K, then we consider $R = \sum_{i=0}^{n-1} Wu^i$ where W is the only valuation ring in E lying over V. By proposition (1.3.5) R is a local ring generating H as left E-space and such that $R \cap K = V$. Moreover since v is discrete of rank 1 W is finitely generated as V-module, hence R is finitely generated as V-module, whence R is a V-order in H. After a rather lengthy proof we show that R is a maximal V-order and since cap R = 1 R becomes a valuation ring in H lying over V.

If K is a global field (see definition in Chapter 1) Corollary (2.1.3) can be strengthened. Before we proceed to our next results we need to recall some remarks and definitions. So let E/K be a finite cyclic extension with Galois group $G = \{1, \sigma, \ldots, \sigma^{n-1}\}$ where $n = [E:K]$. Let v be a valuation on K and let ω be a valuation on L which extends v. By a remark in Chapter I v is discrete of rank 1 and so is ω. Let $\tilde{K}$ (respectively $\tilde{E}$) be the completion of K (resp.E) according to v (resp. ω). Then ([11], Theorem 2.2) yields that $\tilde{E}/\tilde{K}$ is a Galois extension with Galois group isomorphic to the decomposition group of ω.

Recall also that every division algebra H, finite dimensional over

([17] pp 280)

its centre K where K is a global field is cyclic i.e. H is represented by $(E/K;\ \sigma,\ a)$ where E/K is a cyclic extension and $a \in K^*$. Hence the study of the extension problem in H is made much simpler by this representation.

We first have the following lemma which is valid for any K.

Lemma (2.1.6). Let $H = (E/K;\ \sigma,\ a)$ be a cyclic division algebra over the global field K, then $\tilde{H} \cong H \otimes_K \tilde{K} \cong (E\tilde{K}/\tilde{K};\ \sigma,\ a)$ where $\tilde{K}$ is the completion of K according to an indecomposed real valued valuation on K.

Proof. Since v is indecomposed and E/K is Galois, then $E \otimes_K \tilde{K}$ is a field which is isomorphic to the completion of E according to the unique extension ω of v and $E \otimes_K \tilde{K} \cong E\tilde{K}$, hence by the remark above $\mathrm{Gal}(E/K) \cong \mathrm{Gal}(E\tilde{K}/\tilde{K})$.

Now by ([17] pp.261) $\tilde{H} \sim (E\tilde{K}/\tilde{K};\ \sigma,\ a)$ where $\sim$ means equal in the Brauer group B(K). By computing dimensions (over $\tilde{K}$) we see that $\tilde{H} \cong (E\tilde{K}/\tilde{K};\ \sigma,\ a)$.

We observe that if E/K is finite Galois where K is global and if v is a valuation on K with ramification index e then

$e = 1 \Leftrightarrow$ v is unramified in $E \Leftrightarrow T = \{1\}$ (because the residue class field is finite)

We now show that the condition of corollary 2.1.3 is necessary.

Theorem (2.1.7). Let $H = (E/K;\ \sigma,\ a)$ be a cyclic division algebra over the global field K and let v be a normalized unramified valuation on K. Put $n = [E:K]$.

Then v extends to a unique valuation ϕ on H iff

(1) v is indecomposed in E and (2) $(v(a),\ n) = 1$

Proof. The condition is sufficient by a direct application of corollary (2.1.3).

The condition is necessary.

If ϕ exists then (1) is satisfied by corollary (2.1.3) and it remains to prove (2).

We consider $\tilde{H} = H \otimes_K \tilde{K} \cong (E\tilde{K}/\tilde{K};\ \sigma,\ a)$ by lemma (2.1.6). Now $E\tilde{K}/\tilde{K}$ is an unramified extension over a complete field K with finite residue class field because v is unramified in E with finite residue class field. The local class field theory yields that $E\tilde{K} = \tilde{K}(\varepsilon)$ where ε is a primitive (q^n-1)th root of unity where q is the cardinal of the residue class field.

Let $\tau: \varepsilon \to \varepsilon^q$ be the Frobenius automorphism, hence $\tau \in Gal(E\tilde{K}/\tilde{K}) \cong \langle\sigma\rangle$ and τ generates $Gal(E\tilde{K}/\tilde{K}$, whence $\exists\ r \in \mathbb{Z}^+$ such that $\tau = \sigma^r$ and $(r,n) = 1$.

Now by ([17] pp.260) $\tilde{H} \cong (E\tilde{K}/\tilde{K};\ \sigma^r,\ a^r)$ and by ([17] pp.266) $\tilde{H}$ is a skew field iff $(v(a^r),\ n) = 1$. But H is a skew field because v extends to H, hence $(v(a^r),\ n) = 1$, thus $(v(a),\ n) = 1$

and the theorem is proved.

As a corollary we have

<u>Corollary (2.1.8).</u> Let H be a finite dimensional central division algebra over a global field K, then only finitely many valuations on K (if any) can be extended to the whole of H.

<u>Proof.</u> H can be represented by a cyclic algebra $(E/K;\ \sigma,\ a)$. It is well known that almost all the valuations on K are unramified in E, moreover $v(a) = 0$ almost every where (cf.[11] pp.72). Hence applying the theorem yields the corollary.

<u>Remarks and examples (2.1.9).</u> 1) The condition that v is unramified in theorem (2.1.7) cannot be omitted.

<u>Example 1.1.</u> Let $H = \left(\frac{-1,-1}{Q}\right) \cong \left(\frac{Q(i)}{Q};\ \sigma,\ -1\right)$ be the quaternion algebra over the rationals and let ν_2 be the 2-adic valuation on Q. We shall prove that ν_2 extends to H. We note first that ν_2 is indecomposed in Q(i) because $(i-1)^2 = -2i$ implies that the ramification index e is 2 and f = 1, hence by the well known equality $(\Sigma e_i f_i = 1)$, there is one valuation on Q(i) extending ν_2.

Now applying lemma (2.1.6) yields $\tilde{H} = H \underset{Q}{\otimes} Q_2 \simeq (Q_2(i)/Q_2; \sigma, -1)$ where Q_2 is the completion of Q according to ν_2. Let $\tilde{\nu}_2$ be the extension of ν_2 to Q_2. We claim that $\tilde{H}$ is a skew field. We let $\mathbb{Z}_2$ be the valuation ring of $\tilde{\nu}_2$. If $\tilde{H}$ is not a skew field then $\exists\ \alpha,\beta \in \mathbb{Z}_2$ such that (1) $N(\alpha+\beta i) = -1$, hence

$$\alpha^2+\beta^2 = -1,$$

which is impossible.

Now by theorem (0.1.1) ν_2 extends to a valuation on H.

However $(\nu_2(-1), 2) = (0, 2) = 2$ and the condition (2) in theorem (2.1.7) is not necessary, thus the condition that ν is unramified cannot be omitted.

2) Over non-global fields, theorem (2.1.7) is not valid.

Example 2.1. We shall outline briefly the following example since it is a direct application of Chapter III section 1 (to which we refer for details).

Let $E = Q(i)$ with σ: $i \to -i$ and let $R = Q(i)[x;\sigma]$ be the skew polynomial ring with $H = Q(i)(x;\sigma)$ its skew field of fractions.

Any p-adic valuation ν_p on $Q(p \neq 2)$ is unramified and indecomposed in Q(i). Let ω be its unique extension. Since σ preserves ω we can extend it to a Gaussian extension ϕ on H (see chapter III). Now $H \simeq (E(x^2)/Q(x^2); \sigma, -1)$ and ν_2 has a Gaussian extension to $Q(x^2)$ which we call ν_2 and which is unramified in $E(x^2)$. However $(\nu_2(-1), 2) = (0,2) = 2$ and the theorem is not valid because $Z(H) = K = Q(x^2)$ is not global.

3) Let $H = (E/K; \sigma, a)$ be a cyclic division algebra and v be an unramified normalized valuation on K; so far we were mostly interested in the case $(v(a), n) = 1$ where $n = [E:K]$. However there are other cases. If $v(Q) \equiv 0 \pmod n$ then this can be reduced to the case $v(a) = 0$ and the study of the extension problem is achieved by applying (chapter 1, §2).

For example the valuation ring associated to ϕ in the above example 2.1) is the one described by Theorem (1.2.6), while in the example (1.1) if we replace ν_2 by $\nu_p (p \neq 2)$ then $R = \mathbb{Z}_p + i\mathbb{Z}_p + j\mathbb{Z}_p + ij\mathbb{Z}_p$ is matrix local but not local since otherwise it becomes a valuation ring which is impossible by theorem (2.1.7).

In fact when v is normalized, indecomposed and unramified with $v(a) \equiv 0 (\mathrm{mod}\ n)$ then Theorem (1.2.6) gives us a description of the maximal order since if R is the ring constructed by Theorem (1.2.6) then R is clearly a v-order. Now applying ([17] pp.375) yields that R is hereditary and since it is matrix local it becomes a maximal order.

Other results concerning these cases will be obtained in chapter III §1, e.g. the generic cyclic crossed product.

4) Let $H = (E/K; \sigma, a)$ be a division algebra and let v be an unramified, indecomposed and normalized valuation on K such that $(v(a), n) = d$ where $n = [E:K]$; then as before we can assume W.L.O.G that $v(a) = d$. If G is the subgroup of Gal(E/K) of order d with K_G fixed field, then $H_G = (E/K_G; \sigma^s, a)$ is a division subring of H. Now the study of the extension problem in H_G is reduced to the case 3). $(s = n/d)$

Throughout the rest of this section we are given a cyclic division algebra $H = (E/K; \sigma, a)$ with Galois group $G = \{1, \sigma, \ldots, \sigma^{n-1}\}$ and a normalized totally ramified valuation v on K (in E). Our aim is to study the extension problem.

Recall that v is totally ramified precisely when its ramification index $e = n$ and that in this case v is indecomposed in E. We shall have to distinguish between two cases.

Let p be the characteristic of K then either p divides n or $(p,n) = 1$

i) $(p,n) = 1$.

In fact we shall assume that K contains a primitive n-th root of unity which implies that $(p,n) = 1$. We shall show that under certain conditions this case can be reduced to one of the already discussed cases. We recall that in this case E/K is cyclic iff E/K is radical i.e. $E = K(\alpha)$ such that $\alpha^n \in K$. This means that if we write

$$H = \sum_{i=0}^{n-1} Eu^i \; ; \quad u^n = a$$

Then K(u)/K is cyclic Galois. We know that by the Skolem-Noether theorem every K-automorphism of K(u) is induced by an element of H. The following lemma shows that this element can be chosen to be α.

<u>Lemma (2.1.10)</u>. Let $H = (E/K;\ \sigma,\ a)$ be a cyclic division algebra over a field containing a primitive n-th root of unity (where $n = [E:K]$). Let $u \in H$ such that $u^n = a$, then the K-automorphisms of K(u) are realised by inner automorphisms induced by α where α is such that $E = K(\alpha)$ and $\alpha^n \in K$.

<u>Proof</u>. By the remark above $\exists\, \alpha \in H$ such that $E = K(\alpha)$ and $\alpha^n \in K$ and $L = K(u)/K$ is a cyclic Galois extension of K.

We consider $f_\alpha : H \to H$

$$x \to \alpha x \alpha^{-1}.$$

We claim that the restriction of f_α to L is a K-automorphism of L. In fact it is enough to show that $\alpha u \alpha^{-1} \in L$.

Now $\alpha u \alpha^{-1} = \alpha(\alpha^{-1})^\sigma u = \alpha\alpha^{\sigma^{-1}} u = \dfrac{\alpha}{\alpha^\sigma}\, u.$

But $\left(\dfrac{\alpha}{\alpha^\sigma}\right)^n = \dfrac{\alpha^n}{\sigma(\alpha^n)} = \dfrac{\alpha^n}{\alpha^n} = 1.$

Hence α/α^σ is an n-th root of unity, whence $\alpha/\alpha^\sigma \in K$, thus $\alpha u \alpha^{-1} = \alpha/\alpha^\sigma u \in L$.

Put $\alpha u \alpha^{-1} = u^\tau$, then τ is a K-automorphism of L since K centralizes u and we have $Gal(L/K) = \{1, \tau, \ldots, \tau^{n-1}\}$.

We now have

<u>Proposition (2.1.11)</u>. Let $H = \sum_{i=0}^{n-1} K(\alpha)u^i$; $u^n \in K$ be a cyclic division algebra over a field K containing a primitive n-th root of unity. Then α may be chosen such that $\alpha^n \in K$ and then $H = (K(u)/K;\ \tau;\ \alpha^n)$ where τ is

the inner automorphism induced by α.

Proof. Lemma (2.1.10) shows that τ is defined by $\alpha u \alpha^{-1} = u^{\tau}$, so that $A = (K(u)/K;\ \tau;\ \alpha^n)$ with the multiplication and addition induced from H is a subalgebra of H. By computing its dimension over K we find A = H.

Corollary (2.1.12). Let $H = \sum_{i=0}^{n-1} K(\alpha)u^i$; $u^n \in K$ be a cyclic division algebra over a field K containing a primitive n-th root of unity and $\alpha^n \in K$.

Let v be a normalized valuation on K such that $v(\alpha^n) \equiv 1 (\mathrm{mod}\ n)$. If v is indecomposed in K(u) with ramification index e = 1, then v is extendable to H.

Proof. By proposition (2.1.11) $H \cong (K(u)/K;\ \tau;\ \alpha^n)$, hence applying Corollary (2.1.3) yields that v extends to H.

N.B.: $(\alpha^n) \equiv 1 (\mathrm{mod}\ n)$ implies $\exists\ c \in K$ such that $v((c\alpha)^n) = 1$ and $K(\alpha) = K(c\alpha)$. Hence v is totally ramified in $K(\alpha)$ since $v(c\alpha) = \frac{1}{n}$ and e = n.

§2. Azumaya valuations in tensor product division algebras

Throughout this section H is a finite dimensional central division algebra over a field K and v is a real valued valuation on K with associated valuation ring V, maximal ideal $\mathfrak{m}$ and residue class field $\bar{V} = V/\mathfrak{m}$.

A valuation ring W in H lying over V (if it exists) will be called Azumaya valuation ring if W is central separable as V-algebra. Recall that an R-algebra A is separable iff A is projective as left $A \otimes_R A^0$-module where A^0 is the opposite ring and that if A is finitely generated then this is equivalent to saying that A/PA is separable as R/P-algebra where p ranges over the maximal ideals of R. We note that if A is central separable over R, then A is finitely generated over R. The first lemma shows that a central separable R-algebra A over a local ring is a matrix local ring.

Lemma (2.2.1). Let A be a central separable R-algebra where R is a local ring. Then A is a matrix local ring with J(A) = mA where m is the maximal ideal of R.

Proof. By ([9] Chap.2, Cor.3.7), there is a correspondence between ideals $\mathfrak{a}$ of R and two-sided ideals $\mathfrak{v}$ of A given by

$$\mathfrak{a} \to \mathfrak{a}A \text{ and } \mathfrak{v} \to \mathfrak{v} \cap R.$$

Now by lemma (1.2.2) $J(A) \supseteq \mathfrak{m}A$, hence $J(A) = \mathfrak{m}A$ since the above correspondence yields that mA is maximal (as two-sided) ideal. But this just means that R is matrix local.

We note as a first consequence that if W exists and is Azumaya, then

$J(W) = \mathfrak{m}_v W$ e.g. it can be shown that the valuation ring associated to the extension of ν_2 in (2.1.9) example (2.1) is Azumaya while the one associated to the extension of ν_2 in (2.1.9) Example (1.1) is not Azumaya.

More generally, if K is a global field (or the completion of a global field for a non-archimidean valuation i.e. a local field), then H can be represented by a cyclic algebra and theorem (2.1.7) yields that if W exists, then $J(W) \supset \mathfrak{m}_v W$. In fact it can be easily deduced that in this case e = f = n where n = deg H (the reason is that v is discrete and the residue class field is finite), hence by (2.2.1) W is not Azumaya. This shows that we have to assume that K is not global (neither of course local). However in the course of this section, we shall show that this will present no great loss of generalities since our concern will be the case of H being a tensor product.

Recall that a left Bezout domain is an integral domain in which every finitely generated left ideal is principal. Then we have

Lemma (2.2.2). Let A be a left Bezout domain, then every finitely generated torsion free right A-module M is free.

Proof. In fact this is an exercise in ([3] pp.47). The proof consists in embedding M in a free module in a well known manner and applying ([3] Chap. 1 prop.1.4) yields the result.

The next lemma describes W (when it exists) as V-algebra.

Lemma (2.2.3). Let H be a finite dimensional central division algebra over K and let V be a valuation ring in K. Assume that there is a valuation ring W in H lying over V. Then the centre of W is V. If moreover W is finitely generated as V-module, then W generates H as K-space.

Proof. The first part of the lemma is trivial, it suffices to observe that for any $y \in H$, $\exists$ $C \in K$ such that $cy \in W$, hence

$x \in Z(W) \Rightarrow x \in Z(H) = K \Rightarrow x \in K \cap W = V$, whence $Z(W) \subseteq V$ and $Z(W) = V$

since $V \subseteq Z(W)$.

For the second part, we observe that lemma (2.2.2) yields that W is free as V-module, since W is a Bezout domain. Now W has unique rank because V is commutative (it has IBN) so let $W = \sum_{i=1}^{m} Vu_i$ where $\{u_1=1, u_2, \ldots, u_m\}$ is a basis of W over V. We claim that $m = n = \deg H$. We note first that $u_1, \ldots, u_m$ are linearly independent over K because if $\sum_{i=1}^{m} \alpha_i u_i = 0$ where $\alpha_i \in K$ then $\alpha(\Sigma\alpha_i/\alpha\, u_i) = 0$ where $\alpha_i/\alpha \in V$ and $\alpha = \alpha_{i_0}$ is such that $v(\alpha) = \min_{i=1,\ldots,m} v(\alpha_i)$ (v corresponds to V). But this implies that either $\Sigma\alpha_i/\alpha\ u_i = 0$ or $\alpha = 0$.

Now $\Sigma\alpha_i/\alpha\ u_i \neq 0$ because it has coefficient $\alpha/\alpha = 1$, hence $\alpha = 0$ whence $\alpha_i = 0$ $(i = 1,\ldots,m)$. Hence we have

(1) $m \leqslant n$,

(2) $m \geqslant n$, since otherwise $D = \sum_{i=1}^{m} Ku_i$ becomes a division subring of H (because D is finite dimensional over K and has no zero-divisors), hence D becomes the skew field of fractions of W which contradicts the fact that W generates H as its skew field of fractions.

Now (1) and (2) imply that $m = n$. The rest is clear.

We note that if H_1 is a central division subalgebra of H and $H_2 = C_H(H_1)$ then it is well known that $H \cong H_1 \otimes_K H_2$ (cf. [17] p.96).

The next proposition describes matrix local rings in tensor products.

<u>Proposition (2.2.4)</u>. Let $H = H_1 \otimes_K H_2$ be a central division K-algebra, where H_1, H_2 are central division K-subalgebra of H.

Let V be a rank 1 valuation ring in K.

Assume that there exist valuation rings W_i in H_i $(i = 1,2)$ such that $W_i \cap K = V$ and W_i is separable as V-algebra $(i = 1,2)$. Then $W = W_1 \otimes_V W_2$ is a matrix local ring, lying over V and generating H as K-space.

In particular if V is discrete rank 1 then W is a maximal order.

Proof. We note first that by lemma (2.2.3) W_1, W_2 are Azumaya valuations, hence they are both finitely generated over V and by the same lemma W_i generates H as K-space (i = 1,2). Now by ([9] Chap.2, Prop.3.3) $W = W_1 \otimes_V W_2$ is central separable as V-algebra, hence by lemma (2.2.1) W is a matrix local ring.

We now observe that $W_2 = C_W(W_1)$. For

$$C_W(W_1) \subseteq C_H(W_1) \subseteq C_H(H_1) = H_2$$

Hence $\quad C_W(W_1) \subseteq W \cap H_2 = W_2.$

But $\quad W_2 \subseteq C_W(W_1)$, whence $W_2 = C_W(W_1)$.

So applying the commutator theorem (Theorem 2.2.6) yields that the map $f: W_1 \otimes_V W_2 \to H$ defined by $f(a_i \otimes b_i) = a_i b_i$ is an injective homomorphism hence W is torsion-free finitely generated V-module, hence by lemma (2.2.2) W is free V-module. We claim that W generates H as K-space.

For, we consider $g: (W_1 \otimes_V W_2) \otimes_V K \to (W_1 \otimes W_2)K$ defined by

$$g(\sum_\alpha a_\alpha \otimes b_\alpha) = \sum_\alpha a_\alpha b_\beta \quad \text{where} \quad a_\alpha \in W,\ b_\beta \in K.$$

g is an isomorphism because W is torsion free (cf. [17] pp.32). Hence we can identify $(W_1 \otimes W_2) \otimes_V K$ with $(W_1 \otimes W_2)K$ (as K-space). But

$$(W_1 \otimes_V W_2) \otimes_V K \cong W_1 \otimes_V (K \otimes_K W_2) \otimes_V K \cong (W_1 \otimes_V K) \otimes_K (W_2 \otimes_V K)$$

Now $W_i \otimes K \cong W_i K$ (i = 1,2) because W_1, W_2 are torsion free and $W_i K = H_i$ (i = 1,2), W_i generates H_i as K-space (i = 1,2). Hence $WK = (W_1 \otimes W_2)K \cong W_1 K \otimes_K W_2 K \cong H_1 \otimes_K H_2 \cong H$, whence W generates H as K-space and the first part of the theorem is proved.

For the second part it suffices to show that W is hereditary. Let I be any left ideal of W, then I is free as V-module, hence projective and by the lifting property of central separable algebras I is projective as left W-module, whence W is left hereditary and similarly W is right hereditary, thus W is hereditary.

Now W is a matrix local ring and hereditary, hence applying Theorem (0.2.1) yields that W is a maximal order.

The rest of this section is devoted to the representation of valuation rings in tensor product division algebras

By a central division algebra we shall mean a finite dimensional central division algebra.

Recall that if H is a central division algebra over K with $[H:K] = n^2$, then n is called the degree of H which will be denoted deg H. By exp H we shall mean the order of H as an element in the Brauer group Br(K).

A field K is called stable if every central division K algebra of deg n has exp n e.g. global and local fields.

We shall need the following theorem.

Theorem (2.2.6). Let A be a central separable R-algebra. Suppose B is any separable subalgebra of A containing R. Set $S = C_A(B)$. Then S is a separable subalgebra of A and $C_A(S) = B$. If B is also central, so is S and the R-algebra map $B \otimes C \to A$ given by $b \otimes c \to bc$ is an isomorphism.

Proof. ([9] pp.57).

The following proposition reduced the study to the prime power degree case.

Proposition (2.2.7). Let $H = H_1 \underset{K}{\otimes} H_2$ be a central division algebra over K, where H_1, H_2 are central subalgebras and let V be a rank 1 valuation ring V in K. Assume that H_1, H_2 have coprime degrees. Then there exists a valuation ring W lying over V (in H) iff there exist valuation rings W_1, W_2 in H_1 (resp. H_2) lying over V. If moreover W_1, W_2 are Azumaya and V is discrete of rank 1 then W is Azumaya and is given by $W \cong W_1 \underset{V}{\otimes} W_2$.

Proof. Let v be the valuation on K which corresponds to V and let $\tilde{K}$ be the completion of K relative to v. If W_1,W_2 exist, then $\tilde{H}_i = H_i \otimes_K \tilde{K}$ is a skew field (i = 1,2). Hence

$$\tilde{H} = H \otimes_K \tilde{K} \cong (H_1 \otimes_K H_2) \otimes_K \tilde{K} \cong (H_1 \otimes_K \tilde{K}) \otimes_{\tilde{K}} (H_2 \otimes_K \tilde{K}),$$

whence $\tilde{H}$ is a skew field because $\tilde{H}_1,\tilde{H}_2$ have coprime degrees, thus W exists. If W_1,W_2 are Azumaya, then proposition (2.2.4) yields that $W_1 \otimes_V W_2$ is a maximal V-order in H. But since V is discrete of rank 1, W is a maximal V-order, in fact W is the unique maximal V-order, hence $W_1 \otimes_V W_2 \cong W$ and W is Azumaya because W_1 and W_2 are.

Recall that if H is central division K-algebra of degree $n = p_1^{n_1} \dots p_r^{n_r}$ where the p_i's are distinct primes, then $H \cong H_1 \otimes_K \dots \otimes H_r$ where each H_i is a central subalgebra of degree $p_i^{n_i}$ called the p_i-factor, and by the above proposition we have.

Corollary (2.2.8). Let $H = H_1 \otimes H_2 \otimes \dots \otimes H_r$ be the decomposition of H in p_i-factors and let V be a rank 1 valuation ring in K. Then there exists a valuation ring W lying over V (in H) iff there exists W_i in H_i lying over V (i = 1,...,r).

If $W_1,W_2,\dots,W_r$ are Azumaya and V is discrete of rank 1, then W is given by $W \cong W_1 \otimes_V \dots \otimes_V W_r$ and it is Azumaya.

Proof. Repeated applications of proposition (2.2.7).

This corollary shows that it is enough to study central division algebras of prime power degrees.

To start with we consider an abelian crossed product division algebra H of degree p^n; $n \geq 1$ i.e. $H \cong (E/K; f)$ where E/K is a finite abelian extension with abelian Galois group $G \cong S_1 \times \dots \times S_r$. For simplicity we shall assume r = 2 so that $G = S_1 \times S_2$ where $S_1 = \langle\sigma_1\rangle$ of order n_1, $s_2 = \langle\sigma_2\rangle$ of order n_2 and $n = n_1 n_2$.

By the Skolem-Noether theorem each σ_i is induced by an element z_i in H we put $z_i^{p^{n_i}} = a_i$ $(i = 1,2)$. We shall assume that $z_1 z_2 = z_2 z_1$ i.e. H is symmetric. In this case $a_i \in K$.

We let $E_1 = \{x \in E;\ \sigma_2 x = x\}$ and $E_2 = \{x \in E;\ \sigma_1 x = x\}$, hence E_i/K is a cyclic Galois extension with group S_i $(i = 1,2)$ so we can consider the subalgebras $H_i = (E_i/K;\ \sigma_i,\ a_i)$ $(i = 1,2)$ and it is easily seen (cf. [2]) that $H \cong H_1 \underset{K}{\otimes} H_2$.

Throughout what follows K will be assumed non-stable which is justified by the following lemma.

Lemma (2.2.9). Let $H = H_1 \underset{K}{\otimes} H_2$ be a central division algebra of degree $n = n_1 n_2$ over K where H_1, H_2 are central subalgebras of degrees n_1 (resp. n_2). Then exp H = n iff exp $H_i = n_i$ $(i = 1,2)$ and $(n_1, n_2) = 1$.

Proof. The proof of this lemma is readily available once we observe that exp H is the least common multiple of exp H_1 and exp H_2.

N.B.: The above is true for $H = H_1 \otimes \ldots \otimes H_r$ $(r > 2)$.

Throughout the rest of this section the valuation ring V in K will correspond to a Henselian valuation of rank 1 i.e. satisfying Hensel condition; namely.

For any monic polynomials $f \in V[x]$ and $F_1, F_2 \in \bar{V}[x]$ $(\bar{V} \cong V/\mathfrak{m})$ such that $\bar{f} = F_1 F_2$ and F_1, F_2 are coprime, there exist $f_1, f_2 \in V[x]$ such that $\bar{f}_1 = F_1$, $\bar{f}_2 = F_2$ and $f = f_1 f_2$.

We note that this condition is equivalent to saying that v is indecomposed in the algebraic closure of K (cf. [10] pp.117).

We recall that any Henselian valuation on the centre K of a central division algebra H can be extended to the whole of H ([20] theorem 9). We aim to study that extension in tensor products. So we let H be a symmetric abelian crossed product division algebra over E/K. By the remark which preceded Lemma (2.2.9), $H \cong H_1 \otimes \ldots \otimes H_r$ where each H_i is cyclic algebra. We assume r = 2 and we write $H = (E/K;\ \sigma_i;\ z_i,\ b_i,\ (i = 1,2))$.

The following proposition gives a representation of the valuation ring in H lying over V.

Proposition (2.2.10). Let $H = (E/K;\ \sigma_i,\ z_i,\ b_i\ (i = 1,2))$ be a symmetric division algebra and let v be a Henselian rank 1 valuation on K which is unramified in E with associated valuation ring V.

If $v(b_1) = v(b_2) = 0$ then the valuation ring in H lying over V is a tensor product.

Proof. Consider the subalgebra $H_i = (E_i/K;\ \sigma_i,\ b_i\ \ (i = 1,2))$ where $E_1 = \{x \in E,\ \sigma_2(x) = x\}$ and $E_2 = \{x \in E;\ \sigma_1 x = x\}$. Let V_i be the valuation rings in E_i $(i = 1,2)$ lying over V. We consider

$$H_i = \sum_{j=0}^{n_i-1} E_i u^j \quad \text{with} \quad u^{n_i} = b_i \ (i = 1,2) \text{ where } n_i = \deg H_i .$$

We let $$W_i = \sum_{j=0}^{n_i-1} V_i u^j \qquad (i = 1,2).$$

Theorem (1.2.6) yields that W_i is a matrix local ring lying over V and generating H_i as E-space, hence $\bar{W}_i = W_i/J(W_i)$ is simple artinian. Now since v is unramified in E_i ([10] Cor.20.22) yields that v is defectless in E_i and (10 Theorem 18.6 and 18.9) yield that V_i is of finite rank as V-module, hence by ([1] theorem 24) idempotents mod (two sided ideal) can be lifted, whence $\bar{W}_i$ is skew field because otherwise W_i contains non-trivial idempotents which contradicts the fact that H_i has no zero-divisors.

Thus W_i is a local ring and by theorem (1.2.6) W_i is a valuation ring in H_i $(i = 1,2)$.

We now consider $W = W_1 \otimes_V W_2$.

By proposition (2.2.4) W is a matrix local ring generating H as K-space and by a similar proof as above W is a local ring with $J(W) = \mathfrak{m}W$ where $\mathfrak{m} = J(V)$ hence Theorem (1.2.6) yields that W is a valuation ring in H lying over V and since there is only one; the proposition is proved.

N.B.: 1) A necessary condition for the assumption of $v(b_1) = v(b_2) = 0$ is that $Br(\bar{V}) \neq 0$, since otherwise $\bar{W}_i$ splits over $\bar{V}$ as an element of the Brauer group. Now since W_i is of finite rank over V ([1] theorem 24) yields that H_i splits over K which is a contradiction. However since every complete field is Henselian for some valuation, our assumption that K is not stable contains that of $Br(\bar{V}) \neq 0$ by corollary 2 of ([18]) in the case where K is complete.

2) The proposition is valid by induction for $r > 2$.

As another application of theorem (2.2.4) we look at crossed product division algebras with nilpotent Galois groups. We let

$$H = \sum_{\sigma \in G} Eu^{\sigma} \cong (E/K;\ f)$$

where G is a nilpotent group of order $|G| = n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ (the p_i's are distinct primes). It is well known from group theory that

$$G \cong G_1 \times G_2 \times \dots \times G_r \text{ (where the } G_i\text{'s are the } p_i\text{-Sylow subgroups } i = 1,\dots,r)$$

Put $E^j = \{x \in E;\ \sigma x = x \text{ for all } \sigma \in G_j\}$ and $E_i = \bigcap_{j \neq i} E^j$. Then E is the fixed field of G/G_i, and E_i/K is a finite Galois extension with Galois group G_i $(i = 1,\dots,r)$. Accordingly we can decompose E as tensor products i.e.

$$E \cong E_1 \otimes_K \dots \otimes_K E_r$$

The following lemma shows that under some conditions on f the p_i-factors of H are crossed products.

<u>Lemma (2.2.11)</u>. Let $H = (E/K;\ f)$ be a crossed product division algebra with nilpotent Galois group $G = G_1 \times \dots \times G_r$ (the G_i are the p_i-Sylow subgroups where $n = \deg H = p_1^{n_1} \dots p_r^{n_r}$). Assume that f satisfies the following

1) $f_{\sigma,\tau} = f_{\tau,\sigma}$ whenever $\sigma \in G_i$, $\tau \in G_j$ $(i \neq j)$

2) $\rho(f_{\sigma,\tau}) = f_{\sigma,\tau}$ whenever $\sigma,\tau \in G_i$ and $\rho \in \prod_{j\neq i} G_j$.

Then $H \cong H_1 \otimes \ldots \otimes H_r$

where each H_i is a crossed product subalgebra of H with Galois group G_i.

Proof. Write $H = \sum_{\sigma\in G} Eu_\sigma$ where $u_\sigma a = a^\sigma u_\sigma$ for all $a \in E$, $\sigma \in G$

and $u_\sigma u_\tau = f_{\sigma,\tau} u_{\sigma\tau}$ for all $\sigma,\tau \in G$

By the remark above there exist $E_1,\ldots,E_r$ such that E_i/K is Galois with Galois group G_i and $E \cong E_1 \otimes \ldots \otimes E_r$. Consider the subalgebra H_i generated by E_i and the u_σ $(\sigma \in G_i)$; by the condition 2) and the definition of E_i, $f_{\sigma,\tau} \in E_i$ for all $\sigma,\tau \in G_i$; hence $f/G_i \times G_i$ is a factor set from G_i to E_i^*, whence H_i is a crossed product over E_i/K.

Consider the map

$$\phi: H_1 \underset{K}{\otimes} \ldots \underset{K}{\otimes} H_r \to H$$

$$a_1 \otimes \ldots \otimes a_r \to a_i \ldots a_r$$

since G is nilpotent G_i commute with G_j $(i \neq j)$, hence condition 1) yields $u_\sigma u_\tau = u_\tau u_\sigma$ for $\sigma \in G_i$, $\tau \in G_j$ where $i \neq j$. Moreover for any $\sigma \in G_i$ and any $a \in E_j$; $i \neq j$ $au_\sigma = u_\sigma a$ because G_i fixes E_j for $i \neq j$, hence H_i and H_j commute element-wise for $i \neq j$, whence ϕ is a K-homomorphism. It is injective becasue its domain is simple and counting dimensions over K yields surjectivity. Thus

$$H \cong H_1 \otimes H_2 \otimes \ldots \otimes H_r.$$

We are now ready to study representation of valuation rings in this case.

Proposition (2.2.12): Keeping the hypothesis of lemma (2.2.11) and assume that v is a Henselian rank 1 valuation on K which is unramified in E with associated valuation ring V and extension ω to E. Let $H = H_1 \otimes \ldots \otimes H_r$ be the decomposition of H in crossed product p_i-factors (where $\deg H = n = p_1^{n_1} \ldots p_r^{n_r}$).

If $\omega(\mathrm{imf}) = 0$, then the valuation ring in H lying over V is represented by a tensor product.

<u>Proof</u>. Consider $H = \sum_{\sigma \in G} Eu_\sigma$

Then $H_i = \sum_{\sigma \in G_i} E_i u_\sigma$ $(i = 1,\ldots,r)$ (see Lemma (2.2.11).

Let V_i be the valuation ring in E_i lying over V $(i = 1,\ldots,r)$. Then by Theorem (1.2.6) $W_i = \sum_{\sigma \in G_i} V_i u_\sigma$ is a matrix local ring in H_i lying over V and generating H_i as E_i-space. But by an argument similar to that in (2.2.10) V_i has finite rank over V, hence by ([1] theorem 24) idempotents mod (two-sided ideal) can be lifted, whence W_i is local and applying Theorem (1.2.6) again yields that W_i is a valuation ring in H_i lying over V.

Now by corollary (1.2.5) $J(W_i) = \mathfrak{m} W_i$ where $\mathfrak{m} = J(V)$, hence $W_i/\mathfrak{m} W_i$ is separable as $V/\mathfrak{m}$-algebra, whence W_i is separable as V-algebra, thus W_i is Azumaya valuation over V by lemma (2.2.3) $(i = 1,\ldots,r)$.

We now consider

$$W = W_1 \underset{V}{\otimes} \cdots \otimes W_r.$$

First consider $W_{12} = W_1 \otimes W_2$, it is clearly a matrix local ring and by an argument similar to above it is local; applying (proposition (2.2.4)) yields that W_{12} generates $H_1 \otimes H_2$ as K-space with $J(W_{12}) = \mathfrak{m} W_{12}$ and $W_{12} \cap K = V$; hence W_{12} extends V to $H_1 \otimes H_2$ in the sense defined in Chapter 1, whence applying theorem (1.2.6) yields that W_{12} is a valuation ring in $H_1 \otimes H_2$ lying over V.

The rest of the proposition is clear by an easy induction.

§3. <u>Primary algebras</u>

Throughout this section H is a central division algebra over a field K and v is a real valued valuation on K and $\tilde{K}$ is the completion of K relative to v.

We consider $\tilde{H} = H \otimes_K \tilde{K}$.

We aim to study the subalgebras of H and $\tilde{H}$ and relate that accordingly to the extension problem. But first we need to recall a definition. Let A be a central simple algebra over K, then A is said to be a primary algebra if A contains no proper central simple subalgebra over K and $A \neq K$. It is well known that every primary K-algebra is either a division algebra of prime power degree or of the form K_p where p is a prime number. However the converse does not hold in general, it does hold over stable fields as the following lemma shows.

Lemma (2.3.1). Let H be a central simple algebra of degree p^n $(n \neq 0)$ over a stable field K and assume that H is either of the form K_p or else a division algebra, then H is a primary algebra.

Proof. If $H = K_p$ then the lemma is trivial.

If H is division algebra then $\exp H = p^n$.

Now if A is a central simple subalgebra of H, then

$$H \cong A \otimes_K A' \quad \text{where} \quad A' = C_H(A)$$

Applying lemma (2.2.9) yields that (deg A, deg A') = 1 which is a contradiction because deg H = p^n and deg A, deg A' divide p^n. Hence H is a primary algebra.

Proposition (2.3.2). Let H be a central division algebra over a stable field K such that deg H = p^n where p is prime and $n > 1$. Let v be a real valued valuation on K, then v extends to H iff $\tilde{H} = H \otimes_K \tilde{K}$ is a primary algebra.

Proof. If $\tilde{H}$ is primary, then H is not a matrix ring over K since deg $\tilde{H} = p^n$ and $n > 1$, $\tilde{H}$ is a division algebra and applying theorem (0.1.1) yields that v extends to H. The converse is obvious.

Corollary (2.3.3). Let H be a central division algebra over a stable field K such that $\deg H = n = p_1^{n_1} \dots p_r^{n_r}$ where the p_i's are the distinct primes and $n_i > 1$ $(i = 1,\dots,r)$.

Let $H = H_1 \otimes_K \dots \otimes_K H_r$ be the decomposition of H on p_i-factors. Then a real valued valuation v extends to H iff

$$\tilde{H}_i = H_i \otimes_K \tilde{K} \text{ is a primary } \tilde{K}\text{-algebra } (i = 1,\dots,r)$$

Proof. Combining corollary (2.2.8) and proposition (2.3.2) yields the proof.

Remark and example (2.3.4). The condition that $n > 1$ in proposition (2.3.2) can not be omitted as the following example shows. (This example is due to P.M. Cohn see ([5] pp.67.)

Let $H = (\frac{-1,-3}{Q})$ be the rational quaternion algebra then H is a division algebra because $x^2+3y^2+z^2 = 0$ has no solution.

Consider the 2-adic valuation ν_2 on Q and let Q_2 be the field of 2-adic numbers, then Q_2 contains a 2-adic square root of -3, hence $\tilde{H} = H \otimes_Q Q_2 \cong M_2(Q_2)$, thus ν_2 does not extend to H even though $\tilde{H}$ is a primary Q -algebra.

4. A counter example on the extension of valuations in central extensions

Throughout this section D is a finite dimensional central division algebra over a field K and F is a field extension of K. By a central extension of D we shall mean a skew field H generated by D together with the centre of H.

If $D \otimes_K F$ has no zero-divisors, then it is a central extension of D with centre F. So assume that $H = D \otimes_K F$ is a skew field and let ω be a non-trivial real valuation on D with restriction v to K, then v is surely non-trivial (cf.[10]PP.18). We aim to study the extension problem of ω to the whole of H and to prove subsequently that the extension does not always exist.

We note first that if F/K is purely transcendental, then

$H = D \otimes_K F$ is a central extension and ω has an extension to H. For if $F = K(t)$, then $H = D(t)$ and the extension follows as in the commutative case. If $F = K(t_1, t_2, \ldots,)$ then the extension follows by induction. So we shall assume that F/K is an algebraic extension and we shall be mainly interested in the finite case. Let $n = (F:K)$, we shall say <u>v splits in F</u> if there are n distinct valuations extending v to F.

The following proposition determines a subfield of H to which ω extends.

<u>Proposition (2.4.1)</u>. Let $H = D \otimes_K F$ be a central extension where F is a finite abelian Galois extension and let ω be a real valued valuation on D with restriction v to K. Assume that there exist $v_1, \ldots, v_r$ valuations on F extending v with common decomposition field E, then ω extends to $L = D \otimes_K E$. In particular if v splits in F, then ω extends to H.

<u>Proof</u>. $D \otimes_K E \to D \otimes_K \tilde{K} \cong \tilde{D}$ is an embedding where $\tilde{D}$ is the completion of D relative to ω. Now ω extends to $\tilde{\omega}$ on $\tilde{D}$ and $\tilde{\omega}/D \otimes_K E$ is a valuation extending ω.

The second part of the proposition is clear.

<u>Proposition (2.4.2)</u>. Let $H = D \otimes_K F$ be a central extension of D such that F/K is a finite Galois extension. Put $m = [F,K]$ and $n = \deg D$ and assume that $(m,n) = 1$.

Then any real valued valuation ω on D can be extended to r valuations on H where r is the number of valuations extending v to F where $v = \omega|K$.

<u>Proof</u>. Put $F = K(a)$.

Let f be the minimal polynomial of a over K and $\tilde{K}$ be the completion of K relative to v.

Consider $f = f_1 \cdot \ldots \cdot f_r$, the factorization of f over $\tilde{K}$. Let $v_1, v_2, \ldots, v_r$ be the valuation on F extending v and $\tilde{F}^i$ the completion of

F relative to v_i $(i = 1,\ldots,r)$. Then $n_i = (\tilde{F}^i:\tilde{K})$ is the same for all v_i and n_i divides m because F/K is Galois. We now consider $\tilde{H}^i = H \underset{F}{\otimes} \tilde{F}^i = (D \underset{K}{\otimes} F) \underset{F}{\otimes} \tilde{F}^i \cong D \underset{K}{\otimes} \tilde{F}^i \cong (D \underset{K}{\otimes} \tilde{K}) \underset{\tilde{K}}{\otimes} \tilde{F}^i$. We observe that

1) $D \underset{K}{\otimes} \tilde{K}$ is a skew field

2) $(n,n_i) = 1$ otherwise $(n,m) \neq 1$

hence $\tilde{H}^i$ is a skew field, whence v_i extends to a valuation on H whose restriction to D is obviously ω since ω is the only valuation on D extending v.

Since we can repeat the same thing for $i = 1,\ldots,r$, there are exactly r valuations extending ω to H and the proposition is proved.

Remarks and example. 1) The assumption that F/K is Galois was needed to prove that if $(m,n) = 1$, then $(n_i,m) = 1$. However if we omit the normality and assume that v is indecomposed then $[\tilde{F}:\tilde{K}] = [F:K]$ because then $\tilde{F} \cong F \underset{K}{\otimes} \tilde{K}$, hence the condition on normality can be lifted.

Example. Let $D = (\frac{-1,-1}{Q})$ be the quaternion algebra over the rationals and $F = Q(\sqrt[3]{2})$.

Then $H = D \underset{Q}{\otimes} F$ is a central extension because (x^3-2) is irreducible over D.

Consider the 2-adic extension ν_2 on Q which is the only valuation on Q extendable to D (see Example (2.1.9)) and let ω be its extension.

Now ν_2 is clearly indecomposed in F because it is totally ramified and $(\deg D, [F:Q]) = (2,3) = 1$, hence by the proof of the proposition ω extends to H even though F/Q is not normal.

The following proposition is the catalyst for the counter example.

Proposition (2.4.3). Let D be a finite dimensional central division algebra over K and let ω be a real valued valuation on D such that $\omega|K = v$.

Let $F = K(a)$ be a finite separable extension of K with f a minimal polynomial of a over K. Assume that f is irreducible over D so that $H = D \underset{K}{\otimes} F$ is a skew field and that v has a unique extension to F, then

ω extends to H iff f remains irreducible over $\tilde{D} = D \otimes_K \tilde{K}$ where $\tilde{K}$ is the completion of K relative to v.

Proof. The condition is sufficient; let v_F be the extension of v to F then $\tilde{F} = \tilde{K}(a)$.

Now $\tilde{H} = (D \otimes_K F) \otimes_F \tilde{F} = D \otimes_K \tilde{K}(a) \cong (D \otimes_K \tilde{K}) \otimes_{\tilde{K}} \tilde{K}(a) \cong \tilde{D}[X]/f\tilde{D}[X]$,

hence it is a skew field because f is irreducible over $\tilde{D}$, whence by theorem (0.1.1) ω extends to H.

The condition is necessary.

Assume that ω extends to a valuation ϕ on H and call v_F its restriction to F. Let $\tilde{F}$ be the corresponding completion, then

$$\tilde{H} = H \otimes_F \tilde{F} = (D \otimes_K F) \otimes_F \tilde{F} \cong D \otimes_K \tilde{F} \cong (D \otimes_K K) \otimes_{\tilde{K}} \tilde{F} \cong \tilde{D} \otimes_{\tilde{K}} \tilde{F}$$

and $\tilde{H}$ is a skew field.

Now $\tilde{F} \cong \tilde{K}[X]/f\tilde{K}[X]$, hence $\tilde{H} \cong \tilde{D} \otimes_{\tilde{K}} \tilde{K}[X]/f\tilde{K}[X] \cong \tilde{D}[X]/f\tilde{D}[X]$

whence f is irreducible over $\tilde{D}$ otherwise $\tilde{H}$ has zero-divisors.

We now construct the counter example.

Let $D = (\frac{-1,-1}{Q})$ be the quaternion algebra over the rationals and let ν_2 be the 2-adic valuation on Q. We have seen that ν_2 extends to D, we call ν_2 its extension to D.

Consider $H = D \otimes_Q F$ where $F = Q(\sqrt[4]{2})$ then $f(X) = X^4-2$ is the minimal polynomial of $\sqrt[4]{2}$ over Q. f(X) is irreducible over D.

For $f(X) = (X-\lambda)(X+i\lambda)(X+\lambda)(X-i\lambda)$ where $\lambda = \sqrt[4]{2}$ so if f(X) is reducible over D then D must contain an element a such that $a^2 = 2$ i.e. $\exists\ \alpha, \beta, \gamma, \delta \in Q$ such that

$$(\alpha+\beta i+\gamma j+\delta K)^2 = 2$$

hence $$\alpha^2-\beta^2-\gamma^2-\delta^2+2\alpha\beta i+2\alpha\gamma j+2\alpha\delta K = 2$$

this implies that $\alpha^2-\beta^2-\gamma^2-\delta^2 = 2$

$$2\alpha\beta = 0$$

$$2\alpha\gamma = 0$$

$$2\alpha\delta = 0$$

hence $\alpha = 0$, since otherwise $\beta = \gamma = \delta = 0$ and $\alpha^2 = 2$ which is impossible in Q, whence $\beta^2+\gamma^2+\delta^2 = -2$ in Q which is impossible as well and f(X) is irreducible over D.

Thus H is a skew field and H is a central extension of D. We shall prove that f(x) is reducible over $\tilde{D}$ so that ν_2 does not extend to H.

First we recall the following theorem (cf. [17] pp.146).

Theorem: Let D be a central division algebra over a complete field K for a discrete rank 1 valuation v with finite residue class field having q elements. Let n = deg D and let ε be a primitive (q^n-1)-th root of unity, then to any uniformizer π of K corresponds an element π_D of D such that $\pi_D^n = \pi$ and $\pi_D\, \varepsilon\, \pi_D^{-1} = \varepsilon^{q^r}$ where r is a positive integer such that $1 \leq r \leq n$ and $(r,n) = 1$.

<u>Proof</u>. (cf. [17] p.146).

Now 2 is a uniformizer of Q_2 (the field of 2-adic numbers). Consider $\tilde{D} = D \otimes_Q Q_2$. This is a central division algebra over Q_2. Then applying the theorem yields that there exists an element say b in $\tilde{D}$ such that $b^2 = 2$. So $\tilde{D}$ contains a square root of 2. Now f(X) is irreducible over Q_2 since otherwise $\sqrt{2}$ or $\sqrt[4]{2} \in Q_2$ which is impossible because then $\tilde{\nu}_2(\sqrt{2}) = \frac{1}{2}$ where $\tilde{\nu}_2$ is the extension of ν_2 to Q_2 but this leads to a contradiction since the value group of $\tilde{\nu}_2$ is $\mathbb{Z}$. Now f(X) is reducible over $\tilde{D}$ because $\tilde{D}$ contains a square root of 2 hence applying the proposition yields that ν_2 does not extend to H.

CHAPTER III

Extension of valuations in infinite skew field extensions

Let H/D be an infinite skew field extension and let v be a valuation on D. In this chapter we study the extension of v to the skew field H. In section 1. we let H be the skew function field $D(X;\ \sigma,\ \delta)$ where σ is an automorphism on D and δ is a σ-derivation. We prove that v extends to a valuation ω with radical $\mathcal{P}$ such that $\tilde{X} (\mathrm{mod}\ \mathcal{P})$ is transcendental over the residue class field of v iff

(1) σ preserves v   (2) δ is such that $v(a^{\delta}) \geq v(a)$ for all $a \in D$.

The importance of this rather easy theorem lies in its wide application and its repeated use in the rest of this chapter. The applications will include among others i) free algebras, ii) universal associative envelopes of Lie algebras and iii) Generic Crossed product division algebras.

The rest of this chapter is devoted to the following question raised by P.M. Cohn.

Let K_1, K_2 be two skew fields with real valued valuations v_1, v_2 on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ where K is a common subfield.

Let $R = K_1 \underset{K}{\sqcup} K_2$ be the free product of K_1 and K_2 over K and let $H = K_1 \underset{K}{\circ} K_2$ the field coproduct of K_1 and K_2 over K.

Do v_1, v_2 have a common extension to H?

Section 2. recalls some theorems needed later (P.M. Cohn [8] Theorems 5.1,5.4) which are proved here under rather weaker conditions. In section 3. we answer the above question negatively by giving a counter example.

In section 4 we consider the associated epic R-field L constructed in ([8]). We study the centre of L, in particular we show that if K_1/K is a finite abelian Galois extension with E the decomposition field of v_1, then the centre C of L contains E. We then generalize the result of section 2. by showing that in general v_1, v_2 have no extension to any skew field of fractions of R.

In section 5. we show the following.

Let K_1, K_2 be skew fields with centres C_1, C_2 and a common subfield $K \subseteq C_i$ $(i = 1,2)$.

Let v_1, v_2 be real valued valuations on K_1, K_2 such that $v_1|K = v_2|K = v$ and such that v_i is the only real valued valuation on K_i extending v $(i = 1,2)$.

Assume that K_i admits an endomorphism σ_i whose fixed field intersected with C_i is K, then v_1, v_2 have a common extension to $K_1 \underset{K}{\circ} K_2$.

If D is a skew field with centre C and a central subfield K such that D has a family of endomorphisms whose fixed field intersected with C is K. Then any real valued valuation v which uniquely extends its restriction to K can be extended to a valuation ω on $D_K\langle X\rangle$.

We conjecture that ω is real valued.

If the conjecture is true then we have the following theorem.

Let K_1, K_2 be skew fields with centres C_1, C_2 and a common subfield $K \subseteq C_i$ $(i = 1,2)$.

Let v_1, v_2 be real valued valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ and such that v_i is the only real valued valuation on K_i extending v $(i = 1,2)$.

Assume that K_i has a family of endomorphisms whose fixed field intersected with C_i is K.

Then v_1, v_2 have a common extension to $K_1 \underset{K}{\circ} K_2$. The application will be to the non-commutative Galois extensions. Other results concerning other cases will be given in this section as well.

In particular we show that a recent generalization of the specialization lemma by P.M. Cohn entails the generalization of Theorem (3.2.1).

§1. Extension of valuations in skew function fields

Let K be a skew field with an endomorphism σ and a σ-derivation δ and consider the right skew polynomial ring $R = K[x; \sigma, \delta]$ consisting of the elements $\sum_{i=0}^{n} x^i a_i$ where multiplication is defined by $ax = xa^{\sigma}+a^{\delta}$ and the usual addition. It is well known that R is right ore domain and hence it has a skew field of fractions $D = K(X, \sigma, \delta)$ called a skew function field (see chapter O). Let v be any valuation on D, we aim to study the extension of v to D and its applications.

We first need the following lemma.

Lemma (3.1.1). Let $D = K(X; \sigma, \delta)$ be a skew function field and let v be a (not necessarily) abelian valuation on K with associated valuation ring V and radical $\mathfrak{m}$.

Suppose that v extends to ω on D with radical $\mathcal{P}$. Then $\bar{X} (\mathrm{mod}\, \mathcal{P})$ is right transcendental over $V/\mathfrak{m}$ iff

$$(1) \qquad \omega(X^n a_n + \dots + a_0) = \min_{i=0\dots n} v(a_i)$$

Moreover ω is the unique extension for which X remains transcendental over $V/\mathfrak{m}$.

Proof. The proof is exactly the same as in the commutative case (cf. [20] lemma 17).

The extension ω will be called the Gaussian extension. The following theorem isolates the conditions on σ and δ for the extension to be possible. Throughout the section we shall assume σ is an automorphism, hence R is also a left skew polynomial ring and $\bar{X}$ is left transcendental over $V/\mathfrak{m}$. We say $\bar{X}$ is transcendental over $V/\mathfrak{m}$.

Theorem (3.1.2). Let $D = K(X;\ \sigma,\ \delta)$ be a skew function field and let v be a valuation on K with associated valuation ring V and radical $\mathfrak{m}$. Then v extends to a valuation ω with radical $\mathcal{P}$ for which $\bar{X}(\mathrm{mod}\,\mathcal{P})$ is transcendental over $V/\mathfrak{m}$ iff σ preserves v and δ is such that $v(a^{\delta}) \geq v(a)$ for all $a \in K$.

Moreover ω is the unique extension such that X remains transcendental over $V/\mathfrak{m}$ and is given by the Gaussian extension.

Proof. The condition is necessary.

If ω exists then Lemma (3.1.1) yields that ω is given by

$$\omega\Big(\sum_{i=0}^{n} X^i a_i\Big) = \min_{i=0,\dots,n} v(a_i)$$

Now $$ax = xa^{\sigma}+a^{\delta}.$$

Hence 1) $v(a) = \min(v(a^{\sigma}),\ v(a^{\delta}))$ because $\omega(X) = 0$ whence $v(a^{\sigma}) \geq v(a)$ for all $a \in K$.

If $v(a^{\sigma}) > v(a)$ then $-v(a^{\sigma}) < -v(a)$, hence $v({a^{-1}}^{\sigma}) < v(a^{-1})$ which contradicts 1), thus $v(a) = v(a^{\sigma})$ and σ preserves the valuation.

Now it is easily deduced that $v(a^{\delta}) \geq v(a)$.

The condition is sufficient.

We shall consider the right skew polynomial ring $R = K[X;\ \sigma,\ \delta]$ and the map $\omega\colon R \to \Gamma \cup \{\infty\}$ where Γ is the value group of v defined by $\omega\big(\sum_{i=0}^{n} X^i a_i\big) = \min_{i=0,\dots,n} v(a_i)$.

Then ω satisfies the axioms of a valuation on R namely:

v.1 $\omega(f) = \infty \Leftrightarrow f = 0$ for all $f \in R$

v.2 $\omega(f-g) \geq \min(\omega(f),\ \omega(g))$ for $f,g \in R$

v.3 $\omega(fg) = \omega(f) + \omega(g)$ for $f,g \in R$.

For

v.1) is clear because if $f = \sum_{i=0}^{n} X^i a_i = 0$ then $a_i = 0$ for $i = 0,1,\dots,n$ hence $\omega(f) = \infty$ iff $f = 0$

v.2) Let $f = X^m a_m + \dots + a_0$

$$g = X^n b_n + \dots + b_0$$

and assume W.L.O.G. that $m > n$

then $f-g = X^m a_m + \dots + X^n(a_n - b_n) + \dots + a_0 - b_0$

hence $\omega(f-g) = \min\{v(a_m), \dots, v(a_n - b_n), \dots, v(a_0 - b_0)\}$

$$\geq \min\{v(a_m), \dots, \min(v(a_n), v(b_n)); \min(v(a_0), v(b_0)\}$$

$$= \min\begin{cases} \min\{v(a_m), \dots, v(a_0)\} \\ \min\{v(b_n), \dots, v(b_0)\} \end{cases}$$

$$= \min(v(f), v(g))$$

and v.2) is proved.

v.3) Consider $f = X^m a_m + \dots + X^i a_i + \dots + a_0$ and assume that $\omega(f) = v(a_i)$ where a_i is the first coefficient on the left taking the minimum value among the values of the coefficients. Consider

$$g = X^n b_n + \dots + X^j b_j + \dots + b_0$$

and assume that $\omega(g) = v(b_j)$ where b_j is the first coefficient on the right taking the minimum among the values of the coefficients.

Now $fg = \sum_{r=0}^{m+n} X^r C_r$

We first compute aX^t where a is any element in K and t any positive integer.

$$aX = Xa^\sigma + a^\delta$$

$$aX^2 = X^2 a^{\sigma^2} + X(a^{\sigma\delta} + a^{\delta\sigma}) + a^{\delta^2}$$

$$aX^3 = X^3 a^{\sigma^3} + X^2(a^{\sigma^2\delta} + a^{\sigma\delta\sigma} + a^{\delta\sigma^2}) + X(a^{\sigma\delta^2} + a^{\delta\sigma\delta} + a^{\delta^2\sigma}) + a^{\delta^3}$$

$$aX^t = X^t a^{\sigma^t} + X^{t-1}\Big(\sum_{\Sigma t_i = t} a^{\sigma^{t_1}\delta^{t_2}\sigma^{t_3}\delta^{t_4}\dots = \pi\sigma^{t_i}\delta^{t_j}\dots}\Big)_{\substack{t_i = 1,\dots,t-1 \\ t_j = 1,\dots,t-1}} + \dots + a^{\delta^n}$$

We now apply that to compute $X^i a_i X^j b_j$ and so we have

$$X^i a_i X^j b_j = X^i(X^j a_i^{\sigma^j} + X^{j-1}(\sum_{\Sigma j_K = j} a_i^{\sigma^{j_1}\delta^{j_2}\sigma^{j_3}\cdots}) + \ldots + a_i^{\delta^j})b_j$$

$$= X^{i+j} a_i^{\sigma^j} b_j + X^{i+j-1}(\sum_{\Sigma j_K = j} a_i^{\sigma^{j_1}\delta^{j_2}\cdots}) + \ldots + X^i a^{\delta^j} b_j$$

two cases occur.

1. $i < j$ and $i+j = m+p = n+q$

Then $C_{i+j} = \sum_{h=0}^{i-p} a_{i-h}^{\sigma^{j+h}} b_{j+h} + \sum_{K=1}^{j-q} a_{i+k}^{\sigma^{j-k}} b_{j-k}$ + elements of the form

$\Sigma a_s^{\sigma^{s_1'}\delta^{s_2'}\cdots} b_{s'}$ where either s precedes i or s' exceeds j.

We claim that $v(c_{i+j}) = v(a_i) + v(b_j)$.

For if

$h \neq 0$ then $v(a_{i-h}^{\sigma^{j+h}} b_{j+h}) = v(a_{i-h}^{\sigma^{j+h}}) + v(b_{j+h}) = v(a_{i-h}) + v(b_{j+h})$

hence $v(a_{i-h}^{\sigma^{j+h}} b_{j+h}) > v(a_i) + v(b_j)$. Now $v(\sum_{\Sigma s_k' = s} a_s^{\sigma^{s_1'}\delta^{s_2'}\cdots} b_{s'}) > v(a_i)+v(b_j)$ because either s precedes i or s' exceeds j and because σ preserves v and δ is such that $v(a^\delta) \geqslant v(a)$ for any $a \in K$.

Hence $v(C_{i+j}) = v(a_i) + v(b_j)$ and by the definition of ω

$$\omega(fg) = \min_{r=0,\ldots,m+n} v(C_r) = v(a_i) + v(b_j) = \omega(f) + \omega(g).$$

2. $i+j \leqslant m \leqslant n$ or $i+j \leqslant n \leqslant m$.

Then $C_{i+j} = \sum_{h=0}^{i} a_{i-h}^{\sigma^{j+h}} b_{j+h} + \sum_{k=1}^{j} a_{i-k}^{\sigma^{j-k}} b_{j-k}$ + elements of the form

$\sum a_t^{\sigma^{t_1'}\delta^{t_2'}\cdots} b_{t'}$ where either t precedes a_i or t' exceeds j hence an argument similar to that in 1. shows that $\omega(fg) = \omega(f) + \omega(g)$.

In fact these two cases cover all the possibilities, hence v.3 is satisfied, whence ω is a valuation on R.

Now $D = \{f/g;\ f \in R \text{ and } g \in R^*\}$.

Hence ω extends to D by $\omega(f/g) = \omega(f) - \omega(g)$.

For the uniqueness it suffices to apply lemma (3.1.1).

Assume now that K is commutative and is contained in a skew field H, let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be r commuting automorphisms of K fixing the centre of H. Then by the Skolem-Noether theorem, $\sigma_1, \ldots, \sigma_r$ are induced by inner automorphisms of H defined by $X_1, X_2, \ldots, X_r$. Assume that the X_i's are right transcendental over K and that they are right algebraically independent.

We assume furthermore that $u_{ij} = [X_i, X_j] \in K$ for $i,j = 1,\ldots,r$.

Consider $R_1 = K[X_1;\sigma_1]$; the right skew polynomial ring defined by

(1) $aX_1 = X_1 a^{\sigma_1}$ for all $a \in K$.

We define σ_2^* on R_1 by

$$\sigma_2^*(a) = \sigma_2(a) = a^{\sigma_2} \text{ for all } a \in F$$

and $$\sigma_2^*(X_1) = X_1 u_{12}$$

Then σ_2^* preserves the relation (1) because

$$\sigma_2^*(X_1 a^{\sigma_1}) = X_1 u_{12} a^{\sigma_1\sigma_2} = a^{\sigma_1\sigma_2\sigma_1^{-1}} X u_{12} = aX_1 u_{12} = \sigma_2^*(aX_1).$$

Hence σ_2^* is a well defined automorphism on R_1 and we can consider

$$R_2 = R_1[X_2;\ \sigma_2^*].$$

Assume that R_{r-1} is defined so that the following relations hold.

(2) $$aX_i = X_i a^{\sigma_i} \quad i = 2,\ldots,r \quad \text{and} \quad X_i X_j = X_j X_i u_{ij} \quad i,j = 1,2,\ldots,r$$

and define σ_r^* on R_{r-1} as follows.

$$\sigma_r^*(a) = \sigma_r(a) = a^{\sigma_r} \text{ for all } a \in F$$

and $$\sigma_r^*(X_i) = X_i u_{ir} \qquad i = 1,\ldots,r$$

Then σ_r^* preserves (2). For

$$\sigma_r^*(X_i X_j) = X_i u_{ir} X_j u_{jr} = X_i X_j u_{ir}^{\sigma^j} u_{jr} = X_j X_i u_{ij} u_{ir}^{\sigma^j} u_{jr}$$

and

$$\sigma_r^*(X_jX_iu_{ij}) = X_ju_{jr}X_iu_{ir}u_{ij}^{\sigma^r} = X_jX_iu_{jr}^{\sigma^i}u_{ir}u_{ij}^{\sigma^r}$$

Now applying ([2] Lemma 1.2) yields that $u_{ij}u_{ir}^{\sigma^j}u_{jr} = u_{jr}^{\sigma^i}u_{ir}u_{ij}^{\sigma^r}$

hence $\sigma_r^*(X_iX_j) = \sigma_r^*(X_jX_iu_{ij})$

And $\sigma_r^*(aX_i) = \sigma_r^*(X_ia^{\sigma_i})$ as above.

Whence σ_r^* is a well defined automorphism on R_{r-1} and $R = R_{r-1}[X_r;\sigma_r^*]$ is a right skew polynomial ring. It is an ore domain by induction, hence it has a skew field of fractions called the <u>iterated skew function field</u>.

We shall have a corollary about the existence of the Gaussian extension on D.

<u>Corollary (3.1.3)</u>. Let $D = K(X_i;\ \sigma_i\ (i = 1,\dots,r),\ u)$ be the iterated skew function field and let v be any valuation on K with associated valuation ring V and radical $\mathfrak{m}$. Then v extends to a valuation ω (with radical $\mathfrak{P}$) for which $\bar{X}_i \pmod{\mathfrak{P}}$ is transcendental over $V/\mathfrak{m}$ $(i = 1,\dots,r)$ and $\bar{X}_1,\dots,\bar{X}_r$ are right algebraically independent iff (1) σ_i preserves v $(i = 1,\dots,r)$

(2) $v(u_{ij}) = 0$ where $i = 1,2,\dots,r$ and $j = 2,\dots,r$

Moreover ω is the Gaussian extension.

<u>Proof</u>. Consider $R = K[X_i;\ \sigma_i\ (i = 1,\dots,r),\ u]$, then each element f of R can be written as $f = \sum X_{i_1}X_{i_2},\dots,X_{i_s}a_{i_1,\dots,i_s}$ where $X_{ij} \in \{X_1,\dots,X_r\}$ $(j = 1,\dots,s)$ $i_1 \leq i_2 \leq \dots \leq i_s$ and where $a_{i_1\dots i_s} \in K$.

Let Γ be the value group of v and consider the map $\omega: R \to \Gamma \cup \{\infty\}$ defined by

$$\omega(\sum X_{i_1}X_{i_2}\dots X_{i_s}a_{i_1\dots i_s}) = \min v(a_{i_1\dots i_s})$$

ω is clearly a well defined map.

Let $R_1 = K[X_1;\sigma_1]$, then theorem (3.1.2) yields that v extends to

ω_1 on R_1 where ω_1 is the Gaussian extension.

Consider $R_2 = R_1[X_2;\ \sigma_2^*]$. Conditions (1) and (2) yield that σ_2^* preserve ω_1, hence ω_1 extends to a Gaussian extension ω_2 on R_2 and by induction we see that v extends to a Gaussian valuation on D. Now we observe that the same induction process shows that this valuation is given on D by ω. It is also clear by induction that $\bar{X}_i (\mathrm{mod}\,\mathcal{P})$ is transcendental over $V/\mathfrak{m}$ and that $\bar{X}_1,\ldots,\bar{X}_r$ are right algebraically independant. Hence the condition is sufficient.

For necessity we follow the same proof as in the theorem.

Remarks (3.1.4). 1) The condition on δ in theorem (3.1.2) is a necessary one for the existence of the Gaussian extension. However it is not necessary for the solution of the extension problem in general. In fact, if $D = K(X;\ \sigma,\ \delta)$ and **v** is a valuation on K preserved by σ with a value group Γ, then we consider $G = \mathbb{Z} \times \Gamma$ and we order G lexicographically, i.e.

$$(z_1,\gamma_1) < (z_2,\gamma_2) \text{ iff } z_1 < z_2 \text{ or if } z_1 = z_2 \text{ then } \gamma_1 < \gamma_2$$

Let $R = K[X;\ \sigma,\ \delta]$ and consider $\omega\colon R \to G \cup \{\infty\}$ defined by $\omega(f) = (-n, v(a_n))$ where $n = \deg f$. $\omega(0) = \infty$

To prove that ω is a valuation we only need to look at axiom V.3 since the others are easily satisfied.

So let $f = X^n a_n + \ldots + X^i a_i$ where $n \geq i$

$g = X^m b_m + \ldots + X^j b_j$ where $m \geq j$

On multiplying fg we need to know the coefficient of the leading term which is here X^{n+m}.

Now $a_n X^m = X^m a_n^{\sigma^m} + h$ where h is an element of deg < m hence the leading term will have the coefficient

$$a = a_n^{\sigma^m} b_m,\ \text{whence } \omega(fg) = (-(n+m), v(a_n^{\sigma^m} b_m)) = (-n, v(a_n^{\sigma^n}))$$

$+ (-m, v(b_m) = (-n, v(a_n)) + (-m, v(b_m)) = \omega(f)+\omega(g)$

and so the extension does not depend on δ.

ω is called the leading term extension.

2) The importance of theorem (3.1.2) lies in its wide application to the finite and infinite cases as well. It is repeatedly used in field coproducts.

The rest of this section is devoted to the application of theorem (3.1.2) and its corollary (3.1.3).

I) Generic abelian crossed product division algebra

Let $D = K(X; \sigma_i (i = 1,\ldots,r), u)$ be the iterated skew function field constructed above and let k be the fixed field of $\sigma_1,\ldots,\sigma_r$.

Assume that σ_i has a finite order n_i $(i = 1,\ldots,r)$ such that K/k is Galois with Galois group $G = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_r\rangle$ where $\langle\sigma_i\rangle$ is the cyclic group generated by σ_i $(i = 1,\ldots,r)$.

Then by ([2] Theorem 2.3) D is a crossed product over E/F where $E = K(X_1^{n_1},\ldots,X_r^{n_r})$ $F = k(X_1^{n_1}a_1,\ldots,X_r^{n_r}a_r)$ for some $a_1,\ldots,a_r \in K$.

D is called the generic crossed product of K/k. The name is inspired from the fact that every crossed-product algebra over K/k is a homomorphic image of $R = K[X_i; \sigma_1 (i = 1,\ldots,r), u]$. By ([2]) every finite abelian extension has a generic crossed product.

For simplicity we write $D = (K/k; G, a, u)$.

As a first application we have

Corollary (3.1.4). Let $D = (K/k; G, a, u)$ be the generic crossed product of K/k with centre C.

Let v be a valuation on C satisfying the following:

1) v is the Gaussian extension of a non-trivial valuation v_0 on k

2) v_0 has a unique extension ω_0 on K such that

$\omega_0(a_i) = 0$ and $\omega_0(u_{ij}) = 0$ $(i,j = 1,\ldots,r)$ where r = order of G

Then v extends to a valuation ω on D.

Proof. v_0 has a unique extension ω_0 on K, hence G preserves ω_0. ([10] pp.108)

Now $\omega_0(u_{ij}) = 0$ for $i,j = 1,\dots,r$, hence by Corollary (3.1.3) ω_0 extends to the Gaussian extension ω on D.

If $D = K(X_i;\ \sigma_i(i = 1,\dots,r),u)$ then $C = k(X_1^{n_1}a_1,\dots,X_r^{n_r}a_r)$, hence condition 1) and the fact that $\omega_0(a_i) = 0 (i = 1,\dots,r)$ yield that ω is the extension of v to D.

II) Free Algebras

Let $A = k\langle x,y\rangle$ be the free k-algebra on x,y where x,y are k-centralizing indeterminates.

Put $K = k(t)$ where t is a central indeterminate over k and let $R_n = K[z;\ \sigma_n]$ be the right skew polynomial ring where $\sigma_n : t \to t^n$ $(n > 1)$.

Then A can be embedded in R_n where $x = z$ and $y = tz$ (see [12]). $D_n = K(z,\sigma_n)$ is a skew field of fractions of A and from ([13]) the centre of A is precisely K.

Hence as an application of theorem (3.1.2) on free algebras we have

Proposition (3.1.5). Let A be a free K-algebra and let D_n $(n > 1)$ be the skew field of fractions of A arising from the embedding of A in an ore domain. Then any valuation v on the centre K of A can be extended to D_n.

Proof. By the construction above $D_n = k(t)(z;\ \sigma_n)$ where $\sigma_n : t \to t^n$ v extends to a Gaussian extension ω on $k(t)$ $(\omega(\sum_{i=0}^{n} a_i t^i) = \min_{i=0,\dots,n} v(a_i))$ hence σ_n preserves ω and applying theorem (3.1.2) yields that ω extends to D_n.

Remarks. 1) The proposition is true for any $n > 1$.

2) If $A = k\langle x_1,x_2,\dots\rangle$ where $\{x_i\}_{i\in I}$ is a generating set, then we put $K = k(t_{in};\ i \in I,\ n \in \mathbb{Z}^+)$ and $R = K[X;\sigma]$ where $\sigma(t_{in}) = t_{in+1}$ hence by the same reference ([12]) A has a skew field of fractions $D = K(X;\sigma)$. Now any valuation v on k extends to a Gaussian extension ω on K for which $\omega(t_{in}) = 0$ $(i \in I,\ n \in \mathbb{Z}^+)$ hence σ preserves ω and ω

extends to a valuation on D.

3) Let A be any free algebra over k and let U be the universal skew field of fractions of A. In the coming sections we will be looking at the extension problem. In particular we show that every valuation on k extends to U and that there is always a real valued valuation ω on U such that the restriction of ω to k is trivial. N.B. D_n is very much more special than U.

III The universal associative envelope of a Lie algebra

Let K be a field containing i such that $i^2 = -1$ and A be the simple 3-dimensional Lie algebra generated by x,y,z such that

$$[x,y] = z, \ [y,z] = x \ \text{ and } \ [z,x] = y.$$

Let v be any valuation on K.

We aim to construct the universal associative envelope U of A and prove the existence of a valuation ω on the skew field of fractions of U such that $\omega|K = v$.

The following lemma describes U.

Lemma (3.1.6). Let A be a simple 3-dimensional Lie algebra over a field K containing a square root of -1.

Then, the universal associative envelope of A is a skew polynomial ring over K[z].

Proof. Let i denote the square root of -1 and consider the following change of variables.

$$u = x+iy, \quad v = x-iy \quad \text{and} \quad z = z.$$

Then $uz-zu = -y+ix = i(x+iy) = iu$, hence $uz = (z+i)u$

$vz-zv = -y-ix = -(y+ix) = -i(-iy+x) = -iv$

hence $vz = (z-i)v$

Now $uv-vu = (x+iy)(x-iy)-(x-iy)(x+iy) = -2iz$

Put $\mathcal{E} = K[z]$ and let $\sigma: K[z] \to K[z]$

$$f(z) \to f(z+i)$$

Then σ is an automorphism with inverse $\sigma^{-1}(z) = z-i$ we consider the left skew polynomial ring

$$R = \mathcal{E}[u,\sigma] \text{ defined by } uz = (z+i)u$$

σ (hence σ^{-1}) extends to an automorphism on R defined by $\sigma(u) = u$.

Let $\delta: R \to R$ be defined as follows.

1) δ is trivial on $\mathcal{E}$.

2) $\delta(u) = -2iz$.

We note first that δ is a well defined map on R. We claim that δ is a σ^{-1}-derivation on R. It suffices to prove that $\delta(uz) = \delta(z+i)u$.

For $\delta(uZ) = u^{\sigma^{-1}}z^{\delta} + u^{\delta}z = -(2iz)z = -2iz^2$

$$\delta((z+i)u) = (z+i)^{\sigma^{-1}}u^{\delta} + (z+i)^{\delta}u = z(-2iz) = -2iz^2.$$

So we can consider $U = R[v, \sigma^{-1}, \delta]$ and U is the universal associative envelope of A.

N.B. U is an ore domain, hence it has a skew field of fractions $D = L(v, \sigma^{-1}, \delta)$ where L is the skew field of fractions of R. We now deduce easily

Proposition (3.1.7). Let A be a simple 3-dimensional Lie algebra over a field k containing a square root of -1 and let D be the skew field of fractions of its universal associative envelope. Then any valuation on k extends to a valuation on D.

Proof. By the Lemma $D = L(v, \sigma^{-1}, \delta)$ where

$$L = K(u,\sigma) \quad (K = k(z) \quad \text{and} \quad \sigma: Z \to Z+i)$$

v extends to a Gaussian extension ω_1 on K given by $\omega_1(\sum_{i=0}^{n} a_i z^i) = \min_{i=0,\dots,n} v(a_i)$

Now σ preserves ω_1 hence applying theorem (3.1.2) yields that ω_1 extends to a valuation ω_2 on L for which $\omega_2(u) = 0$, hence δ satisfies the condition of theorem (3.1.2) since σ^{-1} preserves $\omega_2(\sigma^{-1}(u) = u)$ the theorem yields that ω_2 extends to a valuation ω on D and ω is surely the extension of v to D.

IV Weyl algebras

Let K be any field and let A be the Weyl algebra generated by x,y such that xy-yx = 1. It is easily seen that A can be written as a skew polynomial ring R[y,1,'] where R = k[X] and ' is the derivation with respect to X. Let D = K(y,1,') be the skew field of fractions where K = k(X), then any valaution on k extends to a valuation on D. It suffices to apply theorem (3.1.2).

Example on the iterated case (3.1.8)

Let k be a field containing ε, η and ξ where ε is a primitive n_1-th root of unity, η is a primitive n_2-th root of unity and ξ is a primitive n_3-th root of unity $(n_1, n_2, n_3 \in \mathbb{Z}^+)$.

Consider the skew field $D = k(x_1, x_2, x_3, x_4)$ where the x_i's are k-centralizing indeterminates satisfying the following relations.

$$x_1x_2 = x_2x_1; \quad x_1x_3 = \varepsilon x_3x_1; \quad x_1x_4 = x_4x_1; \quad x_2x_4 = \eta x_4x_2;$$

$$x_3x_2 = x_2x_3 \quad \text{and} \quad x_3x_4 = \xi x_4x_3.$$

Let v be a valuation on k. We claim that v extends to D.

Consider $E = k(x_1, x_2)$ and let $\sigma: E \to E$ be defined as follows.

$$\sigma(x_1) = \varepsilon x_1 \quad \text{and} \quad \sigma(x_2) = x_2$$

σ is an automorphism of order n_1 with fixed field $E^{\sigma} = k(x_1^{n_1}, x_2)$.

Let $\tau: E \to E$ be defined as follows

$$\tau(x_1) = x_1 \quad \text{and} \quad \tau(x_2) = \eta x_2.$$

τ is an automorphism of order n_2 with fixed field $E^{\tau} = K(x_1, x_2^{n_2})$.
Now $\sigma\tau = \tau\sigma$, hence $G = \langle\sigma\rangle \times \langle\tau\rangle$ is an abelian group of order $n_1 n_2$.
Let F be the fixed field of G. Then $F = E^{\sigma} \cap E^{\tau} = K(x_1^{n_1}, x_2^{n_2})$.
Consider the extension E/F.

We have $(E:K) = (K(x_1,x_2):K(x_1,x_2^{n_2}))(K(x_1,x_2^{n_2}),K(x_1^{n_1},x_2^{n_2})) = n_1 n_2$.
Hence E/F is a Galois extension with Galois group G because there are nF-automorphisms of E.

We let $R_1 = E[x_3;\sigma]$ where multiplication is defined by

(1) $\quad x_3 a = \sigma(a)x_3$ for all $a \in E$.

We define τ^* on R_1 as follows.

$$\tau^*(a) = \tau(a) \text{ if } a \in E$$

and $$\tau^*(x_3) = \xi x_3$$

τ^* is easily seen to preserve (1), hence it is a well-defined automorphism on R_1, whence we can define

$$R = R_1[x_4,\tau^*] \text{ by } x_4 f = f^{\tau^*} x_4 \text{ where } f \in R_1$$

and R is a left skew polynomial ring whose skew field of fractions is D.
Hence by ([2] Theorem 2.3) there exist a_1, a_2 in E such that
$C = F(x_1^n a_1, x_2^n a_2)$ is the centre of D and indeed D is a generic crossed-product abelian division algebra.

Now proving that v extends to D is a simple matter using inductively Theorem (3.1.2).

§2. Some remarks on the extension of valuations in field coproducts

Let K_1, K_2 be two skew fields and consider their coproduct over a subfield K; $R = K_1 \underset{K}{\sqcup} K_2$ by ([4] theorem 5.3.2) R is a fir, hence by ([3] pp.283) it has a universal field of fractions $H = K_1 \underset{K}{\bigcirc} K_2$ called the field coproduct of K_1 and K_2 over K. Let v_1, v_2 be two real valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$.

Our main object in this section and the rest of this chapter is to investigate whether there exists a valuation ω on H such that $\omega|K_1 = v_1$ and $\omega|K_2 = v_2$.

If K_1, K_2 are K-algebras not both 2-dimensional as K-spaces then the centre of R is precisely K (see [14]). Hence applying ([6] Theorem 4.3) yields that the centre of H is K.

Throughout the rest of this chapter K_1, K_2 are not both 2-dimensional over K.

A skew field D with centre C is said to satisfy Amitsur's condition if i) C is infinite, ii) D has infinite degree over C.

Theorem (3.2.1) (P.M. Cohn). Let D be a skew field with centre C satisfying Amitsur's condition. Then any abelian valuation on K has an extension to $D \underset{C}{\circ} C\langle X\rangle = D_C\langle X\rangle$ for any set X.

Proof. ([8] Theorem (5.1)).

Theorem (3.2.2). (P.M. Cohn-Mahdavi-Hezavehi). Let K_1, K_2 be skew fields with common centre C, both satisfying Amitsur's condition, and consider their field coproduct $K_1 \underset{C}{\circ} K_2$. If v_1, v_2 are real valuations on K_1, respectively K_2, agreeing on C, then they have a common extension to $K_1 \underset{C}{\circ} K_2$.

Proof. ([8] Theorem (5.4)).

The following lemma is the first step toward lifting Amitsur's condition.

Lemma (3.2.3). Let D be a finite dimensional central division algebra over a field C. Then there exists a skew field D' containing D and having C as a centre.

Moreover $(D':C) = \infty$ and any real valuation on D can be extended to a real valuation on D'.

Proof. Consider $L = D(t)$ where t is a central indeterminate. Let $\sigma: f(t) \to f(t^2)$ be an endomorphism of L and consider the right skew polynomial ring $R = L[X;\sigma]$. It has a skew field of fractions $D' = L(X;\sigma)$. Now applying ([4] pp.61) yields that the centre of D' is precisely C since L has centre C(t) and C is fixed by σ hence $(D':C) = \infty$.

Now v extends to a Gaussian extension ω on L for which $\omega(\sum_{i=0}^{n} a_i t^i) = \min_{i=0}^{n} v(a_i)$ hence σ preserves ω and applying theorem (3.1.2) yields that ω extends to a valuation on D'.

Moreover this valuation has the same value group as v, hence it is a real valuation and the lemma is proved.

Recall that a matrix A is said to be <u>full</u> if its square, n×n say, and cannot be written as A = PQ, where P is n×r, Q is r×n and r < n. A ring homomorphism $\alpha: R \to R'$ is said to be <u>honest</u> if it preserves the full matrices over R i.e. if A is a full matrix over R then $\alpha(A)$ is a full matrix over R', where the entries of $\alpha(A)$ are the images of the entries of A.

<u>Lemma (3.2.4)</u>. Let $K_1 \subseteq K_2, K_3$ be any skew fields all containing E as a sub-skew field, then the homomorphism $K_1 \sqcup_E K_3 \to K_2 \sqcup_E K_3$ induced by the inclusion $K_1 \subseteq K_2$ is honest.

<u>Proof</u>. ([8] Lemma (5.3)).

We are now ready to lift Amitsur's condition on both theorems.

<u>Proposition (3.2.5)</u>. Let K_1, K_2 be two skew fields with a common centre C, where C is infinite. Let v_1, v_2 be real valuations on K_1, K_2 such that $v_1|C = v_2|C = v$. Then v_1, v_2 have a common extension to $K_1 \circ_C K_2$.

<u>Proof</u>. If $[K_i:C] = \infty$ $(i = 1,2)$, then we apply theorem (3.2.2). So assume W.L.O.G. that $[K_i:C] < \infty$ $(i = 1,2)$. Consider $R = K_1 \sqcup_C K_2$.

By lemma (3.2.5) there exist D_i $(i = 1,2)$ such that $D_i \supset K_i$ $(i = 1,2)$, D_i satisfies Amitsur's condition and v_i extends to a real valuation ω_i on D_i $(i = 1,2)$. Moreover D_i has centre C $(i = 1,2)$. Now the homomorphism $\alpha: K_1 \sqcup_C K_2 \to D_1 \sqcup_C D_2$ induced by the inclusion $K_i \subset D_i$ $(i = 1,2)$ is honest by a double application of lemma (3.2.4).

Hence $K_1 \underset{C}{\circ} K_2 \subseteq D_1 \underset{C}{\circ} D_2$.

Now applying Theorem (3.2.2) yields that ω_1, ω_2 have a common extension ω to $D_1 \underset{C}{\circ} D_2$, whence $\omega | K_1 \underset{C}{\circ} K_2$ is a common extension of v_1, v_2 to $K_1 \underset{K}{\circ} K_2$. N.B: If $[K_i : C] < \infty$. Then.

The condition on C is trivially satisfied since otherwise K_1, K_2 are commutative. We now lift Amitsur's condition from theorem (3.2.1).

<u>Proposition (3.2.6)</u>. Let D be a finite dimensional central division algebra over a field C. Then any abelian valuation v on D extends to a valuation on $H = D_C\langle X \rangle$ for any set X.

<u>Proof</u>. By lemma (3.2.3) we embed D in a skew field D' satisfying Amitsur's condition such that v extends to an abelian valuation ω on D'. Now lemma (3.2.4) yields that the homomorphism $D \underset{C}{\sqcup} C\langle X \rangle \to D' \underset{C}{\sqcup} C\langle X \rangle$ induced by the inclusion $D \subset D'$ is honest. Hence $H = D_C\langle X \rangle \subset D'_C\langle X \rangle$.

Applying Theorem (3.2.1) yields that ω extends to $D'_C\langle X \rangle$ and restricting ω to H yields the result.

The following corollary shows that the extension is always possible to the universal field of fractions of a free algebra.

<u>Corollary (3.2.7)</u>. Let K<X> be a free algebra where X is any set. Then any abelian valuation on K can be extended to the universal field of fractions $K\langle X \rangle$ of K<X>.

<u>Proof</u>. It suffices to observe that K is a finite dimensional central division algebra over its centre K. Hence applying proposition (3.2.6) yields the corollary.

We now consider a free algebra A = K<X> and we let F be the free group on X: then A is embedded in the group algebra KF.

Each element a of F can be written as $a = u_1^{\alpha_1} u_2^{\alpha_2} \ldots$ (possibly an infinite product) where the u_i's are basic commutators (see [16]). We order F lexicographically by the exponent of the u_i's. With this order F becomes a totally ordered group (cf. [15]). Consider K^F, the set of all functions from F to K. Then the subset of K^F consisting of elements having well ordered support is a skew field containing KF, hence containing A (see [15]). It is denoted $K((F))$ and it is called the skew field of Laurent series over K. It is shown in ([13]) that the universal skew field of fractions D of A is the subring of $K((F))$ generated by A.

The following proposition ensures the existence of a real valuation ω on D such that ω restricted to K is trivial.

Proposition (3.2.8). Let $A = K<X_1,\ldots,X_r>$ be a free algebra with universal skew field of fractions D. Then there exists always a non-trivial real valued valuation ω on H such that $\omega|K$ is trivial.

Proof. We consider the free group F on $X_1, X_2, \ldots, X_r$.

As indicated above; each element a of F can be written as $a = u_1^{\alpha_1} u_2^{\alpha_2} \ldots$ (possibly an infinite product) where the u_i's are basic commutators. Order F lexicographically and consider $H = K((F))$, then each element f of H can be written as follows. $f = \sum k_\alpha a_\alpha$ where the k_α's are in K; the a_α's are in F and have a minimal element for the ordering of F, say a_{α_j}, then $k_{\alpha_j} a_{\alpha_j}$ is called the leading term.

Now observe that $X_1,\ldots,X_r$ are the first basic commutators in F and the exponent of the commutators are in $\mathbb{Z}$. Consider the map

$$\omega: H \to \mathbb{Z} \cup \{\infty\}$$

defined as follows.

If $f = \sum k_\alpha a_\alpha$ is a non-zero element of H, then 1) $\omega(f) = \alpha_1$, where α_1 is the exponent of X_1 appearing in the leading term.

2) $\omega(0) = \infty$.

Then ω satisfies the axioms of a valuation on H.

For

V.1) is satisfied by definition.

V.2) is clear from the ordering of F.

It remains to prove V.3) i.e. $\omega(fg) = \omega(f)+\omega(g)$ for all $f,g \in H$. Consider f and assume that the leading term is $k_i X_{i_1}^{\alpha_1} X_{i_2}^{\alpha_2} \ldots X_{i_s}^{\alpha_s} u_{i_{s+1}}^{\alpha_{s+1}} \ldots$

Consider g and assume that the leading term is $k_j X_{j_1}^{\beta_1} X_{j_2}^{\beta_2} \ldots x_{j_t}^{\beta_t} u_{j_{t+1}}^{\beta_{t+1}} \ldots$

Assume further that $X_{i_1} = X_{j_1} = X_1$.

Then on computing fg we can shift X_1 (in the leading term of g) successively to the left using the commutator formulae, hence we get $X_1^{\alpha_1+\beta_1}$ in the leading term of fg, whence $\omega(fg) = \alpha_1+\beta_1$ thus $\omega(fg) = \omega(f)+\omega(g)$.

If X_1 does not appear in the leading term of f or g, then it is clear that $\omega(fg) = \omega(f)+\omega(g)$ and V.3) is satisfied. Hence ω is a valuation on H and since D is contained in H, restricting ω to D finishes the proof of the proposition.

N.B. In the ordering above X_1 is the first basic commutator, hence the exponent of X_j $(j \neq 1)$ in the leading term of an element does not define a valuation, since axiom V.2) is not satisfied in this case (it is possible to define another valuation by taking x_j as a first commutator).

If K is commutative and K_1, K_2 are purely transcendental extensions (commutative) of K, then for some cases Amitsur's condition can be lifted as the following proposition shows.

<u>Proposition (3.2.9)</u>. Let K be an infinite field and let $K_1 = K(t_i, i \in I)$ and $K_2 = K(t_j; j \in J)$ where t_i, t_j $(i \in I, j \in J)$ are central indeterminates (I,J are two sets of indices).

Let v_1, v_2 be two real valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ where v is non-trivial and assume that v_1 and v_2 are the Gaussian extensions of v.

Consider $H = K_1 \underset{K}{\circ} K_2$; then v_1, v_2 have a common extension to H.

<u>Proof</u>. We embed K_1 in $F_1 = K(t_{in}; i \in I \text{ and } n \in \mathbb{Z})$ where $t_{i0} = t_i$ and we consider the map

$$\sigma: F_1 \to F_1 \text{ defined by } \sigma(t_{in}) = t_{in+1}$$

Then σ is an automorphism of infinite order.

We now extend v_1 to ω_1 on F_1 where ω_1 is the inductive Gaussian extension, hence σ preserves ω_1.

Consider the skew function field $D_1 = F_1(X;\sigma)$ and observe that the centre of D_1 is K because K is the fixed field of σ. Moreover ω_1 extends to ϕ_1 on D_1 since σ preserves ω_1. We similarly construct D_2 with centre K and a valuation ϕ_2 extending v_2.

Now applying theorem (3.2.2) yields that ϕ_1, ϕ_2, hence v_1, v_2 have a common extension ϕ to $D_1 \underset{K}{\circ} D_2$.

But the homomorphism $K_1 \underset{K}{\sqcup} K_2 \to D_1 \underset{K}{\sqcup} D_2$ induced by the inclusion $K_i \subset D_i$ $(i = 1,2)$ is honest (see lemma (3.2.4)). Hence $K_1 \underset{K}{\circ} K_2 \subset D_1 \underset{K}{\circ} D_2$ and restricting ϕ to $K_1 \underset{K}{\circ} K_2$ yields the proposition.

§3. The counter example

Let K_1, K_2 be two skew fields with a subfield K and let $R = K_1 \underset{K}{\sqcup} K_2$ be their free product over K.

Given two real valued valuations v_1, v_2 on K_1, respectively K_2 such that $v_1|K = v_2|K = v$ where v is non-trivial.

It has been shown in ([8] theorem (4.4)) that there exists an epic R-field L containing K_1, K_2 and to which v_1, v_2 have a common extension. We shall call L throughout the rest of this chapter the associated epic R-field (associated to v_1, v_2). However L is not unique and whenever L is considered, then L means an arbitrary associated epic R-field. Theorem (3.2.2) and proposition (3.2.5) show that if K is the centre of K_1, K_2 then L can be chosen to be the universal skewfield of fractions of R i.e. $L = K_1 \underset{K}{\circ} K_2$.

In ([8]) P.M. Cohn and Mahdavi-Hezavehi have conjectured that v_1, v_2 have always a common extension to $K_1 \underset{K}{\circ} K_2$.

Our aim is to prove that this conjecture is false and we shall give a counter example.

Before we proceed to our main theorem in this section, we shall need a couple of lemmas.

Lemma (3.3.1). Let E/K be a finite cyclic Galois extension and v a discrete rank 1 valuation on K such that v has ramification index e = 1 in E and v is indecomposed in E.

Then there is a cyclic division algebra D over K to which v can be extended.

Proof. Assume W.L.O.G. that v is normalized and let a be a uniformizer in K, i.e. a is such that v(a) = 1. Let σ be the generator of the Galois group of E/K and assume that σ has order n i.e. $\sigma^n = 1$.

Consider the cyclic algebra D = (E/K; σ, a).

We claim that D is a division algebra. For let t be the exponent of D, then applying ([17] Corollary (30.7)) yields that there exists an $c \in E^*$ such that

$$a^t = c.c^{\sigma}\ldots c^{\sigma^{n-1}}$$

Hence (1) $v(a^t) = tv(a) = t = v(c)+v(c^{\sigma})+\ldots+v(c^{\sigma^{n-1}}) = nv(c)$ because σ preserves the valuation v (v is indecomposed in E). (1) implies $v(c) = t/n$, hence t = n since v has e = 1 in E and t divides n, whence applying ([17] pp.261) yields that D is a division algebra.

Now applying corollary (2.1.3) yields that v extends to a valuation on D and the lemma is proved.

The second lemma describes subalgebras of a central division algebra.

Lemma (3.3.2). Let H be a central division algebra, not necessarily finite dimensional over K, and assume that H contains a field F which is a cyclic extension of K.

Then H contains a cyclic division algebra whose centre is a simple extension of K.

Proof. Let G = $\langle\sigma\rangle$ be the Galois group of F/K and assume that G has order n, i.e. $\sigma^n = 1$. Then, by the Skolem-Noether theorem σ is induced by an inner automorphism of H, hence there exists an element t in H such that

$$at = ta^{\sigma}$$
$$at^2 = t^2a^{\sigma^2}$$
$$\dots$$
$$\dots$$
$$at^n = t^na$$

We now consider $E_1 = K(t^n)$ and $E_2 = F(t^n)$, hence E_2/E_1 is a cyclic Galois extension with Galois group G.

Now consider $D = (E_2/E_1;\ \sigma,\ t^n)$.

D is a cyclic algebra which is a skew field since it has no zero-divisors and the lemma is proved.

We are now ready for the main theorem which we will use to construct the counter example.

Theorem (3.3.3). Let K_1/K be a finite cyclic extension and K_2 a skew field whose centre is K.

Let v_1, v_2 be two real valued valuations on K_1, respectively K_2 such that $v_1|K = v_2|K = v$ where v is discrete of rank 1. Assume that the value group of v_1 is equal to the value group of v. Then v_1, v_2 have a common extension to $H = K_1 \underset{K}{\circ} K_2$ iff v_1 is the only valuation on K_1 extending v i.e. iff v is indecomposed in K_1.

Proof. 1) The condition is sufficient.

If v is indecomposed in K_1, then by lemma (3.3.1) we can find a division algebra K_3 which is cyclic over K_1/K and to which v (hence v_1) can be extended. Let v_3 be the extension of v to K_3.

By lemma (3.2.4) the homomorphism $K_1 \underset{K}{\sqcup} K_2 \to K_3 \underset{K}{\sqcup} K_2$ induced by the inclusion $K_1 \subset K_3$ is honest, hence $H = K_1 \underset{K}{\circ} K_2 \subset K_3 \underset{K}{\circ} K_2$.

Now by the construction of K_3, K is the centre of K_3 and since K is the centre of K_2, applying proposition (3.2.5) yields that v_3, v_2 have a common extension to $K_3 \underset{K}{\circ} K_2$. Restricting ω to H yields the required common extension of v_1, v_2 to H, and the condition is sufficient.

2) The condition is necessary.

Assume that v_1, v_2 have a common extension ω to $H = K_1 \underset{K}{O} K_2$. By the remark in the beginning of §2 the centre of H is K. Hence applying lemma (3.3.2) yields that H contains a cyclic division algebra $D = (E/F;\ \sigma,\ a)$ where σ is the generator of the Galois group of K_1/K. $E = K_1(a)$, $F = K(a)$ where $a = t^n$ (t being the element of H defining the inner automorphism of H inducing σ).

We call ω_D the restriction of ω to D.

We call ω_F the restriction of ω to F.

Hence ω_D is the extension of ω_F to D, whence ω_F is indecomposed in E. Let $\tilde{F}$ be the completion of F relative to ω_F, then it is easily seen that $\tilde{E} = E \underset{F}{\otimes} \tilde{F}$ is a field (from the commutative theory).

Let $\tilde{K}$ be the completion of K relative to v, then $\tilde{K} \subset \tilde{F}$.

Consider the following composition map

$$K_1 \underset{K}{\otimes} \tilde{K} \xrightarrow{i} K_1 \underset{K}{\otimes} \tilde{F} \xrightarrow{\alpha} (K_1 \underset{K}{\otimes} F) \underset{F}{\otimes} \tilde{F} \xrightarrow{\beta} E \underset{F}{\otimes} \tilde{F}$$

i is clearly an embedding and α, β are isomorphisms. Hence $K_1 \underset{K}{\otimes} \tilde{K}$ is embedded in $E \underset{F}{\otimes} \tilde{F}$, whence $K_1 \underset{K}{\otimes} \tilde{K}$ is a field. Hence v is indecomposed in K_1 because it is well known (from the commutative theory of valautions) that when K_1/K is Galois $K_1 \underset{K}{\otimes} \tilde{K} \cong K_1^{(1)} \times \ldots \times K_1^{(r)}$ (direct product) where r is the number of valuations extending v to K_1 and $K_1^{(1)}, \ldots, K_r^{(r)}$ are the relative completions of K_1.

We now construct the counter example.

Example (3.3.4). Consider $K = Q(\sqrt{2})$, the cyclic extension of degree 2. Put $F = Q(t)$ and let σ: $t \to t^2$, then σ is an endomorphism of infinite order on F, hence we can consider the skew polynomial ring $R = F[X;\sigma]$. It has a skew field of fractions $D = F(X;\sigma)$. The centre of D is Q because σ has infinite order and Q is the subfield of F fixed by σ.

Let ν_7 be the 7-adic valuation on Q.

We consider the equation $X^2-2 = 0$ in the residue class field F_7. This equation has two simple zeros in F_7, hence in Q_7 (the field of 7-adic numbers), whence ν_7 splits into two valuations in K, ν_7' and ν_7''. Thus ν_7 decomposes in K.

Let v be the Gaussian extension of ν_7 to F so that $v(t) = 0$, hence σ preserves v, whence v extends to a Gaussian extension ω on D.

Now consider ν_7' and ω which are real valuations on K, respectively D. ν_7 is unramified in K, i.e. ν_7 and ν_7' have the same value group and applying the theorem yields that ν_7' and ω have no common extension to $H = K \underset{Q}{\circ} D$ (since otherwise ν_7 becomes indecomposed in K which is not the case).

§4. <u>On the centre of the associated epic R-field</u>

Let K_1, K_2 be two skew fields with a common subfield K, put $R = K_1 \underset{K}{\sqcup} K_2$ and $H = K_1 \underset{K}{\circ} K_2$.

Let v_1, v_2 be two real valued valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ and let L be an associated epic R-field. Our aim in this section is to study the centre of L and generalize Theorem (3.3.3) so as to show that in general v_1, v_2 have no common extension to any skew field of fractions of R and in particular that the homomorphism $R \to L$ is not even an embedding.

The following lemma is the key element for our results in this section.

<u>Lemma (3.4.1).</u> Let D be any division algebra over a field K and let F be a cyclic extension of K contained in D. Put $C = Z(D)$ and assume that $F \cap C = K$.

Then D contains a cyclic algebra whose centre is a simple extension of C.

<u>Proof.</u> Let σ be the generator of $\mathrm{Gal}(F/K)$ and assume that σ has order n.

Since C is the centre of D and F is commutative CF is a field and by Galois theory CF/C is a cyclic extension with Galois group isomorphic to $Gal(F/C \cap F)$ i.e. to $<\sigma>$.

By the Skolem-Noether theorem σ is induced by an inner automorphism of D; hence $\exists\, t \in D$ such that

$$at = ta^{\sigma}$$
$$at^2 = t^2a^{\sigma^2} \quad \text{for all } a \in CF$$
$$\vdots$$
$$at^n = t^na$$

Put $E = CF(t^n)$ and $\Omega = C(t^n)$, then by Galois theory E/Ω is a Galois extension with Galois group isomorphic to $<\sigma>$. Consider $H = (E/\Omega;\ \sigma,\ t^n)$; then H is a cyclic division algebra contained in D and whose centre is Ω.

The following theorem describes the centre of L and generalizes the theorem (3.3.3) so as to show that v_1, v_2 have no common extension to any skew field of fractions of R.

Theorem (3.4.2). Let K_1/K be a cyclic Galois extension and K_2 a skew field with centre K, put $R = K_1 \underset{K}{\sqcup} K_2$.

Let v_1, v_2 be two real valued valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ and let E be the decomposition field of v_1. Then the centre C of any associated epic R-field L contains E.

Moreover if $E \supset K$, then v_1, v_2 have no common extension to any skew field of fractions of R.

Proof. Assume that $C \cap K_1 = \Omega$ and that $\Omega \not\supseteq E$. Since L is a division algebra over Ω and K_1/Ω is cyclic, we can apply lemma (3.4.1) to obtain a cyclic algebra D in L such that $D = (C_1/C_2;\ \sigma,\ t^n)$ where $C_1 = CK_1(t^n)$, $C_2 = C(t^n)$ and σ generates $Gal(C_1/C_2)$ (note that $Gal(C_1/C_2) \cong Gal(K_1/C \cap K_1)$. t induces σ and n = order of σ.

Let ω be the extension of v_1, v_2 to L

Let ω_D be the restriction of ω to D.

" ω_{C_1} be the restriction of ω to C_1.

" ω_{C_2} " " " " ω " C_2.

Hence ω_{C_2} extends v to C_1.

Now let ω_C be the restriction of ω_{C_2} to C.

and ω_Ω " " " " ω_{C_2} to ω.

Thus ω_Ω extends v to Ω (because $\Omega \supseteq K$).

Now since $\Omega \not\supseteq E$, applying ([10] theorem 15.7) yields that ω_Ω decomposes in K_1, hence if $\tilde{\Omega}$ is the completion of Ω relative to ω_Ω, then $K_1 \otimes_\Omega \tilde{\Omega}$ has zero divisors.

We claim that $K_1 \otimes \tilde{\Omega}$ is a field. For consider the following composition map

$$K_1 \otimes_\Omega \tilde{\Omega} \xrightarrow{i} K_1 \otimes_\Omega \tilde{C} \xrightarrow{\alpha} (K_1 \otimes_\Omega C) \otimes_C \tilde{C} \xrightarrow{\beta} K_1C \otimes_C \tilde{C} \xrightarrow{j} K_1C \otimes_C \widetilde{C(t^n)}$$

$$\downarrow \gamma$$

$$K_1C(t^n) \otimes_{C(t^n)} \widetilde{C(t^n)} \xleftarrow{\delta} (K_1C \otimes_C C(t^n)) \otimes_{C(t^n)} \widetilde{C(t^n)}$$

Note that $K_1C(t^n) = C_1$, $C(t^n) = C_2$ and $K_1 \otimes_\Omega C \cong K_1C$. Note further that $\tilde{C}$ is the completion of C relative to ω_C and $C(t^n)$ is the completion of C_2 relative to ω_{C_2}.

Hence we deduce easily that i,j are embeddings and α, β, γ, δ are isomorphisms. This yields that the composition map is an embedding Thus if $K_1 \otimes_\Omega \tilde{\Omega}$ has zero divisors, then $C_1 \otimes_{C_2} \tilde{C}_2 = K_1C(t^n) \otimes_{C(t^n)} \widetilde{C(t^n)}$ has zero difisors which is a contradiction because ω_{C_2} on C_2 has a unique extension ω_{C_1} to C_1 and the reason is that ω_{C_2} is extendable to D, see (theorem 0.1.1).

So $K_1 \otimes_\Omega \tilde{\Omega}$ has no zero-divisors which is a contradiction; hence $C \cap K_1 \supseteq E$, whence $C \supseteq E$.

For the second part of the theorem, it suffices to observe that if H is any skew field of fractions of R with centre C, then $C \cap K_1 = K$ because K centralizes H and the centre of R is precisely K.

Hence repeating the same argument as above shows that if $E \supset K$, then v_1, v_2 have no common extension to H.

N.B. The centre of L surely contains K since K is the centre of $K_1 \underset{K}{\circ} K_2$ and since there is a specialization from $K_1 \underset{K}{\circ} K_2$ to L i.e. $L \cong T/\mathcal{P}$ where T is a local subring of $K_1 \underset{K}{\circ} K_2$ and $\mathcal{P}$ is its maximal ideal.

As a corollary we have

Corollary (3.4.3). Let K_1/K be a finite abelian extension and K_2 a skew field with centre K. Put $R = K_1 \underset{K}{\sqcup} K_2$.

Let v_1, v_2 be real valued valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ and let E be the decomposition field of v_1, then the centre of any (associated epic R-field) L contains E.

Proof. Let $G = \langle\sigma_1\rangle \times \ldots. \times \langle\sigma_r\rangle$ be the Galois group of K_1/K.

Put $K_1^i = \{x \in K_1;\ \sigma_i(x) = x\}$.

Let $K_1^{(i)} = \bigcap_{j \neq i} K_1^j$ then $K_1^{(i)}/K$ is a cyclic Galois with Galois group isomorphic to $\langle\sigma_i\rangle$ and we have the following decomposition.

$$K_1 \cong K_1^{(1)} \underset{K}{\otimes} K_1^{(2)} \otimes \ldots \underset{K}{\otimes} K_1^{(r)}.$$

Let $v_1^{(i)}$ be the restriction of v_1 to $K_1^{(i)}$ $(i = 1,\ldots,r)$ and $E^{(i)}$ the decomposition field of $v_1^{(i)}$. Then we have the following

$$E \cong E^{(1)} \underset{K}{\otimes} E^{(2)} \underset{K}{\otimes} \ldots \underset{K}{\otimes} E^{(r)}$$

(it suffices to compare dimensions over K).

Let C be the centre of L.

Then by a similar proof as in the theorem we show that $K_1^{(i)} \cap C \supseteq E^{(i)}$.

Hence

$$C \supseteq E^{(i)} \quad \text{for} \quad i = 1,\ldots,r.$$

Whence the homomorphism $f: E^{(1)} \otimes \ldots \otimes E^{(r)} \rightarrow C$ defined by

$$f(\sum e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}) = \sum e_{i_1} e_{i_2} \ldots e_{i_r}$$

is injective because $E^{(1)} \otimes \ldots \otimes E^{(r)}$ is a field and the corollary is proved.

Corollary (3.4.4), Let K_i/K be a cyclic extension of prime power degree (different primes) and let v_i be a real valued valuation on K_i $(i = 1,2)$ such that $v_1|K = v_2|K = v$ and assume that v splits in K_i $(i = 1,2)$. Then there is one and only one associated epic R-field given by $K_1 \otimes_K K_2$.

Proof. We know that there is at least one, say L; first observe that $K_1 \otimes_K K_2$ is a field, hence an epic R-field. Now let C be the centre of L. By the theorem $C \supseteq K_i$ $(i = 1,2)$, hence $C \supseteq K_1 \otimes_K K_2$, whence $L = K_1 \otimes_K K_2$, because L is generated by an image of R. Now any other associated epic R-field is equal to $K_1 \otimes_K K_2$, hence L is unique.

§5. Generalizations

Let K_1, K_2 be two skew fields with centres C_1, C_2 and let K be a common subfield contained in both C_1 and C_2.

Let v_1, v_2 be real valued valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$.

We aim to generalize our previous results to the case where $K \subset C_i$ $(i = 1,2)$. Throughout this section, all skew fields have infinite centres.

We first need a lemma.

Lemma (3.5.1). Let D be a skew field with centre C and let K be a subfield of C.

Let ω be a real valued valuation on D such that $\omega|K = v$. Assume that ω is the unique real valued valuation on D extending v.

Assume further that D admits an endomorphism σ whose fixed field intersected with C is exactly K. Then D can be embedded in a skew-field

D' whose centre is exactly K. Moreover ω extends to a real valued valution on D'.

Proof. Assume first that no power of σ is inner.

Then we consider the right skew polynomial ring $R = D[X; \sigma]$. This is a right ore domain, hence it has a skew field of fractions $D' = D(X; \sigma)$.

Applying ([4] pp.61) yields that the centre of D' is K.

Now since ω is the only real valued valuation extending v to D σ must preserve the valuation, hence applying theorem (3.1.2) yields that extends to a Gaussian extension on D'. Suppose now that σ has an inner power. We put $L = D(t)$ and we extend σ to L by $\sigma(t) = t^2$, hence σ is an endomorphism of L with no inner power.

Now consider $D' = L(y, \sigma)$; by ([4] lemma 6.3.5) the centre of L is C(t), hence applying ([4] pp.61) yields that the centre of D' is K. Moreover ω extends to a Gaussian extension ω_L on L, hence σ preserves ω_L, whence ω_L extends to a Gaussian extension on D' which is real valued since it has the same value group as ω.

We now have the first generalization of theorem (3.2.2).

Theorem (3.5.2). Let K_1, K_2 be two skew fields with centres C_1, C_2 and let K be a common subfield such that $K \subseteq C_i$ $(i = 1,2)$. Let v_1, v_2 be real valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ and such that v_i is the only real valued valuation extending v to K_i $(i=1,2)$.

Assume that K_i admits an endomorphism σ_i whose fixed field intersected with C_i is precisely K $(i = 1,2)$.

Then v_1, v_2 have a common extension to $H = K_1 \underset{K}{\circ} K_2$.

Proof. By lemma (3.4.1) each K_i is contained in a skew field K_i whose centre is precisely K and to which v_i has a Gaussian extension ω_i $(i = 1,2)$.

Now consider the following map

$f_i: K_1 \sqcup_K K_2 \longrightarrow K_1' \sqcup_K K_2'$ induced by the inclusion $K_i \subset K_i'$ $(i = 1,2)$.

By lemma (3.2.4) f is honest, hence $K_1 \underset{K}{\circ} K_2 \subset K_1' \underset{K}{\circ} K_2'$. Now applying theorem (3.2.2) yields that ω_1, ω_2 have a common extension to $K_1' \underset{K}{\circ} K_2'$ and restricting ω to H yields the required extension.

Proposition (3.5.3). Let D be a skew field with centre C and let K be a subfield of C. Assume that D admits an endomorphism whose fixed field intersected with C is precisely K. If V is a real valued valuation on D such that v is the only one on D extending its restriction to K. Then for any set X, v extends to $H = D_K\langle X\rangle$.

Proof. By Lemma (3.4.1) D is contained in D' whose centre is K and to which v extends to a real valued valuation ω. Now the following homomorphism

$$D \underset{K}{\sqcup} K\langle X\rangle \longrightarrow D' \underset{K}{\sqcup} K\langle X\rangle$$

is honest (see lemma 3.2.4). Hence $D_K\langle X\rangle \subseteq D'_K\langle X\rangle$. Applying theorem (3.2.1) yields that ω extends to a valuation ϕ on $D_K(X)$ and restricting ϕ to $D_K\langle X\rangle$ yields the required extension.

Recall that a skew field extension D/K is Galois if K is the fixed field of a group of automorphisms of D. As a corollary we have an application to theorem (3.5.2).

Corollary (3.5.4). Let K_1, K_2 be two skew fields with centres C_1, C_2. Let K be a common subfield of K_1, K_2 such that C_i/K is a finite commutative cyclic extension $(i = 1,2)$ and assume that K_i/K is (a not necessarily commutative) finite Galois extension with group G. Let v_1, v_2 be real valued valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ and v_i is the only valuation on K_i extending v $(i = 1,2)$.

Then v_1, v_2 have a common extension to $H = K_1 \underset{K}{\circ} K_2$.

Proof. Let σ_i be the generator of $Gal(C_i/K)$, then applying ([4] Proposition 3.3.3) yields that σ_i is induced from an automorphism τ_i of

G, hence the fixed field of τ_i intersected with C_i is precisely K (i = 1,2).

So we are in the setting of the theorem, hence applying the theorem yields the corollary.

We shall generalise proposition (3.5.3) by assuming that D admits a family of endomorphism whose fixed field intersected with C is K. The following lemma is the key to our generalization.

Lemma (3.5.5). Let D be a skew field with centre C and let K be a subfield of C.

Given a real valued valuation ω of D such that $\omega|K = v$ and assume that ω is the only real valued valuation on D extending v.

Assume that D admits a family of endomorphisms whose fixed field intersected with C is K. Then D can be naturally embedded in a skew field L whose centre F intersected with D is K and to which ω extends to a real valued valuation.

Proof. Let $\{\sigma_i\}_{i\in I}$ be the given family of endomorphisms of D and let $\{F_i\}_{i\in I}$ be defined as follows.

$$F_i = \{x \in D;\ \sigma_i(x) = x\}$$

Put $C_i = F_i \cap C$ and consider $\{C_i\}_{i\in I}$.

The hypothesis yields that $K = (\bigcap_{i\in I} F_i) \cap C = \bigcap_{i\in I} (F_i \cap C) = \bigcap_{i\in I} C_i$. We assume first that no power of σ_i ($i \in I$) is inner. Consider $L_i = D(X_i;\ \sigma_i)$ the skew function fields ($i \in I$): applying ([4] pp.61) yields that the centre of L_i is precisely C_i, and by the proof of lemma (3.5.1) ω extends to a real valued valuation ω_i on L_i because σ_i preserves ω.

We now consider $R_{12} = L_1 \sqcup_D L_2$ (the free product of L_1 and L_2 over D).

Applying ([8] theorem 4.4) yields that there exists a skew field L_{12} to which ω_1, ω_2 (hence ω) have a common extension ω_{12}.

Let $R_{123} = L_{12} \underset{D}{\sqcup} L_3$.

Applying the same theorem yields the existence of a skew field L_{123} to which ω_{12}, ω_3 have a common extension. Inductively we construct a skew field L containing all the L_i and to which the ω_i's have a common extension, say ϕ.

Now let F be the centre of L.

We claim that $F \cap D = K$. For $F \cap L_i \subseteq C_i$ for all $i \in I$ because C_i is the centre of L_i. Hence $F \cap D \subseteq C_i$ for all i because $D \subseteq L_i$ $(i \in I)$ whence $F \cap D \subseteq \bigcap C_i = K$. Thus $F \cap D = K$ since K is easily proved (by induction) to be a central subfield of L and the lemma is proved in this case.

If some of the $\{\sigma_i\}_{i \in I}$ have inner power we put $D' = D(t)$ and extend each σ_i to D' by $\sigma_i(t) = t^2$, hence we have a family of endomorphisms with no inner power. We proceed exactly as above bearing in mind that if $E = D(t)(X; \sigma_i)$, then the centre of E is $C(t) \cap F_i$ where $F_i = \{X \in D;\ \sigma_i(X) = X\}$. i.e. the centre of E is C_i since $t \notin F_i$ for all $i \in I$.

As a first consequence of this lemma we have the following important generalization of theorem (3.2.1).

<u>Theorem (3.5.6)</u>. Let D be a skew field with centre C and let K be a subfield of C.

Assume that D has a family of endomorphisms whose fixed field intersected with C is K.

Let ω be a real valuation on D such that $\omega|K = v$ and assume that ω is the only real valued valuation on D extending v, then ω extends to $H = D_K\langle X\rangle$ where X is any set.

<u>Proof</u>. By the lemma there exists a skew field L satisfying the following

1) $D \subset L$

2) ω extends to a real valued valuation ϕ on L

3) if F is the centre of L, then $F \cap D = K$ and $(L:F) = \infty$.

Now consider $H' = L_F\langle X\rangle$.

Applying ([4] lemma 6.3.6) yields that $D_K\langle X\rangle \subseteq L_F\langle X\rangle$. Now by theorem (3.2.1) ϕ extends to a valuation ϕ' on H', hence restricting ϕ' to H yields the required result.

Whether ϕ is real valued is not known. However at this stage we shall propose the following Conjecture: Keeping the hypothesis and notation of theorem (3.5.6) and let ω be the real valued valuation on D, then there exists a real valued valuation on $D_K\langle X\rangle$ extending ω.

The conjecture is certainly true if D = K and X is reduced to one element in the case $D_K\langle X\rangle = K\langle X\rangle$ and it suffices to consider the Gaussian extension of ω. Throughout the rest of this section we assume that the conjecture is true. First we have a generalization of theorem (3.5.2).

Theorem (3.5.7). Let K_1, K_2 be two skew fields with centres C_1, C_2 and let K be a common subfield such that $K \subseteq C_i$ $(i = 1,2)$. Let v_1, v_2 be real valued valuations on K_1, K_2 such that $v_1|K = v_2|K = v$ and assume that v_i is the only real valuation on K_i $(i = 1,2)$ extending v.

Assume that K_i has a family of endomorphisms whose fixed field intersected with C_i is K $(i = 1,2)$.

Then v_1, v_2 have a common extension to $H = K_1 \underset{K}{\circ} K_2$.

Proof. By the conjecture v_i extends to a real valued valuation ω_i on ${K_i}_K\langle X\rangle$ for any set X $(i = 1,2)$.

Consider the homomorphism

$$K_1 \underset{K}{\sqcup} K_2 \longrightarrow {K_1}_K(X) \underset{K}{\sqcup} {K_2}_K(Y) \text{ induced by}$$

the inclusion $K_1 \subset {K_1}_K(X)$ and $K_2 \subset {K_2}_K\langle Y\rangle$. By (lemma 3.2.4) this map is honest, hence $K_1 \underset{K}{\circ} K_2 \subset {K_1}_K\langle X\rangle \underset{K}{\circ} {K_2}_K\langle Y\rangle$.

Note that the centre of ${K_1}_K\langle X\rangle$ is K and the centre of ${K_2}_K\langle Y\rangle$ is K. Hence theorem (3.2.2) yields that ω_1, ω_2 have a common extension ϕ to

$K_{1_K}\langle X\rangle \circ K_{2_K}\langle Y\rangle$ and restricting ϕ to $K_1 \underset{K}{\circ} K_2$ yields the result.

As an application we have the following consequence.

Corollary (3.5.8). Let K_1, K_2 be two skew fields with centres C_1, C_2 and a common subfield K such that $K \subseteq C_i$ $(i = 1,2)$. Assume that K_i/K is a (not necessarily commutative) finite Galois extension $(i = 1,2)$. Let v_1, v_2 be real valuations on K_1 and K_2 respectively such that $v_1|K = v_2|K = v$ and v_i is the only real valuation on K_i extending v $(i = 1,2)$. Then v_1, v_2 have a common extension to $K_1 \underset{K}{\circ} K_2$.

Proof. We note first that C_i/K is a Galois extension, it suffices to apply ([4] theorem 3.3.5 (ii)). $(i = 1,2)$. Let G_i be the Galois group of C_i/K $(i = 1,2)$. Then applying ([4] proposition 3.3.3) yields that each $\sigma_j \in G_i$ is induced from the Galois group S_i of K_i/K $(i = 1,2)$. Let $\tau_j \in S_i$ be the extension of σ_j to K_i, then it is easily seen that the fixed field of the τ_j's intersected with C_i is K $(i = 1,2)$. Hence we are in the setting of theorem (3.5.7) and the Corollary is proved by direct application of the theorem.

The rest of this section is devoted to studying the case where K contains the centre C_1, C_2 of K_1, K_2 and where K is not necessarily commutative.

In fact the study of this case arises from the generalization of the specialization lemma.

Generalization of the specialization lemma (3.5.9). (P.M. Cohn)

Let D be s skew field whose centre C is infinite and let E be a subfield of D such that

(1) $E'' = E$ where E'' is the bicentralizer of E

(2) $E c E'$ is infinite dimensional as E-space for any $c \in E^*$ where E' is the centralizer of E in D.

Then any full matrix A over $R = D_E\langle X\rangle$ is non-singular for some set of values of X in E' where X is any set.

Proof. ([7]).

The object of the rest of this section is to see whether the generalization of the specialization lemma entails the generalization of theorems (3.2.1) and (3.2.2).

For simplicity we shall say that (D,E) satisfies (G,A,C) (i.e. generalized Amitsur's condition whenever (D,E) satisfies the hypothesis of lemma (3.5.9).

The basic lemmas for generalization are the following.

Lemma (3.5.10). Let (D,E) be skew fields satisfying (G.A.C.), then any full matrix A over $D_E\langle X,X^{-1}\rangle$ is non-singular for some set of values of X in D.

Proof. Similar to ([8] theorem 3.1).

Lemma (3.5.11). Let (D,E) be skew fields satisfying (G.A.C.) and consider $R = D \underset{E}{\sqcup} D$.

Given any full matrix A over R, there exists an inner automorphism α of D such that $A_{\alpha'}$ is non-singular, where α' is the homomorphism induced by $(1,\alpha)$ on R.

Proof. Similar to ([8] theorem 3.2).

Note that α' is induced from $(1,\alpha)$ by the defining relations of R where 1 is the identity map on K and $\alpha: R \to K$.

We now apply these lemmas to study generalizations.

Theorem (3.5.12). Let (D,E) be skew fields satisfying (G,A,C), then any abelian valuation v on D has an extension ω to $D_E\langle X\rangle$ for any set X.

Proof. Similar to theorem 3.2.1.

Theorem (3.5.13). Let (D,E) be skew fields satisfying (G.A.C.), then for any abelian valuation v on K, there is a valuation on the field coproduct $K \underset{E}{\circ} K$ extending v (on both factors).

Proof. Similar to ([8] theorem 5.2).

Remark. Let K_1, K_2 be skew fields having E as common subfield such that (K_1,E) and (K_2,E) satisfying (G.A.C.).

Consider $H = K_1 \underset{E}{O} K_2$.

Let v_1, v_2 be real valued on K_1 and K_2 respectively such that $v_1|E = v_2|E = v$.

It is an open question whether v_1, v_2 have a commen extension to H.

Let $R = K_1 \underset{E}{\sqcup} K_2$ and let D be an associated epic R-field. If (D,E) satisfy (G.A.C.) then v_1, v_2 have a common extension to H. For the homomorphism

$$K_1 \underset{E}{\sqcup} K_2 \longrightarrow D \underset{E}{\sqcup} D \text{ induced by } K_i \subseteq D \text{ is honest}$$

hence $K_1 \underset{E}{O} K_2 \subseteq D \underset{E}{O} D$ and applying theorem (3.5.13) yields the result.

However such a strong condition [(D,E) satisfy G.A.C.] seems unlikely to be satisfied by D.

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