Simple artinian rings, hereditary rings and skew fields

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Abstract

In this dissertation, Cohn's methods for constructing skew fields from firs are extended to methods for constructing simple artinian rings from hereditary rings; the flexibility this gives is used in order to prove results about the skew field coproduct, and other skew fields originally investigated by Cohn.

Part I is devoted to developing the techniques needed; this has two main themes; the first is a detailed study of the finitely generated projective modules over an hereditary ring; the second is an investigation of the ring construction, universal localisation. The construction of universal homomorphisms from suitable hereditary rings to simple artinian rings leads to the simple artinian coproduct with amalgamation of simple artinian rings, which is the natural generalisation of the skew field coproduct of Cohn.

Part II is a detailed study of the skew fields and simple artinian rings that are constructed in this way from firs or more generally from hereditary rings. The finite dimensional division subalgebras are classified (apart from the case of the skew field coproduct of two quadratic extensions of a skew field), the transcendence degree of the commutative subfields is bounded, and the centralisers of transcendental elements is studied in special cases. In addition, there are isolated results on the isomorphism classes of skew field coproducts.

Part III is distinct from the rest of the dissertation; it consists of a description of generic solutions to the partial splitting for finite dimensional central simple algebras. These are twisted forms of grassmannian varieties; this is used to show that over number fields, these varieties satisfy the Hasse principle.

Acknowledgements

First of all, the author would like to point out that he fails to mention in the introduction that the beginning of section 12 is partly standard and partly due to Bergman and Dicks. Theorems 12.1, 12.2 and 12.3 are well-known: theorems 12.4, 12.5, 12.6 and 12.7 occur in Bergman, Dicks (8).

I owe a great deal to my supervisor, Professor Cohn, who took the first steps in developing this subject and whose interest and optimism have been a constant example.

During many conversations with Warren Dicks, the results of this dissertation have developed. There are innumerable points in it that have benefited from his insight, exactitude and lack of belief.

There are many personal debts that I owe; in particular, I should like to thank Jim and Anne Roseblade, for their encouragement and hospitality on many occasions.

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Contents

Introduction

This dissertation follows on from the work of Cohn, giving a general theory of epimorphisms to skew fields, and Bergman's application of this theory to epimorphisms from hereditary rings to skew fields. At the beginning of the '70s, Cohn was able to characterise epimorphisms from a ring to a skew field in terms of prime matrix ideals. This applied particularly well to firs and Bergman extended this to hereditary rings. Cohn was able to construct the skew field coproduct with amalgamation: given a family of skew fields $\{E_i : i \in I\}$ with specified embeddings $F \hookrightarrow E_i$, where F is a skew field, there is a universal skew field generated by copies of E_i, subject to $E_i \cap E_j = F$.

There are two aspects to this dissertation: the first is to extend the general theory in order to construct the simple artinian coproduct with amalgamation; the second is to investigate the skew fields this general theory allows us to construct.

Correspondingly, this dissertation falls naturally into two halves; in part 1, we develop a theory of epimorphisms to simple artinian rings, which applies particularly well to hereditary rings. It allows us to construct the simple artinian coproduct of simple artinian rings. In order to do this, we have to develop a suitable notion of the size of a projective module; the correct idea is taken from the theory of von Neumann regular ring,and so, as a side issue, we are able to develop a theory of homomorphisms from hereditary rings to von Neumann regular rings. In part II, we study the skew fields and simple artinian rings we construct in part I. We study the finite-dimensional division subalgebras, the transcendence degree of commutative subfields, the centralisers of suitable elements and finally, we discuss the question of isomorphisms between apparently different skew fields.

We need to develop a certain amount of machinery in part I in order to study epimorphisms to simple artinian rings. Chapter 1 consists of a miscellany of results on the connection between hereditary rings and rank functions, which give the notion of size we need later. The basic idea is to find analogies to those results that hold for von Neumann regular rings and, using a different analogy, those that hold for firs with respect to the usual rank on free modules. Lemma 1.1 and theorem 1.8 are due to Bergman and theorem 1.2 is due to Goodearl.

We require the coproduct construction in much of the detail of our later proofs and chapters 2 and 3 are devoted to the study of this construction. Chapter 2 gives a summary of Bergman's coproduct theorems and then his applications of this theory to the study of various universal constructions such as adjoining universal maps between finitely generated projective modules or universal isomorphisms between finitely generated projective modules. There is little due to the author here apart from some examples and also a discussion at the end of the chapter of these universal constructions in the category of commutative rings, which generalises the work of Gabel. Chapter 3 investigates the behaviour of rank functions under the coproduct construction in preparation for the work of chapter 5; on the way we give an application of this theory to accessibility in finitely generated groups. We simplify the proof of a theorem, due to P.A. Linnell (theorem 3.5). The rest of the chapter is original work.

Chapter 4 summarises what is known about the construction of universal localisation. This is a generalisation of Ore localisation and it is the way that we shall construct all the epimorphisms in the following chapter. All the results stated are new; however, the proofs of theorem 4.1 and 4.2 follow, with little alteration, the proofs of special cases already in the literature.

Chapter 5 contains the main results of this first part. We summarise the results of Cohn and Bergman in theorems 5.1, 5.2 and 5.3 on the subject of epimorphisms to skew fields; then we build on this in order to study the universal localisation of an hereditary ring at a rank function to $\frac{1}{n}\mathbb{Z}$, which we are able to show is always a perfect ring in the sense of Bass (3). This result allows us to construct the simple artinian coproduct with amalgamation which generalises Cohn's construction of the skew field coproduct.

In the last chapter of part I we apply the methods we have developed in order to show that any rank function on an hereditary ring arises from a homomorphism to a von Neumann regular ring with rank function. This connects the notion of rank function, where we have used it, to the place from which we originally took it.

Part II is a study of the skew field coproduct. At the beginning of the seventies, Cohn showed that given a couple of skew fields, E_1 and E_2, with common skew subfield F, there is a universal construction of a skew field generated by copies of E_1 and E_2, whose intersection is F. This universal skew field is the skew field coproduct of E_1 and E_2 amalgamating F, and it written as $E_1 \underset{F}{\circ} E_2$. Little was known about these skew fields; it was only recently that the centre of the skew field coproduct was calculated and the question of their isomorphism or non-isomorphism seemed unapproachable, so it seemed reasonable to take the analogy of the group coproduct with amalgamation as a source of interesting questions in the hope that the investigation resulting from this would give some insight into the structure of these skew field coproducts. In the case of the group coproduct with amalgamation, we know that the finite subgroups of such a group are conjugate to a subgroup of one of the factors. This suggested that it would be of some interest to investigate the finite-dimensional division k-algebras lying in $E_1 \underset{F}{\circ} E_2$, where k is a central subfield of F, central in E_1 and E_2.

This question gave rise to the earlier sections of Part II, which present a complete answer to this question except for one very special instance. The method developed is applicable in a much wider context and we present it in this greater generality, illustrating our results for the skew field coproduct by examples and consequences.

We continue by studying in sections 9 to 11 the commutative subfields and centralisers of the skew field coproduct of finite-dimensional division k-algebras over k and of free skew fields over k. Cohn (16) had shown previously that centralisers in the free skew field were commutative. Here we show that the centraliser of an element x transcendental over k in the free skew field or in the coproduct of finite-dimensional division k-algebras over k is finite-dimensional over k(x); so commutative subfields of such a skew field are of transcendence degree 1 over k and finitely generated; further, we are able to bound the dimension of this centraliser over its own centre. In particular, this bound yields Cohn's result that centralisers in the free skew field over k are commutative.

It was unknown until the results of section 12 whether the free skew field over k on an m-element set could be isomorphic to the free skew field on an n-element set for different integers m and n. By examining the universal bimodule of derivations, we show that this can never happen - more strongly, if n elements generate the free skew field over k on an n-alement set, they do so freely. The method we develop here has a number of interesting consequences which we present in the remainder of this section.

In section 13, we find a way of characterising those epimorphisms from right hereditary rings to simple artinian rings that arise as universal localisations. This result was suggested by the work in section 12. We have already had one application of this result in section 3, where we calculated the centre of the simple artinian coproduct and we give a further application to finite-dimensional hereditary k-algebras at the end of this section which we shall use in section 14.

In section 14, we construct and investigate a new class of skew fields that generalises the class of skew field coproducts with amalgamation. The natural methods to apply to them show how useful the general theory, developed in part I, is. It is particularly interesting here because we are able to show that many apparently different skew fields are isomorphic. For example, we show that if D is a finite-dimensional division k-algebra with division sub-algebras E_1 and E_2 such that $[E_1:k] = [E_2:k]$, then $D \underset{k}{\circ} E_1 \cong D \underset{k}{\circ} E_2$. This is in contrast to the case of the group coproduct where analogous results are clearly false. The last section discusses where this essay leaves the isomorphism question for skew field coproducts.

We have given a fairly clear idea of the contents of sections 9 to 15; we shall now give a more detailed summary of the results in the first half of part II.

Section 1 is technical; we present our method for finding the finite-dimensional division subalgebras so as to include all possible cases to which it is later applied.

Section 2 applies the results of section 1 to the skew field coproduct with amalgamation and to the simple artinian coproduct with amalgamation under suitable assumptions about the centres of such rings. These assumptions are verified in section 3 except for one annoying case where at present nothing is known, and for the simple artinian coproduct. The centre of the skew field coproduct was essentially determined by Cohn (19), apart from this one exceptional case. The reader is advised to pass by the calculation of the centre of the simple artinian coproduct on the first reading.

Section 4 illustrates the results of section 2 for the skew field coproduct. It is worth reading this and the following sections as motivation for the difficulties of sections 1 and 3. As well as giving general results, we look for cases where we can show that a finite-dimensional division subalgebra of $D_1 \underset{k}{\circ} D_2$ is conjugate to a subalgebra of one of the factors in analogy to the case of the group coproduct. That this is not true in general is easily seen; however, the cases where it is true and where it fails are of some interest; for example, if D_1 and D_2 are purely inseparable commutative extensions of k, the result is true; if $[D_1:k] = [D_2:k] = p$, a prime not equal to 11 or $(q^s - 1)/(q - 1)$ where q is a prime power and s an integer, again the result is true, but it fails for all other prime numbers. The reason for this is that 11 and primes of the form $(q^s - 1)/(q - 1)$ are those primes p for which we may find two inequivalent permutation representations of some group on a p-element set.

Section 5 discusses the notion of transcendence for skew fields. First, we investigate the coproduct over k of two skew fields having no elements algebraic over k. We show that any finite-dimensional subfield of such a skew field coproduct has dimension over k divisible by at least two primes. We give an example of such a skew field coproduct that contains a subfield of dimension 6. This demonstrates that the most obvious notion of transcendence does not behave well; however, there are a number of useful notions that all behave well for the coproduct construction and we examine them in relation to the ideas of Cohn, Dicks (22). We present a counter-example to a conjecture from this paper.

Section 6 gives an application of the preceding theory to give a counter-example to the analogue for skew field coproducts of the Grushko, Neumann theorem in the theory of group coproducts. A consequence of the Grushko, Neumann theorem is that if G_1 and G_2 require, respectively, n_1 and n_2 generators, then $G_1 * G_2$ requires $(n_1 + n_2)$ generators. We give an example of two non-commutative fields whose skew field coproduct requires only 3 generators, whereas the Grushko, Neumann theorem would force it to require 4 generators.

Given a central division algebra D over k, Amitsur developed the notion of a universal splitting field of D over k; there is an interesting non-commutative analogue of this which we define in section 7 together with some variants whose commutative analogues are studied in part III . Our method allows us to calculate their finite-dimensional division subalgebras.

Section 8 completes the work on finite-dimensional division subalgebras of skew fields by investigating those that lie in the universal skew field of fractions of a k-algebra that is a ring satisfying the weak algorithm with respect to an $\mathbb{N}$-filtration. These rings are a central object of study in Cohn (14), since they are a good source of firs.

Part III is rather a side-issue in relation to the rest of the dissertation. We investigate a particular aspect of the theory of finite-dimensional algebras. Amitsur, Chatelet and Roquette developed the theory of Brauer-Severi varieties and the study of their function fields which are known as generic splitting fields since the Brauer-Severi varieties provide a generic solution to the problem of splitting a central simple algebra. That is, given a central simple k-algebra A, $[A:k] = n^2$, a field K satisfies $A \otimes_k K \cong M_n(K)$ if and only if the Brauer-Severi variety associated to A has a point in K. We try to characterise those fields L such that $A \otimes_k L \cong M_m(A')$ where A' is a central simple L-algebra for a particular m such that m divides n. We find suitable varieties, V, such that V has a point on L if and only if $A \otimes_k L \cong M_m(A')$. We show that if A is a central simple k-algebra where k is an algebraic number field, then these varities V satisfy the Hasse principle.

Definitions and notations

We summarise here the various notations and definitions used in this dissertation. We introduce them here according to chapters of sections in which they appear. There are certain instances (notably in chapter 2) where to introduce all the definitions here would be equivalent to re-writing the chapter, and this we avoid.

Part I

Chapter 1 A left hereditary ring is one such that all left ideals are projective modules. A left semihereditary ring is one such that all finitely generated left ideals are projective. A left $\aleph_0$-hereditary ring is one such that all countably generated left ideals are projective. A weakly semihereditary ring is one such that given a pair of maps between finitely generated projectives: $\alpha: P_1 \longrightarrow P_2$ and $\beta: P_2 \longrightarrow P_3$ such that $\alpha\beta = 0$, $P_2 = Q_1 \oplus Q_2$ where $\mathrm{im}\,\alpha \subseteq Q_1 \subseteq \ker\beta$. A von Neumann regular ring is one such that given any element a R, there exists $x \in R$ such that $axa = a$. A semifir is a ring such that all finitely generated left ideals are free of unique rank; a fir is a ring such that all left ideals and all right ideals are free of unique rank.

Given a ring R, P(R) is the category of finitely generated projective left modules. We can associate a commutative monoid, $P_{\oplus}(R)$ to this, which is generated by isomorphisms classes of finitely generated projective modules subject to the relations $[P] + [Q] = [P + Q]$. The universal abelian group on these relations is known as $K_0(R)$, and the image of the monoid $P_{\oplus}(R)$ in $K_0(R)$ makes $K_0(R)$ in a natural way intc a partially ordered group. A rank function on R is a homomorphism from $K_0(R)$ to $\mathbb{R}$, the real numbers, that sends $[R]$ to 1. A rank function is faithful if the image of no non-zero element of

$P_{\oplus}(R)$ is zero under this homomorphism. A partial rank function is a homomorphism of partially ordered groups from some subgroup of $K_o(R)$ containing $[R^n]$ $\forall n \in \mathbb{N}$, to $\mathbb{R}$ sending $[R]$ to 1.

A ring has unbounded generating number if for all integers n, there exists a finitely generated left R-module requiring at least (n + 1) generators.

Given a rank function ρ on a ring R, we tend to write $\rho(P)$ instead of $\rho([P])$. Let M be a finitely generated R-module; we define the generating number of M with respect to ρ by

$$g.\rho(M) = \inf_P \{\rho(P) : \text{there exists a surjection } P \longrightarrow M\},$$

where P ranges over finitely generated projective modules. Given a map between finitely generated projective modules $\alpha : P \longrightarrow Q$, we define the inner rank of α with respect to ρ to be

$$\rho(\alpha) = \inf_{P'} \{\rho(P') : \text{there exists a commutative diagram } \begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ & \searrow \quad \nearrow & \\ & P' & \end{array} \}$$

α is said to be left full with respect to ρ if $\rho(\alpha) = \rho(P)$; it is right full with respect to ρ if $\rho(\alpha) = \rho(Q)$ and it is full with respect to ρ if both of these occur. A rank function is said to satisfy the law of nullity if given a pair of maps $\alpha : P_1 \longrightarrow P_2$, $\beta : P_2 \longrightarrow P_3$ such that $\alpha\beta = 0$, $\rho(\alpha) + \rho(\beta) \leqslant \rho(P_2)$. If R is a ring such that all projectives are free of unique rank, there can be only one rank function, and if R satisfies the law of nullity with respect to this rank, it is a Sylvester domain.

The map $\alpha : P_1 \longrightarrow Q_1$ is stably associated to the map $\beta : P_2 \longrightarrow Q_2$ if there exists a commutative diagram:

$$\begin{array}{ccc} P_1 \oplus P & \xrightarrow{\alpha \oplus I_P} & Q_1 \oplus P \\ \| \wr & & \| \wr \\ P_2 \oplus Q & \xrightarrow{\beta \oplus I_Q} & Q_2 \oplus Q \end{array}$$

A finitely presented module M is said to be of <u>left full</u>, <u>right full</u> or <u>full presentation</u> with resepct to ρ if we have an exact sequence:

$$0 \longrightarrow P \xrightarrow{\alpha} Q \longrightarrow M \longrightarrow 0$$

where α is left full, right full or full respectively (with respect to ρ).

<u>Chapter 2</u> An <u>R_o-ring</u> is a ring R with a specified homomorphism $\alpha: R_o \longrightarrow R$; it is a <u>faithful R_o-ring</u> if this homomorphism is an embedding. Given a family of R_o-rings $\{R_i, \alpha_i : i \in I\}$, the <u>ring coproduct with amalgamation</u> of these rings is the R_o-ring generated by these rings R_i subject only to the relations identifying the various images of R_o, $\alpha_i(R_o)$ in R_i. R_o is known as the <u>base ring</u>, and the family of rings $\{R_i : i \in I\}$ are the <u>factors</u> of the coproduct. We use the symbol $\coprod_{R_o} R_i$, or, more completely, $\coprod_{\substack{R_o \\ i \in I}} R_i$. An <u>induced module</u> over $\coprod_{R_o} R_i$ is a direct sum of modules $\bigoplus_{i \in I} ((\coprod_{R_o} R_i) \otimes_{R_i} M_i)$. A <u>basic module</u> over one of the factors takes the form $R_i \otimes_{R_o} M_o$, where M_o is an R_o-module. Similarly, a <u>basic module</u> over the coproduct is a module of the form $(\coprod_{R_o} R_i) \otimes_{R_o} M_o$.

A commutative <u>regular</u> ring is one of finite global dimension. A homomorphism of commutative rings $\alpha: R \longrightarrow S$ is said to be geometrically regular when for all prime ideals $P \lhd R$, $\overline{Q(R/P)} \otimes_R S$ is regular, where $\overline{Q(R/P)}$ is the algebraic closure of the quotient field of R/P.

A representable functor on a category with small Hom-sets is one that is naturally equivalent to $\underline{\mathrm{Hom}}(\underline{Q},-)$ if it is a covariant functor or to $\underline{\mathrm{Hom}}(-,\underline{Q})$ if contravariant, where $\underline{Q}$ is an object of the category.

Chapter 3 In lemma 3.1, we show that if R_o is a semisimple artinian ring and R_1, R_2 are a pair of faithful R_o-rings with rank functions ρ_1,ρ_2 respectively that agree on $K_o(R_o)$, they determine uniquely a rank function on $R_1 \underset{R_o}{\sqcup} R_2$, which we denote by (ρ_1,ρ_2).

A graph of groups is given by a finite directed graph (possibly with loops); to each edge and vertex we associate a group; every edge e has a beginning ιe and an end τe and we have homomorphisms $G_e \longrightarrow G_{\iota e}$, $G_e \longrightarrow G_{\tau e}$. We refer the reader to page 54 for the definition of the fundamental group of a graph of groups; also, we refer the reader to page 55 for the definition of accessibility.

$K_o^{q}(KG)$ is that subgroup of $K_o(KG)$ generated by the projective modules induced up from finite subgroups of G.

A finitely generated projective module P over a ring R is said to be stably induced form some family of subrings $\{R_i : i \in I\}$ if there is an equation $P \oplus R^n \cong \bigoplus_{i\in I} P_i \otimes_{R_i} R$, where each P_i is a finitely generated projective module over R_i.

Chapter 4 Given a ring R and a set Σ of maps between finitely generated projective modules there is a ring homomorphism $R \longrightarrow R_\Sigma$ universal with respect to that each element of Σ has an inverse over R_Σ; R_Σ is known as the universal localisation of R at Σ. It is clear that Ore loclaisation is a special case of this process. A set of maps, Σ, between finitely generated projective modules is said to be lower multiplicatively closed if for all maps $\alpha,\beta \in \Sigma$, $\begin{pmatrix} \alpha & 0 \\ * & \beta \end{pmatrix}$ is also in Σ for any choice of lower left hand corner. Similarly, Σ is said to be upper multiplicatively closed when for every pair of maps

$\alpha, \beta \in \Sigma$, $\begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix}$ for any top right hand corner possible.

Chapter 5 A rank function on a ring R is said to be excellent if it satisfies the law of nullity. In the case of an excellent rank function ρ on a ring R, we construct the universal localisation of R at ρ, written R_ρ, by adjoining the universal inverses of all maps between finitely generated projective modules.

The skew field coproduct with amalgamation of a family of skew fields $\{E_i : i \in I\}$ with common skew subfield E, is written $\underset{E}{\circ} E_i$; it is the universal skew field containing the E_i such that $E_i \cap E_j = E$. Similarly, given a family of simple artinian rings $\{S_i : i \in I\}$ such that the size n_i of matrix units in the representation $S_i \cong M_{n_i}(D_i)$ for skew fields D_i is bounded, and with common simple artinian subring S, we construct a simple artinian coproduct with amalgamation $\underset{S}{\circ} S_i$.

We refer the reader to page 70 for the definition of a prime matrix ideal.

A ring is said to be left perfect if it has the descending chain condition on right principal ideals; it is perfect is it is both left and right perfect.

Chapter 6 Let R be a ring with a rank function ρ; it has enough left and right full maps if every map $\alpha : P \longrightarrow Q$ in the category of finitely generated projective modules factors as right full followed by left full. It has enough full maps if every left full map is the left factor of a full map and every right full map is the right factor of a full map.

Part II

Section 1 An extension of commutative fields $k \subset K$ is said to be regular if $K \otimes_k E$ is a domain for all commutative fields E.

Section 2 The function field of a curve of genus 0 over a field k is a commutative field $K \supset k$ such that $K \otimes_k E \cong E(t)$, where t is a transcendental and E is a suitable finite extension of k.

Section 3 Let k be a field and let $\mathcal{A}$ be some object with a k-structure (for example, a k-algebra or a k-variety); $\mathcal{B}$ is a twisted form of $\mathcal{A}$ if there is a finite-dimensional commutative field extension $E \supset k$ such that $\mathcal{A} \otimes_k E \cong \mathcal{B} \otimes_k E$. A finite-dimensional central simple algebra S is always a twisted form of a suitable matrix ring; that is, there are commutative fields $L \supset k$ such that $L \otimes_k S = M_n(L)$ for suitable n; such a commutative field is known as a splitting field for S. The number n, is the PI degree of S. There is a smooth variety, V, defined over k, such that L is a splitting field for S if and only if V has a point in L; the function field of such a variety is a generic splitting field for S.

Section 5 A skew field F is transcendental over a central subfield k if it has no elements algebraic over k.

It is n-transcendental over k if $D \otimes_k F$ is a skew field for all finite-dimensional division algebras of PI degree dividing n; it is totally transcendental over k if it is n-transcendental for all n. 1-transcendental is called regular in Cohn, Dicks (22); a skew field F is defined to be totally algebraically closed over a central subfield k if $F \otimes_k k(\alpha)$ is a skew field for all simple algebraic extensions $k(\alpha)$ of k.

Section 6 Let S be a central simple k-algebra of dimension n^2 over k. Let $m \mid n^2$; then a skew field F is said to be an m-splitting field if $F \otimes_k S \cong M_m(F')$ for some skew field F'. If F is commutative, m must actually divide n.

Section 7 Let F be a skew field with central subfield k; we define the transcendence degree of F over k to be the maximal transcendence degree of commutative subfields of F.

Section 12 We use the notation $\delta: R \to \Omega_{R_0}(R)$ for the universal derivation that vanishes on R_0 from the R_0-ring R to the universal bimodule of derivations.

Given an R-R-bimodule M and a homomorphism of ring $\alpha: R \longrightarrow S$, it is often useful to use the shorthand notation ${}^{\otimes}M^{\otimes}$ for the S-S-bimodule $S \otimes_R M \otimes_R S$. We shall only use this notation when it is clear what rings R and S must be; for example, the formula

$$\Omega_R(R\langle M\rangle) \cong {}^{\otimes}M^{\otimes}$$

quite clearly means that the universal bimodule of derivations of the tensor ring $R\langle M\rangle$ over R must be $R\langle M\rangle \otimes_R M \otimes_R R\langle M\rangle$.

Part I

Chapter 1

In this chapter we introduce the two main subjects of the first part of this book; hereditary rings and rank functions.

A left hereditary ring is one such that all left ideals are projective modules. We shall be interested in a number of variants of this definition; a left semihereditary ring is one such that all finitely generated left ideals are projective and a left $\aleph_0$-hereditary ring is one such that all countably generated left ideals are projective. We shall often need to consider the two-sided properties, whose definitions the reader can formulate for himself; our results tend to work most often in the case of two-sided $\aleph_0$-hereditary rings. However, there is a two-sided condition implied by all of the one-sided conditions above; a ring R is weakly semihereditary if for all pairs of maps $\alpha: P_0 \longrightarrow P_1$, $\beta: P_1 \longrightarrow P_2$ between finitely generated left projective modules such that $\alpha\beta = 0$, $P_1 = P_1' \oplus P_1''$, where the image of α lies in P_1' and the kernel of β contains P_1'. This is a two-sided condition because of the duality between the category of left and the category of right finitely generated projective modules induced by $\mathrm{Hom}_R(-,R)$.

Lemma 1.1 A left semihereditary ring is weakly semihereditary.
Pf Suppose that $\alpha: P_0 \longrightarrow P_1$, $\beta: P_1 \longrightarrow P_2$ are two maps such that $\alpha\beta = 0$, where P_i is a finitely generated left projective module for $i = 0,1,2$. The image of β is a projective module; so $P_1 \cong \mathrm{im}\beta \oplus \ker\beta$ and the image of α lies in the kernel of β.

We shall work for as long as possible with weakly semihereditary rings; however, we shall eventually be forced to restrict our attention to two-sided $\aleph_0$-hereditary rings. This class of rings draws much of its initial interest from the fact that all von

Neumann regular rings satisfy this property. A von Neumann regular ring is one such that all modules are flat; we shall find that there are quite strong connections between these two classes of rings,

Much of the work in this chapter is essentially a study of the category of finitely generated left projective modules over a weakly semihereditary ring. This has been done with a great deal of success for semifirs and firs by Cohn (14); a fir is a ring such that all left ideals (and so all right ideals, too) are free of unique rank and a semifir is a ring such that all finitely generated left ideals (and so all such right ideals too) are free of unique rank. In this case the arguments work well because we have a good notion of the size of a finitely generated projective module, so we should like to have an idea for the size of a projective module over more general rings. The relevant idea comes from the theory of von Neumann regular rings (Goodearl 31),

Given a ring R, we may associate a partially ordered abelian group, $K_0(R)$, to the category of finitely generated left projective modules over R; it is generated by the isomorphism classes of finitely generated left projective modules $[P]$, subject to the relations $[P \oplus Q] = [P] + [Q]$ for every pair of isomorphism classes $[P]$, $[Q]$. The partial order is given by specifying a positive cone, which in this case is the submonoid generated by the elements $[P]$, for every finitely generated left projective module P.

A rank function for the ring R is a homomorphism of partially ordered abelian groups, $\rho : K_0(R) \longrightarrow \mathbb{R}$, the real numbers, such that $\rho(R^n) = n$. By definition, then, $\rho([P]) \geq 0$; we shall call a rank function faithful if $\rho([P]) > 0$ for all non-zero P. We shall often abuse the notation by writing $\rho(P)$ instead of $\rho([P])$. We note that a rank function is a left-right dual notion because of the duality $\mathrm{Hom}_R(-, R)$.

A <u>partial</u> <u>rank</u> <u>function</u> for the ring R is a homomorphism of partially ordered groups $\rho_A : A \longrightarrow \mathbb{R}$, where A is a subgroup of $K_0(R)$ containing $[R']$, and the partial order is that induced from $K_0(R)$ by restriction.

We recall the following version of theorem 18.1 of (Goodearl 31):

Theorem 1.2 Every partial projective rank function extends to a projective rank function on R.

This result allows us to characterise those rings that have a rank function. We say that a ring has <u>unbounded generating number</u> if for every natural number n there is a finitely generated left module requiring at least n generators. It is an easy check that this is equivalent to the condition that for no m is there and equation $R^m \cong R^{(m+1)} \oplus P$; and this is a left-right dual condition, which justifies the two-sided nature of our description.

<u>Theorem 13</u> A ring R has a rank function if and only if it has unbounded generating number.

<u>Pf</u> Certainly, if R has a rank function, it must have unbounded generating number.

Conversely, if R has unbounded generating number, the subgroup of $K_o(R)$ generated by R is isomorphic to $\mathbb{Z}$ and under the isomorphism no stably free projective module can have negative rank, since this would imply an equation of the form $R^m \cong R^n \oplus P$, where $n > m$ and hence $R^m \cong R^{(m+1)} \oplus (R^{(n-m-1)} \oplus P)$. So there is a partial rank function for R, and by Theorem 2, there must be a rank function for R.

We have shown that most rings have a rank function; in fact, rank function arise quite naturally on rings and one is forced to study them in order to solve certain types of problems.

If S is a simple artinian ring it has the form $M_n(D)$ for some skew field D and so $K_0(S)$ can be identified in a natural way with $\frac{1}{n}\mathbb{Z}$; so, in this case, we have a unique rank function. If we have a ring R and a homomorphism $\alpha: R \to S$, this induces a homomorphism from $K_0(R)$ to $K_0(S)$, which is just $\frac{1}{n}\mathbb{Z}$, so a homomorphism to S determines a rank function taking values in $\frac{1}{n}\mathbb{Z}$, and we shall need to study such rank functions when considering homomorphisms to simple artinian rings. More generally, many von Neumann regular rings have rank functions, and so in order to understand homomorphisms to such von Neumann regular rings we may need to study rank functions.

One class of rings with a well-known rank function are the group rings in characteristic 0. We have a trace function on the group ring FG, where F is a field of characteristic 0, given by $\text{trace}(\sum_{g_i \in G} f_i g_i) = f_0$, where g_0 is the identity element of G. We extend this to a trace function on $M_n(FG)$ in the natural way and we define $\rho([P])$ to be the trace of an idempotent $e \in M_n(FG)$ such that $(FG)^n e \cong P$. It is well-known that this is well-defined, taking values in $\mathbb{Q}$, and that it is a faithful rank function; that is, $\rho([P]) \leq 0 \Leftrightarrow P \cong 0$. This will turn out to be of use later on in proving results on accessibility.

If we have a partial rank function, ρ_A, for the ring R, defined on the subgroup A of $K_0(R)$, we define the <u>generating number with respect to</u> ρ_A of a finitely generated left module M over R to be $g.\rho_A(M) = \inf_{[P]} \{\rho_A(P) : [P] \in A, \exists \text{ a surjection } P \to M\}$. If all stably free modules are free of unique rank, and A is the subgroup of $K_0(R)$ generated by $[R^1]$, the generating number with respect to ρ_A, where ρ_A is the only possible partial rank function defined on A, is just the minimal number of generators of a module. So we could hope that our more general notion here is a useful refinement.

We intend to use a rank function ρ to analyse the category of finitely generated left projective modules over a ring R. We have a rank associated to each object of the category so our next aim is to associate a rank to each map in the category. Let $\alpha : P \longrightarrow Q$ be a map between two finitely generated left projective modules; we define the <u>inner rank of α with respect to ρ</u> by

$$\rho(\alpha) = \inf_{[P']}\left\{\rho(P') : \exists \text{ a commutative diagram } \begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr] & & Q \\ & P' \arrow[ur] & \end{tikzcd}\right\} .$$

We may relate the inner rank of a map and the generating number of suitable modules (with respect to ρ).

<u>Lemma 1.4</u> Let R be a ring with a rank function ρ; then the inner rank with respect to ρ of $\alpha : P \longrightarrow Q$ is equal to

$$\inf_M\{g.\rho(M) : \alpha(P) \subseteq M \subseteq Q, \text{ where } M \text{ is a finitely generated submodule of } Q\}$$

In particular, if R is a left semihereditary ring,

$$\rho(\alpha) = \inf_{P'}\{\rho(P') : \alpha(P) \subseteq P' \subseteq Q\} .$$

<u>Pf</u> If there is a commutative diagram $\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr] & & Q \\ & P' \arrow[ur, "\beta"'] & \end{tikzcd}$ then $\alpha(P) \subseteq \beta(P') \subseteq Q$

and $g.\rho(\beta(P')) \leq \rho(P')$, so $\rho(\alpha) \geq \inf_M\{g.\rho(M) : \alpha(P) \subseteq M \subseteq Q\}$.

Conversely, if $\alpha(P) \subseteq M \subseteq Q$, and there exists a surjection $P' \longrightarrow M$, the map $\begin{tikzcd} P \arrow[rr] \arrow[dr] & & Q \\ & P' \arrow[ur] & \end{tikzcd}$ lifts to a commutative diagram since P is projective; so $\rho(\alpha) \leq \inf_M\{g.\rho(M) : \alpha(P) \subseteq M \subseteq Q\}$.

A map $\alpha : P \longrightarrow Q$ is said to be <u>left full with respect to ρ</u> if $\rho(\alpha) = \rho(P)$ and <u>right full</u> if $\rho(\alpha) = \rho(Q)$; it is <u>full with respect to</u>

ρ if $\rho(\alpha) = \rho(P) = \rho(Q)$.

We see that entirely the same theory may be set up on the dual category of finitely generated right projective modules, where the rank of a finitely generated right projective module is the rank of its dual module. It is clear that the dual of a left full map is a right full map and conversely.

The important fact about the inner rank with respect to a rank function ρ on a weakly semihereditary ring R is an analogue of Sylvester's law of nullity. We say that a ring satisfies the <u>law of nullity with respect to the rank function</u> ρ if for every pair of maps $\alpha : P_0 \longrightarrow P_1$, $\beta : P_1 \longrightarrow P_2$ such that $\alpha\beta = 0$, $\rho(\alpha) + \rho(\beta) \leq \rho(P_1)$. If R is a ring such that all projectives are free of unique rank and it satisfies the law of nullity for that rank it is a Sylvester domain.

<u>Theorem 1.5</u> Let R be a weakly semihereditary ring and let ρ be a rank function on R; then R satisfies the law of nullity with respect to ρ.

<u>Pf</u> Recall that if $\alpha\beta = 0$ for $\alpha : P_0 \longrightarrow P_1$, $\beta : P_1 \longrightarrow P_2$ over a weakly semihereditary ring R, then $P_1 \cong P_1' \oplus P_1''$, where $\operatorname{im}\alpha \subseteq P_1'$ and $\ker\beta \supseteq P_1'$. Therefore α factors through P_1' and β factors through P_1'' so $\rho(\alpha) \leq \rho(P_1')$, $\rho(\beta) \leq \rho(P_1'')$ and so $\rho(\alpha) + \rho(\beta) \leq \rho(P_1)$.

There are a few results we can deduce from the fact that R satisfies the law of nullity with respect to a rank function ρ.

<u>Lemma 1.6</u> Let R be a ring satisfying the law of nullity with respect to the rank function ρ; then for any pair of maps $\alpha : P_0 \longrightarrow P_1$, $\beta : P_1 \longrightarrow P_2$ $\rho(\alpha\beta) \geq \rho(\alpha) + \rho(\beta) - \rho(P_1)$. In particular, this holds for weakly semihereditary rings.

<u>Pf</u> Suppose that $\begin{tikzcd} P_0 \arrow[rr, "\alpha\beta"] \arrow[dr, "\gamma"'] & & P_2 \\ & Q \arrow[ur, "\delta"'] & \end{tikzcd}$ is a commutative diagram. We have

the map $(\alpha | \gamma) : P_0 \longrightarrow P_1 \oplus Q$, $\begin{pmatrix} \beta \\ -\delta \end{pmatrix} : P_1 \oplus Q \longrightarrow P_2$ and $(\alpha | \gamma)\begin{pmatrix} \beta \\ -\delta \end{pmatrix} = 0$;

so by the law of nullity, $\rho((\alpha|\gamma((+ \rho\left(\frac{\beta}{-\delta}\right) \leq \rho(P_1) + \rho(Q)$. But $\rho(\alpha) \leq \rho((\alpha|\gamma))$ and $\rho(\beta) \leq \rho\left(\left(\frac{\beta}{-\delta}\right)\right)$; therefore

$$\rho(\alpha) + \rho(\beta) - \rho(P_1) \leq \rho(Q).$$

Taking the infimum on the right we obtain the lemma.

<u>Corollary 17</u> The composition of left full maps with respect to the rank function ρ on a ring satisfying the law of nullity with respect to ρ is left full; dually, the composition of right full maps is right full. In particular, this holds for weakly semihereditary rings.

<u>Pf</u> If $\alpha: P_0 \longrightarrow P_1$, $\beta: P_1 \longrightarrow P_2$ are both left full, then by Lemma 6 $\rho(\alpha\beta) \geqslant \rho(\alpha) + \rho(\beta) - \rho(P_1) = \rho(P_0)$, so $\rho(\alpha\beta) = \rho(P_0)$, as we wished to show. The dual result is entirely similar.

In order to get fairly decisive results, we shall need to restrict our attention to a two-sided $\aleph_0$-hereditary ring R with a faithful projective rank function. Dicks pointed out that the proof of the next main theorem, originally stated for two-sided hereditary rings, actually works for two-sided $\aleph_0$-hereditary rings. Before we prove this result, we recall from Cohn (14 p.12) the following theorem;

<u>Theorem 18</u> A projective module over a weakly semihereditary ring is a direct sum of finitely generated projective modules.

<u>Theorem 19</u> Let R be a two-sided $\aleph_0$-hereditary ring with a projective rank function ρ. Then every map between finitely generated projective left modules factors as a right full followed by a left full map.

<u>Pf</u> First we show that any map $\alpha: P \longrightarrow Q$ has a right factor that is left full.

If $\alpha: P \longrightarrow Q$ is not left full or right full, then, by Lemma 4, there exists P_1, $\mathrm{im}(\alpha) \subseteq P_1 \subseteq Q$ such that $\rho(P_1) < \rho(P), \rho(Q)$ and $\rho(P_1) - \rho(\alpha) < 1$.

If $\alpha_1 : P_1 \to Q$ is not a left full embedding, we may find P_2 such that $P_1 \subset P_2 \subset Q$, $\rho(P_2) < \rho(P_1)$ and $\rho(P_2) - \rho(\alpha_1) < \frac{1}{2}$; in general, at the n^{th} stage, if $\alpha_{n-1} : P_{n-1} \subset Q$ is not left full we choose P_n, $P_{n-1} \subset P_n \subset Q$, such that $\rho(P_n) < \rho(P_{n-1})$ and $\rho(P_n) - \rho(\alpha_{n-1}) < 1/n$; if this process does not terminate we obtain a chain

$$\mathrm{im}\,\alpha \subset P_1 \subset P_2 \subset \ldots\ldots \quad . \subset Q$$

$\bigcup_i P_i$ is a countable generated submodule of Q and so must be projective. It is a countable direct sum $P_1' \oplus P_2' \oplus \ldots\ldots$ where each P_i' is a finitely generated projective module (from Theorem 1.8).

Since $\mathrm{im}\,\alpha$ is finitely generated, it lies in $P_N'' = \bigoplus_{i=1}^{N} P_i'$ for some N. We claim that the embedding $P_N'' \subset Q$ is a left full map.

Since P_N'' is finitely generated, $P_N'' \subset P_n$ for some n and so $P_N'' \subset P_m$ for all $m > n$. Moreover, P_N'' is a direct summand of each such P_m since it is a direct summand of $\bigcup_i P_i$.

Consider any finitely generated projective module, Q', such that $P_N'' \subset Q' \subset Q$; consider the split exact sequence

$$0 \to P_m \cap Q' \to P_m \oplus Q' \to P_m \oplus Q' \to 0 \;;$$

$P_m \cap Q' \supseteq P_N''$, so $P_m \cap Q' \cong P_N'' \oplus Q''$ for some Q'', so that $\rho(P_m \cap Q') \geqslant \rho(P_N'')$.

Also $P_{m-1} \subset P_m \subseteq P_m + Q'$, so $\rho(P_m + Q') \geqslant \rho(P_m) - 1/m$. From the exact sequence, $\rho(P_m) + \rho(Q') = \rho(P_m + Q') + \rho(P_m \cap Q')$ and the right hand side is greater than or equal to $\rho(P_N'') + \rho(P_m) - 1/m$, so that $\rho(Q') \geqslant \rho(P_N'') - 1/m$; but this holds for all $m \geqslant n$ and we deduce that $\rho(Q') \geqslant \rho(P_N'')$. Thus, as stated, $P_N'' \subset Q$ is a left full map that is a right factor of $\alpha : P \to Q$; by construction $\rho(P_N'') \leqslant \rho(P_m) < \rho(Q)$.

By duality, we deduce that we can always find a right full left factor of $\alpha : P \to Q$, $\alpha = \beta_1 \gamma$, where $\beta_1 : P \to Q_1$, where $\rho(Q_1) < \rho(P)$.

Since a sequence of right full maps is right full, by Corollary 7, and a surjection is right full, we may assume that the right full map is to some submodule Q_1 of Q. So we have $\operatorname{im}\alpha \subset Q_1 \subset Q$, where the induced map $\alpha' : P \longrightarrow Q_1$ is right full. If the map $\gamma_1 : Q_1 \subset Q$ is not left full we can find Q_2, $Q_1 \subset Q_2 \subset Q$ such that $\rho(Q_2) < \rho(Q_1)$; and $Q_1 \subset Q_2$ is a right full map. We continue this process and show that if it does not stop we obtain a contradiction; if it continues we have a strictly ascending chain

$$\operatorname{im}\alpha \subset Q_1 \subset Q_2 \subset \cdots\cdots \quad \subset Q \text{, where } \rho(Q_{i+1}) < \rho(Q_i)$$

$\bigcup_i Q_i$ is a countably generated module so it is projective and, by Theorem 8, $\bigcup_i Q_i = \bigoplus_j Q'_j$, where each Q'_j is finitely generated projective. For some m, $Q_1 \subseteq \bigoplus_{j=1}^{m} Q'_j$; again, for some n, $\bigoplus_{j=1}^{m} Q'_j \subseteq Q_n \subset Q_{n+1}$ and $\bigoplus_{j=1}^{m} Q'_j$ is a direct summand of Q_{n+1}; so $\rho(\bigoplus_{j=1}^{m} Q'_j) \leqslant \rho(Q_{n+1}) < \rho(Q_n)$. Since $Q_1 \subset Q_n$ factors through $\bigoplus_{j=1}^{m} Q'_j$ it cannot be right full, but it is the composite of the right full maps $Q_i \subset Q_{i+1}$ (i = 1 to n-1), so it must be right full by Corollary 7. This contradiction shows that at some stage our process must terminate; that is, $Q_m \subset Q$ is left full for some m. But $P \longrightarrow Q_m$ us a composite of right full maps and so must be right full. So we have factorised $\alpha : P \longrightarrow Q$ as a right full followed by a left full map.

<u>Corollary 1.10</u> Let R be a two-sided $\aleph_0$-hereditary ring with a rank function ρ. Then, for any map $\alpha : P \longrightarrow Q$,

$$\rho(\alpha) = \min_{[P']}\{\rho(P') : \operatorname{im}\alpha \subseteq P' \subseteq Q\}.$$

<u>Pf</u> This follows at once from the argument of Theorem 9. For $\alpha : P \longrightarrow Q$ factors as $\alpha_1\alpha_2$, where $\alpha_1 : P \longrightarrow P'$ is right full and $\alpha_2 : P' \subseteq Q$ is a left full map. By Lemma 6, $\rho(\alpha) \geqslant \rho(\alpha_1) + \rho(\alpha_2) - \rho(P') = \rho(P')$; so $\rho(\alpha) = \rho(P')$.

There are occasions when this result is automatically true for a weakly semihereditary ring; for example, if we have the descending chain condition on the numbers $\rho(P)$. It fails in general for weakly semihereditary rings. This result is the key to constructing homomorphisms from two-sided $\aleph_0$-hereditary ring to von Neumann regular rings.

We pass from the study of quite general maps in the category of finitely generated projective modules to the study of the full maps. We shall find it useful to prove a certain type of ascending chain condition but before doing so we present related results that are not strictly relevant here but are of some independent interest.

Theorem 11 Let R be a left $\aleph_0$-hereditary ring with a faithful rank function such that the values of $\rho(P)$ are bounded away from zero for non-zero P; then any finitely generated left projective module satisfies the ascending chain condition on finitely generated submodules of bounded rank.

Pf Let $P_0 \subseteq P_1 \subseteq P_2 \subseteq \dots\dots \subseteq Q$ be and ascending chain of finitely generated submodules of bounded rank; then $\bigcup_i P_i$ is countably generated and so projective. Therefore, by Theorem 8, $\bigcup_i P_i = \bigoplus_j P'_j$, where each P'_j is finitely generated and projective.

Since P'_j is non-zero, $\rho(P'_j) > \varepsilon$ for some ε; so if there are at least n of the P'_j, $\rho(\bigoplus_{j=1}^{n} P'_j) > n\varepsilon$. But $\bigoplus_{j=1}^{n} P'_j$ lies in some $P_{m(n)}$ and must be a direct summand of $P_{m(n)}$ since it is a direct summand of $\bigcup_i P_i$. So $n\varepsilon < \rho(P_{m(n)})$. We conclude that n is bounded, so $\bigcup_i P_i$ is finitely generated and our ascending chain must stop.

We have a similar result independent of rank functions:

Theorem 1.12 Let R be a left $\aleph_0$-hereditary ring such that in any infinite direct sum $\bigoplus_j Q_j$ of finitely generated left projective modules the number of generators of a finite direct subsum is unbounded. Then any left projective module has the ascending chain condition on n-generator submodules.

Pf Let $P_0 \subseteq P_1 \subseteq P_2 \subseteq \ldots\ldots \subset Q$ be an ascending chain of n-generator submodules of the left projective module Q; then $\bigcup_i P_i$ is countably generated and so projective and $\bigcup_i P_i = \bigoplus_j Q_j$, where each Q_j is finitely generated projective, by Theorem 8. Each finite direct sub-sum of this, $\bigoplus_{j=1}^{m} Q_j$, lies in some P'_m and so is a direct summand of P'_m and must be and n-generator module. By the assumption of the theorem, $\bigoplus_j Q_j$ must be a finite direct sum. Therefore $\bigcup_i P_i$ is finitely generated and the chain must stop.

The result we shall actually need for our analysis of full maps has a similar proof to these two last theorems.

Theorem 1.13 Let R be a left $\aleph_0$-hereditary ring with a faithful rank function ρ. Let $P_0 \subseteq P_1 \subseteq P_2 \subseteq \ldots\ldots \subset Q$ be an ascending chain of finitely generated submodules of the left projective module Q, where each map $P_i \subseteq P_{i+1}$ is a full map with respect to ρ; then the chain must stop after finitely many steps.

Pf Consider $\bigcup_i P_i$ which is countably generated and hence projective. Therefore, by Theorem 8, $\bigcup_i P_i \cong \bigoplus_j Q_j$ where each Q_j is finitely generated projective.

Eventually, $P_0 \subseteq \bigoplus_{j=1}^{m} Q_j \subseteq P_n$ for some m and n. $\bigoplus_{j=1}^{m} Q_j$ is a direct summand of $\bigcup P_i$; so $\rho(\bigoplus_{j=1}^{m} Q_j) \leq \rho(P_n)$ with equality only when $\bigoplus_{j=1}^{m} Q_j = P_n$, since the rank function is faithful. However, $P_0 \subseteq P_n$ is a full map with respect to ρ since it is a composite of full maps, so equality must hold. Thus $\bigoplus_{j=1}^{m} Q_j \cong P_n$; but the same argument applies for any $n' > n$, showing that $\bigoplus_{j=1}^{m} Q_j = P_{n'}$ for all $n' \geqslant n$, which completes the proof.

If R is a ring with a faithful rank function ρ and $\alpha: P \longrightarrow Q$ is a full map with respect to ρ, we define it to be an <u>atomic</u> full map if in any non-trivial factorization
$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr] & & Q \\ & P' \arrow[ur] & \end{tikzcd},$$
$\rho(P') > \rho(P) = \rho(Q) = \rho(\alpha)$.

We shall show that, over a two-sided $\aleph_0$-hereditary ring, any full map has a finite factorisation as a product of atomic full maps, that any two such factorisations have the same length and that the atomic full maps in one factorisation may be paired off with those in the other so that the corresponding maps are stably associated; the map $\alpha: P \longrightarrow Q$ is said to be <u>stably associated</u> to $\beta: P' \longrightarrow Q'$ if there exist finitely generated projective modules P_1 and P_2 such that $P \oplus P_1 \cong P' \oplus P_2$, $Q \oplus P_1 \cong Q' \oplus P_2$, and the maps $\alpha \oplus I_{P_1}$, $\beta \oplus I_{P_2}$ are associated maps i.e. there is a commutative diagram

$$\begin{tikzcd} P \oplus P_1 \arrow[r, "\alpha \oplus I_{P_1}"] \arrow[d, "\cong"'] & Q \oplus P_1 \arrow[d, "\cong"] \\ P' \oplus P_2 \arrow[r, "\beta \oplus I_{P_2}"] & Q' \oplus P_2 \end{tikzcd}$$

We shall prove this by considering a certain full sub-category of the category of finitely generated modules over R. A module M is said to be of <u>full presentation</u> if there is an exact sequence

$$0 \longrightarrow P \xrightarrow{\alpha} Q \longrightarrow M \longrightarrow 0,$$

where α is full; if the rank function is faithful and R is weakly semihereditary every full map is injective and so occurs in such a diagram.

<u>Theorem 1.14</u> Let R be a two-sided $\aleph_0$-hereditary ring with faithful rank function ρ. The full subcategory of the category of finitely

generated left modules over R, whose objects are the modules of full presentation is an artinian and noetherian abelian category.

Pf First we show that it is an abelian subcategory. Consider any map $\phi : M_1 \longrightarrow M_2$ between two modules of full presentation; we shall show that $\mathrm{im}\phi$, $\ker\phi$, and $\mathrm{coker}\phi$ are of full presentation, from which this follows.

We obtain the following commutative diagram with exact rows from the projectivity of P_1;

$$\begin{tikzcd}
0 \arrow[r] & Q_1 \arrow[r, "\alpha_1"] \arrow[d] & P_1 \arrow[r] & M_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_2 \arrow[r, "\alpha_2"] & P_2 \arrow[r] & M_2 \arrow[r] & 0
\end{tikzcd}$$

where α_i $(i = 1,2)$ is a full map.

We have two presentations of $\mathrm{im}\phi$;

$$0 \longrightarrow Q_1' \longrightarrow P_1 \longrightarrow \mathrm{im}\phi \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Q_2 \longrightarrow P_2' \longrightarrow \mathrm{im}\phi \longrightarrow 0,$$

where $Q_1 \subseteq Q_1' \subseteq P_1$, $Q_2 \subseteq P_2' \subseteq P_2$ and so $\rho(Q_1') \geqslant \rho(P_1)$, $\rho(P_2') \geqslant \rho(Q_2)$.

By Schanuel's lemma, $Q_1' \oplus P_2' \cong Q_2 \oplus P_1$ and so, in our inequalities, equality must hold; that is $\rho(Q_1') = \rho(P_1)$ and $\rho(P_2') = \rho(Q_2)$. Moreover, any module intermediate between Q_1' and P_1 is between Q_1 and P_1 and so must have rank at least $\rho(Q_1') = \rho(P_1)$. So $\mathrm{im}\phi$ is of full presentation; $\ker\phi \cong Q_1'/Q_1$ and is also of full presentation; $\mathrm{coker}\phi$ is isomorphic to P_2/P_2' and is again of full presentation.

By Theorem 13, it follows that this abelian subcategory must be noetherian. The duality, $\mathrm{Hom}_R(-,R)$ from the category of finitely generated left projective modules to the category of finitely generated right projective modules induces a duality from the category of finitely generated left modules of full presentation to the category of right modules of full presentation, sending $\mathrm{coker}\alpha$, $\alpha : P \longrightarrow Q$ to $\mathrm{coker}\alpha^*$, $\alpha^* : Q^* \longrightarrow P^*$. Since the category of right modules of full presentation is noetherian, the category of left modules of full presentation is artinian as well as noetherian.

Corollary 1.15 Let R be a two-sided χ_0-hereditary ring with a faithful rank function ρ. Then the atomic full maps are exactly those maps which present simple modules in the category of modules of full presentation. Consequently, every map has a factorisation into atoms; if $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ and $\alpha = \beta_1\beta_2 \dots \beta_m$ where α_i, β_j are atoms, then $m = n$ and there is a permutation $\sigma \in S_n$ such that α_i is stably associated to $\beta_{\sigma(i)}$.

Pf It is clear that only atoms can present the simple modules of full presentation and that all atoms do.

Every object in an artinian and noetherian abelian category has a composition series of simple objects and any two such composition series have the smae length and the same set of composition factors. It follows from Schanuel's lemma that if

$$0 \longrightarrow P_1 \xrightarrow{\alpha_1} Q_1 \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow P_2 \xrightarrow{\alpha_2} Q_2 \longrightarrow M \longrightarrow 0$$

are exact sequences where the α_i are full maps, then α_1 is stably associated to α_2. Since a composition series for the module of full presentation N, presented by $0 \longrightarrow P \xrightarrow{\alpha} Q \longrightarrow N \longrightarrow 0$, where α is a full map, corresponds to an atomic factorisation of α, any two factorisations must have the same length and the factors in one may be paired off with the factors in the other so that the two atomic full maps paired together are stably associated.

It follows at once from this that if α is an atomic full map, $\alpha: P \longrightarrow Q$, then the ring of endomorphisms of M, where $0 \longrightarrow P \xrightarrow{\alpha} Q \longrightarrow M \longrightarrow 0$ is exact, is a skew field, since it is a simple object in a full abelian subcategory of the category of modules over R.

Chapter 2

The purpose of this chapter is to present statements of Bergman's coproduct theorems (4), and then to use these results as Bergman does in (5) to study a number of interesting ring constructions. Both of these papers by Bergman are of great interest and this chapter is not designed to obviate the need to read them, but rather to interest the reader in them and, at least, to make them plausible. However, it does provide a summary of all that we shall need in the rest of the book. At the end of the chapter we discuss the analogue of some of these constructions in the category of commutative algebras.

For ease of notation, we write our modules as right modules in this chapter. We begin by running through a few definitions we shall need in order to state the coproduct theorems.

An R_o-ring is a ring, R, with a specified homomorphism from R_o to R; it is a faithful R_o-ring if this homomorphism is an embedding. Given a family $\{R_\lambda : \lambda \in \Lambda\}$ of R_o-rings, there is clearly a coproduct in ths category of R_o-rings, which we write as $R = \coprod_{R_o, \lambda\in\Lambda} R_\lambda$, and call the ring coproduct of $\{R_\lambda : \lambda \in \Lambda\}$, amalgamating R_o; it may well be the trivial ring. We call R_o the base ring and the rings $\{R_\lambda : \lambda \in \Lambda\}$ are known as the factors of the ring coproduct R. It is technically quite convenient to write $M = \Lambda \cup \{0\}$, and to use μ for elements of M. Most of the time, R_o will be a semisimple artinian ring and each R_λ will be a faithful R_o-ring; in this case each R_λ embeds in $\coprod_{R_o, \lambda\in\Lambda} R_\lambda$, as we shall see. In the following we shall often take the indexing set for granted by writing $\coprod_{R_o} R_\lambda$ instead of $\coprod_{R_o, \lambda\in\Lambda} R_\lambda$.

Ideally we should like to reduce all problems about the module theory of $R = \coprod_{R_o} R_\lambda$ to questions about the modules over the factor rings; we are able to go a long way towards doing this when R_0 is semisimple artinian.

An <u>induced module</u> over R is a module of the form $\bigoplus_{\mu\in M} M_\mu \otimes_{R_\mu} R$, where M_μ is a right R_μ-module; an induced R-module of the form $M_0 \otimes_{R_0} R$ or an R_μ-module of the form $M_0 \otimes_{R_0} R_\mu$ is called a <u>basic module</u>. A map, α, between two induced modules $M = \bigoplus_{\mu\in M} M_\mu \otimes_{R_\mu} R$ and $N = \bigoplus_{\mu\in M} N_\mu \otimes_{R_\mu} R$ is an <u>induced map</u> if it takes the form

$$\bigoplus_{\mu\in M} \alpha_\mu \otimes_{R_\mu} R : \bigoplus_{\mu\in M} M_\mu \otimes_{R_\mu} R \longrightarrow \bigoplus_{\mu\in M} N_\mu \otimes_{R_\mu} R,$$

where $\alpha_\mu : M_\mu \longrightarrow N_\mu$ is an R_μ-linear map.

There are certain induced modules which are clearly isomorphic over R; let $M = \bigoplus_{\mu\in M} M_\mu \otimes_{R_\mu} R$ be an induced module, and suppose that for some μ_1, $M_{\mu_1} \cong M'_{\mu_1} \oplus (M_0 \otimes_{R_0} R_{\mu_1})$, where M_0 is an R_0-module. Then $M \cong N$, where $N = \bigoplus_{\mu\in M} N_\mu \otimes_{R_\mu} R$, where $M_\mu = N_\mu$ for $\mu \neq \mu_1, \mu_2$, $N_{\mu_1} \cong M'_{\mu_1}$ and $N_{\mu_2} \cong M_{\mu_2} \oplus (M_0 \otimes_{R_0} R_{\mu_2})$. The isomorphism is the identity map on M for $\mu \neq \mu_1, \mu_2$ and is the natural transfer map from $(M_{\mu_1} \otimes_{R_{\mu_1}} R \oplus M_{\mu_2} \otimes_{R_{\mu_2}} R)$ to $(M'_{\mu_1} \otimes_{R_{\mu_1}} R) \oplus ((M_{\mu_2} \oplus (M_0 \otimes_{R_0} R_{\mu_2})) \otimes_{R_{\mu_2}} R$. We call such an isomorphism a <u>basic transfer map</u>; if R_0 is a skew field, it is often called a free transfer map for the obvious reason.

There is one further type of map that we shall need to consider, a particular sort of isomorphism of an induced module. Suppose that the induced module, M, $\cong \bigoplus M_\mu \otimes_{R_\mu} R$ and, for some μ_1, there is a linear functional $e: M_{\mu_1} \longrightarrow R_{\mu_1}$. Extend e to a linear functional $\bar{e}: M \longrightarrow R$ by setting $\bar{e}(M_\mu) = 0$ $(\mu \neq \mu_1)$ and then extending by linearity; let $a \in R$ and $\ell_a : R \longrightarrow R$ be left multiplication by a; finally for some μ_2, let $x \in M_{\mu_2}$ and let $\gamma : R \longrightarrow M$ take 1 to x. Then if $\mu_1 \neq \mu_2$, the composite map $e\ell_a\gamma$ has square zero, and, if $\mu_1 = \mu_2$ we ensure this by specifying that x must lie in the kernel of e. The map $I_M - e\ell_a\gamma$ (where I_M is the identity map on M) is invertible; we call such an automorphism of M a <u>transvection</u>.

We may now state the main theorems. We shall assume from now on that R_o is semisimple artinian and that $\{R_\lambda : \lambda \in \Lambda\}$ is a family of faithful R_o-rings.

Coproduct Theorem 2.1 Let R_o be a semisimple artinian ring and $\{R_\lambda : \lambda \in \Lambda\}$ a family of faithful R_o-rings. Let $R = \coprod_{\substack{R_o \\ \lambda \in \Lambda}} R_\lambda$. If $M = \bigoplus_\mu M_\mu \otimes_{R_\mu} R$ is an induced module, each M_μ embeds in M, and M is isomorphic as an R_μ-module to a direct sum of M_μ and a basic module.

So we can write $R_\mu \subseteq R$ and $M_\mu \subseteq M$.

Coproduct Theorem 2.2 Let R_o be a semisimple artinian ring and $\{R_\lambda : \lambda \in \Lambda\}$ a family of faithful R_o-rings. Let $R = \coprod_{\substack{R_o \\ \lambda \in \Lambda}} R_\lambda$. Then any R-submodule of an induced R-module is isomorphic to an induced module.

Coproduct Theorem 2.3 Let $R = \coprod_{\substack{R_o \\ \lambda \in \Lambda}} R_\lambda$, where R_o is semisimple artinian and each R_λ is a faithful R_o-ring. Let $f : M \longrightarrow N$ be a surjection of finitely generated induced modules; then there is an isomorphism of induced modules $g : M' \longrightarrow M$ which is a finite composition of basic transfers and transvections such that the composite $gf : M' \longrightarrow N$ is an induced map.

These three theorems allow us to deduce all the rest of the results of this chapter; however, in later work, we shall need technical versions of these theorems and in order to state these results we shall need to set up a certain amount of the machinery for proving the coproduct theorems. We shall do this under the assumption that R_o is a skew field and indicate afterwards the modifications needed to deal with the more general case.

We assume that R_o is a skew field; then each R_μ is a free right

R_o-module, and we choose a basis for R of the form $1 \cup \{T_\lambda\}$. Write $T = \bigcup_{\lambda\in\Lambda} T_\lambda$. For each N_μ in the induced module $\bigoplus N_\mu \otimes_{R_\mu} R$ (where $R = \bigsqcup_{R_o} R_\lambda$), we pick an R_o-basis S_μ. Write $S = \bigcup_{\mu\in M} S_\mu$. If $t \in T_\lambda$, we say that it is <u>associated to</u> λ; and if $s \in S_\lambda$, we say also that it is <u>associated to</u> λ. If $s \in S_o$, it is associated to no index. A monomial is a formal product $st_1 \ldots t_n$, $s \in S$, $t_i \in T$ such that no two successive terms in the series $s, t_1, \ldots, t_n$ are associated to the same index. Let U be the set of monomials; an element of U is <u>associated to</u> λ if and only if its last factor (in S or T) is associated to λ. Every element of U is associated to some index except for those in $S_o \subseteq U$. We denote by $U_{\sim\lambda}$ those elements of U not associated to λ.

<u>Coproduct Theorem 2.4</u> (see coproduct theorem 2.1) Let R_o be a skew field and let $\{R_\lambda : \lambda \in \Lambda\}$ be a family of R_o-rings. Let $R = \bigsqcup_{R_o} R_\lambda$; then the induced module $\bigoplus_{\mu\in M} N_\mu \otimes_{R_\mu} R$ has for a right R_o-basis the set U defined in the foregoing. For each $\lambda \in \Lambda$, N is the direct sum as right R_λ-module of N_λ and a free right R_λ-module with basis $U_{\sim\lambda}$.

Given $\lambda \in \Lambda$ and $u \in U_{\sim\lambda}$, we denote by $c_{\lambda u} : N \longrightarrow R$ the R_λ-linear right co-efficient of u map given by the decomposition of N in this last theorem; for $u \in U$, we denote by $c_{ou} : N \longrightarrow R_o$ the R_o-linear right co-efficient function given by the decomposition of N as R_o-module. For $\lambda \in \Lambda$, the λ-<u>support</u> of an element x in N relative to the decomposition of N in the last theorem is the set of elements $u \in U_{\sim\lambda}$ such that $c_{\lambda u}(x) \neq 0$; x has empty λ-support if and only if it lies in N_λ. The <u>0-support</u> (or <u>support</u>) of an element x consists of those elements $u \in U$ such that $c_{ou}(x) \neq 0$.

The degree of a monomial $st_1 \ldots t_n$ is defined to be $(n+1)$ and the degree of any element of N is the maximum degree of a monomial of an element in its support. We define an element x in N to be <u>λ-pure</u> if all those monomials in its support of maximum length are associated to λ. It is <u>0-pure</u> if it is not λ-pure for any λ.

Next, we well-order the sets S and T in some manner this induces the lexicographic ordering (reading from left to right) on the monomials of fixed degree and so U is well-ordered by regarding elements of greater degree as greater.

The leading term of an element in N is the maximum element of its support. We sometimes call this the <u>0-leading term</u>. If some element x is not λ-pure then some of the elements in its support of maximum degree will lie in $U_{\sim\lambda}$; we define the <u>λ-leading term</u> of x to be the maximal such term.

<u>Coproduct Theorem 2.5</u> (see coproduct theorem 2.2) Let R_o be a skew field and let $R = \underset{\lambda\in\Lambda}{\overset{}{\sqcup_{R_o}}} R_\lambda$. Let $N = \bigoplus_{\mu\in M} N_\mu \otimes_{R_\mu} R$, and let S,T and U be as in the foregoing discussion. Let L be some R-submodule of N. Define L_μ to be the R_μ-submodule of L consisting of those elements of L whose μ-support does not contain the μ-leading terms of some non-μ-pure element of L. Then $L \cong \bigoplus_{\mu\in M} L_\mu \otimes_{R_\mu} R$ and this is induced by the embeddings $L_\mu \hookrightarrow L$.

Now we shall outline the adjustments needed to deal with the more general situation where R_o is a semisimple artinian ring. The first step is to use Morita equivalence to pass from this case to the particular situation that R_o is a finite direct sum of skew fields. This technique is of some interest and use in its own right.

Let R_o be semisimple artinian and let $S_1,\dots,S_n$ be a complete set of simple R_o-modules such that $S_i \cong S_j \Leftrightarrow i = j$. The projective module $P = \bigoplus_{i=1}^{n} S_i$ is a projective generator over R_o; so, by Morita equivalence, the category of modules over R_o is naturally equivalent to the category of projective modules over $\bar{R}_o = \mathrm{End}_{R_o}(P)$. Since each S_i is simple and they are mutually non-isomorphic,

$$\bar{R}_o = \mathrm{End}_{R_o}(\bigoplus_{i=1}^{n} S_i) = \prod_{i=1}^{n} \mathrm{End}_{R_o}(S_i),$$

which is a direct sum of skew fields.

Let R_λ be a faithful R_o-ring; then $P \otimes_{R_o} R_\lambda$ is a projective generator over R_λ, so that the category of modules over R_λ is naturally equivalent to the category of modules over $\bar{R}_\lambda = \mathrm{End}_{R_\lambda}(P \otimes_{R_o} R\ )$ and moreover $\bar{R}_o$ embeds in $\bar{R}_\lambda$.

We give an example to clarify what is happening here. Let $R_o = M_2(k) \times k$; in this case $\bar{R}_o$ is $k \times k$. Let $R_1 = M_3(k)$, where R_1 is an R_o-ring via $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e\right) \longrightarrow \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$. Then $\bar{R}_1 = M_2(k)$ and it is an $\bar{R}_o$-ring via the map $(a,b) \longrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

Returning to the general case, we are given a family of faithful R_o-rings $\{R_\lambda : \lambda \in \Lambda\}$ and we have associated to this family a family of $\bar{R}_0$-rings $\{\bar{R}_\lambda : \lambda \in \Lambda\}$. We wish to study the ring $R = \coprod_{R_o} R_\lambda$; and as before, we form the $\bar{R}_o$-ring, $\bar{R} = \mathrm{End}_R(P \otimes_{R_o} R)$ and this is just the ring coproduct $\bar{R} = \coprod_{\bar{R}_o} \bar{R}_\lambda$. Therefore, in order to study the category of R-modules we may as well study the category of $\bar{R}$-modules, where $\bar{R}$ is a ring coproduct amalgamating a subring which is a direct sum of skew fields. All the statements of Theorems 2.1, 2.2, and 2.3 are Morita invariant and so translate well.

So for the present we assume that R_o is a direct sum of skew fields, $R_o = \prod_{i=1}^{n} K^{(i)}$; let $\{e^{(i)} : i=1,\ldots,n\}$ be the complete set of orthogonal central idempotents. Any R_o-module has a decomposition as a sum of vector spaces over $K^{(i)}$, $M \cong \bigoplus_{i=1}^{n} Me^{(i)}$; we choose a basis $B^{(i)}$ for each $Me^{(i)}$ as a vector space over $K^{(i)}$. We call the n-tuple of bases $\{B^{(i)}\}$ a basis for M over R_o.

If R_λ is an R_o-ring, it decomposes as a right R_λ-module, $R_\lambda \cong \bigoplus_{i=1}^{n} e^{(i)}R_\lambda$; we write ${}^{(i)}R_\lambda = e^{(i)}R_\lambda$. In turn, ${}^{(i)}R_\lambda$ decomposes as a right R_o-module as ${}^{(i)}R_\lambda \cong \bigoplus_{j=1}^{n} {}^{(i)}R_\lambda e^{(j)}$; again we write ${}^{(i)}R_\lambda^{(j)} = e^{(i)}R_\lambda e^{(j)}$. We note that the usual formalism of matrix units works with respect to these superscripts. We pick bases for ${}^{(i)}R_\lambda^{(j)}$ as right $K^{(j)}$-module, ${}^{(i)}T_\lambda^{(j)}$ for $i \neq j$, ${}^{(i)}T^{(i)} \cup \{e_i\}$ for the remaining cases. We note that this gives a basis for R as R_o-ring in the sense defined earlier.

Let $\{N_\mu : \mu \in M = \Lambda \cup \{0\}\}$ be a family of modules such that N_μ is an R_μ-module. We choose an R-basis of each N_μ, $\{S_\mu^{(i)} : i = 1,\dots,n\}$.

Let $S = \cup S_\mu^{(i)}$ and $T = \bigcup {}^{(i)}T_\lambda^{(j)}$. For each $t \in {}^{(i)}T_\lambda^{(j)}$, i and j are respectively the <u>left</u> and <u>right index</u> of t; λ is the <u>Λ-index</u> of t. A member of $S^{(i)}$ has a right index i, and if it does not lie in N_o, it has a Λ-index.

Let U be the set of formal products $st_1 \dots t_n$ where $s \in S$, $t_i \in T$, where adjacent terms do not have the same Λ-index but the right index of any term is equal to the left index of the next term. The right index or Λ-index of any such product is the right index or Λ-index of its last term. So we may partition $U = \bigcup_{i=1}^{n} U^{(i)}$; we form the free $K^{(i)}$-module $N^{(i)}$ on $U^{(i)}$ and consider the R_o-module $N = \bigoplus_{i=1}^{n} N^{(i)}$.

Let $U_{\sim\lambda}^{(i)}$ be those elements of $U^{(i)}$ not associated with λ. We may form the R_λ-module $N_\lambda \oplus (\bigoplus_{i=1}^{n} U_{\sim\lambda}^{(i)} K^{(i)} \otimes_{R_o} R_\lambda)$.

<u>Coproduct Theorem 2.6</u> Let R_o be a direct sum of skew fields, $\bigtimes_{i=1}^{n} K^{(i)}$. Let $\{R_\lambda : \lambda \in \Lambda\}$ be a family of faithful R_o-rings and let $\{N_\mu : \mu \in M = \Lambda \cup \{0\}\}$ be a collection of R_μ-modules. Then $\bigoplus_{\mu \in M} N_\mu \otimes_{R_\mu} R$, where $R = \coprod_{R_o} R_\lambda$, is isomorphic to N (defined above) as R_o-module , and as R_λ-module is isomorphic to $N_\lambda \oplus (\bigoplus_{i=1}^{n} U_{\sim\lambda}^{(i)} K^{(i)} \otimes_{R_o} R_\lambda)$, where $U_{\sim\lambda}^{(i)}$ in this last representation is identified in N as a subset of U.

We note that for $u \in U_{\sim\lambda}^{(i)}$, uR_λ is isomorphic to $e^{(i)}R_\lambda$ via a map sending u to $e^{(i)}$. Consequently, the definition given in Theorem 2.6 allows us to define co-ordinate functions $c_{\mu u} : N \longrightarrow R_\mu$, which take values in $e^{(i)}R_\mu$, where i is the right index of u.

As we did when R_o was simply a skew field, we well-order S and T and then we well-order U length-lexicographically. We define degree, μ-purity, the μ-leading term and so on as we did previously. To recover a version of coproduct theorem 2.5 we need one more concept - that of homogeneity. Given an R_o-module M we define $m \in M$ to be <u>i-homogeneous</u> if $me^{(i)} = m$. An element is <u>homogeneous</u> if it is homogeneous for some i.

<u>Coproduct Theorem 2.7</u> Let R_o be a direct sum of skew fields $\overset{n}{\underset{i=1}{\times}} K^{(i)}$; let $\{R_\lambda : \lambda \in \Lambda\}$ be a family of faithful R_o-rings and let $R = \underset{R_o}{\sqcup} R_\lambda$. Let N be the induced module $\bigoplus_{\mu \in M} N_\mu \otimes_{R_\mu} R$ and let L be an R-submodule of N. Let $L_\mu^{(i)}$ be the R_o-submodule of L consisting of those i-homogeneous elements of L whose μ-support does not contain the μ-leading term of some homogeneous non-μ-pure element of L; then $\bigoplus_{i=1}^{n} L^{(i)}$ is an R_μ-submodule, L_μ, of L and $L \cong \bigoplus_{\mu \in M} L_\mu \otimes_{R_\mu} R$.

In the more general case, where R_o is a semisimple artinian ring; $\{R_\lambda\}$ is a family of faithful R_o-rings, $R = \underset{R_o}{\sqcup} R_\lambda$ and $N = \bigoplus_{\mu \in M} N_\mu \otimes_{R_\mu} R$ with R-submodule L, we pass via Morita equivalence (denoted by bars, $\bar{R}_o$ for R_o and so on) to the case described in the hypotheses of the theorem above. We then produce distinguished submodules $\bar{L}_\mu$ such that $\bar{L} \cong \bigoplus_{\mu \in M} \bar{L}_\mu \otimes_{\bar{R}_\mu} \bar{R}$; by Morita equivalence we have distinguished submodules L_μ such that $L \cong \bigoplus_{\mu \in M} L_\mu \otimes_{R_\mu} R$.

We note a very minor corollary of the details of coproduct theorem 7, which will turn out to have a good deal of consequence later.

<u>Remark 2.8</u> Let $R = \underset{R_o}{\sqcup} R_\lambda$, where R_o is semisimple artinian and each R_λ is a faithful R_o-ring. Let L be an R-submodule of the induced module $\bigoplus_{\mu \in M} N_\mu \otimes_{R_\mu} R$; then in the decomposition of L given in coproduct theorem 7 and the following discussion, $L \cong \bigoplus_{\mu \in M} L_\mu \otimes_{R_\mu} R$, we have $L \cap N_\mu \subseteq L_\mu$.

Pf By the process of Morita equivalence, it is enough to show this when R_o is a direct product of skew fields. In this case, it is clear, for if $\ell \in L \cap N_\mu$, $\ell e^{(i)}$ has empty μ-support and is i-homogeneous, so it satisfies the conditions to lie in $L_\mu^{(i)}$.

It is clear that the information we have built up in the preceding results allows us to answer a number of natural questions about ring coproducts amalgamating a common simple artinian subring. We begin with a few such applications.

Theorem 2.9 Let $R = \coprod_{R_o} R_\lambda$, where R_o is semisimple artinian and each R_λ is a faithful R_o-ring. Then the homological (or weak) dimension of $M_\mu \otimes_{R_\mu} R$ over R is equal to the homological (or weak) dimension of M_μ over R_μ.

Pf Given a resolution $\underline{P} \longrightarrow M_\mu \longrightarrow 0$ over R , $\underline{P} \otimes_{R_\mu} R \longrightarrow M_\mu \otimes_{R_\mu} R \longrightarrow 0$ is a resolution of $M_\mu \otimes_{R_\mu} R$ over R, since, by theorem 2.1, R is a left flat R -module. Again by theorem 2.1, this resolution considered as a sequence of R -modules contains $P \longrightarrow M_\mu \longrightarrow 0$; and it follows trivially that the homological (and by a similar argument the weak) dimension of M_μ over R_μ is equal to that of $M_\mu \otimes_{R_\mu} R$ over R.

From this we are able to determine the global and weak global dimension of a ring coproduct.

Theorem 2.10 Let $R = \coprod_{R_o} R_\lambda$ where R_o is semisimple artinian and each R_λ is a faithful R_o-ring; then the right global dimension of R is equal to $\sup_\lambda\{$global dimension $R_\lambda\}$, provided that at least one of the R_λ has global dimension > 0. If each R_λ is semisimple artinian, however, it may be 0 or 1. A similar result holds for weak global dimension.

Pf Consider any submodule M of a free module; by theorem 2.2 it is an induced module. Moreover, by theorem 2.1, if $M \cong \bigoplus_{\mu \in M} M_\mu \otimes_{R_\mu} R$, each

M_μ is a submodule, as R_μ-module, of a basic module. So each M_μ is a submodule of a free module, too. We deduce from theorem2.9 that the global dimension of R is the supremum of the global dimensions of the R_λ, provided that at least one of these is greater than 0. If each of the R_λ is semisimple artinian, it follows by the same argument that the global dimension is at most 1.

We leave it to the reader to show that if R_0 is a skew field and each R_λ is semisimple artinian then the ring coproduct has global dimension 1. We present an example to show that the global dimension can remain 0 for suitable semisimple R_0 and R_λ.

Let $R_0 = k\times k\ \ k$, $R_1 = M_2(k)\times k$ and $R_2 = k\times M_2(k)$, where $k\times k\times k \longrightarrow M_2(k)\times k$ by $(a,b,c)\longrightarrow\left(\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix},c\right)$ and $k\times k\times k\longrightarrow k\times M_2(k)$ by $(a,b,c)\longrightarrow\left(a,\begin{pmatrix} b & 0\\ 0 & c\end{pmatrix}\right)$.

It is clear that $R_1 \underset{R_0}{\sqcup} R_2 \cong M_3(k)$.

Exactly the same proof as before holds for the weak global dimension. It also show that the coproduct of right semihereditary or right $\aleph_0$-hereditary rings is respectively right semihereditary or right $\aleph_0$-hereditary.

Before passing from the study of the category of modules over the coproduct to the more particular study of the category of finitely generated projective modules over the coproduct, we leave as an exercise for the reader the details of our next example, which answers a question of Bergman.

Example 2.11 We wish to find an extension of induced modules over a coproduct that is not itself an induced module.

Consider the free ring $R = k\langle x,y\rangle \cong k[x]\underset{k}{\sqcup} k[y]$. The module $xyR\backslash R$ is an extension of the induced module $yR\backslash R$ by the induced module $xR\backslash R$, but is not itself an induced module.

We already know all the objects in the category of finitely generated projective modules over a ring coproduct. Specifically:

Theorem 2.12 Let $R = \coprod_{R_0} R_\lambda$, where R_0 is a semisimple artinian ring and $\{R_\lambda : \lambda \in \Lambda\}$ is a family of faithful R_0-rings. Then a projective module over R has the form $\bigoplus_{\mu \in M} P_\mu \otimes_{R_\mu} R$, where each P_μ is a projective module over R_μ. The monoid of isomorphism classes of finitely generated projective modules over R is isomorphic to the commutative monoid coproduct of the monoids of isomorphism classes of finitely generated projective modules over R , amalgamating the monoid of isomorphism classes of finitely generated projective modules over R_0. Consequently, $K_0(R)$ is the commutative group coproduct of $K_0(R_\lambda)$, amalgamating $K_0(R_0)$.

Pf Every projective module, P, lies in a free module and so must be an induced module by theorem 2.2. Moreover, by theorem 2.9, we know that if $P \cong \bigoplus_{\mu \in M} M_\mu \otimes_{R_\mu} R$ each M_μ must be projective.

Theorem 2.3 tells us that any isomorphism between two finitely generated induced modules is the composite of a finite sequence of basic transfers and transvections followed by an induced map. Transvections do not affect the form of an induced module; basic transfers correspond to relations in the monoid of isomorphism classes of induced modules that identify the basic modules over R_{λ_1}, induced up to R, with the basic modules over R_{λ_2} induced up to R. In particular, this applies inside the monoid of isomorphism classes of finitely generated projectives, $P_\oplus(R)$. So

$$P_\oplus(R) \cong \coprod_{P_\oplus(R_0)} P_\oplus(R_\lambda),$$

where '$\sqcup$' is used here for the commutative monoid coproduct.

The corresponding result for $K_0(R)$ holds since $K_0(-)$ is just the universal abelian group functor applied to $P_\oplus(-)$.

We should also like to have information about the maps between finitely generated projective modules; that is, we should like to be able to explain all isomorphisms and all zero-divisors. We already know how to explain all isomorphisms, since theorem 2.3 actually explains all surjections. So that leaves the zero-divisors.

Theorem 2.13 Let $R = \bigcup_{R_0} R_\lambda$, where R_0 is a semisimple artinian ring and each R is a faithful R_0-ring. Let $\alpha: P \longrightarrow P'$ and $\beta: P' \longrightarrow P''$ be maps between finitely generated projectives over R such that $\alpha\beta = 0$. Then there exists a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\alpha"] \arrow[d] & P' \arrow[r, "\beta"] \arrow[d] & P'' \arrow[d] \\
\oplus P_\lambda \otimes_{R_\lambda} R \arrow[r, "\oplus \alpha_\lambda \otimes_{R_\lambda} R"] & \oplus P'_\lambda \otimes_{R_\lambda} R \arrow[r, "\oplus \beta_\lambda \otimes_{R_\lambda} R"] & \oplus P''_\lambda \otimes_{R_\lambda} R
\end{tikzcd}$$

where each direct sum in this diagram is a finite direct sum of finitely generated projective modules and $P_\lambda \xrightarrow{\alpha_\lambda} P'_\lambda \xrightarrow{\beta_\lambda} P''_\lambda$ are homomorphisms of finitely generated projective R_λ-modules that compose to zero.

Pf First, the image of β is an induced module by theorem 2.2; so $P' \longrightarrow \operatorname{im}\beta$ is a surjection of induced modules and, by theorem 2.3, there exists a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\alpha"] \arrow[d] & P' \arrow[r, "\beta"] \arrow[d] & P'' \arrow[d] \\
\oplus \ker\beta_\lambda \otimes_{R_\lambda} R \arrow[r] & \oplus P'_\lambda \otimes_{R_\lambda} R \arrow[r, "\oplus \beta_\lambda \otimes_{R_\lambda} R"] & \oplus \operatorname{im}\beta_\lambda \otimes_{R_\lambda} R \cong \operatorname{im}\beta
\end{tikzcd}$$

So the problem is to replace $\ker\beta_\lambda$, $\operatorname{im}\beta_\lambda$ by suitable finitely generated projective modules. $\operatorname{im}\alpha'$ is a finitely generated submodule of $\oplus \ker\beta_\lambda \otimes_{R_\lambda} R$ and so lies in an R-submodule generated by finitely many elements from $\ker\beta_\lambda$. Thus there is an induced map from $\oplus P_\lambda \otimes_{R_\lambda} R$ to $\oplus \ker\beta_\lambda \otimes_{R_\lambda} R$, where each P_λ is a finitely generated free R_λ-module, such that the image of this induced map contains the image of α'. Since P is a projective module, we find a commutative diagram:

$$\begin{tikzcd}
P \arrow[rr, "\alpha"] \arrow[d, "\alpha''"'] & & P' \arrow[rr, "\beta"] \arrow[d, "u"] & & P'' \\
\oplus P_\lambda \otimes_{R_\lambda} R \arrow[rr, "\oplus\alpha_\lambda\otimes_{R_\lambda}R"] & & \oplus P'_\lambda \otimes_{R_\lambda} R \arrow[rr, "\oplus\beta_\lambda\otimes_{R_\lambda}R"] & & \oplus \mathrm{im}\beta_\lambda \otimes_{R_\lambda} R \arrow[u]
\end{tikzcd}$$

We miss out the bottom right-hand corner of this diagram ($\oplus\, \mathrm{im}\beta_\lambda \otimes_{R_\lambda} R$); dualize the remainder of the digram and then we fill in the hole in the bottom left-hand corner as we did before. The diagram we obtain is the dual of the one we wish to find.

These results on the coproduct construction allow us to study a number of interesting constructions on the category of finitely generated projectives of a ring. Given a k-algebra S, the ring coproduct $S \underset{k}{\sqcup} k[x]$ may be regarded as the S-ring with a universal map on the free module of rank 1 that centralises the k-algebra structure. Suppose that P_1 and P_2 are finitely generated projective modules over S; we are also interested in finding and studying an S-ring T, with a universal map from $P_1 \otimes_S T$ to $P_2 \otimes_S T$ centralising the k-structure. In the language of functors, we are trying to find the S-ring in the category of k-algebras that represents the functor which associates to each S-ring, S', the set $\mathrm{Hom}_{S'}(P_1 \otimes_S S', P_2 \otimes_S S')$. An object $\underline{O}$ in a cagegory $\underline{C}$ represents a covariant functor $F: \underline{C} \to \underline{\mathrm{Sets}}$ if $F(-)$ is naturally equivalent to $\mathrm{Hom}_{\underline{C}}(\underline{O}, -)$. We should like to investigate other universal constructions; thus, we wish to find an S-ring, T_1, in the category of k-algebras with a universal isomorphism between $P_1 \otimes_S T_1$ and $P_2 \otimes_S T_1$. That is, T_1 represents the functor on the category of S-rings that are k-algebras which associates to each object, S', in the category the set of isomorphisms between $P_1 \otimes_S S'$ and $P_2 \otimes_S S'$. Again, we should like to find an S-ring, T_2, that is a k-algebra with a universal idempotent endomorphism of $P \otimes_S T_2$; that is, T_2 represents the functor on the category of S-rings that are k-algebras which associates to each object S' the set of idempotent endomorphisms of $P \otimes_S S'$.

One construction of a slightly different nature that we wish to study is the k-algebra, T', with a universal homomorphism from S to $M_n(T')$. Here T' represents the functor on the categroy of k-algebras $\mathrm{Hom}_{k\text{-alg}}(S, M_n(-))$.

The constructions on the category of finitely generated projective modules are all approached in a similar way. For suitably large n, there are orthogonal idempotents $e_1, e_2 \in M_n(S)$ such that $e_i({}^nS) \cong P_i$; under the Morita equivalence of S and $M_n(S)$, $e_i({}^nS)$ becomes $e_iM_n(S)$, so it is enough to find an $M_n(S)$-ring with a universal homomorphism from $e_1M_n(S)\otimes_{M_n(S)}-$ to $e_2M_n(S)\otimes_{M_n(S)}-$.

Let e_3 be the idempotent such that $e_1 + e_2 + e_3 = 1$ in $M_n(S)$; we have a map $k\times k\times k \longrightarrow M_n(S)$ given by $(a,b,c) \longrightarrow e_1a + e_2b + e_3c$ which makes $M_n(S)$ into a faithful $k\times k\times k$-ring. The projective module induced by the first summand of $k\times k\times k$ is $e_1M_n(S)$ and that induced by the second summand is $e_2M_n(S)$. It is easily seen that the $k\times k$-ring with a universal map from the projective module $k\times 0$ to the projective module $0\times k$ is just the lower triangular matrix ring $T_2(k) = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$, where $k\times k \longrightarrow T_2(k)$ by $(a,b) \rightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and the universal map is induced by the element e_{21}. So, if we take the ring coproduct $T_2(k)\times k \underset{k\times k\times k}{\sqcup} M_n(S)$, we see that we have adjoined a universal map from $e_1M_n(S)$ to $e_2M_n(S)$. $T_2(k)\times k \underset{k\times k\times k}{\sqcup} M_n(S) \cong M_n(T)$, where T is the centraliser of the copy of $M_n(k)$ in the second factor of the coproduct; by Morita equivalence, this S-ring, T, represents the functor which associates to each S-ring S' in the category of k-algebras the set $\mathrm{Hom}_{S'}(P_1\otimes_S S', P_2\otimes_S S')$. We shall use the symbol $T = S_k\langle\alpha : P_1 \longrightarrow P_2\rangle$ for this construction.

<u>Theorem</u> <u>2.14</u> Let S be a k-algebra and let P_1, P_2 be finitely generated projective modules over S. Let $T = S_k\langle\alpha : P_1 \longrightarrow P_2\rangle$ be the k-algebra with a universal map from $P_1\otimes_S T$ to $P_2\otimes_S T$; then T

has the same global (or weak) dimension as S except when the global (or weak) dimension of S is 0; in this case, T has dimension 1. All finitely generated projective modules are induced from S; in fact, $P_{\oplus}(T) \cong P_{\oplus}(S)$.

Pf We retain the notation of the preceding discussion.

We see that $M_n(T) \cong M_n(S) \underset{k\times k\times k}{\sqcup} T_2(k)\times k$; $T_2(k)$ has global (or weak) dimension 1, so $M_n(T)$ has global (or weak) dimension equal to that of S or equal to 1 is S has global dimension (or weak dimension) 0 by theorem 2.10. The monoid of isomorphism classes of finitely generated projective modules over $T_2(k)\times k$ is isomorphic to the monoid of isomorphism classes of finitely generated projective modules over $k\times k\times k$; so from theorem 2.12 we deduce that $P_{\oplus}(M_n(S) \underset{k\times k\times k}{\sqcup} T_2(k)\times k) \cong P_{\oplus}(M_n(S))$. Hence $P_{\oplus}(M_n(T)) \cong P_{\oplus}(M_n(S))$, from which , by Morita equivalence, we deduce that $P_{\oplus}(T) = P_{\oplus}(S)$.

The $k\times k$-ring with a universal isomorphism between the projective module $k\times 0$ and the projective module $0\times k$ is easily seen to be $M_2(k)$, where $k\times k \longrightarrow M_2(k)$ by $(a,b)\longrightarrow\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}$; the elements e_{21} and e_{12} are the universal isomorphism and its inverse. Therefore, if e_1 and e_2 are orthogonal idempotents in $M_n(S)$ such that $e_i({}^nS) \cong P_i$, the $M_n(S)$-ring with a universal isomorphism between $e_1(M_n(S))$ and $e_2(M_n(S))$ is just $M_n(S) \underset{k\times k\times k}{\sqcup} M_2(k)\times k$, where $k\times k\times k \longrightarrow M_n(S)$ via $(a,b,c)\longrightarrow e_1a + e_2b + e_3c$ ($e_3 = 1 - e_1 - e_2$) and $k\times k\times k \longrightarrow M_2(k)\times k$ via $(a,b,c)\rightarrow\left(\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}, c\right)$. By Morita equivalence, we have an S-ring T which has a universal isomorphism between $P_1\otimes_S T$ and $P_2\otimes_S T$ by taking the centraliser of the copy of $M_n(k)$ in the first factor of this coproduct. We shall use the symbol $S_k\langle\alpha,\alpha^{-1};P_1\longrightarrow P_2\rangle$ for this ring.

Theorem 2.15 Let S be a k-algebra and let P_1, P_2 be finitely generated projective modules over k. Then $T = S_k\langle\alpha,\alpha^{-1};P_1\longrightarrow P_2\rangle$ has the same global (or weak) dimension as S, except possibly when S has global (or weak) dimension 0. Here T may have dimension 0 or 1. $P_\oplus(T)$ is isomorphic to the quotient of $P_\oplus(S)$ by the one relation $[P_1] = [P_2]$.

Pf We use the notation of the preceding discussion.

$$M_n(T) \cong M_n(S_k\langle\alpha,\alpha^{-1}:P_1\longrightarrow P_2\rangle \cong M_2(k)\times k \underset{k\times k\times k}{\sqcup} M_n(S).$$

The global (or weak) dimension of $M_n(T)$ must equal the global (or weak) dimension of $M_n(S)$, if this is not 0, by theorem 2.10. If it equals 0, however, the global dimension is at most 1; we see that it may be 1 by adjoining a universal isomorphism of the free module of rank 1 with itself over k, obtaining the Laurent polynomial ring $k[t,t^{-1}]$. On the other hand, it may be 0, as we find for the ring $k\times M_2(k)$, where we adjoin a universal isomorphism between $k\times 0$ and $0\times\begin{pmatrix}1&0\\0&0\end{pmatrix}M_2(k)$, obtaining $M_3(k)$.

For the last statement of the theorem, we note that

$$P_\oplus(M_2(k)\times k) \cong ([k\times 0\times 0] = [0\times k\times 0])\backslash P_\oplus[k\times k\times k];$$

since $P_\oplus(M_n(T))$ is the commutative monoid coproduct of $P_\oplus(M_2(k)\times k)$ and $P_\oplus(M_n(S))$ amalgamating $P_\oplus(k\times k\times k)$, we find that

$$P_\oplus(M_n(T)) \cong ([e_1M_n(S)] = [e_2M_n(S)])\backslash P_\oplus(M_n(S)).$$

By Morita equivalence we deduce that $P_\oplus(T)$ is the quotient of $P_\oplus(S)$ by the relation $[P_1] = [P_2]$.

Next we deal with the adjunction of a universal idempotent map on a finitely generated projective module P. If we adjoin a universal idempotent map to the free module of rank 1 over k, we obtain $k\times k$. Therefore, if the idempotent $e\in M_n(S)$ satisfies $e({}^nS)\cong P$, the S-ring and k-algebra with a universal idempotent on P, $S_k\langle e:e:P\longrightarrow P,\ e^2 = e\rangle$

is obtained by taking the centraliser of $M_n(k)$ in the ring coproduct $M_n(S) \sqcup_{k\times k} (k \times k) \times k$, where $k \times k$ embeds in $M_n(S)$ by $(a,b) \longrightarrow ea + (1-e)b$ and $k \times k$ embeds in $k \times k \times k$ by $(a,b) \longrightarrow (a,a,b)$.

Once again, we may deduce homological information:

<u>Theorem</u> <u>2.16</u> The global (or weak) dimesion of $T = S_k\langle e: e:P \to P,\ e^2 = e\rangle$ is equal to that of S except, possibly, when the dimension of S is 0, where it may be 0 or 1. $P_\oplus(S)$ embeds in $P_\oplus(T)$ and there are two more generators $[Q_1]$ and $[Q_2]$ subject to the relation

$[Q_1] + [Q_2] = [P \otimes_S T]$.

<u>Pf</u> This may be left to the reader; it in no way differs from the proofs of the last two theorems.

It is clear that these constructions may be put together in a number of interesting ways. One construction of some interest is the adjunction to a ring S of a pair of matrices ${}^mA^n$, ${}^nB^m$ such that $({}^mA^n)({}^nB^m) = I_m$ (the identity of $m \times m$ matrices). This may be obtained by adjoining universally an idempotent $n \times n$-matrix, E, to S, obtaining the ring T and then adjoining to this ring T a universal isomorphism between $E({}^nT)$ and ${}^mT$. So we are able to deduce in this case that the global dimension remains the same and that the monoid of isomorphism classes of finitely generated projective modules has exactly one new generator, $[Q]$, subject only to the relation

$[Q] + m = n$.

Our final construction is a k-algebra that represents the functor $\mathrm{Hom}_{k\text{-alg}}(S, M_n(-))$; following Bergman, we denote this ring by $k\langle S \longrightarrow M_n\rangle$.

<u>Theorem</u> <u>2.17</u> Consider the ring T which is the centraliser of the first factor in the ring coproduct $M_n(k) \sqcup_k S$. Then $k\langle S \longrightarrow M_n\rangle \cong T$. The global dimension of T is equal to $\max\{1, \mathrm{gl.dim.} S\}$.

Pf Given a homomorphism $\varphi: T \longrightarrow A$, we have homomorphisms

$$S \longrightarrow M_n(k) \underset{k}{\sqcup} S \cong M_n(T) \xrightarrow{M_n(\varphi)} M_n(A)$$

which determines a homomorphism $\bar{\varphi}: S \longrightarrow M_n(A)$. Conversely, given a homomorphism $\bar{g}: S \longrightarrow M_n(A)$, we have a map $g': M_n(k) \underset{k}{\sqcup} S \longrightarrow M_n(A)$, which maps the matrix units to the matrix units and acts as g on S. Restricting to the centraliser of the matrix units determines a map $\bar{g}: T \longrightarrow A$. These processes are mutually inverse; so they demonstrate the natural equivalence of $\mathrm{Hom}_{k\text{-alg}}(T,-)$ and $\mathrm{Hom}_{k\text{-alg}}(S, M_n(-))$. So $T = k\langle S \longrightarrow M_n \rangle$. By theorem 2.10, we see that the global dimension of $M_n(k) \underset{k}{\sqcup} S$ is equal to that of S except when S is of dimension 0; here it always goes up to 1.

This construction has some interesting properties;

Theorem 2.18 $k\langle S \longrightarrow M_n \rangle$ is a domain and the group of units is just $k^{\times}$.

Pf If $a, b \in k\langle S \longrightarrow M_n \rangle$ and $ab = 0$, we may regard a and b as endomorphisms of the induced projective module $P \otimes_{M_n(k)} (M_n(k) \underset{k}{\sqcup} S)$, where P is the simple $M_n(k)$-module. We write R for $M_n(k) \underset{k}{\sqcup} S$ in the following.

By theorem 2.13, we obtain a commutative diagram

$$\begin{tikzcd} P\otimes_{M_n(k)} R \arrow[r, "b"] \arrow[d] & P\otimes_{M_n(k)} R \arrow[r, "a"] \arrow[d, "\cong u"] & P\otimes_{M_n(k)} R \arrow[d] \\ Q_1\otimes_{M_n(k)} R \arrow[r, "\beta\otimes_{M_n(k)} R"] & P\otimes_{M_n(k)} R \arrow[r, "\alpha\otimes_{M_n(k)} R"] & Q_2\otimes_{M_n(k)} R \end{tikzcd}$$

where β, α are defined over $M_n(k)$ and compose to zero.

We know that the middle term of the bottom row is $P \otimes_{M_n(k)} R$, since this is the unique representation of this module as an induced module.

However, the only way that a pair of map over $M_n(k)$ $\beta: Q_1 \longrightarrow P$, $\alpha: P \longrightarrow Q_2$ can compose to zero is for one of the two maps to be zero itself, which , in turn, implies that either a or b must be zero.

The results on units are a consequence of theorem 2.3. For if we have a unit in $k\langle S \longrightarrow M_n \rangle$, it defines an automorphism of $P \otimes_{M_n(k)} R$; since P is indecomposable over $M_n(k)$, all basic transfers and transvections must be the identity map on $P \otimes_{M_n(k)} R$. Thus by theorem 2.3 the group of automorphisms of $P \otimes_{M_n(k)} R$ must simply be the group of automorphisms of P over $M_n(k)$, which is just $k^{\times}$ as stated.

It is interesting to examine some of these constructions in the category of commutative algebras. We shall not provide detailed proofs since this last section is meant only to be illustrative.

We begin with a commutative ring C and a couple of finitely generated projective modules P_1 and P_2 over C. We wish to find a commutative C-algebra with a universal map between P_1 and P_2.

Alternatively, we wish to find a commutative C-algebra $C[\alpha : P_1 \longrightarrow P_2]$ that represents the functor on the category of C-algebras $A \longmapsto \mathrm{Hom}_{A\text{-mod}}(P_1 \otimes_C A, P_2 \otimes_C A)$. The functor $\mathrm{Hom}_{-}(P_1 \otimes_C -, P_2 \otimes_C -))$ is naturally equivalent to $\mathrm{Hom}_{-}((P_1 \otimes_C P_2^{\times}) \otimes_C -, -)$, where $P_2^{\times}$ is the dual of P_2; but it is clear that the C-algebra representing this functor is just the symmetric algebra over C on the module $P_1 \otimes_C P_2^{\times}$. It is an immediate consequence that this algebra is geometrically regular over C; that is, for each prime ideal p of C, $C[\alpha : P_1 \longrightarrow P_2] \otimes_C \overline{Q(C/p)}$ is a regular ring (where $\overline{Q(C/p)}$ is the algebraic closure of the ring of quotients of C/p.

We shall find that this holds for each construction examined and we shall be able to obtain each construction in a sufficiently explicit form to be able to determine the global dimension of each geometric fibre over C. For example, in the above example, the fibre of ρ is isomorphic to $\overline{Q(C/\rho)}[x_{ij}: i = 1 \to m_1, j = 1 \to m_2]$ where m_i is the local rank of P_i at ρ.

We examine next the construction of adjoining a universal isomorphism from P_1 to P_2. This is only possible when the local rank of P_1 is equal to the local rank of P_2. If such is the case, P_1 becomes isomorphic to P_2 on a suitable Zariski cover and by further refinement we may assume that on each connected component P_i is a free module. Consequently, on each connected component of the Zariski cover the set of isomorphisms from P_1 to P_2 is a principal homogeneous space for $GL_n(-)$, where n is the local rank of P_i on that component. We can see at once that the fibre at any point, ρ, of spec C of the spectrum of the C-algebra representing isomorphisms from P_1 to P_2 is isomorphic to the affine space $GL_n(\overline{Q(C/\rho)})$, where $\overline{Q(C/\rho)}$ is the algebraic closure of C/ρ. It follows that $C[\alpha, \alpha^{-1}: P_1 \to P_2]$ is geometrically regular over C and the dimension of the fibre at any point ρ is n^2, where n is the local rank of P_i at ρ.

If we wish to adjoin an idempotent map $e: P_1 \to P_1$ to C, it is reasonable to specify its local rank also, which must be constant on connected components and dominated by the local rank of P_1. We may as well restrict to a situation where the local rank of P_1 is constant at n and the local rank of the idempotent is constant at m. So we wish to represent idempotents of rank m in $\mathrm{End}_A(P_1 \otimes_C A)$. For a given C-algebra A there is a Zariski cover of spec A on which P becomes a free module of rank n, and, for a particular idempotent $e \in \mathrm{End}_A(P_1 \otimes_C A)$, $e(P_1 \otimes_C A)$ becomes free with free complement; so, with respect to coverings in the Zariske topology, all such elements are conjugate. Moreover, the centraliser of such an idempotent becomes in the Zariski topology a form of $M_m(A) \times M_{n-m}(A)$; the consequence of this is that the set of idempotents of rank m in $\mathrm{End}_A(P_1 \otimes_C A)$ becomes, on a Zariski cover, $GL_m \times GL_{n-m} \backslash GL_n$. Since this last functor is geometrically regular, so is any twisted form of it; in particular, our commutative C-algebra with a universal idempotent of rank m acting on the projective module P_1 of rank n is geometrically regular and it is a quick check that the dimension of a fibre is 2mn.

Chapter 3

The purpose of this chapter is to investigate how rank functions and partial rank functions behave under the coproduct construction. On the way, we shall give an application of this theory to the problem of accessibility for finitely generated groups. However, the main application of the results of this chapter will be to hereditary rings in the following chapters.

When investigating partial rank functions on a ring coproduct, $R = R_1 \underset{R_0}{\sqcup} R_2$, where R_0 is a semisimple artinian ring, it is sensible to assume that it is defined on the image of $K_0(R_0)$ in $K_0(R)$. In this situation, we have the following lemma:

Lemma 3.1 The partial rank functions on a ring coproduct, $R = R_1 \underset{R_0}{\sqcup} R_2$, that are defined on the image of $K_0(R_0)$ in $K_0(R)$ are in 1-1 correspondence with pairs of rank functions (ρ_1, ρ_2) defined on R_1 and R_2 respectively, and defined on the image of $K_0(R_0)$ in $K_0(R_i)$, that agree on $K_0(R_0)$.

Pf This is clear, since by theorem 2.12, we have a pushout diagram:

$$\begin{tikzcd} K_0(R_0) \arrow[r] \arrow[d] & K_0(R_1) \arrow[d] \\ K_0(R_2) \arrow[r] & K_0(R) \end{tikzcd}$$

We shall often describe a partial rank function like those in lemma 3.1 as a pair (ρ_1, ρ_2) of partial rank functions on R_1 and R_2 respectively.

We wish to show that the inner projective rank of a map $\alpha : P \longrightarrow Q$ over R_1 with respect to a rank function ρ_1 is just the same as the inner projective rank of $\alpha \otimes_{R_1} R: P \otimes_{R_1} R \longrightarrow Q \otimes_{R_1} R$ over $R = R_1 \underset{R_0}{\sqcup} R_2$ with respect to the rank function (ρ_1, ρ_2). In order to do this, we need to investigate how the generating number with respect to a rank function of a module over one of the factors behaves. Before we do this, we present a similar result that is independent of rank functions.

In chapter 1, we have already encountered the notion of a ring with unbounded generating number; that is, no equation of the form ${}^{m}R' \cong {}^{m+1}R' \oplus P$ holds. If, however, such an equation holds, it is clearly true that ${}^{m}S \cong {}^{m+1}S \oplus (P \otimes_{R'} S)$ for all R'-rings S. In particular, if $R = R_1 \underset{R_0}{\sqcup} R'$, all finitely generated modules over R require at most m generators, whatever structure R_1 may have. Our next result shows that this is the only way things go wrong.

Theorem 3.2 Suppose that R_0 is a skew field and R_2 has unbounded generating number, where R_1 and R_2 are R_0-rings. Then the minimal number of generators over R_1 of the finitely generated module M over R_1 is equal to the minimal number of generators of $M \otimes_{R_1} R$ over R, where $R = R_1 \underset{R_0}{\sqcup} R_2$.

Pf Suppose that $M \otimes_{R_1} R$ may be generated by m elements over R; then there is a surjection of induced modules $\beta : {}^{m}R_1 \otimes_{R_1} R \longrightarrow M \otimes_{R_1} R$. By theorem 2.3, there is an isomorphism

$$\alpha : P_1 \otimes_{R_1} R \oplus P_2 \otimes_{R_2} R \longrightarrow {}^{m}R_1 \otimes_{R_1} R,$$

which is a composite of free transfers and transvections such that the composite $\alpha\beta$ is an induced map; that is, it maps P_1 onto M and P_2 to 0. We shall show that P_1 is a direct summand of ${}^{m}R_1$ and so M is an m-generator module.

We recall from theorem 1.3 that if R_2 has unbounded generating number, we have a well-defined partial rank function ρ_2 on stably free R_2-modules given by $\rho_2(P) = n - n'$, where $P \oplus {}^{n'}R_2 \cong {}^{n}R_2$. Moreover, since this number must always be non-negative, any free summand of P has rank at most $\rho_2(P)$.

If, after a sequence of free transfers and transvections we have passed from ${}^{m}R_1 \otimes_{R_1} R$ to $P_1 \otimes_{R_1} R \oplus P_2 \otimes_{R_2} R$, we claim that $P_1 \oplus {}^{\rho_2(P_2)}R_1 \cong {}^{m}R_1$. We assume that this is true at the n^{th} stage; if our next step is a transvection, it alters nothing; if it is a free transfer from P_1 to P_2 it is still true; if it is a free transfer from P_1 to P_2 we can only transfer at most $\rho_2(P_2)$ factors, so it is still true. Our claim holds by induction.

We see that if $M \otimes_{R_1} R$ is an m-generator R-module, it must be an m-generator R_1-module. The converse is clearly true, which completes our proof.

We pass to the more technical versions we need for partial rank functions in general.

<u>Theorem 3.3</u> Let $R = R_1 \sqcup_{R_0} R_2$, where R_o is a semisimple artinian ring and each R_i is a faithful R_o-ring; let ρ_1, ρ_2 be partial rank functions for R_1 and R_2 respectively, defined and agreeing on the image of $K_o(R_o)$ in $K_o(R_i)$. Let M_1 and M_2 be finitely generated R_1- and R_2-modules respectively, with generating numbers m_1 and m_2 with respect to ρ_1 and ρ_2 respectively. Then the generating number with respect to (ρ_1, ρ_2) of $M_1 \otimes_{R_1} R \oplus M_2 \otimes_{R_2} R$ is equal to $m_1 + m_2$.

<u>Pf</u> Certainly, it is at most $m_1 + m_2$.

Conversely, suppose that we have a surjection over R

$$(P_1 \otimes_{R_1} R) \oplus (P_2 \otimes_{R_2} R) \longrightarrow (M_1 \otimes_{R_1} R) \oplus (M_2 \otimes_{R_2} R)$$

where $\rho_i(P_i)$ is defined. By theorem 2.3, after a sequence of basic transfers and transvections on $P_1 \otimes_{R_1} R \oplus P_2 \otimes_{R_2} R$, we obtain an induced surjection

$$P_1' \otimes_{R_1} R \oplus P_2' \otimes_{R_2} R \longrightarrow M_1 \otimes_{R_1} R \oplus M_2 \otimes_{R_2} R.$$

Since ρ_i is defined on the image of $K_o(R_o)$ in $K_o(R_i)$ and agree there, the basic transfers produce modules on which ρ_i is defined and

$\rho_1(P_1) + \rho_2(P_2) = \rho_1(P_1') + \rho_2(P_2')$. But $\rho_i(P_i') \geqslant m_i$, since P_i' maps onto M_i; so $\rho_1(P_1) + \rho_2(P_2) \geqslant m_1 + m_2$ as we wished to show.

At this point it is of some interest to show how this may be applied to provide a fairly simple proof of a result of P. Linnell on accessibility for finitely generated groups.

We begin with a summary of the background to the problem and refer the reader to Dicks (24) for the details of the subject.

Let X be a finite connected graph with edge set E and vertex set V; we allow the beginning ie and the end τe of an edge e to be the same vertex. We label the edges and vertices of X with groups, and for each edge $e \in E$ we have embeddings $G_e \hookrightarrow G_{ie}$, $G_e \hookrightarrow G_{\tau e}$; if ie = τe, these embeddings may well be different. This is known as a <u>graph of groups</u>; we associate to this a group, the <u>fundamental group</u> of a graph of groups in the following way.

Let T be a maximal sub-tree of X, which we regard as a subtree of groups; let v be a vertex with only one edge, e, incident with it in T, so v = ie without any loss of generality. We pass to the tree T' with one fewer vertex, obtained by omitting ie and e, which we label by the same groups as before except for τe, which we label with the group coproduct of G_{ie} and $G_{\tau e}$ amalgamating G_e, $G_{ie} \underset{G_e}{*} G_{\tau e}$. By induction, we eventually reach a single point with a group, G_T, associated to it with a specified homomorphism $G_v \longrightarrow G_T$ for each vertex v of T such that $G_{ie} \cap G_{\tau e} = G_e$ for each edge $e \in T$; clearly G_T is universal with respect to this property.

For each edge $e \in X - T$, we have two embeddings of G_e in G_T, $G_e \longrightarrow G_{ie} \longrightarrow G_T$ and $G_e \longrightarrow G_{\tau e} \longrightarrow G_T$; so we form the multiple HNN extension of G_T over all edges of X - T with respect to these pairs of embeddings. The resulting group, G_X, is the fundamental group of our graph of groups; it can be shown that it is independent of the maximal sub-tree chosen. Also if X' is a full connected subgraph of X, we may express G_X as the fundamental group of a graph of groups on X, where X is the graph obtained from X by shrinking X' to a point. The groups associated to edges and vertices are the same as in X for edges and vertices that lie in X; the group associated to the new point is $G_{X'}$, the fundamental group of the subgraph of groups on X'.

In the following, we shall assume that edge groups are finite and also that if there is an edge e such that $G_e = G_{ie}$ or $G_e = G_{\tau e}$, then $ie = \tau e$.

Given a finitely generated group, G, we are interested in the various ways of expressing G as the fundamental group of a graph of groups with finite edge group. It is possible to show that if G is the fundamental group of a graph of groups on X_1 and on X_2, then these two representations have a common refinement; that is, the is a representation of G as the fundamental group of a graph of groups on a graph, X_3, such that the representations on X_1 and X_2 arise by collapsing suitable subgraphs of X_3 to points. So the question arises whether there is a representation from which all others are obtained by collapsing subgraphs to points. If there is the group is said to be accessible. Of course, this is equivalent to finding a representation where all the vertex groups are neither HNN extensions over some finite subgroup nor non-trivial coproducts over some finite subgroup. Since it is also easy to see that the number of generators of the fundamental group of a graph of groups on a graph, X, is at least $|E(X)| - |V(X)| + 1$ (consider the homomorphism to the fundamental group of the graph of groups on X such that $G_e = G_v = 1$ for all e, v in X, which is the free group on $|E(X)| - |V(X)| + 1$ generators), we need in general only consider the possibility that a vertex group G_v in some representation is a group coproduct of two groups, amalgamating a finite subgroup such that one of the factors (say G_1) in turn is a group coproduct of two groups amalgamating a finite subgroup and so on ad infinitum. It is clear that this cannot happen for finitely generated torsion-free groups since the number of generators of $H_1 * H_2$ is the sum of the number of generators of H_1 and the number of generators of H_2 by the Gruško-Neumann theorem. In order to prove a general theorem, we need to find some substitute for the number of generators that grows in a satisfactory way for coproduct with amalgamation $H_1 \underset{F}{*} H_2$, where F is a finite group. Linnell(55) was able to do this for those groups whose subgroups are of bounded order; here we shall present his theorem, using partial rank functions to give us a suitable substitute.

We noted in the first chapter that on every group ring in characteristic 0 there is a faithful rank function on the finitely generated projective modules induced by the trace function. We are interested in the subgroup $K_0^f(KG)$ of $K_0(KG)$ which is generated by the finitely generated projective modules induced from finite subgroups (K a field of characteristic 0). We shall denote by ρ_G^f the partial rank function induced by the trace on $K_0^f(KG)$. If $G = G_1 \underset{F}{*} G_2$, then $KG \cong KG_1 \underset{KF}{\sqcup} KG_2$; if $|F| < \infty$ and K has characteristic 0 then KF is semisimple artinian and we should like to show that ρ_G^f is just the partial rank function $(\rho_{G_1}^f, \rho_{G_2}^f)$.

<u>Lemma 3.4</u> Let $G = G_1 \underset{F}{*} G_2$ where $|F| < \infty$; then $K_0^f(KG)$ is just the subgroup of $K_0(KG)$ generated by the images of $K_0^f(KG_1)$ and $K_0^f(KG_2)$. Consequently, ρ_G^f is the partial rank function induced by $\rho_{G_1}^f$ and $\rho_{G_2}^f$.

<u>Pf</u> If P is a projective module induced up from some finite subgroup H of G, we know that H is a conjugate of some subgroup H' of either

G_1 or G_2; consequently P is isomorphic to a projective module induced up from H' and must lie in the image of $K_o^f(KG_1)$ or $K_o^f(KG_2)$. So $K_o^f(KG)$ is generated by the images of $K_o^f(KG_1)$ and $K_o^f(KG_2)$.

Since $\rho_{G_1}^f$ and $\rho_{G_2}^f$ agree on the image of $K_o(KF)$ and ρ_G^f agrees with all three, lemma 3.1 shows that $\rho_G^f = (\rho_{G_1}^f, \rho_{G_2}^f)$.

This allows us to show Linnell's theorem; we shall consider $g.\rho_G^f(\omega G)$, the generating number with respect to ρ_G^f of the augmentation ideal of G in KG.

Theorem 3.5 Let G be a finitely generated group such that if H is a finite subgroup of G, $|H| < m$; then G is accessible.

Pf We recall that it is sufficient to show that we cannot keep on decomposing such groups as a group coproduct amalgamating a finite subgroup in a non-trivial way.

Since the order of finite subgroups is bounded by m, ρ_G^f takes values in $\frac{1}{m!}\mathbb{Z}$ and we already know that it is a faithful partial rank function.

If $G \cong G_1 \underset{H}{*} G_2$, where $|H| < \infty$, we know that

$$\omega G \cong (\omega G_1)G \oplus (\omega HG_2 \setminus \omega G_2)G$$

where ωG is the augmentation ideal of KG. $KG \cong KG_1 \underset{KH}{\sqcup} KG_2$; $\omega G_1 G \cong \omega G_1 \otimes_{KG_1} KG$ and $(\omega HG_2 \setminus \omega G_2)G \cong (\omega HG_2 \setminus \omega G_2) \otimes_{KG_2} G$. Therefore by theorem 3.3 and lemma 3.4,

$$g.\rho_G^f(\omega G) = g.\rho_{G_1}^f(\ G_1) + g.\rho_{G_2}^f(\omega HG_2 \setminus \omega G_2)$$

and $$\rho_{G_2}^f(\omega HG_2 \setminus \omega G_2) \geqslant \frac{1}{m!}.$$

Consequently, if $g.\rho_G^f(\omega G) = q$, there is no decomposition of G as the group of a tree of groups with finite edge groups having $(m!q)$ vertices, which proves Linnell's theorem.

We return to the general theory. Consider a ring homomorphism $\alpha: R \longrightarrow S$; suppose that we have a rank function ρ_R on R and a rank function ρ_S on S and that $\rho_S(P \otimes_R S) = \rho_R(P)$ for any finitely generated projective module P over R. We shall say that the map $\alpha: R \longrightarrow S$ is <u>honest with respect to the rank functions</u> ρ_R and ρ_S if for any map $\gamma: P_1 \longrightarrow P_2$ in the category of finitely generated projective modules over R, $\rho_S(\gamma \otimes_R S) = \rho_R(\gamma)$. This clearly reduces to the usual notion of honesty for semifirs.

<u>Theorem 3.6</u> Let $R = R_1 \sqcup_{R_0} R_2$, where R_o is a semisimple artinian ring and let ρ_i be rank functions on R_i that agree on the image of $K_o(R_o)$ in $K_o(R_i)$. Let $\rho = (\rho_1, \rho_2)$ be the rank function they define on R. Then the embedding $R_1 \longrightarrow R$ is honest with respect to ρ_1 and ρ.
<u>Pf</u> Let $\alpha: P_1 \longrightarrow Q_1$ be a map between finitely generated projective modules over R_1. It is clear that $\rho_1(\alpha) \geqslant \rho(\alpha \otimes_{R_1} R)$.

So let M be some finitely generated R-module such that

$$\alpha(P_1) \otimes_{R_1} R \subseteq M \subseteq Q_1 \otimes_{R_1} R.$$

By lemma 3.4, we need to show that $g.\rho_1(M) \geqslant \rho_1(\alpha)$.

By remark 2.8, the decomposition of M given by theorem 2.7 and the following remarks takes the form

$$M = (M_o \otimes_{R_o} R) \oplus (M_1 \otimes_{R_1} R) \oplus (M_2 \otimes_{R_2} R),$$

where $\alpha(P_1) \subseteq M_1$. By theorem 3.3 the generating number of M with respect to ρ is at least the generating number of M_1 with respect to ρ_1.

By theorem 2.1, the embedding of R_1-modules $Q_1 \longrightarrow Q_1 \otimes_{R_1} R$ splits; so consider the image M_1' of M_1 in Q_1 under the splitting map. It contains $\alpha(P_1)$ so its generating number with respect to ρ_1 is at least $\rho_1(\alpha)$. We have shown that $g.\rho(M) \geqslant g.\rho_1(M_1) \geqslant g.\rho_1(M_1')$, which is at least $\rho_1(\alpha)$, so $\rho(\alpha \otimes_{R_1} R) \geqslant \rho_1(\alpha)$ and our theorem is proven.

This result will be of great use to us later; for the present we give a couple of interesting consequences.

Theorem 3.7 Let $R = R_1 \underset{R_0}{\sqcup} R_2$, where R_0 is semisimple artinian. Let ρ_i be a rank function on R_i such that ρ_1 and ρ_2 agree on the image of $K_0(R_0)$ in $K_0(R_i)$. Let ρ be the corresponding rank function on R. Then R satisfies the law of nullity with respect to ρ if and only if R_i satisfies the law of nullity with respect to ρ_i for $i = 1,2$.

Pf Suppose that R_i satisfies the law of nullity with respect to ρ_i for $i = 1,2$. Consider a couple of maps $\alpha: P \longrightarrow P'$, $\beta: P' \longrightarrow P''$, defined over R, such that $\alpha\beta = 0$. Then by theorem 2.13, we have a commutative diagram:

$$\begin{tikzcd}
P \arrow[rr, "\alpha"] \arrow[d] & & P' \arrow[rr, "\beta"] \arrow[d, "\cong"] & & P'' \arrow[d] \\
\bigoplus_{i=1}^{2} P_i \otimes_{R_i} R \arrow[rr, "\oplus\alpha_i\otimes_{R_i}R"] & & \bigoplus_{i=1}^{2} P_i' \otimes_{R_i} R \arrow[rr, "\oplus\beta_i\otimes_{R_i}R"] & & \bigoplus_{i=1}^{2} P_i'' \otimes_{R_i} R
\end{tikzcd}$$

where α_i, β_i are maps defined over R_i such that $\alpha_i\beta_i = 0$.

By the law of nullity, $\rho_i(\alpha_i) + \rho_i(\beta_i) \leq \rho_i(P_i')$; summing, we find that $\rho(\alpha) + \rho(\beta) \leq \sum_{i=1}^{2} \rho_i(\alpha_i) + \rho_i(\beta_i) \leq \rho(P')$, so R must satisfy the law of nullity with respect to ρ.

Conversely, suppose that R satisfies the law of nullity with respect to ρ. Let $\alpha: P \longrightarrow P'$, $\beta: P' \longrightarrow P''$ be maps between projective modules over R_1 such that $\alpha\beta = 0$. By the law of nullity for R with respect to ρ,

$$\rho(\alpha \otimes_{R_1} R) + \rho(\beta \otimes_{R_1} R) \leq \rho(P' \otimes_{R_1} R);$$

but, by the last theorem, $\rho(\gamma \otimes_{R_1} R) = \rho_1(\gamma)$ for any map defined over R_1, so

$$\rho_1(\alpha) + \rho_1(\beta) \leq \rho_1(P').$$

This shows that R_1 (and similarly R_2) must satisfy the law of nullity with respect to ρ_1 in the first case and ρ_2 in the second.

Theorem 3.8 In particular, we note that if R_o is a skew field, then R is a Sylvester domain if and only if R_1 and R_2 are Sylvester domains.
Pf This is an immediate consequence of the last theorem, when we notice that all finitely generated projectives over R are free of unique rank if and only if this holds in each factor.

We have seen in theorem 3.6 that if R_1 and R_2 are a couple of R_o-rings with rank functions ρ_1 and ρ_2 respectively, agreeing on the images of $K_o(R_o)$, that full maps with respect to ρ_1 over R_1 remain full with respect to ρ over $R = R_1 \sqcup_{R_o} R_2$ when they are induced up to R. We should like to say more accurately that factorisations of a full map over R_1 as a product of full maps over R essentially comes from R_1.

Theorem 3.9 Let $R = R_1 \sqcup_{R_o} R_2$, where R_o is a semisimple artinian ring and each R_i is a faithful R_o-ring. Assume that each R_i satisfies the law of nullity with respect to a faithful rank function ρ_i and that ρ_1 agrees with ρ_2 on the images of $K_o(R_o)$ in $K_o(R_i)$. Then if $\alpha \otimes_{R_1} R = \beta\gamma$ is a factorisation of a full map as a product of full maps over R, there is an invertible map, ε, such that $\beta\varepsilon$ and $\varepsilon^{-1}\gamma$ are defined over R_1.
Pf Suppose that we have the factorisation over R:

$$\begin{tikzcd}
P \otimes_{R_1} R \arrow[rr, "\alpha \otimes_{R_1} R"] \arrow[dr, "\beta"'] & & Q \otimes_{R_1} R \\
& \bar{P} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

where β, γ are full maps with respect to the rank function $\rho = (\rho_1, \rho_2)$ on R. So $\rho(P) = \rho_1(P) = \rho_1(Q)$. $\gamma(\bar{P})$ is a submodule of $Q \otimes_{R_1} R$ isomorphic to P, since $\alpha(P) \subseteq \gamma(\bar{P})$. α is a full map, so

$$g.\rho(\gamma(\bar{P})) \geqslant \rho_1(P) = \rho(\bar{P});$$

but if there were a kernel of the map $\gamma:\bar{P}\longrightarrow Q\otimes_{R_1}R$, $g.\rho(\gamma(\bar{P})) < \rho(\bar{P})$, since ρ is a faithful rank function.

Since $\gamma(\bar{P}) \supseteq \alpha(P_1)$, we know by remark 2.8 that the decomposition of $\gamma(\bar{P})$ given by theorem 2.7 takes the form:

$$\gamma(\bar{P}) = (M_o\otimes_{R_o}R) \oplus (M_1\otimes_{R_1}R) \oplus (M_2\otimes_{R_2}R)$$

where $\alpha(P) \subseteq M_1$.

If one of M_o or M_2 is non-zero, its rank with respect to ρ_2 is non-zero; consequently, $\rho_1(M_1) < \rho_1(P)$ and so $\alpha\otimes_{R_1}R:P\otimes_{R_1}R\longrightarrow Q\otimes_{R_1}R$ would not be full since $\alpha(P\otimes_{R_1}R) \subseteq M_1\otimes_{R_1}R$; therefore $\gamma(\bar{P}) = M_1\otimes_{R_1}R$.

Now all we need to do is to show that $M_1 \subseteq Q$, for then we take ε to be the identification of $\bar{P}$ with $\gamma(\bar{P})$ and we see that $\beta\varepsilon$ is the induced map $\bar{\beta}\otimes_{R_1}R$, where $\bar{\beta}$ is the map $P\longrightarrow\alpha(P) \subseteq M_1$ and $\varepsilon^{-1}\gamma$ is the induced map $\bar{\gamma}\otimes_{R_1}R$, where $\bar{\gamma}$ is the inclusion of M_1 in Q.

We already know $\bar{\beta}:P\longrightarrow M_1$ is a full map; we consider the map δ over R_1, $\delta:M_1\longrightarrow Q\otimes_{R_1}R\longrightarrow Q\backslash Q\otimes_{R_1}R$ and note that $\bar{\beta}\delta = 0$; hence (since the rank function is faithful) $\delta = 0$, which shows that $M_1 \subseteq Q$ as we wished to show.

Chapter 4

In this chapter, we shall investigate a rather different universal construction from those studied in the previous two chapters. Given a ring R and a collection Σ of maps between finitely generated projective modules over R, we should like to find a ring R_Σ, universal with respect to the property that every element of Σ has an inverse.

Since we shall be working at some stage in this chapter with matrices of maps between finitely generated projective modules, we shall work with finitely generated projective left modules, and write our maps on the right. Of course, all results have a suitable dual formulation.

We begin with a new and possibly clearer approach to the construction of the ring R_Σ. We use the observation that an additive category is a full subcategory of the category of finitely generated projective modules over a ring if there is some object, M, such that every object is a direct summand of some direct sum of copies of M.

Let $\underline{P}(R)$ be the category of finitely generated projective left modules over R. Given a subset Σ of maps in this category, we define a new additive category, $\underline{P}(R)_\Sigma$, whose objects are the same as those of $\underline{P}(R)$; further, it is generated as an additive category by $\underline{P}(R)$ together with maps $\bar{\alpha}: Q_\alpha \longrightarrow P_\alpha$ for each map $\alpha: P_\alpha \longrightarrow Q_\alpha$ in Σ, subject only to the relations $\alpha\bar{\alpha} = I_{P_\alpha}$, $\bar{\alpha}\alpha = I_{Q_\alpha}$. Since its objects are just those of $\underline{P}(R)$, every object is a direct summand of $[R^n]_\Sigma$ for some n, so this additive category, $\underline{P}(R)_\Sigma$, is a full subcategory of the category of finitely generated projective modules over the endomorphism ring of the object $[R]_\Sigma$; we call this endomorphism ring R_Σ. There is a map $R \longrightarrow R_\Sigma$, which by construction has the right universal property. Our category $\underline{P}(R)_\Sigma$ is just the category of finitely generated left projective modules over R_Σ that are induced from R. Of course, we apparently have no way of knowing that this ring is not the trivial ring, but we shall develop criteria which prevent this from happening.

We have two main theorems for studying such a ring; the first theorem, for historical reasons, is known as Cramer's rule, so in order to get the style correct, the second theorem will be called Malcolmson's criterion. They are both way of representing maps in the category of induced projective modules, but they are dual both in form and use.

<u>Theorem</u> <u>4.1</u> (Cramer's rule) Let R be a ring and Σ a collection of maps between finitely generated projective left modules. Then in the category $\underline{P}(R)_\Sigma$ every map $\alpha: R_\Sigma \otimes_R P \longrightarrow R_\Sigma \otimes_R Q$ satisfies an equation of the form:

$$\beta \begin{pmatrix} I & \alpha_1 \\ 0 & \alpha \end{pmatrix} = \beta'$$

where β, β' are both maps defined over R, and $\beta = \begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ * & & \beta_n \end{pmatrix}$, where $\beta_i \in \Sigma$; I is an identity map on a projective module of suitable size and α_1 is some map over R .

<u>Pf</u> We use the generator and relation construction of the category $\underline{P}(R)_\Sigma$ given in the preceding discussion.

Certainly, for $\beta \in \Sigma$, the element $\bar{\beta}$ satisfies

$$\beta\bar{\beta} = I,$$

which is of the required form; for α defined over R, $\alpha: P \longrightarrow Q$,

$$I_P(\alpha) = \alpha$$

is an equation of the required form.

Next, given $\alpha_i: P \longrightarrow Q$ $(i = 1,2)$ satisfying equations:

$$(\beta_i\ \bar{\beta}_i)\begin{pmatrix} I & \alpha_i' \\ 0 & \alpha_i \end{pmatrix} = (\beta_i\ \beta_i') \qquad (i = 1,2)$$

we find that for $\alpha_2 - \alpha_1$, we have the equation

$$\left(\begin{array}{ccc|c} \beta_1 & \bar{\beta}_1 & 0 & 0 \\ 0 & \bar{\beta}_2 & \beta_2 & \bar{\beta}_2 \end{array}\right)\left(\begin{array}{c|c} I & \begin{array}{c}\alpha_1' \\ \alpha_1 \\ \alpha_2'\end{array} \\ \hline 0 & \alpha_2-\alpha_1 \end{array}\right) = \left(\begin{array}{ccc|c} \beta_1 & \bar{\beta}_1 & 0 & \beta_1' \\ 0 & \bar{\beta}_2 & \beta_2 & \beta_2' \end{array}\right)$$

Finally, suppose that we have equations

$$(\beta_i\ \bar{\beta}_i)\begin{pmatrix} I & \alpha_i' \\ 0 & \alpha_i \end{pmatrix} = (\beta_i\ \beta_i') \qquad (i = 1,2)$$

for maps $\alpha_1 : P_1 \longrightarrow P_2$, $\alpha_2 : P_2 \longrightarrow P_3$; then we construct the equation

$$\left(\begin{array}{ccc|c} \beta_2 & \bar{\beta}_2 & 0 & 0 \\ 0 & -\beta_1' & \beta_1 & \bar{\beta}_1 \end{array}\right)\left(\begin{array}{c|c} I & \begin{array}{c}\alpha_2' \\ \alpha_2 \\ \alpha_1'\alpha_2\end{array} \\ \hline 0 & \alpha_1\alpha_2 \end{array}\right) = \left(\begin{array}{ccc|c} \beta_2 & \bar{\beta}_2 & 0 & \beta_2' \\ 0 & -\beta_1' & \beta_1 & 0 \end{array}\right)$$

Since every map in the category $\underline{P}(R)_\Sigma$ may be obtained by successively taking differences of maps previously found, or composing maps previously found (where possible) we see that our theorem is true.

Of course, in the last theorem there are no apparent reasons why the representation of α in the equation $\beta\begin{pmatrix} I & \alpha' \\ 0 & \alpha \end{pmatrix} = \beta'$ should be in any sense unique. We hope to show, however, that we often have very good control over such representations.

In Malcolmson (17), Malcolmson presents a construction of the ring R_Σ where Σ is a collection of matrices over R. The version of Cramer's rule presented above was obtained from known versions simply by replacing matrices with maps between finitely generated projective modules in the statement and proof. Exactly the same can be done with Malcolmson's result.

We give an outline of his construction in our present context and state the main theorem, the proof of which the reader may either omit, or else, by supplying maps for matrices thoughout Malcolmson's paper(37),find.

The first point is that every map in the category $\underline{P}(R)_\Sigma$, where Σ is a collection of maps between finitely generated projective modules, may be represented in the form $f\gamma^{-1}g$, where f,g,γ are maps defined over R and $\gamma = \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix}$ where each $\gamma_i \in \Sigma$. This is clearly true for elements of $\underline{P}(R)$ and for maps $\bar{\alpha}: Q \longrightarrow P$ $(\alpha: P \longrightarrow Q, \alpha \in \Sigma)$.

If $f_1\gamma_1^{-1}g_1$ and $f_2\gamma_2^{-1}g_2$ are maps from $R_\Sigma \otimes_R P$ to $R_\Sigma \otimes_R Q$, then we find that

$$f_1\gamma_1^{-1}g_1 - f_2\gamma_2^{-1}g_2 = (f_1 \quad f_2)\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}^{-1}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

If $f_1\gamma_1^{-1}g_1 : R_\Sigma \otimes_R P_1 \longrightarrow R_\Sigma \otimes_R P_2$ and $f_2\gamma_2^{-1}g_2 : R_\Sigma \otimes_R P_2 \longrightarrow R_\Sigma \otimes_R P_3$ are a couple of maps then

$$f_1\gamma_1^{-1}g_1 . f_2\gamma_2^{-1}g_2 : R_\Sigma \otimes_R P_1 \longrightarrow R_\Sigma \otimes_R P_3$$

is given by

$$f_1\gamma_1^{-1}g_1 . f_2\gamma_2^{-1}g_2 = (f_1 \quad 0)\begin{pmatrix} \gamma_1 & -g_1f_2 \\ 0 & \gamma_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ g_2 \end{pmatrix}$$

As in the proof of Cramer's rule, this shows that all maps in the category $\underline{P}(R)_\Sigma$ take the form $f\gamma^{-1}g$; the main use of this new representation is that we have a criterion for when two of these expressions are equal. We state the result:

<u>Theorem 4.2</u> (Malcolmson's criterion) Let R be a ring and Σ a collection of maps between finitely generated projective modules over R. Then every map in $\underline{P}(R)_\Sigma$ takes the form $f\gamma^{-1}g$ for maps f,g,γ defined over R and $\gamma = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ 0 & & \beta_n \end{pmatrix}$ for $\beta_i \in \Sigma$; further, $f_1\gamma_1^{-1}g_1 = f_2\gamma_2^{-1}g_2$ if and only if there is an equation where all maps are in R :

$$\begin{pmatrix} \gamma_1 & 0 & 0 & 0 & -g_1 \\ 0 & \gamma_2 & 0 & 0 & g_2 \\ 0 & 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & g_4 \\ f_1 & f_2 & f_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} P \\ u \end{pmatrix} \begin{pmatrix} Q & v \end{pmatrix}$$

where γ_3, γ_4, P, Q all have the form $\begin{pmatrix} \beta_1 & & * \\ & \ddots & \\ 0 & & \beta_n \end{pmatrix}$ for suitable $\beta_i \in \Sigma$ and $n \in \mathbb{N}$.

We refer the reader to Malcolmson(27), where he will find the proof in the case of elements of R_Σ and matrices; the translation to this present form is mechanical.

In general, one needs to be dealing with a specific situation in order to apply these results with any power; however, Cramer's rule has a corollary which is quite surprising and holds for any universal localisation.

Theorem 4.3 Let R be a ring and Σ a collection of maps between finitely generated projecgive modules over R; then every finitely presented module over R_Σ is an induced 4.1. module.

Pf Let M be a finitely presented module over R_Σ; then it is the cokernel of some map between free modules of finite rank over R_Σ; by Cramer's rule we know that any such map is stably associated to an induced map. However, two maps that are stably associated have isomorphic cokernels; but since $R_\Sigma \otimes_R -$ is right exact, the cokernel of an induced map is an induced module.

In particular, this result applies to finitely generated projective modules over R_Σ; however, although every projective module is an induced module, it need not be induced from a projective module over R. We give a simple example, using results we shall prove later, to show it need not be a stably induced projective module.

Let $k \subset K$ be an inseparable extension of fields such that $[K:k] = p = \text{char } k$. The ring coproduct $K \sqcup_k k[x]$ is a fir by the coproduct theorems; it has a universal skew field of fractions (see chapter 5 or Cohn(Ch.7)), which we write as $K \circ_k k(x)$. Cohn (Ch.7) or chapter 5 shows that this has centre k; consequently, $K \otimes_k (K \circ_k k(x)) \cong M_n(F)$ for some skew field F. However, $K \otimes_k (K \circ_k k(x))$ is the universal localisation of $K \otimes_k (K \sqcup_k k[x])$ at the set of matrices over $K \sqcup_k k[x]$ that are full; $K \otimes_k (K \sqcup_k k[x]) \cong (K \otimes_k K) \sqcup_k k[x]$. All projectives over $K \otimes_k K$ are free, since $K \supset k$ is purely inseparable; so all projectives over $K \otimes_k (K \sqcup_k k[x])$ are free, although not all projectives over $K \otimes_k (K \circ_k k(x))$ are stably free.

Recently, the author has found another general result about universal localisation that generalises an exact sequence in K-theory due to Bass and Murthy on central localisation and recent results of Révész and Cohn on firs. We shall not give a proof, since it is long and calculational and will not be applied in this dissertation; however, we state the result.

First of all, we need to define a certain category. Let R be a ring and Σ a collection of maps between finitely generated projective modules such that R embeds in R_Σ. Let $\bar{\Sigma}$ be the collection of maps between finitely generated projective modules over R that have inverses in R_Σ; clearly $R_{\bar{\Sigma}} \cong R_\Sigma$. Further, all elements of $\bar{\Sigma}$ are injective maps. Let I be the full subcategory of the category of finitely generated modules over R, whose objects are those modules $\{M_\alpha : \alpha \in \bar{\Sigma}\}$ with presentations

$$0 \longrightarrow P \longrightarrow Q \longrightarrow M_\alpha \longrightarrow 0.$$

We have already considered a category of this form in chapter 1, where we showed that the category of modules with full presentation with respect to a rank function ρ on a two-sided $\aleph_0$-hereditary ring was a noetherian and artinian abelian category.

In general, the category is closed under extensions in the category of R-modules.

Theorem 4.4 Let R be a ring and Σ a collection of maps between finitely generated projective modules over R such that R embeds in R_Σ; let $\bar{\bar{\Sigma}}$ and I be defined as above. Then there is an exact sequence;

$$K_1(R) \longrightarrow K_1(R_\Sigma) \longrightarrow K_0(I) \longrightarrow K_0(R) \longrightarrow K_0(R_\Sigma).$$

It is clear how to define α, γ, δ; β is defined using Cramer's rule; specifically, if $a \in GL_n(R\ )$, Cramer's rule gives us an equation:

$$(A \quad A_1) \begin{pmatrix} I & * \\ 0 & a \end{pmatrix} = (A \quad A')$$

where $(A \quad A_1)$ and $(A \quad A')$ are in $\bar{\bar{\Sigma}}$.

We attempt to define $\beta(a) = [M_{(A \ A')}] - [M_{(A \ A_1)}]$ where $M_{(A \ A')}$ is the cokernel of $(A \quad A')$ and $M_{(A \ A_1)}$ is the cokernel of $(A \quad A_1)$.

The difficulty lies in showing that β is well-defined and induces a map on $K_1(R_\Sigma)$; once this has been done, it is a simple exercise to show that our sequence is exact. It produces the Bass-Murthy sequence, when we note that central localisation (indeed, Ore localisation) is just a special case of universal localisation.

Finally, there is some reason to suppose that this should extend to a long exact sequence of Quillen K-theory generalising Gersten's extension of the Bass-Murthy result.

Chapter 5

The purpose of this chapter is to study epimorphisms from weakly semihereditary rings to simple artinian ring. As one would suspect, this theory is intimately related to the possible rank functions taking values in $\frac{1}{n}\mathbb{Z}$ for suitable n. If R is a weakly semihereditary ring with a rank function ρ, it satisfies the law of nullity with respect to ρ; in fact, for a particular rank function ρ, the most important fact is just the law of nullity and so much of the theory goes through in this generality. In order to save the reader's eyes and the author's hand, therefore, we make the following definition: let ρ be a rank function on a ring R, taking values in $\mathbb{R}$; it is said to be excellent if R satisfies the law of nullity with respect to ρ.

The case of an excellent rank function ρ on a ring R taking values in $\mathbb{Z}$ has already been considered by Bergman; he shows that the ring obtained by adjoining universal inverses to all maps between finitely generated projective modules that are full with respect to ρ is a skew field. He also considered the case of an excellent rank function ρ, taking values in $\frac{1}{n}\mathbb{Z}$, on a k-algebra R; here, he showed that the rank function arose from a homomorphism to a simple artinian ring. However, the simple artinian ring in question was not closely related to the original ring R. The author considers the universal localisation of R at the set of full maps with respect to ρ, which he shows is always a perfect ring (in the sense of Bass(3)) with an excellent rank function extending that on R. There are examples to show that this is almost as much as one can hope for; however, this is sufficiently good to be able to deduce in many cases that the resulting ring is, in fact, simple artinian. This allows us to construct, for example, the simple artinian coproduct of two simple artinian rings, amalgamating a common simple artinian ring; this construction is the natural extension of Cohn's skew field coproduct with amalgamation. In a later chapter we shall consider the interaction of this theory with the representation theory of finite dimensional hereditary algebras.

Given a ring R with an excellent rank function ρ, we shall often wish to consider the ring obtained by adjoining universal inverses to all maps which are full with respect to ρ; we shall use the symbol R_ρ for this ring, and refer to it as the universal localisation of R at ρ.

We begin by summarising Cohn's theory of homomorphisms to skew fields for arbitrary rings; we refer the reader to Cohn (4 Ch.7) or to Malcolmson (38) for the presentation of part of it.

The epimorphisms from a ring R to skew fields are in 1-1 correspondence with the prime matrix ideals of the ring R; a <u>prime matrix ideal</u> is a set of square matrices, P, over R satisfying the following four properties;

(i) The 1×1 identity matrix is not in P

(ii) For square matrices A, B, $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in P$ if and only if one of A or B is in P.

(iii) If $(A|b)$ and $(A|c)$ lie in P, then $(A|b-c)$ lies in P; and the analogous condition for rows.

(iv) Every non-full matrix lies in P; an $n\times n$-matrix is said to be non-full if it factors as a product of an $n\times(n-1)$ and an $(n-1)\times n$ matrix.

The 1-1 correspondence is given by taking $P_{(F,\alpha)}$ for an epimorphism $\alpha : R \longrightarrow F$ to be the collection of matrices over R that are singular over F.

It is clear that given a descending chain of prime matrix ideals, $P_\alpha \supset P_\beta \supset \ldots$, $\bigcap_{\gamma\in\Gamma} P_\gamma$ is also a prime matrix ideal and is non-zero (since it contains the non-full matrices); so there are minimal prime matrix ideals inside any given one. Our aim is to classify the minimal prime matrix ideals, which we are able to do for weakly semihereditary rings; this theory is due to Bergman, who showed that they were in 1-1 correspondence with the rank functions taking values in $\mathbb{Z}$. Here, the skew field corresponding to a particular minimal prime matrix ideal arises in a particularly interesting way; it is the universal localisation at the rank function associated to it.

We present this theory, following the proofs given in Dicks(27) for most of the time.

<u>Theorem 5.1</u> Let R be a ring with an excellent rank function, ρ, taking values in $\mathbb{Z}$; then the set of matrices, P_ρ, non-full with respect to the rank function ρ is a minimal prime matrix ideal; in fact, the universal localisation of R at ρ is the skew field associated to this prime matrix ideal. The homomorphism $R \longrightarrow R_\rho$ induces the rank function ρ on R and is an honest map.

<u>Pf</u> First we show that P_ρ is a prime matrix ideal. Next, we show that the skew field, F, associated to P_ρ induces the rank function ρ on R and that the homomorphism $R \longrightarrow F$ is an honest map with respect to the rank function ρ and the usual rank on a skew field. A final argument shows that R_ρ is a skew field and so P_ρ is shown to be a prime matrix ideal.

We prove conditions (i)-(iv) for P_ρ to be a prime matrix ideal. Certainly (i) holds. Next we show that $\rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \rho(A) + \rho(B)$, from which condition (ii) follows.

Let Q be a projective module of minimal rank through which $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ factors; we partition our factorisation as

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} X \\ X' \end{pmatrix}\begin{pmatrix} Y & Y' \end{pmatrix},$$

so $XY' = 0$. Hence, by the law of nullity,

$$\rho\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix} = \rho(Q) \geqslant \rho(X) + \rho(Y') \geqslant \rho(XY) + \rho(X'Y') = \rho(A) + \rho(B) \geqslant \rho\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$$

from which equality holds throughout.

We come to condition (iii); we shall prove only the column version, since the row version follows by duality as the law of nullity is a left-right invariant condition on a ring with a rank function.

If $(A \mid b)$ and $(A \mid c)$ are $n \times n$-matrices that are non-full with respect to ρ, we have two cases; if A has rank less than $(n-1)$ with respect to ρ, all matrices $(A \mid *)$ are non-full with respect to ρ, including $(A \mid b-c)$, so we consider the second case where A has inner projective rank $(n-1)$ with respect to ρ and so do $(A \mid b)$ and $(A \mid c)$.

We generalise slightly to consider the case where A, $(A \mid B)$ and $(A \mid C)$ all have the same rank; we wish to show that this is the rank of $(A \mid B \mid C)$. We take minimal factorisations $(A \mid B) = D(E \mid E')$ and $(A \mid C) = F(G \mid G')$; then $A = DE$, $A = FG$ are both minimal factorisations of A. The rank of the codomain of $(D \mid F)$ is $2\rho(A)$ and $(D \mid F)\begin{pmatrix}E\\ -G\end{pmatrix} = 0$, so by the law of nullity we have the equations:

$$2\rho(A) \geqslant \rho(D \mid F) + \rho\begin{pmatrix}E\\ -G\end{pmatrix} \geqslant \rho(D) + \rho(G) = 2\rho(A),$$

hence equality holds everywhere. So $(D \mid F) = H(J \mid K)$, where the rank of the codomain of H is $\rho(D \mid F) = \rho(A)$. Therefore we construct the following factorisation;

$$(A \mid B \mid C) = (DE \mid DE' \mid FG') = (HJE \mid HJE' \mid HKG') = H(JE \mid JE' \mid KG')$$

and so $\rho(A \mid B \mid C) \leqslant \rho(H) = \rho(A) \leqslant \rho(A \mid B \mid C)$, from which equality follows.

Returning to our problem, we find that $\rho(A) = \rho(A \mid b) = \rho(A \mid c)$ $(= n-1)$, so $\rho(A \mid b \mid c) = n-1$ and therefore

$$\rho(A \mid b-c) = \rho\left[(A \quad b \quad c)\begin{pmatrix}I & 0\\ 0 & 1\\ 0 & -1\end{pmatrix}\right] \leqslant \rho(A \mid b \mid c) = n-1$$

so $(A \mid b-c)$ must also lie in P_ρ, which proves condition (iii).

Finally, for condition (iv), it is clear that non-full matrices are non-full with respect to ρ. So we have shown that P_ρ is a prime matrix ideal.

Next we shall show that the homomorphism to a skew field F induced by the prime matrix ideal P_ρ preserves the inner projective rank of matrices; consequently it induces the rank function ρ on the projective modules, and it is an honest map.

Over a skew field, the rank of a matrix is the maximal rank of square invertible submatrices; under the homomorphism from R to F, those matrices that become inverted are exactly those not in P_ρ, so we wish to show that the inner projective rank of a matrix is the maximal rank of a squre submatrix that is full with respect to ρ.

Suppose, for a given matrix M, that this maximal rank is q; then every $(q+1)\times(q+1)$-submatrix of M that contains a square matrix of rank q that is full with respect to ρ must itself have inner projective rank q. By induction, and the earlier result that if $\rho(A)$, $\rho(A \mid B)$ and $\rho(A \mid C)$ are all equal, then they are equal to $\rho(A \mid B \mid C)$, we see that the rank of any $(q+1)\times n$-submatrix of M with respect to ρ, that contains a square $q\times q$-matrix, full with respect to ρ, must be q. By induction, and the dual result, we see that the inner projective rank of M must be q, as we wished to show.

This shows that the map from R to F must induce ρ on projective modules; for, given any projective module Q over R, there is an idempotent $n\times n$-matrix E such that R^nE is isomorphic to Q, and so $F^nE \cong F\otimes_R Q$; moreover, the inner projective rank of E is just $\rho(Q)$, so $\rho(Q)$ is equal to the rank of E over F, which is just $[F\otimes_R Q:F]$ as we wished to show. Finally, given any map $\alpha: Q_1 \longrightarrow Q_2$ over R between finitely generated projectives, let $p: R^n \longrightarrow Q_1$, $i: Q_2 \longrightarrow R^m$ be a split surjection and a split injection respectively; then

$\rho(\alpha) = \rho(p\alpha i)$; $p\alpha i$ is represented by some matrix and the rank of over F is just the rank of this matrix over F, but this is just $\rho(p\alpha i) = \rho(\alpha)$ which shows that $R \longrightarrow F$ is an honest map with respect to ρ on R and the usual rank on F.

An immediate consequence of this is that all maps between projective modules over R that are full with respect to ρ become invertible in F.

Let Σ_ρ be the collection of matrices full with respect to ρ; then, by Cohn(14 Ch.7), we know that R_{Σ_ρ}, the universal localisation of R at the set of matrices Σ_ρ is a local ring with residue class skew field F. Since every map $\alpha : Q_1 \longrightarrow Q_2$ that is full with respect to ρ becomes invertible over F, it is already invertible over R_{Σ_ρ}: so $R_{\Sigma_\rho} = R_\rho$.

Let $x \in R_\rho$, and suppose that the image of x in F is 0; then, by Cramer's rule, there is an equation

$$(A \mid a) \begin{pmatrix} I & * \\ 0 & x \end{pmatrix} = (A \mid b),$$

where $(A \mid a)$ is a full matrix with respect to ρ from some projective module P, $\rho(P) = n$, to a projective of rank n; $(A \mid b)$ is a map defined over R.

Since x has image 0 in F, $(A \mid b)$ cannot be full with respect to ρ. Hence there is a factorisation $(A \mid b) = B(C \mid c)$ where the codomain of B is a projective module Q, $\rho(Q) = n-1$.

We use this to re-write our first equation over R ;

$$(B \mid a)\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & * \\ 0 & x \end{pmatrix} = (B \mid a) \begin{pmatrix} C & c \\ 0 & 0 \end{pmatrix}$$

Since $(B \mid a)$ is a map from P, where $\rho(P) = n$ to $Q \oplus R$, where $\rho(Q) = n-1$, and since it is a left factor of the full map $(A \mid a)$, it must be full and so invertible in R_ρ; we cancel it and deduce that $x = 0$ already in R_ρ. So $R_\rho = F$ as we wished to show.

Now suppose that P is a prime matrix ideal contained in P_ρ; all maps in Σ_ρ become invertible in the skew field associated to P, so there is an epimorphism from the skew field R_{Σ_ρ} to the skew field associated to P. But the only possible such epimorphism is an isomorphism between them , so $P = P_\rho$, which shows that P_ρ is a minimal prime matrix ideal.

This applies equally well to the case of weakly semihereditary rings, for any rank function on such a ring must be excellent.

Theorem 5.2 Let R be a weakly semihereditary ring; then the minimal prime matrix ideals are in 1-1 correspondence with the rank functions taking values in $\mathbb{Z}$.

Pf Given a rank function ρ on R, we know that is is excellent, so, by theorem 5.1, the set of matrices non-full with respect to ρ is a minimal prime matrix ideal. Moreover, the rank function is recoverable from the prime matrix ideal, since ρ is the rank function induced by the map to the associated skew field.

Conversely, if P is a minimal prime matrix ideal, the epimorphism from R to the associated skew field induces some rank function ρ on R and the set of matrices non-full with respect to ρ, P_ρ, must lie in P; since ρ is excellent, P_ρ is a minimal prime matrix ideal and hence since P is minimal, $P = P_\rho$.

The last two theorems give a fairly clear picture of the interaction between rank functions to $\mathbb{Z}$ and epimorphisms to skew fields for weakly semihereditary rings. One of the most interesting points is that the universal localisation of a weakly semihereditary ring at a rank function taking values in $\mathbb{Z}$ is always a skew field. As soon as we pass to rank functions taking values in $\frac{1}{n}\mathbb{Z}$, we find quite quickly that the universal localisation at the rank function need not be simple artinian; we give a few examples.

We begin with a fairly trivial example. Consider the ring $k \oplus k$ and the rank function that assigns the rank $\frac{1}{2}$ to each direct summand. Clearly, all full maps with respect to this rank function are already invertible, so it is its own universal localisation at the rank function. Next, consider the ring $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subseteq M_2(k)$; there is a rank function assigning the value $\frac{1}{2}$ to $\begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$ and to $\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$. The element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ defines a full map from the second to the first projective; moreover, the ring with a universal inverse to this map is just $M_2(k)$. However, if we assign the rank 2/3 to $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ and the value 1/3 to $\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$, we see that all full maps with respect to this rank function are already invertible; so $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ is its own universal localisation at this rank function. In a later chapter, we shall examine more general finite-dimensional hereditary algebras in greater detail.

Given a k-algebra R with an excellent rank function ρ taking values in $\frac{1}{n}\mathbb{Z}$, we show that this rank function is induced by a homomorphism to a simple artinian ring. This was first shown by Bergman for weakly semihereditary rings.

<u>Theorem 5.3</u> Let R be a ring with an excellent rank function ρ taking values in $\frac{1}{n}\mathbb{Z}$; then there is a homomorphism from R to a simple artinian ring, inducing the rank function ρ on R.

<u>Pf</u> Consider the ring $M_n(k) \underset{k}{\sqcup} R = M_n(T)$; since the unique rank function on $M_n(k)$ is excellent, taking values in $\frac{1}{n}\mathbb{Z}$, the rank function ρ' defined on $M_n(k) \underset{k}{\sqcup} R$ by the unique one on $M_n(k)$ and ρ on R must also be excellent by theorem 3.7, and it takes values in $\frac{1}{n}\mathbb{Z}$. By Morita equivalence, there is a rank function $\bar{\rho}$ on T, corresponding to ρ', taking values in $\mathbb{Z}$, and it must be excellent. Consequently, the universal localisation of T at $\bar{\rho}$, $T_{\bar{\rho}}$, is a skew

field by theorem 5.1 and the homomorphism

$$R \longrightarrow M_n(k) \underset{k}{\sqcup} R \;=\; M_n(T) \longrightarrow M_n(T_{\bar{\rho}})$$

induces ρ on R.

Although this result is interesting, we should prefer to have information on epimorphisms rather than just homomorphisms, and the simple artinian ring we've constructed is rather a long way from R. However, it will be of great use to use, mainly because it is an honest map;

Theorem 5.4 Let R be a k-algebra with an excellent rank function taking values in $\frac{1}{n}\mathbb{Z}$; then the homomorphism from R to a simple artinian ring constructed in theorem 5.3 is an honest map.

Pf In theorem 3.6, we showed that the map $R \longrightarrow M_n(k) \underset{k}{\sqcup} R \cong M_n(T)$ is an honest map; in theorem 5.1, we showed that the map $T \longrightarrow T_{\bar{\rho}}$ is an honest map, so the map $M_n(T) \longrightarrow M_n(T_{\bar{\rho}})$ is honest. Since the composition of honest maps is honest, this completes our proof.

In order to develop the theory of universal localisations of rings at excellent rank functions taking values in $\frac{1}{n}\mathbb{Z}$, we need to examine the case where the rank function takes values in $\mathbb{Z}$ in greater detail; in fact, we need to examine the localisations intermediate between R and R_ρ. The next theorem first appeared in joint work with Cohn (Cohn, Schofield 23). We define a set of maps Σ between finitely generated projectives to be lower multiplicatively closed if $\gamma, \gamma' \in \Sigma \Rightarrow \begin{pmatrix} \gamma & 0 \\ * & \gamma' \end{pmatrix} \in \Sigma$.

Theorem 5.5 Let R be a ring with an excellent rank function ρ taking values in $\mathbb{Z}$. Let Σ be a collection of full maps with respect to ρ that is lower multiplicatively closed, and such that every left factor in R, that is full with respect to ρ, of an element of Σ

is invertible in R_Σ; then R_Σ embeds in the universal localisation of R at ρ, R_ρ.

Pf By Cramer's rule, for any element x of R_Σ, we have an equation:

$$(\alpha|\alpha_1)\left(\begin{array}{c|c} I & \begin{matrix} x_1 \\ \vdots \\ x_{n-1} \end{matrix} \\ \hline 0 & x \end{array}\right) = (\alpha|\beta)$$

where $(\alpha|\alpha_1):P\longrightarrow R^n$, $(\alpha|\beta):P\longrightarrow R^n$ are maps defined over R and $(\alpha|\alpha_1)\in\Sigma$. If the image in R_Σ of x is 0, $(\alpha|\beta)$ is singular in R_ρ and so $(\alpha|\beta) = \gamma(\alpha'|\beta')$ in R, where $\gamma:P\longrightarrow Q$ is a map from P to a projective of rank (n-1). Using this equation, we may re-write our first equation:

$$(\gamma|\alpha_1)\begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & \begin{matrix} x_1 \\ \vdots \\ x_{n-1} \end{matrix} \\ 0 & x \end{pmatrix} = (\gamma|\alpha_1)\begin{pmatrix} \alpha' & \beta' \\ 0 & 0 \end{pmatrix}$$

Since $(\gamma|\alpha_1)$ is a map from a projective of rank n to a projective of rank n, and since it is a left factor of $(\alpha|\alpha_1)$, we deduce first that it must be full, and then by assumption it must be invertible in R_Σ; cancelling it from our second equation shows that x = 0 in R_Σ; which proves that R_Σ embeds in R_ρ.

This theorem allows us to give another necessary and sufficient condition for a universal localisation R_Σ of a ring R at a set of maps between finitely generated projective modules full with respect to an excellent rank function, to embed in the universal localisation R_ρ of R.

Since R_ρ is a skew field, the map $R_\Sigma\longrightarrow R_\rho$ induces a rank function ρ' on the ring R_Σ taking values in $\mathbb{Z}$. We use this rank function; clearly $\rho'(R_\Sigma\otimes_R P) = \rho(P)$.

<u>Theorem 5.6</u> Let R be a ring with an excellent rank function ρ taking values in $\mathbb{Z}$; let Σ be a collection of full maps with respect to ρ. Then R_Σ embeds in R_ρ if and only if the rank function ρ' induced by the map $R_\Sigma\longrightarrow R_\rho$ is faithful.

Pf If there is a projective Q over R_Σ such that $\rho'(Q) = 0$, then the trace ideal of Q must be in the kernel of the map $R_\Sigma \longrightarrow R_\rho$.

Conversely, if $R_\Sigma \longrightarrow R_\rho$ is not faithful, then, by the last theorem, there is a map $\gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ * & & \gamma_n \end{pmatrix}$ in the lower multiplicative closure of Σ and a factorisation $\gamma = \alpha\beta$ where α, β are full maps with respect to ρ such that α is not invertible in R_Σ; let $\alpha : P \longrightarrow P'$, where $\rho(P) = \rho(P') = n$; then in R_Σ, α has a right inverse $\beta\gamma^{-1}$, so

$$R_\Sigma \otimes_R P' \cong R_\Sigma \otimes_R P' \oplus Q$$

for some non-zero Q. But then

$$\rho'(Q) = \rho'(R_\Sigma \otimes_R P') - \rho'(R_\Sigma \otimes_R P) = \rho(P') - \rho(P) = 0.$$

We wish to have slightly more precise information about a ring of the form R_Σ, when R_Σ embeds in R_ρ in the situation of the preceding theorem; it turns out that we can relate much of the theory of such a ring to that of R.

Theorem 5.7 Let R be a ring with an excellent rank function taking values in $\mathbb{Z}$; let Σ be a collection of full maps with respect to ρ such that R_Σ embeds in R_ρ; then all finitely generated projective modules over R_Σ are stably induced from R, that is, $Q \oplus R^m \cong R_\Sigma \otimes_R P$. Further the rank function induced by $R_\Sigma \longrightarrow R_\rho$ on R_Σ is excellent.

Pf Let P be a finitely generated projective module over R_Σ, the image of an idempotent $e \in M_n(R_\Sigma)$; by Cramer's rule, e is stably associated to a map β defined over R. This gives the following diagram, which we explain below;

$$\begin{tikzcd}
R_\Sigma \otimes_R Q_1 \arrow[r, "R_\Sigma \otimes_R \beta_1"] \arrow[rr, bend left, "R_\Sigma \otimes_R \beta"] \arrow[d, "\gamma"', "\cong"] & R_\Sigma \otimes_R Q_2 \arrow[r, "R_\Sigma \otimes_R \beta_2"] & R_\Sigma^{n+m} \arrow[d, "\delta", "\cong"'] & \\
R_\Sigma^{n+m} \arrow[r, "p"] & P \oplus R_\Sigma^m \arrow[r, "i"] & R_\Sigma^{n+m} \arrow[r, "p"] & P \oplus R_\Sigma^m
\end{tikzcd}$$

Here the maps γ, δ are isomorphisms defined over R_Σ; β is a map defined over R; β_1 and β_2 are defined over R and $\beta = \beta_1\beta_2$ is a factorisation of β as a right full followed by a left full map, so $\rho(Q_2)$ is the inner projective rank of β. p is the projection from R_Σ^{n+m} onto the image $P \oplus R_\Sigma^m$ of $e \oplus I_m$; i is the injection of this image in R_Σ^{m+n}.

Since β and $e + I_m$ are associated over R_Σ, they have the same rank over R_ρ; so $\rho(Q_2) = m + \rho(P)$, where by $\rho(P)$ we mean $[R_\rho \otimes_{R_\Sigma} P : R]$

The map $\beta_2 \delta p$ is surjective, so over R_Σ,

$$R_\Sigma \otimes_R Q_2 \cong (P \quad R^m) \oplus P',$$

where $P' \cong \ker(\beta_2 \delta p)$. However, over R_ρ,

$$R_\rho \otimes_R Q_2 \cong (R_\rho \otimes_{R_\Sigma} P) \oplus R_\rho^m$$

which implies that $R_\rho \otimes_{R_\Sigma} P' = 0$; since R_Σ embeds in R_ρ, P' must be zero, so $P \oplus R_\Sigma^m = R_\Sigma \otimes_R Q_2$, where Q_2 is defined over R, which is what we wished to show.

We have the information available to us now to prove our main result on universal localisations of a k-algebra R at an excellent rank function taking values in $\frac{1}{n}\mathbb{Z}$. First we set up the terminology of our next theorem.

Let R be a k-algebra with an excellent rank function, ρ, taking values in $\frac{1}{n}\mathbb{Z}$; let T be the ring such that $M_n(k) \sqcup_k R \cong M_n(T)$. Then T is a ring with excellent rank function, $\bar{\rho}$, as we saw in theorems 5.3 and 5.4, so $T_{\bar{\rho}}$ is a skew field and we saw that the homomorphism

$$R \longrightarrow M_n(k) \sqcup_k R \cong M_n(T) \longrightarrow M_n(T_{\bar{\rho}})$$

induces the rank function ρ on R and is an honest map. Consequently the full maps with respect to ρ over R become invertible over $M_n(T_{\bar{\rho}})$, so we have a homomorphism $R_\rho \longrightarrow M_n(T_{\bar{\rho}})$ and, in fact, a homomorphism from the ring $M_n(k) \sqcup_k R_\rho$ to $M_n(T_{\bar{\rho}})$. By Morita equivalence,

$M_n(k) \sqcup_k R_\rho \cong M_n(T_\rho)$, where T_ρ is a universal localisation of T at a collection of maps between finitely generated projectives, full with respect to $\bar{\rho}$.

<u>Theorem 5.8</u> Let R be a k-algebra with an excellent rank function taking values in $\frac{1}{n}\mathbb{Z}$. Then R_ρ, the universal localisation of R at ρ, is a perfect ring with nilpotent radical; all its finitely generated projective modules are stably induced from R and the rank function ρ extends to an excellent faithful rank function ρ' on R_ρ.

<u>Pf</u> We use the terminology of the preceding discussion.

$$M_n(k) \sqcup_k R \longrightarrow M_n(k) \sqcup_k R_\rho \cong M_n(T_\rho) \longrightarrow M_n(T_{\bar{\rho}})$$

induces a rank function ρ' on R_ρ extending that on R.

T_ρ is a universal localisation of T and we wish to show that T_ρ and hence R_ρ embeds in $M_n(T_{\bar{\rho}})$; by theorem 5.6 it embeds provided there are no projective modules of rank 0 over T_ρ with respect to the homomorphism to $M_n(T_{\bar{\rho}})$. But if there is a projective of rank 0 over T_ρ, there is a projective of rank 0 over $M_n(k) \sqcup_k R_\rho$ with respect to the rank function induced by the homomorphism to $M_n(T_{\bar{\rho}})$. Since this projective module decomposes as

$$P_1 \otimes_{M_n(k)} M_n(T_\rho) \oplus P_2 \otimes_{R_\rho} M_n(T_\rho),$$

we see that P_1 must be zero and P_2 a projective module over R_ρ such that $\rho'(P_2) = 0$. Let E be an idempotent $n \times n$-matrix over R_ρ such that $E({}^nR_\rho) \cong P_2$; then I - E becomes invertible over $M_n(T_{\bar{\rho}})$.

By Cramer's rule, we have an equation:

$$(\alpha\ \alpha_1)\begin{pmatrix} I & * \\ 0 & I-E \end{pmatrix} = (\alpha\ \alpha_2)$$

where α, α_1, and α_2 are defined over R and $(\alpha\ \alpha_1)$ is full with respect to ρ.

Since I-E becomes invertible and $(\alpha\ \alpha_1)$ becomes invertible over

$M_n(T_{\bar\rho})$, so does $(\alpha' \alpha_2)$: therefore $(\alpha' \alpha_2)$ is a full map with respect to ρ, so it is invertible over R_ρ and I-E must also be invertible over R_ρ. Hence E = 0 and, following the argument back, we find that all projective modules over T_ρ are of rank greater than 0 with respect to the homomorphism $T_\rho \longrightarrow T_{\bar\rho}$. We deduce that all projective modules over T_ρ are stably induced from T and that the rank function induced on T_ρ by $T_\rho \longrightarrow T_{\bar\rho}$ is a faithful excellent rank function.

By Morita equivalence, all projective modules over $M_n(T_\rho)$ ($\cong M_n(k) \sqcup_k R_\rho$) are stably induced from $M_n(k) \sqcup_k R$ and the rank function induced by $M_n(T_\rho) \longrightarrow M_n(T_{\bar\rho})$ on $M_n(T_\rho)$ is a faithful excellent rank function. We deduce that all projective modules over R_ρ are stably induced from R and that the rank function ρ' on R_ρ is a faithful excellent rank function with values in $\frac{1}{n}\mathbb{Z}$.

We need to show that perfection of our ring and the nilpotence of the radical. A ring is said to be <u>left perfect</u> if it has the descending chain condition on principal right ideals. We shall show the descending chain condition on right ideals of bounded generating number with respect to the rank function ρ' in the ring R .

First, however, we note that all full maps with respect to ρ' in R_ρ are invertible. For, if $\alpha : P \longrightarrow Q$ is a map in R_ρ, the projectives P,Q are stably induced from R, so $\alpha \oplus I_{R_\rho} : P \oplus {}^nR_\rho \longrightarrow Q \oplus {}^nR_\rho$ is a map between induced projective modules. By Cramer's rule, $\alpha \oplus I$ is stably associated to a map, $\beta \otimes_R R_\rho$, induced from R. Since α is full with respect to ρ', it becomes invertible in $M_n(T_{\bar\rho})$, so β must also become invertible there; so β is invertible in R_ρ already and so α must be also.

Suppose that we have an infinite descending chain of right ideals of bounded generating number with respect to ρ'; then we may find an infinite descending chain where all right ideals have the same generating number, q, and this number is minimal for such a chain to exist; let $I_0 \supsetneqq I_1 \supsetneqq I_2 \supsetneqq \dots$ be such a chain. We construct a diagram

$$\begin{tikzcd}
Q_0 \arrow[d, "p_0"'] & Q_1 \arrow[l, dashed, "\alpha_0"'] \arrow[d, "p_1"'] & \cdots & Q_i \arrow[d, "p_i"] & Q_{i+1} \arrow[l, dashed, "\alpha_i"'] \arrow[d, "p_{i+1}"] & \cdots \\
I_0 \arrow[r, phantom, "\supsetneqq"] & I_1 \arrow[r, phantom, "\supsetneqq"] & \cdots & I_i \arrow[r, phantom, "\supsetneqq"] & I_{i+1} & \cdots
\end{tikzcd}$$

where $Q_i \xrightarrow{p_i} I_i$ is a minimal presentation of I_i, so $\rho(Q_i) = q$. Since each Q_{i+1} is projective, we may construct maps $\alpha_i : Q_{i+1} \longrightarrow Q_i$ which make the diagram commute. This map α_i cannot be surjective since $I_{i+1} \subsetneqq I_i$, so it cannot be full by our preceding argument; suppose it factors as

$$\begin{tikzcd}
Q_{i+1} \arrow[rr, "\alpha_i"] \arrow[dr, "\beta_i"'] & & Q_i \\
& Q_i' \arrow[ur, "\gamma_i"'] &
\end{tikzcd}$$

where $\rho(Q_i') \lneqq \rho(Q_i) = q$.

Let I_i' be the image of Q_i' in I_i; then the generating number of I_i' is less than q and

$$I_0 \supsetneqq I_0' \supsetneqq I_1 \supsetneqq I_1' \supsetneqq \dots \supsetneqq I_i \supsetneqq I_i' \supsetneqq I_{i+1} \supsetneqq \dots$$

so that $I_0' \supsetneqq I_1' \supsetneqq \dots$ is a strictly descending chain of right ideals of generating number less than q, which contradicts our earlier assumption about q.

So R_ρ must be left perfect; it is right perfect by symmetry. Since R_ρ is perfect, its radical is nil and, modulo the radical, it is semisimple artinian; however, R_ρ embeds in the simple artinian ring $M_n(T_{\bar{\rho}})$ and all nil subrings of an artinian ring are nilpotent, so the radical of R is nilpotent as claimed.

First, we shall use this theorem to construct the simple artinian coproduct of simple artinian rings amalgamating a common simple artinian subring. At the end of the chapter, there will be a discussion of how near to the total truth theorem 5.8 approaches.

Our next theorem gives one set of circumstances under which we can conclude that the universal localisation of a ring R at an excellent rank function ρ taking values in $\frac{1}{n}\mathbb{Z}$ is a simple artinian ring; we remark that theorem 6.1 of the next chapter shows that this is also true, provided R has enough full maps (see next chapter for the definition of this).

Theorem 5.9 Let R be a ring with an excellent rank function ρ onto $\frac{1}{n}\mathbb{Z}$ and assume that ρ is unique; then the universal localisation of R at ρ, R_ρ, is a simple artinian ring.

Pf We use the notation and results of theorem 5.8. We have shown that R_ρ is a perfect ring; therefore it is a semisimple artinian ring modulo its radical. It is clear that a semisimple artinian ring has a unique rank function if and only if it is simple artinian. But R has a unique rank function, and all projective modules over R_ρ are stably induced from R, which implies that any rank function on R_ρ is determined by its values on R; therefore R is a simple artinian ring modulo its radical. Since we may lift idempotents modulo the radical in this case, $R_\rho \cong M_n(S)$ for some ring S.

In the course of theorem 5.8, we showed that R_ρ is a ring with an excellent rank function ρ', taking values in $\frac{1}{n}\mathbb{Z}$ and that all full maps with respect to ρ' between finitely generated projectives are invertible. By Morita equivalence, S is a ring with an excellent rank function ρ_S, taking values in $\mathbb{Z}$, such that all full maps with respect to ρ_S are invertible; consequently $S \cong S_{\rho_S}$. But by theorem 5.1, S_{ρ_S} is a skew field, so we find that $R_\rho \cong M_n(S)$ is a simple artinian ring.

Theorem 5.10 Let S_1 and S_2 be a couple of simple artinian rings with common simple artinian subring S. Then $S_1 \sqcup_S S_2$ is an hereditary ring with a unique rank function ρ. Therefore, $(S_1 \sqcup_S S_2)_\rho$ is a simple artinian ring, which we call the simple artinian coproduct of S_1 and S_2, amalgamating S. If $S_i \cong M_{n_i}(D_i)$ for skew fields D_i, $(S_1 \underset{S}{\sqcup} S_2)_\rho \cong M_n(D)$, where $n = \text{l.c.m.}\{n_1, n_2\}$.

Pf From theorem 3.1 we know that $S_1 \underset{S}{\sqcup} S_2$ has a unique rank function ρ; if $S_i \cong M_{n_i}(D_i)$, where D_i is a skew field, the image of ρ is $\frac{1}{n}\mathbb{Z}$, where $n = \text{l.c.m.}\{n_1, n_2\}$.

By theorem 5.9, the universal localisation of $S_1 \underset{S}{\sqcup} S_2$ is a simple artinian ring, $M_n(D)$, for some skew field D.

Following Cohn's notation for skew field coprodcuts, we shall denote the simple artinian coproduct of S_1 and S_2, amalgamating S_o, as $S_1 \underset{S}{\circ} S_2$. Of course, we can construct the simple artinian coproduct of finitely many simple artinian rings, S_i, amalgamating S in exactly the same way; for infinitely many simple artinian rings, we have to be slightly more careful. In this case, the last theorem is proved in exactly the same way, provided that if $S_i \cong M_{n_i}(D_i)$, D_i a skew field, then $\text{l.c.m.}\{n_i\}$ exists; in the contrary case, our universal localisation at ρ does exist, but it is a von Neumann regular ring that is not simple artinian.

In theorem 5.8, we showed that the universal localisation of R at an excellent rank function, ρ, taking values in $\frac{1}{n}\mathbb{Z}$ is a perfect ring. There is a certain amount of evidence to suggest that we should be able to prove artinian rather than perfect; however, at present, this seems hard to approach. We have already given examples of hereditary artinian rings that are complete localisations at a rank function; we should like to show that not all such complete localisations need to be hereditary.

So let R be the quotient ring of the 3×3 upper triangular matrix ring over a field k modulo the square of its radical; so it is

$$\begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we assign the rank $\frac{1}{4}$ to the projectives $\begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}$ and the rank $\frac{1}{2}$ to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & k & k \\ 0 & 0 & 0 \end{pmatrix}$, we find that this is an excellent rank function and that the only full maps are invertible, as we wished to show.

There is a related direction in which we should be able to better theorem 5.8. If R is a right hereditary ring, theorem 5.8 shows that R_ρ is a right hereditary perfect ring, from which we can deduce that it is two-sided hereditary. We may investigate such rings using the classification of perfect hereditary rings due to Chase (11) and Bergman (6). If R is weakly semihereditary, we should like to deduce that R_ρ is actually two-sided hereditary; we would be able to deduce this if we knew that R_ρ is weakly semihereditary. Bergman has conjectured that any localisation of a weakly semihereditary ring is still weakly semihereditary.

Chapter 6

In the last chapter, we investigated the universal localisation of a ring R at an excellent rank function ρ, taking values in $\frac{1}{n}\mathbb{Z}$; the resulting ring is always a perfect ring. Since the degree of nilpotence of the radical may grow with the integer n, it is likely that if the rank function takes values in the real numbers, we are not going to be able to say a great deal about the universal localisation at the rank function. This suggests that rather than investigating epimorphisms to von Neumann regular rings that induce the rank function, we should look for homomorphisms; and we shall be able to prove that every rank function on a two-sided $\mathfrak{X}_o$-hereditary ring arises from a homomorphism to a von Neumann regular ring with a suitable rank function. Under suitable hypotheses, we are able to show that the universal localisation at the rank function is von Neumann regular; however, it is not a particularly easy matter to check this condition.

In the first chapter, we showed that if R is a two-sided $\mathfrak{X}_o$-hereditary ring with a rank function ρ, then every map between finitely generated projective modules factors as a right full followed by a left full map; we described this condition as having enough right full and left full maps. We shall need a complementary condition. We shall say that R has enough full maps with respect to ρ if every left full map is a left factor of a full map and every right full map is a right factor of a full map.

Theorem 6.1 Let R be a ring with an excellent rank function ρ such that R has enough right full, left full and full maps with respect to ρ. Then the universal localisation of R at ρ, R_ρ, is a von Neumann regular ring. All projective modules are stably induced from

R and the rank function extends to R_ρ. The kernel of the homomorphism from R to R_ρ is the trace ideal of the projectives of rank 0.

<u>Pf</u> Consider the category of induced finitely generated projective modules over R_ρ; we know, by Cramer's rule, that if $\alpha: R_\rho\otimes_R P \longrightarrow R_\rho\otimes_R Q$ is in this category, it is stably associated to a map

$$R_\rho\otimes_R\alpha' : R_\rho\otimes_R P' \longrightarrow R_\rho\otimes_R Q',$$

induced from the category of finitely generated projective modules over R. Over R, we may factor α' as $\alpha' = \alpha'_r\alpha'_i$ $(\alpha'_r : P' \longrightarrow P_1$, $\alpha'_i : P_1 \longrightarrow Q')$, where α'_r is right full and α'_i is left full. However, since α'_r is a right factor of a full map with respect to ρ in R and all full maps over R are invertible in R_ρ, α'_r has a left inverse over R_ρ; dually, α'_i has a right inverse. Therefore, over R_ρ, $R_\rho\otimes_R P_1$ is isomorphic to the image of $R_\rho\otimes_R\alpha$ and the image is a direct summand of $R_\rho\otimes_R Q'$. We conclude that the image of $\alpha : P \longrightarrow Q$ must also be a direct summand of Q, since this property remains invariant under stable association. So R_ρ is von Neumann regular, since every finitely generated left ideal of R_ρ is the image of some map $\alpha : R^n \longrightarrow R$ and consequently, a direct summand of R_ρ.

Next, we show that all projective modules are stably induced from R. Let $e \in M_n(R_\rho)$ be an idempotent presenting a finitely generated projective module $\bar{P}$ over R_ρ; as before, e is stably associated to a map $R_\rho\otimes_R\alpha' : R_\rho\otimes_R P \longrightarrow R_\rho\otimes_R Q$ induced from R. Moreover, the image of $R_\rho\otimes_R\alpha'$ is isomorphic to $R_\rho\otimes_R P_1$, where $\alpha' = \alpha'_r\alpha'_i$ $(\alpha'_r : P \longrightarrow P_1$, $\alpha'_i : P_1 \longrightarrow Q$ is a minimal factorisation of α'. But the image of $e \oplus I_n$ is just $\bar{P} \oplus R^n$; so $\bar{P} \oplus R^n = R_\rho\otimes_R P_1$.

So far we have not used the law of nullity at all; however, if our rank function is not excellent, the kernel of the map from R to R_ρ will be large. We use Malcolmson's criterion together with the law of nullity to determine the kernel of the map $R \longrightarrow R_\rho$.

Before continuing, we recall that the nullity with respect to ρ of a map $\alpha: P \longrightarrow Q$ between projectives such that $\rho(P) = \rho(Q)$ is equal to $\rho(P) - \rho(\alpha)$. We recall also that the nullity of a map $\begin{pmatrix} \alpha & 0 \\ /// & \beta \end{pmatrix}$ where α is a full map with respect to an excellent rank function ρ is equal to the nullity of β; a similar statement holds for $\begin{pmatrix} \alpha & /// \\ 0 & \beta \end{pmatrix}$

Suppose that $r \in R$ has image 0 in R_ρ. Malcolmson's criterion states that there is an equation of matrices of maps over R:

$$M = \left(\begin{array}{ccc|c} 1 & 0 & 0 & r \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & \beta_1 \\ \hline 1 & \alpha_1 & 0 & 0 \end{array}\right) = \left(\begin{array}{c} \gamma \\ \hline \gamma_1 \end{array}\right) \left(\begin{array}{c|c} \delta & \delta_1 \end{array}\right)$$

where $\alpha, \beta, \gamma, \delta$ are full maps over R. This states that the nullity of M is 1. Since the row through α and the column through β both have only zeros apart from the named elements, we may conclude that $\begin{pmatrix} 1 & r \\ 1 & 0 \end{pmatrix}$ has nullity 1. Again, we may conclude that r has nullity 1, which implies that right multiplication by r factors through a projective module of rank 0. So R lies in the trace ideal of the projectives of rank 0. Conversely, if P is a projective of rank 0 with respect to ρ over R, the image of a matrix $e \in M_n(R)$, then $(1-e)$ is a full matrix so e becomes zero in R , and $R_\rho \otimes_R P \cong 0$. So the trace ideal of the projectives of rank zero with respect to ρ is exactly the kernel of $R \longrightarrow R_\rho$.

Finally, we wish to show that the rank function ρ extends to R_ρ; we know that every projective module is stably induced from R, so we wish to define a rank function ρ', by $\rho'(P) = \rho(P') - n$, where $P \oplus R_\rho^n \cong R_\rho \otimes_R P'$, where P' is a projective module over R. There are two problems to deal with; ρ' may take negative values with this definition; and there may be two different equations $P \oplus R_\rho^n \cong R_\rho \otimes_R P_i'$ $(i = 1,2)$, giving different values for $\rho'(P)$. We shall show that neither problem arises under the assumption that there can be no isomorphism over R_ρ, $\beta: R_\rho \otimes_R Q_1 \longrightarrow R_\rho \otimes_R Q_2$ if $\rho(Q_1) \neq \rho(Q_2)$; then we show that this last statement is true.

First, ρ' must be well-defined given the assumption; for if $P \oplus R_\rho^{n_1} \cong R_\rho \otimes_R P_1'$ and $P \oplus R_\rho^{n_2} \cong R_\rho \otimes_R P_2'$, we may assume that $n_1 = n_2$, by changing the first equation to $P \oplus R^{n_2} \cong R_\rho \otimes_R (P_1' \oplus R^{n_2 - n_1})$ if $n_1 \leqslant n_2$ (or the corresponding change if $n_2 \leqslant n_1$). Then $R_\rho \otimes_R P_1'$ and $R_\rho \otimes_R P_2'$ are isomorphic over R_ρ; so $\rho(P_1') = \rho(P_2')$ and $\rho'(P)$ is well-defined.

Next, if $\rho'(P) \leqslant 0$, we find that $P \oplus R_\rho^n \cong R_\rho \otimes_R P'$, where $\rho(P') \leqslant n$, so over R_ρ there is a surjection $\alpha : R_\rho \otimes_R P' \longrightarrow R_\rho^n$; by Cramer's rule, this is stably associated to a map $R_\rho \otimes_R \alpha' : R_\rho \otimes_R Q \longrightarrow R_\rho \otimes_R Q'$ defined over R such that $\rho(Q) \leqslant \rho(Q')$ and $R_\rho \otimes_R \alpha'$ is surjective with kernel P over R_ρ.

Let $\begin{tikzcd} Q \arrow[rr, "\alpha'"] \arrow[dr, "\beta"'] & & Q' \\ & Q'' \arrow[ur, "\gamma"'] & \end{tikzcd}$ be a factorisation of α' as right full by left full; the first argument of this theorem shows that over R_ρ, $R_\rho \otimes_R \gamma$ induces an isomorphism between $R_\rho \otimes_R Q''$ and the image of $R_\rho \otimes_R \alpha'$, which is $R_\rho \otimes_R Q'$. Consequently, our assumption tells us that $\rho(\alpha) = \rho(Q'') = \rho(Q')$; since $\rho(Q) \leqslant \rho(Q')$, $\rho(Q) = \rho(Q')$ and α' is full; so it becomes invertible over R_ρ and its kernel P is zero, which completes our proof, given the assumption that the only isomorphisms between induced projective modules are between those of the same rank with respect to ρ.

So let $\alpha : R_\rho \otimes_R P_1 \longrightarrow R_\rho \otimes_R P_2$ be an isomorphism over R_ρ; we wish to show that $\rho(P_1) = \rho(P_2)$. By Cramer's rule,

$$\beta \begin{pmatrix} I & /\!/\!/ \\ 0. & \alpha \end{pmatrix} = \beta'$$

where β is full $\beta : P \longrightarrow R^n \oplus P_1$, $\beta' : P \longrightarrow R^n \oplus P_2$. So β' becomes an isomorphism over R_ρ and $\rho(P_1) = \rho(P_2) \Longleftrightarrow \rho(P) = \rho(R^n \oplus P_2)$. We factor β':

$$\begin{tikzcd} P \arrow[rr, "\beta'"] \arrow[dr, "\beta_1"'] & & R^n \oplus P_2 \\ & Q \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

where $\beta_1 : P \longrightarrow Q$ is right full, $\beta_2 : Q \longrightarrow R^n \oplus P_2$ is left full, and recall that the first argument of this theorem shows that β_2 induces an isomorphism of Q with the image of β', which is $R^n \oplus P_2$; so $\beta_1 : P \longrightarrow Q$ is also an isomorphism.

We have a left full map $\beta_2 : Q \longrightarrow R^n \oplus P_2$; it is a left factor of a full map $\beta_2 \gamma = \delta : Q \longrightarrow Q'$. So, a right inverse of β_2 over R_ρ is $\gamma \delta^{-1}$; however, β_2 is an isomorphism over R_ρ, so $\gamma \delta^{-1} \beta_2$ is the identity map on $R^n \oplus P_2$. We apply Malcolmson's criterion, which gives us the following matrix of maps equation over R:

$$M = \left(\begin{array}{cccc|c} \delta & 0 & 0 & 0 & \beta_2 \\ 0 & I & 0 & 0 & -I \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta & \beta_1 \\ \hline \gamma & I & \alpha_1 & 0 & 0 \end{array}\right) = \left(\begin{array}{c} \mu \\ \hline \mu_1 \end{array}\right) \left(\begin{array}{c|c} \nu & \nu_1 \end{array}\right)$$

where α, β, μ, ν are full maps over R with respect to ρ.

This equation shows that the nullity of M is equal to $\rho(R^n \oplus P_2)$; since α, β are full maps and the row through α and the column through β contain only 0s (other than α and β), we deduce that the nullity of $\begin{pmatrix} \delta & 0 & \beta_2 \\ 0 & I & -I \\ \gamma & I & 0 \end{pmatrix}$ is $\rho(R^n \oplus P_2)$. Adding the second column to the last, we deduce that the nullity of $\begin{pmatrix} \delta & 0 & \beta_2 \\ 0 & I & 0 \\ \gamma & I & I \end{pmatrix}$ is $\rho(R^n \oplus P_2)$, so, since the middle row of this is $(0 \quad I \quad 0)$, we deduce that the nullity of $\begin{pmatrix} \delta & \beta_2 \\ \gamma & I \end{pmatrix}$ is equal to $\rho(R^n \oplus P_2)$. However, $\begin{pmatrix} \delta & \beta_2 \\ \gamma & I \end{pmatrix}$ is a map from $Q \oplus R^n \oplus P_2$ to $Q' \oplus R^n \oplus P_2$ and restricted to $R^n \oplus P_2$ induces the identity isomorphism to $R^n \oplus P_2$; since the nullity plus the inner rank with respect to ρ must equal $\rho(Q \oplus R^n \oplus P_2) = \rho(Q' \oplus R^n \oplus P_2)$, we deduce that $\rho(Q) = \rho(Q') \geqslant \rho(R^n \oplus P_2)$. However, $\rho(Q) \leqslant \rho(R^n \oplus P_2)$ since $\beta_2 : Q \longrightarrow R^n \oplus P_2$ is left full; therefore $\rho(Q) = \rho(R^n \oplus P_2)$.

An entirely similar argument shows that the right full map $\beta_1: P \longrightarrow Q$ induces an isomorphism $R_\rho \otimes_R \beta_1 : R_\rho \otimes_R P \longrightarrow R_\rho \otimes_R Q$ if and only if $\rho(P) = \rho(Q)$; so $\rho(P) = \rho(R^n \oplus P_2) \Leftrightarrow \rho(P_1) = \rho(P_2)$, which shows that there can be an isomorphism between induced projective modules only when $\rho(P_1) = \rho(P_2)$.

Of course, it may be rather hard to check the condition that there should be enough full maps; however, we have already seen that over a two-sided χ_0-hereditary ring there are enough right full and left full maps with respect to a rank function. We should like to be able to embed any k-algebra with an excellent rank function ρ honestly in another k-algebra with an excellent rank function ρ' such that the second k-algebra has enough full maps. It turns out that we can do this simply by adjoining a large number of generic maps between finitely generated projectives.

<u>Theorem 6.2</u> Let R be a k-algebra with an excellent rank function ρ, taking values in $\mathbb{R}$; then the embedding of R in $R \sqcup_k k\langle X\rangle$, where X is an infinite set is an honest map with respect to the pair of rank functions ρ on R and (ρ, r) on $R \sqcup_k k\langle X\rangle$ (where r is the unique rank function on $k\langle X\rangle$). Further, $R \sqcup_k k\langle X\rangle$ has enough full maps with respect to (ρ, r).

<u>Pf</u> Since all projectives over $R \sqcup_k k\langle X\rangle$ are induced from R, we write P for $(R \sqcup_k k\langle X\rangle)\otimes_R P$. The idea is fairly simple; let $\alpha: P \longrightarrow Q$ be a left full map with respect to ρ over $R \sqcup_k k\langle X\rangle$ defined over $R \sqcup_k k\langle X\rangle$. Then it is actually defined over a finite subset Y of X; that is, it is induced from a map $\alpha': P' \longrightarrow Q'$ over $R \sqcup_k k\langle Y\rangle$. We may use the elements of X - Y to define a generic map from Q to P over $R \sqcup_k k\langle X\rangle$ and the composition of this with α should be full with respect to the rank function. An entirely dual argument should show that right full maps are right factors of full maps.

There is some integer n for which P and Q are n-generator projective modules, and neither is of rank n with respect to ρ; that is, there are idempotents e_P, e_Q in $M_n(R)$ such that $R^n e_P \cong P$, $R^n e_Q \cong Q$, so our map $\alpha: P \longrightarrow Q$ defined over $R \sqcup_k k\langle Y\rangle$ may be represented by a matrix $a \in M_n(R \sqcup_k k\langle Y\rangle)$ such that $e_P a = a = a e_Q$. Under the Morita equivalence of $M_n(R)$ and R, $M_n(R)e_P$ is the module corresponding to P and $M_n(R)e_Q$ is that corresponding to Q.

There is a unique extension of ρ on R to a rank function ρ_Y on $R \sqcup_k k\langle Y\rangle$, and, by Morita equivalence, this induces a rank function ρ_1 on $R_1 = M_n(R \sqcup_k k\langle Y\rangle)$. We see that $\rho_1(R_1 e_P) < 1$, since it equals $\rho(P)/n$ and similarly $\rho_1(R_1 e_Q) < 1$. Further, since the left full map $\alpha: P \longrightarrow Q$ is represented by the element $a \in R_1$, which is the map from $R_1 e_P$ to $R_1 e_Q$ Morita equivalent to α, and since the embedding of $R_1 e_Q$ in R_1 is left full, the map from $R_1 e_P$ to R_1, defined by right multiplication by a is left full because it is a composition of left full maps; so, for all left ideals I of R containing a, $\rho_1(I) \geqslant \rho_1(Re_P)$ by 1.4

Over the ring $S = R_1 \sqcup_k k[x]$ we have the map defined by right multiplication by $e_Q x e_P$ is, in a suitable sense, a generic map from $R_1 e_Q$ to $R_1 e_P$. However, we note that

$$S = R_1 \sqcup_k k[x] \cong M_n(R \sqcup_k k\langle Y\rangle) \sqcup_k k[x] \cong M_n(R \sqcup_k k\langle Y\rangle \sqcup_k k\langle x_{ij}\rangle)$$

where $i,j = 1 \longrightarrow n$; the map sends x to the matrix (x_{ij}). So, if we can show that $a e_Q x e_P$ is a full map with respect to the rank function on S extending ρ, Morita equivalence shows that $\alpha: P \longrightarrow Q$ is the left factor of a full map over the ring $R \sqcup_k k\langle Y\rangle \sqcup_k k\langle x_{ij}\rangle$; since $|Y| < \infty$ and $|X| = \infty$, we may choose x_{ij} $(i,j = 1 \rightarrow n)$ to be elements of $X - Y$ and by theorem 3.6 we know that this full map remains full over $R \sqcup_k k\langle X\rangle$; so we need to show that $axe_P = ae_Q xe_P$ is a full map over S with respect to ρ', the rank function extending ρ_1 on R_1. In order to prove this, we need some of the details on the coproduct theorems.

What we need to show is that any left module inside $Se_p \cong S\otimes_{R_1} R_1e_p$ containing axe_p has generating number at least $\rho'(Se_p)$ with respect to the rank function ρ' on S extending ρ_1 on R_1. Let M be such a submodule of Se_p; then, if $\rho'(M) \leq \rho'(Se_p) < 1$, we find that in the decomposition given by theorem 2.7,

$$M \cong S\otimes_k M_o \oplus S\otimes_{R_1} M_1 \oplus S\otimes_{k[x]} M_2$$

M_o and M_2 are zero, for if they are not $\rho'(M) \geq 1$. Therefore $M \cong S\otimes_{R_1} M_1$.

We choose the basis $1, x, x^2, \ldots.$ for $k[x]$ and order it by $1 < x < x^2 < \ldots.$, We well-order some basis for R_1 over k containing the element e_p. Then, by theorem 2.7, M_1 is the R_1-submodule of M consisting of those elements whose 1-support does not contain the 1-leading term of some non-1-pure term; if $axe_p \notin M$, its 1-support, $\{xe_p\}$ must be the 1-leading term of some element and the nature of the ordering of $k[x]$ forces this element to be of the form $xe_p + r_1$, where $r_1 \in R_1e_p$. However, I_2 is empty, so we deduce that $e_p \in M$ implies that $M = Se_p$ and $\rho'(M) = \rho'(Se_p)$.

This leaves the case $axe_p \in M_1$; we recall from theorem 2.6 that the structure of Se_p as an R_1-module has the form

$$R_1e_p \oplus R_1xe_p \oplus B,$$

where B is a basic R_1-module. We project M_1 onto the direct summand R_1xe_p, which is a free module on the generator xe_p; its image contains axe_p. We have already noted that any left R_1-ideal containing a has generating number at least $\rho_1(R_1e_p)$, since a defines a left full map from R_1e_p to R_1; so the generating number of M_1 is at least $\rho(R_1e_p)$ and therefore M has generating number at least $\rho_1(Re_p)$ with respect to ρ', which shows that left multiplication by axe_p is a full map on Se_p.

Therefore, as we noted earlier, it follows that every left full map with respect to the rank function extending ρ on $R \sqcup_k k\langle X\rangle$ is a left factor of a full map. By duality (or by a dual argument) every right full map is a right factor of a full map.

This calculation allows us to prove;

<u>Theorem</u> <u>6.3</u> Let R be a two-sided $\aleph_0$-hereditary k-algebra with rank function ρ taking values in $\mathbb{R}$; then R has an honest homomorphism to a von Neumann regular k-algebra with rank function.

<u>Pf</u> First, the homomorphism $R \longrightarrow R \sqcup_k k\langle X\rangle$ is honest. Next, by theorem 2.2, $R \sqcup_k k\langle X\rangle$ is two-sided $\aleph_0$-hereditary and so, by theorem 1.9, $R \sqcup_k k\langle X\rangle$ has enough right full and left full maps; by theorem 6.2, it has enough full maps. By theorem 6.1, the universal localisation of $R \sqcup_k k\langle X\rangle$ at the rank function is a von Neumann regular ring, V, with rank function and the homomrphism $R \sqcup_k k\langle X\rangle \longrightarrow V$ is honest; so the homomorphism $R \longrightarrow V$ is honest.

It is possible to show that the monoid of finitely generated projectives over the ring V constructed in the last theorem is the positive cone of the subgroup of $\mathbb{R}$ that is the image of ρ.

Part II

Section 1

In this section, we shall develop the techniques we need to find the finite-dimensional k-algebras lying in skew field coproducts with amalgamation. Since the method can be applied in a much wider context, we shall work in this greater generality.

For the purposes of this section, the rings R,F,K,k will be assumed to satisfy the following assumptions:

<u>Assumptions</u>: R is a left semihereditary k-algebra, where k is a commutative field. ρ is a rank function on the finitely generated projective modules over R taking values in $\frac{1}{m}\mathbb{Z}$ such that the universal two-sided localisation of R at ρ, R_ρ, is a simple artinian ring $M_n(F)$, where F is a skew field with centre K, a regular commutative extension of k.

An extension $K \supset k$ of commutative fields is said to be <u>regular</u> if $K \otimes_k L$ is a domain for all field extensions L of k.

In future sections we shall specialise and adapt the results of this section to those cases we are interested in.

First of all, it is useful to reformulate our problem; we wish to find whether a finite-dimensional division algebra D over k embeds in a skew field F with centre K, a regular extension of k. We can broaden this question a little and ask; for what natural numbers, m, does D embed in $M_m(F)$? This question has a useful reformulation.

<u>Lemma 1.1</u> Under the previous assumptions, $D^o \otimes_k F$ is a simple artinian ring with simple module S. Let $n = [S:F]$. Then D embeds in $M_m(F)$ if and only if n divides m.

<u>Pf</u> Let C be the centre of D^o; then $D^o \otimes_k F$ is isomorphic to $D^o \otimes_C (C \otimes_k F)$, so any non-zero ideal of $D^o \otimes_k F$ intersects non-trivially

with $C\otimes_k F$.

In turn, $C\otimes_k F = (C\otimes_k K)\otimes_K F$ and so any non-zero ideal of $C\otimes_k F$ intersects non-trivially with $C\otimes_k K$, which is a field, since K is a regular extension of k and $[C:k]<\infty$. Therefore, $D^o\otimes_k F$ is simple. It must be artinian because $[D^o:k]$ is finite.

We are left with the last sentence of the lemma. If D embeds in $M_m(F)$, D^o embeds in $M_m(F^o)$, which is the endomorphism ring of the F-module F^m. Consequently, we can give F^m the structure of a left $F\otimes_k D^o$-module and so F^m is a direct sum of copies of S and n must divide m. The argument is reversible.

It is important to bear this lemma in mind, since most of the embedding theorems of the early part of these sections will be stated in the form of the dimension of a simple $F\otimes_k D^o$-module, as, by lemma 1.1, this is a succinct expression of all possible embeddings of D in matrix rings over F. A further point to remember is that the same information is conveyed by the rank as $M_m(F)$-module of a simple $M_m(F)\otimes_k D^o$-module, since the Morita equivalence of $M_m(F)\otimes_k D^o$ and $F\otimes_k D^o$ shows that the dimension of a simple $F\otimes_k D^o$-module over F is m times the rank of a simple $M_m(F)\otimes_k D^o$-module over $M_m(F)$.

Our next theorem reduces statements about the module theory of $R_\rho\otimes_k D^o$ to statements about the modules of $R\otimes_k D^o$; as usual, for this type of result, we shall use a version of Cramer's rule.

<u>Theorem 1.2</u> Let R satisfy the assumptions stated earlier, and let D be a finite-dimensional division k-algebra; then the rank of a simple $R_\rho\otimes_k D^o$-module as an R_ρ-module is equal to $\text{h.c.f.}_M\{\rho(M)\}$ as M runs over the finitely generated left $R\otimes_k D^o$-submodules of free $R\otimes_k D^o$-modules.

<u>Pf</u> That the rank of a simple $R_\rho\otimes_k D^o$-module as R_ρ-module must divide

this highest common factor is easily seen by considering the $R_\rho \otimes_k D^o$-modules, $R_\rho \otimes_R M$, where M is an $R \otimes_k D^o$-module.

Our first point is that $R_\rho \otimes_k D^o$ is the universal two-sided localisation of $R \otimes_k D^o$ at those maps between projective modules induced up from maps over R, that are full with respect to ρ. This allows us to use Cramer's rule.

Let $e \in R_\rho \otimes_k D^o$ be an idempotent such that $(R_\rho \otimes_k D^o)e$ is isomorphic to the unique simple module S over $R_\rho \otimes_k D^o$.

Then by Cramer's rule I.4.1, there is an equation

$$(\alpha_1, \ldots, \alpha_n) \begin{pmatrix} I_{n-1} & \begin{matrix} x_1 \\ \vdots \\ x_{n-1} \end{matrix} \\ 0 & e \end{pmatrix} = (\alpha_1, \ldots, \alpha_{n-1}, \beta)$$

where $\alpha_1, \ldots, \alpha_n$ are maps from some projective module P to $(R \otimes_k D^o)^n$ induced up from maps defined over R, β is defined over $R \otimes_k D^o$ and is a map from P to $R \otimes_k D^o$; and $x_1, \ldots, x_{n-1}$ are elements of $R_\rho \otimes_k D^o$. Moreover, the map $(\alpha_1, \ldots, \alpha_n)$ is a full map with respect to the rank function ρ over R and consequently, invertible over $R_\rho \otimes_k D^o$.

We can rewrite this equation as:

$$(\alpha_1, \ldots, \alpha_n) \begin{pmatrix} I & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} I & \begin{matrix} x_1 \\ \vdots \\ x_{n-1} \end{matrix} \\ 0 & 1 \end{pmatrix} = (\alpha_1, \ldots, \alpha_{n-1}, \beta).$$

Since $(\alpha_1, \ldots, \alpha_n)$ and $\begin{pmatrix} I & \begin{matrix} x_1 \\ \vdots \\ x_{n-1} \end{matrix} \\ 0 & 1 \end{pmatrix}$ are invertible over $R_\rho \otimes_k D^o$, the images of the maps over $R_\rho \otimes_k D^o$ defined by $\begin{pmatrix} I & 0 \\ 0 & e \end{pmatrix}$ and $(\alpha_1, \ldots, \alpha_{n-1}, \beta)$ are isomorphic. The image of $\begin{pmatrix} I & 0 \\ 0 & e \end{pmatrix}$ is $(R_\rho \otimes_k D^o)^{n-1} \oplus S$, where S is the unique simple $R_\rho \otimes_k D^o$-module. The second map is induced from a map over $R \otimes_k D^o$ from P to $(R \otimes_k D^o)^n$.

Let $M \subseteq (R \otimes_k D^o)^n$ be the image of this map over $R \otimes_k D^o$. Then the image of $(\alpha_1, \ldots, \alpha_{n-1}, \beta)$ as a map over $R_\rho \otimes_k D^o$ will be

$$(R_\rho \otimes_k D^o)M \subseteq (R_\rho \otimes_k D^o)^n.$$

Further, $(R_\rho \otimes_k D^o)M = R_\rho M$ since D^o centralises R_ρ and M is already a D^o-module.

The following lemma becomes useful at this point:

<u>Lemma 1.3</u> Let $N \subseteq R^s$ be a finitely generated submodule of the free module R^s; then the rank of $R_\rho N$, lying in R_ρ^s, is equal to the minimal rank of R-modules N' such that $N \subseteq N' \subseteq R^s$.

<u>Pf</u> Let N be an n-generator module; then the rank of $R_\rho N$ lying in R_ρ^s is just the inner rank of a map $\alpha : R^n \longrightarrow R^s$ where the image of α is N. By lemma I.1.4, this is equal to the infimum of the ranks of finitely generated projective modules, N', such that $N \subseteq N' \subseteq R^s$. Since the rank takes values in $\frac{1}{m}\mathbb{Z}$, we may change infimum to minimum.

In the case we are examining, we have an $R\otimes_k D^o$-module $M \subseteq (R\otimes_k D^o)^n$ and we wish to find the rank as an R_ρ-module of $R_\rho M \subseteq (R_\rho \otimes_k D^o)^n$, since we know that $R_\rho M \cong (R_\rho \otimes_k D^o)^{n-1} \oplus S$ and we shall be able to deduce the rank of S as an R_ρ-module. We know by lemma 1.3 that the rank of $R_\rho M$ as R_ρ-module is the minimal rank of an R-module M' such that $M \subseteq M' \subseteq (R\otimes_k D^o)^n$. Our final step is to show that there is an $R\otimes_k D^o$-module having this minimal rank.

Let N be an R-module such that $M \subseteq N \subseteq (R\otimes_k D^o)^n$ and N has minimal rank $q \in \frac{1}{m}\mathbb{N}$. We shall show that D^oN, an $R\otimes_k D^o$-module, has this same rank as R-module.

Let $\{1=d_0, d_1, \ldots, d_t\}$ be a basis of D^o over k. Then $D^oN = \sum_{i=0}^{t} d_iN$ is a finitely generated $R\otimes_k D^o$-submodule of $(R\otimes_k D^o)^n$ containing M.

First, d_iN is isomorphic to N as an R-module since d_i is invertible and centralises R. Moreover, $d_iN \supseteq M$ since M is an $R\otimes_k D^o$-module.

If M_1 and M_2 are both R-submodules of minimal rank containing M inside $(R\otimes_k D^o)^n$, then from the exact sequence

$$0 \longrightarrow M_1 \cap M_2 \longrightarrow M_1 \oplus M_2 \longrightarrow M_1 + M_2 \longrightarrow 0$$

which is split as a sequence of R-modules, we deduce that

$$\rho(M_1 \cap M_2) + \rho(M_1 + M_2) = 2q$$

But $\rho(M_1 \cap M_2) \geqslant q \leqslant \rho(M_1 + M_2)$, since both contain M. Therefore $\rho(M_1 + M_2) = q$. By an easy induction, we see that

$$\rho(D^o N) = \rho(\sum_{i=0}^{t} d_i N) = q.$$

What we have shown so far is that the rank as an R_ρ-module of $(R_\rho \otimes_k D^o)^{n-1} \oplus S$, where S is the unique simple $R_\rho \otimes_k D^o$-module, is equal to the rank as an R-module of the $R \otimes_k D^o$-module $D^o N$, which is a finitely generated $R \otimes_k D^o$-submodule of a free module. Consequently, $\text{h.c.f.}_M \{\rho(M)\}$ as M runs over those finitely generated $R \otimes_k D^o$-submodules of free modules must divide the rank of S as an R_ρ-module.

Conversely, let M be a finitely generated $R \otimes_k D^o$-module, projective as an R-module. Then $R_\rho \otimes_R M$ is an $R_\rho \otimes_k D^o$-module, since the action of D^o on M centralises the action of R on M. Moreover, the rank of M as an R-module equals the rank of $R_\rho \otimes_R M$ as an R_ρ-module. So the rank of S as an R_ρ-module must divide $\text{h.c.f.}_M \{\rho(M)\}$ as M runs over those finitely generated $R \otimes_k D^o$-modules projective as R-modules, which is just the class of finitely generated submodules of free $R \otimes_k D^o$-modules since R is left semihereditary. So the rank of S as an R_ρ-module must be equal to $\text{h.c.f.}_M \{\rho(M)\}$ as we wished to show.

Section 2

If we knew the centre of the simple artinian coproduct of two simple artinian rings S_1 and S_2, amalgamating a common simple artinian subring S, we would be able to apply theorem 1.3. Unfortunately, the results are incomplete. Below we state the conjectural result and we follow that with a summary of those cases where the result is known.

Conjecture 2.1 The centre of $S_1 \circ_S S_2$ lies in S except possibly when both S_1 and S_2 are of rank 2 over S. In this case, the centre lies in S or is the function field of a curve of genus 0 over its intersection with S.

The function field of a curve of genus 0 over a field K is a field F such that $L \otimes_K F = L(t)$, where L is a finite-dimensional extension of K and t is a transcendental.

This conjecture is know to be true when S is a common central subfield of S_1 and S_2 as we shall show in section 3. It is also true for skew field coproducts (Cohn 19), except, possibly, when both factors are two-dimensional over the amalgamated skew subfield. We shall prove a generalisation of this result is section 3. We shall also show in section 3 that all function fields of curves of genus 0 may arise.

In this section, we shall find the dimension of a simple $(S_1 \circ_S S_2) \otimes_C D^o$-module over $S_1 \circ_S S_2$, where C is the intersection of the centre of $S_1 \circ_S S_2$ with S, and D is a finite-dimensional C-algebra, under the assumption that conjecture 2.1 holds for this coproduct. Thus when we demonstrate that conjecture 2.1 holds for skew field coproducts (except for the one annoying case where conjecture 2.1 has not yet been proven), we shall be able to describe the finite-dimensional division C-algebras lying in our coproduct (where C is the intersection of the centre of the coproduct with the amalgamated skew field). Finally, if k is a commutative subfield of C, there is an easy reduction from the problem of finite-dimensional division k-algebras in the coproduct to the problem of finite-dimensional C-algebras in the coproduct, which we present at the end of this section.

There is a spurious generality in dealing with the amalgamation over a common simple artinian ring, since this can be reduced to amalgamation over a common skew subfield. Whilst performing this quite simple reduction, we shall set up the names of the rings used in this section.

Let $S \cong M_n(E)$, $S_i = M_{n_i}(E)$ $(i = 1,2)$, where S, S_i are simple artinian k-algebras and S embeds in S_i respecting the k-algebra structure. E and E_i are skew fields.

Since S embeds in S_i, n divides n_i; let $m_i = n_i/n$; then on taking the centraliser of $M_n(k)$ in $S_1 \underset{S}{\sqcup} S_2$, we see that

$$S_1 \underset{S}{\sqcup} S_2 \cong M_n(M_{m_1}(E_1) \underset{E}{\sqcup} M_{m_2}(E_2)),$$

so we need only consider the ring $M_{m_1}(E_1) \underset{E}{\circ} M_{m_2}(E_2)$ is just the $n \times n$-matrix ring over it.

We shall call the rank function on $M_{m_i}(E_i)$-modules ρ_i and the unique rank function on projective R-modules, where $R = M_{m_1}(E_1) \underset{E}{\sqcup} M_{m_2}(E_2)$ will be called ρ.

Let $m = \text{l.c.m.}\{m_1, m_2\}$; then ρ takes values in $\frac{1}{m}\mathbb{Z}$ and the universal two-sided localisation of R at ρ, R_ρ, was shown in theorem I.5.9 to be isomorphic to $M_m(F)$ for some skew field F. We shall use $\bar{\rho}$ for the rank function on $M_m(F)$-modules. Let C be the intersection of the centre of $M_m(F)$ with the amalgamated subfield E; then $k \subseteq C$. We determine the finite-dimensional division C-algebras in F, which allows us to determine the finite-dimensional k-algebras in F.

Theorem 2.2 We use the preceding names for our rings; further we assume that the coproduct $M_{m_1}(E_1)\underset{E}{\circ}M_{m_2}(E_2)$ satisfies conjecture 2.1. Let D be a finite-dimensional C-algebra. Then the rank of a simple $D^o\otimes_C M_m(F)$-module as $M_m(F)$-module is given by the formula h.c.f.$_{i,j}\{\rho_i(A_{ij})\}$ as A_{ij} ranges over finitely generated $M_{m_i}(E_i)\otimes_C D^o$-modules.

Pf We have $R = M_{m_1}(E_1)\underset{E}{\sqcup}M_{m_2}(E_2)$ and $R_\rho = M_{m_1}(E_1)\underset{E}{\circ}M_{m_2}(E_2) \cong M_m(F)$. Moreover, our conditions are such as to fulfil the conditions of theorem 1.3 (where we must take the present 'C' as the 'k' of that theorem).

Therefore, by theorem 1.3, the rank of a simple $M_m(F)\otimes_C D^o$-module is h.c.f.$_M\{\rho(M)\}$ as M ranges over finitely generated $R\otimes_C D^o$-submodules of free $R\otimes_C D^o$-modules.

We need to deal with a special case first.

Case 1 E has centre C.

In this case,
$$R\otimes_C D^o \cong (M_{m_1}(E_1)\underset{E}{\sqcup}M_{m_2}(E_2))\otimes_C D^o$$
$$\cong (M_{m_1}(E_1)\otimes_C D^o)\underset{E\otimes_C D^o}{\sqcup}(M_{m_2}(E_2)\otimes_C D^o).$$

Since E has centre C, $E\otimes_C D^o$ is simple artinian, which allows us to use Bergman's coproduct theorems.

By theorem I.2.2, any $R\otimes_C D^o$-submodule of a free module is of the form;

$$(M_1\otimes_{(M_{m_1}(E_1)\otimes_C D^o)}(R\otimes_C D^o)\oplus(M_2\otimes_{(M_{m_2}(E_2)\otimes_C D^o)}(R\otimes_C D^o),$$

where M_i is a submodule of a free $M_{m_i}(E_i)\otimes_C D^o$-module. We see that $\rho(M) = \rho_1(M_1) + \rho_2(M_2)$ and so, since the rank of a simple $M_m(F)\otimes_C D^o$-module as $M_m(F)$-module equals h.c.f.$_M\{\rho(M)\}$ as M runs over the finitely generated $R\otimes_C D^o$-submodules of free modules, it is equal to h.c.f.$_{i,j}\{\rho_i(B_{ij})\}$ as B_{ij} runs over finitely generated submodules of free $M_{m_i}(E_i)\otimes_C D^o$-modules.

Finally, given any finitely generated $M_{m_i}(E_i)\otimes_C D^o$-module, it has a presentation:

$$0 \longrightarrow B_{ij} \longrightarrow {}^{s}(M_{m_i}(E_i) \otimes_C D^o) \longrightarrow A_{ij} \longrightarrow 0$$

and so $\rho_i(A_{ij}) = s[D:C] - \rho_i(B_{ij})$, from which we deduce that h.c.f.$_{i,j}\{\rho_i(A_{ij})\}$ as A_{ij} runs through the finitely generated $M_{m_i}(E_i) \otimes_C D^o$-modules is equal to h.c.f.$_{i,j}\{\rho_i(B_{ij})\}$ as B_{ij} runs through the finitely generated $M_{m_i}(E_i) \otimes_C D^o$-submodules of free modules.

Before passing to the general case, we note the following corollary which we shall need for the more general case.

<u>Corollary 2.3</u> Let S be a simple artinian C-algebra and let D be a finite-dimensional division algebra over C; then the rank of a simple $(S \circ_C C(x)) \otimes_C D^o$-module as $S \circ_C C(x)$-module is equal to h.c.f.$\{\rho_S(A_j)\}$ where ρ_S is the rank function on S-modules and A_j runs over finitely generated $S \otimes_C D^o$-modules. In particular, if the centre of S is a regular extension of C, the rank of a simple $(S \circ_C C(x)) \otimes_C D^o$-module over $S \circ_C C(x)$ equals the rank of a simple $S \otimes_C D^o$-module as S-module.

<u>Pf</u> In section 3, we show that $S \circ_C C(x)$ satisfies conjecture 2.1, so we can apply case 1 of this theorem to prove our corollary. The last sentence follows because $S \otimes_C D^o$ is a simple artinian ring.

We are able to deal with the general case of our theorem using this corollary.

<u>Case 2</u> The general case.

We assumed that the coproduct $M_{m_1}(E_1) \circ_E M_{m_2}(E_2)$ satisfied conjecture 2.1, so that its centre is either C or else the function field of a curve of genus 0. In either case, the conditions for the last part of corollary 2.3 are satisfied and it follows that the rank of a simple $(M_{m_1}(E_1) \circ_E M_{m_2}(E_2)) \otimes_C D^o$-module as $M_{m_1}(E_1) \circ_E M_{m_2}(E_2)$-module is equal to the rank of a simple $(M_{m_1}(E_1) \circ_E M_m (E_2) \circ_C C(x)) \otimes_C D^o$-

module as $(M_{m_1}(E_1) \underset{E}{\circ} M_{m_2}(E_2) \underset{C}{\circ} C(x))$-module.

We shall show that $M_{m_1}(E_1) \underset{E}{\circ} M_{m}(E_2) \underset{C}{\circ} C(x)$ is isomorphic to $(M_{m_1}(E_1) \underset{C}{\circ} C(x))_{(E \underset{C}{\circ} C(x))}^{\circ}(M_{m_2}(E_2) \underset{C}{\circ} C(x))$.

We note that the two factors in this last coproduct and the amalgamated common skew subfield are all coproducts that satisfy conjecture 2.1, since their centres are C (see section 3). Moreover, since the centre of $M_{m_1}(E_1) \underset{E}{\circ} M_{m_2}(E_2) \underset{C}{\circ} C(x)$ is C, once we have shown the isomorphism above, we shall know that the last coproduct satisfies conjecture 2.1. To each factor, we can apply corollary 2.3 and to the coproduct as a whole we can apply case 1 of our theorem in order to find the rank of a simple $M_m(F) \underset{C}{\circ} C(x) \otimes_C D^o$-module as $M_m(F) \underset{C}{\circ} C(x)$-module, which equals the rank of a simple $M_m(F) \otimes_C D^o$-module as $M_m(F)$-module, the number we wish to find.

We begin by proving the stated isomorphism. $M_{m_1}(E_1) \underset{E}{\circ} M_{m_2}(E_2) \underset{C}{\circ} C(x)$ is the universal localisation of $T = M_{m_1}(E_1) \underset{E}{\sqcup} M_{m_2}(E_2) \underset{C}{\sqcup} C[x]$ at the unique rank function on this ring.

Consider the ring $M_{m_i}(E_i) \underset{C}{\sqcup} C[x]$. It is isomorphic to $M_{m_i}(E_i) \underset{E}{\sqcup} (E \underset{C}{\sqcup} C[x])$ which has the universal two-sided localisation $M_{m_i}(E_i) \underset{E}{\sqcup} (E \underset{C}{\circ} C(x))$. Since $M_{m_i}(E_i) \underset{C}{\sqcup} C[x]$ has a unique universal localisation that is simple artinian, we have shown that $M_{m_i}(E_i) \underset{C}{\circ} C(x) \cong M_{m_i}(E_i) \underset{E}{\circ} (E \underset{C}{\circ} C(x))$.

Let Σ_i be the collection of full maps between projective modules with respect to the unique rank function of $M_{m_i}(E_i) \underset{C}{\sqcup} C[x]$, and let Σ be the union of these maps induced up to T.

Consider the ring T_Σ and the ring T', where

$$T' = (M_{m_1}(E_1) \underset{C}{\circ} C(x)) \underset{E \underset{C}{\circ} C(x)}{\sqcup} (M_{m_2}(E_2) \underset{C}{\circ} C(x)).$$

We shall show that $T' \cong T_\Sigma$.

There is a homomorphism from T to T', sending $M_{m_i}(E_i)$ to $M_{m_i}(E_i)$ by the identity and x to x. Under this homomorphism all elements of

Σ are inverted, so this extends to a homomorphism from T_Σ to T' which must be surjective, since its image contains $M_{m_i}(E_i) \circ_C C(x)$, $i = 1, 2$.

Conversely, T_Σ is generated by a copy of $M_{m_1}(E_1) \circ_C C(x)$ and a copy of $M_{m_2}(E_2) \circ_C C(x)$, amalgamating the common skew subfield $E \circ_C C(x)$, therefore there is a homomorphism from T' to T_Σ which is clearly inverse to the homomorphism of the previous paragraph.

T' has the universal two-sided localisation

$$(M_{m_1}(E_1) \circ_C C(x)) \underset{(E \circ_C C(x))}{\circ} (M_{m_2}(E_2) \circ_C C(x))$$

which must be the universal two-sided localisation of T that is simple artinian. We know this to be $M_{m_1}(E_1) \circ_E M_{m_2}(E_2) \circ_C C(x)$ and so our isomorphism is proven.

Since $E \circ_C C(x)$ has centre C, we may apply case 1 to conclude that a simple $((M_{m_1}(E_1) \circ_C C(x)) \underset{(E \circ_C C(x))}{\circ} (M_{m}(E_2) \circ_C C(x)) \otimes_C D^o$-module has rank as $(M_{m_1}(E_1) \circ_C C(x)) \underset{(E \circ_C C(x))}{\circ} (M_{m_2}(E_2) \circ_C C(x))$-module equal to h.c.f.$\{\bar{\rho}_1(A_1), \bar{\rho}_2(A_2)\}$ where A_i is the unique simple $(M_{m_i}(E_i) \circ_C C(x)) \otimes_C D^o$-module and $\bar{\rho}_i$ is the rank function on $M_{m_i}(E_i) \circ_C C(x)$-modules. We know that there is a unique simple $(M_{m_i}(E_i) \circ_C C(x)) \otimes_C D^o$-module, since $M_{m_i}(E_i) \circ_C C(x)$ has centre C.

In turn, by corollary 2.3, $\bar{\rho}_i(A_i)$ is equal to h.c.f.$_j\{\rho_i(A_{ij})\}$ as A_{ij} runs over finitely generated $M_{m_i}(E_i) \otimes_C D^o$-modules.

Putting all this together, we find that the rank of a simple $(M_m(F) \circ_C C(x)) \otimes_C D^o$-module as $M_m(F) \circ_C C(x)$-module is equal to h.c.f.$_{i,j}\{\rho_i(A_{ij})\}$ as A_{ij} runs over finitely generated $M_{m_i}(E_i) \otimes_C D^o$-modules. We have already remarked that this number is equal to the rank of a simple $M_m(F) \otimes_C D^o$-module when conjecture 2.1 holds. This completes the proof of the theorem.

As we saw in lemma 1.1, this theorem allows us to decide whether or not a particular finite-dimensional division C-algebra embeds in F, where $M_m(F) = M_{m_1}(E_1) \underset{E}{\circ} M_{m_2}(E_2)$. However, our original question was whether a particular finite dimensional k-algebra embeds in F, where E and $M_{m_i}(E_i)$ are k-algebras and the embedding of E in $M_{m_i}(E_i)$ respects this k-algebra structure. The next lemma settles our problem.

<u>Lemma</u> <u>2.4</u> Let F be a skew field with central subfield C containing k. Let D be a finite-dimensional division k-algebra and let $\{D_i : 1 = 1 \text{ to } n\}$ be the (possibly empty) set of division C-algebras that are images of $D \otimes_k C$. Then D embeds in F if and only if some D_i embeds in F.

<u>Pf</u> Obvious.

Section 3

The purpose of this section is to verify conjecture 2.1 in a number of interesting cases. The theorem that we shall need to do this is the following due to Cohn (19)

Theorem 3.1 Let T be a non-Ore fir with universal skew field of fractions U. Then the centre of U is the centre of T.

We refer the reader to Cohn (19) for a proof of this result. Our first case is a fairly direct application of this theorem.

Theorem 3.2 The centre of $E_1 \underset{E}{\circ} M_m(E_2)$ lies in E, provided that $[E_1:E]$ or $[M_m(E_2):E]$ is greater than 2.

Pf $E_1 \underset{E}{\circ} M_m(E_2)$ is the universal simple artinian ring of fractions of $E_1 \underset{E}{\sqcup} M_m(E_2) \cong M_m(R)$, where R is a fir, since, by Bergman's coproduct theorems ((4) and see section I.2) and the Morita equivalence of $M_m(R)$ and R, every submodule of a free R-module is free. Further, the assumptions on dimensions imply that R is not an Ore domain and so, by theorem 3.1, the centre of its skew field of fractions is its own centre; in particular, it consists of units in R.

R^o is the ring of endomorphisms of the module $S \otimes_{M_m(E_2)} M_m(R)$, where S is the unique simple $M_m(E_2)$-module, and the units of R^o induce the automorphisms of $S \otimes_{M_m(E_2)} M_m(R)$. However, by Bergman's coproduct theorem (theorem I.2.18), the group of automorphisms of this module are induced by the automorphisms of S as $M_m(E_2)$-module; so the units of R^o lie in the copy of $E_2{}^o$ inside it. Therefore the centre of R is contained in the copy of E_2 inside it, and the centre of $M_m(R) \cong E_1 \underset{E}{\sqcup} M_m(E_2)$ must also lie in E_2. However, if e is an element of $M_m(E_2) - E$ and $f \in E_1 - E$, $ef \neq fe$, so cannot be central. Therefore, the centre of $M_m(R)$, which must lie in E_2, must lie in E.

The centre of U, the universal skew field of fractions of R, is equal to the centre of R; so the centre of $M_m(U) = E_1 \underset{E}{\circ} M_m(E_2)$ is equal to the centre of $M_m(R)$ and so lies in E, as we wished to show.

Since this is all that will be used in the sections that follow immediately, the reader would be well-advised to pass the remainder of this section, except for theorems 3.5 and 3.6, on the first reading. These two theorems present examples to show that all the possibilities envisaged in conjecture 2.1 do occur.

We wish to find the centre of the simple artinian coproduct of two simple artinian ring over a common central subfield. The next two theorems deal first with the non-Noetherian, and then the Noetherian, case. The proof of the next theorem is dependent on section 13, the results of which that we use do not depend on the remainder of this section.

Theorem 3.3 The centre of $M_{m_1}(E_1) \underset{k}{\circ} M_{m_2}(E_2)$ is equal to k, provided that one of $[M_{m_1}(E_1):k]$ or $[M_{m_2}(E_2):k]$ is greater than 2.

Pf The idea of the proof is to reduct this theorem to the previous one, by replacing $M_{m_1}(E_1)$ by a skew field closely related to it; the skew field will be a twisted form (see Waterhouse (46)) of it.

We begin with an important special case.

Special case There exists a central division k-algebra D such that $[D:k] = m_1^2$ and $D\otimes_k E_1$ is a skew field.

In this case, our idea is to show that

$$((D\otimes_k E_1) \underset{k}{\circ} M_{m_2}(E_2))\otimes_k L \cong (M_{m_1}(E_1) \underset{k}{\circ} M_{m_2}(E_2))\otimes_k L,$$

where L is a Galois splitting field of D. From theorem 3.2 we know that $(D\otimes_k E_1) \underset{k}{\circ} M_{m_2}(E_2)$ has centre k. If C is the centre of $M_{m_1}(E_1) \underset{k}{\circ} M_{m_2}(E_2)$, $C\otimes_k L \cong L$, so C must be k.

Let G be the Galois group of L over k; the isomorphism

$D\otimes_k L \cong M_{m_1}(L)$ gives two different actions of G on $M_{m_1}(L)$ extending the action of G on L, one of which has fixed ring E, the other fixed ring $M_{m_1}(k)$. This determines an element α of $H^1(G,PGL_m(L))$, since $PGL_{m_1}(L)$ is the automorphism group of $M_{m_1}(L)$ over L.

Since $(D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2)$ has centre k, $L\otimes_k((D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2))$ is simple artinian: it is also a universal two-sided localisation of

$$L\otimes_k((D\otimes_k E_1)\underset{k}{\sqcup} M_{m_2}(E_2)) \cong (L\otimes_k D\otimes_k E_1)\underset{L}{\sqcup} M_{m_2}(E_2\otimes_k L)$$
$$= M_{m_1}(E_1\otimes_k L)\underset{L}{\sqcup} M_{m_2}(E_2\otimes_k L).$$

Since L is a Galois extension of k, $M_{m_i}(E_i\otimes_k L)$ is a semisimple artinian ring for i = 1 and 2, and so $M_{m_1}(E_1\otimes_k L)\underset{L}{\sqcup} M_{m_2}(E_2\otimes_k L)$ is an hereditary ring. Therefore, by theorem I.5.8, the simple artinian universal two-sided localisation $L\otimes_k((D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2))$ is a universal two-sided localisation at a rank function. The action of $PGL_{m_1}(L)$ on $M_{m_1}(E_1\otimes_k L)\underset{L}{\sqcup} M_{m_2}(E_2\otimes_k L)$ defined by making it fix $M_{m_2}(E_2\otimes_k L)$ and act as the group of inner automorphisms of $M_{m_1}(L)$ on $M_{m_1}(E_1\otimes_k L)$ sends any projective module of $M_{m_1}(E_1\otimes_k L)\underset{L}{\sqcup} M_{m_2}(E_2\otimes_k L)$ to an isomorphic projective module, and so must preserve the rank function. Therefore this action extends to the universal two-sided localisation at this rank function, which is $L\otimes_k((D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2))$.

Therefore, $\alpha\in H^1(G,PGL_m(L))$ defines an element α' of $H^1(G,\mathrm{Aut}_L(L\otimes_k((D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2)))$. We have an action of G on $L\otimes_k((D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2))$ with fixed ring $(D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2)$. We use the derivation α' to define a different action of G on this ring with fixed ring S such that $S\otimes_k L \cong ((D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2))\otimes_k L$. We wish to show that S is $M_{m_1}(E_1)\underset{k}{\circ} M_{m_2}(E_2)$.

Certainly, it contains a copy of $M_{m_2}(E_2)$ and $M_{m_1}(E_1)$ for the action of $PGL_{m_1}(L)$ was extended so as to fix $M_{m_2}(E_2)$ and therefore G fixes $M_{m_2}(E_2)$, whilst the action of $PGL_{m_1}(L)$ restricts to the inner automorphisms of $M_{m_1}(L)$ on $M_{m_1}(E_1\otimes_k L)$, and so the twisted action of G is our original twisted action giving as fixed ring $M_{m_1}(E_1)$.

As often happens, in order to show that $M_{m_1}(E_1) \sqcup_k M_{m_2}(E_2) \xrightarrow{\phi} S$ induces an isomorphism of S and $M_{m_1}(E_1) \circ_k M_{m_2}(E_2)$, we do well to linearise; that is, we look at the induced map

$$d\phi : {}^{\otimes}\Omega_k(M_{m_1}(E_1) \sqcup_k M_{m_2}(E_2))^{\otimes} \longrightarrow \Omega_k(S)$$

where $\Omega_k(A)$ is the universal bimodule of derivations of a k-algebra A over k (see section 12).

Since $(D\otimes_k E_1) \circ_k M_{m_2}(E_2)$ is a universal two-sided localisation of $(D\otimes_k E_1) \sqcup_k M_{m_2}(E_2)$, the map

$$D\otimes_k E_1 \sqcup_k M_{m_2}(E_2) \longrightarrow (D\otimes_k E_1) \circ_k M_{m_2}(E_2)$$

induces an isomorphism

$$({}^{\otimes}\Omega_k(D\otimes_k E_1)^{\otimes}) \oplus ({}^{\otimes}\Omega_k(M_{m_2}(E_2)^{\otimes}) \cong \Omega_k((D\otimes_k E_1) \circ_k M_{m_2}(E_2))$$

by theorem 12.6

Tensoring with L, we obtain an isomorphism

$$({}^{\otimes}\Omega_L(D\otimes_k E_1 \otimes_k L)^{\otimes}) \oplus ({}^{\otimes}\Omega_L(M_{m_2}(E_2\otimes_k L)^{\otimes}) \cong \Omega_L(L\otimes_k((D\otimes_k E_1) \circ_k M_{m_2}(E_2))).$$

Using the twisted action of G on the rings $M_{m_1}(E_1 \otimes_k L)$ and $L\otimes_k((D\otimes_k E_1) \circ_k M_{m_2}(E_2))$ (having fixed points $M_{m_1}(E_1)$ and S respectively) and on this last isomorphism, we obtain the isomorphism of fixed points

$$({}^{\otimes}\Omega_k(M_m(E_1))^{\otimes}) \oplus ({}^{\otimes}\Omega_k(M_{m_2}(E_2))^{\otimes}) \cong \Omega_k(S),$$

where the isomorphism is induced by the ring homomorphism

$$\phi : M_{m_1}(E_1) \sqcup_k M_{m_2}(E_2) \longrightarrow S.$$

First of all, we deduce that ϕ must be an epimorphism since the induced map $d\phi$ is surjective. Next, we see that

$$\Omega_k(S) \cong {}^{\otimes}(\Omega_k(M_{m_1}(E_1) \sqcup_k M_{m_2}(E_2)))^{\otimes}$$

so by theorem 13.4, $S \cong M_{m_1}(E_1) \circ_k M_{m_2}(E_2)$.

We have shown the isomorphism:

$$L\otimes_k((D\otimes_k E_1)\underset{k}{\circ} M_{m_2}(E_2)) \cong L\otimes_k(M_{m_1}(E_1)\underset{k}{\circ} M_{m_2}(E_2))$$

Therefore the centre of $M_{m_1}(E_1)\underset{k}{\circ} M_{m_2}(E_2)$ can only be k by our earlier reasoning.

General case We use this special case to solve the general problem. Our method is to extend k in order to reduce to the previous case.

Let K be the commutative field $k(x_1,\dots,x_{m_1})$, where $x_1,\dots,x_{m_1}$ is a set of commuting indeterminates. There is an automorphism σ of this field of order m_1, given by the cyclic permutation of the indices. We form the skew field $D = K(y:\sigma)$; let C be the centre of D.

It is easy to see that $D\otimes_k E$ is a Noetherian domain for any skew field E and so $D\otimes_k E$ has a skew field of fractions.

Let E_i' be the skew field of fractions of $E_1\otimes_k C$ for $i = 1,2$ and let F' be the skew field of fractions of $F\otimes_k C$, where $M_m(F) \cong M_{m_1}(E_1)\underset{k}{\circ} M_{m_2}(E_2)$.

Then $M_m(F') = M_{m_1}(E_1')\underset{C}{\circ} M_{m_2}(E_2')$. If L is the centre of $M_{m_1}(E_1)\underset{k}{\circ} M_{m_2}(E_2)$, then $L\otimes_k C$ is central in $M_{m_1}(E_1')\underset{C}{\circ} M_{m_2}(E_2')$. So once we know that the centre of $M_{m_1}(E_1')\underset{C}{\circ} M_{m_2}(E_2')$ is just C, we shall know that $M_{m_1}(E_1)\underset{k}{\circ} M_{m_2}(E_2)$ has centre k.

However, $[D:C] = m_1^2$ and $D\otimes_C E_1'$ is the skew field of fractions of $D\otimes_C(C\otimes_k E_1) \cong D\otimes_k E_1$, a noetherian domain. So, by case 1, the centre of $M_{m_1}(E_1')\underset{C}{\circ} M_{m_2}(E_2')$ is C and by the previous paragraph, the centre of $M_{m_1}(E_1)\underset{k}{\circ} M_{m_2}(E_2)$ is k which completes the proof of our theorem.

We wish to deal with the remaining case of the coproduct of two simple artinian rings S_1 and S_2 such that $[S_i:k] = 2$. Clearly, S_1 and S_2 are commutative fields.

Theorem 3.4 Let E_1, E_2 be a pair of quadratic extensions of k. Then the centre of $E_1 \underset{k}{\circ} E_2$ is purely transcendental of transcendence degree 1 over k.

Pf This was first shown by Cohn (unpublished) by a different method.

By Bergman (4), the centre of $E_1 \underset{k}{\sqcup} E_2$ is of the form $k[t]$ for some transcendental t in $E_1 \underset{k}{\sqcup} E_2$ and $E_1 \underset{k}{\sqcup} E_2$ is a free module of rank 4 over $k[t]$. $E_1 \underset{k}{\circ} E_2$ must be the central localisation of $E_1 \underset{k}{\sqcup} E_2$ and must be four-dimensional over the central subfield $k(t)$. Since it is non-commutative, this must be the whole centre.

Now that we have these results, we are able to apply the results of the previous sections and the following sections are devoted to such applications. We complete this section by showing that the function field of a curve of genus 0 over k occurs as the centre of a simple artinian coproduct with amalgamation of suitable division k-algebras and simple artinian k-algebras. We shall need the next theorem in the course of this demonstration.

Theorem 3.5 $M_2(L) \underset{L\oplus L}{\sqcup} M_2(L) \cong M_2(L[t,t^{-1}])$ for a commuting indeterminate t.

Pf We note that all embeddings of $L \oplus L$ in $M_2(L)$ are conjugate to the embedding along the diagonal. So we may assume that the first factor has matrix units $e_{11}, e_{12}, e_{21}, e_{22}$ and the second factor has matrix units $e_{11}, f_{12}, f_{21}, e_{22}$.

Since f_{12} lies in $e_{11}(M_2(L) \underset{L\oplus L}{\sqcup} M_2(L))$, it takes the form $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$ when written as a matrix over the centraliser of the elements $\{e_{ij} : i,j = 1,2\}$, and similarly f_{21} must take the form $\begin{pmatrix} 0 & 0 \\ t^{-1} & 0 \end{pmatrix}$. Since the centraliser of the first factor is generated by the entries of these two matrices, the centraliser takes the form $L[t,t^{-1}]$ and as we wished to show, $M_2(L) \underset{L\oplus L}{\sqcup} M_2(L) \cong M_2(L[t,t^{-1}])$ where t is an indeterminate, since $[M_2(L) \underset{L\oplus L}{\sqcup} M_2(L) : L]$ is clearly infinite.

Suppose that F is the function field of a curve of genus 0 over k. That is, $F \otimes_k F \cong K(x)$ for some finite Galois extension K of k with Galois group G.

Then F corresponds to a suitable element α_F of $H^1(G, PGL_2(K))$ since $PGL_2(K)$ is the automorphism group of $K(x)$ over K (see Waterhouse(46)). Since $PGL_2(K)$ is also the automorphism group of $M_2(K)$ over K, this element α_F of $H^1(G, PGL_2(K))$ corresponds to some central quarternion algebra D_F over k; moreover, Roquette (43) shows that F is the universal splitting field of D_F. Let L be a maximal separable commutative subfield of D_F; $[L:k] = 2$.

Theorem 3.6 Name everything as in the last paragraph; then $M_2(k) \circ_L D_F \cong M_2(F)$. In particular, F arises as the centre of $M_2(k) \circ_L D_F$.

Pf $(M_2(k) \sqcup_L D_F) \otimes_k L \cong M_2(L) \sqcup_{L \oplus L} M_2(L) \cong M_2(L[t,t^{-1}])$ by the last theorem.

So $(M_2(k) \circ_L D_F) \otimes_k L \cong M_2(L(t))$, since both sides are obtained by central localisation of the prime PI-rings in the previous isomorphism.

Let F' be the centre of $M_2(k) \circ_L D_F$. Then $F' \otimes_k L$ is isomorphic to $L(t)$ so F' is either purely transcendental or, by the same argument that we used for F before the statement of the theorem, F' is the universal splitting field of some central quarternion algebra $D_{F'}$.

However, $M_2(k) \circ_L D_F \cong F' \otimes_k D_F$; so F' cannot be purely transcendental since it splits D_F. Roquette (43) shows that the universal splitting field of a central quarternion algebra can split no other central simple algebra, so F' must be isomorphic to F. That is, $M_2(k) \circ_L D_F \cong M_2(F)$, as we wished to show.

Section 4

Our main goal in this section is to demonstrate some of the consequences of theorem 2.2. As well as proving a number of general results, we shall also try to find conditions under which a finite-dimensional division subalgebra of a skew field coproduct must be conjugate to a division subalgebra of one of the factors.

We begin with a few general results:

Theorem 4.1 Let E be a skew field and k-algebra and let $\{E_i : i \in I\}$ be a collection skew fields and k-algebras having the k-algebra E as a skew subfield. Assume that the centre of $F = \circ_E E_i$ is a regular extension of k; then if D is a finite-dimensional division k-algebra, the dimension of a simple $D^o \otimes_k F$-module over F is $\text{h.c.f.}_{i,j}\{[A_{ij} : E_i]\}$ where A_{ij} runs over the finitely generated $D^o \otimes_k E_i$-modules.

Pf The techniques of sections 1 and 2 apply in entirely the same way for an arbitrary indexing set rather than just a two-element indexing set, and this result is the consequence.

Corollary 4.2 With assumptions as in theorem 4.1, D embeds in F if and only if it embeds in $\circ_k E_i$.

Pf The dimension of a simple $D^o \otimes_k F$-module is given by the same number as the dimension of a simple $D^o \otimes_k (\circ_k E_i)$-module over $\circ_k E_i$. So D embeds in F if and only if D embeds in $\circ_k E_i$.

A consequence of this corollary is that we need only consider coproducts of the form $\circ_k E_i$. In this section, we shall in general limit ourselves further to the case where each E_i is finite-dimensional over k or else a rational function field in one variable over k.

From section 3, we recall that the centre of $\circ_k E_i$ is k except when i = 1,2 and $[E_i : k]$ is also 2, in which case the centre is purely transcendental in one variable over k. So theorem 4.1 applies without need to state the hypothesis.

Theorem 4.3 Let $\{E_i : i = 1 \text{ to } n\}$ be finite-dimensional division algebras and let $t = \text{l.c.m.}\{[E_i : k]\}$. If D embeds in $\underset{k}{\circ} E_i$ and $[D:k] < \infty$, $[D:k]$ divides t.

Pf $[D:k]$ divides $[A_{ij}:k]$ as A_{ij} runs through $D^o \otimes_k E_i$-modules. $[A_{ij}:k] = [A_{ij}:E_i][E_i:k]$, which divides $[A_{ij}:E_i]\,t$. If D embeds in $\underset{k}{\circ} E_i$, $\text{h.c.f.}\{[A_{ij}:E_i]\} = 1$ and so $[D:k]$ divides t.

The next result has slightly more general hypotheses, but clearly applies to the cases we are considering at the moment.

Theorem 4.4 Let $\{E_i : i = 1 \text{ to } n\}$ be skew fields satisfying a polynomial identity and let D be a finite-dimensional division k-algebra lying in $\underset{k}{\circ} E_i$; then the polynomial identity degree of D divides $\text{l.c.m.}\{\text{PI degree } E_i\}$.

Pf Let $p = \text{l.c.m.}\{\text{PI degree } E_i\}$.

Let A_{ij} be a $D^o \otimes_k E_i$-module and let $m_{ij} = [A_{ij}:E_i]$.

Then the centraliser of the action of E_i on A_{ij} is $M_{m_{ij}}(E_i{}^o)$ and D^o embeds in it, so by Bergman, Small(4), PI degree D = PI degree D^o divides PI degree $(M_{m_{ij}}(E_i{}^o))$ which is $m_{ij}(\text{PI degree } E_i)$.

Since $\text{h.c.f.}\{m_{ij}\} = 1$ and $\text{l.c.m.}\{\text{PI degree } E_i\} = p$, we find that PI degree D divides p, as we wished to show.

Our next theorem is more in the nature of an example to show that the bounds in theorems 4.3 and 4.4 are best possible; also it shows that the hope that finite-dimensional division subalgabras of $\underset{k}{\circ} E_i$ are conjugate to a division subalgebra of one of the factors is ill-founded.

Theorem 4.5 Let D_1 and D_2 be finite-dimensional division algebras over k such that $[D_1:k]$ is coprime to $[D_2:k]$; then $D_1 \otimes_k D_2$ embeds in $D_1 \underset{k}{\circ} D_2$.

Pf We have a simple $(D_1 \otimes_k D_2)^{\circ} \otimes_k D_1$-module of dimension $[D_2:k]$ over D_1 and a simple $(D_1 \otimes_k D_2)^{\circ} \otimes_k D_2$-module of dimension theorem 4.1, we have a simple $(D_1 \otimes_k D_2)^{\circ} \otimes_k (D_1 \underset{k}{\circ} D_2)$-module of dimension 1 over $D_1 \underset{k}{\circ} D_2$; so $D_1 \otimes_k D_2$ embeds in $D_1 \underset{k}{\circ} D_2$.

We wish to investigate the special cases where division subalgebras of a coproduct are conjugate to a division subalgebra of one of the factors. The next theorem is one of a number of such theorems.

Theorem 4.6 Let E be a finite-dimensional (possibly non-commutative) Galois extension of k. Then any finite-dimensional division subalgebra of $E \underset{k}{\circ} k(x)$, for x transcendental over k, is conjugate to a division subalgebra of E.

Pf Let C be the centre of E; then C is a Galois extension of k with Galois group G where G is the group of automorphisms of E over k.

$D^{\circ} \otimes_k C \cong \bigoplus_{i=1}^{n} S_i$, where $\{S_i : i = 1 \text{ to } n\}$ are simple algebras over C and the Galois group of C over k acts transitively on this set.

$D^{\circ} \otimes_k E \cong (D^{\circ} \otimes_k C) \otimes_C E \cong (\bigoplus_{i=1}^{n} S_i) \otimes_C E \cong \bigoplus_{i=1}^{n} (S_i \otimes_C E)$, and the outer automorphisms of E again, by their action on C permute the simple algebras $S_i \otimes_C E$ transitively.

Therefore, $[A_i:E] = [A_j:E]$ where A_k is the simple $S_k \otimes_C E$-module. So $\text{h.c.f.}_i \{[A_i:E]\} = [A_1:E]$ which implies that D embeds in $E \underset{k}{\circ} k(x)$ if and only if it embeds in E.

Since the Noether-Skolem theorem states that all possible embeddings of a finite-dimensional division k-algebra in a central simple artinian k-algebra are conjugate, and since $E \underset{k}{\circ} k(x)$ has centre k, any embedding of D in $E \underset{k}{\circ} k(x)$ is conjugate to its embedding in E.

We shall restrict our attention for the time being to the coproduct of commutative finite-dimensional extensions of k and of rational function fields in one variable over k. We note that theorem

4.4 tells us that any finite-dimensional subfield of such a coproduct is commutative. In this case, we can extend theorem 4.6 a little.

Theorem 4.7 Let E be a normal extension of k; then any finite extension of k lying in $E \circ_k k(x)$ is conjugate to a subfield of E.
Pf Let C be a commutative extension of k lying in $E \circ_k k(x)$ such that $[C:k] < \infty$.

Then h.c.f. $\{[A_i:E]\} = 1$ as A_i runs over field joins of C and E inside the algebraic closure of k. But since E is a normal extension of k, there is a unique such field join up to k-isomorphism and so $[A_i:E] = [A_j:E]$. Therefore h.c.f. $\{[A_i:E]\} = 1$ implies that C embeds in E. The conjugacy follows from the Noether-Skolem theorem as before.

We can put these results together to obtain a useful restriction on the finite-dimensional extensions of k lying in the coproduct of finite-dimensional commutative extensions.

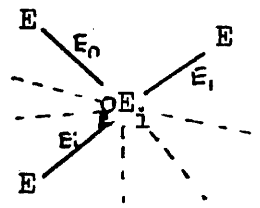
Theorem 4.8 Let $\{E_i : i = 1 \text{ to } n\}$ be a finite collection of finite-dimensional commutative extensions of k and let D be a finite-dimensional division subalgebra of $\circ_k E_i$. Then D is commutative, its dimension over k divides l.c.m. $\{[E_i:k]\}$ and it lies in the normal closure E of a join of the fields E_i in the algebraic closure of k.
Pf The first two conditions have already been shown in theorems 4.3 and 4.4, which leaves us with the third restriction. We shall show that $\circ_k E_i$ embeds in $E \circ_k k(x)$ and then apply theorem 4.7.

Sublemma 4.9 $\circ_k E_i$ embeds in $E \circ_k k(x)$.
Pf $E \circ_k k(x)$ is the universal skew field of fractions of the fir $E \sqcup_k k[x,x^{-1}]$ which is isomorphic to the skew polynomial ring $R[x,x^{-1};\sigma]$ where R is the fir $\sqcup^k_{j\in\mathbb{Z}} E$, the coproduct over k of copies of E indexed by $\mathbb{Z}$, and σ is the shift automorphism. σ extends to a

shift automorphism on $\underset{j\in\mathbb{Z}}{\overset{}{\underset{k}{\circ}}} E$, the skew field of fractions of $\underset{j\in\mathbb{Z}}{\underset{k}{\sqcup}} E$, and so the universal skew field of fractions of $E \underset{k}{\sqcup} k[x,x^{-1}]$, $E \underset{k}{\circ} k(x)$, which is isomorphic to $(\underset{j\in\mathbb{Z}}{\underset{k}{\circ}} E)(x,x^{-1};\sigma)$.

So $\underset{j\in\mathbb{Z}}{\underset{k}{\circ}} E$ embeds in $E \underset{k}{\circ} k(x)$.

The fir $\underset{k}{\sqcup} E_i$ embeds in $\underset{k}{\sqcup} E$, where the second coproduct is on the same indexing set as the first. The universal two-sided localisation of $\underset{k}{\sqcup} E$ at the full matrices over the subring $\underset{k}{\sqcup} E_i$ is the tree product of rings shown below:



and this is a fir, since it is an iterated coproduct of firs amalgamating skew subfields. So it has a universal skew field of fractions which must be $\underset{k}{\circ} E$. So $\underset{k}{\circ} E_i$ embeds in $\underset{k}{\circ} E$, which embeds in $\underset{j\in\mathbb{Z}}{\underset{k}{\circ}} E$, which lies in $E \underset{k}{\circ} k(x)$.

We return to the proof of theorem 4.8. We have shown that $\underset{k}{\circ} E_i$ can be embedded in $E \underset{k}{\circ} k(x)$ where E is the normal closure of a join of the fields E_i. So any subfield of $\underset{k}{\circ} E_i$ lies in $E \underset{k}{\circ} k(x)$, and so, by theorem 4.7, is isomorphic to a subfield of E.

We present an example to show that these conditions are not sufficient.

<u>Example</u> Consider $Q(\sqrt{2}) \underset{Q}{\circ} Q(\sqrt{3})$; the normal closure of $Q(\sqrt{2},\sqrt{3})$ is $Q(\sqrt{2},\sqrt{3})$ and contains a square root of 6. But

$$Q(\sqrt{6}) \otimes_Q Q(\sqrt{2}) \cong Q(\sqrt{6}) \otimes_Q Q(\sqrt{3}) \cong Q(\sqrt{2},\sqrt{3})$$

which is a field, and so $Q(\sqrt{6})$ cannot lie in $Q(\sqrt{2}) \underset{Q}{\circ} Q(\sqrt{3})$ by theorem 4.1.

As has become clear, the coproduct of separable extensions does not behave particularly well and so it comes as a pleasant surprise that purely inseparable extensions behave extremely well in our present study. We can work in full generality for our next theorem.

<u>Theorem</u> <u>4.10</u> Let $\underset{k}{\circ} E_i$ be the skew field coproduct of the skew fields $\{E_i : i = 1 \text{ to } n\}$. If C is a finite purely inseparable extension of k lying in $\underset{k}{\circ} E_i$, it is conjugate to a subfield of one of the factors.

<u>Pf</u> Let p be the characteristic of k; we shall show that the dimension over E_i of a simple $C \otimes_k E_i$-module is a power of p and so the highest common factor of these dimensions is 1 if and only if C embeds in E_i for some i.

Let C_i be the centre of E_i; then the centre of $C \otimes_k E_i$ is $C \otimes_k C_i$, which is a local artinian ring with residue class ring C_i', the unique field join of C and C_i. C is a purely inseparable extension of C_i, so $[C:C_i]$ is a power of p.

$C_i' \otimes_C E_i$ is the simple artinian image of $C \otimes_k E_i$ modulo its radical and so the dimension of a simple $C \otimes_k E_i$-module divides $[C_i' : C_i]$ and must be a power of p. We conclude the proof as previously indicated, followed by the Noether-Skolem theorem.

We can put this together with earlier results in order to prove:

<u>Theorem</u> <u>4.11</u> Let $\underset{k}{\circ} E_i$ be a coproduct of finitely many purely inseparable extensions of k. Then any finite-dimensional subfield of $\underset{k}{\circ} E_i$ is conjugate to a subfield of one of the factors.

<u>Pf</u> By theorem 4.1, any subfield must lie in the normal closure of a field join of the fields E_i. But the fields E_i have a unique purely inseparable field join which is therefore normal. Hence any such subfield of $\underset{k}{\circ} E_i$ is purely inseparable and, by theorem 4.10, is conjugate to a subfield of one of the factors.

There is a further situation where one could hope that a finite-dimensional subfield of a coproduct of commutative fields had to embed in one of the factors. This is when each factor is an extension of k of dimension p, a prime, over k. This turns out to be false in general; however, it is true for many primes and the exact condition is of some interest.

The following theorem depends on the classification of finite simple groups.

<u>Theorem 4.12</u> Let p be a prime not equal to 11 or $(q^t-1)/(q-1)$ where q is a prime power. Then if $\{E_i : i = 1 \text{ to } n\}$ are finitely many extensions of k of dimension p over k, any finite-dimensional subfield of $\underset{k}{\circ} E_i$ is isomorphic to one of the factors. For the proscribed primes, there are counterexamples.

<u>Pf</u> Suppose that E embeds in $\underset{k}{\circ} E_i$, $[E:k] < \infty$ and that $E \not\cong E_i$ for any i. By theorem 4.8, $[E:k] = p$, and E cannot be purely inseparable by theorem 4.10. So it is a separable extension of k.

Since E embeds in $\underset{k}{\circ} E_i$, there is some i for which $E \otimes_k E_i \cong \oplus F_{ij}$ $(n \geq 2)$. Since $E \not\cong E_i$, $[F_{ij}:E_i] \neq 1$ for any j.

Because $\sum [F_{ij}:E_i] = p$, which is prime, $\text{h.c.f.}_j\{[F_{ij}:E_i]\} = 1$ and so E lies in $E_i \underset{k}{\circ} E_i$ and must lie in the normal closure of E_i over k by theorem 4.8. Therefore E_i is separable and its normal closure N is actually a Galois extension with Galois group G. Since G acts faithfully on the roots of an irreducible polynomial of an element of E_i over k, p divides $|G|$ but p^2 does not. Since $[E_i:k] = [E:k] = p$ and $E \subseteq N$, we see that the subgroups H' and H that fix E_i and E respectively are inconjugate p-complements in G (since $E \not\cong E_i$). Therefore, by Hall's theorem, G must be insoluble.

The actions of G on the right cosets of H and on the right cosets of H' define non-isomorphic faithful transitive actions of G on a p-element set. By Burnside(10), both actions are doubly transitive. Cameron(13), shows that this can happen only when $p = 11$ or is of the form $(q^t-1)/(q-1)$ for a prime power q. For $p = 11$, there are two distinct actions of $PSL(2,11)$ on 11 points. For $p = (q^t-1)/(q-1)$, the actions are given by the action of $PGL(q,t)$ on the points, and dually on the hyperplanes of projective space over the finite field $GF(q)$.

We have reached a contradiction if $p = 11$ or $(q^t-1)/(q-1)$, so we need to examine what happens in this case.

Given a group G having two such faithful actions on a p-element set, we have two inconjugate p-complements in G.

If $N \supset k$ is a Galois extension of k with Galois group G, the fixed fields E and E' of the inconjugate p-complements are non-isomorphic extensions of k, both of dimension p, and E must embed in $E' \circ_k E'$, because $E \otimes_k E'$ is not a field (p^2 does not divide $|G|$).

Our first example occurs when $p = 7 = (2^3-1)/(2-1)$.

Cameron's work(13) uses the classification of finite simple groups; hence the remark before the statement of the theorem.

Section 5

One question that arises naturally is whether the coproduct of skew fields having no elements algebraic over k also has no elements algebraic over k. We shall present a counter-example to this in the course of this section, but we begin with a positive result.

We say that the skew field E is transcendental over a central subfield k if it contains no elements algebraic over k.

Theorem 5.1 Let C be a commutative subfield of $\circ_k E_i$ such that $[C:k] = p^n$ for some prime p. Then one of the skew fields E_i is not transcendental over k.

Pf We lose nothing by assuming that C is a primitive extension of k, $C = k(\alpha)$.

Let L_i be the centre of E_i. If $C \otimes_k L_i$ is not a field, the irreducible polynomial of α over k splits over L_i, and the coefficients of the factors in L_i generated an algebraic extension of k in L_i.

If $C \otimes_k L_i$ is a field for each L_i, then

$$C \otimes_k E_i \cong (C \otimes_k L_i) \otimes_{L_i} E_i,$$

which is simple artinian, and so the dimension of a simple $C \otimes_k E_i$-module divides $[C:k] = p^n$, and must itself be a power of p. Since the highest common factor of these dimensions is 1 when C embeds in $\circ_k E_i$, there is an i for which $C \otimes_k E_i$ has a simple module of dimension 1 over E_i, which implies that C embeds in E_i.

So in both cases, one of the skew fields E_i cannot be transcendental over k.

Of course, we cannot conclude that C must embed in one of the E_i even when $[C:k]$ is a prime, as we saw at the end of the last section.

We can turn the statement of this last theorem round:

Corollary 5.2 An algebraic extension of k lying in the skew field coproduct of transcendental skew fields over k has dimension over k divisible by at least two primes.

We shall find an example of such a subfield of dimension 6 over k.

Example 5.3 First of all, it is possible to find an extension of fields $E \supset k$, $[E:k] = 6$, having no intermediate fields. This occurs, for example, when the Galois group of the Galois closure of E over k is isomorphic to A_6 or S_6.

We construct a couple of transcendental skew fields E_1 and E_2 such that E embeds in $E_1 \circ E_2$.

Let E_1 be the skew field such that $M_2(E_1)$ is isomorphic to $M_2(k) \underset{k}{\circ} E$, and let E_2 be the skew field such that $M_3(E_2)$ is isomorphic to $M_3(k) \underset{k}{\circ} E$.

Certainly, E embeds in $E_1 \underset{k}{\circ} E_2$, since by construction a simple $E \otimes_k E_1$-module of dimension 2 over E_1 and a simple $E \otimes_k E_2$-module of dimension 3 over E_2 exist; since h.c.f.$\{2,3\} = 1$, by theorem 4.1, E embeds in $E_1 \underset{k}{\circ} E_2$.

It remains to show that both E_1 and E_2 are transcendental over k. Let C be a commutative finite-dimensional extension of k lying in E_1. Let $[C:k] = n$. Since C lies in E_1 and $M_2(E_1) \cong M_2(k) \underset{k}{\circ} E$, by theorem 2.2, $\frac{1}{2}$ = h.c.f.$\left\{\frac{n}{2}, [C_j:E]\right\}$ as C_j runs through simple $C \otimes_k E$-modules. $\frac{n}{2}$ is the rank of a simple $M_2(C)$-module as $M_2(k)$-module. So n must be odd, and the equation 1 = h.c.f.$\{n, [C_j:E]\}$ holds.

Since $[C \otimes_k E:E] = n$, n is a sum of the numbers $[C_j:E]$ and so 1 = h.c.f.$\{[C_j:E]\}$, which is equivalent to the hypothesis that C embeds in $E \underset{k}{\circ} E$.

By theorem 4.3, $[C:k] \mid [E:k]$ and since $[C:k]$ is odd, $[C:k] = 1$ or 3. Since h.c.f.$\{[C_j:E]\} = 1$, $[C_j:E] = 1$ for some j, which implies that C embeds in E, and since E contains no intermediate extension of k, C must be k. Therefore E_1 is transcendental.

An entirely similar argument shows that E_2 is also transcendental over k.

Thus $E_1 \underset{k}{\circ} E_2$ is a coproduct of transcendental skew fields that is not transcendental itself, since it contains a copy of E. It is not hard to see that all algebraic subfields of $E_1 \underset{k}{\circ} E_2$ are conjugate to E.

We give some variants of the definition of transcendence that do behave well with respect to the coproduct construction. Further, we discuss that connections with related concepts in Cohn, Dicks (22) and present a counter-example to a conjecture in that paper.

We define a skew field E to be <u>n-transcendental</u> over the central subfield k if, for all finite dimensional division algebras D over k of PI degree dividing n, $D \otimes_k E$ is a skew field. We define a skew field E to be <u>totally transcendental</u> over the central subfield k if it is n-transcendental for all natural numbers n.

It is clear that n-transcendental implies transcendental, for if E contains an algebraic element e over k, then $k(e) \otimes_k E$ has zero-divisors, but k(e) has PI degree 1, which divides n. The skew fields E_1, E_2 of example 5.3 are transcendental skew fields, but are clearly not even 1-transcendental.

<u>Theorem 5.4</u> The coproduct of n-transcendental skew fields is n-transcendental. Therefore the coproduct of totally transcendental skew fields in totally transcendental.

<u>Pf</u> Let $\{E_i : i = 1 \text{ to } m\}$ be n-transcendental skew fields and let D

be a finite-dimensional division k-algebra of PI degree dividing n.

By definition, $D\otimes_k E_i$ is a skew field and so the dimension of a simple $D\otimes_k E_i$-module over E_i is $[D:k]$.

By theorem 4.1, the dimension of a simple $D\otimes_k(\underset{k}{\circ} E_i)$-module is $\text{h.c.f.}_i\{[D\otimes_k E_i : E_i]\} = [D:k]$. So $D\otimes_k(\underset{k}{\circ} E_i)$ is a skew field.

The last sentence is an immediate corollary.

We come to the connections with the ideas in Cohn, Dicks (22). A skew field is said to be <u>regular</u> over k, if $E\otimes_k K$ is a domain for all commutative field extensions K of k. In Cohn, Dicks (22), it is shown that E is regular over k if $E\otimes_k \bar{k}$ is a skew field, where $\bar{k}$ is the algebraic closure of k. So 'regular' is equivalent to '1-transcendental'.

Cohn and Dicks define k to be <u>totally algebraically closed</u> in E if $E\otimes_k L$ is a skew field for all simple algebraic extensions L of k. Further, they ask whether this condition and the hypothesis that $E\otimes_k k$ is a skew field (where p is the characteristic of k) imply that E is a regular extension of k. We demonstrate counter-examples to this as corollaries of a perhaps more natural positive result.

<u>Theorem 5.5</u> Let $F\supset k$ be a normal extension of commutative fields such that $[F:k] = p^n$, a prime power. Let D be the skew field such that $M_q(D)\cong M_q(k)\underset{k}{\circ} F$, where $q = p^{n-1}$. Then, for any algebraic extension $E\supset k$, $D\otimes_k E$ has zero-divisors if and only if $E\supseteq F$.

<u>Pf</u> Let $[E:k] = e$. As usual we use theorem 2.2. The rank of a simple $M_q(E)$-module as $M_q(k)$-module is e/q. The simple modules for $E\otimes_k F$ correspond to field joins of E and F, and there is a unique such field join up to isomorphism since F is a normal extension, and their dimensions are all equal to $[EF:F]$.

$$\begin{tikzcd}
& EF \arrow[dl, dash, "p^t"'] \arrow[dr, dash, "e/p^m"] & \\
E \arrow[dr, dash, "e"'] & & F \arrow[dl, dash, "p^n"] \\
& k &
\end{tikzcd}$$

We see that $[EF:F] = e/p^m$ and so the rank of a simple $M_q(D)\otimes_k E$-module is equal to h.c.f.$\{\frac{e}{q}, \frac{e}{p^m}\}$. $D\otimes_k E$ has zero-divisors if and only if the rank of a simple $M_q(D)\otimes_k E$-module is less than e/q, but since either $(e/q)|(e/p^m)$ or $(e/p^m)|(e/q)$, this rank is less than e/q if and only if $(e/p^m)|(e/q)$. Since $m \leq n$, this can happen if and only if $m = n$ (for $q = p^{n-1}$). If $p^m = p^n$,

$$[EF:k] = (e/p^m)p^n = e,$$

so $EF = E$ or $F \subseteq E$, as we wished to show.

<u>Corollary 5.6</u> Let $F \supset k$ be a normal extension of k that is not a simple extension not a purely inseparable extension of k, and assume that $[F:K] = p^n$. Let D be the skew field such that $M_q(D) \cong M_q(k) \circ_k F$ where $q = p^{n-1}$. Then k is totally algebraically closed in D and $D\otimes_k k^{p^{-\infty}}$ is a skew field, where $k^{p^{-\infty}}$ is the maximal purely inseparable extension of k. However, $D\otimes_k F$ is not a skew field, that is D is not 1-transcendental or regular over k.

<u>Pf</u> In theorem 5.5, we showed that $E\otimes_k D$ has zero-divisors, where E is a finite-dimensional commutative extension of k, if and only if $E \supseteq F$; however, if $E \supset F$, E cannot be a simple extension of k, nor can E be purely inseparable. Therefore, k is totally algebraically closed in D and $D\otimes_k k^{p^{-\infty}}$ is a skew field.

Section 6

Amitsur( ) and Roquette(43) developed the notion of the universal splitting field of an central simple algebra. In part III , this is generalised to the notion of a generic m-splitting field (explained below) for a central simple algebra. This construction has a natural and interesting non-commutative analogue which we shall develop and explain in the course of this section. We shall investigate its finite-dimensional division subalgebras using our techniques from section 2.

Let D be a central division k-algebra, $[D:k] = n^2$, and let m be a number dividing n^2.

A skew field E is said to be an m-splitting skew field of D if $D\otimes_k E = M_m(S)$ for some simple artinian ring S. A commutative field E can only be an m-splitting field for D if m divides n, and when it is an m-splitting field, S in the above formula is a central simple E-algebra.

Our method of construction in part III is to consider certain functors that we showed were representable. We use this same idea here.

We call a covariant functor from the category of k-algebras to the category of sets, F: k-alg $\longrightarrow$ Sets, representable if there is a k-algebra R such that F(-) is naturally equivalent to Hom(R,-).

Theorem 6.1 The functor $\mathrm{Hom}(M_m(k), D\otimes_k -)$ is representable by the k-algebra R, where $D \sqcup_k M_m(k)$ is isomorphic to $D\otimes_k R$. In particular, E is an m-splitting skew field for D if and only if Hom(R,E) is not empty.

Pf Let $\alpha \in \mathrm{Hom}(M_m(k), D\otimes_k A)$ for some k-algebra A. Then $1\sqcup\alpha : D \sqcup_k M_m(k) \longrightarrow D\otimes_k A$ is a homomorphism and must take the centraliser of D in the first ring to A, giving a homomorphism

$\bar{\alpha}: R \longrightarrow A$ by restriction.

Conversely, given $\beta: R \longrightarrow A$, we have a homomorphism $1 \otimes \beta: D \otimes_k R \longrightarrow D \otimes_k A$, giving a homomorphism

$$\beta': M_m(k) \longrightarrow M_m(k) \underset{k}{\sqcup} D \cong D \otimes_k R \xrightarrow{1 \otimes \beta} D \otimes_k A$$

where β' is the composite of these maps.

The passage from α to $\bar{\alpha}$ and from β to β' are clearly mutually inverse, so R does represent the functor $\mathrm{Hom}(M_m(k), D \otimes_k -)$.

The last sentence is a triviality.

We look at R a little more closely, but first we make our notation a little more precise; the ring representing $\mathrm{Hom}(M_m(k), D \otimes_k -)$ is denoted by $R(D,m)$.

__Theorem 6.2__ $R(D,m)$ is an hereditary domain, whose monoid of projectives is generated by the free module of rank 1 and a projective P satisfying $R^{n^2} \cong P^m$, and no other relations. Therefore, it has a unique rank function on its projective modules and so has a universal skew field of fractions.

__Pf__ $D \otimes_k R \cong D \underset{k}{\sqcup} M_m(k)$, so

$$M_{n^2}(R) \cong D^o \otimes_k (D \otimes_k R) \cong D^o \otimes_k (D \underset{k}{\sqcup} M_m(R)) \cong M_{n^2}(k) \underset{D^o}{\sqcup} M_m(D^o) ,$$

where in this isomorphism R is the centraliser of the first factor.

By Bergman's coproduct theorems (theorem I.2.18), R is a domain and all projective modules are induced up from the factors in the coproduct representation $M_{n^2}(k) \underset{D^o}{\sqcup} M_m(D^o)$. Using the Morita equivalence of $M_{n^2}(R)$ and R, the monoid of projectives must be as stated. Moreover, by Bergman's coproduct theorems, R must be hereditary.

We have shown all but the last sentence. Since $P^m \cong R^{n^2}$, its rank can only be n^2/m so there is a unique rank function and R must have a universal skew field of fractions by Dicks, Sontag(27).

Before passing to an investigation of the universal skew field of fractions of $R(D,m)$, we note that the skew field spectrum of $R(D,m)$ (Cohn [21]) is the non-commutative analogue of the twisted homogeneous spaces for $PGL_n(k)$ considered in part III. Also this space is irreducible since $R(D,m)$ has a universal skew field of fractions.

Let $U(D,m)$ be the universal skew field of fractions of $R(D,m)$, which we shall call the universal m-splitting skew field of D.

<u>Theorem 6.3</u> Let $U(D,m)$ be the universal m-splitting skew field of D. Then the finite-dimensional division k-algebra E embeds in $U(D,m)$ if and only if it embeds in D^o and h.c.f.$\{[E:k],\ n^2/m\} = 1$. In particular, if $p \mid n \Leftrightarrow p \mid (n^2/m)$ for all primes p, $U(D,m)$ is transcendental over k.

<u>Pf</u> We recall from the proof of theorem 6.2 that $U = U(D,m)$ is the universal skew field of fractions of R where $M_{n^2}(R)$ is isomorphic to $M_{n^2}(k) \underset{k}{\sqcup} M_m(D^o)$. Hence

$$M_{n^2}(U(D,m)) \cong M_{n^2}(k) \underset{D^o}{\circ} M_m(D^o).$$

We can apply theorem 2.2 to conclude that the rank of a simple $E^o \otimes_k M_{n^2}(U)$-module over $M_{n^2}(U)$ is given by h.c.f.$\left\{\frac{[E:k]}{n^2},\ \frac{[S:D]}{m}\right\}$ where S is the unique simple $D^o \otimes_k E^o$-module. Hence $[S:D^o]$ divides $[E:k]$.

If E embeds in U, we know that this highest common factor is $1/n^2$, which happens if and only if $[S:D^o] = 1$ and h.c.f.$\left\{\frac{[E:k]}{n^2},\ \frac{1}{m}\right\}$ is $1/n^2$, which is equivalent to the statements:

(i) E embeds in D^o, and

(ii) h.c.f.$\{[E:k],\ n^2/m\} = 1$,

which proves all but the last statement, which is an easy corollary. It occurs, for example, when $n = m$.

Section 7

As a different type of application of the preceding theory we shall show that the analogue of the Grushko-Neumann theorem(47) cannot hold for skew field coproducts. The Grushko, Neumann theorem states that if $\{g_i : i \in I\}$ is a generating set for the group coproduct $A * B$, then by Neilsen transformations we obtain another generating set $\{h_j : j \in J\}$ such that h_j lies in A or in B. A corollary of this is that if A and B are finitely generated and require m and n generators respectively, then $A * B$ requires m+n generators. We give a counter-example to the analogue of this corollary.

Example 7.1 Let D_1 and D_2 be central division k-algebras of dimension 4 and 9 respectively over k. Then $D_1 \otimes_k D_2$ embeds in $D_1 \circ_k D_2$ by theorem 4.5; we can find an embedding that extends the embedding of D_1 in $D_1 \circ_k D_2$ as the first factor, since all copies of D_1 in $D_1 \circ_k D_2$ are conjugate because $D_1 \circ_k D_2$ has centre k.

The embedding of D_2 in $D_1 \circ_k D_2$ given by the embedding of $D_1 \otimes_k D_2$ is conjugate to the embedding of D_2 as the second factor. Therefore $D_1 \circ_k D_2$ is generated by this copy of $D_1 \otimes_k D_2$ and an element y that conjugates the two different embeddings of D_2 in $D_1 \circ_k D_2$ above.

Every central division k-algebra finite-dimensional over k may be generated by two elements as a ring over k. Hence $D_1 \circ_k D_2$ can be generated by 3 elements, two of which generate $D_1 \otimes_k D_2$ inside $D_1 \circ_k D_2$; the third element is y. But D_1 and D_2 both require two generators since they are non-commutative.

Section 8

The purpose of this section is to extend our results on the finite-dimensional division subalgebras of skew field coproducts to a related construction.

The tensor E-ring on the free E-E-bimodule over k on the set X is isomorphic to $E \underset{k}{\sqcup} k\langle X\rangle$. Thus the tensor E-ring satisfies the weak algorithm (Cohn 14 Ch.2) with respect to the filtration given by $E \mapsto 0$, $X \mapsto 1$, so it is natural to consider the universal skew field of fractions of a ring satisfying the weak algorithm with respect to an $\mathbb{N}$-filtration.

We recall that if R satisfies the weak algorithm with respect to an $\mathbb{N}$-filtration, R is a fir whose universal skew field of fractions we call U. R_o, the set of elements whose filtration is 0, is a skew field and R is a two-sided Ore domain only when it takes the form $R_o[x;\alpha,\delta]$ where α is an automorphism and δ a $(1,\alpha)$ derivation on R_o. When R is not a two-sided Ore domain, the centre of U lies in R_o by theorem 3.1, since R_o^* is the group of units of R; when R is a two-sided Ore domain we shall have to use different techniques so we shall deal with this case at the end of the section.

In the case where R is not a two-sided Ore domain, let $K \subseteq R_o$ be the centre (both of R and of U).

Theorem 8.1 Let D be a finite-dimensional division K-algebra. Then the dimension over U of a simple $D^o \otimes_K U$-module is equal to h.c.f.$\{[N_i:R_o]\}$ as N_i runs through the finitely generated $D^o \otimes_K R_o$-modules.

Pf By theorem 1.2, the dimension of a simple $D^o \otimes_K U$-module is given by h.c.f.$\{[M_j:R]\}$ as M_j runs through finitely generated $D^o \otimes_K R$-submodules of free modules. For ease of notation, we shall consider left D, right R bimodules on K-centralising generators which are the objects of an equivalent category to the category of $D^o \otimes_K R$-modules.

We recall that R is filtered by

$$R_o \subseteq R_1 \subseteq R_2 \subseteq \ldots\ldots \subseteq R, \quad \cup R_i = R.$$

Let F be a free left D, right R bimodule on the set X which we filter by

$$DXR_o \subseteq DXR_1 \subseteq \ldots\ldots \subseteq DXR = F$$

For $f \in F$, we define $v(f) = \min\{r : f \in DXR_r\}$. Let M be a finitely generated left D right R submodule of F. We construct a basis for M as an R-module, using the weak algorithm, from which it will be clear that the dimension of M as R-module is a sum of the numbers $[N_i : R_o]$ for left D, right R_o modules N_i.

We choose our basis by an induction. Let $M_o = 0$. Suppose that at the i^{th} stage, we have a left D, right R submodule of M generated as R-module by elements $\{a_j\}_{i-1} \subseteq DXR_n$ such that $\{a_j\}_{i-1}$ is a basis of this submodule, M_{i-1} and no element of $M - M_{i-1}$ lies in DXR_n.

Let $A_i' = \{a : a \in M - M_{i-1},\ v(a) \text{ is minimal}\}$. Since M_{i-1} and M are invariant under D, A_i' must be a left D-set, that is, if $a \in A_i'$, then so is da for all $d \in D - \{0\}$.

Therefore $M_{i-1} \setminus (A_i'R_o + M_{i-1})$ is a left D, right R_o module; moreover, it is a consequence of the weak algorithm that if $M_i = A_i'R + M_{i-1}$, then a basis for $M_{i-1} \setminus (A_i'R_o + M_{i-1})$ over R_o together with a basis for M_{i-1} over R gives a basis for M_i over R. The module M_i together with such an extended basis $\{a_j\}_i$ satisfy the induction hypothesis.

Since M is finitely generated over R, $M = M_n$ for some n and $[M_i : R] < \infty$ for each i.

However, by construction $[M_{i+1}:R] - [M_i:R]$ is equal to $[M_i \setminus (A'_{i+1}R_o + M_i) : R_o]$ where $(M_i \setminus (A'_{i+1}R_o + M_i))$ is a left D, right R_o module.

So $[M:R] = \sum_{i=1}^{n} [N_i:R_o]$ for suitable left D, right R_o modules N_i. Referring back to the first paragraph, we see that the dimension of a simple $D^o \otimes_K U$-module over U is given by h.c.f. $\{[M_j:R]\}$ as M_j runs over the left D, right R sub-bimodules of free left D, right R bimodules, and so must be divisible by h.c.f.$\{[N_i:R_o]\}$ as N_i runs over finitely generated left D, right R_o bimodules. However, given a left D, right R_o bimodule N, we construct the left D, right U bimodule $N \otimes_{R_o} U$ and $[N:R_o] = [N \otimes_{R_o} U:U]$, so equality must hold.

We are left with the two-sided Ore case; we recall that if R is a fir with weak algorithm that is a two-sided Ore domain, it takes the form $R_o[x:\alpha,\delta]$ where R_o is a skew field, α is an automorphism of R_o, δ is a $(1,\alpha)$ derivation and for all $r \in R_o$, $rx - xr^\alpha = r^\delta$.

<u>Theorem</u> <u>8.2</u> Let U be the universal skew field of fractions of $R_o[x:\alpha,\delta]$. Let K be the intersection of the centre of U with R_o. Then any finite-dimensional K-algebra lying in U is isomorphic to a skew subfield of R_o

<u>Pf</u> We shall show that there is a valuation on U with residue class skew field R_o. This valuation will be trivial on R_o and so on K, and if D is a finite-dimensional division K-algebra lying in U, the valuation must be trivial on D and so there is an embedding of D as a K-algebra in R_o.

The construction of the valuation essentially occurs on p.18 of (Cohn 15). We summarise it briefly here.

Set $y = x^{-1}$ and write out our commutation formula;

$$rx = xr^\alpha + r^\delta \quad \text{as} \quad yr = r^\alpha y + yr^\delta y.$$

The set of power series over R_o in y with coefficients on the left and the commutation rule given above is a principal valuation domain, and $R_o[x;\alpha,\delta]$ embeds in its skew field of fractions by $x \longrightarrow y^{-1}$.

The residue class field of the skew field of power series is just R_o and the valuation is trivial on R_o.

This completes the proof of the theorem.

Section 9

Given a commutative field k and a skew field E such that k lies in the centre of E, it is interesting to investigate the transcendence degree over k of maximal commutative subfields of E. If these numbers are bounded, we call the maximal such number the transcendence degree of E over k.

The work of Resco(38) shows that there is a close connection between the transcendence degree of E over k and the global dimension of the rings $E\otimes_k k(x_1,\ldots,x_n)$ for varying n. We show that given a collection $\{D_i : i = 1 \text{ to } n\}$ of skew fields containing the central subfield k, and a common finite-dimensional division k-algebra D, then if each $D_i\otimes_k k(x_1,x_2)$ has global dimension $\leqslant 1$, $(\circ_D D_i)\otimes_k k(x_1,x_2)$ has global dimension 1 and therefore, by Resco(38), the transcendence degree of $\circ_D D_i$ over k is 1. Similar techniques may be use to show that, in characteristic 0, if $D_i\otimes_k W_1$ has global dimension 1, where W_1 is the skew field of fractions of the Weyl algebra, then so does $(\circ_D D_i)\otimes_k W_1$ and consequently W_1 cannot embed in $\circ_D D_i$. Both of these results apply to the free skew field and to the skew field coproduct of finite-dimensional division k-algebras.

Theorem 9.1 Let $\{D_i : i\in I\}$ be a collection of skew fields with common central subfield k, and let D be a common finite-dimensional division k-algebra. Suppose that E is a skew field with central subfield k, such that $D\otimes_k E$ is semisimple artinian, and that $D_i\otimes_k E$ has global dimension at most 1, then $(\circ_D D_i)\otimes_k E$ has global dimension 1.

Pf $(\sqcup_D D_i)\otimes_k E \cong \sqcup_{D\otimes_k E}(D_i\otimes_k E)$. Since $D\otimes_k E$ is semisimple artinian and $D_i\otimes_k E$ has global dimension at most 1, Bergman's coproduct theorems imply that $(\sqcup_D D_i)\otimes_k E$ has global dimension 1.

$(\underset{D}{\circ} D_i) \otimes_k E$ is the universal two-sided localisation of $(\underset{D}{\sqcup} D_i) \otimes_k E$ at the full matrices of $\underset{D}{\sqcup} D_i$. By theorem 12.7, this universal two-sided localisation of a ring of global dimension 1 must itself have global dimension 1; so $(\underset{D}{\circ} D_i) \otimes_k E$ has global dimension 1.

Our next two results are immediate corollaries of this theorem.

<u>Corollary 9.2</u> Let $\{D_i : i \in I\}$ be a set of skew fields with central subfield k and common finite-dimensional division k-algebra D. If $D_i \otimes_k k(x_1, x_2)$ has global dimension 1, where x_1 and x_2 are commuting indeterminates, then the transcendence degree of $\underset{D}{\circ} D_i$ is 1.

<u>Pf</u> By theorem 9.1, the global dimension of $(\underset{D}{\circ} D_i) \otimes_k k(x_1, x_2)$ is 1; so the result follows from Resco(38).

<u>Corollary 9.3</u> Let $\{D_i : i \in I\}$ be a collection of skew fields with central subfield k of characteristic 0, and common finite-dimensional division k-algebra D. If $D_i \otimes_k W_1$ has global dimension 1 for each i, where W_1 is the skew field of fractions of the Weyl algebra $k\{x, y : xy - yx = 1\}$ there is no embedding of W_1 in $\underset{D}{\circ} D_i$.

<u>Pf</u> Suppose that W_1 embeds in $\underset{D}{\circ} D_i$; then $(\underset{D}{\circ} D_i) \otimes_k W_1$ is a faithfully flat Noetherian ring extension of $W_1 \otimes_k W_1$, which is a ring of global dimension 2 and so $(\underset{D}{\circ} D_i) \otimes_k W_1$ must have global dimension at least 2. That $W_1 \otimes_k W_1$ has global dimension 2 is proven in Stafford(45).

However, we assumed that $D_i \otimes_k W_1$ has global dimension 1 from which, by theorem 9.1, it follows that $(\underset{D}{\circ} D_i) \otimes_k W_1$ has global dimension 1.

<u>Corollary 9.4</u> The transcendence degree of the free skew field is 1. In characteristic 0, the Weyl skew field cannot embed in the free skew field.

<u>Pf</u> The free skew field can be written as the skew field coproduct $\underset{k}{\circ} k(x_i)$. Certainly, $k(x_i) \otimes_k E$ is hereditary for any skew field E

and so corollary 9.2 and corollary 9.3 apply.

Corollary 9.5 Let $\{D_i : i \in I\}$ be a collection of finite-dimensional division k-algebras with common division subalgebra D. The transcendence degree of $\underset{D}{\circ} D_i$ is 1. In characteristic 0, the Weyl skew field cannot embed in $\underset{D}{\circ} D_i$.

Pf $D_i \otimes_k E$ is a skew field where E is either $k(x_1, x_2)$ or the Weyl skew field in characteristic 0. Once again, corollaries 9.2 and 9.3 apply.

The natural generalisation of the result on transcendence degree is:

Conjecture 9.6 Let $\{D_i : i \in I\}$ be a set of skew fields with central subfield k and let D be a common skew subfield. Suppose that the supremum of the global dimensions of $D \otimes_k k(x_1, \ldots, x_m)$ is (n-1) for varying m and that the supremum of the global dimensions of $D_i \otimes_k k(x_1, \ldots, x_m)$ is n for varying i and m. Then the transcendence degree of $\underset{D}{\circ} D_i$ is at most n.

If we knew that the universal two-sided localisation of a ring R of global dimension n was either bounded by n or infinite, we could prove this conjecture by similar reasoning to what has gone before. We give an outline:

$$D_i \otimes_k k(x_1, \ldots, x_m) \cong \underset{D \otimes_k k(x_1, \ldots, x_m)}{\sqcup} (D_i \otimes_k k(x_1, \ldots, x_m)) .$$

$D \otimes_k k(x_1, \ldots, x_m)$ has global dimension at most (n-1) and $D_i \otimes_k k(x_1, \ldots, x_m)$ has global dimension at most n. So, by Dicks (25), the global dimension of $(\underset{D}{\sqcup} D_i) \otimes_k k(x_1, \ldots, x_m)$ is at most n.

$(\underset{D}{\circ} D_i) \otimes_k k(x_1, \ldots, x_m)$ is a universal two-sided localisation of $(\underset{D}{\sqcup} D_i) \otimes_k k(x_1, \ldots, x_m)$ and has finite global dimension. Given the previously mentioned conjecture on the global dimension of universal two-sided localisations, we could conclude that $(\underset{\mathcal{D}}{\circ} D_i)\otimes_k k(x_1,\dots,x_m)$ has global dimension at most n for all values of m, and so, by Resco (38), the transcendence degree of $\underset{\mathcal{D}}{\circ} D_i$ is at most n.

Conjecture 9.6 has now been proven essentially by adapting the above program.

Section 10

There have been a number of interesting results on centralisers in rings and skew fields. The most interesting of these is due to Bergman, who showed that the centraliser of a non-central element of $k\langle X\rangle$ is a polynomial ring in one variable (Cohn 41); another result, due to Cohn (16), is that the centraliser of a non-central element in the free skew field is commutative: together with the results of the last section we know that a centraliser in the free skew field is commutative of transcendence degree 1 over the centre. We prove further that it is a finitely generated extension, as a special case of results on centralisers in skew subfields of $M_n(k\langle X\rangle)$ which we show are finitely generated of PI degree dividing n and transcendence degree 1 over k. We show that a number of interesting skew fields may be embedded in an $n \times n$-matrix ring over a free skew field: our main examples are the skew field coproduct of finitely many finite-dimensional division k-algebras over k, and again, the skew field $D \circ k\langle X\rangle$, where D is a finite-dimensional division k-algebra. In turn, these results imply that centralisers in the ring coproduct of finite-dimensional division k-algebras also satisfy polynomial identities.

Theorem 10.1 Let C be an arbitrary commutative field extension of k; then $C \otimes_k M_n(k\langle X\rangle)$ is an hereditary noetherian prime ring.

Pf Theorem 12.10 shows that $k\langle X\rangle$ is the skew subfield of $C\langle X\rangle$ generated by the set X over k.

Since $k\langle X\rangle$ has centre k, $C \otimes_k k\langle X\rangle$ is a simple ring and so it is a domain because it must embed in $C\langle X\rangle$. It must be an Ore domain because C is commutative (Cohn, Dicks 22).

$C \otimes_k k\langle X\rangle$ is a universal two-sided localisation of the free algebra over C, $C\langle X\rangle$, and since $C\langle X\rangle$ is hereditary, $C\otimes_k k\langle X\rangle$ must be hereditary by Bergman, Dicks (8). However, an hereditary Ore domain must be noetherian by Robson (42).

$M_n(C\otimes_k k\langle X\rangle)$ must be an hereditary, noetherian prime ring and $C\otimes_k M_n(k\langle X\rangle) \cong M_n(C\otimes_k k\langle X\rangle)$.

It is this result that allows us to prove our next result.

<u>Theorem</u> <u>10.2</u> Any commutative subfield of $M_n(k\langle X\rangle)$ is finitely generated of transcendence degree 1 over k.

<u>Pf</u> Let C be a commutative subfield of $M_n(k\langle X\rangle)$; by our last theorem, $C\otimes_k M_n(k\langle X\rangle)$ is an hereditary, noetherian ring. So $C\otimes_k C$ must be noetherian, since $C\otimes_k M_n(k\langle X\rangle)$ is faithfully flat over $C\otimes_k C$ (Resco, Small, Wadsworth (39)). Therefore, C must be finitely generated.

Because $k(x_1,x_2)\otimes_k k\langle X\rangle$ is hereditary, where x_1,x_2 are commuting indeterminates, C must be of transcendence degree 1 by Resco (38).

We have already shown that centralisers of non-central elements in $k\langle X\rangle$ are finitely generated and so finite-dimensional over the field generated by the element they centralise. We extend this a little.

<u>Theorem</u> <u>10.3</u> Let E be a skew subfield of $M_n(k\langle X\rangle)$ whose centre contains an element y transcendental over k; then $[E:k(y)]<\infty$.

<u>Pf</u> Let M be a maximal commutative subfield of E. $M\supseteq k(y)$ and so is finitely generated of transcendence degree 1 over k. Therefore, M is a finitely generated algebraic extension of $k(y)$ and $[M:k(y)]<\infty$.

If C is the centre of E, $[M:C]\leq[M:k(y)]$ and so must be finite. Therefore, if M' is the centraliser of M in E, $[E:M']<\infty$.

Since M is a maximal commutative subfield of E, $M' = M$ and $[E:k(y)] = [E:M][M:k(y)]<\infty$.

We have shown that skew subfields of $M_n(k\langle X\rangle)$ whose centres are transcendental over k are finite-dimensional over their centre. It turns out that we can get a good bound on the polynomial identity degree of such skew subfields.

Theorem 10.4 Let E be a skew subfield of $M_n(k\langle X\rangle)$; then the polynomial identity degree of E divides n, when it is finite.

Pf Let $\bar{k}$ be the algebraic closure of k; E embeds in $M_n(k\langle X\rangle)$ and so $E\otimes_k\bar{k}$ embeds in $M_n(k\langle X\rangle)\otimes_k\bar{k} \cong M_n(\bar{k}\langle X\rangle)$.

If E has finite PI degree, E is finite-dimensional over its centre, which is finitely generated over k by theorem 10.2, and so $E\otimes_k\bar{k}$ is an artinian ring.

If $E\otimes_k\bar{k}/\mathrm{rad}(E\otimes_k\bar{k}) \cong \bigoplus_{i=1}^{t} S_i$, where each S_i is simple artinian, each S_i is a central extension of E and so has the same PI degree. Further, the centre C_i of S_i is of transcendence degree at most 1 over k; therefore, by Tsen's theorem, S_i is isomorphic to $M_m(C_i)$, where m is the PI degree of E.

So $E\otimes_k\bar{k}/\mathrm{rad}(E\otimes_k\bar{k}) \cong M_m(\bigoplus_{i=1}^{t} C_i)$, and since matrix units lift over nilpotent ideals, $E\otimes_k\bar{k} \cong M_m(A)$ for some artinian ring A.

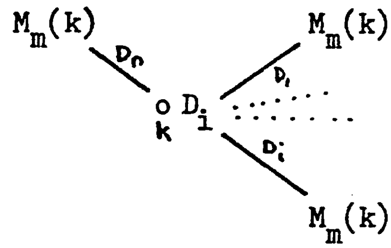
Therefore there is a unit-preserving embedding of $M_m(k)$ in $M_n(k)$, where m is the PI degree of E. So m divides n, as we wished to show.

It remains to give some interesting examples to which these results may be applied.

Theorem 10.5 Let y be a transcendental element over k in $\circ_k D_i$, where $\{D_i : i = 1,\dots,n\}$ is a finite set of finite-dimensional division k-algebras. Let m be the l.c.m. of $[D_i:k]$ (i = 1 to n). Then the centraliser of y in $\circ_k D_i$ is finite-dimensional over k(y) and its PI degree divides m.

Pf We shall embed $\circ_k D_i$ in $M_m(k\langle X\rangle)$ for some set X.

D_i embeds in $\underset{k}{\sqcup} M_m(k)$, since each D_i embeds in $M_m(k)$. This embedding is easily seen to induce an embedding of $\underset{k}{\circ} D_i$ in $\underset{k}{\circ} M_m(k)$, since the universal two-sided localisation of $\underset{k}{\sqcup} M_m(k)$ at the full matrices over $\underset{k}{\sqcup} D_i$ is the tree coproduct of rings



which is an hereditary ring with a rank function on its projectives taking values in $\frac{1}{n}\mathbb{N}$. So it has a universal two-sided localisation that is simple artinian. Since this is a universal two-sided localisation of $\underset{k}{\sqcup} M_m(k)$, it can only be $\underset{k}{\circ} M_m(k)$.

$\underset{k}{\sqcup} M_m(k)$ embeds in $\underset{\substack{k\\ j\in\mathbb{Z}}}{\sqcup} M_m(k)$ which has as universal two-sided localisation $\underset{\substack{k\\ j\in\mathbb{Z}}}{\circ} M_m(k)$. Both $\underset{\substack{k\\ j\in\mathbb{Z}}}{\sqcup} M_m(k)$ and $\underset{\substack{k\\ j\in\mathbb{Z}}}{\circ} M_m(k)$ have a shift automorphism σ; further, $(\underset{\substack{k\\ j\in\mathbb{Z}}}{\circ} M_m(k))[x,x^{-1};\sigma]$ is a universal two-sided localisation of $(\underset{\substack{k\\ j\in\mathbb{Z}}}{\sqcup} M_m(k))[x,x^{-1};\sigma] \cong M_m(k) \underset{k}{\sqcup} k[x,x^{-1}]$. The only univesal two-sided localisation of this is $M_m(k) \underset{k}{\circ} k(x)$.

Finally, the ring $M_m(k) \underset{k}{\circ} k[x]$ is isomorphic to $M_m(k\langle X\rangle)$, where X is the set of entries when x is written as a matrix over the centraliser of $M_m(k)$ in $M_m(k) \underset{k}{\sqcup} k[x]$, and it is clear that there can be no relations between the elements if X. $M_m(k) \underset{k}{\circ} k(x)$ is a universal two-sided localisation of $M_m(k) \underset{k}{\sqcup} k[x] \cong M_m(k\langle X\rangle)$, so $M_m(k) \underset{k}{\circ} k(x) \cong M_m(k\langle X\rangle)$.

We have embedded $\underset{k}{\circ} D_i$ in $M_m(k\langle X\rangle)$, so our result follows from the previous theorem.

Section 11

It is interesting to illustrate the results of the last section by finding skew subfields of $E \underset{k}{\circ} k(x)$ for a finite-dimensional field extension E of k, whose centres have transcendence degree 1 over k and whose PI degree does not equal 1. These constructions are quite simple and presumably do not exhaust the possibilities for such skew subfields; however, they do help to understand these skew field coproducts.

On the way we develop the theory of the D-D-bimodule structure of a skew field F with centre D and finite-dimensional division subalgebra D; F must always be a free D-D-bimodule, and this allows us to give an explicit method for finding the centraliser in F of D. We begin with these results.

Theorem 11.1 Let k be a commutative field, F a skew field with centre k and let D be a finite-dimensional division subalgebra of F. Then as a D-D-bimodule, F is free.

Pf Let $[D:k] = n$; the right action of D on itself induces an embedding of D in $M_n(k)$ and it is well-known that $M_n(k)$ as a D-D-bimodule via this embedding is a free bimodule.

Therefore, $M_n(F)$ considered as a D-D-bimodule via the embedding of D in $M_n(k)$ is a free D-D-bimodule on a basis of F as a vector space over k.

Since any two embeddings of D in $M_n(F)$ are conjugate by the Noether-Skolem theorem, the structure of $M_n(F)$ as a D-D-bimodule via the embedding of D in F must also be a free D-D-bimodule.

However, $M_n(F)$ as a D-D-bimodule via this embedding is just $\bigoplus_{i,j=1}^{n} e_{ij}F$, where e_{ij} are the matrix units of $M_n(k)$ and $e_{ij}F$ is isomorphic to F as D-D-bimodule, since e_{ij} centralises the D-D-bimodule structure. Since $D^o \otimes_k D$ is an artinian ring, any right $D^o \otimes_k D$-module M such that $n^2 M$ is a free module must itself be free; so, by the equivalence of the category of right $D^o \otimes_k D$-modules and D-D-bimodules, we deduce that F is free as a D-D-bimodule.

From this theorem we deduce an algorithm for calculating the centraliser of D in F.

<u>Algorithm</u> <u>11.2</u> Given a skew field F with centre k, we wish to find the centraliser of a finite-dimensional division subalgebra D in F.

We consider the following chains of subfields of D,

$$k \subseteq S \subseteq C \subseteq D,$$

where C is the centre of D and S the separable closure of k in C.

Since $S \supset k$ is a separable extension, the centraliser of S in F is the image of the separability idempotent of S over k, acting on S considered as an S-S-bimodule.

The centraliser of S in F is a skew field F' with centre S and the centraliser of D must lie in F'.

The extension $C \supset S$ is purely inseparable, and in order to find the centraliser of C in F', we need only find an algorithm for finding the centraliser of S(x) over S in F where $x^p \in S$, $x \notin S$.

F' has centre S and so is a free S(x)-S(x)-bimodule by theorem 11.1. Moreover, the centraliser of C in the free S(x)-S(x)-bimodule of rank 1 is the image of d_x^{p-1} where $d_x(t) = xt - tx$ for t an element of the free C-C-bimodule. That the bimodule is free on 1 generator is unimportant since the result clearly extends to free on an arbitrary generating set. Therefore, the centraliser of S(x) in F' is just the image of d_x^{p-1}. Further, the centraliser of S(x) in F' must have centre S(x), so induction applies.

Since we can climb up the purely inseparable extension $C \supset S$ by induction, we are left with finding the centraliser of D in F'', the centraliser of C. F'' has centre C and D is a central division C-subalgebra of F''. In this case, the centraliser of D is just the image of the separability idempotent of D over C.

From the information built up in the previous pages of this section, we can construct our example.

Example 11.3 Let $E \supset k$ be a finite-dimensional separable extension and let σ be an automorphism of E over k. There is a submodule of the free E-E- bimodule generated by an element t such that $et = te^{\sigma}$.

Since $E \underset{k}{\circ} k(x)$ has centre k, it is a free E-E-bimodule and there are many such elements $t \neq 0$. The skew subfield of $E \underset{k}{\circ} k(x)$ generated by E and t is generated by an image of the skew Laurent ring $E[y,y^{-1};\sigma]$ and so has PI degree equal to the order of σ as an automorphism of E. If the skew field is not isomorphic to $E(y,y^{-1};\sigma)$ the kernel of $E[y,y^{-1};\sigma] \longrightarrow E \underset{k}{\circ} k(x)$ given by $y \longmapsto t$ is non-zero as the image must be finite-dimensional over k. However, all finite-dimensional division subalgebras of $E \underset{k}{\circ} k(x)$ are commutative by theorem 4.7. So we have found a large number of embeddings of $E(y;\sigma)$ in $E \underset{k}{\circ} k(x)$.

Example 11.4 Suppose that $E \supset k$ is a purely inseparable extension of dimension p over k.

Recall from the course of the algorithm 11.2 that the centraliser of E in $E \underset{k}{\circ} k(x)$ is the image of d_e^{p-1} for some element $e \in E - k$. Therefore, there is some element t such that $et - te = 1$. Indeed, there are clearly many such elements.

The skew subfield of $E \underset{k}{\circ} k(x)$ generated by E and such an element t is generated by the image of the differential operator ring $E[y;1,\delta]$ where δ is the derivation of E sending e to 1 and k to 0. Any image of $E[y;1,\delta]$ has PI degree p, and if the kernel is non-zero, the image is finite-dimensional over k, which cannot happen in $E \underset{k}{\circ} k(x)$, since by theorem 4.7 all finite-dimensional division subalgebras of $E \underset{k}{\circ} k(x)$ are commutative. So we have many embeddings of $E(y;1,\delta)$ in $E \underset{k}{\circ} k(x)$.

Section 12

In the study of commutative rings we frequently find that in order to study a homomorphism of rings, it is useful to look at the associated map on the tangent spaces. Surprisingly, the non-commutative analogue turns out to be a powerful tool as we shall see in this and future sections.

In the commutative case, we look at derivations from a commutative ring to modules over this ring. The natural non-commutative analogue is to look at derivations to bimodules over the ring; thus if R is an R_o-ring for some ring R_o, we are interested in those derivations that vanish on R_o. On general principles, there is a universal derivation to a bimodule $\delta : R \longrightarrow \Omega_{R_o}(R)$; that is, the functor that associates to a bimodule the set of derivations from R to it that vanish on R_o, $\mathrm{Der}_{R_o}(R,M)$ is naturally isomorphic to $\mathrm{Hom}_{R\text{-bimod}}(\Omega_{R_o}(R),M)$, where the derivation associated to a particular bimodule homomorphism is its composite with δ. We construct this bimodule $\Omega_{R_o}(R)$ as follows: for each element $r \in R$, we associate a bimodule generator δr and impose the relations

$$\delta s = 0 \text{ for all } s \in R_o \quad \text{and} \quad \delta(r_1 r_2) = r_1(\delta r_2) + (\delta r_1) r_2.$$

It is clear that the map $\delta : R \longrightarrow \Omega_{R_o}(R)$ has the correct universal properties; however the nature of this bimodule is quite opaque from such a description, so we look for another description. It is easy to see from this, however, that if R has a presentation over R_o

$$R = \langle R_o,\ x_i (i \in I) : \ f_j \ (j \in J) \rangle$$

then $\Omega_{R_o}(R)$ is generated by dx_i subject to the relations obtained by taking the differential of the elements f_j (see theorem 12.3)

<u>Theorem</u> <u>12.1</u> There is an exact sequence of bimodules

$$0 \longrightarrow \Omega_{R_o}(R) \longrightarrow R\otimes_{R_o} R \xrightarrow{m} R \longrightarrow 0$$

where m is the multiplication map and with respect to this representation takes the form $\delta(r) = r\otimes 1 - 1\otimes r$.

<u>Pf</u> Certainly the map $d:R\longrightarrow \ker m$ given by $d(r) = r\otimes 1 - 1\otimes r$ is a derivation vanishing on R_o. So the composite of this derivation with any bimodule homomorphism $\mathrm{Hom}_{R\text{-bimod}}(\ker m, -)$ defines a natural transformation from $\mathrm{Hom}_{R\text{-bimod}}(\ker m, -)$ to $\mathrm{Der}_{R_o}(R,-)$, which must be injective since $\ker m$ is generated as bimodule by the elements $r\otimes 1 - 1\otimes r$. Given any derivation vanishing on R_o, $d':R\longrightarrow M$, where M is an R-bimodule, we define a bimodule homomorphism from ker m to M by $\alpha_{d'}(\Sigma x_i\otimes y_i) = \Sigma d'(x_i)y_i = -\Sigma x_i d'(y_i)$ (since d' is a derivation and $\Sigma x_i y_i = 0$). From these formulae, it is clear that this map $\alpha_{d'}$ is a bimodule homomorphism and that $d' = d\alpha_{d'}$; so the pair $d:R\longrightarrow \ker m$ has the right universal property and by the uniqueness of an object representing a functor, we see that our exact sequence holds.

It is useful to have various change of rings theorems. We begin by changing the ring over which derivations occur.

<u>Theorem</u> <u>12.2</u> Let $R_1 \longrightarrow R_2 \longrightarrow R$ be ring homomorphisms. We have an exact sequence;

$$^{\otimes}\Omega_{R_1}(R_2)^{\otimes} \longrightarrow \Omega_{R_1}(R) \longrightarrow \Omega_{R_2}(R) \longrightarrow 0.$$

<u>Pf</u>

$$\begin{tikzcd}
 & & R\otimes_{R_2}\Omega_{R_1}(R_2)\otimes_{R_2}R \arrow[dl, dashed] \arrow[d] & & \\
0 \arrow[r, dash] & \Omega_{R_1}(R) \arrow[r, dash] \arrow[d] & R\otimes_{R_1}R \arrow[r, dash] \arrow[d] & R \arrow[r, dash] & 0 \\
0 \arrow[r, dash] & \Omega_{R_2}(R) \arrow[r, dash] & R\otimes_{R_2}R \arrow[r, dash] \arrow[d] & R \arrow[r, dash] & 0 \\
 & & 0 & &
\end{tikzcd}$$

In the above commutative diagram, all rows and columns are exact, since they come from those of theorem 1; a simple diagram chase shows that the image of ${}^{\otimes}\Omega_{R_1}(R_2)^{\otimes}$ lies in $\Omega_{R_1}(R)$ and that the resulting sequence

$$ {}^{\otimes}\Omega_{R_1}(R_2)^{\otimes} \longrightarrow \Omega_{R_1}(R) \longrightarrow \Omega_{R_2}(R) \longrightarrow 0 $$

is exact.

The majority of our change of rings theorems involve the ring R rather than R_o. We shall be interested in how the universal bimodule of derivations varies under the universal constructions considered earlier and in most cases we shall fix our base ring to be semisimple artinian.

<u>Theorem</u> <u>12.3</u> Let K be a semisimple artinian ring. Let R be a K-ring and let I be an ideal of R. There is an exact sequence

$$ 0 \longrightarrow I/I^2 \longrightarrow {}^{\otimes}\Omega_K(R)^{\otimes} \longrightarrow \Omega_K(R/I) \longrightarrow 0 $$

where the map $I/I^2 \longrightarrow {}^{\otimes}\Omega_K(R)^{\otimes}$ is given by $i \longmapsto \delta(i)$

<u>Pf</u> From the sequence

$$ 0 \longrightarrow \Omega_K(R) \longrightarrow R \otimes_K R \longrightarrow R \longrightarrow 0 $$

which must be split exact as a sequence of right modules (since R is a free module), we obtain the exact sequence:

$$ 0 \longrightarrow \Omega_K(R) \otimes_R R/I \longrightarrow R \otimes_K R/I \longrightarrow R/I \longrightarrow 0 $$

From the long exact sequence of $\mathrm{Tor}_i^R(R/I,-)$ we extract the sequence:

$$ \mathrm{Tor}_1^R(R/I, R \otimes_K R/I) \longrightarrow \mathrm{Tor}_1^R(R/I, R/I) \longrightarrow {}^{\otimes}\Omega_K(R)^{\otimes} \longrightarrow $$

$$ \longrightarrow R/I \otimes_K R/I \longrightarrow R/I \longrightarrow 0 $$

Since K is semisimple artinian, R/I is a projective K-module, so $R \otimes_K R/I$ is projective and the first term is zero. The second term is I/I^2, so we obtain the sequence;

$$0 \longrightarrow I/I^2 \longrightarrow {}^{\otimes}\Omega_K(R)^{\otimes} \longrightarrow R/I \otimes_K R/I \longrightarrow R/I \longrightarrow 0.$$

The map $R/I \otimes_K R/I \longrightarrow R/I$ is still the multiplication map and the map ${}^{\otimes}\Omega_K(R)^{\otimes} \longrightarrow R/I \otimes_K R/I$ is just the map $\delta(r) \longrightarrow r \otimes 1 - 1 \otimes r$; by theorem 12.1 we pull out the exact sequence:

$$0 \longrightarrow I/I^2 \longrightarrow {}^{\otimes}\Omega_K(R)^{\otimes} \longrightarrow \Omega_K(R/I) \longrightarrow 0$$

To check that the map $I/I^2 \longrightarrow {}^{\otimes}\Omega_K(R)^{\otimes}$)is as stated, we consider the map of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "x \mapsto \delta(x)\otimes 1"'] & R \arrow[r] \arrow[d, "x \mapsto x \otimes 1"'] & R/I \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_K(R) \otimes_K R/I \arrow[r] & R \otimes_K R/I \arrow[r] & R/I \arrow[r] & 0
\end{tikzcd}$$

Applying $\mathrm{Tor}_i^R(R/I,-)$ to top and bottom we obtain the commutative square:

$$\begin{tikzcd}
\mathrm{Tor}_1^R(R/I,R/I) \arrow[r] \arrow[d, equal] & R/I \otimes_R I \arrow[d] \\
I/I^2 = \mathrm{Tor}_1^R(R/I,R/I) \arrow[r, "\beta"'] & {}^{\otimes}\Omega_K(R)^{\otimes}
\end{tikzcd}$$

β is the map we are interested in and we see that it is as we stated.

Next we consider the coproduct construction;

<u>Theorem</u> <u>12.4</u> Let $\{R_i : i \in I\}$ be a family of R_o-rings; then

$$\Omega_{R_o}\Big(\coprod_{\substack{R_o \\ i \in I}} R_i \Big) \cong \bigoplus_i \big({}^{\otimes}\Omega_{R_o}(R_i)^{\otimes} \big)$$

<u>Pf</u> This is best seen by recalling our original construction of the universal bimodule of derivations as the bimodule on elements ds subject to the relations $d(s) = 0$ for all $s \in R_o$ and $d(s_1 s_2) = s_1 d(s_2) + d(s_1)s_2$. We see that $\Omega_{R_o}(\coprod_{R_o} R_i)$ is generated by copies of $\Omega_{R_o}(R_i)$ and there are no further relations imposed, so the theorem follows.

<u>Theorem 12.5</u> Let M be an R-R-bimodule; then $\Omega_R(R\langle M\rangle) \cong {}^{\otimes}M^{\otimes}$, where the derivation restricted to M induces the isomorphism.

<u>Pf</u> Again this is best looked at in terms of the original construction. Since the bimodule M generates $R\langle M\rangle$ as R-ring, d(M) generates $\Omega_R(R\langle M\rangle)$ and the only relations on this are given by $d(rm) = rd(m)$ and $d(mr) = d(m)r$ for $r \in R, m \in M$, so the theorem follows.

There is one further construction where it is important to know the universal bimodule of derivations.

<u>Theorem 12.6</u> Let Σ be a collection of maps between finitely generated porjective modules over an R_o-ring R; then $\Omega_{R_o}(R_\Sigma) \cong {}^{\otimes}\Omega_R(R)^{\otimes}$

<u>Pf</u> Given a bimodule M over a ring T we can form a ring on $T \oplus M$ by $(t,m)(t',m') = (tt', tm' + mt')$. We write this ring as (T,M).

We wish to construct a derivation from R_Σ to ${}^{\otimes}\Omega_R(R)^{\otimes}$ that extends the universal derivation.

Consider the ring homomorphisms:

$$\begin{tikzcd} R \arrow[r, "r \mapsto (r,dr)"] \arrow[d] & (R, \Omega_{R_o}(R)) \arrow[d] \\ R_\Sigma \arrow[r, dashed] & (R_\Sigma, {}^{\otimes}\Omega_{R_o}(R)^{\otimes}). \end{tikzcd}$$

We wish to complete the square with a map from R_Σ to $(R_\Sigma, {}^{\otimes}\Omega_{R_o}(R)^{\otimes})$, since such a map must take the form $s \mapsto (s,ds)$ where d is a derivation extending the universal derivation on R. However, the set of maps over R are invertible in $(R_\Sigma, {}^{\otimes}\Omega_{R_o}(R)^{\otimes})$, since they are invertible modulo the nilpotent ideal $(0, {}^{\otimes}\Omega_{R_o}(R)^{\otimes})$ so there is a unique ring homomorphism from R_Σ to $(R_\Sigma, {}^{\otimes}\Omega_{R_o}(R)^{\otimes})$ completing the diagram.

We wish to show that this is a universal derivation. Given a derivation $\delta: R_\Sigma \longrightarrow M$, vanishing on the image of R_o, we know from the universal property of $\Omega_{R_o}(R)$ that $\delta|R$ factors through $\Omega_{R_o}(R)$, and by extension of scalars we obtain a diagram:

$$\begin{tikzcd}
R \arrow[r, "d"] \arrow[d] & \Omega_{R_o}(R) \arrow[d] \arrow[ddr, bend left] & \\
R_\Sigma \arrow[r, "d"] \arrow[drr, "\delta"'] & {}^{\otimes}\Omega_{R_o}(R)^{\otimes} \arrow[dr, "\alpha"] & \\
 & & M
\end{tikzcd}$$

where we have shown that the top triangle is commutative and wish to show that the bottom one is also commutative to complete the proof. However, we recall that R R_Σ is an epimorphism in the category of rings; we have two homomorphisms from R_Σ to (R_Σ, M) the first via s $(s, \delta s)$ and the second via s $(s, (d\alpha)(s))$. These agree on R and so must be equal; that is, $\delta = d\alpha$ as we wanted to show.

One use for the universal bimodule of derivations is for the study of how homological properties behave under the various constructions. We generally have to impose flatness assumptions but there is one case where this is not needed:

<u>Theorem</u> <u>12.7</u> Let R be a K-ring for some semisimple artinian ring K; if R is hereditary then so is R_Σ for any collection Σ of maps between finitely generated projective modules.

<u>Pf</u> We begin with the exact sequence , split as a sequence of left modules:

$$0 \longrightarrow \Omega_K(R\ ) \longrightarrow R_\Sigma \otimes_K R_\Sigma \longrightarrow R_\Sigma \longrightarrow 0$$

Since it is split as a sequence of left modules

$$0 \longrightarrow M \otimes_{R_\Sigma} \Omega_R(R_\Sigma) \longrightarrow M \otimes_K R_\Sigma \longrightarrow M \longrightarrow 0$$

is a resolution of M, because $M \otimes_K R_\Sigma$ must be projective.

However, $\Omega_K(R_\Sigma) \cong R_\Sigma \otimes_R \Omega_K(R) \otimes_R R_\Sigma$ so $M \otimes_{R_\Sigma} \Omega_K(R_\Sigma) \cong M \otimes_R \Omega_K(R) \otimes_R R_\Sigma$. But $M \otimes_R \Omega_K(R)$ is a projective

R-module as we see from the exact sequence of R-modules:

$$0 \longrightarrow M \otimes_R \Omega_K(R) \longrightarrow M \otimes_K R \longrightarrow M \longrightarrow 0$$

Thus $M \otimes_{R_\Sigma} \Omega_K(R_\Sigma)$ is projective and R_Σ must be hereditary.

This theorem can also be shown even when R is not a k-algebra (see Bergman, Dicks (8)). For applications of the universal bimodule of derivations and related matters to the homological properties of certain universal constructions and rings, the reader is referred to the papers of Dicks (25, 26)

We pass from the general theory to its applications in studying skew fields; there are a number of questions whose origins are in the cominatorial theory of groups that we are able to attack using these methods.

For a free group on n generators, it is well known and easy to see that any generating set of n generators is a free generating set; moreover, the corresponding result for a free k-algebra was shown by Cohn(18). It is natural to ask whether the analogous result holds for a free skew field over k.

Given a couple of groups G_1 and G_2 with common subgroup H and subgroups G_1', G_2' of G_1 and G_2 respectively, such that $G_1' \cap H = G_2' \cap H = H'$, it is easy to see by a normal form argument that the subgroup of $G_1 *_H G_2$ generated by G_1' and G_2' is isomorphic to $G_1' *_{H'} G_2'$. Once again, we ask whether the analogous result holds for skew fields. In this generality, it becomes clear that the result is false. However, under a natural restriction, it becomes true, which allows us to prove, for example, that if $D \subseteq E$ then the skew subfield of $E\langle X \rangle$ generated by D and the set X is isomorphic to $D\langle X \rangle$.

In the first of our problems, we shall show that if n elements $t_1, \ldots, t_n$ generate $k\langle x_1, \ldots, x_n \rangle$ as a skew field over k, the homomorphism $\alpha : k\langle y_1,\dots,y_n\rangle \longrightarrow k\langle x_1,\dots,x_n\rangle$ given by $\alpha(y_i) = t_i$ induces an isomorphism

$$\delta\alpha : {}^{\otimes}\Omega_k(k\langle y_1,\dots y_n\rangle)^{\otimes} \longrightarrow \Omega_k(k\langle x_1,\dots,x_n\rangle).$$

(For the notation, see the section on notation.)

That this map is an isomorphism implies that α extends to an isomorphism of $k\langle y_1,\dots,y_n\rangle$ with $k\langle x_1,\dots,x_n\rangle$ as we wished to show.

The proof of the second result will also follow from the examination of the 'differential' or 'linearisation' of a suitable map. Its proof comes first since the first result depends on the corollary of the second result mentioned earlier.

The first theorem is the heart of this section: it gives us a method for recognising those epimorphisms from a right hereditary ring to a skew field that are universal two-sided localisations at a rank function.

<u>Theorem</u> <u>12.8</u> Let E be a skew field and let R be an E-ring and a right hereditary ring. If the ring homomorphism $\alpha : R \longrightarrow F$ to the skew field F induces an isomorphism on the universal bimodules of derivations

$$d\alpha : {}^{\otimes}\Omega_E(R)^{\otimes} \cong \Omega_E(F),$$

F must be the universal two-sided localisation of R at the rank function on the finitely generated projective modules induced by the homomorphism to F.

<u>Pf</u> From the exact sequence of theorem 12.2,

$${}^{\otimes}\Omega_E(R)^{\otimes} \xrightarrow{d} \Omega_E(F) \longrightarrow \Omega_R(F) \longrightarrow 0$$

we deduce that $\Omega_R(F) = 0$ and so $\alpha : R \longrightarrow F$ is an epimorphism. Therefore, by Cohn (14, Ch.7), if Σ is the collection of square matrices over R that become invertible over F and R_Σ is the universal Σ-inverting ring, $\alpha : R \longrightarrow F$ factors as $R \longrightarrow R_\Sigma \longrightarrow F$, where R_Σ is a local ring with residue class field F.

Since R_Σ is a universal two-sided localisation of a right hereditary ring, it is right hereditary by theorem 12.7 and so the kernel of $R_\Sigma \longrightarrow F$, I, must be a free right ideal, and $I \neq I^2$, except when $I = 0$.

From the exact sequence of theorem 12.3,

$$0 \longrightarrow I/I^2 \longrightarrow {}^{\otimes}\Omega_E(R_\Sigma)^{\otimes} \longrightarrow \Omega_E(F) \longrightarrow 0$$

and from the isomorphisms $\Omega_E(F) \cong {}^{\otimes}\Omega_E(R)^{\otimes}$ and $\Omega_E(R_\Sigma) \cong {}^{\otimes}\Omega_E(R)^{\otimes}$ we deduce that $I/I^2 = 0$, which implies that $I = 0$. So R_Σ is isomorphic to F, as we wished to show.

We may often apply this result: we begin our applications here by giving a sufficient condition for a particular type of skew subfield of a skew field coproduct with amalgamation tow be a skew field coproduct with amalgamation in the natural way. There are examples to suggest that if our condition fails, it will be wholly accidental if the skew field they generate is a skew field coproduct with amalgamation.

Let F be a skew field with a couple of skew subfields D and E; we say that D and E are <u>linearly disjoint</u> in F if $DE = \{\Sigma d_i e_i : d_i \in D, e_i \in E\}$ as left D, right E bimodule is isomorphic to $D \otimes_{D\cap E} E$, and similarly, ED as left E, right D bimodule is isomorphic to $E \otimes_{D\cap E} E$.

That this may be an important concept is shown by a counterexample due to Bergman (unpublished) which shows that is D,E are not linearly disjoint skew subfields of F, the skew subfield of $F \underset{E}{\circ} E(x)$ generated by D and x is not, in general, isomorphic to $D \underset{D\cap E}{\circ} (D \cap E(x))$.

Theorem 12.9 Let $F_1 \underset{E}{\circ} F_2$ be the skew field coproduct of F_1 and F_2 amalgamating the common skew subfield E. Let D_1 and D_2 be skew subfields of F_1 and F_2 respectively, whose intersections with E are identical, $D_1 \cap E = D_2 \cap E = E'$. If D_i and E are linearly disjoint in F_i for i = 1,2 then the skew subfield of $F_1 \underset{E}{\circ} F_2$ generated by D_1 and D_2 is isomorphic to $D_1 \underset{E'}{\circ} D_2$.

Pf There is a natural ring homomorphism

$$\alpha: D_1 \underset{E'}{\sqcup} D_2 \to F_1 \underset{E}{\sqcup} F_2 \to F_1 \underset{E}{\circ} F_2.$$

Let the skew subfield of $F_1 \underset{E}{\circ} F_2$ generated by the image of $D_1 \underset{E'}{\sqcup} D_2$ be K; we wish to show that the induced map

$$d\alpha: {}^{\otimes}\Omega_{E'}(D_1 \underset{E'}{\sqcup} D_2)^{\otimes} \longrightarrow \Omega_{E'}(K)$$

is an isomorphism. Theorem 12.8 then show that K must be $D_1 \underset{E'}{\circ} D_2$ since $D_1 \underset{E}{\sqcup} D_2$ is a fir and so has a unique univesal two-sided localisation that is a skew field.

By theorem 12.4,

$$\Omega_E(F_1 \underset{E}{\sqcup} F_2) \cong {}^{\otimes}\Omega_E(F_1)^{\otimes} \oplus {}^{\otimes}\Omega_E(F_2)^{\otimes} .$$

Further, for any ring R and universal two-sided localisation R_Σ of R, we recall from theorem 12.6 that $\Omega_E(R_\Sigma) \cong {}^{\otimes}\Omega_E(R)^{\otimes}$. So

$$\Omega_E(F_1 \underset{E}{\circ} F_2) \cong {}^{\otimes}\Omega_E(F_1)^{\otimes} \oplus {}^{\otimes}\Omega_E(F_2)^{\otimes} .$$

Consider the K-K-sub-bimodule of $\Omega_E(F_1 \underset{E}{\circ} F_2)$ generated by the elements dD_1 and dD_2. dD_1 lies in $\Omega_E(F_1)$ and dD_2 lies in $\Omega_E(F_2)$; so if M_i is the K-K-bimodule generated by dD_i, $M_1 \oplus M_2$ is the bimodule we are interested in.

Further, this bimodule is the natural image of $\Omega_{E'}(K)$ in $\Omega_E(F_1 \underset{E}{\circ} F_2)$ induced by the embeddings:

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[d, hook] & E \arrow[d, hook] \\ K \arrow[r, hook] & F_1 \underset{E}{\circ} F_2 \end{tikzcd}$$

Recall from theorem 12.1 that the universal bimodule of derivations of F_i over E is the kernel of the multiplication map:

$$0 \longrightarrow \Omega_E(F_i) \longrightarrow F_i \otimes_E F_i \xrightarrow{m} F_i \longrightarrow 0$$

where the universal derivation is given by $df = f\otimes 1 - 1\otimes f$.

Since D_i is linearly disjoint with E in F_i, the vertical arrows in the following diagram are embeddings for the second and third column. Therefore, by the five lemma, so is the arrow in the first column.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_{E'}(D_i) \arrow[r] \arrow[d] & D_i \otimes_{E'} D_i \arrow[r] \arrow[d] & D_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_E(F_i) \arrow[r] & F_i \otimes_E F_i \arrow[r] & F_i \arrow[r] & 0
\end{tikzcd}$$

Hence the K-K-bimodule $M_1 \oplus M_2$ must be ${}^{\otimes}\Omega_{E'}(D_1)^{\otimes} \oplus {}^{\otimes}\Omega_{E'}(D_2)^{\otimes}$.

However,

$$\Omega_{E'}(D_1 \sqcup_{E'} D_2) \cong {}^{\otimes}\Omega_{E'}(D_1)^{\otimes} \oplus {}^{\otimes}\Omega_{E'}(D_2)^{\otimes},$$

so $\Omega_{E'}(K) \cong {}^{\otimes}\Omega_{E'}(D_1 \sqcup_{E'} D_2)^{\otimes}$ and this isomorphism is just given by $d\alpha$.

By theorem 12.8, K is isomorphic to $D_1 \circ_{E'} D_2$.

We note for the purpose of induction that $D_1 \circ_{E'} D_2$ is linearly disjoint from E in $F_1 \circ_E F_2$ and then leave the statement and proof of the generalisation to the interested reader. Alternatively, we could perform the same proof inside $\circ_E F_i$ for an arbitrary indexing set.

We give a simple application of the theorem.

<u>Theorem</u> <u>12.10</u> Let $D \subset E$ be an extension of skew fields. Then the skew subfield of $E\langle X\rangle$, the free skew field over E on the set of E-centralising generators X, generated by D and X, is isomorphic to $D\langle X\rangle$.

<u>Pf</u> Since E and D(x) are linearly disjoint in E(x), we prove our result by considering $E\langle X\rangle$ as $\circ_E E(x_i)$.

We pass on to our first problem, the question of whether n elements that generate the free skew field $k\langle x_1,\dots,x_n\rangle$ must generate it freely. We shall look at a more general problem.

First, we recall that the enveloping algebra of a k-algebra R is the ring $R^{o}\otimes_{k}R$. Since our earlier investigations led us to consider certain bimodules (universal bimodules of derivations) it is natural that enveloping algebras should arise.

<u>Theorem 12.11</u> Let E be a skew field with central subfield k. Let $t_1,\dots,t_n$ be elements of $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ that generate the skew field $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ over E. Then if the enveloping algebra of $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ is weakly finite, the natural map from $E \underset{k}{\sqcup} k\langle y_1,\dots,y_n\rangle$ to $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ sending y_i to t_i extends to an isomorphism of $E \underset{k}{\circ} k\langle y_1,\dots,y_n\rangle$ with $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$.

<u>Pf</u> The universal bimodule of derivations of $E \underset{k}{\sqcup} k\langle x_1,\dots,x_n\rangle$ over E is the free k-centralising bimodule over $E \underset{k}{\sqcup} k\langle x_1,\dots,x_n\rangle$ on generators $dx_1,\dots,dx_n$ since $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ is a universal two-sided localisation of $E \underset{k}{\sqcup} k\langle x_1,\dots,x_n\rangle$, $\Omega_E(E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle)$ is isomorphic to ${}^{\otimes}\Omega_E(E \underset{k}{\sqcup} k\langle x_1,\dots,x_n\rangle)^{\otimes}$, and so $\Omega_E(E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle)$ is the free bimodule on k-centralising generators $dx_1,\dots,dx_n$.

Since $t_1,\dots,t_n$ generate $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ as a skew field over E, $dt_1,\dots,dt_n$ generate the universal bimodule of derivations of $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ over E. Therefore they are free generators since the enveloping algebra of $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ is weakly finite.

Hence the map $\alpha : E \underset{k}{\sqcup} k\langle y_1,\dots,y_n\rangle \longrightarrow E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ given by $y_i \longrightarrow t_i$ induces an isomorphism

$$d\alpha : {}^{\otimes}\Omega_E(E \underset{k}{\sqcup} k\langle y_1,\dots,y_n\rangle)^{\otimes} \longrightarrow \Omega_E(E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle)$$

Therefore, by theorem 12.8, the map α extends to an isomorphism of $E \underset{k}{\circ} k\langle y_1,\dots,y_n\rangle$ with $E \underset{k}{\circ} k\langle x_1,\dots,x_n\rangle$ sending y_i to t_i.

No example is known of a tensor product of skew fields that is not weakly finite, so it is possible that this theorem can be applied in full generality. Here we shall only prove that the enveloping algebra of $E \underset{k}{\circ} k\langle x_1,\ldots,x_n\rangle$ is weakly finite, where $[E:k] < \infty$.

Theorem 12.12 The enveloping algebra of $E \underset{k}{\circ} k\langle x_1,\ldots,x_n\rangle$ is weakly finite if $[E:k] < \infty$. Consequently, when $[E:k] < \infty$, any n element set that generates $E \underset{k}{\circ} k\langle x_1,\ldots,x_n\rangle$ over E is a free generating set over E.

Pf Recall that if $[E:k] = m$, $E \underset{k}{\sqcup} k\langle X\rangle$ embeds in $M_m(k) \underset{k}{\sqcup} k\langle X\rangle$, which is isomorphic to $M_m(k\langle Y\rangle)$ for some set Y. Further, by an argument exactly similar to that in theorem 10.5, this induces an embedding of $E \underset{k}{\circ} k\langle X\rangle$ in $M_m(k\langle Y\rangle)$. So the enveloping algebra of $E \underset{k}{\circ} k\langle X\rangle$ embeds in $M_{m^2}(k\langle Y\rangle^{\circ} \otimes_k k\langle Y\rangle)$.

However, by theorem 12.10, the skew subfield of $k\langle Y\rangle^{\circ}\langle Z\rangle$ generated by Z over k is just $k\langle Z\rangle$. Therefore, there is a homomorphism from $k\langle Y\rangle^{\circ} \otimes_k k\langle Z\rangle$ to $k\langle Y\rangle^{\circ}\langle Z\rangle$, and since the centre of $k\langle Y\rangle^{\circ}$ is k, the first ring is simple and the homomorphism must be an embedding. In particular, $k\langle Y\rangle^{\circ} \otimes_k k\langle Y\rangle$ embeds in a skew field. Therefore, the enveloping algebra of $E \underset{k}{\circ} k\langle X\rangle$ embeds in a simple artinian ring and must be weakly finite, since a subring of a weakly finite ring is weakly finite.

The last sentence of the theorem is just a corollary of theorem 12.1.

There is one more result that we can prove using the techniques of this section which we shall need in a later section.

Theorem 12.7 Let $M \subset N$ be an extension of bimodules over the skew field E. The skew subfield of the universal skew field of fractions of the tensor ring of N, generated by M over E, is isomorphic to the universal skew field of fractions of the tensor ring on M.

Pf Write $T(M)$ and $T(N)$ for the tensor rings on M and N respectively. Both are firs (Cohn [4 Ch. 2) and the embedding of M in N extends to an embedding of $T(M)$ in $T(N)$. The universal bimodule of derivations of $T(M)$ over E, $\Omega_E(T(M))$ is isomorphic to ${}^{\otimes}M^{\otimes}$; similarly, $\Omega_E(T(N)) \cong {}^{\otimes}N^{\otimes}$ by theorem 12.5.

Let K be the skew subfield of $E\langle N\rangle$, the skew field of fractions of $T(N)$, generated by M over E; we wish to find $\Omega_E(K)$ and we know that this is the K-K-bimodule of $\Omega_E(E\langle N\rangle)$ generated by dK. This follows from the commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_E(K) \arrow[r] & K\otimes_E K \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_E(E\langle N\rangle) \arrow[r] & E\langle N\rangle \otimes_E E\langle N\rangle \arrow[r] & E\langle N\rangle \arrow[r] & 0
\end{tikzcd}$$

where the rows and columns are exact, from which it follows that $\Omega_E(K)$ embeds in $\Omega_E(E\langle N\rangle)$.

However, $\Omega_E(E\langle N\rangle) \cong {}^{\otimes}\Omega_E(T(N))^{\otimes}$, since $E\langle N\rangle$ is a universal two-sided localisation of $T(N)$, so $\Omega_E(E\langle N\rangle) \cong {}^{\otimes}N^{\otimes}$, where one checks from Bergman, Dicks (8) that d on $N \subseteq E\langle N\rangle$ is the identity map to $N \subseteq {}^{\otimes}N^{\otimes}$.

The K-sub-bimodule generated by dK must actually be generated by dM since M generates K; but the sub-bimodule generated by dM is just ${}^{\otimes}M^{\otimes}$, which is $\Omega_E(T(M))$. Our theorem follows from theorem 12.8.

Section 13

In this section, we return to the general theory of epimorphisms to simple artinian rings; we find a criterion for a particular epimorphism from a right hereditary ring to a simple artinian ring to be a universal localisation at a suitable rank function. This is the natural generalisation of our earlier result that characterised those universal localisations that are skew fields. This criterion suits well to the study of finite-dimensional hereditary k-algebras and we end the chapter with an application to this theory. This will be applied in the following section to prove certain isomorphism theorems.

First of all, we prove a couple of results that belong in chapter 1 of Part I, but were proven too recently. We shall need them in the following study.

Theorem 13.1 Let R be weakly semihereditary and let $T \triangleleft R$ be the trace ideal of some set of finitely generated projective modules over R; then $T\backslash R$ is a weakly semihereditary ring and $P_{\oplus}(T\backslash R)$, the monoid of finitely generated projective modules over $T\backslash R$ is naturally isomorphic to the quotient of $P_{\oplus}(R)$ by the relation $P \sim P'$ if and only if there exist Q, Q' such that the trace ideals of Q and Q' lie in T, and $P \oplus Q \cong P' \oplus Q'$.

Pf We denote passage to $T\backslash R$ from R by bars, $\bar{R} = T\backslash R$.

Let $\bar{\alpha}:\bar{P}\longrightarrow\bar{P}'$, $\bar{\beta}:\bar{P}'\longrightarrow\bar{P}''$ be maps between induced projectives over R such that $\bar{\alpha}\bar{\beta} = 0$; then over R, $\alpha\beta = \gamma\delta$, $\gamma:P\longrightarrow Q$, $\delta:Q\longrightarrow P''$, where Q is a finitely generated projective whose trace ideal lies in T. Thus we have the equation over R:

$$(\alpha\ \gamma)\begin{pmatrix}\beta\\ -\delta\end{pmatrix} = 0$$

and we note that $\overline{(\alpha\gamma)} = \bar{\alpha}$ and $\overline{\binom{\beta}{-\delta}} = \bar{\beta}$.

Since R is weakly semihereditary, $P' \oplus Q \cong P_1 \oplus P_2$, where $\operatorname{im}(\alpha\gamma) \subseteq P_1$ and $\ker\binom{\beta}{-\delta} \supseteq P_1$; therefore, $\bar{P}' \cong \bar{P}_1 \oplus \bar{P}_2$ where $\operatorname{im}\bar{\alpha} \subseteq \bar{P}_1 \subseteq \ker\bar{\beta}$; so two maps $\bar{\alpha}, \bar{\beta}$ between induced projectives that compose to zero do so because the codomain of α decomposes so the $\operatorname{im}\bar{\alpha}$ is contained in an induced direct summand which lies in the kernel of $\bar{\beta}$.

Let $e: R^n \longrightarrow R^n$ be a map such that $\bar{e}^2 = \bar{e}$ over $\bar{R}$; then $\bar{e}(\overline{I - e}) = 0$, so by our previous argument, $\bar{R}^n = \bar{P}_1 \oplus \bar{P}_2$ where $\operatorname{im}\bar{e} \subseteq P_1$ and $P_1 \supseteq \ker(\overline{I - e})$; but $\operatorname{im}\bar{e} = \ker(\overline{I - e})$, so $\operatorname{im}\bar{e} = \bar{P}_1$, which shows that all projective modules are induced, and, inconsequence, R must be weakly semihereditary.

Let $\bar{\alpha}: P \longrightarrow P'$ be an isomorphism over $\bar{R}$; then over R, $\operatorname{im}\alpha \oplus TP' = P'$, so there is a surjection:

$\binom{\alpha}{\beta}: P \oplus Q \longrightarrow P'$, where Q is a projective whose trace ideal lies in T. Also $\overline{\binom{\alpha}{\beta}} = \bar{\alpha}$ and so it is an isomorphism over R. We have the split exact sequence:

$$0 \longrightarrow \ker\binom{\alpha}{\beta} \longrightarrow P \oplus Q \xrightarrow{\binom{\alpha}{\beta}} P' \longrightarrow 0$$

so since $\overline{\binom{\alpha}{\beta}}$ is an isomorphism, $\overline{\ker\binom{\alpha}{\beta}} = 0$; it must be a finitely generated projective Q', whose trace ideal lies in T; so if $\alpha: \bar{P} \longrightarrow \bar{P}'$ is an isomorphism, there is an isomorphism over R, $P \oplus Q \cong P' \oplus Q'$, where Q, Q' have trace ideals in T, as we wished to show.

This allows us to pass from a weakly semihereditary ring with a rank function to one where the rank function is faithful simply by killing the projectives of rank 0.

<u>Theorem 13.2</u> Let R be a weakly semihereditary ring with a rank function ρ; let T be the trace ideal of projectives of rank zero; then $T\backslash R$ is a weakly semihereditary ring and ρ induces a faithful rank

function $\bar{\rho}$ on $T\backslash R$; if ρ takes values in $\frac{1}{n}\mathbb{Z}$, then $T\backslash R$ is right and left semihereditary.

Pf All is clear except for the last remark. In this case, $T\backslash R$ is a weakly semihereditary ring with a faithful rank function $\bar{\rho}$ taking values in $\frac{1}{n}\mathbb{Z}$; let $M \subseteq R$ be a finitely generated left ideal and let P be a projective module of minimal rank with respect to ρ such that there is a surjection $\alpha: P \longrightarrow M$. If $x \in \ker\alpha$, we have a sequence:

$$R \xrightarrow{x} P \xrightarrow{\alpha} M \hookrightarrow R$$

whose composite is zero; so $P \cong P_1 \oplus P_2$, where $x \in P_1$ and $\ker\alpha \supseteq P_1$. Thus $\alpha|P_2 : P_2 \longrightarrow M$ is a surjection; but $\rho(P_2) < \rho(P)$, so the kernel of $\alpha: P \longrightarrow M$ must be zero and $P \cong M$. Therefore, R is left semihereditary; by symmetry, it is right semihereditary.

Using this result, we see that when considering a rank function ρ to $\frac{1}{n}\mathbb{Z}$ on a weakly semihereditary ring R, we may assume that R is two-sided semihereditary and ρ is faithful, which is often a useful saving of effort.

In the last chapter, we saw that if R is a right hereditary k-algebra, then we may characterise a skew field F that is a universal localisation of R at a rank function by the property that $\Omega_k(F) \cong {}^{\otimes}\Omega_k(R)^{\otimes}$; it turns out that this has an interesting reformulation. Let $R \longrightarrow S$ be an epimorphism. From the exact sequence, split as a sequence of left R-modules,

$$0 \longrightarrow \Omega_k(R) \longrightarrow R \otimes_k R \longrightarrow R \longrightarrow 0$$

we obtain

$$0 \longrightarrow S \otimes_R \Omega_k(R) \longrightarrow S \otimes_k R \longrightarrow S \longrightarrow 0$$

where $S \otimes_k R$ is clearly a free right R-module. The long exact sequence of Tor's gives us

$$0 \longrightarrow \mathrm{Tor}_1^R(S,S) \longrightarrow {}^{\otimes}\Omega_k(R)^{\otimes} \longrightarrow S \otimes_k S \longrightarrow S \longrightarrow 0;$$

consequently, $\Omega_k(S) \cong {}^{\otimes}\Omega_k(R)^{\otimes}$ is equivalent to $\mathrm{Tor}_1^R(S,S) = 0$.

Dicks (8) has shown that for a universal localisation $R \longrightarrow R_\Sigma$, $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$ whether or not R contains a field; in view of this and of theorem 12.8, it seems reasonable to attempt to characterise those epimorphisms from a right hereditary ring to simple artinian rings that are universal localisations at rank functions by the property that $\mathrm{Tor}_1^R(S,S) = 0$. In doing this we are able to describe the R-module structure of S in some detail.

Theorem 13.3 Let R be a right hereditary ring and let $R \longrightarrow S$ be an epimorphism from R to a simple artinian ring S; then S is a universal localisation of R at the rank function ρ induced by $R \longrightarrow S$ if and only if $\mathrm{Tor}_1^R(S,S) = 0$.

Pf Universal localisations of a ring R at a set of maps Σ between finitely generated projective modules always satisfy the condition $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$ (Bergman, Dicks (8)).

The strategy for proving the converse is to find the structure of S as a right and left R-module. In fact, we could get by with only half this information, but the R-module structure of S is of great interest to us.

First of all, we can divide out by the trace ideal T of all projectives of rank zero with respect to ρ; this is a universal localisation at suitable maps all of which become invertible over S, for if $e \in M_n(R)$ is such that $e^nR \cong P$ where $\rho(P) = 0$, $I - e$ becomes invertible over S and $R_{(I-e)} = T_P \backslash R$, where T_P is the trace ideal of P. So by Bergman, Dicks (8), $T \backslash R$ is still right hereditary, By theorem 13.1, all projective modules over $T\backslash R$ are induced from R and the kernel of the natural map $K_0(R) \longrightarrow K_0(T\backslash R)$ is generated by projectives of rank zero; so ρ extends to $T\backslash R$ and it is a faithful rank function. Further, since T lies in the kernel of $R \longrightarrow S$ and $\mathrm{Tor}_1^R(S,S) = 0$, clearly $\mathrm{Tor}_1^{T\backslash R}(S,S) = 0$. Therefore, we lose nothing by assuming from now on that ρ is a faithful rank function on R.

Let M be a right R-submodule of S; then from the exact sequence

$$0 \longrightarrow M \longrightarrow S \longrightarrow S/M \longrightarrow 0$$

we get a long exact sequence of Tor's, from which we extract:

$$0 = \mathrm{Tor}_2^R(S/M,S) \longrightarrow \mathrm{Tor}_1^R(M,S) \longrightarrow \mathrm{Tor}_1^R(S,S) = 0.$$

Thus $\mathrm{Tor}_1^R(M,S) = 0$.

Let $0 \longrightarrow P \xrightarrow{\alpha} Q \longrightarrow M \longrightarrow 0$ be a presentation of M where Q is a finitely generated projective and P is some projective and so a direct sum of finitely generated projectives by theorem I.1.8 . Then, since $\mathrm{Tor}_1^R(M,S) = 0$,

$$0 \longrightarrow P \otimes_R S \longrightarrow Q \otimes_R S \longrightarrow M \otimes_R S \longrightarrow 0$$

is an exact sequence; so, since ρ is a faithful rank function and $M \otimes_R S = 0$, P must be a finitely generated projective and $\rho(P) < \rho(Q)$. We note that $\alpha : P \longrightarrow Q$ must be a left full map, for if it were not, we would find that $\alpha \otimes_R S : P \otimes_R S \longrightarrow Q \otimes_R S$ factors through a projective of rank smaller than that of $P \otimes_R S$, and so $\alpha \otimes_R S$ could not be injective.

We define the presentation rank of a finitely presented right or left module M by $p.\rho(M) = \rho(Q) - \rho(P)$, where

$$0 \longrightarrow P \longrightarrow Q \longrightarrow M \longrightarrow 0$$

is a presentation of M; it is well-defined by Schanuel's lemma. Our aim is to show that S as a right module is a directed union of finitely presented modules of left full presentation of presentation rank 1 having no submodules of presentation rank 0. There is an analogous result on the left.

In order to carry this out, we need to see how left full maps with respect to ρ behave under $\otimes_R S$. We wish to show that they become injective; this is a little harder to show than that right full maps become surjective, which is our next step.

Let $\alpha: P \longrightarrow Q$ be a right full map between finitely generated right projective modules; then all finitely generated submodules of Q containing P have rank at least equal to that of Q, and so, by our first step, cannot occur as the kernel of a map from Q to S. Since all kernels of maps from Q to S must be finitely generated we deduce that $\mathrm{Hom}(\mathrm{coker}\,\alpha, S) = 0$; so $\mathrm{coker}\,\alpha \otimes_R S = 0$ and $\alpha \otimes_R S$ must be surjective.

Since all right full maps with respect to ρ between right projectives become split surjective over S, all left full maps with respect to between left projective modules become split injective over S by duality. If we had assumed two-sided hereditary, we would know that left full maps on the right become injective and right full maps on the left become surjective and we could carry on in the manner of the latter half of this proof; as it is, we are able to deduce these properties in the right hereditary case from the fact that left full maps on the left become injective.

So let M be a finitely generated left R-submodule of S; then by the long exact sequence of Tor's associated to

$$0 \longrightarrow M \longrightarrow S \longrightarrow S/M \longrightarrow 0$$

we deduce that $\mathrm{Tor}_1^R(S,M) = 0$. Let

$$0 \longrightarrow F \longrightarrow Q \longrightarrow M \longrightarrow 0 \qquad \text{(i)}$$

be a presentation of M, where Q is finitely generated projective and, since R is left semihereditary (because ρ is a faithful rank function and so theorem 13.2 applies), all finitely generated submodules of F are projective.

We investigate F a little more closely; we wish to show that F is a directed union of finitely generated submodules where all inclusions are left full maps, which will be helpful in looking at $S\otimes_R F$, since all left full maps become injective.

Let $\{P_{1i}: i \in I_1\}$ be the set of submodules of F of minimal possible rank q_1; if there are finitely generated submodules of F that lie in no such P_{1i}, we consider the set of submodules $\{P_{2i}: i \in I_2\}$ where $P_{2i} \nsubseteq P_{1j}$ for any i, j and the rank of each P_{2i} is q_2, the minimal rank possible. In general, at the n^{th} stage, either all finitely generated submodules of F lie in some P_{ki} for $k < n$, $i \in I_k$; if not, we consider the set of submodules $\{P_{ni}: i \in I_n\}$ of F, where $P_{ni} \nsubseteq P_{kj}$ for $k < n$, any i, j and the rank of each P_n is q_n, the minimal possible. We see that every submodule must lie in some P_{ni}; for if $P \subseteq F$, then since $q_1 \lneq q_2 \lneq q_3 \lneq \dots\dots$, some q_n is greater than $\rho(P)$ and so P lies in some P_k, $k < n$, by definition.

So F is the directed union of the P_{ki}; moreover, if $P_{k_1 i_1} \subset P_{k_2 i_2}$ it is a left full map since no submodule of F lying over $P_{k_1 i_1}$ can have rank less than $q_{k_1} = \rho(P_{k_1 i_1})$ by the definition of $P_{k_1 i_1}$.

Consequently, $S\otimes_R F$ is the directed union of the system of modules $\{S\otimes_R P_{k_i}: k \in \mathbb{N}, i \in I_k\}$ and all the maps are injective; since $\mathrm{Tor}_1^R(S,M) = 0$, we deduce from (i) that the sequence

$$0 \longrightarrow S\otimes_R F \longrightarrow S\otimes_R Q \longrightarrow S\otimes_R M \longrightarrow 0$$

is exact.

So $S\otimes_R F$ must be a finitely generated module and of rank less than that of Q. Consequently, the rank of the P_{k_i} are bounded by $\rho(Q)$ and there must be some finite n such that all finitely generated submodules of F lie in some P_{k_i}, $k < n$, $i \in I_k$; further, $q_n = \rho_S(S\otimes_R F) < \rho(Q)$.

Let $\alpha: P \longrightarrow Q$ be a right full map between left finitely generated projectives; then all finitely generated submodules of Q containing $\mathrm{im}\,\alpha$ have rank at least that of Q, so no submodule of Q containing $\mathrm{im}\,\alpha$ can be expressed as a union of finitely generated submodules of rank less than $\rho(Q)$. Therefore $\mathrm{Hom}_R(\mathrm{coker}\,\alpha, S) = 0$ and $S \otimes_R \mathrm{coker}\,\alpha = 0$. Thus right full maps between left projective modules become split surjective; so left full maps between right projective modules become injective.

Let $M \subseteq N$ be a pair of finitely presented right modules of left full presentation such that the presentation rank of any module M_1 such that $M \subseteq M_1 \subseteq N$ is at least $\mathrm{p.}\rho(M)$; then we claim that $M \otimes_R S \longrightarrow N \otimes_R S$ is still an embedding.

For consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{\alpha} & Q_1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \cap\mid & & \cap\mid & & \\
0 & \longrightarrow & P & \xrightarrow{\beta} & Q & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

where α, β are left full maps, and if $Q_1 \subseteq Q' \subseteq Q$, $\rho(Q') > \rho(Q_1)$ since the presentation rank of the image of Q' in N is at least the presentation rank of M by assumption. So $Q_1 \subseteq Q$ is left full. We tensor our diagram with S:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P\otimes_R S & \longrightarrow & Q_1 \otimes_R S & \longrightarrow & M \otimes_R S & \longrightarrow & 0 \\
 & & \| & & \cap\mid & & \downarrow & & \\
0 & \longrightarrow & P\otimes_R S & \longrightarrow & Q \otimes_R S & \longrightarrow & N \otimes_R S & \longrightarrow & 0
\end{array}$$

where the rows are exact and the middle column is an embedding because left full maps become injective over S. So $M \otimes_R S \longrightarrow N \otimes_R S$ must also be an embedding.

Now we use this to examine S as a right R-module in the same way as we examined F as a left R-module earlier in the proof.

First of all, however, we note that every submodule M of S containing R has presentation rank at least 1; for on tensoring $R \subseteq M \subseteq S$ with S, we obtain $S = R \otimes_R S \longrightarrow M \otimes_R S \longrightarrow S \otimes_R S = S$, where the composite is the identity; but since $\mathrm{Tor}_1^R(M,S) = 0$, we know that $\rho_S(M \otimes_R S)$ is equal to $p.\rho(M)$, so $p.\rho(M) \geqslant 1$.

So we consider the set $\{M_{1i} : i \in I_1\}$ of all finitely generated right R-submodules of S of minimal presentation rank q_1; if there are finitely generated submodules of S lying in no M_{1i}, we let $\{M_{2i} : i \in I_2\}$ be the set of finitely generated submodules of S such that $M_{2i} \nsubseteq M_{1j}$ for any i,j and $p.\rho(M_{2i}) = q_2$, the minimal possible. In general, at the n^{th} stage, we let $\{M_{ni} : i \in I_n\}$ be the set of finitely generated submodules of S such that $M_{ni} \nsubseteq M_{kj}$ for $k < n$ and all i,j and $p.\rho(M_{ni}) = q_n$, the minimal possible. As we have seen when analyzing F, every finitely generated submodule of S must lie in some M_{ni}; so S is the directed union of the $\{M_{ni} : n \in \mathbb{N}, i \in I_n\}$. Also, if $M_{n_1 i_1} \subseteq M_{n_2 i_2}$ there is no finitely generated submodule M' with $M_{n_1 i_1} \subseteq M' \subseteq M_{n_2 i_2}$ such that $p.\rho(M') < p.\rho(M_{n_1 i_1})$, so as we have recently shown, $M_{n_1 i_1} \otimes_R S \longrightarrow M_{n_2 i_2} \otimes_R S$ is an injective map. Consequently we may represent $S = S \otimes_R S$ as the directed union of $\{M_{n_i} \otimes_R S : n \in \mathbb{N}, i \in I_n\}$ where each map is an embedding; therefore $p.\rho(M_{n_i}) = \rho_S(M_{n_i} \otimes_R S)$ must be at most 1. Therefore our process must stop at some finite stage, say the n^{th} stage; moreover $q_n = 1$. It also follows that we may dispense with the modules $\{M_{ki}\}$ for $k < n$, since if M is a finitely generated submodule, so is $(M + R)$, and this must be contained in some M_{r_i}, $r < n$; but $p.\rho(M_{k_i}) = q_k < 1$, if $k < n$; so $M + R \subseteq M_{n_i}$, since no module in S containing R can have presentation rank less than 1.

Let $s \in S$ and let M be a right R-submodule of S of presentation rank 1 containing R and s; we consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & P \oplus R & \longrightarrow & R & \longrightarrow & 0 \\
 & & \| & & \cap\!\mid & & \cap\!\mid & & \\
0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

We know that $\rho(P \oplus R) = \rho(Q)$ and since all finitely generated modules between R and M have presentation rank at least 1, $P \oplus R \subseteq Q$ is a full map; we also know that full maps become invertible over S, so there is a homomorphism from the universal localisation of R at ρ, R_ρ, to S, which induces the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & P\otimes_R S \arrow[r] \arrow[dd, dashed] & (P\oplus R)\otimes_R S \arrow[r] \arrow[dd, dashed, equal] & S \arrow[r] \arrow[dd, dashed] & 0 \\
0 \arrow[r] & P\otimes_R R_\rho \arrow[r] \arrow[u] \arrow[dd, equal] & (P\oplus R)\otimes_R R_\rho \arrow[r] \arrow[u] \arrow[dd, equal] & R_\rho \arrow[r] \arrow[u] \arrow[dd] & 0 \\
0 \arrow[r, dashed] & P\otimes_R S \arrow[r, dashed] & Q\otimes_R S \arrow[r, dashed] & M\otimes_R S \arrow[r, dashed] & 0 \\
0 \arrow[r] & P\otimes_R R_\rho \arrow[r] \arrow[u, dashed] & Q\otimes_R R \arrow[r] \arrow[u, dashed] & M\otimes_R R_\rho \arrow[r] \arrow[u, dashed] & 0
\end{tikzcd} \tag{ii}$$

where the dotted lines are dotted only to increase clarity. All rows are exact.

We deduce that $R_\rho \longrightarrow M\otimes_R R_\rho$ is an isomorphism and this induces a commutative diagram

$$\begin{tikzcd}
R \arrow[d] \arrow[dr] \arrow[drr] & & \\
M \arrow[r] & R_\rho \arrow[r] & S
\end{tikzcd}$$

such that the composition $M \longrightarrow R_\rho \longrightarrow S$ is just the original embedding of M in S as we can read off from diagram (ii).

Therefore, there is a map from our directed system of modules of presentation rank 1 containing R to R_ρ and the composite of this with the homomorphism $R_\rho \longrightarrow S$ is the representation of S as the direct limit; so R_ρ must also be the direct limit and is isomorphic to S, as we wished to show.

It is useful to abstract from this the information that we have on the right and left R-module structure of S under the hypotheses of theorem 13.3.

Let M be a finitely presented right R-module of left full presentation of presentation rank 1 with respect to the rank function ρ. Further, we assume that all submodules of M have non-zero (and so positive) presentation rank. We have already seen that S is the directed union of such modules. We define an injective map $\alpha: R \longrightarrow M$ to be a stone if M is such a module and all modules M' such that $R \subseteq M' \subseteq M$ have presentation rank at least 1. We shall show that the ring S is built from all these stones.

Let $\alpha: R \longrightarrow M$ be a stone; then we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & P \oplus R & \longrightarrow & R & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \bar{\alpha}} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where our assumptions force $\bar{\alpha}$ to be a full map, so over S, $\alpha \otimes_R S: S \longrightarrow M \otimes_R S$ is an isomorphism and the map $R \longrightarrow S$ extends uniquely to an embedding of M in S. If $\alpha_1: R \longrightarrow M_1$, $\alpha_2: R \longrightarrow M_2$ are two stones it is easy to check that their push-out is also a stone; therefore, the proof of theorem 13.3 shows that S is just the directed union of all the stones.

In order to describe the left structure of S as an R-module, we need to alter our definition of stones a little.

We have already seen that if M is a finitely generated left submodule of S, it has a presentation:

$$0 \longrightarrow F \longrightarrow Q \longrightarrow M \longrightarrow 0$$

where F is a flat module, Q is projective and F is the directed union of finitely generated projective modules $\{P_i : i \in I\}$ where $\rho(P_i) + \rho_S(S \otimes_R M) = \rho(Q)$. In fact $P_i \subseteq Q$ is left full for all i. We consider those finitely generated modules M of this sort such that $\rho_S(S \otimes_R M) = 1$, and such that if $N \subseteq M$, then $S \otimes_R N \neq 0$. We define a stone to be a map $\alpha: R \longrightarrow M$ where M is such a module and if $R \subseteq M' \subseteq M$, then M' has a presentation

$$0 \longrightarrow F' \longrightarrow Q' \longrightarrow M' \longrightarrow 0$$

where F' is the directed union of projective modules P_i' such that $P_i' \longrightarrow Q'$ is left full, $\rho(P_i') + \rho_S(S\otimes_R M') = \rho(Q)$ and $\rho_S(S\otimes_R M') \geqslant 1$.

One can show by a slight adaption of the arguments in theorem 13.3 that S is the directed union of all the stones on the left in exactly the same way as on the right.

We are able to rephrase theorem 13.3 in the case where R is an E-ring, where E is a skew field.

<u>Theorem 13.4</u> Let R be an E-ring that is right hereditary and let $R \longrightarrow S$ be an epimorphism to a simple artinian ring S. Then R is the universal localisation of R at the rank function induced on R by the epimorphism if and only if $\Omega_E(S) \cong {}^{\otimes}\Omega_E(R)^{\otimes}$.

<u>Pf</u> We have the sequence

$$0 \longrightarrow \Omega_E(R) \longrightarrow R\otimes_E R \longrightarrow R \longrightarrow 0$$

which is split as a sequence of left R-modules. So

$$0 \longrightarrow S\otimes_R \Omega_E(R) \longrightarrow S\otimes_E R \longrightarrow S \longrightarrow 0$$

is exact. Since $S\otimes_E R$ is a free R-module, the long exact sequence of Tor's gives us

$$0 \longrightarrow \mathrm{Tor}_1^R(S,S) \longrightarrow {}^{\otimes}\Omega_E(R)^{\otimes} \longrightarrow S\otimes_E S \longrightarrow S\otimes_R S = S \longrightarrow 0$$

Therefore, $\mathrm{Tor}_1^R(S,S) = 0 \Longleftrightarrow \Omega_E(S) = {}^{\otimes}\Omega_E(R)^{\otimes}$ and we may apply theorem 13.3.

Before applying this theory to epimorphisms from finite-dimensional hereditary algebras to simple artinian ring, we need to link the notion of epimorphisms on rings to the structure of the modules. The next result is due to Silver (44).

<u>Theorem 13.5</u> Let $\alpha : R \longrightarrow S$ be a homomorphism of rings; it is an epimorphism if and only if the forgetful functor $\bar{\alpha}$: mod $S \longrightarrow$ mod R is full.

Pf If $\alpha:R\longrightarrow S$ is an epimorphism, then for any S-module M, $M\otimes_R S \cong M\otimes_S(S\otimes_R S) \cong M$; consequently, for any pair of S-modules M and N any element β of $\mathrm{Hom}_R(M,N)$ must actually be S-linear as we see by considering $\beta\otimes_R S = \beta:M\longrightarrow N$.

Conversely, if $\bar{\alpha}$: mod $S\longrightarrow$ mod R is full, $\mathrm{Hom}_S(S,S\otimes_R S) = \mathrm{Hom}_R(S,S\otimes_R S)$; we have a map in $\mathrm{Hom}_R(S,S\otimes_R S)$ given by $\beta(s) = s\otimes 1$; so

$$s\otimes 1 = \beta(s) = \beta(1)s = 1\otimes s$$

which implies that $S\otimes_R S = S$; that is, $\alpha:R\longrightarrow S$ is an epimorphism.

The following consequence is noted by Ringel (41).

Theorem 13.6 Let R be a ring and let M be a module over R such that $\mathrm{End}_R(M)$ is a skew field D and $[M:D] = n$; then the corresponding map $R\longrightarrow M_n(D)$ is an epimorphism and all epimorphisms to simple artinian ring arise in this way. M is the simple right $M_n(D)$-module.

Pf Let $R\longrightarrow S$ be an epimorphism from R to the simple artinian ring S; then if M is the simple S-module, $\mathrm{End}_R(M) = \mathrm{End}_S(M) = D$, a skew field, and $[M:D] = n$, where $S = M_n(D)$.

Conversely, if M is an R-module such that $\mathrm{End}_R(M) = D$, a skew field and $[M:D] = n$, we have a map $R\longrightarrow M_n(D)$ and, given any $M_n(D)$-module, its structure as an R-module is just a direct sum of copies of M; so it is clear that the map mod-$M_n(D)\longrightarrow$ mod R is full as we wished to show.

Of course there is an entirely similar theory using the simple left modules over epic simple artinian R-rings. The passage between the two theories is quite simple, for if M is a simple right S-module over the simple artinian ring S with $\mathrm{End}_S(M) = D$, $\mathrm{Hom}_D(M,D)$ is the simple left S-module.

If R is a ring and M a module with $\mathrm{End}_R(M) \cong D$, a skew field, $[M:D] = n$, then the right R-module structure of $M_n(D)$ under the homomorphism $R \longrightarrow M_n(D)$ is clearly just $\bigoplus_{i=1}^{n} M$ and its left module structure must be $\bigoplus_{i=1}^{n} \mathrm{Hom}\,(M,D)$. Consequently, $\mathrm{Tor}_1^R(S,S) = 0$ is equivalent to $\mathrm{Tor}_1^R(M,\mathrm{Hom}\,(M,D)) = 0$.

If R is a finite-dimensional k-algebra, there are a number of finite-dimensional modules M over R whose endomorphism rings are finite-dimensional division algebras E, and clearly $[M:E] = n$; these clearly give all the epimorphisms from R to simple artinian finite dimensional k-algebras. Further, in this situation, $\mathrm{Hom}_E(M,E) = \mathrm{Hom}_k(M,k)$, which is usually written as DM, as left R-module; consequently, the condition we wish to check is whether $\mathrm{Tor}_1^R(M,DM) = 0$. This is unlikely to occur except when M is a pre-projective or pre-injective module in the sense used by Dlab, Ringel and Auslander for hereditary k-algebras. We explain the meaning of pre-projective and pre-injective in the way that is useful to us here.

We have already used one duality between the category of finite-dimensional right R-modules, where R is a finite-dimensional hereditary k-algebra. This is $M \longrightarrow DM = \mathrm{Hom}(M,k)$. The other duality we shall use is the transpose. Let M be any finite-dimensional R-module having no projective direct summand. Let

$$0 \longrightarrow P \longrightarrow Q \longrightarrow M \longrightarrow 0 \qquad \text{(i)}$$

be a presentation of M where P,Q are finitely generated projective modules; then we have the exact sequence:

$$0 = \mathrm{Hom}(M,R) \longrightarrow Q^{*} \longrightarrow P^{*} \longrightarrow \mathrm{Ext}^1(M,R) \longrightarrow 0$$

We call $\mathrm{Ext}^1(M,R) = \mathrm{Tr}\,M$, the transpose of M. On the category of modules with no projective summands, this defines a duality with the similar category on the other side of the ring, as one can check quite easily.

A pre-projective module M is an indecomposable module such that $(D\ Tr)^n M$ is projective for some non-negative integer n; a pre-injective module M is an indecomposable module such that $(Tr\ D)^n M$ is injective for some n. It is clear that these are dual notions with respect to the duality D.

Theorem 13.7 Let R be a finite-dimensional hereditary k-algebra and let M be a pre-projective or a pre-injective module over R; then $\mathrm{Tor}_1^R(M, DM) = 0$. Consequently, the epimorphism to a finite-dimensional simple artinian k-algebra, S, associated to the module M is the universal localisation at the rank function induced by the homomorphism.

Pf Due to the symmetry of the problem, it is enough to prove the theorem when M is pre-injective.

First, we show that $\mathrm{Tor}_1^R(M,-)$ is naturally equivalent to $\mathrm{Hom}_R(Tr\ M,-)$ for a module having no projective direct summands. Let

$$0 \longrightarrow P \longrightarrow Q \longrightarrow M \longrightarrow 0$$

be a presentation of M; then

$$0 \longrightarrow Q^* \longrightarrow P^* \longrightarrow Tr\ M \longrightarrow 0$$

is a presentation of Tr M. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(Tr\ M,-) \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_R(P^*,-) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_R(Q^*,-) \arrow[d, "\wr"] & & \\
0 \arrow[r] & \mathrm{Tor}_1^R(M,-) \arrow[r] & P\otimes_R - \arrow[r] & Q\otimes_R - \arrow[r] & M\otimes_R - \arrow[r] & 0
\end{tikzcd}$$

consequently, $\mathrm{Hom}(Tr\ M,-) = \mathrm{Tor}_1^R(M,-)$.

So we need to show that $\mathrm{Hom}_R(Tr\ M, DM) = 0$ if M is pre-injective. If M is injective, DM is projective and certainly $\mathrm{Tor}_1^R(M,DM) = \mathrm{Hom}_R(Tr\ M,DM) = 0$; so we may assume that DM is pre-projective non-projective. Hence, there is some natural number n such that

$(D\ Tr)^n(DM)$ is projective and $(D\ Tr)^k(DM)$ is not projective for $k < n$: in particular, $(D\ Tr)^{n-1}(DM)$ is not projective. Since $(D\ Tr)^k(DM)$ is indecomposable for all $k < n$, we deduce that $\mathrm{Hom}_R(Tr\ M, DM)$ is isomorphic to $\mathrm{Hom}_R((D\ Tr)^n Tr\ M, (D\ Tr)^n DM) = \mathrm{Hom}_R((D\ Tr)^{n-1}DM, (D\ Tr)^n DM)$, which is zero since $(D\ Tr)^{n-1}DM$ has no projective direct summand.

We have shown that if M is pre-injective, $\mathrm{Tor}_1^R(M,DM) = 0$; the same clearly holds if DM is pre-injective; that is, if M is pre-projective. By our previous discussion, this is equivalent to $\mathrm{Tor}_1^R(S,S) = 0$, where $R \longrightarrow S$ is the epimorphism associated to the module M. So, by theorem 13.3, S must be the universal localisation of R at the rank function.

Section 14

One of the purposes of this section is to construct a new class of skew fields generalising the construction of the skew field coproduct with amalgamation. They are interesting to us for two reasons; many arise naturally as skew subfields of skew field coproducts; but also, the methods that apply naturally to this class of skew fields apply just as well to the skew field coproducts in order to show that apparently different skew field coproducts are isomorphic.

Given a couple of skew fields E_1 and E_2, we define a pointed cyclic left E_1, right E_2 bimodule to be a pair (M,x) where M is a cyclic left E_1, right E_2 bimodule and x is a bimodule generator of M. For example, if F is a common skew subfield of E_1 and E_2, the pair $(E_1 \otimes_F E_2, 1\otimes 1)$ is a pointed cyclic left E_1, right E_2 bimodule on an F-centralising generator.

The importance of this idea for us lies in the observation that the ring coproduct $E_1 \sqcup_F E_2$ is the universal ring containing a copy of E_1 and of E_2 such that the pointed cyclic bimodule $(E_1E_2,1)$ is a quotient of $(E_1 \otimes_F E_2, 1\otimes 1)$.

If (M,x) is a pointed cyclic left E_1, right E_2 bimodule and the relations for the generator x are $\{\sum e_{ij}xe'_{ij} = 0;\ e_{ij}\in E_1,\ e'_{ij}\in E_2\}$, then the universal ring containing a copy of E_1 and E_2 such that $(E_1E_2,1)$ is a quotient of (M,x) is clearly the ring

$$E_1 \underset{(M,x)}{\sqcup} E_2 = \langle E_1, E_2 : \sum_i e_{ij}e'_{ij} = 0 \text{ for all } j\rangle$$

The first question to arise is whether this ring is not the trivial ring. We shall show below that inside this ring $(E_1E_2,1)$ is isomorphic to (M,x) and so it cannot be trivial if M is non-zero. Further, it is possible to show that $E_1 \underset{(M,x)}{\sqcup} E_2$ is a fir; we shall not demonstrate this, but we shall show that it is an hereditary domain such that all non-zero projectives are stably free of unique positive rank, and so it has a universal skew field of fractions, which we shall call $E_1 \underset{(M,x)}{\circ} E_2$.

It is clear that the isomorphism class of the ring $E_1 \underset{(M,x)}{\sqcup} E_2$ depends on the generator x of M. Surprisingly, the skew field of fractions $E_1 \underset{(M,x)}{\circ} E_2$ is isomorphic to $E_1 \underset{(M,y)}{\circ} E_2$ for different generators x and y. This gives us a number of interesting isomorphism theorems.

Finally, the method we develop is quite as applicable to the study of simple artinian coproducts. So our policy will be to work in this generality whilst pointing out what occurs in the skew field case.

Let S_1 and S_2 be a couple of simple artinian rings and let (M,x) be a pointed cyclic left S_1, right S_2 bimodule. A good way to study such a situation is to consider the upper triangular matrix ring: $R = \begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}$. This ring is a particularly pleasant sort of hereditary ring; so we could hope that if we can pull the ring $S_1 \underset{(M,x)}{\sqcup} S_2$ out of it in some way, then this will also have good properties.

<u>Theorem</u> <u>14.1</u> Let (M,x) be a pointed cylcic left S_1, right S_2 bimodule, where S_1 and S_2 are simple artinian rings; let $R = \begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}$ and let α_x: $\begin{pmatrix} 0 & 0 \\ 0 & S_1 \end{pmatrix} \longrightarrow \begin{pmatrix} S_1 & M \\ 0 & 0 \end{pmatrix}$ be left multiplication by $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$. Then $R_{\alpha_x} \cong M_2(S_1 \underset{(M,x)}{\sqcup} S_2)$. If $R' = \begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}$ where N is some left S_1, right S_2 bimodule, then $R'_{\alpha_x} \cong M_2(T)$, where T is the tensor ring over $S_1 \underset{(M,x)}{\sqcup} S_2$ on the bimodule

$$(S_1 \underset{(M,x)}{\sqcup} S_2) \otimes_{S_1} N \otimes_{S_2} (S_1 \underset{(M,x)}{\sqcup} S_2).$$

Therefore, $S_1 \underset{(M,x)}{\sqcup} S_2$ and also this tensor ring are hereditary.

<u>Pf</u> In the ring R_{α_x}, the elements $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and those representing

the maps α_x and α_x^{-1} form a set of 2x2 matrix units; therefore $R_{\alpha_x} = M_2(\bar{R})$, where $\bar{R} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} R_{\alpha_x}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$. $\bar{R}$ is clearly generated by S_1 and $\alpha_x S_2 \alpha_x^{-1}$ and the only relations arise because $(S_1\alpha_x S_2\alpha_x^{-1}, 1)$ as pointed left S_1, right $\alpha_x S_2 \alpha_x^{-1}$ bimodule arises as a quotient of $(S_1 x S_2, x) \cong (M, x)$; therefore, $\bar{R} \cong S_1 \underset{(M,x)}{\sqcup} S_2$. Of course, we do not know whether $\bar{R}$ is trivial.

If $R' = \begin{pmatrix}S_1 & M \oplus N\\0 & S_2\end{pmatrix}$, then $R' = M_2(\bar{R}')$, where $\bar{R}'$ is generated by S_1, $\alpha_x S_2 \alpha_x^{-1}$ and $N\alpha_x^{-1}$, where the only relations that occur state that S_1 and $\alpha_x S_2 \alpha_x^{-1}$ generate a copy of $S_1 \underset{(M,x)}{\sqcup} S_2$ and that $N\alpha_x^{-1}$ is isomorphic as left S_1, right $\alpha_x S_2 \alpha_x^{-1}$ bimodule to N as left S_1, right S_2 bimodule. This is clearly just the tensor ring over $S_1 \underset{(M,x)}{\sqcup} S_2$ on the bimodule $(S_1 \underset{(M,x)}{\sqcup} S_2) \otimes_{S_1} N \otimes_{S_2} (S_1 \underset{(M,x)}{\sqcup} S_2)$.

Since $M_2(S_1 \underset{(M,x)}{\sqcup} S_2)$ arises as a universal localisation of an hereditary ring it must be hereditary by theorem 12.7 . So $S_1 \underset{(M,x)}{\sqcup} S_2$ and similarly our tensor ring are hereditary.

Our original ring $\begin{pmatrix}S_1 & M\\0 & S_2\end{pmatrix}$ has a rank function on its finitely generated projective modules defined by $\rho\left(\begin{pmatrix}S_1 & M\\0 & 0\end{pmatrix}\right) = \frac{1}{2} = \rho\left(\begin{pmatrix}0 & 0\\0 & S_2\end{pmatrix}\right)$. If the map defined by $\begin{pmatrix}0 & x\\0 & 0\end{pmatrix}$ is a full map with respect to this rank function, then R_{α_x} will inherit many of the good properties of $\begin{pmatrix}S_1 & M\\0 & S_2\end{pmatrix}$ as we shall now show. In most applications, it will be fairly clear that is a full map; for example, this is obvious if both S_1 and S_2 are skew fields.

<u>Theorem</u> <u>14.2</u> Let (M,x) be a pointed cyclic left S_1, right S_2 bimodule and let N be some other left S_1, right S_2 bimodule. Let $R = \begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}$ and let ρ be the rank function that assigns the value $\frac{1}{2}$ to $\begin{pmatrix} S_1 & M \oplus N \\ 0 & 0 \end{pmatrix}$ and to $\begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}$. Then if $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ defines a full map from $\begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}$ to $\begin{pmatrix} S_1 & M \oplus N \\ 0 & 0 \end{pmatrix}$, the tensor ring T, over $S_1 \underset{(M,x)}{\sqcup} S_2$ on the bimodule $(S_1 \underset{(M,x)}{\sqcup} S_2) \otimes_{S_1} N \otimes_{S_2} (S_1 \underset{(M,x)}{\sqcup} S_2)$ is an hereditary ring such that all non-zero projective modules are stably induced from the subrings S_1 and S_2 and have a unique positive rank.

<u>Pf</u> We know that $M_2(T) \cong \begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}_{\alpha_x}$ where

$\alpha_x : \begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix} \longrightarrow \begin{pmatrix} S_1 & M \oplus N \\ 0 & 0 \end{pmatrix}$ is left multiplication by $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$.

We assume that this map is full. It is clearly an atomic map and it is easy to check the lower multiplicative closure of this map (that is, the set of all map of the form $\begin{pmatrix} \alpha_x & & 0 \\ & \alpha_x & \\ & & \ddots \\ & & & \alpha_x \end{pmatrix}$) satisfies the condition of theorem I.5.5; so $\begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}_{\alpha_x}$ embeds in $\begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}_{\rho}$.

Theorem I.5.7 shows that all finitely generated projective modules over $\begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}_{\alpha_x}$ are stably induced from $\begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}$. However, over $\begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}_{\alpha_x}$ the projective $\begin{pmatrix} S_1 & M \oplus N \\ 0 & 0 \end{pmatrix}_{\alpha_x}$ is isomorphic to $\begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}_{\alpha_x}$, so over the ring T (where $M_2(T) \cong \begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}_{\alpha_x}$) all projectives are stably induced from the subrings S_1 and S_2. This implies that if there is a rank function it must be unique, and there clearly is a rank function since we have an embedding $\begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}_{\alpha_x} \hookrightarrow \begin{pmatrix} S_1 & M \oplus N \\ 0 & S_2 \end{pmatrix}_{\rho}$. Theorem I.5.6 shows that this rank function is faithful as we wished to show.

In the case where $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ defines a full map in the ring $\begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}$ from $\begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}$ to $\begin{pmatrix} S_1 & M \\ 0 & 0 \end{pmatrix}$, we deduce that $S_1 \sqcup_{(M,x)} S_2$ has a unique universal localisation that is simple artinian, since it has a unique rank function, and so we may apply theorem I.5.9. We denote this universal localisation by $S_1 \circ_{(M,x)} S_2$.. Further, we see that $(S_1S_2,1)$ is isomorphic to (M,x) as pointed bimodule. For $\begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}$ embeds in $\begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}_\rho$ since the rank function ρ is faithful; so it certainly embeds in $\begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}_{\alpha_x}$; but now we have an isomorphism $(S_1\alpha_x S_2\alpha_x^{-1},1) \cong (S_1\alpha_x S_2,\alpha_x) \cong (M,x)$ as we wished to show. We state what we now know in the skew field case.

<u>Theorem</u> <u>14.3</u> Let E_1 and E_2 be a couple of skew fields and let (M,x) be a pointed cyclic left E_1, right E_2 bimodule; then $E_1 \sqcup_{(M,x)} E_2$ is an hereditary domain such that all non-zero projectives are stably free of positive rank. Also $(E_1E_2,1) \cong (M,x)$ as pointed bimodule.

<u>Pf</u> Theorems 14.1 and 14.2 and the last discussion.

We are able at once to prove our first isomorphism theorem.

<u>Theorem</u> <u>14.4</u> Let S_1 and S_2 be simple artinian ring and let M be a cyclic bimodule. Assume that x and y are both generators of M as a bimodule such that $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ both define full maps from $\begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}$ to $\begin{pmatrix} S_1 & M \\ 0 & 0 \end{pmatrix}$ on the ring $\begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}$ with respect to the rank function assigning the value $\frac{1}{2}$ to $\begin{pmatrix} S_1 & M \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}$. Then

$S_1 \underset{(M,x)}{o} S_2 = S_1 \underset{(M,y)}{o} S_2$.

Pf $\begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}_{\alpha_x} \cong M_2(S_1 \underset{(M,x)}{\sqcup} S_2)$.

The universal localisation of this at its unique rank function is $M_2(S_1 \underset{(M,x)}{o} S_2)$; but it must also be $\begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}_{\rho}$. A similar statement holds with y in place of x.

So $M_2(S_1 \underset{(M,x)}{o} S_2) = M_2(S_1 \underset{(M,y)}{o} S_2)$ which implies that $S_1 \underset{(M,x)}{o} S_2 = S_1 \underset{(M,y)}{o} S_2$ since they are simple artinian.

For skew fields, this theorem simply becomes:

Theorem 14.5 Let E_1 and E_2 be skew fields; let M be a cyclic left E_1, right E_2 bimodule; then $E_1 \underset{(M,x)}{o} E_2 = E_1 \underset{(M,y)}{o} E_2$ for any bimodule generators x and y.

If $y = \Sigma e_i x e_i'$, it is possible to calculate that a specific map from $E_1 \underset{(M,y)}{\sqcup} E_2$ to $E_1 \underset{(M,x)}{o} E_2$ which extends to an isomorphism of $E_1 \underset{(M,y)}{o} E_2$ with $E_1 \underset{(M,x)}{o} E_2$ is given by $E_2 \longrightarrow E_2$, $E_1 \longrightarrow E_1^{\Sigma e_i e_i'}$

We see from the last theorem that if F_1 and F_2 are both common skew subfields of E_1 and E_2 such that the bimodules $E_1 \otimes_{F_1} E_2$ and $E_1 \otimes_{F_2} E_2$ are isomorphic, then $E_1 \underset{F_1}{o} E_2 = E_1 \underset{F_2}{o} E_2$. Our next result simply states some interesting special cases of this result.

Theorem 14.6 Let S_1 be a finite-dimensional simple k-algebra and let S_2 be a simple artinian ring with central subfield k such that $S_1^o \otimes_k S_2$ is simple. If T_1 and T_2 are common finite-dimensional simple k-algebras of S_1 and S_2 such that $[T_1:k] = [T_2:k]$, then

$S_1 \underset{T_1}{o} S_2 = S_1 \underset{T_2}{o} S_2$.

<u>Pf</u> The bimodule $S_1 \otimes_{T_1} S_2$ is isomorphic to the bimodule $S_1 \otimes_{T_2} S_2$, since the ring $S_1^o \otimes_k S_2$ is simple artinian and both bimodules are free S_2-modules of the same rank. So

$$R = \begin{pmatrix} S_1 & S_1 \otimes_{T_1} S_2 \\ 0 & S_2 \end{pmatrix} \cong \begin{pmatrix} S_1 & S_1 \otimes_{T_2} S_2 \\ 0 & S_2 \end{pmatrix}$$

We see that the element $\begin{pmatrix} 0 & 1 \otimes_{T_1} 1 \\ 0 & 0 \end{pmatrix}$ defines a full map α_{T_1} from $\begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}$ to $\begin{pmatrix} S_1 & S_1 \otimes_{T_1} S_2 \\ 0 & 0 \end{pmatrix}$, since $R_{\alpha_{T_1}} \cong M_2(S_1 \underset{T_1}{\sqcup} S_2)$.

Similarly, the element $\begin{pmatrix} 0 & 1 \otimes_{T_2} 1 \\ 0 & 0 \end{pmatrix}$ defines a full map.

Therefore, by theorem 14.4, $S_1 \underset{T_1}{\circ} S_2 \cong S_1 \underset{T_2}{\circ} S_2$.

Our last theorem gives a large number of case of simple artinian coproducts with the same factors but different amalgamated simple artinian subrings that must be isomorphic. Our next theorem gives examples where the factors are different but we amalgamate over a common central subfield.

<u>Theorem 14.7</u> Let S be a finite-dimensional simple k-algebra and let S_1 and S_2 be simple artinian subalgebras such that $[S_1:k] = [S_2:k]$; then $S \underset{k}{\circ} S_1 = S \underset{k}{\circ} S_2$.

<u>Pf</u> $S \underset{k}{\circ} S_1 \cong S \underset{S_2}{\circ} (S_2 \underset{k}{\circ} S_1) \cong S \underset{S_1}{\circ} (S_2 \underset{k}{\circ} S_1)$ by theorem 14.6, since $S^o \otimes_k (S_2 \underset{k}{\circ} S_1)$ is simple artinian (the centre of $S_1 \underset{k}{\circ} S_2$ is k or else purely transcendental over k of degree 1). However, $S \underset{S_1}{\circ} (S_2 \underset{k}{\circ} S_1) = S \underset{k}{\circ} S_2$, which completes the proof.

We can prove a rather stronger theorem than 14.7 in the case where S is a central simple k-algebra.

<u>Theorem 14.8</u> Let S be a central simple k-algebra with $[S:k] = n^2$; let S_1 be a simple artinian k-subalgebra with $[S_1:k] = m$. Then

$S \underset{k}{\circ} S_1 \cong S \otimes_k k\langle X\rangle$, where $|X| = n^2(\frac{m-1}{m})$.

<u>Pf</u> $M_2(S \underset{k}{\circ} S_1)$ is the universal localisation of the ring $\begin{pmatrix} S & S\otimes_k S_1 \\ 0 & S_1 \end{pmatrix}$ at the rank function ρ given by $\rho\begin{pmatrix} 0 & 0 \\ 0 & S_1 \end{pmatrix} = \rho\begin{pmatrix} S & S\otimes_k S_1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}$

by theorems 14.1 and 14.2 and the remarks following the proof of theorem 14.1. As a bimodule, $S\otimes_k S_1 \cong \bigoplus_{i=1}^{m} M_i$, where $M_i \cong M_j \cong {}_S S_{S_1}$.

We write this as ${}_S S^m_{S_1} \cong S\otimes_k S_1$; by theorem 14.1, the universal localisation of $\begin{pmatrix} S & {}_S S^m_{S_1} \\ 0 & S_1 \end{pmatrix}$ at the map α from $\begin{pmatrix} 0 & 0 \\ 0 & S_1 \end{pmatrix}$ to $\begin{pmatrix} S & {}_S S^m_{S_1} \\ 0 & 0 \end{pmatrix}$

defined by $\begin{pmatrix} 0 & (1,0,..,0) \\ 0 & 0 \end{pmatrix}$ is $M_2(T)$, where T is the tensor ring over $S \underset{S_1}{\circ} S_1 \cong S$ on the bimodule $S\otimes_S({}_S S^{(m-1)}_{S_1})\otimes_{S_1} S \cong (S\otimes_{S_1} S)^{m-1}$. This bimodule is just the direct sum of $n^2(\frac{m-1}{m})$ simple bimodules; so $T \cong S\otimes_k k\langle X\rangle$ where $|X| = n^2(\frac{m-1}{m})$ (since it is clear that $S\langle {}_S S_S\rangle \cong S[x]$, so $S\langle {}_S S^r_S\rangle \cong S\otimes_k k\langle Y\rangle$, where $|Y| = r$).

Hence the universal localisation of T at the unique rank function is just $S\otimes_k k\langle X\rangle$ where $|X| = n^2(\frac{m-1}{m})$. Therefore, the universal localisation of $\begin{pmatrix} S & S\otimes_k S_1 \\ 0 & S_1 \end{pmatrix}$ at the rank function ρ must be $M_2(S\otimes_k k\langle X\rangle)$. So $S\otimes_k k\langle X\rangle \cong S \underset{k}{\circ} S_1$.

At this point, we can use our methods in a rather more complicated way in order to investigate simple artinian coproducts of the form $M_m(k) \underset{k}{\circ} M_n(k)$; we should like to show that these are always suitably sized matrices over a free skew field on suitably many generators. We are not able to deal with this degree of generality, but there are a number of special cases of interest, which we can settle. Of course, if $m|n$, our last theorem applies and this allows us to reduce to the case where m and n are co-prime in the following way.

Let $p = \text{h.c.f.}\{m,n\}$; then

$$M_m(k) \circ_k M_n(k) = M_m(k) \underset{M_p(k)}{\circ} (M_p(k) \circ_k M_n(k))$$

$$= M_m(k) \underset{M_p(k)}{\circ} M_n(k\langle X\rangle),$$

where $X = n^2(\frac{p^2-1}{p^2})$.

In turn, $M_m(k) \underset{M_p(k)}{\circ} M_n(k\langle X\rangle)$ is isomorphic to

$M_p(M_{\bar{m}}(k) \circ_k M_{\bar{n}}(k\langle X\rangle))$ where $\bar{m} = m/p$, $\bar{n} = n/p$ and so $\bar{m}$ and $\bar{n}$ are coprime.

If $M_{\bar{m}}(k) \circ_k M_{\bar{n}}(k) = M_{\bar{m}\bar{n}}(k\langle Y\rangle)$ for some set Y, we find that

$$\begin{aligned} M_{\bar{m}}(k) \circ_k M_{\bar{n}}(k\langle X\rangle) &= M_{\bar{m}}(k) \circ_k M_{\bar{n}}(k) \underset{M_{\bar{n}}(k)}{\circ} M_{\bar{n}}(k\langle X\rangle) \\ &= M_{\bar{m}\bar{n}}(k\langle Y\rangle) \underset{M_{\bar{n}}(k)}{\circ} M_n(k\langle X\rangle) \\ &= M_{\bar{n}}(M_{\bar{m}}(k\langle Y\rangle) \circ_k k\langle X\rangle) \end{aligned}$$

and it is an easy matter to check that this is the $\bar{m}\bar{n}$ by $\bar{m}\bar{n}$ matrices over a free skew field. In order to attack these problems, the methods of this chapter suggest that we consider the ring $\begin{pmatrix} M_m(k) & M_m(k)\otimes_k M_n(k) \\ 0 & M_n(k)\end{pmatrix}$ which is a finite-dimensional hereditary k-algebra. We shall use theorem 13.7 of the last chapter to attack this ring.

We recall that the epimorphism from a finite-dimensional hereditary algebra R associated to a pre-projective or pre-injective right module M is always a universal localisation.

For our purposes, we take R to be the ring $\begin{pmatrix} k & k^s \\ 0 & k\end{pmatrix}$ where $s \geq 2$.

We wish to find the rank function associated to a particular pre-injective or pre-projective module M. It is clear that this is given by

$$\rho(\begin{pmatrix}1&0\\0&0\end{pmatrix}R) = [M\begin{pmatrix}1&0\\0&0\end{pmatrix}:k]/[M:k]$$

$$\rho(\begin{pmatrix}0&0\\0&1\end{pmatrix}R) = [M\begin{pmatrix}0&0\\0&1\end{pmatrix}:k]/[M:k].$$

From Dlab, Ringel (28), we find that the possible rank functions associated to pre-projective or pre-injective modules over the ring R are just multiples of the positive roots associated to the bilinear form on the 2-dimensional vector space given by $B(\underline{x},\underline{y}) = x_1y_1 + x_2y_2 - \frac{s}{2}(x_1y_2 + x_2y_1)$. These are generated in the following way; begin with the pair (0,1); then our inductive procedure is to pass from the pair (a,b) to the pair (b,sb-a); finally, after we have constructed all such pairs (a,b), all of which satisfy $a < b$, it is clear that all pairs (b,a) where (a,b) is a positive root are positive roots; these are all of them. We note that a and b are coprime. If (a,b) is a particular positive root associated with the module M, $[M:k] = a + b$, $\rho(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}R) = \frac{a}{a+b}$, $\rho(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}R) = \frac{b}{a+b}$. This implies that we have an epimorphism:

$$\begin{pmatrix}k & k^s\\ 0 & k\end{pmatrix} \longrightarrow M_{(a+b)}(k)$$

which must be a universal localisation at ρ, where

$$\rho\begin{pmatrix}k & k^s\\ 0 & 0\end{pmatrix} = \frac{a}{a+b}, \quad \rho\begin{pmatrix}0 & 0\\ 0 & k\end{pmatrix} = \frac{b}{a+b}.$$

By Morita equivalence, we have an epimorphism from

$\begin{pmatrix}M_b(k) & M_b(k)S^s M_a(k)\\ 0 & M_a(k)\end{pmatrix}$ to $M_{2ab}(k)$, which must be a universal localisation.

(S is the simple left $M_b(k)$, right $M_a(k)$ bimodule). The rank function associated to this universal localisation is $\rho\begin{pmatrix}M_b(k) & S^s\\ 0 & 0\end{pmatrix} = \frac{1}{2}$,

$$\rho\begin{pmatrix}0 & 0\\ 0 & M_a(k)\end{pmatrix} = \frac{1}{2}.$$

Consequently, we consider the ring $\begin{pmatrix}M_b(k) & M_b(k)\otimes_k M_a(k)\\ 0 & M_a(k)\end{pmatrix}$ as

$\begin{pmatrix}M_b(k) & S^s \oplus S^{ab-s}\\ 0 & M_a(k)\end{pmatrix}$. We see from the above that we have a

universal localisation of this which must be $M_2(T)$, where
$T \cong M_{ab}(k)\langle M_{ab}(k)\otimes_{M_b(k)}S^{ab-s}\otimes_{M_a(k)}M_{ab}(k)\rangle$ by theorem 14.1.

A dimension check shows that the bimodule

$$M_{ab}(k)\otimes_{M_b(k)}S^{ab-s}\otimes_{M_a(k)}M_{ab}(k) \cong {}_{M_{ab}(k)}M_{ab}(k)^{ab(ab-s)}_{M_{ab}(k)}.$$

Consequently, $T \cong M_{ab}(k)\otimes_k k\langle X\rangle$, where $|X| = ab(ab-s)$. Therefore, the universal localisation of T at the rank function is $M_{ab}(k\langle X\rangle)$. So $M_2(M_{ab}(k\langle X\rangle))$ is the universal localisation of
$\begin{pmatrix} M_b(k) & M_b(k)\otimes_k M_a(k) \\ 0 & M_a(k)\end{pmatrix}$ at the rank function; but so is
$M_2(M_b(k)\underset{k}{\circ}M_a(k))$, so $M_a(k)\underset{k}{\circ}M_b(k) \cong M_{ab}(k\langle X\rangle)$.

We summarise what we have shown in the next theorem:

Theorem 14.9 Let (a,b) be a positive integral root associated to the bilinear form on two-dimensional space $B(\underline{x},\underline{y})$ where a and b are coprime and $B(\underline{x},\underline{y}) = x_1y_1 + x_2y_2 - \frac{s}{2}(x_1y_2 + x_2y_1)$. Then

$$M_a(k)\underset{k}{\circ}M_b(k) \cong M_{ab}(k\langle X\rangle),$$

where $|X| = ab(ab-s)$.

It is unclear to the author whether $M_m(k)\underset{k}{\circ}M_n(k)$ is always isomorphic to a suitably sized matrix ring over a free skew field; however, if this is false, we know that it is a matrix ring over a skew field, which is not free, but can be embedded in a free skew field which would be of some interest since we have no such examples at present.

Section 15

In this section, we shall round off our present results on the possible isomorphisms of skew field coproducts over k of finite-dimensional division k-algebras by demonstrating a way of telling some of them apart. Our technique is related to that of section 12. We shall consider the universal bimodule of derivations of such a skew field. After we have shown that the enveloping algebra of a skew field coproduct over k of finite-dimensional division k-algebras is a weakly finite hereditary ring with a unique rank function on its finitely generated projective modules that takes values in $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N}$, it will follow that the universal bimodule of derivations of such a skew field is a projective bimodule, and the rank of this projective bimodule is an invariant of the skew field. Since this number can be computed without difficulty from any coproduct representation, this invariant is quite useful.

Theorem 15.1 Let $\{D_i : i = 1 \text{ to } n\}$ be a finite collection of finite-dimensional division k-algebras over k; then the enveloping algebra of $\underset{k}{\circ} D_i$ is a weakly finite hereditary ring with a unique rank function on the finitely generated projective modules taking values in $\frac{1}{n}\mathbb{Z}$ for some n .

Pf In the course of the proof of theorem 10.5, we showed that $\underset{k}{\circ} D_i$ embeds in $M_m(k\langle X\rangle)$ for some set X. So the enveloping algebra of $\underset{k}{\circ} D_i$ emebds in $M_m(k\langle X\rangle)^{\circ} \otimes_k M_m(k\langle X\rangle)$ which we showed was weakly finite in theorem 12.12.

Next, the enveloping algebra of $\underset{k}{\circ} D_i$ is a universal two-sided localisation of $(\underset{k}{\circ} D_i) \otimes_k (\underset{k}{\sqcup} D_i)$, which is isomorphic to

$$\underset{(\underset{k}{\circ} D_i)^{\circ}}{\sqcup} ((\underset{k}{\circ} D_i)^{\circ} \otimes_k D_i)$$

Each ring $(\underset{k}{\circ} D_i)^{\circ} \otimes_k D_i$ is simple artinian, because the centre of $(\underset{k}{\circ} D_i)^{\circ}$ is k or k(t) for transcendental t; so $\underset{(\underset{k}{\circ} D_i)^{\circ}}{\sqcup} (\underset{k}{\circ} D_i)^{\circ} \otimes_k D_i$ is an hereditary ring with a unique rank function taking values in $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N}$.

Since our enveloping algebra is a universal two-sided localisation of this, it must be hereditary by theorem 12.7. Since it is weakly finite, it is possible to show by an adaption of the techniques of chapter 5 of Part I that all projective modules over the enveloping algebra are stably induced from the ring $\underset{(\underset{k}{\circ} D_i)^{\circ}}{\sqcup} (\underset{k}{\circ} D_i)^{\circ} \otimes_k D_i$; that is, if P is a projective module over the enveloping algebra, then for some finite rank free module F, $P \oplus F$ is induced from a projective module over $\underset{(\underset{k}{\circ} D_i)^{\circ}}{\sqcup} ((\underset{k}{\circ} D_i)^{\circ} \otimes_k D_i)$. Moreover, the rank of P is well-defined from such equations.

Once we have this result, we know that the universal bimodule of derivations of $\underset{k}{\circ} D_i$ over k is a projective bimodule, since it is a subbimodule of a free bimodule; by theorem 12.1, the following sequence is exact:

$$0 \longrightarrow \Omega_k(F) \longrightarrow F \otimes_k F \longrightarrow F \longrightarrow 0$$

for any skew field F (or ring).

Therefore, we may associate to the skew field $\underset{k}{\circ} D_i$ the rank of $\Omega_k(\underset{k}{\circ} D_i)$ as a projective $\underset{k}{\circ} D_i$ bimodule. This number is an invariant of $\underset{k}{\circ} D_i$. The next result gives a simple formula for this number.

<u>Theorem 15.2</u> $\Omega_k(\circ D_i) \cong \overset{n}{\underset{i=1}{\oplus}} {}^{\otimes}\Omega_k(D_i)^{\otimes}$. Consequently, the rank of $\Omega_k(\underset{k}{\circ} D_i)$ is equal to $\sum_{i=1}^{n} \left(\frac{m_i - 1}{m_i}\right)$ where $m_i = [D_i : k]$.

<u>Pf</u> $\underset{k}{\circ} D_i$ is the universal two-sided localisation of $\underset{k}{\sqcup} D_i$. So $\Omega_k(\circ D_i) \cong {}^{\otimes}\Omega_k(\sqcup D_i)^{\otimes}$ by theorem 12.6. By theorem 12.4,

$$\Omega_k(\underset{k}{\sqcup} D_i) \cong \overset{n}{\underset{i=1}{\oplus}} {}^{\otimes}\Omega_k(D_i)^{\otimes} .$$

Thus $\Omega_k(\underset{k}{\circ} D_i) \cong \oplus {}^{\otimes}\Omega_k(D_i)^{\otimes}$.

Next, we have the exact sequence of D_i-bimodules

$$0 \longrightarrow \Omega_k(D_i) \longrightarrow D_i \otimes_k D_i \longrightarrow D_i \longrightarrow 0$$

Tensoring on the left over D_i with $\underset{k}{\circ} D_i$ gives the exact sequence of left $\underset{k}{\circ} D_i$, right D_i bimodules

$$0 \longrightarrow (\underset{k}{\circ} D_i) \otimes_{D_i} \Omega_k(D_i) \longrightarrow (\underset{k}{\circ} D_i) \otimes_k D_i \longrightarrow \underset{k}{\circ} D_i \longrightarrow 0$$

Since $(\underset{k}{\circ} D_i)^{\circ} \otimes_k D_i$ is simple artinian, this sequence is split exact; the middle module is free of rank 1, the right module has rank $1/m_i$; so $(\underset{k}{\circ} D_i) \otimes_{D_i} \Omega_k(D_i)$ has rank $\frac{m_i - 1}{m_i}$. In fact, ${}^{m_i}((\underset{k}{\circ} D_i) \otimes_{D_i} \Omega_k(D_i))$ is a free module of rank $(m_i - 1)$.

So the $\underset{k}{\circ} D_i$ bimodule ${}^{\otimes}\Omega_k(D_i)^{\otimes}$ must have rank $\frac{m_i - 1}{m_i}$, and the rank of $\Omega_k(\underset{k}{\circ} D_i)$ must be $\sum_{i=1}^{n} (\frac{m_i - 1}{m_i})$, as we wished to show.

The interested reader may construct many particular non-isomorphism theorems from this result. We content ourselves here by finding examples that it does not help us with.

From the equation $\frac{2}{3} + \frac{3}{4} = \frac{1}{2} + \frac{11}{12}$, we construct our first example. Let $E \supset k$ be a cubic extension and $F \supset k$ a biquadratic extension; that is, $[E:k] = 3$, $[F:k] = 4$, where $F \supsetneq F_1, F_2, F_3 \supsetneq k$ and $[F_i:k] = 2$.

We know from theorem 13.6 that $(E \otimes_k F) \underset{k}{\circ} F_i$ is independent of the extension F_i taken. However, is $(E \otimes_k F) \underset{k}{\circ} F_i$ isomorphic to $E \underset{k}{\circ} F$? The rank of the univesal bimodule of derivations of both is the same. Moreover, they have the same finite-dimensional commutative subfields as one check using the results of section 4.

We recall from section 4 that we found a pair of seven-dimensional extensions $E, E' \supset k$ such that E embeds in $E' \underset{k}{\circ} E'$ and E' in $E \underset{k}{\circ} E$. Is $E \underset{k}{\circ} E$ isomorphic to $E' \underset{k}{\circ} E'$?

These two questions are, at present, unapproachable. An answer in either case would presumably lead to a significant advance in our understanding. My hope is that they will be isomorphic.

Part III

Amitsur (1) constructed the generic splitting field of a central simple k-algebra and later Roquette (43) showed a construction of the generic splitting fields using the Galois cohomology of the projective general linear group. In fact, he was working on Brauer-Severi varieties, and this was made more explicit and generalised by Grothendieck (32) who constructed the Brauer-Severi scheme associated to a sheaf of Azumaya algebras on a scheme. He used the Čech cohomology of the projective general linear group.

The Brauer-Severi variety provides a generic solution to the problem of when $A \otimes_k K \cong M_n(K)$; it seemed that the following related problem ought to have a similar generic solution. The problem is: for what fields K is $A \otimes_k K \cong M_m(A')$ for some central simple K-algebra A' ? There are several generic solutions to this problem, each of which has some individual interest. All arise as twisted forms of homogeneous spaces for the projective general linear group in the way that the Brauer-Severi schemes appear. We present two such solutions one affine, the other projective; the affine solution is to consider the twisted forms of the homogeneous spaces $PGL_{n/m} \backslash PGL_n$; the projective solution arises by considering twisted forms of suitable Grassmannian spaces.

Since the theory is no harder to develop for sheaves of Azumaya algebras on schemes than for central simple algebras over a field k, we shall work in that generality. We shall use the language of schemes throughout. First, however, we summarise what we shall prove simply in the case of a central simple algebra A over a field k.

Suppose that $[A:k] = n^2$; then $A \otimes_k K \cong M_m(A')$ for some central simple K-algebra A' if and only if $A \otimes_k K$ has a right ideal of dimension n^2/m over A. This suggests that for our projective solution, we should take the space of all right ideals of dimension n^2/m over A.

We show that the function field of this variety is a generic m-splitting field of A; it is also easy to see that on extending scalars to a splitting field of A, this variety simply becomes the Grassmannian space of n/m-dimensional subspaces of n-dimensional space.

Our affine solution is simply to consider the space of homomorphisms from $M_m(-)$ to $A\otimes_k -$; alternatively, we look at the commutative ring R with a universal R-homomorphism $M_m(R) \longrightarrow A\otimes_k R$.

Finally, we note that these varieties associated to central simple algebras over algebraic number fields satisfy the Hasse principle.

We now return to the general treatment. We give a brief summary of notation.

For a scheme S, O_S is the structure sheaf on S. Given an object A over S (scheme, coherent module, sheaf of algebras etc.) an object B of similar type is said to be a twisted form of A split by the faithfully flat cover $T \longrightarrow S$ if the pullback of A to T is isomorphic to the pullback of B to T. This pullback is denoted by $A|T$. We recall from Grothendieck (35) that the twisted forms of A split by $T \longrightarrow S$ each give rise to a unique cohomology class in $\check{H}^1(T/S, \mathrm{Aut}(A)|T)$, where Aut(A) is the sheaf of automorphisms of A and $\check{H}^1(T/S,-)$ is the first Čech cohomology set with respect to the faithfully flat map $T \longrightarrow S$. A T'-point in T for a pair of S-schemes $T' \longrightarrow S$, $T \longrightarrow S$ is an S-scheme map $T' \longrightarrow T$.

Let S be a quasicompact separated scheme. We may assume that it is connected, since the techniques of the following may be applied to each of its components. Every locally free sheaf on S, therefore, has a well-defined rank.

Let A be an Azumaya algebra over S of rank n^2. Then for any vector bundle, E, of rank n on S, A is a twisted form of End(E) with respect to the étale or flat topology. Let T be a faithfully flat cover of S such that $A|T = \mathrm{End}(E)|T$. The automorphisms of $\mathrm{End}(E)$, $P\mathrm{Aut}(E)$ is the sheaf associated to the pre-sheaf $U \longrightarrow \mathrm{Aut}(E)|U/\mathcal{O}_U^*$ for affine U. By the theory of descent (Grothendieck (35) Section 4), A determines uniquely an element α of $\check{H}^1(T/S, \mathrm{Aut}(E)|T)$. Throughout this part, we shall use these names without further explanation.

We begin by constructing an affine solution to our general problem.

Suppose that $E \cong E_1 \otimes E_2$, where E_1 and E_2 are vector bundles on S of rank m, n/m respectively. Then $\mathrm{End}(E)$ is isomorphic to $\mathrm{End}(E_1) \otimes \mathrm{End}(E_2)$. The embedding of $\mathrm{End}(E_2)$ in $\mathrm{End}(E)$ induces an embedding of $P\mathrm{Aut}(E_2)$ in $P\mathrm{Aut}(E)$ and we may construct the sheaf associated to the pre-sheaf given by $P\mathrm{Aut}(E_2) \backslash P\mathrm{Aut}(E)$, which is an homogeneous space for $P\mathrm{Aut}(E)$. The embedding of $P\mathrm{Aut}(E)$ in the sheaf of automorphisms of this space induces a map from $\check{H}^1(T/S, \mathrm{Aut}(E)|T)$ to $H^1(T/S, \mathrm{Aut}(P\mathrm{Aut}(E_2) \backslash P\mathrm{Aut}(E)))$. Let $\bar{\alpha}$ be the image of α; we wish to determine whether $\bar{\alpha}$ arises from a twisted form of $P\mathrm{Aut}(E_2) \backslash P\mathrm{Aut}(E)$, and if it does, we wish to find what functor this scheme represents. Before doing this, we note that $\mathrm{End}(E)$ becomes $M_n(\mathcal{O}_{S'})$ for a suitable Zariski cover S' of S, and $P\mathrm{Aut}(E_2) \backslash P\mathrm{Aut}(E)$ becomes $PGL_{n/m}(\mathcal{O}_{S'}) \backslash PGL_n(\mathcal{O}_{S'})$; so we are constructing twisted forms of homogeneous spaces for the projective general linear group.

<u>Theorem 1</u> $P\mathrm{Aut}(E_2) \backslash P\mathrm{Aut}(E)$ represents the functor $S_1 \longmapsto \mathrm{Hom}_{S\text{-algebras}}(\mathrm{End}(E_1)\ S_1, \mathrm{End}(E)\ S_1)$. Consequently, $\bar{\alpha}$ arises from a twisted form of $P\mathrm{Aut}(E_2) \backslash P\mathrm{Aut}(E)$ which must represent the functor $S_1 \longmapsto \mathrm{Hom}_{S\text{-algebras}}(\mathrm{End}(E_1)|S_1, A|S_1)$.

<u>Pf</u> Consider a homomorphism from $\mathrm{End}(E_1)|S_1$ to $\mathrm{End}(E)|S_1$. By the Noether-Skolem theorem for Azumaya algebras (Grothendieck (82)I.5.10),

there is a Zariski cover S_1' of S_1 on which this homomorphism arises by conjugation from the embedding of $\mathrm{End}(E_1)$ given by $\mathrm{End}(E) = \mathrm{End}(E_1) \otimes \mathrm{End}(E_2)$. Further, if the homomorphism is given by conjugation by x_1' on S_1' and by x_1'' on S_1'', then $(x_1')^{-1}x_1''$ defines an invertible element in $\mathrm{End}(E_2)$ on $S_1' \times_{S_1} S_1''$, since $\mathrm{End}(E_2)$ is the centraliser of $\mathrm{End}(E_1)$ in $\mathrm{End}(E)$. So certainly a homomorphism determines uniquely an element of $\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)$. The converse follows, since an element of $\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)$ on a suitable Zariski cover determines a right coset of the automorphisms of E_2 in the automorphisms of E. This coset determines a unique embedding of $\mathrm{End}(E_1)$ in $\mathrm{End}(E)$.

The action of $\rho\mathrm{Aut}(E)$ on $\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)$ given by right multiplication is induced by the action of $\rho\mathrm{Aut}(E)$ on $\mathrm{Hom}_{S\text{-algebras}}(\mathrm{End}(E_1),\mathrm{End}(E))$ as the automorphisms of $\mathrm{End}(E)$.

We noted before the statement of the theorem that $\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)$ is a twisted form with respect to the Zariski topology of $PGL_{n/m}(\mathcal{O}_S)\backslash PGL_n(\mathcal{O}_S)$; this last scheme is affine over S; therefore $\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)$ is affine and by theorem 2 of (Grothendieck ( )) all cohomology classes of $H^1(T/S,\mathrm{Aut}(\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)))$ arise from twisted forms of $\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)$.

Since $\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)$ represents the functor $\mathrm{Hom}_{S\text{-algebras}}(\mathrm{End}(E_1),\mathrm{End}(E))$ and the action of $\rho\mathrm{Aut}(E)$ is induced by the automorphisms of $\mathrm{End}(E)$, the twisted form of $\rho\mathrm{Aut}(E_2)\backslash\rho\mathrm{Aut}(E)$ corresponding to $\bar{\alpha}$ must represent the functor $\mathrm{Hom}_{S\text{-algebras}}(\mathrm{End}(E_1),A)$.

This has the following consequence:

Theorem 2 If $A|T' = \mathrm{End}(E') | A'$ for some scheme T' over S, where E' is a vector bundle of rank m over T' then the twisted form of $\rho\mathrm{Aut}(E_2)\backslash \rho\mathrm{Aut}(E)$ corresponding to A has a T''-point for some Zariski cover T'' of T'.

Pf There is a Zariski cover T'' of T' on which $E'|T'' = E_1|T''$ and so, if $A|T' = \mathrm{End}(E') | A'$, then $A|T'' = \mathrm{End}(E_1)\otimes A'|T''$ and, by the fact that our twisted form represents $\mathrm{Hom}(\mathrm{End}(E_1), A)$, T'' must have a point in the twisted form.

In the special case where $E_1 = E$, our twisted form becomes simply the left principal homogeneous space for $\rho\mathrm{Aut}(E)$ with respect to .

We also consider the case where $S = \mathrm{spec}(k)$ for a field k. Our scheme representing $R \longrightarrow \mathrm{Hom}(M_m(R), A\otimes_k R)$ is the twisted form of the homogeneous space $PGL_{n/m}\backslash PGL_n$ corresponding to A. Since $PGL_{n/m}\backslash PGL_n$ is affine and geometrically irreducible, so is the twisted form. The co-ordinate ring of the variety is the generic total m-splitting ring of A; and the function field is the generic m-splitting field of A. We call the co-ordinate ring $R(A,m)$ a <u>total m-splitting ring</u>, since $A\otimes_k R(A,m) = M_m(A')$, whereas a ring R is an <u>m-splitting ring</u> if $A\otimes_k R = \mathrm{End}(P\otimes_R A')$ for a projective module P of rank m, which is a weaker statement. It is clear that a field $K \supset k$ is an m-splitting field if and only if $R(A,m)$ has a homomorphism to it, or, equivalently, if there is a specialisation from our generic m-splitting field to it.

Our next aim is to generalise the Brauer-Severi varieties. These arise as twisted forms of projective space; closely related spaces are the Grassmannian varieties which are also homogeneous spaces for the projective general linear group. Their twisted forms again provide us with a generic solution to our problem.

Let E be the dual vector bundle to E; so it is the sheaf

$Hom(E,\mathcal{O}_S)$. We recall from Kleiman (34) that the Grassmannian space $Gr_{n/m}(\check{E})$ over S represents the functor $T' \longmapsto$ the set of local direct summands of rank n/m of $E|T'$ as $\mathcal{O}_{T'}$-modules. We see that $\rho Aut(E)$ acts naturally on this space and that this action makes $Gr_{n/m}(E)$ a homogeneous space for $\rho Aut(E)$. If $\bar{\alpha}$ is the image of α in $\check{H}^1(T/S, Aut(Gr_{n/m}(\check{E})))$ induced by the embedding of $\rho Aut(E)$ in $Aut(Gr_{n/m}(E))$, we wish to know whether $\bar{\alpha}$ comes from a twisted form of $Gr_{n/m}(\check{E})$ and, if it does, what functor this space represents. First, we rephrase what functor $Gr_{n/m}(\check{E})$ represents :

Using the Morita equivalence of $\mathcal{O}_S$ and End(E), we find that the set of rank n/m local direct summands of E is in 1-1 correspondence with the set of rank n^2/m local direct summands as $\mathcal{O}_S$-modules of End(E), that are left End(E)-modules. So $Gr_{n/m}(\check{E})$ represents the functor $T' \longmapsto$ the set of local direct summands of End(E) of rank n^2/m as $\mathcal{O}_{T'}$-modules that are left End(E)-modules. The action of $\rho Aut(E)$ on the functor is given by $M \longmapsto Mx$ for $x \in Aut(E)$ and $M \in End(E)$. Since $M^x = x^{-1}Mx = Mx$, we see that the action of $\rho Aut(E)$ induced by its action as the automorphisms of the algebra End(E) is just the natural action of $\rho Aut(E)$ on $Gr_{n/m}(\check{E})$.

<u>Theorem 3</u> Let $\bar{\alpha}$ be the image of the cohomology class $\alpha \in H^1(T/S, \rho Aut(E))$ in $H^1(T/S, Aut(Gr_{n/m}(\check{E})))$. $\bar{\alpha}$ arises from a twisted form of $Gr_{n/m}(\check{E})$ and this twisted form represents the functor that assigns to $T' \longrightarrow S$, the set of left A-module direct summands of $\mathcal{O}_{T'}$-rank n^2/m of $A|T'$.

<u>Pf</u> We shall show that this functor in the last sentence of the theorem is representable. It is then clear that it is a twisted form of $Gr_{n/m}(\check{E})$ and must correspond to the element $\bar{\alpha}$ in $H^1(T/S, Aut(Gr_{n/m}(\check{E})))$, since the action of $\rho Aut(E)$ on $Gr_{n/m}(E)$ is that induced by $\rho Aut(E)$ on End(E).

End(E) is a rank n^2 vector bundle on S; so the functor that assigns to $T' \longrightarrow S$, the set of rank n^2/m local direct summands of $\mathrm{End}(E)|T'$ is represented by $Gr_{n^2/m}(\breve{\mathrm{End}}(E))$, where $\breve{\mathrm{End}}(E)$ is the dual vector bundle to $\mathrm{End}(E)$; that an $\mathcal{O}_{T'}$-submodule should be a left $\mathrm{End}(E)$-submodule is given by a finite number of equations, so our functor is representable by a closed subscheme of $Gr_{n^2/m}(\breve{\mathrm{End}}(E))$.

In the case where $n = m$, this gives a construction for the Brauer-Severi schemes.

In the case where $S = \mathrm{spec}(k)$ for some field k, and A is a central simple k-algebra, we find that for a field $L \supset k$, $L \otimes_k A \cong M_m(A')$ for some central simple K-algebra A' if and only if the twisted form of Grassmannian space constructed in theorem 3 has a K-point.

We have constructed both an affine and a projective solution to our basic problem. Presumably, other homogeneous spaces have twisted forms that represent interesting functors associated to A. We give a couple of examples without proof (the reader will find no difficulty in constructing proofs in the manner of the foregoing work).

Let $[E;k] = m$, where E is a commutative field extension of k; the centraliser of E in $M_n(k)$ is $M_{n/m}(k)$; if $\alpha \in H^1(L/k, M_n(k))$ corresponding to a central simple algebra A split by L, then the twisted form of $PGL_{n/m}(E) \backslash PGL_n(k)$ corresponding to α represents the R-algebra embeddings of $E \otimes_k R$ in $A \otimes_k R$. So E embeds in A if and only if this twisted form has a k-point. This twisted form is an affine scheme.

Again, let B be a Borel subgroup of $PGL_n(k)$; the twisted form corresponding to α of $B \backslash PGL_n(k)$ represents the maximal tori in the algebraic group $R^* \backslash (A \otimes_k R)^*$ (by which we mean the sheaf associated to the pre-sheaf on the Zariski topology on $\mathrm{spec}(R)$). So K has a point in the variety if and only if it splits A. The function field of this variety is a generic splitting field of A of transcendence degree $\frac{1}{2}n(n-1)$.

We return to the examples investigated in theorems 1 and 3 in the case where A is a central simple algebra over an algebraic number field in order to demonstrate the Hasse principle for these varieties.

Theorem 4 Let A be a central simple algebra of dimension n^2 over an algebraic number field k; let V be a variety defined over k such that V has a K-point if and only if $A \otimes_k K \cong M_m(A')$ for some central simple K-algebra A'; then V satisfies the Hasse principle.

Pf V has a k-point if and only if $A \cong M_m(A')$ where A' is a central simple k-algebra. Since the order of a central simple algebra equals its index, this is equivalent to its order dividing n/m.

We have an embedding given by class field theory: $Br(k) \longrightarrow \bigoplus_p Br(k_p)$ as p ranges over all finite and infinite primes. This shows that our condition is equivalent to the statement that the order of $A \otimes_k k_p$ divides n/m for all primes p; since the order is equal to the index for all central simple algebras over local fields, this is equivalent to $A \otimes_k k_p \cong M_m(B_p)$ for some central simple k_p-algebra B_p for all primes p. By the condition on V, this is equivalent to V having a k_p-point for all primes p, which proves the Hasse principle for V.

This is of most interest for the twisted forms of Grassmannian space constructed in theorem 3, since these are complete varieties.

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