Phase Transitions In Relativistic Systems

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Abstract

The BCS free energy for $^3P_2$ paired neutron matter is derived taking account of relativistic effects. It is found that the values taken by the Ginzburg-Landau parameters are always in the region of the phase diagram correponding to a unitary phase.

Phase transitions in the early universe are also discussed with inclusion of the effects of Higgs scalar chemical potentials as well as fermionic chemical potentials. The conditions for equilibrium, and the critical density to prevent symmetry restoration at high temperatures are studied. It is observed that the decay of pre-existing Higgs scalar asymmetries could greatly reduce baryon number and lepton number to entropy ratios from their initial values.

Phase transitions in supersymmetric theories and the phenomenom of symmetry anti-restoration in a supersymmetric model with a U(1) gauge symmetry are studied at finite density.

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PART I: Neutron Star Matter

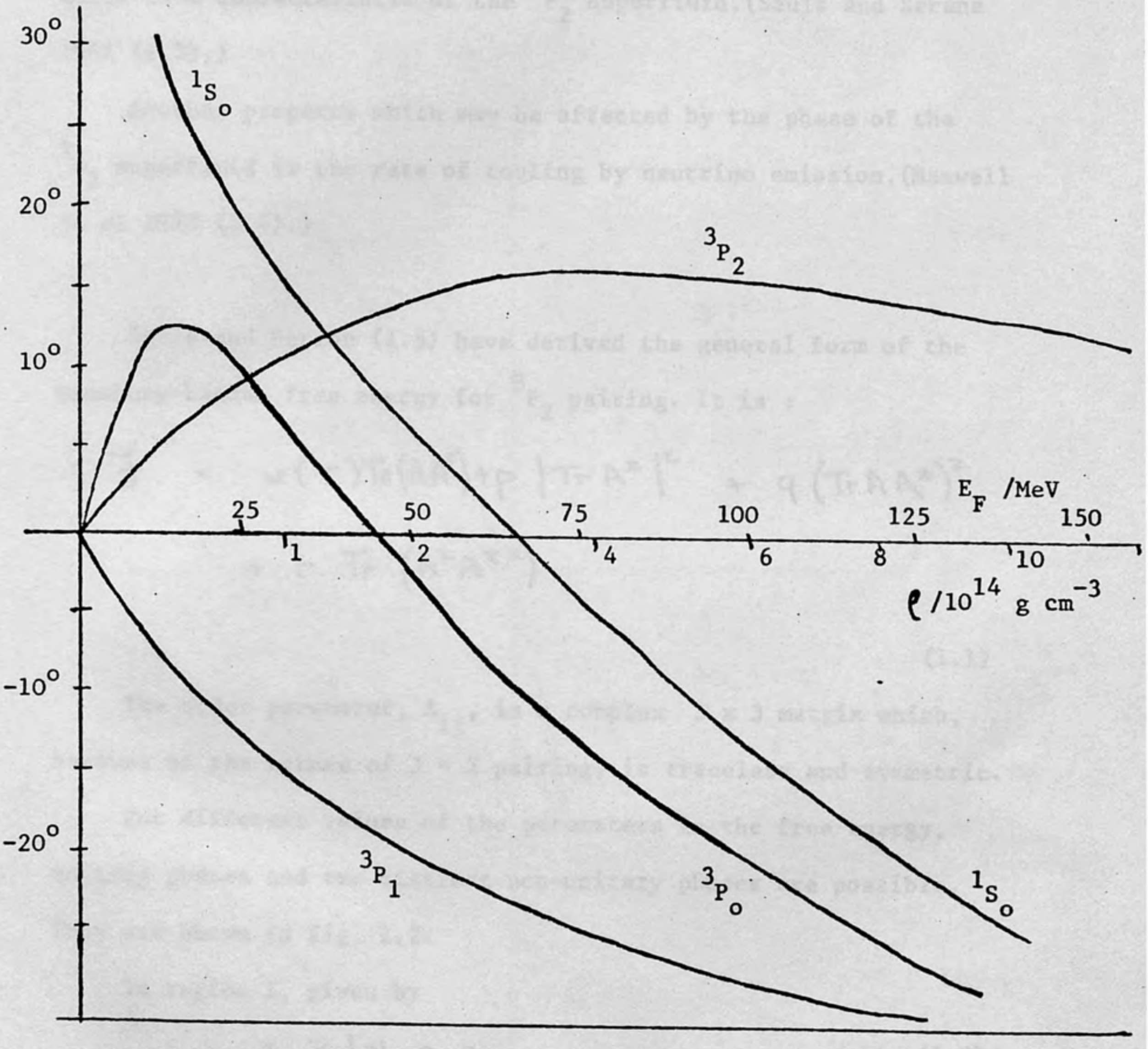
The derivation of gap equations and Ginsburg-Landau free energies is reviewed. The case of superfluid neutron matter is described in detail.

Chapter 1: Introduction

In this section the phase diagram for a $^3P_2$ paired neutron superfluid is examined, taking into account relativistic effects. Such a superfluid is thought to exist within the cores of neutron stars where the density is in the range 5×10^{13} g cm^{-3} $< \rho <$ 6×10^{14} g cm^{-3} (about 1/3 to 4 times the density of neutrons in a nucleus).

The attraction between appropriately paired neutrons at the top of their Fermi sea causes a BCS type superfluid behaviour. The particular type of superfluid is very dependant upon density: the Fermi energy, E_F , is proportional to $\rho^{2/3}$ and the known neutron - neutron phase shifts (fig 1.1) depend strongly upon energy. At densities lower than 1.5×10^{14} g cm^{-3} (about 50 MeV) the dominant interaction is the attractive $^1S_0$ one, which leads to a convention -ally Cooper paired superfluid state. At greater densities than 1.5×10^{14} g cm^{-3}, however, the $^1S_0$ interaction becomes repulsive due to the repulsive core, whereas the ^3P_J effective interactions turn out to be strong. Owing to a short range negative spin-orbit force the $^3P_2$ interaction is attractive, whereas the $^3P_{J=0,1}$ inter -actions are repulsive at high energies. Hence the ordinary $^1S_0$ pairing disappears and the significant attraction is in the $^3P_2$ state. (Hoffberg et al 1970 (1.1), Tamagaki 1970 (1.2).) As we shall see, this $^3P_2$ superfluid, which we expect to be found in the cores of neutron stars, can exist in one of three separate phases. It is important to determine which of these superfluid phases is selected as the anistropic $^3P_2$ paired superfluid can affect the observable properties of a neutron star. One such observable is the relaxation time for the transfer of momentum between the interior and the surface of the star through interactions of elect -rons with vertex cores. The $^3P_2$ superfluid is threaded by an

Figure 1.1: Nucleon-nucleon scattering phase shifts versus Fermi energy, E_F, and density, ρ.



array of quantized vertices; after a discontinuous change in the rotational speed of the star (a glitch) angular momentum is transferred to the superfluid via electrons scattering off vortices. The relaxation time for this process depends strongly upon the gap, which is a characteristic of the $^3P_2$ superfluid. (Sauls and Serene 1981 (1.3).)

Another property which may be affected by the phase of the $^3P_2$ superfluid is the rate of cooling by neutrino emission. (Maxwell et al 1978 (1.4).)

Sauls and Serene (1.5) have derived the general form of the Ginzburg-Landau free energy for $^3P_2$ pairing. It is :

$$\mathcal{F} = \alpha(T)\,\mathrm{Tr}(AA^*) + p\,|\mathrm{Tr}\,A^2|^2 + q\,(\mathrm{Tr}\,AA^*)^2 + r\,\mathrm{Tr}\,(A^2A^{*2}) \tag{1.1}$$

The order parameter, A_{ij}, is a complex 3 x 3 matrix which, because of the nature of J = 2 pairing, is traceless and symmetric.

For different values of the parameters in the free energy, unitary phases and two distinct non-unitary phases are possible. They are shown in fig. 1.2.

In region I, given by

$$r > |p| - p \tag{1.2}$$

the order parameter is unique and non-unitary.

In region II, given by

$$0 > r > -6p \tag{1.3}$$

the order parameter is again unique and non- unitary.

Region III is that in which

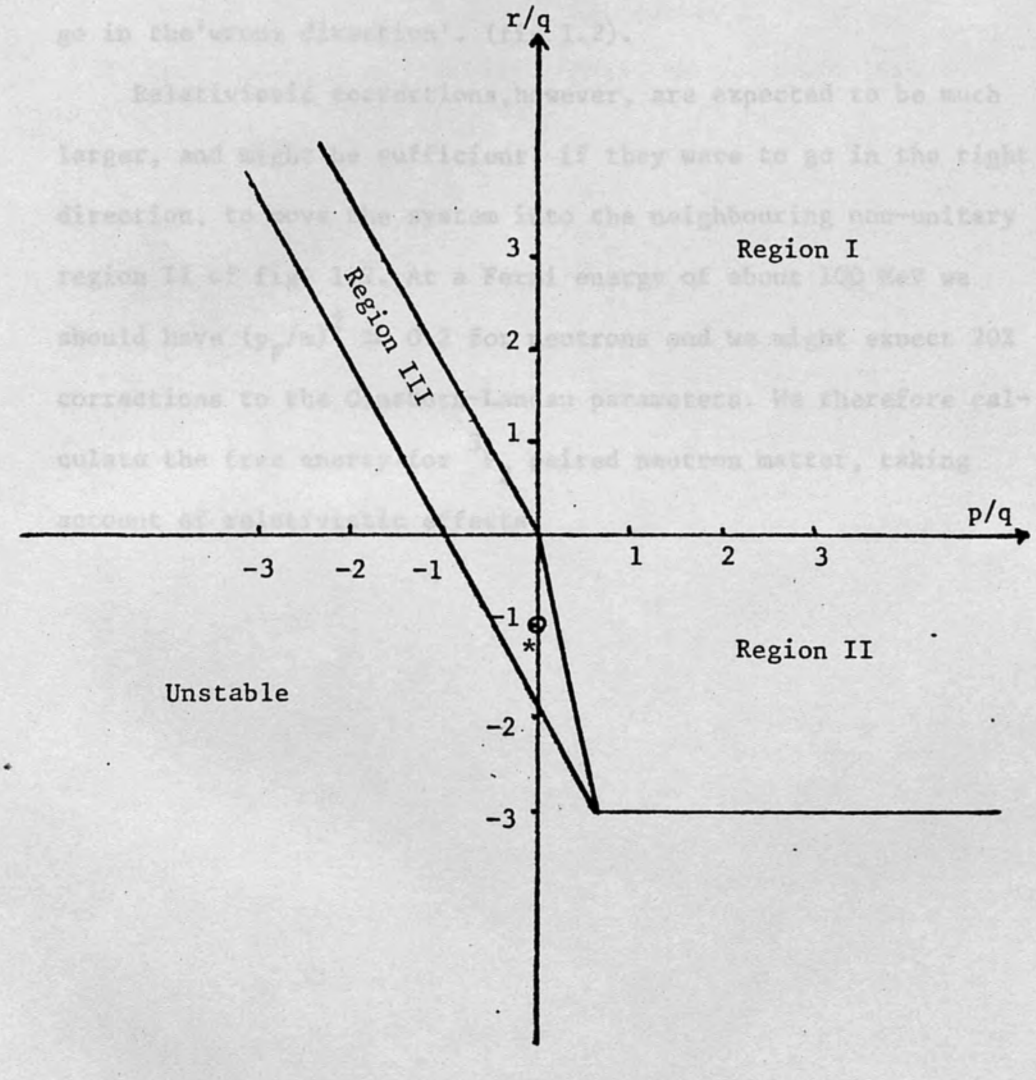
$$r < -4p - 2|p| \tag{1.4}$$

In this region the Ginzburg-Landau functional is minimized by any real, traceless, symmetric A_{ij}

Figure 1.2: Phase diagram for the $^3P_2$ neutron superfluid.

The BCS (non-relativistic limit) point is indicated by O

The strong coupling point by *.



The BCS point falls in region III and is given by

$$p = 0 \ , \quad q = -r \tag{1.5}$$

Sauls and Serene (1.5) investigated the possibility that strong coupling corrections might instead select one of the non-unitary phases, but found the corrections to be too small and to go in the 'wrong direction'. (fig 1.2).

Relativistic corrections, however, are expected to be much larger, and might be sufficient, if they were to go in the right direction, to move the system into the neighbouring non-unitary region II of fig. 1.2. At a Fermi energy of about 100 MeV we should have $(p_F/m)^2 \simeq 0.2$ for neutrons and we might expect 20% corrections to the Ginsburg-Landau parameters. We therefore calculate the free energy for ${}^3P_2$ paired neutron matter, taking account of relativistic effects.

Chapter 1 : References.

1.1 Hoffberg et al (1970) Phys.Rev.Lett. 24 775

1.2 Tamagaki (1970) Prog.Theor.Phys 44 905

1.3 Sauls and Stein (1981) Physica 107B 55

1.4 Maxwell et al (1978) Astrophys. J. 216 77

1.5 Sauls and Serene (1978) Phys. Rev. D17 1524

Chapter 2: The Relativistic Gap Equation

In order to reach our goal, namely the free energy for a $^3P_2$ paired relativistic neutron superfluid, we must find an expression for the relativistic gap matrix. This is the purpose of this chapter; to review the derivation of an equation for the gap matrix for a general relativistic fermion superfluid. The method here described was developed for non-relativistic superfluids by Nambu (2.1) and extended to relativistic systems by Barrois (2.2,2.3) and Bailin and Love (2.4,2.5,2.6). (For a review of non-relativistic fermion superfluids see Leggett (2.7).) The gap equation is derived from the Dyson equation for the proper self-energy of the fermion.

The origin of superfluidity in a fermion system is a non-zero expectation value for a product of two fermion fields, describing Cooper pairing.

This is introduced as an effective Lagrangian term

$$\mathcal{L}_\Delta = \left[\bar{\psi}_c(x) \Delta(x,y) \psi(y) + \text{h.c.} \right] \tag{2.1}$$

where $\psi(x)$ is a relativistic fermion field and $\psi_c(x)$ its charge conjugate field. $\Delta(x,y)$ is the gap matrix: it is a 4 x 4 matrix in spinor indices and may also be a matrix in other indices (for J = 2 pairing Δ is a 3 x 3 matrix, Δ_{ij}). It is this object which we shall calculate self consistently by means of the Dyson equation for the self energy.

The remaining quadratic terms in the Lagrangian are of the form

$$\mathcal{L}_{free} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \ + \mu \bar{\psi}\gamma^0\psi \tag{2.2}$$

assuming Cooper pairing between fermions of equal mass, m, at non-zero density with chemical potential μ .

Following Nambu we write the inverse propagator for the fermions as a 2 x 2 matrix acting upon the column vector

$$\Psi = \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}$$

and transform to momentum space. In the first instance we shall assume that the superfluid is homogenous (ie $\Delta(x,y)$ depends only upon the relative position x -y and not upon the centre of mass co-ordinate x + y). Later we shall extend the discussion to non-homogeneous systems whereupon gradient terms will appear in the Ginzburg-Landau free energy.

We write

$$\Delta(q) = \int d^4z \, e^{iqz} \, \Delta(z) \tag{2.3}$$

where

$$z = x - y \tag{2.4}$$

The momentum space inverse propagator acting upon $\begin{pmatrix} \psi \\ \psi_c \end{pmatrix}$ is

$$S^{-1}(q) = \begin{pmatrix} \not{q} + \not{\mu} - m & \tilde{\Delta}(q) \\ \Delta(q) & \not{q} - \not{\mu} - m \end{pmatrix} \tag{2.5}$$

where

$$\not{\mu} = \mu \gamma^0 \tag{2.6}$$

and

$$\tilde{\Delta}(q) = \gamma^0 \Delta^{\dagger}(q) \gamma^0. \tag{2.7}$$

Let the momentum space propagator acting upon $\begin{pmatrix} \psi \\ \psi_c \end{pmatrix}$ be

$$S(q) = \begin{pmatrix} A(q) & B(q) \\ C(q) & D(q) \end{pmatrix} \tag{2.10}$$

Inverting (2.5) gives

$$C(q) = (\not{q} - \not{\mu} - m)^{-1} \Delta \left[\tilde{\Delta} (\not{q} - \not{\mu} - m)^{-1} \Delta - (\not{q} - \not{\mu} - m) \right]^{-1} \tag{2.11}$$

We will not need other entries.

We assume an interaction Lagrangian

$$\mathcal{L}_{int} = g\bar{\psi}\Gamma^{A}\psi\phi_{A} \tag{2.12}$$

ϕ_A is the field of the exchange field which has a propagator $D_{AB}(k - q)$. A, B denote a set of spin and internal symmetry indices.

In Ψ space

$$\mathcal{L}_{int} = g\bar{\Psi}\begin{pmatrix}\Gamma^{A} & 0 \\ 0 & \tilde{\Gamma}^{A}\end{pmatrix}\Psi\phi_{A} \tag{2.13}$$

where

$$\tilde{\Gamma}^{A} = C\,(\Gamma^{A})^{T}C^{-1} \tag{2.14}$$

and C is the charge conjugation matrix. The proper self energy is separated out by writing

$$S^{-1}(q) = S_0^{-1}(q) - \Sigma(q) \tag{2.15}$$

with

$$S_0^{-1}(q) = \begin{pmatrix}\not{q} + \not{\mu} - m & 0 \\ 0 & \not{q} - \not{\mu} - m\end{pmatrix} \tag{2.16}$$

and the proper self-energy, $\Sigma(q)$, is

$$\Sigma(q) = -\begin{pmatrix}0 & \tilde{\Delta}(q) \\ \Delta(q) & 0\end{pmatrix} \tag{2.17}$$

The Dyson equation then gives, at finite temperature, (fig 2.1)

$$\Sigma(q) = -g^{2}\int d^{3}q\,\frac{1}{\beta}\sum_{n\ \mathrm{odd}} D_{AB}(k-q)\cdot\begin{pmatrix}\Gamma^{A} & 0 \\ 0 & \tilde{\Gamma}^{A}\end{pmatrix}S(q)\begin{pmatrix}\Gamma^{B} & 0 \\ 0 & \tilde{\Gamma}^{B}\end{pmatrix} \tag{2.18}$$

where

$$\beta = (k_B T)^{-1} \tag{2.19}$$

The summation is over Matsubara frequencies

$$\omega_n = n\pi/\beta \tag{2.20}$$

and

$$q = (i\omega_n, \mathbf{q}) \tag{2.21}$$

Using (2.10) and (2.17), (2.18) gives

$$\Delta(k) = g^2 \beta^{-1} \int d^3q \sum_{n\ \text{odd}} D_{AB}(k-q)\, \tilde{\Gamma}^A C(q)\, \Gamma^B \tag{2.22}$$

We now make use of the fact that for any function $f(q_o)$ we can write

$$\beta^{-1} \sum_{n\ \text{odd}} f(i\omega_n) = \frac{i}{4\pi} \oint dq_0\, f(q_0) \tanh\left(\tfrac{1}{2}\beta q_0\right) \tag{2.23}$$

to remove the Matsubara frequency sum. The q_o integration is around a contour which includes the poles of $f(q_o)$ but not those of $\tanh(\frac{1}{2}\beta\ q_o)$.

Then (2.22) becomes

$$\Delta(k) = \tfrac{1}{2} i g^2 \oint d^4q\, D_{AB}(k-q)\, \tilde{\Gamma}^A C(q) \tanh\left(\tfrac{1}{2}\beta q_0\right) \Gamma^B \tag{2.24}$$

which gives

$$\Delta(k) = \tfrac{1}{2} g^2 \int d^3q\ D_{AB}(\underline{k}-\underline{q}) .$$
$$\cdot\ \tilde{\Gamma}^A \left\{ \hat{\Delta}(q) \left[\varepsilon^2 I + \tilde{\Delta}(q) \hat{\Delta}(q) \right]^{-\frac{1}{2}} \tanh \tfrac{1}{2}\beta \left[\varepsilon^2 I + \tilde{\Delta}(q) \Delta(q) \right]^{\frac{1}{2}} \right\} \Gamma^B \tag{2.25}$$

where $$D_{AB}(\underline{k}-\underline{q}) = D_{AB}(k_0 - q_0 = 0,\ \underline{k} - \underline{q}) \tag{2.26}$$

$$\Delta(\underline{k}) = \Delta\left(k_0 = \sqrt{\underline{k}^2 + m^2} - \mu,\ \underline{k}\right) \tag{2.27}$$

$$\hat{\Delta}(\underline{q}) = \frac{1}{4\mu^2} \left(\mu\gamma^0 + \underline{q}.\underline{\gamma} - m\right) \Delta(\underline{q}) \left(\mu\gamma^0 - \underline{q}.\underline{\gamma} + m\right) \tag{2.28}$$

and $$\varepsilon = \sqrt{\underline{q}^2 + m^2} - \mu \tag{2.29}$$

To zeroth order in g^2 we have

$$\mu^2 = p_F^2 + m^2 = E_F^2 \tag{2.30}$$

We have made the assumption that only those momenta close to the Fermi surface are important, via

$$\varepsilon\ ,\ \sqrt{\underline{k}^2 + m^2} - \mu \ll \mu \tag{2.31}$$

The $d\!\!\!{}^{-}{}^3q$ integration is separated into radial and angular parts

$$d\!\!\!{}^{-}{}^3q = \frac{d^3q}{(2\pi)^3} = \frac{1}{2\pi^2}(\varepsilon+\mu)\left[(\varepsilon+\mu)^2-m^2\right]^{1/2} d\varepsilon \frac{d\Omega}{4\pi} \tag{2.32}$$

As in the non-relativistic case (2.7) the ε integration is cut off at $|\varepsilon| = \varepsilon_0$ (in order to approximate the integral) with

$$\beta^{-1} << \varepsilon_0 << \mu \tag{2.33}$$

Then

$$d\!\!\!{}^{-}{}^3q \simeq \frac{1}{2}\frac{dn}{d\varepsilon} d\varepsilon \frac{d\Omega}{4\pi}. \tag{2.34}$$

$$\frac{dn}{d\varepsilon} = \frac{\mu p_F}{\pi^2} \tag{2.35}$$

and is the density of states at the Fermi surface.

We assume that the propagator $D_{AB}(\underline{k} - \underline{q})$ is slowly varying in the sense that its variation is on a scale large compared with β^{-1}, corresponding to the assumption of a short ranged potential in the non-relativistic case. We can then have $\Delta(\underline{k})$ to be a function of $\underline{n}' = \hat{\underline{k}}$ only and $\Delta(\underline{q})$ a function of $\underline{n} = \hat{\underline{q}}$ only.

The gap equation for a homogeneous relativistic superfluid is thus

$$\Delta(\underline{n}') = \frac{1}{4} g^2 \frac{dn}{d\varepsilon}\int d\varepsilon \int \frac{d\Omega}{4\pi} D_{AB}(\underline{n},\underline{n}') \tilde{\Gamma}^A R \Gamma^B \tag{2.36}$$

where

$$R = \hat{\Delta}\left[\varepsilon^2 I + \tilde{\Delta}\hat{\Delta}\right]^{-\frac{1}{2}} \tanh\left[\tfrac{1}{2}\beta(\varepsilon^2 I + \tilde{\Delta}\hat{\Delta})^{\frac{1}{2}}\right] \tag{2.37}$$

$$\tilde{\Delta}(\underline{n}) = \gamma^0 \Delta^{+}(\underline{n})\gamma^0 \tag{2.38}$$

$$\hat{\Delta}(\underline{n}) = \frac{1}{4\mu^2}(\mu\gamma^0 + p_F \underline{n}\cdot\underline{\gamma} - m)\Delta(\mu\gamma^0 - p_F\underline{n}\cdot\gamma^0 + m) \tag{2.39}$$

Gradient Terms.

We must now generalize to the case of an inhomogeneous superfluid, leading us to gradient terms in the gap equation.

$\Delta(x,y)$ now depends on x+y as well as on x-y. We write the Fourier transform as

$$\Delta(x,y) = \int d^4p'\, d^4p\, e^{-ip'x} e^{ipy}\, \Delta(p',p) \tag{2.40}$$

The momentum space inverse propagator acting upon $\Psi = \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}$ is now

$$S^{-1}(p',p) = \begin{pmatrix} (\not{p}+\not{\mu}-m)\,\delta\!\!\!^{-}(p'-p) & \tilde{\Delta}(p',p) \\ \Delta(p',p) & (\not{p}-\not{\mu}-m)\,\delta\!\!\!^{-}(p'-p) \end{pmatrix} \tag{2.41}$$

with $$\delta\!\!\!^{-}(p'-p) = (2\pi)^4\,\delta(p'-p) \tag{2.42}$$

and $$\tilde{\Delta}(p',p) = \gamma^0 \Delta^{\dagger}(p',p)\gamma^0 \tag{2.43}$$

We rewrite (2.10) by

$$S(p',p) = \begin{pmatrix} A(p',p) & B(p',p) \\ C(p',p) & D(p',p) \end{pmatrix} \tag{2.44}$$

Inverting, as before, leads to

$$C(p',p)(\not{p}+\not{\mu}-m) + (\not{p}'-\not{\mu}-m)^{-1}\Delta(p',p)$$
$$-\int d^4p''\int d^4p'''\, C(p',p''')\,\Delta(p''',p'')(\not{p}''-\not{\mu}-m)^{-1}\Delta(p'',p) = 0 \tag{2.45}$$

Since we only need to derive the Ginzburg-Landau free energy in the Ginzburg-Landau region we need only keep spatial derivatives acting upon the lowest order in Δ . We may therefore carry out the inversion to order Δ giving

$$C(p',p) = -(\not{p}'-\not{\mu}-m)^{-1}\Delta(p',p)(\not{p}+\not{\mu}-m)^{-1} \tag{2.46}$$

Order Δ^3 and higher non-gradient terms are evaluated as before.

The Dyson equation for the proper self-energy (fig 2.2) gives

$$\Delta(p',p) = \tfrac{1}{2}ig^2\oint d^4q\, D_{AB}(p-q)\tilde{\Gamma}^A C(p'-p+q,q).$$
$$\cdot\ \Gamma^B \tanh\tfrac{1}{2}\beta q_0 \tag{2.47}$$

where we have again used the 'trick' (2.23) to convert the Matsubara frequency sum to the q_o integration. After performing the contour integration we arrive at the gap equation with gradient terms:

$$\Delta(\underline{\Omega}',\underline{k}) = \tfrac{1}{4}g^2\frac{dm}{d\varepsilon}\int d\varepsilon\int\frac{d\Omega}{4\pi}\, D_{AB}(\underline{k}-q)\tilde{\Gamma}^A R\,\Gamma^B \tag{2.48}$$

where

$$R = \hat{\Delta}(q,\underline{k})\left[\varepsilon^2 I+\tilde{\Delta}(q,\underline{k})\hat{\Delta}(q,\underline{k})\right]^{-\frac{1}{2}}\tanh\left[\tfrac{1}{2}\beta(\varepsilon^2 I+\tilde{\Delta}\hat{\Delta})^{\frac{1}{2}}\right]$$
$$-\frac{1}{16\mu^2}(q.\underline{k})^2\beta^2\varepsilon^{-1}\tanh\tfrac{1}{2}\beta\varepsilon\,\operatorname{sech}^2\tfrac{1}{2}\beta\varepsilon\;\hat{\Delta}(q.\underline{k}) \tag{2.49}$$

with $\quad \underline{K} = \underline{p}' - \underline{p}$ (2.50)

In the Ginzburg-Landau region, correct to order Δ^3 the gap equation is

$$\Delta(\underline{\Omega}',\underline{k}) = \int\frac{d\Omega}{4\pi}\, D_{AB}(\underline{\Omega}.\underline{\Omega}')\tilde{\Gamma}^A\Big\{a\,\hat{\Delta}(\underline{\Omega},\underline{k})$$
$$+b\hat{\Delta}(\underline{\Omega},\underline{k})\hat{\Delta}^+(\underline{\Omega},\underline{k})\hat{\Delta}(\underline{\Omega},\underline{k}) - c\frac{\beta^2p_F^2}{16\mu^2}(\underline{\Omega}.\underline{k})^2\hat{\Delta}(\underline{\Omega},\underline{k})\Big\}\Gamma^B \tag{2.51}$$

where
$$a = \frac{1}{4} g^2 \frac{dn}{d\varepsilon} \int \frac{d\varepsilon}{\varepsilon} \tanh \tfrac{1}{2}\beta\varepsilon \tag{2.52}$$

$$b = \frac{1}{4} g^2 \frac{dn}{d\varepsilon} \int d\varepsilon \, \frac{d}{d\varepsilon^2}\left[\frac{1}{\varepsilon} \tanh \tfrac{1}{2}\beta\varepsilon \right] \tag{2.53}$$

and
$$c = \frac{1}{4} g^2 \frac{dn}{d\varepsilon} \int \frac{d\varepsilon}{\varepsilon} \tanh \tfrac{1}{2}\beta\varepsilon \, \mathrm{sech}^2 \tfrac{1}{2}\beta\varepsilon \tag{2.54}$$

Figure 2.1: Single particle exchange contribution to the off-diagonal component of the Dyson equation. Cross-hatching denotes the proper self-energy,and diagonal shading marks the exact propagator of the fermion.

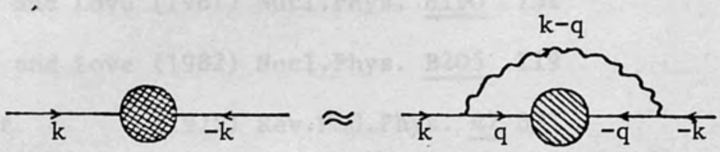
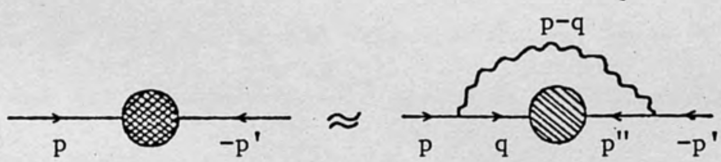


Figure 2.2: Single particle exchange contribution to the off - diagonal component of the Dyson equation for pairing with non-zero centre of mass momentum.



$p'' = -p'+p-q$

Chapter 2: References.

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2.6 Bailin and Love (1982) Nucl.Phys. B205 119

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Chapter 3: Helicity Amplitudes

The gap equations of chapter 2 are highly model dependent, involving the detailed form of the pairing force assumed. However, the gap equations for the possible order parameters may be expressed in terms of the helicity amplitudes for neutron-neutron scattering in a form which is independent of the particular pairing force.In the non-relativistic case it is the interaction potential between the fermions which enters the gap equation: it only affects the value of the critical temperature, T_c (Leggett (3.1)). In the relativistic case the helicity amplitudes affect the detailed form of the gap matrix as well as the value of T_c, but not the Ginzburg-Landau free energy other than through T_c.

Thus we are able to write our results in terms of the helicity amplitudes, as it is the Ginzburg-Landau free energy only that we need to study in order to determine the phase behaviour of our superfluid.

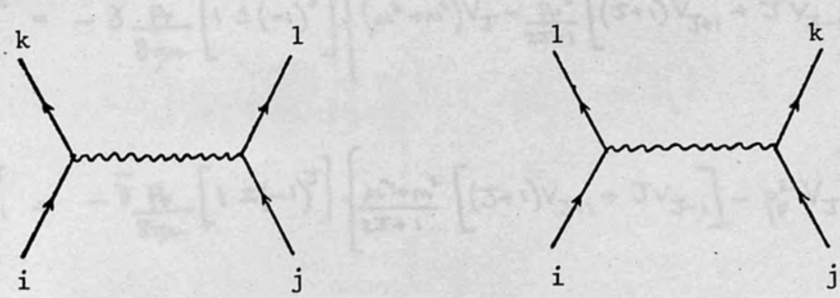
In this chapter we present the helicity amplitudes for spin $\frac{1}{2}$ scattering for both scalar and vector exchange. In each case we consider first the general case and then the particular case of J = 2 neutron scattering.

Scalar Exchange

To leading order, we must calculate the scattering amplitude from the one scalar exchange diagram and the crossed diagram (fig. 3.1). The indices i, j, k, l refer to the possibility that the fermions may have internal symmetry indices.

If the coupling at each interaction vertex is g and the propagator associated with the scalar exchange when the two fermions are on the Fermi surface is $D(\cos\theta)$, where θ is the angle of

Figure 3.1 : Single scalar exchange diagrams.



scattering, then, following Goldberger et al (3.2) the helicity amplitudes at the Fermi surface are:

$$f_0^J = -\tilde{\gamma}\frac{p_F}{8\pi\mu}\left[1 \pm (-1)^J\right]\left\{(\mu^2+m^2)V_J - \frac{p_F^2}{2J+1}\left[(J+1)V_{J+1} + J V_{J-1}\right]\right\} \tag{3.1}$$

$$f_{11}^J = -\tilde{\gamma}\frac{p_F}{8\pi\mu}\left[1 \pm (-1)^J\right]\left\{\frac{\mu^2+m^2}{2J+1}\left[(J+1)V_{J+1} + J V_{J-1}\right] - p_F^2 V_J\right\} \tag{3.2}$$

and for $J \geq 1$

$$f_{12}^J = \tilde{\gamma}\frac{m p_F}{4\pi}\left[1 \pm (-1)^J\right] \frac{\sqrt{J(J+1)}}{2J+1}\left(V_{J+1} - V_{J-1}\right) \tag{3.3}$$

$$f_{22}^J = \tilde{\gamma}\frac{p_F}{8\pi\mu}\left[1 \pm (-1)^J\right]\left\{p_F^2 V_J - \frac{\mu^2+m^2}{2J+1}\left[J V_{J+1} + (J+1)V_{J-1}\right]\right\} \tag{3.4}$$

$$f_1^J = -\tilde{\gamma}\frac{p_F}{8\pi\mu}\left[1 \mp (-1)^J\right]\left\{(\mu^2+m^2)V_J - \frac{p_F^2}{2J+1}\left[J V_{J+1} + (J+1)V_{J-1}\right]\right\} \tag{3.5}$$

where
$$V_J = \frac{1}{2}\int_{-1}^{1} dz\, P_J(z)\, D(z) \tag{3.6}$$

$$\mu^2 = p_F^2 + m^2 \tag{3.7}$$

and
$$\tilde{\gamma} = g^2/4\pi \tag{3.8}$$

The upper or lower sign is to be taken according as the wave function of the pair of fermions scattering is symmetric or anti-symmetric in any internal symmetry indices.

J=2 Neutron scattering through scalar exchange.

The J = 2 helicity amplitudes we shall require are:

$$f_{11}^2 = \left(\frac{1}{4}g^2\right)\left(\frac{1}{2\mu^2}\right)\left(\frac{\mu p_F}{2\pi^2}\right)\left[p_F^2 V_2 - \frac{1}{5}(\mu^2+m^2)(3V_3 + 2V_1)\right] \tag{3.9}$$

$$f_{12}^{2} = \left(\frac{1}{4}g^2\right)\left(\frac{mp_F}{2\pi^2}\right)\frac{1}{5}\sqrt{6}\left(V_3 - V_1\right) \tag{3.10}$$

$$f_{22}^{2} = \left(\frac{1}{4}g^2\right)\left(\frac{1}{3\mu^2}\right)\left(\frac{\mu p_F}{2\pi^2}\right)\left[p_F^2 V_2 - \frac{1}{5}(\mu^2+m^2)\left(2V_3 + 3V_1\right)\right] \tag{3.11}$$

Vector Exchange.

We now consider the case where the scattering is due to vector exchange with a coupling $-ig\gamma^{A}$ at each interaction vertex. We assume the propagator for this vector exchange on the Fermi surface to have the form

$$D^{00} = -D^{E}(\cos\theta) \tag{3.12}$$

$$D^{\alpha\beta} = \delta^{\alpha\beta} D^{M}(\cos\theta) \tag{3.13}$$

The diagrams we need to calculate, to leading order are shown in fig 3.2.

In this case the helicity amplitudes are

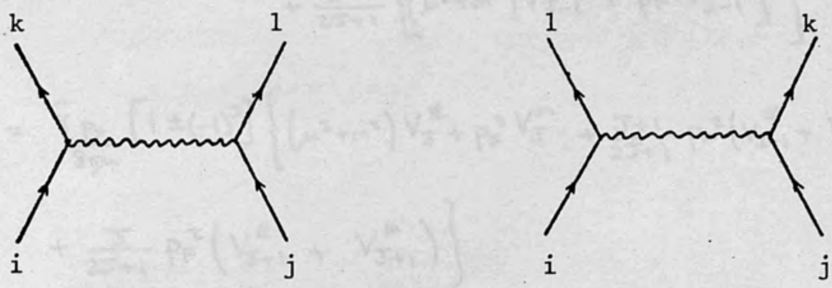
$$f_0^J = \tilde{\gamma}\frac{p_F}{8\pi\mu}\left[1 \pm (-1)^J\right]\Big\{(\mu^2+m^2)V_J^E + 3p_F^2 V_J^M + \frac{J+1}{2J+1}p_F^2\left(V_{J+1}^E - V_{J+1}^M\right) + \frac{J}{2J+1}p_F^2\left(V_{J-1}^E - V_{J-1}^M\right)\Big\} \tag{3.14}$$

$$f_{11}^J = \tilde{\gamma}\frac{p_F}{8\pi\mu}\left[1 \pm (-1)^J\right]\Big\{p_F^2\left(V_J^E + 3V_J^M\right) + \frac{J+1}{2J+1}\left[(\mu^2+m^2)V_{J-1}^E - p_F^2 V_{J+1}^M\right] + \frac{J}{2J+1}\left[(\mu^2+m^2)V_{J-1}^E - p_F^2 V_{J-1}^M\right]\Big\} \tag{3.15}$$

and for $J \geqslant 1$

$$f_{12}^J = \tilde{\gamma}\frac{mp_F}{4\pi}\left[1 \pm (-1)^J\right]\frac{\sqrt{J(J+1)}}{2J+1}\left(V_{J-1}^E - V_{J+1}^E\right) \tag{3.16}$$

Figure 3.2: Single Vector Exchange Diagrams.



$$f_{22}^J = \tilde{\delta}\frac{p_F}{8\pi\mu}\left[1 \pm (-1)^J\right]\left\{p_F^2\left(V_J^E + V_J^m\right) + \frac{J}{2J+1}\left[(\mu^2+m^2)V_{J+1}^E + p_F^2 V_{J+1}^m\right]\right.$$

$$\left. + \frac{J+1}{2J+1}\left[(\mu^2+m^2)V_{J-1}^E + p_F^2 V_{J-1}^m\right]\right\}$$

(3.17)

$$f_1^J = \tilde{\delta}\frac{p_F}{8\pi\mu}\left[1 \pm (-1)^J\right]\left\{(\mu^2+m^2)V_J^E + p_F^2 V_J^m + \frac{J+1}{2J+1}p_F^2\left(V_{J-1}^E + V_{J-1}^m\right)\right.$$

$$\left. + \frac{J}{2J+1}p_F^2\left(V_{J+1}^E + V_{J+1}^m\right)\right\}$$

(3.18)

where $$V_J^m = \frac{1}{2}\int_{-1}^{1} dz\ P_J(z) D^m(z) \qquad (3.19)$$

and $$V_J^E = \frac{1}{2}\int_{-1}^{1} dz\ P_J(z) D^E(z) \qquad (3.20)$$

J=2 Neutron scattering through vector exchange.

The J = 2 helicity amplitudes we shall require are:

$$f_{11}^2 = \left(\frac{g^2}{4}\right)\left(\frac{1}{2\mu^2}\right)\left(\frac{mp_F}{2\pi^2}\right)\left[p_F^2\left(V_2^E + 3V_2^m\right) + \frac{3}{5}\left\{(\mu^2+m^2)V_3^E - p_F^2 V_3^m\right\}\right.$$

$$\left. + \frac{2}{5}\left\{(\mu^2+m^2)V_1^E - p_F^2 V_1^m\right\}\right] \qquad (3.21)$$

$$f_{12}^2 = \left(\frac{g^2}{4}\right)\left(\frac{mp_F}{2\pi^2}\right)\frac{1}{5}\sqrt{6}\left(V_1^E - V_3^E\right)$$

(3.22)

$$f_{22}^2 = \left(\frac{g^2}{4}\right)\left(\frac{1}{2\mu^2}\right)\left(\frac{mp_F}{2\pi^2}\right)\left[p_F^2\left(V_2^E + V_2^m\right) + \frac{3}{5}\left\{(\mu^2+m^2)V_1^E + p_F^2 V_1^m\right\}\right.$$

$$\left. + \frac{2}{5}\left\{(\mu^2+m^2)V_3^E + p_F^2 V_3^m\right\}\right] \qquad (3.23)$$

Chapter 3: References

3.1 Leggett (1975) Rev.Mod.Phys. 47 331

3.2 Goldberger et al (1960) Phys.Rev. 120 2250

J = 2^- Pairing.

The gap equation is, to order Δ^3 in the Ginzburg-Landau region:

$$\Delta(\underline{n}',\underline{K}) = \int \frac{d\Omega}{4\pi} D_{AB}(\underline{n},\underline{n}') \tilde{\Gamma}^A \Big[\bar{a}\,(2\mu^2 \hat{\Delta})$$

$$+ \bar{b}(2\mu^2)^3 \hat{\Delta}\hat{\bar{\Delta}}\hat{\Delta} - \bar{c}\,(\underline{n}.\underline{K})^2 (2\mu^2 \hat{\Delta}) \Big] \Gamma^B \qquad (4.1)$$

where $$\bar{a} = \frac{1}{2\mu^2}\frac{1}{4}g^2 \frac{dn}{d\varepsilon}\int_{-\varepsilon_0}^{\varepsilon_0} \frac{d\varepsilon}{\varepsilon} \tanh \tfrac{1}{2}\beta\varepsilon \qquad (4.2)$$

$$\bar{b} = \frac{1}{(2\mu^2)^3}\frac{1}{4}g^2 \frac{dn}{d\varepsilon}\int_{-\varepsilon_0}^{\varepsilon_0} d\varepsilon \left[\frac{d}{d\varepsilon^2}\left(\frac{1}{2\varepsilon}\tanh \tfrac{1}{2}\beta\varepsilon\right)\right] \qquad (4.3)$$

$$\bar{c} = \frac{1}{2\mu^2}\frac{\beta^2 p_F^2}{16\mu^2}\frac{1}{4}g^2 \frac{dn}{d\varepsilon}\int_{-\varepsilon_0}^{\varepsilon_0} \frac{d\varepsilon}{\varepsilon} \tanh \tfrac{1}{2}\beta\varepsilon \,\mathrm{sech}^2 \tfrac{1}{2}\beta\varepsilon \qquad (4.4)$$

and $$\hat{\Delta} = \frac{1}{4\mu^2}(\mu\gamma^0 + p_F \underline{n}.\underline{\gamma} - m)\,\Delta\,(\mu\gamma^0 - p_F\underline{n}.\underline{\gamma} + m) \qquad (4.5)$$

The redefinitions of the constants a, b, and c have been made to avoid an annoying recurring factor of $2\mu^2$ arising from the calculation of $\hat{\Delta}$.

We proceed by writing down the most general possible form of the gap matrix consistent with Fermi statistics and angular momentum requirements.

We have an order parameter Δ_{ij} , which is 3 x 3, traceless and symmetric and which couples with n_i, S, $\underline{v}$ and ε_{ijk} where S is any Dirac scalar covariant and $\underline{v}$ is any vector Dirac covariant.

Fermi statistics require that we keep only those terms which do not vanish when we anti-symmetrise in the neutron fields.

Anti-symmetrising:

$$\bar{\psi}_c \Delta \psi - \psi(\bar{\psi}_c \Delta) = \psi^T C \Delta \psi - \psi^T(\psi^T C \Delta)$$

$$= \psi^T \left(C\Delta - \Delta^T C^T\right) \psi$$

$$= 0$$

for $$C \Delta C^{-1} = - \Delta^T \tag{4.6}$$

where C is the charge conjugation matrix.

Thus our gap matrix must have the property

$$C \Delta C^{-1} = \Delta^T \tag{4.7}$$

There is, however, another requirement: Δ is a function of $\underline{n}$ which corresponds to derivatives acting upon the neutron field in co-ordinate space and we must allow $\underline{n} \rightarrow - \underline{n}$.

Δ must therefore have the property:

$$C \Delta(\underline{n}) C^{-1} = \Delta^T(-\underline{n}) \tag{4.8}$$

Table 4.1 shows all the possible structures to which Δ_{ij} may couple. (For the properties of Dirac covariants see, for example, Itzykson and Zuber (4.1).)

Thus the most general form of the gap matrix for $J^P = 2^-$ pairing consistent with Fermi statistics is

$$\Delta(\underline{n}) = \Delta^{(1)}_{ij} T_{ij} + \Delta^{(2)}_{ij} S_{ij} + \Delta^{(3)}_{ij} Y_{ij} + \Delta^{(4)}_{ij} \tilde{S}_{ij} + \Delta^{(5)}_{ij} \tilde{Y}_{ij} + \Delta^{(6)}_{ij} X_{ij} \tag{4.9}$$

where the covariants T, S, $\tilde{S}$, Y, $\tilde{Y}$ and X have been defined, with definite values of L, by

$$T_{ij} = n_i n_j - \frac{1}{3} \delta_{ij} \tag{4.10}$$

$$S_{ij} = \frac{1}{2}\left(n_i \gamma^0 \gamma^j + n_j \gamma^0 \gamma^i\right) - \frac{1}{3} \underline{n} \cdot \gamma^0 \underline{\gamma}\, \delta_{ij} \tag{4.11}$$

$$\tilde{S}_{ij} = \frac{1}{2}\left(n_i \gamma^j + n_j \gamma^i\right) - \frac{1}{3} \underline{n} \cdot \underline{\gamma}\, \delta_{ij} \tag{4.12}$$

Table 4.1 : Allowed structures to which Δ_{ij} may couple. Those consistent with Fermi statistics are marked by ticks.

	Parity	Fermi statistics
$n_i n_j$	$-$	✓
$n_i n_j \gamma_5$	$+$	✓
$n_i n_j \gamma_o \gamma_5$	$+$	✓
$n_i n_j \gamma_o$	$-$	×
$n_i n_j \underline{n} \cdot \underline{\gamma}$	$-$	✓
$n_i n_j \underline{n} \cdot \underline{\gamma} \gamma_5$	$+$	×
$n_i n_j \underline{n} \cdot \gamma_o \underline{\gamma} \gamma_5$	$+$	✓
$n_i n_j \underline{n} \cdot \gamma_o \underline{\gamma}$	$-$	✓
$n_i \gamma_j$	$-$	✓
$n_i \gamma_j \gamma_5$	$+$	×
$n_i \gamma_o \gamma_j \gamma_5$	$+$	✓
$n_i \gamma_o \gamma_j$	$-$	✓
$\epsilon_{ikl} n_l n_j \gamma_k$	$+$	×
$\epsilon_{ikl} n_l n_j \gamma_k \gamma_5$	$-$	✓
$\epsilon_{ikl} n_l n_j \gamma_o \gamma_k \gamma_5$	$-$	×
$\epsilon_{ikl} n_l n_j \gamma_o \gamma_k$	$+$	×

$$Y_{ij} = \left[n_i n_j n_k - \frac{1}{5}\left(n_i \delta_{jk} + n_k \delta_{ij} + n_j \delta_{ik}\right)\right]\gamma^0\gamma^k \tag{4.13}$$

$$\tilde{Y}_{ij} = \left[n_i n_j n_k - \frac{1}{5}\left(n_i \delta_{jk} + n_k \delta_{ij} + n_j \delta_{ik}\right)\right]\gamma^k \tag{4.14}$$

$$X_{ij} = \frac{1}{2} i \left(\underline{n} \times \underline{\gamma}\gamma_5\right)_i n_j + \frac{1}{2} i \left(\underline{n} \times \underline{\gamma}\gamma_5\right)_j n_i \tag{4.15}$$

It will be noted that although $^3P_2$ pairing means L = 1, we have included terms with L other than 1. The explanation rests with the fact that we are considering relativistic effects. In the relativistic regime the only 'good' quantum numbers are J and P, the parity.

The most general form of the gap matrix for $J^P = 2^+$ pairing is:

$$\Delta(\underline{n}) = \Delta^{(7)}_{ij} P_{ij} + \Delta^{(8)}_{ij} \tilde{P}_{ij} + \Delta^{(9)}_{ij} Q_{ij} + \Delta^{(10)}_{ij} R_{ij} \tag{4.16}$$

where

$$P_{ij} = \left(n_i n_j - \frac{1}{3}\delta_{ij}\right)\gamma_5 \tag{4.17}$$

$$\tilde{P}_{ij} = \left(n_i n_j - \frac{1}{3}\delta_{ij}\right)\gamma_0\gamma_5 \tag{4.18}$$

$$Q_{ij} = \left[n_i n_j n_k - \frac{1}{5}\left(n_i \delta_{jk} + n_k \delta_{ij} + n_j \delta_{ik}\right)\right]\gamma_0\gamma_k\gamma_5 \tag{4.19}$$

$$R_{ij} = \left[\frac{1}{2}\left(n_i \gamma^0\gamma^j + n_j \gamma^0\gamma^i\right) - \frac{1}{3}\underline{n}\cdot\gamma^0\underline{\gamma}\right]\gamma_5 \tag{4.20}$$

We leave the case of $J^P = 2^+$ until later.

Here we proceed to solve the gap equation for $J^P = 2^-$. Initially we solve to first order only in each of two cases: scalar exchange and vector exchange. The purpose of this is to demonstrate that the result may be expressed in terms of the helicity amplitudes in a form which is independant of the specific pairing force.

Having satisfied ourselves that this is indeed the case we can then solve the full gap equation to third order in Δ ,with gradient terms, for scalar exchange only. The answer will be expressed in terms of helicity amplitudes and we can assume that the calculation for vector exchange will give the same result provided that the answer were expressed in a similar manner.

<u>Scalar Exchange To Order Δ</u>

The gap equation, to first order in Δ , is

$$\Delta(\underline{n}') = \bar{a}\int\frac{d\Omega}{4\pi}\,D(\underline{n},\underline{n}')\left(2\mu^2\hat{\Delta}\right) \tag{4.21}$$

where
$$\bar{a} = \frac{1}{2\mu^2}\frac{1}{4}g^2\frac{dn}{dE}\int_{-\varepsilon_0}^{\varepsilon_0}\frac{d\varepsilon}{\varepsilon}\tanh\tfrac{1}{2}\beta\varepsilon \tag{4.22}$$

and
$$2\mu^2\hat{\Delta} = \tfrac{1}{2}\left(\mu\gamma^0 + p_F\,\underline{n}.\underline{\gamma} - m\right)\Delta\left(\mu\gamma^0 - p_F\,\underline{n}.\underline{\gamma} + m\right) \tag{4.23}$$

For Δ given by (4.9) we find

$$\begin{aligned} 2\mu^2\hat{\Delta} = {} & p_F\, d^{(1)}_{ij} T_{ij} + m\, d^{(2)}_{ij} S_{ij} - \left(\mu d^{(1)}_{ij} + m\, d^{(2)}_{ij}\right) Y_{ij} \\ & - \mu d^{(2)}_{ij}\,\tilde{S}_{ij} + \left(m d^{(1)}_{ij} + \mu d^{(2)}_{ij}\right)\tilde{Y}_{ij} + p_F\, d^{(2)}_{ij} X_{ij} \end{aligned} \tag{4.24}$$

where
$$d^{(1)}_{ij} = p_F\,\Delta^{(1)}_{ij} + \mu\left(\Delta^{(2)}_{ij} + \tfrac{3}{5}\Delta^{(3)}_{ij}\right) - m\left(\Delta^{(4)}_{ij} + \tfrac{3}{5}\Delta^{(5)}_{ij}\right) \tag{4.25}$$

and
$$d^{(2)}_{ij} = -m\left(\Delta^{(2)}_{ij} - \tfrac{2}{5}\Delta^{(3)}_{ij}\right) + \mu\left(\Delta^{(4)}_{ij} - \tfrac{2}{5}\Delta^{(5)}_{ij}\right) + p_F\,\Delta^{(6)}_{ij} \tag{4.26}$$

In order to perform the angular integration
$$\int\frac{d\Omega}{4\pi}\,D(\underline{n}.\underline{n}')\left(2\mu^2\hat{\Delta}\right)$$
we shall require expressions for $\int\frac{d\Omega}{4\pi}D(\underline{n}.\underline{n}')\,n_i$, $\int\frac{d\Omega}{4\pi}D(\underline{n},\underline{n}')\,n_i n_j$ and $\int\frac{d\Omega}{4\pi}D(\underline{n}.\underline{n}')\,n_i n_j n_k$.

These are deduced by tensor arguments and the results are expressed in terms of

$$V_l = \int \frac{d\Omega}{4\pi}\, D(\underline{\Omega}\cdot\underline{\Omega}')\, P_l(\underline{\Omega}\cdot\underline{\Omega}') \tag{4.27}$$

A list of angular integrals is given in Appendix IA.

After these integrals have been performed, coefficients of covariants are compared and we obtain

$$\Delta^{(1)}_{ij} = \bar{a}\, p_F V_2 d^{(1)}_{ij} \tag{4.28}$$

$$\Delta^{(2)}_{ij} = \frac{1}{5}\bar{a} m d^{(2)}_{ij}(3V_1 + 2V_3) - \frac{1}{5}\bar{a}\mu d^{(1)}_{ij}(2V_1 - 2V_3) \tag{4.29}$$

$$\Delta^{(3)}_{ij} = -\bar{a}V_3(\mu d^{(1)}_{ij} + m d^{(2)}_{ij}) \tag{4.30}$$

$$\Delta^{(4)}_{ij} = \frac{1}{5}\bar{a}\left[2m d^{(1)}_{ij}(V_1 - V_3) - \mu d^{(2)}_{ij}(3V_1 + 2V_3)\right] \tag{4.31}$$

$$\Delta^{(5)}_{ij} = \bar{a}V_3\left[m d^{(1)}_{ij} + \mu d^{(2)}_{ij}\right] \tag{4.32}$$

$$\Delta^{(6)}_{ij} = \bar{a}\, p_F V_2 d^{(2)}_{ij} \tag{4.33}$$

The definitions (4.25) and (4.26) for $d^{(1)}$ and $d^{(2)}$ lead us to

$$\left(\frac{1}{\bar{a}}\right) d^{(1)}_{ij} = \left[V_2 p_F^2 - \frac{1}{5}(\mu^2 + m^2)(2V_1 + 3V_3)\right] d^{(1)}_{ij} + \left[\frac{6}{5}\mu m(V_1 - V_3)\right] d^{(2)}_{ij} \tag{4.34}$$

$$\left(\frac{1}{\bar{a}}\right) d^{(2)}_{ij} = \left[\frac{4}{5}\mu m(V_1 - V_3)\right] d^{(1)}_{ij} + \left[V_2 p_F^2 - \frac{1}{5}(\mu^2 + m^2)(3V_1 + 2V_3)\right] d^{(2)}_{ij} \tag{4.35}$$

Using the expressions of chapter 3 for the helicity amplitudes at the Fermi surface for scalar exchange equations (4.34) and (4.35) may be written

$$\begin{pmatrix} d_{ij}^{(1)} \\ \\ d_{ij}^{(2)} \end{pmatrix} = A \cdot \frac{24\pi^2}{\mu p_F} \begin{pmatrix} f_{11}^2 & -\sqrt{\frac{3}{2}}\, f_{12}^2 \\ \\ \sqrt{\frac{2}{3}}\, f_{12}^2 & f_{22}^2 \end{pmatrix} \begin{pmatrix} d_{ij}^{(1)} \\ \\ d_{ij}^{(2)} \end{pmatrix} \tag{4.36}$$

where $$A = \frac{1}{6} \frac{dn}{d\varepsilon} \int_{-\varepsilon_0}^{\varepsilon_0} \frac{d\varepsilon}{2\varepsilon} \tanh \tfrac{1}{2}\beta\varepsilon \tag{4.37}$$

$$= \frac{1}{6} \frac{dn}{d\varepsilon} \ln\left(\xi \beta \varepsilon_0\right)$$

$$\left(\quad \xi = \tfrac{1}{2} \exp - \int_0^\infty \ln z \, \mathrm{sech}^2 z \, dz \simeq 1.14 \quad \right) \tag{4.38}$$

$\frac{dn}{d\varepsilon} = \frac{\mu p_F}{\pi^2}$ is the density of states at the Fermi surface.

Vector Exchange To First Order

The gap equation is, to first order in Δ ,

$$\Delta(\underline{n}') = \bar{a}\int \frac{d\Omega}{4\pi}\, D_{AB}(\underline{n},\underline{n}')\,\tilde{\Gamma}^A\,(2\mu^2\hat{\Delta})\,\Gamma^B \tag{4.39}$$

where

$$\bar{a} = \frac{1}{2\mu^2}\frac{1}{4}g^2\frac{dn}{d\mathcal{E}}\int\frac{d\mathcal{E}}{\mathcal{E}}\tanh\tfrac{1}{2}\beta\mathcal{E} \tag{4.40}$$

For the case where there is vector exchange we put $\Gamma^A = \gamma^A$ and assume that the propagator for vector exchange has the form:

$$D^{oo}\;(\underline{n},\ \underline{n}') = -\,D_E(\underline{n}.\underline{n}') \tag{4.41}$$

$$D^{\alpha\beta}\;(\underline{n},\ \underline{n}') = \delta^{\alpha\beta}D_M(\underline{n}.\underline{n}') \tag{4.42}$$

Then

$$D_{AB}(\underline{n},\underline{n}')\,\tilde{\Gamma}^A\hat{\Delta}\Gamma^B = D_E(\underline{n}.\underline{n}')\,\gamma^0\hat{\Delta}\gamma^0 - D_M(\underline{n}.\underline{n}')\,\gamma^i\hat{\Delta}\gamma^i \tag{4.43}$$

We define

$$V_l^E = \int\frac{d\Omega}{4\pi}\,D_E(\underline{n}.\underline{n}')\,P_l(\underline{n}.\underline{n}') \tag{4.44}$$

$$V_l^M = \int\frac{d\Omega}{4\pi}\,D_M(\underline{n}.\underline{n}')\,P_l(\underline{n}.\underline{n}') \tag{4.45}$$

The calculation is similar to that carried out above except that in this case the E (electric) and M (magnetic) parts are treated separately.

We arrive at

$$\begin{aligned}(1/\bar{a})\,d_{ij}^{(1)} = \Big[p_F^2\left(V_2^E + 3V_2^M\right) &+ \tfrac{3}{5}\left\{(\mu^2+m^2)V_3^E - p_F^2V_2^M\right\} \\ &+ \tfrac{2}{5}\left\{(\mu^2+m^2)V_1^E - p_F^2V_1^M\right\}\Big]\; d_{ij}^{(1)} \\ &- \tfrac{6}{5}\mu m\left(V_1^E - V_3^E\right)\, d_{ij}^{(2)}\end{aligned} \tag{4.46}$$

$$\left(1/\tilde{a}\right) d_{ij}^{(2)} = -\frac{4}{5}\mu m\left(V_1^E - V_3^E\right) d_{ij}^{(1)}$$

$$+ \left[p_F^2\left(V_2^E + V_2^m\right) + \frac{3}{5}\left\{(\mu^2+m^2)V_1^E + p_F^2 V_1^m\right\} + \frac{2}{5}\left\{(\mu^2+m^2)V_3^E + p_F^2 V_3^m\right\} \right] d_{ij}^{(2)} \qquad (4.47)$$

From chapter 3 we see that the helicity amplitudes for vector exchange are such that this result may be written

$$\begin{pmatrix} d_{ij}^{(1)} \\ d_{ij}^{(2)} \end{pmatrix} = A.\frac{24\pi^2}{\mu p_F} \begin{pmatrix} f_{11}^2 & -\sqrt{\frac{3}{2}}f_{12}^2 \\ -\sqrt{\frac{2}{3}}f_{12}^2 & f_{22}^2 \end{pmatrix} \begin{pmatrix} d_{ij}^{(1)} \\ d_{ij}^{(2)} \end{pmatrix} \qquad (4.48)$$

with

$$A = \frac{1}{6}\frac{dn}{d\varepsilon} \ln \zeta\beta\varepsilon_0 \qquad (4.49)$$

Precisely the result which we obtained for scalar exchange in equation (4.36).

Thus, although gap equations for the individual matrices are highly model dependent, the gap equations for the possible order parameters $d_{ij}^{(1)}$ and $d_{ij}^{(2)}$ may be expressed in terms of the helicity amplitudes for neutron-neutron scattering in a way which is independant of the specific pairing force.

We are therefore able to work to higher orders for scalar exchange only assuming that the more complicated vector exchange cal -culation would give the same result when expressed in terms of the helicity amplitudes.

Scalar Exchange to Order Δ^3

The gap equation for scalar exchange is

$$\Delta(\underline{n}',\underline{K}) = \int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\Big[\bar{a}(3\mu^2\hat{\Delta}) + \bar{b}(3\mu^2)^3\hat{\Delta}\tilde{\Delta}\hat{\Delta}$$
$$- \bar{c}(\underline{n}.\underline{K})^2(3\mu^2\hat{\Delta})\Big] \tag{4.50}$$

with $\bar{a}$, $\bar{b}$, $\bar{c}$ as in (4.2),(4.3), (4.4).

Constuction of $\hat{\Delta}\tilde{\Delta}\hat{\Delta}$ involves terms including upto 9 factors of n's (i.e. terms like $n_i n_j n_k n_l n_m n_n n_p n_q n_r$). Angular integrals are again found in terms of

$$V_l = \int \frac{d\Omega}{4\pi} D(\underline{n}.\underline{n}')P_l(\underline{n}.\underline{n}') \tag{4.51}$$

and are listed in appendix IA.

The angular integration of gradient terms takes the form

$$K_p K_q \int \frac{d\Omega}{4\pi} D(\underline{n}.\underline{n}')\hat{\Delta}\, n_p n_q \tag{4.52}$$

and so contains upto 5 factors of n's.

After all these angular integrations have been performed the results will contain contributions from J values other than J = 2. As in the non-relativistic case (4.2) these admixtures are assumed to be small and we project out the dominant J = 2 part. The J = 2 projections are listed in appendix IB.

The calculations are straightforward but extremely long; we merely quote the results here.

To order Δ^3 we find

$$d^{(\theta)} = F^{\theta\phi}\Big[A d^{(\phi)} + B D^{(\phi)} + C G^{\phi}\Big] \qquad \theta,\phi = 1,2 \tag{4.53}$$

where

$$F^{\phi\phi} = \frac{24\pi^2}{\mu p_F}\begin{pmatrix} f_{11}^2 & -\sqrt{\frac{3}{2}}\,f_{12}^2 \\ -\sqrt{\frac{2}{3}}\,f_{12}^2 & f_{22}^2 \end{pmatrix} \tag{4.54}$$

is the matrix of helicity amplitudes for scalar exchange. (Chapter 3).

$D^{(1)}$ and $D^{(2)}$ are 3 x 3 matrices cubic in $d^{(1)}$ and $d^{(2)}$ defined by

$$\begin{aligned} 63\mu^2 D^{(1)} = &-7\,\mathrm{Tr}\left[(d^{(2)})^2\right] d^{(1)*} + 14\,\mathrm{Tr}\left(d^{(2)*}d^{(2)}\right)d^{(1)} \\ &+ 2\,\mathrm{Tr}\left[(d^{(1)})^2\right] d^{(1)*} + 4\,\mathrm{Tr}\left(d^{(2)}\,d^{(1)*}\right)d^{(2)} \\ &- 4\,\mathrm{Tr}\left(d^{(1)}\,d^{(2)}\right)d^{(2)*} - 4\,\mathrm{Tr}\left(d^{(2)*}\,d^{(1)}\right)d^{(2)} \\ &+ 4\,\mathrm{Tr}\left(d^{(1)*}\,d^{(1)}\right)d^{(1)} - 20\,(d^{(2)})^2\,d^{(1)*} \\ &+ 20\,d^{(2)*}d^{(2)}\,d^{(1)} \quad + 20\,d^{(2)}d^{(2)*}\,d^{(1)} \\ &+ 8\,d^{(2)}\,d^{(1)*}\,d^{(2)} \quad + 8\,d^{(1)}\,d^{(1)*}\,d^{(1)} \\ &+ 16\,d^{(1)*}(d^{(1)})^2 \quad + 16\,d^{(2)*}d^{(1)}\,d^{(2)} \end{aligned} \tag{4.55}$$

$$\begin{aligned} 189\mu^2 D^{(2)} = &\ 62\,\mathrm{Tr}\left(d^{(2)*}d^{(2)}\right)d^{(2)} - 23\,\mathrm{Tr}\left[(d^{(2)})^2\right]d^{(2)*} \\ &+ 28\,\mathrm{Tr}\left(d^{(1)*}\,d^{(1)}\right)d^{(2)} - 8\,\mathrm{Tr}\left(d^{(2)}d^{(1)*}\right)d^{(1)} \\ &- 8\,\mathrm{Tr}\left(d^{(1)}d^{(2)}\right)d^{(1)*} + 8\,\mathrm{Tr}\left(d^{(1)}d^{(2)*}\right)d^{(1)} \\ &- 14\,\mathrm{Tr}\left[(d^{(1)})^2\right]d^{(2)*} + 32\,d^{(2)*}(d^{(2)})^2 \\ &- 2\,d^{(2)}d^{(2)*}d^{(2)} \quad + 40\,d^{(1)*}d^{(1)}\,d^{(2)} \\ &+ 40\,d^{(1)}d^{(1)*}\,d^{(2)} \quad - 32\,d^{(1)*}d^{(2)}\,d^{(1)} \\ &- 40\,(d^{(1)})^2\,d^{(2)*} \quad + 16\,d^{(1)}d^{(2)*}\,d^{(1)} \end{aligned} \tag{4.56}$$

$$G_{ij}^{(1)} = \frac{1}{7}\left(d_{ij}^{(1)}\,\underset{\sim}{k}^2 + 4\,d_{ip}^{(1)}\,k_j k_p \right) \tag{4.57}$$

$$G_{ij}^{(2)} = \frac{1}{21}\left(5\,d_{ij}^{(2)}\,\underset{\sim}{k}^2 + 6\,d_{ip}^{(2)}\,k_j k_p \right) \tag{4.58}$$

$$A = \frac{1}{6}\frac{dn}{d\varepsilon}\int_{-\varepsilon_0}^{\varepsilon_0}\frac{d\varepsilon}{2\varepsilon}\tanh\frac{1}{2}\beta\varepsilon \tag{4.59}$$

$$= \frac{1}{6}\frac{dn}{d\varepsilon}\ln\zeta\beta\varepsilon_0 \quad \text{(Leggett)} \tag{4.60}$$

$$B = \frac{1}{6}\frac{dn}{d\varepsilon}\int_{-\varepsilon_0}^{\varepsilon_0}d\varepsilon\,\frac{d}{d\varepsilon^2}\left[\frac{1}{2\varepsilon}\tanh\frac{1}{2}\beta\varepsilon\right] \tag{4.61}$$

$$= \frac{-7\,\zeta(3)}{48(\pi k_B T_c)^2}\frac{dn}{d\varepsilon} \quad \text{(Leggett)} \tag{4.62}$$

and
$$C = -\frac{\beta^2}{16\mu^2}\;\frac{7\,\zeta(3)}{6\pi^2}\frac{dn}{d\varepsilon} \tag{4.63}$$

We proceed by decoupling equations (4.53) by diagonalizing the matrix of helicity amplitudes, $F^{\theta\phi}$.

This may be achieved by means of the matrix

$$S^{-1} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \frac{1}{2}\left[1 + z^2 + \sqrt{1+z^2}\right]^{-\frac{1}{2}}\begin{pmatrix} 1+\sqrt{1+z^2} & -\sqrt{\frac{3}{2}}\,z \\ \sqrt{\frac{2}{3}}\,z & 1+\sqrt{1+z^2} \end{pmatrix} \tag{4.64}$$

where
$$z = \frac{2f_{12}^2}{f_{22}^2 - f_{22}^1} \tag{4.65}$$

Then
$$SFS^{-1} = \begin{pmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{pmatrix} \tag{4.66}$$

where
$$\lambda^{(1)} = \frac{12\pi^2}{\mu p_F}\left[f_{11}^2 + f_{22}^2 + \left(f_{11}^2 - f_{22}^2\right)\sqrt{1+z^2}\right] \tag{4.67}$$

and
$$\lambda^{(2)} = \frac{12\pi^2}{\mu p_F}\left[f_{11}^2 + f_{22}^2 - \left(f_{11}^2 - f_{22}^2\right)\sqrt{1+z^2}\right]. \tag{4.68}$$

The order parameters which diagonalize the gap equations are then

$$e^{(\theta)} = S^{\theta\phi}\,d^{(\phi)} \quad \text{and we write} \tag{4.69}$$

$$E^{(\theta)} = S^{\theta\phi}\,D^{(\phi)} \tag{4.70}$$

and
$$H^{(\theta)} = S^{\theta\phi}\,G^{(\phi)}. \tag{4.71}$$

The gap equations become

$$e^{(\theta)} = \lambda^{(\theta)} \left(A\, e^{(\theta)} + B\, E^{(\theta)} + C\, H^{(\theta)} \right) \qquad (4.72)$$

$$\theta = 1,2$$

i.e.

$$e^{(\theta)} \left[\frac{1}{\lambda^{(\theta)}} - A \right] = B E^{(\theta)} + C H^{(\theta)} \qquad (4.73)$$

$$\theta = 1,2$$

From (4.60)

$$A = \frac{1}{6} \frac{dn}{d\varepsilon} \ln (\zeta \beta \varepsilon_0) \qquad (4.74)$$

Now at $T_c^{(\theta)}$ $e^{(\theta)}$ vanishes and we must have

$$\left(\frac{1}{\lambda^{(\theta)}} - A \right) \Big|_{T_c} = 0$$

and so

$$A\big|_{T_c} = \frac{1}{6} \frac{dn}{d\varepsilon} \ln \left(\zeta \beta_c^{(\theta)} \varepsilon_0 \right) = \frac{1}{\lambda^{(\theta)}} \qquad (4.75)$$

(4.73) then becomes

$$e^{(\theta)} \frac{1}{6} \frac{dn}{d\varepsilon} \left[\ln \left(\zeta \beta_c^{(\theta)} \varepsilon_0 \right) - \ln (\zeta \beta \varepsilon_0) \right] = B E^{(\theta)} + C H^{(\theta)} \qquad (4.76)$$

Now $\ln \zeta \beta_c^{(\theta)} \varepsilon_0 - \ln \zeta \beta \varepsilon_0 = \ln \frac{\beta_c^{(\theta)}}{\beta}$

$$= \ln \frac{T}{T_c^{(\theta)}} = \ln \left[1 - \left(1 - \frac{T}{T_c^{(\theta)}} \right) \right]$$

$$\simeq \frac{T - T_c^{(\theta)}}{T_c^{(\theta)}} \qquad (4.77)$$

The approximation is valid since we are in the Ginzburg-Landau region where T is close to T_c and so $1 - \frac{T}{T_c}$ is small.

We may now rewrite (4.73) by

$$\frac{1}{6} \frac{dn}{d\varepsilon} t^{(\theta)} e^{(\theta)} = B E^{(\theta)} + C H^{(\theta)} \qquad (4.78)$$

$$\theta = 1,2$$

where $t^{(\vartheta)} = \dfrac{T - T_c^{(\vartheta)}}{T_c^{(\vartheta)}}$ $\qquad \vartheta = 1,2 \qquad$ (4.79)

The order parameter $e^{(\vartheta)}$ with the higher critical temperature is the one which orders at the phase transition. Then in (4.78), in the gap equation for that particular order parameter, the other should be set to zero.

In the meantime, however, we treat both possibilities, $e^{(1)}$ or $e^{(2)}$ ordering in a single formula.

We find

$$\begin{aligned}\frac{1}{6}\frac{dn}{d\varepsilon}\, t e_{ij} = \frac{Bk}{63\mu^2}\Big[& 6(x^4 + x^2y^2 - 2y^4)\,\mathrm{Tr}(e^2)e^*_{ij} \\ & + 6(2x^4 + 14x^2y^2 + 5y^4)\,\mathrm{Tr}(e^*e)e_{ij} \\ & - 9(8x^2y^2 - y^4)(e^2e^* + e^*e^2)_{ij}\Big] \\ + \frac{Ck}{14}\Big[& e_{ij}\underline{k}^2(2x^2 + 5y^2) \\ & + e_{ip}k_jk_p(8x^2 + 6y^2)\Big]\end{aligned} \qquad (4.80)$$

where $k^{(\vartheta)} = 1,\ 2/3$ for $\vartheta = 1,2$.

The ϑ index has been suppressed in (4.80)

We have used the identity for 3 x 3 traceless matrices:

$$2ee^*e + 2e^2e^* + 2e^*e^2 = \mathrm{Tr}(e^2)e^* + 2\,\mathrm{Tr}(e^*e)e \qquad (4.81)$$

to eliminate ee^*e (Mermin 1974 (4.3)).

The gap equation (4.80) is that equation which would result when the Ginzburg-Landau free energy for the system is minimised with respect to the order parameter, e.

We may therefore deduce the Ginzburg-Landau free energy from the gap equation, upto a constant of proportionality.

The Ginzburg-Landau free energy corresponding to the gap equation (4.80) is

$$\mathcal{F} \propto \frac{1}{6}\frac{dn}{d\varepsilon}\, t\, \mathrm{Tr}(e^* e) - \frac{Bk}{63\mu^2}\Big[p\,|\mathrm{Tr}(e^2)|^2 + q\{\mathrm{Tr}(e^* e)\}^2 + r\,\mathrm{Tr}\,(e^{*2} e^2) \Big] + \frac{Ck}{14}\Big[s(\underline{\nabla} e^*.\underline{\nabla} e) + t\,\nabla^2 \mathrm{Tr}\,(ee^*) \Big] \quad (4.82)$$

$$(\underline{\nabla} e^*.\underline{\nabla} e \equiv \nabla_p e_{ip} \nabla_q e^*_{iq}) \quad (4.83)$$

with

$$p = 3(x^4 + x^2y^2 - 2y^4) \quad (4.84)$$

$$q = 3(2x^4 + 14x^2y^2 + 5y^4) \quad (4.85)$$

$$r = -9(8x^2y^2 - y^4) \quad (4.86)$$

$$s = 8x^2 + 6y^2 \quad (4.87)$$

$$t = 2x^2 + 5y^2 \quad (4.88)$$

$x^{(\sigma)}$ and $y^{(\sigma)}$, $\sigma = 1,2$ are given by (4.64).

To identify the order parameter for the realistic case of neutron star matter we consider the non-relativistic limit in which $z \rightarrow 2\sqrt{6}$.

Then using (4.69) we find that

$$e^{(1)} \rightarrow \sqrt{\tfrac{3}{5}}\, m\,(\Delta^{(3)} - \Delta^{(5)}) \quad (4.89)$$

and

$$e^{(2)} \rightarrow \sqrt{\tfrac{5}{3}}\, m\,(\Delta^{(4)} - \Delta^{(2)}) \quad (4.90)$$

Now we see from (4.9) that in the non-relativistic limit $e^{(1)}$ and $e^{(2)}$ are pure L = 3 and pure L = 1 order parameters respectively Thus the realistic $^3P_2$ pairing is described by the order parameter $e^{(2)}$. Accordingly we restrict our attention hereafter to the Ginzburg -Landau free energy for the order parameter $e^{(2)}$.

For this case the non-relativistic limit of (4.84),(4.85) and

(4.86) gives

$$p = 0, \quad r = -q \tag{4.91}$$

in agreement with Sauls and Serene (4.4). The system is in region III of fig 1.2 corresponding to a unitary phase. In general the criterion for region III is

$$4p + 2p + r < 0$$

and this is always satisfied by the p, q, and r of (4.84),(4.85) and (4.86) for the allowed values of z:

$$0 \leq z \leq 2\sqrt{6}$$

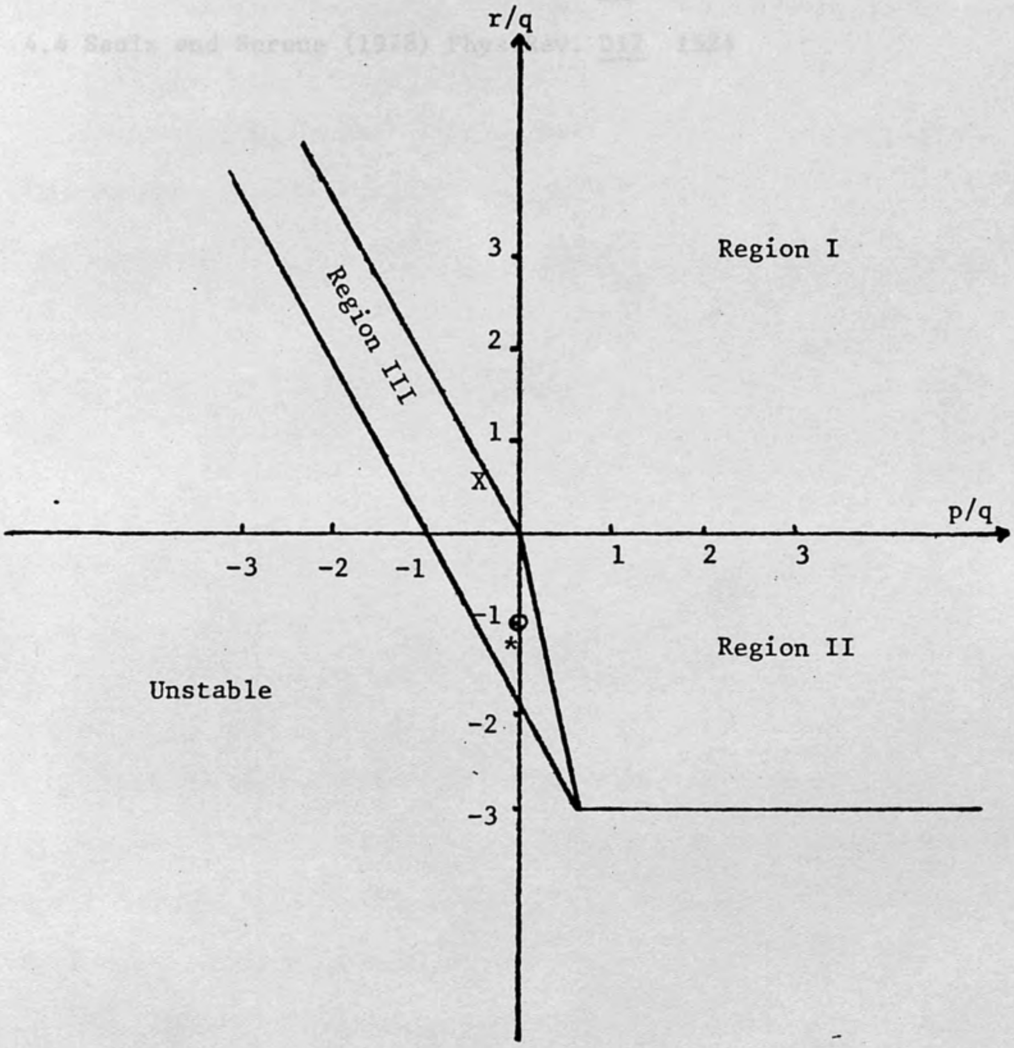
Thus,even after taking account of relativistic effects, the system is always in a unitary phase. (fig 4.1). The corrections due to relativistic considerations can indeed be large: in the ultra-relativistic limit ($z \rightarrow 0$) we find

$$r:q:p = 3:5:-2$$

in contrast to (4.91), the non-relatistic limit. However, as in the case of strong coupling corrections they are in the 'wrong direction' to move the system into another phase.

Figure 4.1: Phase diagram for the $^3P_2$ paired neutron superfluid.

The BCS (non-relativistic limit) point is indicated by O, the strong coupling point by * and the ultra-relativistic point by X.



Chapter 4: References.

4.1 Itzykson and Zuber (1980) "Quantum Field Theory" McGraw Hill.

4.2 Leggett (1975) Rev.Mod.Phys. 47 331

4.3 Mermin (1974) Phys. Rev. A9 868

4.4 Sauls and Serene (1978) Phys.Rev. D17 1524

Chapter 5 : $J^P = 2^+$ Paired Neutron Superfluid Matter

While $J^P = 2^-$ pairing corresponds to the $^3P_2$ paired superfluid which is believed to exist in the cores of neutron stars, $J = 2^+$ pairing corresponds to a D wave paired superfluid and no such existence has been predicted. However, for completeness we here solve the gap equation for the case of $J^P = 2^+$ pairing.

As we saw in chapter 4 (equation (4.16)) the most general form of the gap matrix for $J = 2^+$ pairing is

$$\Delta(\underline{n}) = \Delta_{ij}^{(7)} P_{ij} + \Delta_{ij}^{(8)} \tilde{P}_{ij} + \Delta_{ij}^{(9)} Q_{ij} + \Delta_{ij}^{(10)} R_{ij} \tag{5.1}$$

with
$$P_{ij} = (n_i n_j - \tfrac{1}{3}\delta_{ij})\gamma_5 \tag{5.2}$$

$$\hat{P}_{ij} = (n_i n_j - \tfrac{1}{3}\delta_{ij})\gamma_0\gamma_5 \tag{5.3}$$

$$Q_{ij} = \left[n_i n_j n_k - \tfrac{1}{5}(n_i\delta_{jk} + n_j\delta_{ik} + n_k\delta_{ij})\right]\gamma_0\gamma_k\gamma_5 \tag{5.4}$$

$$R_{ij} = \left[\tfrac{1}{2}(n_i\gamma_0\gamma_j + n_j\gamma_0\gamma_i) - \tfrac{1}{3}\underline{n}\cdot\gamma_0\underline{\gamma}\,\delta_{ij}\right]\gamma_5 \tag{5.5}$$

This was obtained by considering those structures which couple to a 3 x 3 traceless symmetric gap matrix (consistent with J = 2 pairing) with positive parity and which are allowed by Fermi Statistics. (See table 4.1).

We proceed, as before, by solving the gap equation to third order in Δ for scalar exchange.

The gap equation for this case is

$$\Delta(\underline{n}', \underline{K}) = \int \frac{d\Omega}{4\pi} D(\underline{n}, \underline{n}')\Big[\bar{a}(2\mu^2\hat{\Delta}) + \bar{b}(2\mu^2)^3 \hat{\Delta}\tilde{\Delta}\hat{\Delta} - \bar{c}\,(\underline{n}\cdot\underline{K})^2 (2\mu^2\hat{\Delta})\Big] \tag{5.6}$$

$\bar{a}, \bar{b}$ and $\bar{c}$ are as in equations (4.2),(4.3) and (4.4).

We find

$$2\mu^2\hat{\Delta} = d_{ij}\left[-\mu P_{ij} + m\tilde{P}_{ij} + p_F Q_{ij}\right] \tag{5.7}$$

where

$$d_{ij} = \mu \Delta_{ij}^{(7)} - m \Delta_{ij}^{(8)} + p_F \left(\frac{3}{5} \Delta_{ij}^{(9)} + \Delta_{ij}^{(10)} \right) \tag{5.8}$$

$$(3\mu^2)^3 \hat{\Delta}\tilde{\Delta}\hat{\Delta} = 4\mu^2 d_{ij} \left[-\mu P_{ij} + m \tilde{P}_{ij} + p_F Q_{ij} \right] d_{k\ell} d_{mn}^* n_k n_\ell n_m n_n \tag{5.9}$$

$$= 4\mu^2 d d^* \hat{\Delta} \tag{5.10}$$

where $$d = d_{ij} n_i n_j \tag{5.11}$$

After the angular integrations and J = 2 projections have been performed we arrive at the gap equation:

$$d_{ij} = F \Big[a\, d_{ij} + \frac{2}{21} b \left\{ d_{ij}^* \operatorname{tr}(d^2) + 2 d_{ij} \operatorname{tr}(d d^*) \right\}$$
$$- \frac{1}{7} c \left\{ d_{ij} \underset{\sim}{k}^2 + 4\, d_{ip} k_p k_j \right\} \Big] \tag{5.12}$$

where

$$a = \frac{1}{6} \frac{dn}{d\varepsilon} \ln(\zeta \beta \varepsilon_0) \tag{5.13}$$

$$b = -\frac{7}{48} \frac{dn}{d\varepsilon} (\pi k T_c)^{-2} \zeta(3) \tag{5.14}$$

$$c = \frac{p_F^2 \beta^2}{16\mu^2} \frac{7\zeta(3)}{6\pi^2} \frac{dn}{d\varepsilon} \tag{5.15}$$

$$F = \frac{24\pi^2}{\mu p_F} f_0^2 \tag{5.16}$$

and $$f_0^2 = \frac{1}{4} g^2 \frac{1}{3\mu^2} \frac{\mu p_F}{2\pi^2} \left[(\mu^2 + m^2) V_2 - \frac{p_F^2}{5} (3V_3 + 2V_1) \right] \tag{5.17}$$

The gap equation (5.12) leads to the Ginzburg-Landau free energy:

$$\mathcal{F} \propto \frac{1}{6} \frac{dn}{d\varepsilon}\, t \operatorname{Tr}(d^* d)$$
$$+ \frac{2}{21} b \left[|\operatorname{tr}(d^2)|^2 + 2 \{ \operatorname{tr}(d^* d) \}^2 \right]$$
$$- \frac{1}{7} c \left[4 \underset{\sim}{\nabla} d^* \cdot \underset{\sim}{\nabla} d + \nabla^2 \operatorname{tr}(d d^*) \right] \tag{5.18}$$

Referring to the discussion of the definitions of the various phase regions of chapter 1 we see that this corresponds to

$$q = 2, \qquad p = 1, \qquad r = 0. \tag{5.19}$$

Thus such a D wave paired superfluid with $J = 2^+$ pairing would exist on the boundary between regions I and II of fig. 4.1.

Appendix IA: Angular Integrations

In solving the gap equation angular integrals of the form

$$\int \frac{d\Omega}{4\pi}\, D(\underline{\Omega},\underline{\Omega}')\, n_i n_j \ldots n_k \qquad \text{are required.}$$

These are evaluated by tensor arguements and are expressed in terms of V_l defined by

$$V_l = \int \frac{d\Omega}{4\pi}\, D(\underline{\Omega},\underline{\Omega}')\, P_l(\underline{\Omega}\cdot\underline{\Omega}') \tag{A.1}$$

For example $\int \frac{d\Omega}{4\pi} D(\underline{\Omega},\underline{\Omega}') n_i n_j$ must take the form

$$\int \frac{d\Omega}{4\pi}\, D(\underline{\Omega},\underline{\Omega}')\, n_i n_j = A\,\delta_{ij} + B n_i' n_j' \tag{A.2}$$

'Multiplying' both sides by $n_i' n_j'$ gives

$$\int \frac{d\Omega}{4\pi}\, D(\underline{\Omega},\underline{\Omega}')\, n_i n_j n_i' n_j' = A\delta_{ij} n_i' n_j' + B n_i' n_j' n_i' n_j' \tag{A.3}$$

$$\Rightarrow \int \frac{d\Omega}{4\pi}\, D(\underline{\Omega},\underline{\Omega}')\, (\underline{\Omega}\cdot\underline{\Omega}')^2 = A + B \tag{A.4}$$

$$\Rightarrow \quad \frac{2}{3} V_2 + \frac{1}{3} V_0 = A + B \tag{A.5}$$

Contracting i and j in (A.2) gives

$$V_0 = \int \frac{d\Omega}{4\pi}\, D(\underline{\Omega},\underline{\Omega}') = 3A + B \tag{A.6}$$

Equations (A.5) and (A.6) then give

$$A = \frac{1}{3}(V_0 - V_2)\,, \quad B = V_2 \tag{A.7}$$

Thus

$$\int \frac{d\Omega}{4\pi}\, D(\underline{\Omega},\underline{\Omega}')\, n_i n_j = V_2\, n_i' n_j' + \frac{1}{3}(V_0 - V_2)\,\delta_{ij} \tag{A.8}$$

Below are listed all the angular integrations required in the solution of the gap equation to order Δ^3. V_l for $l \geq 4$ have been neglected as we are only interested in order parameters with $J = 2$.

$$\int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}') = V_o \tag{A.9}$$

$$\int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\underline{n} = V_1 \underline{n}' \tag{A.10}$$

$$\int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')n_i n_j = V_2 n_i' n_j' + \frac{1}{3}(V_o - V_2)\delta_{ij} \tag{A.11}$$

$$\int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')n_i n_j n_k = V_3 n_i' n_j' n_k' + \frac{1}{5}(V_1 - V_3)(\delta_{ij} n_k' + 2 \text{ perms}) \tag{A.12}$$

$$\begin{aligned}\int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')n_i n_j n_k n_l &= \frac{1}{7}V_2(\delta_{ij} + 5 \text{ perms}) \\ &\quad + \left(\frac{1}{15}V_o - \frac{2}{21}V_2\right)(\delta_{ij}\delta_{kl} + 2 \text{ perms})\end{aligned} \tag{A.13}$$

$$\begin{aligned}\int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')n_i n_j n_k n_l n_m &= \frac{1}{9}V_3(\delta_{ij} n_k' n_l' n_m' + 9 \text{ perms}) \\ &\quad + \left(\frac{1}{35}V_1 - \frac{2}{45}V_3\right)(\delta_{ij}\delta_{kl} n_m' + 14 \text{ perms})\end{aligned} \tag{A.14}$$

$$\begin{aligned}\int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')n_i n_j n_k n_l n_m n_n &= \frac{1}{63}V_2(\delta_{ij}\delta_{kl} n_m' n_n' + 44 \text{ perms}) \\ &\quad + \left(\frac{1}{105}V_o - \frac{1}{63}V_2\right)(\delta_{ij}\delta_{kl}\delta_{mn} + 14 \text{ perms})\end{aligned} \tag{A.15}$$

$$\begin{aligned}\int \frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')n_i n_j n_k n_l n_m n_n n_p &= \frac{1}{99}V_3(\delta_{ij}\delta_{kl} n_m' n_n' n_p' + 104 \text{ perms}) \\ &\quad + \left(\frac{1}{315}V_1 - \frac{1}{165}V_3\right)(\delta_{ij}\delta_{kl}\delta_{mn} n_p' \\ &\quad + 104 \text{ perms})\end{aligned} \tag{A.16}$$

Appendix IB J = 2 Projections

We list here the projections of the J = 2 parts of covariants which are required in the solving of the gap equation. The covariants are as defined in equations (4.10) to (4.15). a_{ij}, b_{ij} and c_{ij} are traceless symmetric 3 x 3 matrices and we define

$$a_i = a_{ij}n_j \qquad \text{etc.} \tag{B.1}$$

V_l is as defined in (A.1).

$$\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\underline{a}.\underline{n}\ \underline{b}.\underline{n}\ \underline{c}.\underline{\gamma} \rightarrow \frac{1}{189}V_3(4abc + 4bac + 40acb + 10b\mathrm{Tr}\ ac + 10a\mathrm{Tr}\ bc - 8c\mathrm{Tr}\ ab)_{ij}\tilde{Y}_{ij} + \frac{1}{35}V_1(4abc + 4bac + 2c\mathrm{Tr}\ ab)_{ij}\tilde{S}_{ij} \tag{B.2}$$

$$\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\underline{a}.\underline{b}\ \underline{c}.\underline{\gamma} \rightarrow \frac{1}{21}V_3(3abc + 3bac - 2c\mathrm{Tr}\ ab)_{ij}\tilde{Y}_{ij} + \frac{1}{5}V_1(abc + bac + c\mathrm{Tr}\ ab)_{ij}\tilde{S}_{ij} \tag{B.3}$$

$$\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\underline{a}.\underline{b}\ \underline{c}.\underline{n}\ \underline{n}.\underline{\gamma} \rightarrow (2bac + 2abc + c\ \mathrm{Tr}\ ab)_{ij}(\tfrac{1}{7}V_3\tilde{Y}_{ij} + \tfrac{2}{35}V_1\tilde{S}_{ij}) \tag{B.4}$$

$$\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\underline{a}.\underline{n}\ \underline{b}.\underline{n}\ \underline{c}.\underline{n}\ \underline{n}.\underline{\gamma} \rightarrow (8abc+8bac+8acb+2c\ \mathrm{Tr}\ ab+2b\ \mathrm{Tr}\ ac +2a\ \mathrm{Tr}\ bc)_{ij}(\tfrac{1}{63}V_3\tilde{Y}_{ij} + \tfrac{2}{315}V_1\tilde{S}_{ij}) \tag{B.5}$$

$$\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')(\underline{n}.\underline{a} \times \underline{b})(\underline{n} \times \underline{c}.\underline{\gamma}) \rightarrow (abc-bac+3b\ \mathrm{Tr}\ ac-3a\ \mathrm{Tr}\ bc)_{ij} . (\tfrac{1}{35}V_1\tilde{S}_{ij} - \tfrac{1}{21}V_3\tilde{Y}_{ij}) \tag{B.6}$$

$$i\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\underline{a}.\underline{b}(\underline{c} \times \underline{n}.\underline{\gamma}\gamma_5) \rightarrow (3cab + 3cba + 5c\ \mathrm{Tr}\ ab)_{ij} . \frac{1}{21}V_2X_{ij} \tag{B.7}$$

$$i\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')(\underline{n}.\underline{a} \times \underline{b})\underline{a}.\underline{\gamma}\gamma_5 \rightarrow (7b\ \mathrm{Tr}\ a^2 - 4a\ \mathrm{Tr}\ ab - 6a^2b)_{ij} . \frac{1}{42}V_2X_{ij} \tag{B.8}$$

$$i\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\underline{n}.\underline{a}\ \underline{n}.\underline{b}(\underline{c} \times \underline{n}.\underline{\gamma}\gamma_5) \rightarrow (10abc + 10bac - 8acb - 2b\ \mathrm{Tr}\ ac - 2a\ \mathrm{Tr}\ bc + 7c\ \mathrm{Tr}\ ab)_{ij}\frac{2}{189}V_2X_{ij} \tag{B.9}$$

$$i\int\frac{d\Omega}{4\pi} D(\underline{n},\underline{n}')\underline{n}.\underline{a}(\underline{b} \times \underline{c}.\underline{\gamma}\gamma_5) \rightarrow (2acb - 2abc + b\ \mathrm{Tr}\ ac - c\ \mathrm{Tr}\ ab)_{ij} . \frac{2}{21}V_2X_{ij} \tag{B.10}$$

$$i\int\frac{d\Omega}{4\pi}D(\underline{n},\underline{n}')\underline{n}.\underline{a}(\underline{n}.\underline{b}\times\underline{c})\underline{n}.\underline{\gamma}\gamma_5 \longrightarrow 0 \qquad \text{(B.11)}$$

PART II: Grand Unified Theories At Non-Zero Temperature And Density

Spontaneously broken gauge theories provide an elegant framework for the unification of the weak and electromagnetic interactions in an SU(2) x U(1) gauge theory (6.1) and for the unification of the strong, weak and electromagnetic interactions in a Grand Unified Theory, of which the simplest is SU(5) (6.2). In these theories the symmetry breakdown is accomplished via the Higgs Mechanism, in which scalar fields are introduced, some components of which acquire non-zero expectation values, thus breaking the gauge symmetry. (6.3)

The effects of high temperatures in spontaneously broken gauge symmetries have been widely studied. As in the cases of ferromagnetism and superconductivity it has been shown (6.4, 6.5, 6.6, 6.7, 6.8) that, in most cases, the symmetry is restored at high temperatures. Therefore in the standard big-bang cosmology, the grand unified symmetry is manifest at some very high temperature and then the universe, as it cools, undergoes one or more phase transitions to an $SU(3)_{QCD}$ x SU(2) x U(1) gauge symmetry and later the transition SU(2) x U(1) $\longrightarrow$ U(1) occurs leaving only $SU(3)_{QCD}$ x U(1) as unbroken symmetries.

There has been much interest in these phase transitions, most confined to the case of finite temperature but zero chemical potentials. (e.g. (6.4) to (6.13).) The restriction of zero chemical potential is by no means a necessary restriction, however.

While the present baryon asymmetry is estimated to be small, with

$$\frac{n_b - n_{\bar{b}}}{n_\gamma} \sim 10^{-10}$$

(n_b = baryon density, $n_{\bar{b}}$ = anti-baryon density and n_γ = photon number density.) bounds on the lepton number

Chapter 6: Introduction

Spontaneously broken gauge theories provide an elegant framework for the unification of the weak and electromagnetic interactions in an SU(2) x U(1) gauge theory (6.1) and for the unification of the strong, weak and electromagnetic interactions in a Grand Unified Theory, of which the simplest is SU(5) (6.2). In these theories the symmetry breakdown is accomplished via the Higgs Mechanism, in which scalar fields are introduced,some components of which acquire non-zero expectation values, thus breaking the gauge symmetry. (6.3)

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While the present baryon asymmetry is estimated to be small, with

$$\frac{n_b - n_{\bar{b}}}{n_\gamma} \sim 10^{10}$$

(n_b = baryon density, $n_{\bar{b}}$ = anti-baryon density and n_γ = photon number density.) bounds on the lepton number asymmetry (due to an excess of neutrinos over anti-neutrinos) are weak:

$$\frac{n_{\nu} - n_{\bar{\nu}}}{n_{\gamma}} \quad 8 \times 10^{4}$$

David and Reeves (6.14) deduced from calculations of helium production in the early universe that the electron number density is small but that the muon number density is only constrained by

$$n_{\mu} \lesssim 2.5\ T^{3}$$

or equivalently

$$n_{\mu}/n_{\gamma} \lesssim 10$$

Thus it is quite possible that there is a very large lepton number asymmetry.

However, in SU(5) and SO(10) grand unified theories B, the baryon number, and L, the lepton number, are not absolutely conserved quantities and so there is a tendency for B + L to relax to zero. This, coupled with our earlier comment that B is small would seem to imply that L is also small.

While this may be true it is possible that larger asymmetries in the early universe have been diluted to their present levels.

Moreover, as Harvey and Kolb (6.15) pointed out, it may be that a lepton asymmetry could have survived through to the present to give $L \gg B$.
Thus it would appear to be sensible to consider the effects of finite chemical potentials upon phase transitions in the early universe.

Earlier works (6.16 to 6.19) have discussed the effects of non-zero fermion chemical potentials in Electroweak theory (6.6), and grand unified theories (6.16). In general it is found that there is a tendency for non-zero chemical potentials to suppress symmetry restoration at high temperatures, or at least to raise the temperature at which symmetry restoration occurs.

Kapusta(6.20) also considered the effect of bosonic chemical potentials upon phase transitions in the early universe.

In what follows the effects of including a 'complete set' of chemical potentials (i.e. both bosonic and fermionic) on phase transitions is discussed. In a gravitationally closed universe it is important that we include such a complete set, since long range forces due to massless gauge fields would require the 'charges' coupled to these fields to be zero in equilibrium. It may therefore be necessary for fermionic densities in the early universe to have been balanced by bosonic densities carrying the same 'charge'.

In an open universe this may not be necessary and a 'charge' imbalance may be stabilized by a fictitious external source.

In chapters 7 to 11 we discuss phase transitions at finite bosonic and fermionic chemical potential and finite temperature in the Higgs model, electroweak theory and a sample grand unified theory, SU(5) x U(1) (as a subgroup of the SO(10) theory). It will be seen that the inclusion of the bosonic chemical potential alters the results of previous authors discussing these models by an order of magnitude at most.

In later chapters phase transitions in supersymmetric grand unified theories are considered.

Chapter 6: References

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Chapter 7: Field Theories At Finite Temperatures

Introduction

In order to study the symmetry properties of a field theory at finite temperatures it is necessary to calculate the effective potential. We start,as always, by deriving an expression for the partition function for scalar and fermion fields. Temperature Green functions are then defined. Finally the Higgs model is used to demonstrate the calculation of the effective potential at finite temperature.

Much of what follows is in analogy to the situation at zero temperature. At finite temperatures we here restrict ourselves to the equilibrium properties of the system. The temperature dependence is introduced through an imaginary time. At zero temperature the dynamical properties are described by the time dependence. On going to finite temperatures this time dependence is 'exchanged' for temperature dependence. The important difference lies in the boundary conditions; periodic boundary conditions are relevant at finite temperatures.

The Partition Function

We consider first the case of a scalar field theory with Hamiltonian density, H. The partition function Z is given by

$$Z = \operatorname{tr} \exp - \beta H \qquad (7.1)$$

where the trace means the sum of all matrix elements of exp- β H between all independent states of the system and

$$\beta = (k_B T)^{-1}. \qquad (7.2)$$

H is a function of $\hat{\phi}(\underline{x},t)$, the Heisenberg picture field operator, and of $\pi(\underline{x},t)$, its conjugate momentum.

The Schrödinger picture field operator is $\hat{\phi}(\underline{x},0)$. Let $|\phi_0\rangle$ and $|\phi_1\rangle$ be the eigenstates of $\hat{\phi}(\underline{x},0)$ with eigenvalues $\phi_0(\underline{x})$ and $\phi_1(\underline{x})$ respectively.

Thus $$\hat{\phi}(\underline{x},0)|\phi_0\rangle = \phi_0(\underline{x})|\phi_0\rangle \tag{7.3}$$

and $$\hat{\phi}(\underline{x},0)|\phi_1\rangle = \phi_1(\underline{x})|\phi_1\rangle \tag{7.4}$$

The transition amplitude to go from $|\phi_0\rangle$ at t=0 to $|\phi_1\rangle$ at $t=t_1$ is given by

$$\langle\phi_1|e^{-iHt_1}|\phi_0\rangle$$
$$= N\int \mathcal{D}\pi\,\mathcal{D}\phi \exp\left\{i\int_0^{t_1} dt\int d^3x\left[\pi\dot{\phi} - H(\pi,\phi)\right]\right\} \tag{7.5}$$

where the integral over classical fields $\int\mathcal{D}\phi$ runs over all possible configurations with $\phi_0(\underline{x})$ at t=0 and $\phi_1(\underline{x})$ at $t=t_1$. The momentum integral is unrestricted.

N is a normalisation constant and $\dot{\phi} = \partial/\partial t\, \phi(\underline{x},t)$

Let $$it_1 = \beta \tag{7.6}$$
$$it = \tau \tag{7.7}$$

in the integrand. This substitution makes (7.5) begin to look very much like the partition function we require.

We have

$$\langle\phi_1|e^{-\beta H}|\phi_0\rangle$$
$$= N\int\mathcal{D}\pi\int\mathcal{D}\phi \exp\int_0^\beta d\tau\int d^3x\left[i\pi\dot{\phi} - H(\pi,\phi)\right] \tag{7.8}$$

where now $$\dot{\phi} = \partial\phi/\partial\tau \tag{7.9}$$

Now from (7.1)

$$Z = \mathrm{Tr}\, e^{-\beta H} \tag{7.1}$$

Thus to find Z we need only to let the ϕ integration of (7.8) run over all paths which have the same classical field at $\tau=\beta$ as at $\tau=0$.

Hence

$$Z = N\int\mathcal{D}\pi\int_{\text{Periodic}}\mathcal{D}\phi \exp\int_0^\beta d\tau\int d^3x\left[i\pi\dot{\phi} - H(\pi,\phi)\right] \tag{7.10}$$

If, as is usually the case, H is a quadratic function in π then the π integration can easily be performed by completing the square, to give

$$Z = N'(\beta)\int_{\text{Periodic}}\mathcal{D}\phi \exp\int_0^\beta d\tau\int d^3x\, \mathcal{L}(\phi,\partial_\mu\phi) \tag{7.11}$$

where $$\bar{\partial}_\mu \phi = \left(i \frac{\partial \phi}{\partial \tau}, \nabla \phi \right) \tag{7.12}$$

and $N'(\beta)$ is a temperature dependent normalisation constant arising from the operator determinant when the π path integral is performed. $\mathcal{I}$ is the effective Lagrangian density.

The situation for fermions is slightly, but significantly, different. The eigenstates $|\pm \psi(\underline{x}), t=0\rangle$ of a Schrödinger picture operator $\hat{\psi}(t=0, \underline{x})$ correspond to the same values of the physical observables and describe the same state. There is therefore some ambiguity in deriving a path integral formulation of the partition function. To obtain a prescription which is consistent with Fermi statistics it is necessary to start from:

$$Z = \sum_{\psi(\underline{x})} \langle \psi(\underline{x}), t=0 | e^{-\beta H} | -\psi(\underline{x}), t=0 \rangle \tag{7.13}$$

The analogue of (7.11) for fermions is then

$$Z = N'(\beta) \int_{\text{Anti-Periodic}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \int_0^\beta d\tau \int d^3x \, \mathcal{I}(\psi) \tag{7.14}$$

where in $\mathcal{I}$ the field ψ is understood to be a function of τ and $\underline{x}$ anti-periodic in $0 < \tau < \beta$,

that is $$\psi(\tau=0, \underline{x}) = -\psi(\tau=\beta, \underline{x}) \tag{7.15}$$

<u>Temperature Green Functions and the Effective Potential</u>

In analogy with zero temperature theory we define Green functions

$$\mathcal{G}^{(N)}(\bar{x}_1, \dots \bar{x}_N) = \langle T_\tau (\hat{\phi}(\bar{x}_1) \dots \hat{\phi}(\bar{x}_N) \rangle \tag{7.16}$$

where $$\bar{x} = (-i\tau, \underline{x}) \tag{7.17}$$

and T_τ is a 'τ - ordering' operator.

$\langle \dots \rangle$ here means a thermal average.

i.e.

$$\zeta^{(N)}(\bar{x}_1 \ldots \bar{x}_N) \equiv \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\, T_\tau\left(\hat{\phi}(\bar{x}_1)\ldots\hat{\phi}(\bar{x}_N)\right)\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]} \tag{7.18}$$

which by following similar steps to those above leads to

$$\zeta^{(N)}(\bar{x}_1 \ldots \bar{x}_N) = \frac{\int_{\text{Periodic}} \mathcal{D}\phi \;\; \phi(\bar{x}_1)\ldots\phi(\bar{x}_N) \exp\int_0^\beta d\tau \int d^3x\, \mathcal{L}(\phi, \bar{\partial}_\mu\phi)}{\int_{\text{Periodic}} \mathcal{D}\phi \;\; \exp\int_0^\beta d\tau \int d^3x\, \mathcal{L}(\phi, \bar{\partial}_\mu\phi)} \tag{7.19}$$

We may now introduce a generating functional for temperature Green functions

$$\bar{W}[J] = \frac{\int_{\text{Periodic}} \mathcal{D}\phi \exp\int_0^\beta d\tau \int d^3x \left(\mathcal{L}(\phi, \bar{\partial}_\mu\phi) + J\phi\right)}{\int_{\text{Periodic}} \mathcal{D}\phi \exp\int_0^\beta d\tau \int d^3x \;\; \mathcal{L}(\phi, \bar{\partial}_\mu\phi)} \tag{7.20}$$

where $\qquad J = J(\bar{x})$

Then

$$\zeta^{(N)}(\bar{x}_1 \ldots \bar{x}_N) = \left.\frac{\delta^N \bar{W}[J]}{\delta J(\bar{x}_N)\ldots\delta J(\bar{x}_1)}\right|_{J=0} \tag{7.21}$$

and conversely

$$\bar{W}[J] = \sum_{N=0}^{\infty} \frac{1}{N!}\int d\bar{x}_1 \ldots d\bar{x}_N \; \zeta^{(N)}(\bar{x}_1 \ldots \bar{x}_N) \tag{7.22}$$

where we have written

$$\int d\bar{x} = \int_0^\beta d\tau \int d^3x \tag{7.23}$$

We may also define a generating functional $\bar{X}[J]$ for the connected Green functions $G^N(\bar{x}_1 \ldots \bar{x}_N)$ by

$$\bar{W}[J] = e^{\bar{X}[J]} \tag{7.24}$$

A classical field $\phi_c(\bar{x})$ may be defined by

$$\phi_c(\bar{x}) = \frac{\delta\bar{X}[J]}{\delta J(\bar{x})} \tag{7.25}$$

Now

$$\frac{\delta\bar{W}[J]}{\delta J(\bar{x})} = \langle\hat{\phi}(\bar{x})\rangle_J \tag{7.26}$$

hence $$\langle \hat{\phi}(\bar{x}) \rangle_J = \phi_c(\bar{x}) \bar{W}[J] \tag{7.27}$$

At J=0 $\quad \bar{W}[J] = 1$

and $$\phi_c(\bar{x}) = \langle \hat{\phi}(\bar{x}) \rangle_{J=0} \tag{7.28}$$

Thus $$\phi_c(\bar{x}) = \frac{\mathrm{Tr}\, e^{-\beta \hat{H}} \hat{\phi}(0,x)}{\mathrm{Tr}\, e^{-\beta \hat{H}}} \tag{7.29}$$

Thus for zero source $\phi_c(\bar{x})$ is the expectation value of $\hat{\phi}(0,\underline{x})$, the Schrödinger picture operator.

An effective action is defined by

$$\bar{\Gamma}(\phi_c) = \bar{X}[J] - \int d\bar{x}\, J(\bar{x}) \phi_c(\bar{x}) \tag{7.30}$$

$J(\bar{x})$ is then given by

$$J(\bar{x}) = - \frac{\delta \bar{\Gamma}[\phi_c]}{\delta \phi_c(\bar{x})} . \tag{7.31}$$

The finite temperature effective potential, $\bar{V}(\phi_c)$, is defined by

$$\bar{\Gamma}[\phi_c] = \int d\bar{x} \left(-\bar{V}(\phi_c) + \frac{\bar{A}(\phi_c)}{2} \bar{\partial}_\alpha \phi_c \bar{\partial}^\alpha \phi_c + \dots \right). \tag{7.32}$$

Since we do not expect translational invariance to be broken we may assume that ϕ_c is independent of x.

Then (7.31) and (7.32) lead us to

$$J = \frac{d\bar{V}}{d\phi_c} \tag{7.33}$$

At zero source ϕ_c is the expectation value (thermal average) of the field operator and

$$\frac{d\bar{V}}{d\phi_c} = 0 \tag{7.34}$$

Thus ϕ_c may be obtained by minimizing the finite temperature effective potential.

Symmetry breaking occurs when

$$\frac{d\bar{V}}{d\phi_c} = 0 \quad \text{for} \quad \phi_c \neq 0 . \tag{7.35}$$

For our purposes it is necessary only to calculate the effective potential to one loop order. This contribution is obtained by direct analogy with zero temperature field theory. The scalar fields are shifted by their expectation values (which are now thermal averages) and retain only those terms remaining in the Lagrangian which are quadratic in the shifted fields.

Then

$$\exp \bar{\Gamma}_1(\phi_c) = \exp - \int_0^\beta d\tau \int d^3x \; \bar{V}_1(\phi_c) \tag{7.36}$$

$$= \underset{\text{Periodic}}{\int \mathcal{D}\phi \, \mathcal{D}A^\mu \mathcal{D}\eta^* \mathcal{D}\eta} \; \underset{\text{Anti-Periodic}}{\int \mathcal{D}\bar{\psi} \mathcal{D}\psi} \exp \int_0^\beta d\tau \int d^3x \; \mathcal{L}_{quad}(\phi_c) \tag{7.37}$$

We have included, for completeness, the gauge field A^μ and Fadeev-Popov ghosts η .

The Higgs Model At Finite Temperature

To illustrate the calculation of the effective potential to one loop order we consider the Higgs model extended by the inclusion of fermions.

The Lagrangian we use is

$$\begin{aligned} \mathcal{L} = {} & (\bar{D}_\mu \phi)^* (\bar{D}_\mu \phi) - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \\ & - \tfrac{1}{4} \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} - \tfrac{1}{2\xi} (\bar{\partial}_\mu A^\mu)^2 + \partial_\mu \eta^* \partial^\mu \eta \\ & + \bar{\psi}(i \gamma^\mu \partial_\mu - m_f)\psi - e \bar{\psi} \gamma_\mu \psi A^\mu \end{aligned} \tag{7.38}$$

$$\bar{D}_\mu = (\bar{\partial}_\mu - ieA_\mu)\phi \tag{7.39}$$

$$\bar{F}_{\mu\nu} = \bar{\partial}_\mu A_\nu - \bar{\partial}_\nu A_\mu \tag{7.40}$$

$$\bar{\partial}_\mu = (i\partial/\partial\tau , \underset{\sim}{\nabla}) \tag{7.12}$$

m^2 is negative.

The fields η are Fadeev-Popov ghosts needed to cancel contributions from unphysical degrees of freedom of the gauge field A_μ . They are treated as having the same periodicity in τ as the gauge fields.

Following our prescription we shift the scalar field by the expectation value ϕ_c by writing

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_c + \phi_1 + i\phi_2\right) \tag{7.41}$$

After making this shift the quadratic terms in the Lagrangian are

$$\begin{aligned}\mathcal{L}_{quad} = {} & \tfrac{1}{2}(\bar{\partial}_\mu\phi_1)^2 + \tfrac{1}{2}(\bar{\partial}_\mu\phi_2)^2 - \tfrac{1}{2}(m^2 + 3\lambda\phi_c^2)\phi_1^2 \\ & - \tfrac{1}{2}(m^2 + \lambda\phi_c^2)\phi_2^2 \qquad - \tfrac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} \\ & + \tfrac{1}{2}e^2\phi_c^2 A_\mu A^\mu \qquad - \frac{1}{2\xi}(\bar{\partial}_\mu A^\mu)^2 \\ & + \bar{\partial}_\mu \eta^* \bar{\partial}^\mu \eta \quad + \bar{\psi}(i\gamma^\mu\partial_\mu - M_\psi)\psi\end{aligned}$$

. (7.42)

We have adopted the Landau gauge, $\xi \to 0$, so as to remove an $A^\mu\bar{\partial}_\mu\phi_2$ cross term.

The tree terms are

$$\mathcal{L}_{tree} = \tfrac{1}{2}m^2\phi_c^2 + \tfrac{1}{4}\lambda\phi_c^4 \tag{7.43}$$

From (7.36) we have

$$\exp\left[-\int_0^\beta d\tau\int d^3x\ \bar{V}_1(\phi_c)\right]$$

$$= \underbrace{\int \mathcal{D}\phi\,\mathcal{D}A^\mu\mathcal{D}\eta^*\,\mathcal{D}\eta}_{\text{Periodic}}\ \underbrace{\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi}_{\text{Anti-Periodic}}\ \exp\int_0^\beta d\tau\int d^3x\ \mathcal{L}_{quad}(\phi_c) \tag{7.36}$$

$$\begin{aligned}\int_0^\beta d\tau\int d^3x\,\mathcal{L}_{quad} = {} & -\tfrac{1}{2}\int d\bar{x}'d\bar{x}\ \ \phi_i(\bar{x}')A_{ij}(\bar{x}',\bar{x})\phi_j(\bar{x}) \\ & -\tfrac{1}{2}\int d\bar{x}'d\bar{x}\ \ A_\mu(\bar{x}')B^{\mu\nu}(\bar{x}',\bar{x})A_\nu(\bar{x}) \\ & -\int d\bar{x}'d\bar{x}\ \ \eta^*(\bar{x}')C(\bar{x}',\bar{x})\eta(\bar{x}) \\ & -\int d\bar{x}'d\bar{x}\ \ \bar{\psi}(\bar{x}')D(\bar{x}',\bar{x})\psi(\bar{x})\end{aligned} \tag{7.44}$$

where we have defined

$$A_{ij}(\bar{x}', \bar{x}) = \begin{pmatrix} -\partial'_\mu \partial^\mu + m^2 + 3\lambda\phi_c^2 & 0 \\ 0 & -\partial'_\mu \partial^\mu + m^2 + \lambda\phi_c^2 \end{pmatrix} \delta(\bar{x}' - \bar{x}) \quad (7.45)$$

(writing $\partial'_\mu = \partial_{x'_\mu}$)

$$B_{\mu\nu}(\bar{x}', \bar{x}) = \left[g^{\mu\nu} \partial'_\rho \partial^\rho - (1 - \xi^{-1}) \partial'_\nu \partial^\nu - g^{\mu\nu} e^2 \phi_c^2 \right] \delta(\bar{x}' - \bar{x}) \quad (7.46)$$

$$C(\bar{x}', \bar{x}) = \partial'_\rho \partial^\rho \;\; \delta(\bar{x}' - \bar{x}) \quad (7.47)$$

$$D(\bar{x}', \bar{x}) = \left[i\gamma^\mu \partial_\mu - m_\psi \right] \delta(\bar{x}' - \bar{x}) \quad (7.48)$$

Using the well known result

$$\int \mathcal{D}\phi \, \exp - \tfrac{1}{2} \int d\bar{x}' d\bar{x} \; \phi(\bar{x}') \, A(\bar{x}', \bar{x}) \phi(\bar{x}) = \exp - \tfrac{1}{2} \mathrm{Tr} \ln A \quad (7.49)$$

and similar results for the other filds gives

$$- \int_0^\beta d\tau \int d^3x \;\; \bar{V}_1(\phi_c) = -\tfrac{1}{2} \mathrm{Tr} \ln A - \tfrac{1}{2} \mathrm{Tr} \ln B + \mathrm{Tr} \ln C + \mathrm{Tr} \ln D \quad (7.50)$$

In evaluating the traces we make use of the (anti-) periodicity of the fields in the range $0 < \tau < \beta$ to express the fields as Fourier series.

Thus to find trlnA we first write

$$\phi(\bar{x}) = \frac{1}{\beta} \sum_n \int \bar{d}^3p \; e^{-ip.\bar{x}} \, \tilde{\phi}(p) \quad (7.51)$$

where $\quad p \equiv (i\omega_n, \underline{p}) \quad (7.52)$

$$p.\bar{x} = \omega_n \tau - \underline{p}.\underline{x} \quad (7.53)$$

$$\bar{d}^3p = \frac{d^3p}{(2\pi)^3} \quad (7.54)$$

$\omega_n = \frac{2\pi n}{\beta}$, $n \in \mathbb{Z}$ are the Matsubara frequencies for bosons. (7.55)

Using

$$\delta(\bar{x}' - \bar{x}) = \frac{1}{\beta} \sum_n \int \bar{d}^3p \; e^{-ip(\bar{x}' - \bar{x})} \quad (7.56)$$

we arrive at

$$A(\bar{x}'-\bar{x}) = \frac{1}{\beta}\sum_n \int \bar{d}^3p \, e^{-ip(\bar{x}'-\bar{x})}\hat{A}(p) \tag{7.57}$$

where

$$\hat{A}(p) = \begin{pmatrix} \omega_n^2+p^2+m^2+3\lambda\phi_c^2 & 0 \\ & \\ 0 & \omega_n^2+p^2+m^2+\lambda\phi_c^2 \end{pmatrix} \tag{7.58}$$

Setting $\bar{x}' = \bar{x}$ and integrating over $\bar{x}$ gives

$$\mathrm{Tr}\ln A = \int_0^\beta d\tau \int d^3x \frac{1}{\beta}\sum_n \int \bar{d}^3p \left[\ln(\omega_n^2+p^2+m^2+3\lambda\phi_c^2) + \ln(\omega_n^2+p^2+m^2+\lambda\phi_c^2)\right] \tag{7.59}$$

$$= \int d^3x \sum_n \int \bar{d}^3p \left[\ln(\omega_n^2+p^2+m^2+3\lambda\phi_c^2) + \ln(\omega_n^2+p^2+m^2+\lambda\phi_c^2)\right] \tag{7.60}$$

From Appendix IIA we see that the temperature dependent contribution from

$$\frac{1}{2}\int \bar{d}^3p \sum_n \left[\ln\{(\omega_n+ia)^2+p^2+R\}\right]$$

$$= -\frac{\pi^2T^4}{90} + \frac{RT^2}{24} - \frac{a^2T^2}{12} \tag{7.61}$$

(The one loop zero temperature contributions to the effective potential are negligible compared to the zero temperature contributions from the tree terms provided that $e^4 \ll \lambda$)

Thus the contribution to V_1^T , the one loop temperature dependent effective potential, from the ' ϕ sector' is

$$-2\frac{\pi^2T^4}{90} + (m^2+2\lambda\phi_c^2)\frac{T^2}{12}$$

The gauge vector boson sector is

$$B_{\mu\nu}(\bar{x}',\bar{x}) = \left[g^{\mu\nu}\partial_\rho'\partial^\rho - (1-\xi^{-1})\partial_\nu'\partial^\nu - g^{\mu\nu}e^2\phi_c^2\right]\delta(\bar{x}'-\bar{x}) \tag{7.46}$$

Again we Fourier transform to give

$$B_{\mu\nu}(\bar{x}',\bar{x}) = \frac{1}{\beta}\sum_n \int \bar{d}^3p \, e^{-ip(\bar{x}'-\bar{x})} \times \left[p^2(g^{\mu\nu}-p^{-2}p^\mu p^\nu)+p^2\xi^{-1}p^{-2}p^\mu p^\nu - e^2\phi_c^2 g^{\mu\nu}\right] \tag{7.62}$$

where we have separated $\tilde{B}_{\mu\nu}(p)$ into projection operators. The logarithm may now be found by taking the logarithm of each coefficient in turn. The trace must be taken in both $\bar{x}$ space and in the space of Lorentz indices. We get

$$\mathrm{Tr}\ln B = \int d^3x \sum_n \int d^3p \Big[3\ln(w_n^2 + p^2 + e^2\phi_c^2) + \ln(w_n^2 + p^2 + \xi e^2\phi_c^2) \Big] \tag{7.63}$$

with $\xi \to 0$ for Landau gauge.

Again using (7.61) we get the gauge boson sector contribution to V_1^T :

$$-4\,\frac{\pi^2 T^4}{90} + 3e^2\phi_c^2\frac{T^2}{24}$$

The Fadeev-Popov sector is

$$C(\bar{x}', \bar{x}) = \partial'_\mu \partial' \,\delta(\bar{x}' - \bar{x}) \tag{7.47}$$

Fourier transforming:

$$C(\bar{x}' - \bar{x}) = \frac{1}{\beta}\sum_n \int d^3p \, e^{-ip(\bar{x}' - \bar{x})} p^2 \tag{7.64}$$

Hence $$\mathrm{Tr}\ \ln\ C = \int d^3x \sum_n \int d^3p \ln(w_n^2 + p^2) \tag{7.65}$$

Using (7.61) the contribution of the Fadeev-Popov sector to V_1^T is:

$$+\,2\,\frac{\pi^2 T^4}{90}$$

The fermion sector is

$$D(\bar{x}', \bar{x}) = \left[i\gamma^\mu \partial_\mu - m_\mu \right] \delta(\bar{x}' - \bar{x}) \tag{7.48}$$

In the light of the anti-periodicity of ψ the appropriate Fourier transform is

$$\psi(\bar{x}) = \frac{1}{\beta}\sum_n \int d^3p \, e^{-ip.\bar{x}}\, \tilde{\psi}(p) \tag{7.66}$$

where the Matsubara frequencies for fermions are

$$w_n = \frac{(2n+1)\pi}{\beta}, \quad n \in \mathbb{Z} \tag{7.67}$$

Hence

$$D(\bar{x}', \bar{x}) = \frac{1}{\beta}\sum_n \int d^3p \, e^{-ip(\bar{x}' - \bar{x})} (-\not{p} + m) \tag{7.68}$$

In evaluating tr ln D the trace must be taken in Dirac indices as well as in x. We find

$$\mathrm{Tr}\ln D = 2\int d^3x \sum_n \int d^3p \ln(\omega_n^2 + p^2 + M_\psi^2) \tag{7.69}$$

Appendix IIA gives the result for fermions

$$\frac{1}{2}\int d^3p \sum_n \left[\ln(\omega_n + ia)^2 + p^2 + R\right]$$
$$= \frac{7}{8}\frac{\pi^2 T^4}{90} + \frac{a^2 T^2}{24} - \frac{R T^2}{48} \tag{7.70}$$

Thus the fermion sectcr contributes to V_1^T

$$-4\cdot\frac{7}{8}\frac{\pi^2 T^4}{90} + M_\psi^2\frac{T^2}{12}$$

For massless fermions (with only one helicity state) a similar calculation, using Weyl spinors, gives half the above answer.

Adding all the contributions to $V_1^T(\phi_c)$ gives

$$\begin{aligned} V_1^T(\phi_c) &= -2\frac{\pi^2 T^4}{90} + (m^2 + 2\lambda\phi_c^2)\frac{T^2}{12} \\ &\quad -4\frac{\pi^2 T^4}{90} + 3e^2\phi_c^2\frac{T^2}{24} \\ &\quad +2\frac{\pi^2 T^4}{90} \\ &\quad -4\cdot\frac{7}{8}\frac{\pi^2 T^4}{90} + m_\psi^2\frac{T^2}{12} \\ &= -\frac{\pi^2 T^4}{90}\left(N_B + \frac{7}{8}N_F\right) + (4\lambda + 3e^2)\phi_c^2\frac{T^2}{24} \end{aligned} \tag{7.71}$$

for $m^2, m_\psi^2 \ll T^2$.

Notice that the Fadeev-Popov sector cancels the contribution of the two unphysical states of freedom of the gauge field.

We have written N_B as the number of bosonic degrees of freedom and N_F the number of fermionic degrees of freedom. ($N_F = 4$ for a Dirac field : two helicity states for the particle and two for its anti-particle, $N_F = 2$ for a Weyl field.)

Including tree terms (7.43) we arrive at

$$V(\phi_c) \simeq -\left(N_B + \frac{7}{8}N_F\right)\frac{\pi^2 T^4}{90} + \frac{1}{2}m^2(T)\phi_c^2 + \frac{1}{4}\lambda\phi_c^4 \tag{7.72}$$

where $$m^2(T) = m^2 + \frac{4\lambda + 3e^2}{12} \cdot T^2 \tag{7.73}$$

Minimizing $V(\phi_c)$ gives

$$\phi_c^2 = 0 \ , \ \phi_c^2 = -\frac{1}{\lambda}\left[m^2 + \frac{4\lambda+3e^2}{12} T^2\right] \tag{7.74}$$

Thus there is symmetry breaking and the critical temperature is given by

$$T_c^2 = -\frac{12\, m^2}{4\lambda + 3e^2} \tag{7.75}$$

For $T > T_c$ $m^2(T)$ is negative and at the minimum of the effective potential $\phi_c = 0$: the system is in the symmetric phase. For $T < T_c$ $m^2(T)$ is positive, the $\phi_c = 0$ minimum becomes a maximum and the system is in the energetically preferred anti-symmetric phase

$$\phi_c^2 = -\frac{1}{\lambda}\left[m^2 + \frac{4\lambda + 3e^2}{12} T^2\right]$$

The system passes continuously from one phase to the other at $T = T_c$ and there is a second order phase transition.

Chapter 7: References

Bailin and Love. "Introduction to Gauge Field Theories"
Adam Hilger:to be published.

Fetter and Welecka (1971) "Quantum Theory of Many Particle Systems" McGraw Hill, New York.

Chapter 8: Field Theory At Finite Density

Introduction

Having discussed the problem of finite temperature we can now extend our discussion to include finite bosonic and fermionic densities. Several authors (see chapter 6) have extensively treated the inclusion of fermion densities,fewer the inclusion of bosonic, and in particular Higgs scalar densities.

We follow Kapusta (8.1) by first examining a simple non-inter -acting scalar theory to show how the introduction of a chemical potential μ multiplying the conserved number density in the Hamiltonian leads to a simple prescription for its introduction into the effective Lagrangian.

The case of a fermionic chemical potential is rather more easy and is discussed briefly. Finally the symmetry behaviour,at finite temperature and density,of the Higgs model is studied.

A Non-Interacting Scalar Theory

We first consider a simple non-interacting scalar theory.

The partition function is (7.10)

$$Z = N \int \mathcal{D}\pi \int_{\text{Periodic}} \mathcal{D}\phi \, \exp \int_0^\beta d\tau \int d^3x \left[i\pi\dot{\phi} - H(\pi, \phi) \right] \tag{8.1}$$

where $\dot{\phi} = \partial\phi/\partial\tau$ as before (7.9).

Now, let the system admit a set of mutally commuting,con- served, addative observables $\underline{N}$, then we may associate with them a set of chemical potentials, $\underline{\mu}$.

The partition function is then

$$Z = \mathrm{Tr} \; \exp -\beta(H + \underline{\mu}\cdot\underline{N}) \tag{8.2}$$

$$= N \int \mathcal{D}\pi \int_{\text{Periodic}} \mathcal{D}\phi \, \exp \int_0^\beta d\tau \int d^3x \left[i\pi\dot{\phi} - H(\pi,\phi) + \underline{\mu}\cdot\underline{N} \right] \tag{8.3}$$

We consider a model with Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{8.4}$$

where $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

There is a global U(1) symmetry with conserved current

$$J_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \tag{8.5}$$

Hence $N = J_0 = i(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*)$ (8.6)

We have

$$\pi_1 = \frac{\partial \phi_1}{\partial t}, \quad \pi_2 = \frac{\partial \phi_2}{\partial t} \tag{8.7}$$

$$H = \tfrac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)\right] \tag{8.8}$$

and $N = \phi_2 \pi_1 - \phi_1 \pi_2$ (8.9)

Thus

$$Z = \int \mathcal{D}\pi_1 \mathcal{D}\pi_2 \int_{\text{Periodic}} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp \int_0^\beta d\tau \int d^3x \Big\{ i\pi_1 \dot{\phi}_1 + i\pi_2 \dot{\phi}_2$$

$$+ \mu(\phi_2\pi_1 - \phi_1\pi_2) - \tfrac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1^2) + (\nabla\phi_2^2) + m^2(\phi_1^2 + \phi_2^2)\right]\Big\} \tag{8.10}$$

$$= N \int_{\text{Periodic}} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp \int_0^\beta d\tau \int d^3x \left\{ -\tfrac{1}{2}\left[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)\right]\right\}$$

$$\times \int \mathcal{D}\pi_1 \exp \int_0^\beta d\tau \int d^3x \, (i\pi_1\dot{\phi}_1 - \tfrac{1}{2}\pi_1^2 + \mu\phi_2\pi_1)$$

$$\times \int \mathcal{D}\pi_2 \exp \int_0^\beta d\tau \int d^3x \, (i\pi_2\dot{\phi}_2 - \tfrac{1}{2}\pi_2^2 - \mu\phi_1\pi_2) \tag{8.11}$$

Performing the momentum integrals by completing the square gives

$$Z = N'(\beta) \int_{\text{Periodic}} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp \int_0^\beta d\tau \int d^3x \, -\tfrac{1}{2}\left[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)\right]$$

$$\cdot \exp \tfrac{1}{2}\int_0^\beta d\tau \int d^3x \left[\mu^2\phi_2^2 + 2i\mu\phi_2 \frac{\partial\phi_1}{\partial\tau} - \left(\frac{\partial\phi_1}{\partial\tau}\right)^2\right]$$

$$\cdot \exp \tfrac{1}{2}\int_0^\beta d\tau \int d^3x \left[\mu^2\phi_1^2 - 2i\mu\phi_1 \frac{\partial\phi_2}{\partial\tau} - \left(\frac{\partial\phi_2}{\partial\tau}\right)^2\right] \tag{8.12}$$

$$= N'(\beta) \int_{\text{Periodic}} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp -\tfrac{1}{2}\int_0^\beta d\tau \int d^3x \Big[\left(\frac{\partial\phi_1}{\partial\tau}\right)^2 + \left(\frac{\partial\phi_2}{\partial\tau}\right)^2$$

$$+ (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2) + 2i\mu\left(\phi_1 \frac{\partial\phi_2}{\partial\tau} - \phi_2 \frac{\partial\phi_1}{\partial\tau}\right)$$

$$- \mu^2(\phi_1^2 + \phi_2^2)\Big] \tag{8.13}$$

Defining

$$\mathcal{L}' = (\partial^\mu + i\mu\delta^{\mu 0})\phi^*(\partial_\mu - i\mu\delta_{\mu 0})\phi \quad - m^2\phi^*\phi \tag{8.14}$$

(i.e. making the substitution $\partial_0 \to \partial_0 - i\mu$ in (8.4)) we get

$$Z = N'(\beta)\int_{\text{Periodic}} \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp -\tfrac{1}{2}\int_0^\beta d\tau \int d^3x\, \mathcal{L}'(\phi) \tag{8.15}$$

$\mathcal{L}'$ is obtained from $\mathcal{L}$ by the substitution $\partial_0 \to \partial_0 - i\mu$

Although we have only shown this to be the case for a simple non-interacting theory, it holds in general provided that there is a conserved number density of the form $i(\phi^+ D_0\phi - D_0\phi^+\phi)$ Then we introduce the term $i\mu(\phi^+ D_0\phi - D_0\phi^+\phi)$ to the Hamiltonian. The momentum integrations in this case are slightly more complicated but the result holds true: the introduction of the bosonic chemical potential μ leads to the substitution in the Lagrangian:

$$\partial_0 \to \partial_0 - i\mu \quad \text{in} \quad D_\mu\phi^+ D^\mu\phi.$$

In the case of fermions the conserved number density normally depends only on the fields and not on their conjugate momenta. This will certainly be the case in the theories we will be considering. Thus $\underline{\mu}.\underline{N}$ will not enter into the momentum integrations which may be carried out in the usual way giving

$$Z = N'(\beta)\int_{\text{Anti-Periodic}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\int_0^\beta d\tau\int d^3x\left[\mathcal{L}(\psi,\partial_\mu\psi) + \underline{\mu}.\underline{N}(\psi)\right] \tag{8.16}$$

The Higgs Model At Finite Temperature and Density

As in chapter 7 we use the Lagrangian (7.38)

$$\mathcal{L} = (\bar{D}_\mu \phi)^* (\bar{D}^\mu \phi) - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} - \frac{1}{2\xi} (\bar{\partial}_\mu A^\mu)^2 + \bar{\partial}_\mu \eta^* \bar{\partial}^\mu \eta + \bar{\psi} (i \gamma^\mu \partial_\mu - m_\psi) \psi - e \bar{\psi} \gamma_\mu \psi A^\mu$$

$\bar{F}_{\mu\nu}$, $\bar{D}_\mu$, $\bar{\partial}_\mu$ are as in (7.39),(7.40) and (7.41).

There are conserved currents

$$J_\mu^\phi = ie \left[\phi (\bar{D}_\mu \phi^*) - \phi^* (\bar{D}_\mu \phi) \right] \quad (8.17)$$

and

$$J_\mu^\psi = - e \bar{\psi} \gamma^\mu \psi \quad (8.18)$$

We associate chemical potentials $\tilde{\mu}$, for the Higgs field, and μ, for the fermion field with J_0^ϕ and J_0^ψ respectively.

Using the results of the previous sections we get

$$\mathcal{L}' = (\bar{\partial}_\mu + ieA_\mu + i\tilde{\mu}\delta_{\mu 0}) \phi^* (\bar{\partial}^\mu - ieA^\mu - i\tilde{\mu}\delta^{\mu 0}) \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} \bar{F}^{\mu\nu} \bar{F}_{\mu\nu} - \frac{1}{2\xi} (\bar{\partial}_\mu A^\mu)^2 + \bar{\partial}_\mu \eta^* \bar{\partial}^\mu \eta + \bar{\psi} (i\gamma^\mu \partial_\mu - m_\psi) \psi - e \bar{\psi} \gamma_\mu \psi A^\mu + \mu \bar{\psi} \gamma^0 \psi \quad (8.19)$$

As in chapter 7 we shift the scalar field by its expectation value, writing

$$\phi = \frac{1}{\sqrt{2}} (\phi_c + \phi_1 + i\phi_2) \quad (8.20)$$

In this case,however, we must also shift the gauge field since it will also develop an expectation value as a result of the intro -duction of the chemical potentials. (μ is coupled to $\langle \bar{\psi} \psi \rangle$, the number density of fermions, this is in turn linked to the gauge field A by terms of the form $\bar{\psi} \psi A_\mu$ in the Lagrangian. Since $\bar{\psi} \psi$ has a non-zero expectation value A_0 will have also, through the tadpole diagrams associated with $\bar{\psi} \psi \, A_0$.)

So we write

$$A_\mu \rightarrow (A_c, \underline{0}) + B_\mu \quad (8.21)$$

Performing these shifts gives the tree terms

$$\frac{1}{2}m^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4 - \frac{1}{2}(\tilde{\mu} + eA_c)^2\phi_c^2 \tag{8.22}$$

and the quadratic terms in the Lagrangian are

$$\begin{aligned}\mathcal{L}_{quad} = &\frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2 \\ &+ (\tilde{\mu} + eA_c)(\phi_2\partial_0\phi_1 - \phi_1\partial_0\phi_2) + \frac{1}{2}e^2\phi_c^2 B_\mu B^\mu \\ &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu B^\mu)^2 + \partial_\mu\eta^*\partial^\mu\eta \\ &+ \overline{\psi}\left[i\gamma^\mu\partial_\mu - m_\psi + \gamma^0(\mu + eA_c)\right]\psi\end{aligned} \tag{8.23}$$

where $\quad m_1^2 = m^2 + 3\lambda\phi_c^2 - (\tilde{\mu} + eA_c)^2$ (8.24)

and $\quad m_2^2 = m^2 + \lambda\phi_c^2 - (\tilde{\mu} + eA_c)^2$ (8.25)

As before (chapter 7) we have neglected cross terms

$e\phi_c B_\mu\partial_\mu\phi_2$ and $2e\phi_c(\tilde{\mu} + eA_c)B_0\phi_1$.

We proceed as before, separating $\mathcal{L}_{quad}$ into the 4 sectors (Higgs,gauge bosons, Fadeev-Popov ghosts and fermions).The gauge boson and ghost sectors are identical to those of chapter 7.

The scalar sector: after Fourier transforming the matrix A has the transform

$$\tilde{A} = \begin{pmatrix} \omega_n^2 + p^2 + m^2 + 3\lambda\phi_c^2 - \tilde{\alpha}^2 & 2\omega_n\tilde{\alpha} \\ -2\omega_n\tilde{\alpha} & \omega_n^2 + p^2 + m^2 + \lambda\phi_c^2 - \tilde{\alpha}^2 \end{pmatrix} \tag{8.26}$$

where $\quad \tilde{\alpha} = \tilde{\mu} + eA_c$

Using tr ln A = ln det A we obtain

$$\begin{aligned}\mathrm{tr}\ln A = \int d^3x \sum_n \int d^3p \Big\{ &\ln\left[(\omega_n + i\tilde{\alpha})^2 + p^2 + m^2 + 3\lambda\phi_c^2\right] \\ &+ \ln\left[(\omega_n - i\tilde{\alpha})^2 + p^2 + m^2 + \lambda\phi_c^2\right]\Big\}\end{aligned} \tag{8.27}$$

which gives a contribution to V_1^T (after using (7.61)) of

$$-2\frac{\pi^2 T^4}{90} + \frac{m^2 + 2\lambda\phi_c^2}{12}T^2 - \tilde{\alpha}^2\frac{T^2}{6}$$

The fermion sector is

$$\bar{\psi}\left[i\gamma^\mu\partial_\mu - m_\psi + \gamma^0\alpha\right]\psi \tag{8.28}$$

where $\alpha = \mu + eA_c$

The matrix $\tilde{D}$ is now

$$\tilde{D} = -\not{\partial} - m_\psi + (\alpha\gamma^0) \tag{8.29}$$

Hence

$$\mathrm{tr}\ln D = 2\int d^3x \sum_n \int d^3p \ln\left[(\omega_n - i\alpha)^2 + p^2 + m_\psi^2\right] \tag{8.30}$$

which gives a contribution to V_1^T of

$$-4\cdot\frac{7}{8}\frac{\pi^2T^4}{90} + m_\psi^2\frac{T^2}{12} - \alpha^2\frac{T^2}{6}$$

From chapter 7 we find the contribution of the gauge vector bosons and Fadeev-Popov ghosts to be

$$-2\frac{\pi^2T^4}{90} + 3e^2\phi_c^2\frac{T^2}{24}$$

V_1^T is therefore given by

$$V_1^T \simeq -\left(N_B + \frac{7}{8}N_F\right)\frac{\pi^2T^4}{90} + (4\lambda + 3e^2)\phi_c^2\frac{T^2}{24} - (\hat{\alpha}^2 + \alpha^2)\frac{T^2}{6} \tag{8.32}$$

for $m^2, m_\psi^2 << T^2$

Together with the tree terms (8.22) we find

$$V(\phi_c) \simeq \frac{1}{2}m^2(T)\phi_c^2 + \frac{1}{4}\lambda\phi_c^4 - \frac{1}{2}\hat{\alpha}^2\phi_c^2 - \frac{1}{6}(\hat{\alpha}^2 + \alpha^2)T^2 - \left(N_B + \frac{7}{8}N_F\right)\frac{\pi^2T^4}{90} \tag{8.33}$$

where

$$m^2(T) = m^2 + \frac{4\lambda + 3e^2}{12}T^2 \tag{8.34}$$

and $\hat{\alpha} = \hat{\mu} + eA_c$, $\alpha = \mu + eA_c$

Minimising $V(\phi_c)$:

$$\frac{\partial V}{\partial\phi_c} = 0 \Rightarrow \phi_c = 0,\ \phi_c^2 = -\frac{1}{\lambda}\left[m^2(T) - \hat{\alpha}^2\right] \tag{8.35}$$

The critical temperature, T_c, is thus given by

$$T_c^2 = -\frac{12}{4\lambda + 3e^2}\left[m^2 - (\hat{\mu} + eA_c)^2\right] \tag{8.36}$$

i.e.

$$m^2(T_c) = \hat{\alpha}^2 \tag{8.37}$$

The number densities for bosons and fermions are given by

$$\tilde{n} = -\partial V/\partial\tilde{\mu} \tag{8.38}$$

$$n = -\partial V/\partial\mu \tag{8.39}$$

Hence

$$\tilde{n}\Big|_{\phi_c=0} = \frac{1}{3}T^3(\tilde{\mu}+eA_c) \tag{8.40}$$

and

$$n\Big|_{\phi_c=0} = \frac{1}{3}T^3(\mu+eA_c) \tag{8.41}$$

We also have the condition for equilibrium

$$\frac{\partial V}{\partial A_c} = 0 \tag{8.42}$$

which gives at $\phi_c = 0$,

$$-e\frac{T^2}{3}\left[(\tilde{\mu}+eA_c)+(\mu+eA_c)\right] = 0 \tag{8.43}$$

Hence, at $\phi_c = 0$

$$\tilde{n} + n = 0 \tag{8.44}$$

That is, the total 'charge' coupled to the U(1) gauge field is zero.

(8.37) and (8.40) give

$$T_c^2 = -\frac{12}{4\lambda+3e^2}\left(m^2 - \frac{9\tilde{n}^2}{T_c^4}\right) \tag{8.45}$$

T_c is the temperature at which symmetry is restored. From (8.45) we can now see that for

$$|\tilde{n}| = |n| \sim \frac{1}{6}T^3\left(\frac{4\lambda}{3}+e^2\right)^{1/2} \tag{8.46}$$

symmetry restoration will be prevented. Thus a high enough density can ensure that there is no restoration of symmetry.

Chapter 8: References

8.1 Kapusta (1981) Phys.Rev. D24 426

Chapter 9: Electroweak Theory

Having discussed the behaviour of the simple Higgs model at finite temperature and density we move on in this chapter to study the more realistic model of the Salam-Weinberg-Glashow standard electroweak theory.

A bosonic chemical potential $\bar{\mu}$ is coupled to the U(1) of weak hypercharge. Fermionic chemical potentials are introduced coupled to the various $SU_L(2) \times U(1)$ invariants which may be constructed from the quark and lepton doublets and singlets.

The terms to be introduced into the Lagrangian are then

$$\mathcal{L}_{\bar{\mu}} = (\bar{D}_0 + i\bar{\mu})\Phi^+(\bar{D}_0 - i\bar{\mu})\Phi \tag{9.1}$$

(Φ is the Higgs scalar doublet) and

$$\begin{aligned}\mathcal{L}_{\mu} &= \mu_1(\nu_L^+\nu_L + e_L^+e_L) + \mu_2(e_R^+e_R) \\ &+\mu_1^q(u_{Li}^+u_{Li} + d_{Li}^+d_{Li}) + \mu_2^q(u_{Ri}^+u_{Ri}) \\ &+\mu_3^q(d_{Ri}^+d_{Ri}) \quad +\text{(other generations)}\end{aligned} \tag{9.2}$$

i = 1, 2, 3 label the quark colours.

For simplicity we assume that the same chemical potentials couples to all generations. The generalization to a situation in which different chemical potentials are coupled to each generation is simple.

The rest of the Lagrangian is

$$\begin{aligned}\mathcal{L} &= (D_\mu\Phi^+)(D_\mu\Phi) - m^2\Phi^+\Phi - \lambda(\Phi^+\Phi)^2 \\ &- \tfrac{1}{4}\underline{W}_{\mu\nu}\cdot\underline{W}^{\mu\nu} - \tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ &+ (\bar{\nu}_L\ \bar{e}_L)(i\gamma^\mu D_\mu)\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} + \bar{e}_R(i\gamma^\mu D_\mu)e_R \\ &+ (\bar{u}_L\ \bar{d}_L)(i\gamma^\mu D_\mu)\begin{pmatrix}u_L\\ d_L\end{pmatrix} + \bar{u}_R(i\gamma^\mu D_\mu)u_R \\ &+ \bar{d}_R(i\gamma^\mu D_\mu)d_R\end{aligned}$$

$$+ \text{(other generations)} + \text{(Fadeev- Popov terms)} \tag{9.3}$$

As usual we make the shift

$$\Phi \rightarrow \Phi + \begin{pmatrix} 0 \\ \phi_c \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_c + \chi_1 + i\chi_2 \end{pmatrix} \tag{9.4}$$

As before we expect the U(1) gauge field to develop a non-zero expectation value and so we make the shift

$$B_\mu \rightarrow B_\mu + (B_c , \underline{0}) \tag{9.5}$$

The W_a^μ fields will not develop non-zero expectation values because the chemical potentials are U(1) factors affecting only the U(1) field, B_μ .

After these shifts we get

$$\mathcal{L}_{tree} = \tfrac{1}{2}(\tilde{\mu} - \tfrac{1}{2}g' B_c)^2 \phi_c^2 - \tfrac{1}{2} m^2 \phi_c^2 - \tfrac{1}{4}\lambda \phi_c^4 \tag{9.6}$$

and the relevant quadratic terms

$$\begin{aligned}
\mathcal{L}_{quad} = {} & -\tfrac{1}{4} \underline{W}^{\mu\nu} \cdot \underline{W}_{\mu\nu} - \tfrac{1}{4} B^{\mu\nu} B_{\mu\nu} \\
& - (\bar{\nu}_L \;\; \bar{e}_L)\left[-i\gamma^\mu \partial_\mu + \gamma^0 (\mu_1 + \tfrac{1}{2} g' B_c)\right] \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
& - (\bar{u}_L \;\; \bar{d}_L)\left[-i\gamma^\mu \partial_\mu + \gamma^0 (\mu_1^q - \tfrac{1}{6} g' B_c)\right] \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\
& - \bar{u}_R \left[-i\gamma^\mu \partial_\mu + \gamma^0 (\mu_2^q - \tfrac{2}{3} g' B_c)\right] u_R \\
& - \bar{d}_R \left[-i\gamma^\mu \partial_\mu + \gamma^0 (\mu_3^q + \tfrac{1}{3} g' B_c)\right] d_R \\
& + \tfrac{1}{2}(\partial_\alpha \phi_1)^2 + \tfrac{1}{2}(\partial_\alpha \phi_2)^2 + \tfrac{1}{2}(\partial_\alpha \chi_1)^2 + \tfrac{1}{2}(\partial_\alpha \chi_2)^2 \\
& - \tfrac{1}{2}\left[m^2 + \lambda \phi_c^2 - (\tilde{\mu} - \tfrac{1}{2} g' B_c)^2 \right] (\phi_1^2 + \phi_2^2) \\
& - \tfrac{1}{2}\left[m^2 + 3\lambda \phi_c^2 - (\tilde{\mu} - \tfrac{1}{2} g' B_c)^2 \right] \chi_1^2 \\
& - \tfrac{1}{2}\left[m^2 + \lambda \phi_c^2 - (\tilde{\mu} - \tfrac{1}{2} g' B_c)^2 \right] \chi_2^2 \\
& + (\tilde{\mu} - \tfrac{1}{2} g' B_c)(\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) \\
& + (\tilde{\mu} - \tfrac{1}{2} g' B_c)(\chi_1 \partial_0 \chi_2 - \chi_2 \partial_0 \chi_1) \\
& + \tfrac{1}{8} g^2 \underline{W}^\mu \cdot \underline{W}_\mu \phi_c^2 + \tfrac{1}{8} g'^2 B_\mu B^\mu \phi_c^2 \\
& + \partial_\mu \eta_1^* \gamma^\mu \eta_1 + \partial_\mu \eta_2^* \gamma^\mu \eta_2 \\
& - \tfrac{1}{2\alpha}(\partial_\mu B^\mu)^2 - \tfrac{1}{2\alpha}(\partial_\mu \underline{W}^\mu) \cdot (\partial_\nu \underline{W}^\nu)
\end{aligned}$$

(9.7)

As before we separate $\mathcal{L}_{quad}$ into its various sectors. The methods and results of chapter 8 give

$$V_{eff} = \frac{1}{2} m^2(T)\phi_c^2 + \frac{1}{4}\lambda\phi_c^4 - \xi\frac{\pi^2 T^4}{90}$$

$$- N_G \frac{T^2}{12}\Big[2(\mu_1 + \tfrac{1}{2}g'B_c)^2 + (\mu_2 + g'B_c)^2$$

$$+ 6(\mu_1^q - \tfrac{1}{6}g'B_c)^2 + 3(\mu_2^q - \tfrac{2}{3}g'B_c)^2 + 3(\mu_3^q + \tfrac{1}{3}g'B_c)^2\Big]$$

$$- \frac{1}{3}T^2(\tilde{\mu} - \tfrac{1}{2}g'B_c)^2 - \frac{1}{2}(\tilde{\mu} - \tfrac{1}{2}g'B_c)^2\phi_c^2$$

(9.8)

where $$\xi = N_B + \frac{7}{8}N_F \qquad (9.9)$$

and
$$m^2(T) = m^2 + \frac{3g^2 + g'^2 + 8\lambda}{16}.T^2 \qquad (9.10)$$

N_G is the number of generations.

Number densities $\tilde{n}$, n_1,n_2, n_1^q , n_2^q , n_3^q are given by

$$n_p = -\partial V_{eff}/\partial\mu_p \qquad (9.11)$$

The effective potential is minimised with respect to ϕ_c to give an expression for the critical temperature, T_c :

$$m^2(T_c) = (\mu^2 - \tfrac{1}{2}g'B_c)^2\Big|_{\phi_c=0} \qquad (9.12)$$

We also have the condition for equilibrium

$$\partial V_{eff}/\partial B_c = 0 \qquad (9.13)$$

(9.11),(9.12) and (9.13) give

$$\frac{1}{2}n_1 + n_2 - \frac{1}{6}n_1^q - \frac{2}{3}n_2^q + n_3^q - \frac{1}{2}\tilde{n} = 0 \qquad (9.14)$$

i.e. the total weak hypercharge is zero.

The critical scalar density $\tilde{n}_c$ required to prevent the restoration of symmetry at high temperatures is

$$|\tilde{n}_c| \sim \frac{1}{6}T^3\left[3g^2 + g'^2 + 8\lambda\right]^{1/2} \qquad (9.15)$$

$\tilde{n}$ is related to the fermion densities by (9.14). In particular if we assume that the only large asymmetry was due to neutrinos then we could set n_2, n_1^q, n_2^q, and n_3^q to zero and would have

$$\tilde{n} \simeq n_1 \simeq n_\nu - n_{\bar{\nu}}$$

Then (9.15) gives the critical neutrino asymmetry to prevent symmetry restoration at high temperatures. This result is in order of magnitude agreement with previous estimates (9.1,9.2,9.3) though not in precise agreement because a complete set of chemical potentials was not included in previous calculations.

The Higgs Scalar Density

The Higgs scalar density was introduced to balance the weak hypercharge of the neutrinos. In this section we discuss the possibility that this density could be a real density in a closed universe, rather than a fictitious density, to simulate zero charge coupled to the massless U(1) gauge field,in an open universe.

At finite temperature,the decay rate Γ of a scalar field of mass M_ϕ into light particles is given by (ref 9.4)

$$\Gamma \sim \alpha_\phi M_\phi^2 \sqrt{T^2 + M_\phi^2} \tag{9.16}$$

where $$\alpha_\phi = g_\phi^2 / 4\pi \tag{9.17}$$

where g_ϕ is the coupling constant for the decay vertex. Most scalars decay when

$$\Gamma \sim H \tag{9.18}$$

where H is the Hubble constant given by

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 \simeq \frac{8\pi}{3} G \rho \tag{9.19}$$

with ρ the energy density and G the gravitational constant. If Higgs scalar decay occurs for $T \gg m_\phi$ (and we shall see that this is the case) then the energy density is radiation dominated and

$$\rho = \frac{\pi^2 \zeta T^4}{30} \tag{9.20}$$

with ζ as in (9.9).

If we take $\zeta \sim 100$ (9.21)

then

$$H \sim 16 \frac{T^2}{M_{pl}} \tag{9.22}$$

where m_{pl} is the Planck mass,

$$M_{pl} = G^{-\frac{1}{2}} \sim 10^{19}\ \mathrm{GeV} \tag{9.23}$$

Using (9.16,9.18,9.22) for $T \gg m_\phi$ we estimate that most Higgs scalars decay when $T = T_D$ where

$$T_D^3 \simeq \frac{\alpha_\phi M_{pl} M_\phi^2}{16} \tag{9.24}$$

Also if $\phi \to f\bar{f}$ is the dominant ϕ decay mode, where f is a quark or lepton, then

$$\alpha_\phi \sim \frac{\alpha M_f^2}{M_W^2} \tag{9.25}$$

where α is the fine structure constant.

Taking $m \sim 10$ GeV for the Higgs scalar, and $m_f \sim m_\tau \sim$ 2 GeV leads to

$$T_D \sim 6 \times 10^4\ \mathrm{GeV} \tag{9.26}$$

(with a larger Higgs scalar mass, and decay into,say, top quarks, the value for T_D would be even larger).

It is thus not consistent to assume a Higgs scalar asymmetry persisting to temperatures of the order of the electroweak scale ($\sim$ 100 GeV).

We cannot therefore assume that the weak hypercharge of a neutrino asymmetry was neutralized by a real Higgs scalar density (in a closed universe) at temperatures less than T_D of (9.26).

In equilibrium, thermal effects can recreate densities of Higgs scalars and their anti-particles of order T^3, but no asymmetry,i.e. excess of particles over anti-particles can arise in this way.

It is not possible to assume the existance of a large neutrino asymmetry in a closed universe surviving at electroweak scale. This is because the Higgs scalar asymmetry needed to neutralize the weak hypercharge of the neutrinos would have long since decayed. Then drastic cosmological inflation would have to occur, driven by the net weak hypercharge, which would provide a long-range force in the symmetric phase. This would dilute any existing baryon asymmetry to a negligible value, with no possibility of regenerating it at these low temperatures ($\sim$ 100 GeV). A large neutrino asymmetry would, however, be acceptable in an open universe where 'charge' neutrality is not required (9.5), and could prevent symmetry restoration at high temperatures, as discussed above and by several earlier authors (9.1,9.2,9.3,9.6).

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9.2 Linde (1979) Phys.Lett. <u>86B</u> 39

9.3 Bailin and Love (1983) Nucl.Phys. <u>B226</u> 493

9.4 Weinberg (1979) Phys.Rev.Lett <u>42</u> 850

9.5 Haber and Weldon (1982) Phys.Rev. <u>D25</u> 502

9.6 Linde (1979) Rep.Prog.Phys. <u>42</u> 389
and references therein.

Chapter 10: Grand Unified Theories

We illustrate the situation in grand unified theories by considering the SU(5) x U(1) $\longrightarrow$ SU(5) transition in the chain of symmetry breaking:

SO(10) $\longrightarrow$ SU(5) x U(1) $\longrightarrow$ SU(5) $\longrightarrow$ SU(3)xSU(2)xU(1)

Let the spontaneous symmetry breaking be produced by a Higgs scalar ϕ which couples to the U(1) gauge field with strength k. (ϕ may belong, for example, to a 16 or 126 of SO(10)) (Rajpoot (10.1)).

$$D^\mu\phi = \partial^\mu\phi + ikA^\mu\phi \tag{10.1}$$

where in terms of SO(10) gauge fields (ref. 10.1)

$$A^\mu = A^\mu_{12} + A^\mu_{34} + A^\mu_{9\,10} - A^\mu_{56} - A^\mu_{78} \tag{10.2}$$

A bosonic chemical potential is introduced in the usual way, that is we make the substitution

$$\partial_0 \rightarrow \partial_0 - i\tilde{\mu} \tag{10.3}$$

in $D^\mu\phi$.

Fermionic chemical potentials are introduced via

$$\begin{aligned} J_\mu = {} & \mu_5\left(d^+_{Ri}d_{Ri} - \nu^+_L\nu_L - e^+_L e_L\right) \\ & + \mu_{10}\left(u^+_{Li}u_{Li} + d^+_{Li}d_{Li} - u^+_{Ri}u_{Ri} - e^+_R e_R\right) \\ & - \mu_1\left(\nu^+_R\nu_R\right) \\ & + \text{other generations} \end{aligned} \tag{10.4}$$

Again we make the assumption, for simplicity, that the same chemical potentials couple to all generations. A generalisation to the case where different chemical potentials couple to each generation is simple. For example the μ_1 term of J_μ would become

$$-\sum_g \mu_1^g\left(\nu_R^{g+}\nu_R^g\right) \tag{10.5}$$

where the generations are labelled by g.

ν_R is the 'right-handed neutrino' from the 16 of SO(10).

After performing the usual calculations we obtain

$$V_{eff} = m^2(T)\phi_c^2 + \frac{\lambda}{4}\phi_c^4 - \phi_c^2(kA_c - \tilde{\mu})^2$$
$$- \frac{T^2}{12}N_G\left[5(\mu_5 - \tfrac{3}{2}gA_c)^2 + 10(\mu_{10} - \tfrac{1}{2}gA_c)^2\right.$$
$$\left. + (\mu_1 - \tfrac{5}{2}gA_c)^2\right] - \frac{T^2}{6}(kA_c - \tilde{\mu})^2 \qquad (10.6)$$

N_G is the number of generations, g the gauge coupling constant for A^μ, A_c is the expectation value of A^0 and

$$m^2(T) = m_\phi^2 + \left(\frac{\lambda}{12} + \frac{25}{16}g^2\right)T^2 \qquad (10.7)$$

$$(m_\phi^2 < 0). \qquad (10.8)$$

Generalising to the case in which different generations have different chemical potentials coupled to them would give

$$V_{eff} = m^2(T)\phi_c^2 + \frac{\lambda}{4}\phi_c^4 - \phi_c^2(kA_c - \tilde{\mu})^2$$
$$- \frac{T^2}{12}\sum_g\left[5(\mu_5^g - \tfrac{3}{2}gA_c)^2 + 10(\mu_{10}^g - \tfrac{1}{2}gA_c)^2\right.$$
$$\left. + (\mu_1^g - \tfrac{5}{2}gA_c)^2\right] - \frac{T^2}{6}(kA_c - \tilde{\mu})^2$$

(10.9)

Minimising V_{eff} with respect to ϕ_c and using the equilibrium condition

$$\frac{\partial V_{eff}}{\partial \phi_c} = 0 \qquad (10.10)$$

gives us the critical scalar asymmetry

$$|\tilde{n}_c| \sim \frac{T^3}{12}\left(\frac{4}{3}\lambda + 25g^2\right)^{1/2} \qquad (10.11)$$

and the equilibrium condition ;

$$\frac{1}{2}n_{10} + \frac{3}{2}n_5 + \frac{5}{2}n_1 + \frac{k}{g}\tilde{n} = 0 \qquad (10.12)$$

where $$\tilde{n} = -\partial V_{eff}/\partial\tilde{\mu} \qquad (10.13)$$

and $$n_\alpha = -\partial V_{eff}/\partial\mu_\alpha, \quad \alpha = 1,5,10 \qquad (10.14)$$

For ϕ in the 16 of SO(10) $k = \frac{5}{2}g$ (10.15)

and for ϕ in the 126 of SO(10) $k = 5g$ (10.16)

For SU(5) invariant asymmetries, the difference of the baryon number density, n_b, and the lepton number density, n_L, is given by (ref 10.2)

$$n_b - n_L = \frac{3}{5} n_5 + \frac{1}{5} n_{10} + n_1 \qquad (10.17)$$

Thus the critical density to prevent symmetry restoration at high temperatures is

$$|(n_b - n_L)_c| = \frac{1}{30} \frac{k}{g} T^3 \left(\frac{4\lambda}{3} + 25 g^2 \right)^{1/2} \qquad (10.18)$$

This is in agreement with the result of ref.(10.2) when $k = \frac{5}{2}$ g, as appropriate to the 16 of SO(10). It is a general feature that omission of the chemical potential for the symmetry breaking Higgs scalar does not significantly alter the calculated value for the critical fermion density, provided a complete set of fermion chemical potentials has been included. However, the physical interpretation changes if the Higgs scalar density is a real one. Inclusion of different chemical potentials would not alter the condition (10.18).

As in the case of electroweak theory we must now determine whether the Higgs scalar density balancing the U(1) charge of the fermions can be a real density, or whether it must be regarded as a fictitious density to simulate zero 'charge' coupled to the massless U(1) gauge field.

The temperature, T_D, at which this Higgs scalar asymmetry would decay can be estimated as follows.(Thereafter there could at most be thermal densities of Higgs scalars and their anti-particles but no asymmetry,i.e. no excess of particles over anti-particles.) We assume $T_D \ll m_\phi$ as we shall see shortly that this is the case.

We can use (9.16) and (9.18) in the calculation of T_D except that now we can no longer assume that the energy density ρ is radiation dominated as in (9.20). Instead we assume that ρ is dominated by the Higgs scalar density. (This resembles the decay of out-of-equilibrium gravitinos (10.3))

Thus

$$\rho \sim M_\phi T^3 \qquad (10.19)$$

for an asymmetry of the order of the critical asymmetry of (10.11).

We obtain $T_D \sim \frac{1}{2}\left(\alpha_\phi^2 \, M_\phi \, M_{pl}^2\right)^{1/3}$

A Higgs scalar in the 126 of SO(10) (which can give a mass to ν_R as well as breaking the SU(5) x U(1) symmetry) will undergo the decay

$$\phi \rightarrow \nu_R \, \nu_R \tag{10.20}$$

with the 'right-handed neutrinos' subsequently decaying to neutrinos and photons. The coupling g_ϕ for this vertex is related to the Majorana mass m_R of the right- handed neutrino and the mass m_A of the U(1) gauge field in the SU(5) symmetric phase by

$$\alpha_\phi = \frac{g_\phi^2}{4\pi} = \alpha_k \frac{M_R^2}{M_A^2} \tag{10.21}$$

with
$$\alpha_k = \frac{k^2}{4\pi} \tag{10.22}$$

and k as in (10.1)

Thus

$$T_D \sim \frac{1}{2}\left(\alpha_k^2 \, M_\phi \, M_{pl}^2\right)^{1/3} \left(M_R/M_A\right)^{4/3} \tag{10.23}$$

For a Higgs scalar belonging to the 126 of SO(10), k coincides with the SU(5) coupling constant in a standard normalisation, so we estimate

$$\alpha_k \sim 1/40 \tag{10.24}$$

Even for $m_R = m_A$ (and in general m_R may be very much less than m_A) (10.23) implies that T_D is less than m_ϕ provided m_ϕ is greater than

$$\frac{\alpha M_{pl}}{\sqrt{8}} \sim 10^{-2} M_{pl} \tag{10.25}$$

We therefore expect T_D to be less than m_ϕ and it is consistent to suppose that there existed a real Higgs scalar asymmetry, at least down to the temperature at which the SU(5)xU(1) $\longrightarrow$ SU(5) transition would have occurred at zero density. Below this temperature the question of symmetry restoration does not arise.

When Higgs scalar decay occurs entropy is generated by the reheating of the universe.

Let T_f be the temperature after the entropy released by Higgs decay has been thermalized and assume a Higgs scalar energy density as in (10.19). Then

$$\pi^2 \xi T_f^4 / 30 \sim M_\phi T_D^3 \tag{10.26}$$

ξ refers to the degrees of freedom light compared with T_D. Thus the increase in entropy is

$$\frac{S_f}{S_i} = \left(\frac{T_f}{T_D}\right)^3 \sim \left(\frac{30 \, m_\phi}{\xi \pi^2 T_D}\right)^{3/4} \tag{10.27}$$

or, using (10.26)

$$\frac{S_f}{S_i} = \left(\frac{T_f}{T_D}\right)^3 \sim \left(\frac{60}{\xi \pi^2}\right)^{3/4} \left(\frac{m_\phi}{M_{pl} \, \kappa_k}\right)^{1/2} \frac{m_A}{m_R} \tag{10.28}$$

For a grand unified scale close to the Planck mass, and a comparatively light 'right-handed neutrino' this can be rather large. For $m_\phi \sim m_A \sim m_{pl}$ and $\xi \sim 100$ we can achieve

$$S_f/S_i \gtrsim 10^{10} \quad \text{for} \quad m_R \lesssim 10^9 \, \text{GeV} \tag{10.29}$$

A Majorana mass for ν_R of this order is compatible (ref 10.4) with a mass for ν_L of $\lesssim 10^{-1}$ eV. It is therefore possible that entropy generation, through the decay of a grand unified Higgs scalar asymmetry, may have diluted the baryon number and lepton number to photon number ratio by as much as 10^{-10}. In that case, the present very small baryon number asymmetry may have arisen from an asymmetry of order 1, before the Higgs scalar decay took place. The considerations of this last paragraph apply whether or not the Higgs scalar asymmetry was large enough to prevent symmetry restoration, provided it was of order T^3. They might apply in the absence of a Higgs scalar asymmetry, if the Higgs scalars dropped out of equilibrium for $T > m_\phi$.

In theories, like SU(5) and SO(10) grand unified theories, where B - L is conserved or nearly conserved, an initial $n_b - n_L$ of order T^3, as in (10.18), would normally (refs. 10.5, 10.6) result in a final $|n_b|$ and $|n_L|$ both of order T^3. For example, when B - L is conserved but B + L is thermalized efficiently (ref. 10.6) the final values $(n_b)_f$ and $(n_L)_f$ of n_b and n_L are related to the initial value $(n_b - n_L)_i$ of $n_b - n_L$ by

$$(n_b)_f = -(n_L)_f = \tfrac{1}{2}(n_b - n_L)_i \tag{10.30}$$

In the present context there is some B - L violation from the 'right-handed neutrino' mass, but this will be insignificant when $m_R << m_A$, as assumed above.

If, on the other hand, the 'right-handed neutrino' were to be heavy ($m_R \sim m_A$), then the entropy generation discussed above would not be large. In that case, the B - L violation from the 'right-handed neutrino' mass would be more important, and we might expect n_b to relax to zero before a small value was regenerated at the SU(5) grand unification scale, in the usual way.

To conclude; it may be that a fermionic asymmetry of order T^3 was neutralized by a grand unified Higgs scalar asymmetry,so that there was no 'charge' density coupled to massless gauge fields. In that case the decay of the Higgs scalar asymmetry at a later stage may have greatly diluted the baryon number asymmetry, perhaps by as much as 10^{-10} . Thus the present small baryon number may have arisen from an initial asymmetry of order T^3.

This is an alternative to the usual scenario where a small baryon number is generated from zero by baryon number violating interactions. Here, a large baryon number density (of order T^3) is diluted to its present value by entropy production. As has been particularly emphasised by Wilczek (10.5) and by Dolgov and Linde (10.6), it is entirely possible, with appropriate initial conditions, for there to have been baryon number and lepton number asymmetries of order T^3 surviving when the baryon and lepton number violating interactions froze out.

Chapter 10: References

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Chapter 11: Supersymmetry

This chapter, and those following are devoted to the question of phase transitions in supersymmetric theories. The notion of supersymmetry is a relatively new one and in the interests of completeness we present, in this chapter, a brief review of the ideas and techniques which go to make up supersymmetric theories.

The supersymmetry is a symmetry between integer spin particles, bosons, and half-integer spin particles, fermions.

As a simple first example we construct an action with scalar and spinor fields and supersymmetry involving kinetic terms only.

The kinetic term for a scalar field, S, is

$$\mathcal{L}_S = \frac{1}{2}\partial_\mu S \partial^\mu S \tag{11.1}$$

and that for a fermion field, ψ_L, is

$$\mathcal{L}_\psi = \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L \tag{11.2}$$

There are obvious differences: $\mathcal{L}_S$ contains two derivatives, $\mathcal{L}_\psi$ only one; ψ_L is a Grassman field, S a normal field; and finally $\mathcal{L}_\psi$ has the phase invariance

$$\psi_L \rightarrow e^{i\alpha}\psi_L \tag{11.3}$$

while $\mathcal{L}_S$ has none.

The task we have set ourselves is to find a set of transformations between, in this case, spinless and spin 1/2 fields which leaves the sum of their kinetic terms invariant.

Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu S \partial^\mu S + \frac{1}{2}\partial_\mu P \partial^\mu P + \frac{1}{4}\bar{\chi}\gamma^\mu \overleftrightarrow{\partial_\mu}\chi \tag{11.4}$$

where S and P are two scalar fields and χ is a Majorana spinor field.

There are two global phase invariances:

$$\chi \rightarrow e^{i\alpha\gamma_5}\chi \tag{11.5}$$

and $$(S + iP) \rightarrow e^{i\beta}(S + iP) \tag{11.6}$$

Any further invariance will involve transformations changing the spinless fields S and P into the spinor field χ. Such a transformation has two characteristics:

1) the transformation parameter must be a Grassman spinor field, $\bar{\alpha}$, say, a global infinitesimal Majorana spinor parameter.

2) the transformation of S and P must involve no derivative operator and that of χ must involve one since the fermion kinetic term has one less derivative than the scalar kinetic term.

We are led to

$$\delta\,(S \text{ or } P) = \bar{\alpha} M \chi \tag{11.7}$$

where M is some 4 x 4 matrix. As no 4 vector indices are involved it can only contain 1 or γ_5. We fix it to be

$$\delta S = a\bar{\alpha}\chi \tag{11.8}$$

$$\delta P = ib\bar{\alpha}\gamma_5\chi \tag{11.9}$$

where a and b are unknown coefficients.

Then

$$\delta\left[\tfrac{1}{2}\partial_\mu S \partial^\mu S + \tfrac{1}{2}\partial_\mu P \partial^\mu P\right] = (a\partial^\mu S\bar{\alpha} + ib\partial^\mu P\bar{\alpha}\gamma_5)\chi \tag{11.10}$$

To find the variation of χ we first note that

$$\delta\left[\tfrac{1}{4}\bar{\chi}\gamma^\mu\partial_\mu\chi\right] = \delta\bar{\chi}\gamma^\mu\partial_\mu\chi \tag{11.11}$$

upto surface terms.

The variation of the Lagrangian (11.4) is

$$\delta\mathcal{L} = (\delta\bar{\chi}\gamma^\mu + a\partial_\mu S\bar{\alpha} + ib\partial_\mu P\bar{\alpha}\gamma_5)\partial^\mu\chi \quad + \text{s.t.} \tag{11.12}$$

$$= -(\partial_\mu\delta\bar{\chi}\gamma^\mu + a\partial_\mu\partial^\mu S\bar{\alpha} + ib\partial_\mu\partial^\mu P\bar{\alpha}\gamma_5)\chi \quad + \text{s.t.}' \tag{11.13}$$

where a partial integration has been performed in going from (11.12) to (11.13).

Thus $\mathcal{L}$ changes only by a total divergence if $\delta\bar{\chi}$ obeys

$$\partial_\mu\delta\bar{\chi}\gamma^\mu + a\Box S\bar{\alpha} + ib\Box P\bar{\alpha}\gamma_5 = 0 \tag{11.14}$$

A solution is

$$\delta\chi = a\gamma_\ell \alpha \partial^\ell S - ib\gamma_\ell\gamma_5\alpha\partial^\ell P \tag{11.15}$$

We have found a set of transformations between spin 0 and spin 1/2 fields which leaves the sum of their kinetic terms invariant.We must now test that these transformations close to form a group.

Consider

$$[\delta_1,\delta_2]S = a\bar{\alpha}_2\delta_1\chi - a\bar{\alpha}_1\delta_2\chi \tag{11.16}$$

$$= a\bar{\alpha}_2\left[a\gamma_\ell\alpha_1\partial^\ell S - ib\gamma_\ell\gamma_5\alpha\partial^\ell P\right] - (1\leftrightarrow 2)$$

$$= 2a^2\bar{\alpha}_2\gamma_\ell\alpha_1\partial^\ell S$$

Similarly

$$[\delta_1,\delta_2]P = 2b^2\bar{\alpha}_2\gamma_\ell\alpha_1\partial^\ell P \tag{11.17}$$

Thus,since the transformations msut be the same for S, P and χ, we find

$$a^2 = b^2 \tag{11.18}$$

We see that the effect of two symmetry transformations on S and P is a translation by an amount $2ia^2\bar{\alpha}_2\gamma_\ell\alpha_1$.

$$[\delta_1,\delta_2]\chi = a^2\gamma_\ell\alpha_2\bar{\alpha}_1\partial^\ell\chi + b^2\gamma_\ell\gamma_5\alpha_2\bar{\alpha}_1\gamma_5\partial^\ell\chi - (1\leftrightarrow 2) \tag{11.19}$$

$$= a^2\bar{\alpha}_2\gamma^\ell\alpha_1\gamma_\mu\gamma_\ell\partial^\mu\chi \tag{11.20}$$

i.e.

$$[\delta_1,\delta_2]\chi = 2a^2\bar{\alpha}_2\gamma^\mu\alpha_1\partial_\mu\chi - a^2\bar{\alpha}_2\gamma^\mu\alpha_1\gamma_\mu\gamma^\ell\partial_\ell\chi \tag{11.21}$$

(after using the anti-commutator of γ matrices).

The first term on the right-hand side of (11.21) is that for which we were looking, but we have acquired an extra term. To eliminate this term we must enlarge the definition of $\delta\chi$ in (11.15). Consider the redefinition:

$$\delta\chi = a\gamma_\ell\alpha\partial^\ell S - ib\gamma_\ell\gamma_5\alpha\partial^\ell P + (F + i\gamma_5 G)\alpha \tag{11.22}$$

where

$$\delta_{extra}\chi = (F + i\gamma_5 G) \tag{11.23}$$

The relations (11.16) and (11.17) are unchanged. By choosing

$$\delta_1 F = a^2 \bar{\alpha}_1 \gamma^\rho \partial_\rho \chi \tag{11.24}$$

and

$$\delta_1 G = -ia^2 \bar{\alpha}_1 \gamma_5 \gamma^\rho \partial_\rho \chi \tag{11.25}$$

we find

$$[\delta_1, \delta_2](F \text{ or } G) = 2a^2 \bar{\alpha}_2 \gamma^\mu \alpha_1 \partial_\mu (F \text{ or } G) \tag{11.26}$$

and, using our new definition of $\delta\chi$ (11.22), that

$$[\delta_1, \delta_2]\chi = 2a^2 \bar{\alpha}_2 \gamma^\mu \alpha_1 \partial_\mu \chi \tag{11.27}$$

as required.

Now, though, we must modify the Lagrangian (11.4) to make that invariant. The Lagrangian

$$\mathcal{L}^{WZ} = \frac{1}{2}\partial_\mu S \partial^\mu S + \frac{1}{2}\partial_\mu P \partial^\mu P + \frac{1}{4}\bar{\chi}\gamma^\rho \partial_\rho \chi + \frac{1}{2a^2}(F^2 + G^2) \tag{11.28}$$

is invariant under the supersymmetry transformations

$$\delta S = a\bar{\alpha}\chi \tag{11.29}$$

$$\delta P = ia\bar{\alpha}\gamma_5 \chi \tag{11.30}$$

$$\delta F = a^2 \bar{\alpha}\gamma^\rho \partial_\rho \chi \tag{11.31}$$

$$\delta G = -ia^2 \bar{\alpha}\gamma_5 \partial_\rho \chi \tag{11.32}$$

and

$$\delta\chi = a\gamma_\rho \alpha \partial^\rho S - ib\gamma_\rho \gamma_5 \alpha \partial^\rho P + (F + i\gamma_5 G)\alpha \tag{11.33}$$

which all satisfy

$$[\delta_1, \delta_2] = 2a^2 \bar{\alpha}_2 \gamma^\mu \alpha_1 \partial_\mu \tag{11.34}$$

The effect of two supersymmetry transformations (11.34) is a translation. Since the supersymmetry parameters are spinors the generators of the supersymmetry transform are spinors. The Poincare group is therefore enlarged to include the supersymmetry generators.

The F and G fields have no kinetic terms; they serve as auxiliary fields which are totally uncoupled for the free theory.

We have found a group of transformations between spin 0 and spin 1/2 fields which leaves their kinetic Lagrangian part in - variant. This was achieved at the expense of introducing auxiliary fields F and G. These, however, are easily eliminated in practice by using the equations of motion.

Having demonstrated the existance of supersymmetry transformations we move on to introduce superfields, leaving more rigorous treatment of the derivation of supersymmetry transformations,their algebra and superfields to the reviews of Fayet and Ferrara (11.1) and Wess and Bagger (11.2).

Superfields

A superfield may be thought of as a power series in θ, the transformation parameter, a two component Weyl spinor field.

A scalar superfield, S, has the form

$$\begin{aligned} S &= \tfrac{1}{2}(A - iB) + \theta^T \varepsilon \psi + \theta^T \varepsilon \theta \, \tfrac{1}{2}(F + iG) \\ &= \tfrac{1}{2}\mathcal{Q} + \theta^T \varepsilon \psi + \theta^T \varepsilon \theta \, \tfrac{1}{2}\mathcal{F} \end{aligned} \qquad (11.35)$$

where

$$\varepsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (11.36)$$

$\mathcal{Q}$ = A - iB is a scalar field, ψ is a Weyl spinor field and $\mathcal{F}$ = F + iG is an auxiliary field.

The useful property of superfields is that under the supersymmetry transformations the coefficient of the highest power of θ is mapped onto a total derivative.

Suppose that the field corresponding to the highest power of θ is ϕ_{LAST} then

$$\delta\phi_{LAST} = i\delta\chi \qquad (11.37)$$

where χ is some Fermi field.

Then it follows that the quantity

$$\int d^4x \ \phi_{LAST} \tag{11.38}$$

is invariant under the supersymmetry transformations. This property allows us to construct Lagrangians which are invariant under the supersymmetry transformations.

It also means that we shall only require the last component field obtained in products of superfields.

Full details of superfields and their full expansions can be found in (11.1) and (11.2). We quote the necessary results for scalar superfields here:

$$\tfrac{1}{2}\left[S_i + h.c \right]_F = F_i = \tfrac{1}{2}\left(\mathfrak{f}_i^* + \mathfrak{f}_i \right) \tag{11.39}$$

$$\begin{aligned} \tfrac{1}{2}\left[S_iS_j + h.c.\right]_F &= \left(\tfrac{1}{4} a_i \mathfrak{f}_j + \tfrac{1}{4} a_j \mathfrak{f}_i + h.c.\right) \\ &\qquad - \left(\tfrac{1}{2} \psi_i^T \varepsilon \psi_j + h.c \right) \\ &= \tfrac{1}{2}\left(A_iF_j + A_jF_i + B_iG_j + B_jG_i \right) \\ &\qquad - \tfrac{1}{2} \bar{\Psi}_i \Psi_j \end{aligned} \tag{11.40}$$

where $\Psi = \begin{pmatrix} \varepsilon\psi^* \\ \psi \end{pmatrix}$ (11.41) is a 4 component Majorana field with the property

$$\bar{\Psi}_i \Psi_j = \psi_i^T \varepsilon \psi_j + h.c. \tag{11.42}$$

$$\begin{aligned} \tfrac{1}{2}\left[S_iS_jS_k + h.c\right]_F = &\ \tfrac{1}{8}\left(a_i a_j \mathfrak{f}_k + a_j a_k \mathfrak{f}_i + a_k a_i \mathfrak{f}_j + h.c.\right) \\ &- \tfrac{1}{4}\left(A_i \bar{\Psi}_j \Psi_k + A_j \bar{\Psi}_k \Psi_i + A_k \bar{\Psi}_i \Psi_j \right) \\ &+ \tfrac{i}{4}\left(B_i \bar{\Psi}_j \gamma_5 \Psi_k + B_k \bar{\Psi}_i \gamma_5 \Psi_j + B_j \bar{\Psi}_k \gamma_5 \Psi_i \right) \end{aligned} \tag{11.43}$$

The subscript F indicates that the last term in the expansion is the $\theta^T \varepsilon \theta$ term.

We shall also need

$$[S_i^* S_i]_D = \frac{1}{2}(F_i^2 + G_i^2) + \frac{1}{2}(\partial_\mu A_i)^2 + \frac{1}{2}(\partial_\mu B_i)^2 + \frac{i}{2}\bar{\Psi}_i \gamma^\mu \partial_\mu \Psi_i \tag{11.44}$$

where the subscript D indicates that the last term is the $(\theta^T \varepsilon\, \theta)(\theta^{T*} \varepsilon\, \theta^*)$ term. (The F part is the coefficient of $\frac{1}{2}\theta^T \varepsilon\, \theta$; the D part the coefficient of $\frac{1}{2}(\theta^T \varepsilon\, \theta)(\theta^{T*} \varepsilon\, \theta^*)$.)

Using (11.40), (11.43) and (11.44) we can construct the most general supersymmetric renormalizable Lagrangian involving only scalar superfields. It is

$$\mathcal{L} = [S_i^* S_i]_D + \frac{1}{2}\left[m_{ij} S_i S_j + \frac{4}{3} g_{ijk} S_{ijk} + \lambda_i S_i + \text{h.c.}\right] \tag{11.45}$$

In terms of component fields this becomes

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\mathcal{F}_i^* \mathcal{F}_i + \frac{1}{2}\partial_\mu a_i^* \partial^\mu a_i + \frac{i}{2}\bar{\Psi}_i \gamma^\mu \partial_\mu \Psi_i \\ & + \frac{1}{2}(\lambda_i \mathcal{F}_i + \text{h.c}) + \frac{1}{2} m_{ij}(a_i \mathcal{F}_j + \text{h.c}) \\ & - \frac{1}{2} m_{ij} \bar{\Psi}_i \Psi_j + \frac{1}{2} g_{ijk}(a_i a_j \mathcal{F}_k + \text{h.c.}) \\ & - g_{ijk}(A_i \bar{\Psi}_j \Psi_k - i B_i \bar{\Psi}_j \gamma_5 \Psi_k)\end{aligned} \tag{11.46}$$

where the coupling constants m_{ij} and g_{ijk} are symmetric in their indices.

The auxiliary fields are eliminated by means of their Euler equation:

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}_k^*} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}_k} = 0 \tag{11.47}$$

$$\Rightarrow \frac{1}{2}\mathcal{F}_k^* + \frac{1}{2}\lambda_k + \frac{1}{2} m_{ik} a_i + \frac{1}{2} g_{ijk} a_i a_j = 0 \tag{11.48}$$

Then $\mathcal{L}$ becomes

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu a_i^* \partial^\mu a_i \quad - \tfrac{i}{2}\bar{\Psi}_i \gamma^\mu \partial_\mu \Psi_i$$
$$- \tfrac{1}{2} m_{ij} \bar{\Psi}_i \Psi_j \quad - g_{ijk}\left(A_i \bar{\Psi}_j \Psi_k - iB_i \bar{\Psi}_j \gamma_5 \Psi_k\right)$$
$$- \tfrac{1}{2}\left| \lambda_i + m_{ij} a_j + g_{ijk} a_j a_k \right|^2 \tag{11.49}$$

The potential is

$$\tfrac{1}{2} F_i^* F_i = \tfrac{1}{2}\left| \lambda_i + m_{ij} a_j + g_{ijk} a_j a_k \right|^2 \tag{11.50}$$

It is always greater than, or equal to, zero; this is one consequence of supersymmetry.

A vector superfield has the form

$$V = -\theta\sigma_\mu\bar{\theta}\, V^\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta} D$$

where we have defined

$$\theta\theta = \theta^T \varepsilon\, \theta \tag{11.52}$$

$$\bar{\theta}\bar{\theta} = \theta^{T*} \varepsilon\, \theta^* \quad \text{etc.} \tag{11.53}$$

V^μ is a vector field, λ a Weyl spinor field and D is an auxiliary field.

It has the free supersymmetric Lagrangian

$$\mathcal{L}_0 = -\tfrac{1}{4} V_{\mu\nu} V^{\mu\nu} + \tfrac{i}{2}\bar{\lambda}\gamma^\mu \partial_\mu \lambda + \tfrac{1}{2} D^2 \tag{11.54}$$

where now λ is a Majorana spinor.

Gauge Invariant Interactions

Consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{KE} + \mathcal{L}_{PE} \tag{11.55}$$

where $$\mathcal{L}_0 = -\frac{1}{4} V^{\mu\nu} V_{\mu\nu} + \frac{i}{2} \bar{\lambda} \gamma^\mu \partial_\mu \lambda + \frac{1}{2} D^2 \tag{11.56}$$

$$\mathcal{L}_{PE} = \frac{1}{2} \left[m_{ij} S_i S_j + \frac{4}{3} g_{ijk} S_{ijk} + \lambda_i S_i \right]_F + \text{h.c.} \tag{11.57}$$

and $$\mathcal{L}_{KE} = - \left[S_i^+ e^{-2gV} S_i \right]_D \tag{11.58}$$

It is invariant under the U(1) gauge transformation

$$S_i \rightarrow e^{2ig_iV} S_i \tag{11.59}$$

$$V \rightarrow V + i(\Lambda - \Lambda^+) \tag{11.60}$$

where the parameter, Λ , behaves as a scalar superfield under supersymmetry transformations.

$\mathcal{L}_{KE}$ is evaluated in the Wess-Zumino gauge (ref 11.1) where $V^3 = 0$, so that

$$\left[S_i^+ e^{-2g_iV} S_i \right]_D = \left[S_i^+ S_i - 2g_i S_i^+ S_i V + 2g_i^2 S_i^+ S_i V^2 \right] \tag{11.61}$$

In terms of component fields

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{i}{2} \bar{\lambda} \gamma^\mu \partial_\mu \lambda + \frac{1}{2} D^2 \\
& + \frac{1}{2} \mathcal{F}_i^* \mathcal{F}_i + \frac{1}{2} D_\mu a_i^+ D^\mu a_i + i \bar{\Psi}_{L,i} \gamma^\mu D_\mu \Psi_{L,i} \\
& - i g_i (a_i^+ \bar{\lambda} \Psi_{L,i} + \text{h.c.}) + \frac{1}{2} g_i a_i^+ D a_i \\
& + \frac{1}{2} (\lambda_i \mathcal{F}_i + \text{h.c.}) \\
& + \frac{1}{2} m_{ij} (a_i \mathcal{F}_j + \text{h.c.}) - \frac{1}{2} m_{ij} \bar{\Psi}_{L,i} \Psi_{L,j} \\
& + \frac{1}{2} g_{ijk} (a_i a_j \mathcal{F}_k + \text{h.c.}) \\
& - g_{ijk} (A_i \bar{\Psi}_{L,j} \Psi_{L,k} - i B_i \bar{\Psi}_{L,j} \gamma_5 \Psi_{L,k})
\end{aligned} \tag{11.62}$$

where now the $\Psi_{L,i}$ are left-handed Majorana spinors.

The covariant derivatives are defined by

$$D_\mu = \partial_\mu + i g_i V_\mu \tag{11.63}$$

The auxiliary fields are eliminated by their Euler equations:

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}^*} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \frac{\partial \mathcal{L}}{\partial D} = 0 \tag{11.64}$$

As before (11.48)

$$\mathcal{F}^*_k + \lambda_k + m_{ik} a_i + g_{ijk} a_i a_j = 0 \tag{11.65}$$

and now

$$D + \tfrac{1}{2} g_i a_i^+ a_i = 0 \tag{11.66}$$

We arrive at the most general possible Lagrangian with supersymmetric invariance and U(1) gauge invariance

$$\begin{aligned} \mathcal{L} = & -\tfrac{1}{4} V^{\mu\nu} V_{\mu\nu} + \tfrac{i}{2} \bar{\lambda} \gamma^\mu \partial_\mu \lambda \\ & + \tfrac{1}{2} D_\mu a_i^+ D^\mu a_i + i \bar{\Psi}_{L,i} \gamma^\mu D_\mu \Psi_{L,i} \\ & - i g_i (a_i^+ \bar{\lambda} \Psi_{L,i} + h.c) \\ & - \tfrac{1}{2} m_{ij} \bar{\Psi}_{L,i} \Psi_{L,j} \\ & - g_{ijk} (A_i \bar{\Psi}_{L,j} \Psi_{L,k} - i B_i \bar{\Psi}_{L,j} \gamma_5 \Psi_{L,k}) \\ & - \tfrac{1}{2} \left| \lambda_i + m_{ij} a_j + g_{ijk} a_j a_k \right|^2 \\ & - \tfrac{1}{8} \left(g_i a_i^* a_i \right)^2 \end{aligned} \tag{11.67}$$

(there is a summation over i)

The covariant derivatives are defined as

$$D_\mu a_i = \partial_\mu a_i + i g_i V_\mu a_i \tag{11.67a}$$

$$D_\mu \Psi_{L,i} = \partial_\mu \Psi_{L,i} + i g_i V_\mu \Psi_{L,i} \tag{11.67b}$$

Note that if U(1) symmetry is not to be broken then

$m_{ij} = 0$ for $g_i + g_j \neq 0$ and

$g_{ijk} = 0$ for $g_i + g_j + g_k \neq 0$.

General Supersymmetric Grand Unified Theory

Having shown the situation for a U(1) gauge theory we now consider the case of a general gauge theory.

Let the vector supermultiplet be V_a, and let S_i be a left chiral supermultiplet corresponding to the representation of the gauge group with generators t_a. Let the generators of the adjoint representation be

$$(T_a)_{bc} = -i f_{abc} \tag{11.68}$$

The generalised gauge transformation is

$$S \rightarrow e^{2ig\Lambda} S \tag{11.69}$$

and

$$e^{V} \rightarrow e^{-i\Lambda^{+}} e^{V} e^{i\Lambda} \tag{11.70}$$

where

$$\Lambda = \Lambda_a t_a \tag{11.71}$$

and

$$V = V_a t_a \tag{11.72}$$

(11.70) is an extension of the gauge transformation (11.60).

The Lagrangian is

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{PE} \tag{11.73}$$

where

$$\begin{aligned}\mathcal{L}_{gauge} = & -\frac{1}{4} V_a^{\mu\nu} V^a_{\mu\nu} + \frac{i}{2} \bar{\lambda}_a \gamma^\mu D_\mu \lambda_a + \frac{1}{2} D_a D^a \\ & + \frac{1}{2} \mathcal{F}_i^* \mathcal{F}_i + \frac{1}{2} D_\mu a_i^+ D^\mu a_i + i \bar{\Psi}_{L,i} \gamma^\mu D_\mu \Psi_{L,i} \\ & - ig\left(a_i^+ \bar{\lambda} \Psi_{L,i} + h.c.\right) + \frac{g}{2} a^+ D a\end{aligned} \tag{11.74}$$

We have defined

$$D = D_a t_a \tag{11.75}$$

$$\lambda = \lambda_a t_a \tag{11.76}$$

and the covariant derivatives are

$$D_\mu \lambda_a = \partial_\mu \lambda_a - g f_{abc} (V_\mu)_b \lambda_c \tag{11.77}$$

$$D_\mu \lambda = \partial_\mu \lambda + ig\left[V_\mu, \lambda\right] \tag{11.78}$$

$$D_\mu a_i = \partial_\mu a_i + ig V_\mu a_i \tag{11.79}$$

$$D_\mu \Psi_{L,i} = \partial_\mu \Psi_{L,i} + ig V_\mu \Psi_{L,i} \tag{11.80}$$

and
$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + ig \left[V_\mu, V_\nu \right] \tag{11.81}$$

In general we would expect $\mathcal{I}_{PE}$ to have the form

$$\mathcal{L}_{PE} = m \left[S_i S_i \right]_F + \frac{4h}{3} d_{ijk} \left[S_i S_j S_k \right]_F \tag{11.82}$$

$$\begin{aligned} &= \frac{m}{2} \left(a_i \mathcal{F}_i + h.c. \right) - \frac{m}{2} \bar{\Psi}_{L,i} \Psi_{L,i} \\ &+ \frac{h}{2} \left(d_{ijk}\, a_i a_j \mathcal{F}_k + h.c. \right) \\ &- h\, d_{ijk} \left(\bar{\Psi}_{L,i} \Psi_{L,j} A_k - i \bar{\Psi}_{L,i} \gamma_5 \Psi_{L,j} B_k \right) \end{aligned} \tag{11.83}$$

where the d_{ijk} are totally symmetric invariant tensors with respect to the internal symmetry group.

Chapter 11: References

11.1 Wess and Bagger (1983) " Supersymmetry and Supergravity"

Princeton University Press

11.2 Fayet and Ferrara (1977) Phys.Rep. 32 249

Chapter 12: Spontaneous Symmetry Breaking in Supersymmetric Theories

As in previous chapters dealing with non-supersymmetric theories we are primarily concerned with theories exhibiting the spontaneous breaking of a U(1) gauge symmetry.

In this chapter we discuss symmetry breaking in two models in which a U(1) gauge symmetry is broken.

In the first breaking is produced by a ξD term. However, as we shall see, supersymmetry may also be broken in this model.

Consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 - [S^+ e^{-2eV} S]_D + [\xi V]_D \tag{12.1}$$

This is the most general Lagrangian we can construct containing a single charged Higgs scalar. $[S, S^2, S^3]_F$ would break the U(1) symmetry automatically.

Since $[V]_D = D$ is neutral we are able to introduce the $[\xi V]_D$ term.

In terms of component fields the Lagrangian (12.1) becomes

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}V_{\mu\nu}V^{\mu\nu} + i\bar{\lambda}_R\gamma^\mu\partial_\mu\lambda_R + \tfrac{1}{2}D^2 + \xi D \\ &+ \tfrac{1}{2}\mathcal{F}^*\mathcal{F} + D_\mu\phi^+ D^\mu\phi + i\bar{\Psi}_L\gamma^\mu D_\mu\Psi_L \\ &- i\sqrt{2}(\phi^+\bar{\lambda}\Psi_L + h.c.) + g\phi^+ D\phi\end{aligned} \tag{12.2}$$

where, following the convention of Fayet and Ferrara (12.1) we have defined the complex field

$$\phi = \frac{1}{\sqrt{2}}(A - iB) = \frac{1}{\sqrt{2}}a . \tag{12.3}$$

The covariant derivative is

$$D_\mu = \partial_\mu + ieV_\mu \tag{12.4}$$

Using the equations of motion for $\mathcal{F}$ and D we find

$$\mathcal{F} = 0 , \quad D + \xi + e\phi^+\phi = 0 . \tag{12.5}$$

(12.2) becomes

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + i\bar{\lambda}_R \gamma^\mu \partial_\mu \lambda_R + D_\mu \phi^+ D^\mu \phi$$
$$+ i \bar{\Psi}_L \gamma^\mu D_\mu \Psi_L - ie\sqrt{2}\,(\phi^+ \bar{\lambda}\, \Psi_L + h.c.)$$
$$- \frac{1}{2} \left| \xi + e\phi^+ \phi \right|^2 \qquad (12.6)$$

Consider the potential

$$V(\phi) = \frac{1}{2} \left| \xi + e\phi^+ \phi \right|^2 \qquad (12.7)$$

If ϕ has the expectation value a , then a is given by

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi = a} = 0 \qquad (12.8)$$

i.e. $$(\xi + e a^* a)\, a = 0 \qquad (12.9)$$

i.e. $$a = 0 \;, \quad a a^* = -\frac{\xi}{e} \qquad (12.10)$$

If $\xi e > 0$ then $a = 0$ and $\langle D \rangle \neq 0$ (12.11)

Thus for $\xi e > 0,\ \phi$ has a vanishing vacuum expectation value but the auxiliary field D has not: supersymmetry is broken but the U(1) gauge symmetry is preserved.

When $\xi e < 0, a \neq 0$ and $D = 0$ .The U(1) gauge symmetry is broken while supersymmetry is conserved.

The second model exhibiting spontaneous symmetry breaking of a U(1) gauge symmetry has three scalar superfields, one neutral, one positive and one negative.

We employ the Lagrangian used by Wess and Bagger (12.2) namely

$$\mathcal{L} = \mathcal{L}_0 - \left[S_+^+ e^{2eV} S_+ + S_-^+ e^{-2eV} S_- + S^+ S \right]_D$$
$$+ \frac{1}{2} \left[\frac{1}{2} m_0 S^2 + m S_+ S_- + \lambda S + 4g\, S S_+ S_- + h.c. \right]_F \qquad (12.12)$$

In terms of component fields this becomes

$$
\begin{aligned}
\mathcal{L} = & -\tfrac{1}{4}V^{\mu\nu}V_{\mu\nu} + i\bar{\lambda}_R\gamma^\mu\partial_\mu\lambda_R + \tfrac{1}{2}D^2 \\
& + \tfrac{1}{2}\mathcal{F}^*\mathcal{F} + \tfrac{1}{2}\partial_\mu\mathcal{A}^+\partial^\mu\mathcal{A} + i\bar{\psi}\gamma^\mu\partial_\mu\psi \\
& + \tfrac{1}{2}\mathcal{F}_+^*\mathcal{F}_+ + \tfrac{1}{2}D_\mu\mathcal{A}_+^+D^\mu\mathcal{A}_+ + i\bar{\psi}_+\gamma^\mu D_\mu\psi_+ \\
& + \tfrac{1}{2}\mathcal{F}_-^*\mathcal{F}_- + \tfrac{1}{2}D_\mu\mathcal{A}_-^+D^\mu\mathcal{A}_- + i\bar{\psi}_-\gamma^\mu D_\mu\psi_- \\
& + ie(\mathcal{A}_+^*\bar{\lambda}_R\psi_+ + h.c.) - ie(\mathcal{A}_-^*\bar{\lambda}\psi_- + h.c.) \\
& + \tfrac{1}{2}e\mathcal{A}_+^+D\mathcal{A}_+ - \tfrac{1}{2}e\mathcal{A}_-^+D\mathcal{A}_- \\
& + \tfrac{1}{2}(\lambda\mathcal{F} + \lambda^*\mathcal{F}^*) \\
& + \tfrac{1}{2}m_0(\mathcal{A}\mathcal{F} + h.c.) - \tfrac{1}{4}m_0\bar{\psi}\psi \\
& + \tfrac{1}{2}m(\mathcal{A}_+\mathcal{F}_- + \mathcal{A}_-\mathcal{F}_+ + h.c.) - \tfrac{1}{2}m\bar{\psi}_+\psi_- \\
& + \tfrac{1}{2}g(\mathcal{A}_+\mathcal{A}_-\mathcal{F} + \mathcal{A}_-\mathcal{A}\mathcal{F}_+ + \mathcal{A}\mathcal{A}_+\mathcal{F}_- + h.c.) \\
& - g(A\bar{\psi}_+\psi_- + A_+\bar{\psi}_-\psi + A_-\bar{\psi}\psi_+) \\
& - ig(B\bar{\psi}_+\gamma_5\psi_- + B_+\bar{\psi}_-\gamma_5\psi + B_-\bar{\psi}\gamma_5\psi_+)
\end{aligned}
\tag{12.13}
$$

$$\mathcal{A} = A - iB \tag{12.14}$$

$$D_\mu\mathcal{A}_+ = \partial_\mu\mathcal{A}_+ - ieV_\mu\mathcal{A}_+ \tag{12.15}$$

$$D_\mu\mathcal{A}_- = \partial_\mu\mathcal{A}_- + ieV_\mu\mathcal{A}_- \tag{12.16}$$

and similarly for the $\psi_\pm$ fields, which are left-handed Majorana spinors. The notation has been simplified.

The equations of motion give:

$$
\begin{aligned}
\mathcal{F}^* + \lambda + m_0\mathcal{A} + g\mathcal{A}_+\mathcal{A}_- &= 0 \\
\mathcal{F}_+^* + m\mathcal{A}_- + g\mathcal{A}_-\mathcal{A} &= 0 \\
\mathcal{F}_-^* + m\mathcal{A}_+ + g\mathcal{A}_+\mathcal{A} &= 0 \\
D + \tfrac{1}{2}e(\mathcal{A}_+^+\mathcal{A}_+ - \mathcal{A}_-^+\mathcal{A}_-) &= 0
\end{aligned}
\tag{12.17}
$$

The Lagrangian becomes:

$$
\begin{aligned}
\mathcal{L} = & -\tfrac{1}{4}V^{\mu\nu}V_{\mu\nu} \quad + i\bar{\lambda}_R\gamma^\mu\partial_\mu\lambda_R \\
& + \tfrac{1}{2}\partial_\mu\mathcal{A}^+\gamma^\mu\mathcal{A} + \tfrac{1}{2}D_\mu\mathcal{A}_+^* D^\mu\mathcal{A}_+ + \tfrac{1}{2}D_\mu\mathcal{A}_-^* D^\mu\mathcal{A}_- \\
& + i\bar{\psi}\gamma^\mu\partial_\mu\psi \quad + i\bar{\psi}_+\gamma^\mu D_\mu\psi_+ \quad + i\bar{\psi}_-\gamma^\mu D_\mu\psi_- \\
& + ie\left(\mathcal{A}_+^*\bar{\lambda}_R\psi_+ + h.c.\right) - ie\left(\mathcal{A}_-^*\bar{\lambda}_R\psi_- + h.c.\right) \\
& - \tfrac{1}{4}m_0\bar{\psi}\psi \qquad\qquad -\tfrac{1}{2}m\bar{\psi}_+\psi_- \\
& - g\left(A\bar{\psi}_+\psi_- + A_+\bar{\psi}_-\psi + A_-\bar{\psi}\psi_+\right) \\
& + ig\left(B\bar{\psi}_+\gamma_5\psi_- + B_+\bar{\psi}_-\gamma_5\psi + B_-\bar{\psi}\gamma_5\psi_+\right) \\
& - \tfrac{1}{2}\left|\lambda + m_0\mathcal{A} + g\mathcal{A}_+\mathcal{A}_-\right|^2 \\
& - \tfrac{1}{2}\left(|\mathcal{A}_+|^2 + |\mathcal{A}_-|^2\right)\left|m + g\mathcal{A}\right|^2 \\
& - \tfrac{1}{8}e^2\left(|\mathcal{A}_+|^2 - |\mathcal{A}_-|^2\right)^2
\end{aligned}
$$

(12.18)

Vacuum expectation values a, a_+, a_- of $\mathcal{A}$, $\mathcal{A}_+$, $\mathcal{A}_-$ for which $\mathcal{F}$, $\mathcal{F}_+$, $\mathcal{F}_-$ = 0 signal supersymmetric minima of the potential.

Thus we obtain

$$
\begin{aligned}
\lambda + m_0 a + g a_+ a_- &= 0 \\
a_-(m + ga) &= 0 \\
a_+(m + ga) &= 0
\end{aligned}
\qquad (12.19)
$$

with solutions

1) $a_+ = a_- = 0 \quad , \quad a = -\frac{\lambda}{m_0}$

(12.20)

2) $a_+ a_- = -\frac{1}{g}\left(\lambda - \frac{m_0 m}{g}\right), \; a = -\frac{m}{g}$

The first solution does not break the U(1) symmetry while the second does.

Also for conservation of supersymmetry we require

$$\langle D \rangle = 0 \qquad (12.21)$$

which yeilds $\quad |a_+|^2 = |a_-|^2 \qquad$ (12.22)

A Supersymmetric Higgs Model At Finite Temperature and Density

The Lagrangian (12.1)

$$\mathcal{L} = \mathcal{L}_0 - \left[S^+ e^{-2eV} S\right]_D + \left[\xi D\right] \tag{12.23}$$

exhibits spontaneous symmetry breaking of the U(1) gauge symmetry when $e\xi < 0$.

In analogy with the non-supersymmetric case of chapter 8 we study the symmetry behaviour of this model at high densities and temperatures.

In terms of component fields the Lagrangian (12.23) is

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{4} V^{\mu\nu} V_{\mu\nu} + i \bar{\lambda}_R \gamma^\mu \partial_\mu \lambda_R + D_\mu \phi^+ D^\mu \phi \\ & + i \bar{\Psi}_L \gamma^\mu D_\mu \Psi_L - ie\sqrt{2}\left(\phi^+ \bar{\lambda} \Psi_L + h.c.\right) \\ & - \tfrac{1}{2}\left(\xi + e\phi^+\phi\right)^2\end{aligned} \tag{12.24}$$

where $\phi = \frac{1}{\sqrt{2}}$ (A - iB) and Ψ_L are the complex scalar field and left-handed Majorana spinors associated with the superfield S.

λ_R is a right-handed Majorana spinor.

The covariant derivatives are

$$\begin{aligned} D_\mu \phi &= \partial_\mu \phi + ie V_\mu \phi \\ D_\mu \Psi_L &= \partial_\mu \Psi_L + ie V_\mu \Psi_L \end{aligned} \tag{12.25}$$

A chemical potential may be coupled to the Noether current of the U(1) gauge symmetry:

$$j^\mu = \bar{\Psi}_L \gamma^\mu \Psi_L + i\left(\phi^+ D^\mu \phi - D^\mu \phi^+ . \phi\right) \tag{12.26}$$

Because of the supersymmetry of the Lagrangian there is no conserved (or approximately conserved) current density involving only fermions. Consequently, we have a chemical potential coupled to both fermions and bosons together. We therefore introduce, into the Hamiltonian, the term $-\mu j_0$. As for the case of non-super-symmetric theories this results in the shift

$$\partial_0 \to \partial_0 - i\mu \quad \text{in} \quad D_\mu \phi \tag{12.27}$$

for the boson part and the inclusion of a term

$$\mu \bar{\Psi}_L \gamma^0 \Psi_L \tag{12.28}$$

for the fermion part.

The calculation of V_{eff} is carried out in the same manner as for non-supersymmetric theories. We allow an expectation value V_c for the time component of the gauge field

$$\langle V^0 \rangle = V_c \tag{12.29}$$

and shift the scalar field by its expectation value

$$\langle \phi \rangle = \phi_c \tag{12.30}$$

The zero temperature effective potential is

$$V^0_{eff} = -(\mu - eV_c)^2 \phi_c^2 + \tfrac{1}{2} e^2 \phi_c^4 + e\xi \phi_c^2 \tag{12.31}$$

and the one loop temperature effective potential is

$$V^T_{eff} = -\left\{ \frac{\pi^2 T^4}{90} + \frac{7}{12} e^2 T^2 \phi_c^2 - \frac{1}{4}(\mu - eV_c)^2 T^2 \right. \tag{12.32}$$

We find

$$\begin{aligned} V_{eff} = -\Big\{ \frac{\pi^2 T^4}{90} &+ \left[e\xi + \tfrac{7}{12} e^2 T^2 \right] \phi_c^2 - (\mu - eV_c)^2 \phi_c^2 \\ &+ \tfrac{1}{2} e^2 \phi_c^4 - \frac{T^2}{4}(\mu - eV_c)^2 \end{aligned} \tag{12.33}$$

where $\qquad \xi = N_B + \frac{7}{8} N_F \qquad$ (11.34)

where N_B and N_F are the numbers of degrees of freedom of boson and fermion fields respectively.

It should be noted here that in the non-supersymmetric theories we neglected radiative corrections on the grounds that they were at least an order of magnitude smaller than other terms in the effective potential to one loop order. In the case of super-symmetric theories, however, the problem does not arise at all. Exact supersymmetry does not allow radiative corrections, they can not arise when supersymmetry is exact.

We now carry out the familar treatment of V_{eff}. Minimising V_{eff} with respect to ϕ_c :

$$\frac{\partial V_{eff}}{\partial \phi_c} = 0 = 2\phi_c\left[\left(e\xi + \frac{7}{12}e^2T^2\right) - (\mu - eV_c)^2\right] + 2e^2\phi_c^3 \tag{12.35}$$

$$\Rightarrow \phi_c = 0, \quad \phi_c^2 = -\frac{1}{e^2}\left[\left(e\xi + \frac{7}{12}e^2T^2 - (\mu - eV_c)^2\right)\right]$$

The critical temperature is given by

$$e\xi + \frac{7}{12}e^2T_c^2 - (\mu - eV_c)^2 = 0 \tag{12.36}$$

$$\Rightarrow T_c^2 = \frac{12}{7e^2}\left[-e\xi + (\mu - eV_c)^2\right] \tag{12.37}$$

For $T > T_c$ the only solution is $\phi_c = 0$ and the system is in the symmetric phase.

For $T < T_c$

$$\phi_c = \pm\frac{1}{e}\left[(\mu - eV_c)^2 - \left(e\xi + \frac{7}{12}e^2T^2\right)\right]^{1/2} \tag{12.38}$$

becomes the value of ϕ_c giving the minimum of V_{eff}: the system is in the asymmetric phase, the U(1) gauge symmetry is broken.

The number density, n, is given by

$$n = -\frac{\partial V_{eff}}{\partial \mu}$$

$$= 2\phi_c^2(\mu - eV_c) + \frac{1}{2}(\mu - eV_c)T^2 \tag{12.39}$$

At T_c $\phi_c = 0$ so

$$T_c^2 = \frac{12}{7e^2}\left[-e\xi + \frac{4n^2}{T_c^4}\right] \tag{12.40}$$

and symmetry restoration will be prevented if

$$n \sim \frac{e}{4}\left(\frac{7}{3}\right)^{1/2}T^3 \tag{12.41}$$

For equilibrium

$$\frac{\partial V_{eff}}{\partial V_c} = -J_c$$

$$= 2e\phi_c^2(\mu - eV_c) + \frac{1}{2}e(\mu - eV_c)T^2$$

$$= en$$

where J_c is an external source introduced to stabilize the system.

It is a device to take account of the fact that 'charge' neutrality is not required by long range forces in an open universe. (Haber and Weldon 12.3 and Kapusta 12.4)

The Three Field Model At Finite Temperature and Density

The model described by the Lagrangian (12.12) also exhibits U(1) gauge symmetry breaking. In this section we study the behaviour of such a model at high temperatures and densities.

We make a simple extension of the model of Wess and Bagger (12.12) to the most general possible, with

$$\mathcal{L}_{PE} = \frac{1}{2}\left[\frac{1}{2}m_0 S^2 + m_+ S_+ S_- + fS + hSS_+S_- + \frac{\tilde{h}}{3}S^3 + h.c.\right]_F \tag{12.42}$$

where the chiral superfields S_+ , S_- carry opposite charges of the U(1) gauge symmetry and S is neutral under this symmetry.

A chemical potential may be coupled to the Noether current of the U(1) gauge symmetry:

$$\begin{aligned} j_\mu = \; & \bar{\Psi}_{+,L}\gamma^\mu \Psi_{+,L} + i\left(\phi_+^+ D^\mu \phi_+ - D_\mu \phi_+^+ \phi_+\right) \\ & - \bar{\Psi}_{-,L}\gamma^\mu \Psi_{-,L} - i\left(\phi_-^+ D^\mu \phi_- - D_\mu \phi_-^+ \phi_-\right) \end{aligned} \tag{12.43}$$

$\phi_\pm$ and $\Psi_{\pm,L}$ are the complex scalar fields

$\left(\phi_\pm = \frac{1}{\sqrt{2}}(A_\pm - iB_\pm) = \frac{1}{\sqrt{2}}\mathcal{A}_\pm\right)$ and left-handed Majorana spinors associated with the superfields $S\pm$.

The gauge covarinat derivative is

$$D_\mu \phi_\pm = \left(\partial_\mu \mp ieV_\mu\right)\phi_\pm \tag{12.44}$$

and V_μ is the gauge field. Thus we introduce the term

$$-\mu j_0 \tag{12.45}$$

to the Hamiltonian density, where μ is the chemical potential. Because of the supersymmetry of the Lagrangian there is no conserved (or approximately conserved) current density involving fermions only. Consequently we have a chemical potential μ, coupled to both fermions and bosons.

The Lagrangian is then modified by the shifts

$$D^0 \phi_+ \rightarrow (D^0 - i\mu)\phi_+$$
$$D^0 \phi_- \rightarrow (D^0 + i\mu)\phi_- \qquad (11.46)$$

and the addition of terms

$$\mu\left(\bar{\Psi}_{+L}\Psi_{+L} - \bar{\Psi}_{-L}\Psi_{-L}\right) \qquad (12.47)$$

(see chapter 8)

As usual, the introduction of the chemical potential μ means that it is necessary to allow an expectation value V_c to the time component of the gauge field.

In terms of component fields $\mathcal{L}$ becomes (including the μ terms)

$$\begin{aligned}
\mathcal{L} = & -\tfrac{1}{4}V^{\mu\nu}V_{\mu\nu} + i\bar{\lambda}_R\gamma^\mu\partial_\mu\lambda_R + \tfrac{1}{2}D^2 \\
& + \tfrac{1}{2}F^*F + \partial_\mu\phi^+\partial^\mu\phi + i\bar{\psi}\gamma^\mu\partial_\mu\psi \\
& + \tfrac{1}{2}F_+^*F_+ + D_\mu\phi_+^+D^\mu\phi_+ + i\bar{\psi}_+\gamma^\mu D_\mu\psi_+ \\
& + \tfrac{1}{2}F_-^*F_- + D_\mu\phi_-^*D^\mu\phi_- + i\bar{\psi}_-\gamma^\mu D_\mu\psi_- \\
& + \sqrt{2}\,ie\,(\phi_+^*\bar{\lambda}_R\psi_+ + h.c.) \\
& - \sqrt{2}\,ie\,(\phi_-^*\bar{\lambda}_R\psi_- + h.c.) \\
& + e\,(\phi_+^+D\phi_+ - \phi_-^+D\phi_-) \\
& + \tfrac{1}{2}(fF + f^*F^*) \\
& + \tfrac{1}{4}\sqrt{2}\,m_0(\phi F + h.c) - \tfrac{1}{4}m_0\bar{\psi}\psi \\
& + \tfrac{1}{4}\sqrt{2}\,m(\phi_+F_- + \phi_-F_+ + h.c.) - \tfrac{1}{2}m\bar{\psi}_+\psi_- \\
& + \tfrac{1}{4}h(\phi_+\phi_-F + \phi_-\phi F_+ + \phi\phi_+F_- + h.c.) \\
& - \tfrac{1}{4}h(A\bar{\psi}_+\psi_- + A_+\bar{\psi}_-\psi + A_-\bar{\psi}\psi_+)
\end{aligned}$$

(continued)

$$+ \frac{1}{4} i h \left(B \bar{\Psi}_+ \gamma_5 \Psi_- + B_+ \bar{\Psi}_- \gamma_5 \Psi + B_- \bar{\Psi} \gamma_5 \Psi_+ \right)$$
$$+ \frac{1}{4} \hat{h} \left(\phi \phi \mathcal{F} + h.c. \right)$$
$$- \frac{1}{4} \hat{h} \left(A \bar{\Psi} \Psi - i B \bar{\Psi} \gamma_5 \Psi \right)$$
$$+ \mu \left(\bar{\Psi}_+ \Psi_+ - \bar{\Psi}_- \Psi_- \right)$$

(12.48)

Eliminating auxiliary fields by means of their Euler equations:

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}^*} = 0 \Rightarrow \frac{1}{2} \mathcal{F}^* + \frac{1}{2} f + \frac{m_0}{\sqrt{2}} \phi + \frac{1}{4} h \phi_+ \phi_- + \frac{1}{4} \hat{h} \phi \phi = 0 \qquad (12.49)$$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}_+^*} = 0 \Rightarrow \frac{1}{2} \mathcal{F}_+^* + \frac{1}{4} \sqrt{2} m \phi_- + \frac{1}{4} h \phi_- \phi = 0 \qquad (12.50)$$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}_-^*} = 0 \Rightarrow \frac{1}{2} \mathcal{F}_-^* + \frac{1}{4} \sqrt{2} m \phi_+ + \frac{1}{4} h \phi_+ \phi = 0 \qquad (12.51)$$

$$\frac{\partial \mathcal{L}}{\partial D} = 0 \Rightarrow D + e \phi_+^* \phi_+ - e \phi_-^* \phi_- = 0 \qquad (12.52)$$

we arrive at

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + i \bar{\lambda}_R \gamma^\mu \partial_\mu \lambda_R$$
$$+ \partial_\mu \phi^* \partial^\mu \phi + D_\mu \phi_+^* D^\mu \phi_+ + D_\mu \phi_-^* D^\mu \phi_-$$
$$+ i \bar{\Psi} \gamma^\mu \partial_\mu \Psi + i \bar{\Psi}_+ \gamma^\mu D_\mu \Psi_+ + i \bar{\Psi}_- \gamma^\mu D_\mu \Psi_-$$
$$+ \sqrt{2} ie \left(\phi_+^* \bar{\lambda}_R \Psi_+ + h.c. \right) - \sqrt{2} ie \left(\phi_-^* \bar{\lambda}_R \Psi_- + h.c. \right)$$
$$- \frac{1}{2} e^2 \left(\phi_+^* \phi_+ - \phi_-^* \phi_- \right)^2$$
$$- \frac{1}{4} m_0 \bar{\Psi} \Psi - \frac{1}{2} m \bar{\Psi}_+ \Psi_- + \mu \left(\bar{\Psi}_+ \Psi_+ - \bar{\Psi}_- \Psi_- \right)$$
$$- \frac{h}{4} \left(A \bar{\Psi}_+ \Psi_- + A_+ \bar{\Psi}_- \Psi + A_- \bar{\Psi} \Psi_+ \right)$$
$$+ \frac{ih}{4} \left(B \bar{\Psi}_+ \gamma_5 \Psi_- + B_+ \bar{\Psi}_- \gamma_5 \Psi + B_- \bar{\Psi} \gamma_5 \Psi_+ \right)$$
$$- \frac{\hat{h}}{4} A \bar{\Psi} \Psi + \frac{i \hat{h}}{4} B \bar{\Psi} \gamma_5 \Psi$$
$$- \frac{1}{8} \left| 2f + \sqrt{2} m_0 \phi + h \phi_+ \phi_- + \hat{h} \phi \phi \right|^2$$
$$- \frac{1}{8} \left| \sqrt{2} m + h \phi \right|^2 \left(|\phi_+|^2 + |\phi_-|^2 \right) \qquad (12.53)$$

The zero temperature effective potential, V^{o}_{eff} , derived from this Lagrangian is

$$V^{o}_{eff} = \frac{1}{8}\left(\sqrt{2}m_{+} + ha\right)^{2}\left(a_{+}^{2} + a_{-}^{2}\right) + \frac{1}{8}\left(2f + \sqrt{2}m_{0}a + ha_{+}a_{-} + \hat{h}a^{2}\right)^{2} + \frac{1}{2}e^{2}\left(a_{+}^{2} - a_{-}^{2}\right)^{2} - \left(\mu + eV_{c}\right)^{2}\left(a_{+}^{2} + a_{-}^{2}\right) \tag{12.54}$$

where

$$a = \langle \phi \rangle \; , \; a_{\pm} = \langle \phi_{\pm} \rangle \tag{12.55}$$

are the expectation values of ϕ_{+} , ϕ_{-} , and ϕ ,the scalar fields associated with S_{+}, S_{-} and S respectively. We can assume a, a_{+} and a_{-} to be real, without loss of generality.

V^{T}_{eff}, the finite temperature effective potential, may be calculated by the usual method:

$$V^{T}_{eff} = \frac{T^{2}}{32}\left[\left(16e^{2} + h^{2}\right)\left(a_{+}^{2} + a_{-}^{2}\right) + 2\left(m_{0} + a\hat{h}\right)^{2} + \left(2m_{+} + ah\right)^{2}\right] - \frac{1}{2}T^{2}\left(\mu + eV_{c}\right)^{2} \tag{12.56}$$

apart from T^{4} terms not involving the expectation values.

The $(\mu + eV_{c})^{4}$ terms have cancelled and one loop corrections to $(\mu + gV_{c})^{2}(a_{+} + a_{-})^{2}$ have been dropped; they are negligible for our purposes because,as we shall see, the critical value of μ to prevent symmetry breaking is of order hT or eT.

Chapter 12: References

12.1 Fayet and Ferrara (1977) Phys. Rep. 32 249

12.2 Wess and Bagger (1983) "Supersymmetry and Supergravity" Princeton University Press

12.3 Haber and Weldon (1982) Phys.Rev. D25 502

12.4 Kapusta (1981) Phys.Rev. D24 426

Chapter 13: Supersymmetric Grand Unified Theories

For the case of non-supersymmetric grand unified theories we illustrated the situation by considering the SU(5) x U(1) $\rightarrow$ SU(5) transition in the chain of symmetry breaking:

SO(10) $\rightarrow$ SU(5) x U(1) $\rightarrow$ SU(5) $\rightarrow$ SU(3) x SU(2) x U(1)

For supersymmetric grand unified theories we shall also use the SU(5) x U(1) $\rightarrow$ SU(5) transition but for each of two cases: SU(5) x U(1) embedded in SO(10) and SU(5) x U(1) NOT embedded in SO(10).

We consider first the case of a 'free' SU(5) x U(1) supersymmetric grand unified theory, i.e. one that is not embedded in a larger gauge group such as SO(10).

To break the symmetry to SU(5) a Higgs field which is an SU(5) singlet but charged under U(1) is required. Then $[S^2]_F$ and $[S^3]_F$ terms are disallowed, but a ξD term is possible, because the D term of the Abelian vector supermultiplet V associated with the U(1) factor is neutral with respect to the U(1) charge.

If SU(5) x U(1) is embedded in SO(10) a ξD term would break the SO(10) gauge invariance (the U(1) of SU(5) x U(1) is non-Abelian when embedded in SO(10)). However, it is possible to arrange symmetry breaking using three Higgs fields with positive, negative and neutral charges under the U(1) field. Then we can construct terms like

$$[S_+ S_-] \quad , [S_+ S S_-], \quad [S^3]$$

which have zero charge under the U(1) of SU(5) x U(1) and will not break SO(10) gauge invariance.

We shall consider each of these two cases at finite temperature and potential. The first employing ξD breaking, the second 'F-type' breaking.

D Breaking of SU(5) x U(1) $\longrightarrow$ SU(5)

The part of the supersymmetric Lagrangian involving the super-field S (for ϕ, ψ_L), the U(1) gauge field (S is an SU(5) singlet) is

$$\mathcal{L}_S = -\left[S^* e^{2kV} S\right]_D + \xi\left[V\right]_D \tag{13.1}$$

where V is the U(1) vector supermultiplet (V_μ, λ , D) and k is the coupling strength of ψ_L and ϕ to the U(1) gauge field.

$$\begin{aligned} \mathcal{L}_S = {} & i\,\bar{\Psi}_L \gamma^\mu D_\mu \Psi_L + D_\mu \phi^+ D^\mu \phi \\ & + \sqrt{2}\,ik\left(\phi^+ \bar{\lambda}\Psi_L - \bar{\Psi}_L \lambda \phi\right) \\ & + \tfrac{1}{2} D^2 + k\,\phi^+ D \phi + \xi D \end{aligned} \tag{13.2}$$

where $$D_\mu \phi = \partial_\mu \phi + ik A_\mu \phi \tag{13.3}$$

and $$D_\mu \Psi_L = \partial_\mu \Psi_L + ik A_\mu \Psi_L \tag{13.4}$$

In constructing the quark-lepton sector Lagrangian we note that only couplings to the U(1) gauge field A_μ will be relevant - other couplings cannot affect the quadratic Lagrangian which is all that we need for a one loop calculation.

The supersymmetric Lagrangian we shall need is

$$\begin{aligned} \mathcal{L}_{quad} = {} & -\left[S_p^{5*} e^{2k_5 V} S_p^5\right]_D \\ & -\left[S_{10}^{pq*} e^{2k_{10} V} S_{10}^{pq}\right]_D \\ & -\left[S_1^* e^{2k_1 V} S_1\right]_D \end{aligned} \tag{13.5}$$

V is the U(1) vector supermultiplet (V_μ , λ , D)

S_p^5 is the superfield (ϕ_p^5 , $\Psi_{p,L}^5$) where $\Psi_{p,L}^5$ is the usual fermion $\underline{\bar{5}}$,

$$\Psi_{p,L}^5 = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ \nu_L \end{pmatrix} \tag{13.6}$$

S_{10}^{pq} is the superfield (ϕ_{pq}^{10}, $\left(\Psi_{10}^{pq}\right)_L$) where $\left(\Psi_{10}^{pq}\right)_L$ is the usual fermion $\underline{10}$

$$\left(\Psi_{10}^{pq}\right)_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & d_3 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{pmatrix}$$

S_1 is the superfield $(\phi_1, (\psi_1^c)_L)$ where $(\psi_1)_R = \nu_R$ the 'right-handed neutrino'.

After eliminating auxiliary fields we arrive at:

$$\begin{aligned} \mathcal{L}_{ql} = \; & i(\overline{\Psi}_L^5)_p \gamma^\mu D_\mu (\psi_L^5)_p + D_\mu \phi_p^{5+} D^\mu \phi_p^5 \\ & + \sqrt{2}\, i k_5 (\phi_p^{5+} \bar{\lambda} \psi_{L,p}^5 - \overline{\Psi}_{L,p}^5 \lambda \phi_p^5) \\ & + i \overline{\psi}_{L,pq}^{10} \gamma^\mu D_\mu \psi_{L,pq}^{10} + D_\mu \phi_{pq}^{10+} D^\mu \phi_{pq}^{10} \\ & + \sqrt{2}\, i k_{10} (\phi_{pq}^{10+} \bar{\lambda} \psi_{L,pq}^{10} - \overline{\Psi}_{L,pq}^{10} \lambda \phi_{pq}^{10}) \\ & + i \overline{\Psi}_L^1 \gamma^\mu D_\mu \psi_L^1 + D_\mu \phi^{1+} D_\mu \phi^1 \\ & + \sqrt{2}\, i k_1 (\phi^{1+} \bar{\lambda} \psi_L^1 - \overline{\Psi}_L^1 \lambda \phi^1) \\ & + \tfrac{1}{2} D^2 + D(k_5 \phi_p^{5+} \phi_p + k_{10} \phi_{pq}^{10+} \phi_{pq}^{10} + k_1 \phi^{1+} \phi^1) \end{aligned} \tag{13.8}$$

where the covariant derivatives are simply those for the U(1) factor i.e.

$$D_\mu = \partial_\mu + i k_5 A_\mu \quad \text{on } \underline{\overline{5}} \tag{13.9}$$

$$= \partial_\mu + i k_{10} A_\mu \quad \text{on } \underline{10} \tag{13.10}$$

$$= \partial_\mu + i k_1 A_\mu \quad \text{on } \underline{1} \tag{13.11}$$

These, for our purposes, are the only terms which are relevant. Other terms will not contribute to the quadratic Lagrangian since only ϕ, the scalar field, develops an expectation value.

The Lagrangian we use is:

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{4}V_{\mu\nu}V^{\mu\nu} + i\bar{\lambda}\gamma^{\mu}D_{\mu}\lambda \\ & + i\bar{\Psi}_L\gamma^{\mu}D_{\mu}\Psi_L + D_{\mu}\phi^{+}D^{\mu}\phi \\ & + \sqrt{2}\, ik(\phi^{+}\bar{\lambda}\Psi_L - \bar{\Psi}_L\lambda\phi) \\ & + i\bar{\Psi}^{5}_{L,p}\gamma^{\mu}D_{\mu}\Psi^{5}_{L,p} + D_{\mu}\phi^{5+}_{p}D^{\mu}\phi^{5}_{p} \\ & + \sqrt{2}\, ik_5(\phi^{5+}_{p}\bar{\lambda}\Psi^{5}_{L,p} - \bar{\Psi}^{5}_{L,p}\lambda\phi^{5}_{p}) \\ & + i\bar{\Psi}^{10}_{L,pq}\gamma^{\mu}D_{\mu}\Psi^{10}_{L,pq} + D_{\mu}\phi^{10+}_{pq}D^{\mu}\phi^{10}_{pq} \\ & + \sqrt{2}\, ik_{10}(\phi^{10+}_{pq}\bar{\lambda}\Psi^{10}_{L,pq} - \bar{\Psi}^{10}_{L,pq}\lambda\phi^{10}_{pq}) \\ & + i\bar{\Psi}'_L\gamma^{\mu}D_{\mu}\Psi'_L + D_{\mu}\phi'^{+}D^{\mu}\Psi' \\ & + \sqrt{2}\, ik_1(\phi'^{+}\bar{\lambda}\Psi'_L - \bar{\Psi}'_L\lambda\phi') \\ & - \tfrac{1}{2}\Big[\xi + k\phi^{+}\phi + k_5\phi^{5+}_{p}\phi^{5}_{p} \\ & \qquad + k_{10}\phi^{10+}_{pq}\phi^{10}_{pq} + k_1\phi'^{+}\phi'\Big]^2\end{aligned} \tag{13.12}$$

where the last term arises through the elimination of D terms via

$$\frac{\partial\mathcal{L}}{\partial D} = 0 \tag{13.13}$$

The Noether current for the global phase symmetry

$$\Psi_L \to e^{i\alpha}\Psi_L, \quad \phi \to e^{i\alpha}\phi$$

is

$$j^{\mu} = -\bar{\Psi}_L\gamma^{\mu}\Psi_L + i(D^{\mu}\phi^{+}\phi - \phi^{+}D^{\mu}\phi) \tag{13.14}$$

and so we introduce the chemical potential term μj^0 in the Hamiltonian. This leads, in the manner described in chapter 8, to the substitution in the Lagrangian

$$\partial_0 \to \partial_0 - i\mu \quad \text{in} \quad D_{\mu}\phi^{+}D^{\mu}\phi \tag{13.15}$$

and the addition of the term

$$-\mu\bar{\Psi}_L\gamma^{0}\Psi_L \tag{13.16}$$

to the Lagrangian.

The Noether current for the phase symmetry

$$\Psi^{5}_{L,p} \to e^{i\alpha}\Psi^{5}_{L,p}, \quad \phi^{5}_{p} \to e^{i\alpha}\phi^{5}_{p} \tag{13.17}$$

is $$j^{\mu}_{S} = -\bar{\Psi}^{5}_{L,p}\gamma^{\mu}\Psi^{5}_{L,p} + i\left(D^{\mu}\phi^{5+}_{p}\phi^{5}_{p} - \phi^{5+}_{p}D^{\mu}\phi^{5}_{p}\right) \tag{13.18}$$

The introduction of a term $-\mu_5 j^0_S$ to the Hamiltonian leads to the substitution

$$\partial_0 \to \partial_0 + i\mu_5 \quad \text{in} \quad D_\mu\phi^{5+}_p D^\mu\phi^5_p \tag{13.19}$$

in the Lagrangian and an extra term

$$+\mu_5\,\bar{\Psi}^5_{L,p}\gamma^0\Psi^5_{L,p} \tag{13.20}$$

For the phase symmetry

$$\Psi^{10}_{L,pq} \to e^{i\alpha}\Psi^{10}_{L,pq}\,,\quad \phi^{10}_{pq} \to e^{i\alpha}\phi^{10}_{pq} \tag{13.21}$$

we similarly alter the Lagrangian by

$$\partial_0 \to \partial_0 - i\mu_{10} \quad \text{in} \quad D_\mu\phi^{10+}_{pq}D^\mu\phi^{10}_{pq} \tag{13.22}$$

and add a term

$$-\mu_{10}\,\bar{\Psi}^{10}_{L,pq}\gamma^0\Psi^{10}_{L,pq} \tag{13.23}$$

For the phase symmetry

$$\Psi^1_L \to e^{i\alpha}\Psi^1\,,\quad \phi^1 \to e^{i\alpha}\phi^1 \tag{13.24}$$

we make the substitution

$$\partial_0 \to \partial_0 - i\mu_1 \quad \text{in} \quad D_\mu\phi^{1+}D^\mu\phi^1 \tag{13.25}$$

and add a term

$$-\mu_1\,\bar{\Psi}^1\gamma^0\Psi^1 \tag{13.26}$$

to the Lagrangian.

(The different sign on μ_5 is in agreement with conventions used by other authors e.g. (13.1))

For a U(1) gauge symmetry there is the symmetry

$$\Psi_L \to e^{ik\alpha}\Psi_L\,,\quad \phi \to e^{ik\alpha}\phi \tag{13.27}$$

$$\Psi^5_{L,p} \to e^{ik_5\alpha}\Psi^5_{L,p}\,,\quad \phi^5 \to e^{ik_5\alpha}\phi^5 \tag{13.28}$$

$$\Psi^{10}_{L,pq} \to e^{ik_{10}\alpha}\Psi^{10}_{L,pq}\,,\quad \phi^{10} \to e^{ik_{10}\alpha}\phi^{10} \tag{13.29}$$

$$\Psi^1_L \to e^{ik_1\alpha}\Psi^1_L\,,\quad \phi^1 \to e^{ik_1\alpha}\phi^1 \tag{13.30}$$

and there is the Noether current

$$
\begin{aligned}
j^{\mu} = {} & k\left[-\bar{\psi}_L \gamma^{\mu} \psi_L + i\left(D^{\mu}\phi^{+}\phi - \phi^{+}D^{\mu}\phi\right)\right] \\
& + k_5\left[-\bar{\psi}^5_{L,p}\gamma^{\mu}\psi^5_{L,p} + i\left(D^{\mu}\phi^{5+}_p\,\phi^5_p - \phi^{5+}_p D^{\mu}\phi^5_p\right)\right] \\
& + k_{10}\left[-\bar{\psi}^{10}_{L,pq}\gamma^{\mu}\psi^{10}_{L,pq} + i\left(D^{\mu}\phi^{10+}_{pq}\,\phi^{10}_{pq} - \phi^{10+}_{pq}D^{\mu}\phi^{10}_{pq}\right)\right] \\
& + k_1\left[-\bar{\psi}'_L\gamma^{\mu}\psi'_L + i\left(D^{\mu}\phi'^{+}\phi' - \phi'^{+}D^{\mu}\phi'\right)\right]
\end{aligned}
\tag{13.31}
$$

we introduce to the Hamiltonian the chemical potential $\tilde{\mu}$ by

$$\tilde{\mu} j^{0} \tag{13.32}$$

However, this is just a linear combination of the four chemical potentials μ_1, μ_5, μ_{10}, μ introduced above and so need not be introduced separately.If we were keeping chemical potentials corresponding to <u>exact</u> symmetries only then $\tilde{\mu}$ would be the only chemical potential required.

V_{eff} may be evaluated in the usual manner. We allow an expectation value for the time component of the U(1) vector gauge field,

$$V_c = \langle V_0 \rangle \tag{13.33}$$

and for the U(1) scalar field

$$\phi_c = \langle \phi \rangle \tag{13.34}$$

The effective potential to one loop order is found to be:

$$
\begin{aligned}
V_{eff} = {} & -\xi\frac{\pi^2 T^4}{90} + \frac{1}{2}\left(\xi + k\phi_c^2\right)^2 - \left(\mu - kV_c\right)^2\phi_c^2 \\
& + \frac{T^2}{12}\left[k\left(\xi + 7k\phi_c^2\right) + \left(5k_5 + 10k_{10} + k_1\right)\left(\xi + k\phi_c\right)^2\right] \\
& - \frac{T^2}{4}\left[\left(\mu - kV_c\right)^2 + 5\left(\mu_5 + k_5 V_c\right)^2 + 10\left(\mu_{10} - k_{10}V_c\right)^2 \right. \\
& \left. + \left(\mu_1 - k_1 V_c\right)^2\right]
\end{aligned}
\tag{13.35}
$$

where $$\xi = N_B + \frac{7}{8}N_F \tag{13.36}$$

where N_B, N_F are the numbers of degrees of freedom of boson and fermion fields respectively.

Neglecting terms like $k_i \xi T^2$ (they are at least an order smaller than other terms) and ξ^2 (a constant) we may rewrite V_{eff} by

$$
\begin{aligned}
V_{\text{eff}} = - \Big\{ & \frac{\pi^2 T^4}{90} - (\mu - kV_c)^2 \phi_c^2 + \tfrac{1}{2} k^2 \phi_c^4 \\
& + \Big[\xi k + \frac{T^2}{12} k (7k + 5k_5 + 10k_{10} + k_1) \Big] \phi_c^2 \\
& - \frac{T^2}{4} \Big[(\mu - kV_c)^2 + 5(\mu_5 + k_5 V_c)^2 + 10(\mu_{10} - k_{10} V_c)^2 \\
& + (\mu_1 - k_1 V_c)^2 \Big]
\end{aligned}
\tag{13.37}
$$

Minimising with respect to ϕ_c yields

$$
\phi_c = 0 \,, \quad \phi_c^2 = \frac{1}{k^2} \Big[-\xi k - \frac{T^2}{12} (7k + 10k_{10} + 5k_5 + k_1) k + (\mu - kV_c)^2 \Big]
\tag{13.38}
$$

For $T > T_c$ where

$$
T_c^2 = 12 . \frac{-\xi k + (\mu - kV_c)^2}{(7k + 10k_{10} + 5k_5 + k_1) k}
\tag{13.39}
$$

the only solution is $\phi_c = 0$ and the system is in the symmetric phase.

For $T < T_c$ the minimum of V_{eff} is for

$$
\phi_c^2 = \frac{1}{k^2} \Big[-\xi k + (\mu - kV_c)^2 - \frac{T^2}{12} . k (7k + 10k_{10} + 5k_5 + k_1) \Big]
\tag{13.40}
$$

and the U(1) symmetry is broken.

Number densities are given by

$$
- \frac{\partial V_{\text{eff}}}{\partial \mu_i} = n_i
\tag{13.41}
$$

and we have the condition for equilibrium

$$
\frac{\partial V_{\text{eff}}}{\partial V_c} = 0
\tag{13.42}
$$

These yield

$$
k n - k_5 n_5 + k_{10} n_{10} + k_1 n_1 = 0
\tag{13.43}
$$

Now

$$
k_5 = -\tfrac{3}{2} g \,, \quad k_{10} = \tfrac{1}{2} g \,, \quad k_1 = \tfrac{5}{2} g
\tag{13.44}
$$

and we obtain the same equilibrium condition as we did for the non- supersymmetric case (10.12), namely

$$
\frac{1}{2} n_{10} + \frac{3}{2} n_5 + \frac{5}{2} n_1 + \frac{k}{g} n = 0
\tag{13.45}
$$

'F - Type' Breaking of SU(5) x U(1) $\longrightarrow$ SU(5)

In this case the quark-lepton sector of the Lagrangian will be the same as for the ξ D breaking model.

We introduce a triplet of U(1) scalar superfields S, S_+ and S_- as in chapter 12 and the superpotential

$$\mathcal{L}_{PE} = fS + hSS_+S_- + \frac{\tilde{h}}{3}S^3 \qquad (13.46)$$

The Lagrangian for the scalar sector is derived from (12.53) except that now the D term is

$$-\tfrac{1}{2}\left(k|\phi_+|^2 - k|\phi_-|^2 + k_1|\phi_1|^2 + k_5|\phi_5|^2 + k_{10}|\phi_{10}|^2\right) \qquad (13.47)$$

The chemical potentials for the 5,10 and 1 are as in the previous section. For the scalar sector we introduce a chemical potential μ into the Hamiltonian via the term

$$\mu\left[-\bar{\psi}_{+L}\gamma^0\psi_{+L} + i(D^0\phi_+^{\dagger}\phi_+ - \phi_+^{\dagger}D^0\phi_+\right]$$
$$-\mu\left[-\bar{\psi}_{-L}\gamma^0\psi_{-L} + i(D^0\phi_-^{\dagger}\phi_- - \phi_-^{\dagger}D^0\phi_-\right] \qquad (13.48)$$

The evaluation of V_{eff} follows as usual:

we define $\quad V_c := \langle V_o \rangle \qquad (13.49)$

$$\langle\phi_+\rangle = \langle\phi_-\rangle = v \qquad (13.50)$$

$$\langle\phi\rangle = 0 \qquad (13.51)$$

and find

$$\begin{aligned} V_{eff} = {} & \tfrac{1}{8}(2f + hv^2)^2 + \tfrac{1}{4}h^2v^4 - 2v^2(\mu - kV_c)^2 \\ & + \tfrac{T^2}{16}(16k^2 + h^2)v^2 - \tfrac{1}{2}T^2(\mu - kV_c)^2 \\ & - \tfrac{T^2}{4}\left[5(\mu_5 + k_5V_c)^2 + 10(\mu_{10} - k_{10}V_c)^2 + (\mu_1 - k_1V_c)^2\right] \end{aligned} \qquad (13.52)$$

As usual we minimise V_{eff} with respect to v to give

$$v = 0 \qquad (13.53a)$$

or

$$v^2 = \frac{2}{3h^2}\left[4(\mu - kV_c)^2 - f - \frac{T^2}{8}(16k^2 + h^2)\right] \qquad (13.53b)$$

The critical temperature is then given by

$$T_c^2 = \frac{8}{16k^2+h^2}\left[4(\mu - kV_c)^2 - f\right] \qquad (13.54)$$

For $T > T_c$ $v = 0$ and the system is in the symmetric phase. For $T < T_c$ the solution (13.53b) for v^2 is energetically advantageous.

As before we can obtain the equilibrium condition

$$k\,n - k_5\,n_5 + k_{10}\,n_{10} + k_1\,n_1 = 0 \qquad (13.55)$$

as in (13.43).

Chapter 13: References

13.1 Rajpoot (1980) Phys.Rev. D22 2244

Chapter 14: Symmetry Anti-Restoration

In a recent paper Masiero et al (14.1) showed that the simple supersymmetric model, with a global U(1) symmetry, described by the superpotential

$$h\,S\,S_+\,S_- \quad - f\,S \quad + \frac{\tilde{h}}{3}\,S^3 \tag{14.1}$$

exhibits interesting symmetry properties.

The usual effect of high temperatures is to favour the restoration of symmetries which were broken at zero temperature.

The initial effect in the model of Masiero et al is to lift a degeneracy of symmetric and anti-symmetric minima which existed at zero temperature (because of supersymmetry) in favour of an anti-symmetric phase rather than a symmetric phase. (Both phases do, however, correspond to non-zero expectation values for some of the fields.) Masiero et al refer to this phenomenon as symmetry anti-restoration. At a higher temperature symmetry is eventually restored.

The evaluation of V_{eff} in a previous chapter now allows us to examine property for the gauged theory and to study the effect of finite chemical potential, both for the theory with global U(1) symmetry and for the corresponding gauge theory.

The usual effect of chemical potentials is to favour symmetry restoration for a global symmetry and symmetry breaking for a gauge symmetry.

From (12.54) and (12.56) we see that the Lagrangian

$$\begin{aligned}\mathcal{L} = \mathcal{L}_0 &- \left[S_+^* e^{2eV} S_+ + S_-^* e^{-2eV} S_- + S^* S\right]_D \\ &+ \left[h\,S S_+ S_- + \frac{\tilde{h}}{3} S^3 - f S\right]_F \end{aligned} \tag{14.2}$$

leads to the effective potential

$$
\begin{aligned}
V_{eff} = \; & \frac{1}{8} h^2 a^2 (a_+^2 + a_-^2) \\
& + \frac{1}{8} (-2f + h a_+ a_- + \tilde{h} a^2)^2 \\
& + \frac{1}{2} e^2 (a_+^2 - a_-^2)^2 - (\mu + eV_c)^2 (a_+^2 + a_-^2) \\
& + \frac{T^2}{32} \left[(16e^2 + h^2)(a_+^2 + a_-^2) + (2\tilde{h}^2 + h^2) a^2 \right] \\
& - \frac{T^2}{2} (\mu + eV_c)^2
\end{aligned} \tag{14.3}
$$

where μ, V_c, a, a_+, a_- are as defined in chapter 12.

Global U(1) Theory

We can recover the model used by Masiero et al by considering the limit $e \to 0$, $\mu = 0$.

Then

$$
\begin{aligned}
V_{eff} = \; & \frac{1}{8} h^2 a^2 (a_+^2 + a_-^2) \\
& + \frac{1}{8} (-2f + h a_+ a_- + \tilde{h} a^2)^2 \\
& + \frac{T^2}{32} \left[h^2 (a_+^2 + a_-^2) + (2\tilde{h}^2 + h^2) a^2 \right]
\end{aligned} \tag{14.4}
$$

$$V_{eff} = V^{o}_{eff} + V^{T}_{eff}$$

$$V^{o}_{eff} = \frac{1}{8} h^2 a^2 (a_+^2 + a_-^2) + \frac{1}{8} (-2f + h a_+ a_- - \tilde{h} a^2)^2 \tag{14.5}$$

$$V^{T}_{eff} = \frac{T^2}{32} \left[h^2 (a_+^2 + a_-^2 + a^2) + 2\tilde{h}^2 a^2 \right] \tag{14.6}$$

$$V^{o}_{eff} = 0 \quad \text{when} \quad a_+ = a_- = 0, \; a^2 = \frac{2f}{h} \tag{14.7}$$

$$\text{and} \quad a_+ a_- = \frac{2f}{g}, \quad a = 0 \tag{14.8}$$

The minimum in (14.7) corresponds to the U(1) unbroken phase, that in (14.8) breaks U(1).

Consider first the asymmetric phases of V_{eff} where $a_+ = a_- = 0$.

Minimising $V_{eff}(a, 0, 0)$ with respect to a yields, for a,

$$a = 0 \quad \text{for} \quad T^2 > T_1^2 = \frac{16 \tilde{h} f}{h^2 + 2\tilde{h}^2} \tag{14.9}$$

$$\text{and} \quad a = 0 \;\text{ or }\; a = A = \pm \left[\frac{2}{\tilde{h}^2} \left(\tilde{h} f - \frac{h^2 + 2\tilde{h}^2}{16} T^2 \right) \right]^{1/2} \tag{14.10}$$

for $T < T_1$

$$V_{eff}(0, 0, 0) = \tfrac{1}{2} f^2 \tag{14.12}$$

$$V_{eff}(A, 0, 0) = \tfrac{1}{2} f^2 - \frac{1}{2\tilde{h}^2}\left[\tilde{h}f - \frac{T^2}{16}(h^2 + 2\tilde{h}^2)\right] \tag{14.13}$$

Hence in the unbroken phase

$$\begin{aligned} V_{eff} &= V_{eff}(0, 0, 0) = \tfrac{1}{2} f^2 \qquad \text{for } T > T_1 \\ &= V_{eff}(A, 0, 0) = \tfrac{1}{2} f^2 - \left[\tilde{h}f - \frac{T^2}{16}(h^2 + 2\tilde{h}^2)\right]^2 \\ &\qquad \text{for } T < T_1 \end{aligned} \tag{14.14}$$

Now we consider the U(1) broken phase

$$a_+ = a_- = v \ , \quad a = 0 \tag{14.15}$$

Minimising $V_{eff}(0, v, v)$ with respect to v^2 yields, for v^2,

$$v^2 = \frac{2f}{h} - \frac{T^2}{4} \quad \text{for } T^2 < T_2^2 = \frac{8f}{h} \tag{14.16}$$

$$V_{eff}(0, v, v) = \frac{f^2}{2} - \frac{1}{2}\left(f - \frac{hT^2}{8}\right)^2 \quad T < T_2 \tag{14.17}$$

Now $V_{eff}(0, v, v) < V_{eff}(0, 0, 0)$ so in the region $T_1 < T < T_2$ the system is in the U(1) broken phase.

Now consider the region $T < T_1$:

$V_{eff}(0, v, v) < V_{eff}(A, 0, 0)$ for

$$T^2 < \frac{32\tilde{h}f}{h^2 + 2\tilde{h}^2 + 2h\tilde{h}} \tag{14.18}$$

However, for all f, h, $\tilde{h}$

$$\frac{32\tilde{h}f}{h^2 + 2h\tilde{h} + 2\tilde{h}^2} > \frac{16\tilde{h}f}{h^2 + 2\tilde{h}^2} = T_1^2 \tag{14.19}$$

so $V_{eff}(0, v, v) < V_{eff}(A, 0, 0)$ for $T < T_1$ (14.20)

The situation is as follows:

at $T = 0$ $a_+ = a_- = 0$, $a \neq 0$ and there is U(1) symmetry.

for $T < T_2 = \left(\frac{8f}{h}\right)^{1/2}$ the U(1) symmetry is broken and

$a_+ = a_- = v$, $a = 0$ gives the minimum of V_{eff}.

for $T > T_2$ the U(1) symmetry is restored and $a_+ = a_- = a = 0$.

Gauged U(1) Theory

From (14.3) we find, for a gauged U(1) theory at zero density

$$
\begin{aligned}
V_{eff} = \; & \frac{1}{8} h^2 a^2 (a_+^2 + a_-^2) \\
& + \frac{1}{8}\left(-2f + h a_+ a_- + \hat{h} a^2\right)^2 \\
& + \frac{1}{2} e^2 (a_+^2 - a_-^2) \\
& + \frac{T^2}{32}\left[(16e^2 + h^2)(a_+^2 + a_-^2) + (2\hat{h}^2 + h^2) a^2\right]
\end{aligned}
\tag{14.21}
$$

$$
V_{eff}(a,0,0) = \frac{1}{8}(-2f + \hat{h} a^2)^2 + \frac{T^2}{32}(2\hat{h}^2 + h^2) a^2 \tag{14.22}
$$

which as before gives

$$
\begin{aligned}
V_{eff} &= V_{eff}(0,0,0) = \frac{1}{2} f^2 && \text{for } T > T_1 \\
&= V_{eff}(A,0,0) = \frac{1}{2} f^2 - \frac{1}{2\hat{h}^2}\left[\hat{h} f - \frac{T^2}{16}(h^2 + 2\hat{h}^2)\right]^2 && \text{for } T < T_1
\end{aligned}
\tag{14.23}
$$

where
$$
T_1^2 = \frac{16\hat{h} f}{h^2 + 2\hat{h}^2} \tag{14.24}
$$

and
$$
A^2 = \frac{2}{\hat{h}^2}\left(\hat{h} f - \frac{h^2 + 2\hat{h}^2}{16} T^2\right) \tag{14.25}
$$

$$
\begin{aligned}
V_{eff}(0,v,v) = \; & \frac{1}{8}(-2f + h v^2)^2 \\
& + \frac{T^2}{16}(16e^2 + h^2) v^2
\end{aligned}
\tag{14.26}
$$

$$
\partial V_{eff} / \partial v^2 = 0
$$

$$
\Rightarrow v^2 = \frac{2f}{h} - \frac{h^2 + 16e^2}{4h^2} T^2 \quad \text{for } T < T_2 \tag{14.27}
$$

where
$$
T_2^2 = \frac{8fh}{h^2 + 16e^2} \tag{14.28}
$$

and

$$V_{eff}(0, v, v) = \frac{1}{2}f^2 - \frac{1}{2}\left(f - \frac{h^2 + 16e^2}{8h}T^2\right)^2 . \tag{14.29}$$

Now in $T_1 < T < T_2$

$$V_{eff}(0, v, v) < V_{eff}(0, 0, 0)$$

For $T < T_1$

$$V_{eff}(0, v, v) < V_{eff}(A, 0, 0)$$

provided

$$T^2 < \frac{32\tilde{h}hf}{h(h^2 + 2\tilde{h}^2) + 2\tilde{h}(h^2 + 16e^2)} \tag{14.30}$$

Also

$$\frac{32\tilde{h}hf}{h(h^2 + 2\hat{h}^2) + 2\tilde{h}(h^2 + 16e^2)} > T_1^2 = \frac{16\tilde{h}f}{h^2 + 2\tilde{h}^2}$$

provided

$$e^2 < \frac{1}{32}\frac{h}{\tilde{h}}\left[\tilde{h}^2 + (h - \tilde{h})^2\right] \tag{14.31}$$

Provided that this relationship is satisfied the system exhibits the same unusual behaviour as the global theory.

The asymmetric phase ($a_+ = a_- = v$, $a = 0$) and the symmetric phase ($a_+ = a_- = 0$, $a = A$) are degenerate at zero temperature. Finite temperature lifts the degeneracy in favour of the asymmetric phase for $T < T_2$. (Symmetry anti-restoration)

At $T = T_2$, there is a second order phase transition to the phase ($a_+ = a_- = a = 0$) which is the phase for all $T > T_2$.

Finite Chemical Potential

We move on to study the model of Masiero et al with the intro -duction of finite chemical potentials as well as the U(1) gauge.

From (14.3)

$$\begin{aligned} V_{eff} &= \tfrac{1}{8} h^2 a^2 (a_+^2 + a_-^2) \\ &\quad + \tfrac{1}{8}(-2f + h a_+ a_- + \tilde{h} a^2)^2 \\ &\quad + \tfrac{1}{2} e^2 (a_+^2 - a_-^2)^2 - (\mu + eV_c)^2 (a_+^2 + a_-^2) \\ &\quad + \frac{T^2}{32}\left[(16e^2 + h^2)(a_+^2 + a_-^2) + (2\tilde{h}^2 + h^2) a^2\right] \\ &\quad - \frac{T^2}{2}(\mu + eV_c)^2 \end{aligned} \qquad (14.32)$$

$$\begin{aligned} V_{eff}(a,0,0) &= \tfrac{1}{8}(-2f + \tilde{h} a^2)^2 \\ &\quad + \frac{T^2}{32}(2\tilde{h}^2 + h^2) a^2 - \frac{T^2}{2}(\mu + eV_c)^2 \end{aligned} \qquad (14.33)$$

which leads to

$$V_{eff} = V_{eff}(0,0,0) = \tfrac{1}{2} f^2 - \tfrac{1}{2} T^2 (\mu + eV_c)^2$$

for $T > T_1$ (14.34)

$$= V_{eff}(A,0,0) = \frac{1}{2\tilde{h}^2}\left[\tilde{h} f - \frac{T^2}{16}(h^2 + 2\tilde{h}^2)\right]^2 - \tfrac{1}{2} T^2 (\mu + eV_c)^2$$

for $T < T_1$ (14.35)

where
$$T_1^2 = \frac{16 \tilde{h} f}{h^2 + 2\tilde{h}^2} \qquad (14.36)$$

and
$$A^2 = \frac{2}{\tilde{h}^2}\left(\tilde{h} f - \frac{h^2 + 2\tilde{h}^2}{16} T^2\right). \qquad (14.37)$$

$$\begin{aligned} V_{eff}(0,v,v) &= \tfrac{1}{8}(-2f + h v^2)^2 - 2(\mu + eV_c)^2 v^2 \\ &\quad + \frac{T^2}{16}\left[(16e^2 + h^2) v^2\right] - \tfrac{1}{2} T^2 (\mu + eV_c)^2 \end{aligned} \qquad (14.38)$$

$$\frac{\partial V_{eff}}{\partial v^2} = 0$$
$$\Rightarrow v^2 = \frac{1}{h^2}\left[2fh - \frac{16e^2+h^2}{4}T^2 + 8(\mu+eV_c)^2\right]$$

(14.39)

for $T < T_2$

where $$T_2^{\,2} = 4.\,\frac{2fh + 8(\mu+eV_c)^2}{16e^2+h^2}\,. \quad (14.40)$$

and $V_{eff}(0,v,v)$
$$= \frac{1}{2}f^2 - \frac{1}{8h^2}\left[2fh - \frac{16e^2+h^2}{4}T^2 + 8(\mu+eV_c)^2\right]^2 - \frac{1}{2}T^2(\mu+eV_c)^2. \quad (14.41)$$

Now, in the range $T_1 < T < T_2$

$$V_{eff}(0,\ v,\ v) < V_{eff}(0,\ 0,\ 0) \quad (11.42)$$

For $T < T_1$

$$V_{eff}(0,\ v,\ v) < V_{eff}(A,\ 0,\ 0) \quad (14.43)$$

provided that the inequality (14.31) holds.

Thus there is a range of temperature, starting from T = 0 over which the system is in the anti-symmetric phase, and the symmetry anti-restoration phenomenon still occurs in the pres ence of the chemical potential.

We next consider whether symmetry restoration at T_2 can be prevented by the pres ence of the chemical potential.

The density, n, associated with the chemical potential, , is

$$n = -\frac{\partial V_{eff}}{\partial \mu} = (\mu+eV_c)\left[T^2 + 2(a_+^2 + a_-^2)\right]$$

(14.44)

For equilibrium

$$\frac{\partial V_{eff}}{\partial V_c} = -J_c = -e(\mu+eV_c)\left[T^2+2(a_+^2+a_-^2)\right] \quad (14.45)$$

where J_c is an external source introduced to stablize the system.

Then

$$J_c = e\ n \quad (14.46)$$

From (14.40)

$$T_2^2 = 4 \cdot \frac{2fh + 8(\mu + eV_c)^2}{16e^2 + h^2} \quad . \tag{14.47}$$

Eliminating $\mu + eV_c$ by using (14.44) which gives, for n at T_2,

$$n = (\mu + eV_c) T^2 \tag{14.48}$$

then we can see that a number density

$$n_c \sim \left(\frac{16e^2 + h^2}{32} \right)^{1/2} T^3 \tag{14.49}$$

will prevent symmetry restoration from ever taking place.

As we have seen in other models, this is the usual effect of a chemical potential in a gauge theory.

For the model discussed by Masiero et al (ref 14.1) with only global U(1) symmetry, a chemical potential may be introduced as in (12.45), replacing the covariant derivatives by ordinary derivatives.

The effective potential is then given by (14.32) with e = 0, again neglecting one loop corrections to the last term.

An exactly analogous discussion to the one given above for the gauged theory shows that the asymmetric minimum is the lowest minimum wherever it exists, so that the symmetry anti-restoration phenomenon continues in the prescence of the chemical potential. Also, for a critical density n_c given by (14.49) with e = 0, symmetry restoration is prevented from occuring even at very high temperatures.

This is very different to the usual situation in theories with a global symmetry (Linde ref.(14.2)). Normally one has a chemical potential coupled to a conserved density which involves only fermions. Then there is no contribution of the chemical potential to V_{eff} at the tree level and terms like

$$(\mu + eV_c)^2 (a_+^2 + a_-^2)$$

do not arise.

Instead the leading term of the form $\mu^2(a_+^2+a_-^2)$ arises at one loop order, and is suppressed by a factor of e^2. Moreover, the contribution is of opposite sign to the one occuring here and promotes symmetry restoration rather than preventing it.

To summarize; provided that the gauge coupling constant obeys the inequality (14.31), the model discussed by Masiero et al continues to exhibit symmetry anti-restoration when it is gauged. It also occurs in the prescence of a chemical potential.

Symmetry restoration can be prevented by a sufficiently large chemical potential,both in the gauged and un-gauged theories. This would normally be expected for the gauge theory (Linde (14.2)), but is the reverse of the usual behaviour with only a global symmetry. It should be a general feature of supersymmetric theories at finite chemical potential, resulting from the necessary prescence of bosonic chemical potentials as well as fermionic chemical potentials.

Chapter 14: References

14.1 Masiero et al (1983) CERN preprint TH 3757

14.2 Linde (1979) Rep.Prog.Phys. 42 289

Appendix IIA: Matsubara Frequency Sums and Momentum Integrals

Matsubara Frequency Sums

In evaluating the one loop effective potential by the path integral method (refs. A.1,A.2,A.3,A.4 and A.5) we encounter Matsubara frequency sums for bosons of the type

$$\sum_{n\,\mathrm{even}} \ln\left[(\omega_n + ia)^2 + x^2\right] \tag{A.1}$$

where

$$x^2 = \mathbf{p}^2 + R \tag{A.2}$$

$$\omega_n = n\pi/\beta \,, \quad n \text{ even} \tag{A.3}$$

and

$$\beta = 1/k_B T \tag{A.4}$$

Now

$$\frac{d}{dx}\sum_{n\,\mathrm{even}} \ln\left[(\omega_n + ia)^2 + x^2\right] = \sum_n \frac{2x}{(\omega_n + ia)^2 + x^2}$$

$$= \sum_n \frac{i}{\omega_n + ia + ix} - \sum_n \frac{i}{\omega_n + ia - ix} \tag{A.5}$$

It may be shown by contour integration (Fetter and Walecka ref. A.6) that

$$\sum_n \frac{1}{i\omega_n - A} = -\frac{\beta}{e^{\beta A} - 1} \tag{A.6}$$

i.e.

$$\sum_n \frac{i}{\omega_n + iA} = \frac{\beta}{e^{\beta A} - 1} \tag{A.7}$$

Hence (A.5) becomes

$$\frac{d}{dx}\sum_n \ln\left[(\omega_n + ia)^2 + x^2\right] = \frac{\beta}{e^{\beta(a+x)} - 1} - \frac{\beta}{e^{\beta(a-x)} - 1} \tag{A.8}$$

Performing the x integration we find

$$\sum_n \ln\left[(\omega_n + ia)^2 + x^2\right] = \beta(x - a) + \ln\left[1 - e^{\beta(a-x)}\right] + \ln\left[1 - e^{-\beta(a+x)}\right] + x\text{ - independent constant} \tag{A.9}$$

The constant in (A.9) is temperature dependent and finite. Fortunately it cancels exactly with the temperature dependent part of N'(β) when we evaluate Z (as discussed in detail by Bernard ref A.4).

For the case of fermions the result

$$\sum_n \frac{1}{i\omega_n - A} = \frac{\beta}{e^{\beta A} - 1} \tag{A.10}$$

where $\omega_n = n\pi/\beta \quad , \quad n \text{ odd}$ (A.11)

leads us (through similar steps) to the result

$$\sum_{n\,\text{odd}} \ln\left[(\omega_n + ia)^2 + x^2\right] = \beta(x - a) + \ln\left[1 + e^{\beta(a-x)}\right] + \ln\left[1 + e^{-\beta(a+x)}\right] \tag{A.12}$$

The temperature dependent constant cancels against $N'(\beta)$ when Z is calculated.

Momentum Integrals

In calculating the one loop effective potential we require expressions for

$$\frac{1}{2}\int d^3p \sum_n \left[\ln\left\{(\omega_n + ia)^2 + p^2 + R\right\}\right] \quad , \quad n \text{ even} \tag{A.13}$$

for bosons and

$$\frac{1}{2}\int d^3p \sum_n \left[\ln\left\{(\omega_n + ia)^2 + p^2 + R\right\}\right] \quad , \quad n \text{ odd} \tag{A.14}$$

for fermions.

We use the results of Elze et al (ref.A.7) who performed the integrations analytically to give, for bosons,

$$\frac{1}{2}\int d^3p \sum_n \left[\ln\left\{(\omega_n + ia)^2 + p^2 + R\right\}\right] \cong -\frac{\pi^2 T^4}{90} + \frac{RT^2}{24} - \frac{a^2 T^2}{12} \tag{A.15}$$

and, for fermions,

$$\frac{1}{2}\int d^3p \sum_n \left[\ln\left\{(\omega_n + ia)^2 + p^2 + R\right\}\right] \cong \frac{7}{8}\frac{\pi^2 T^4}{90} + \frac{a^2 T^2}{24} - \frac{RT^2}{48} \tag{A.16}$$

In both cases we have ignored an infinite term which contributes to the renormalization of the source term in the connected generating functional and does not enter the effective potential.

Appendix IIA: References

A.1 Kirzhnits and Linde (1972) Phys.Lett. 42B 471

A.2 Weinberg (1974) Phys.Rev. D9 3357

A.3 Dolan and Jackiw (1974) Phys.Rev. D9 3320

A.4 Bernard (1974) Phys.Rev. D9 3312

A.5 Iliopoulus,Itzykson and Martin (1975) Rev.Mod.Phys. 47

A.6 Fetter and Walecka (1971) "Quantum Theory of Many Particle Systems" McGraw Hill, New York.

A.7 Elze, Greiner and Rafelski (1980) J.Phys G:Nucl.Phys. 6 L149

Acknowledgements

I am extremely grateful to my supervisor, Dr. Alex Love for all his help and patience, and to the Science and Engineering Research Council for their financial support.

I would also like to thank the staff of the Physics Department of Bedford College, especially Mrs Pearson for her help with the production of this thesis; and Melanie Harrison for all her help and encouragement.